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Geometria algebrica. — *Theta loci and deformation theory.* Nota (*) di Claudio Fontanari, presentata dal Socio E. Arbarello.

Abstract. — We investigate deformation-theoretical properties of curves carrying a half-canonical linear series of fixed dimension. In particular, we improve the previously known bound on the dimension of the corresponding loci in the moduli space and we obtain a natural description of the tangent space to higher theta loci.

Key words: Theta characteristics; Infinitesimal deformations; Moduli of curves.

Riassunto. — *Luoghi definiti da serie lineari semicanoniche e teoria delle deformazioni.* In questo lavoro si prendono in esame alcune proprietà infinitesimali delle curve con una serie lineare semicanonica di dimensione prefissata. In particolare, si migliora la stima presente in letteratura sulla dimensione dei luoghi corrispondenti nello spazio dei moduli e si perviene a una naturale descrizione dello spazio tangente ai luoghi definiti da serie lineari sottocanoniche di ordine superiore.

1. Introduction

The slang expression in the title refers to some very natural but still mysterious subvarieties of the moduli space of curves. Namely, define $\mathfrak{Th}_g^r = \{C :$ there exists a line bundle L on C with $h^0(C, L) = r + 1$ such that $L \otimes L = K_C\} \subseteq \mathfrak{M}_g$. The first steps towards a purely algebraic theory of theta-characteristics were moved as usual by David Mumford, in the paper [6] going back to 1971. About ten years later, Joe Harris laid once for all in [5] the fundations of the subject, so giving a strong impulse to further investigations. In particular, he posed the following question: «What is the dimension of the subvarieties $\mathfrak{Th}_g^r$ of moduli? Is it the case that if $g >> r$ then $\mathfrak{Th}_g^r$ has codimension $r(r+1)/2$ in $\mathfrak{M}_g$?» (see [5, p. 617]). In [9] Montserrat Teixidor i Bigas was able to answer affirmatively Harris' question at least for $r \leq 4$; indeed, the solution of the case $r = 4$ was obtained as a by-product of the following more general result:

Theorem 1 [9, (2.13)]. *If* $r \geq 4$ *and* $g \geq \max(12r - 22\, , \frac{1}{2}(r^2 + 3r + 2))$, *then* $\dim \mathfrak{Th}_g^r \leq 3g - 4r + 3$.

In the paper [9] it is also observed that «the bounds given are not the best possible and could be improved by ad hoc methods». Of course this fact is not relevant in order to address Harris' question, but the search for sharper numerical hypotheses, which is essentially the subject of the present *Note*, turns out to be interesting from other points of view. Here it is our best result in this direction:

Theorem 2. *If* $r \geq 3$ *and* $g \geq 6r - 11$, *then* $\dim \mathfrak{Th}_g^r \leq 3g - 4r + 3$.

(*) Pervenuta in forma definitiva all'Accademia il 29 ottobre 2001.

Although we apply the same deformation-theoretical methods, developed by Enrico Arbarello and Maurizio Cornalba in [1] and [2], our approach is quite different from Teixidor's; the spirit of our proof is perhaps closer to the old paper [8] of Beniamino Segre. Moreover, we make essential use of the following result, proved in Section 3, which generalizes previous work by Nagaraj (see [7]):

Theorem 3. *Set* $\mathfrak{T}\mathfrak{h}^r_{g,m} = \{C : C$ *is a smooth curve of genus* g *with a line bundle* $\widetilde{L}$ *such that* $h^0(C, \widetilde{L}) = r+1$ *and* $2m\widetilde{L} = K\}$*. Define*

$$\widetilde{\mu} : H^0(C, \widetilde{L}) \otimes H^0(C, K - \widetilde{L}) \longrightarrow H^0(C, 2K)$$

$$\sigma \otimes \tau \longmapsto (2m-1)\sigma d\tau - \tau d\sigma \,.$$

Then $T_C\left(\mathfrak{T}\mathfrak{h}^r_{g,m}\right) = (\operatorname{Coker} \widetilde{\mu})^*$.

We work over the field $\mathbb{C}$ of complex numbers.

2. Notation and preliminaries

Almost all the technical tools needed in the present work are already contained in the well-known paper [1] by Enrico Arbarello and Maurizio Cornalba. Here we limit ourselves to briefly recall the set-up, by translating into English (almost verbatim) some salient passages from [1]. We will be interested in the deformation theory of couples (C, L), where C is a smooth, complex, irreducible algebraic curve and L is a line bundle on C. By an *infinitesimal deformation* of $L \to C$ we will mean the datum of a deformation of C, $X \to \operatorname{Spec} \mathbb{C}[\varepsilon]$, and of a line bundle $\mathcal{L}$ on X whose restriction to C is isomorphic to L. Now, let Σ_L be the rank 2 locally free $\mathcal{O}_C$-module whose sections are the differential operators, of order at most equal to one, acting on sections of L. To every infinitesimal deformation of $L \to C$ is associated an element of $H^1(C, \Sigma_L)$, which is called its *Kodaira-Spencer class*. This association induces a bijective correspondence between the set of equivalence classes of infinitesimal deformations of $L \to C$ and $H^1(C, \Sigma_L)$. Now, given $\sigma \in H^1(C, \Sigma_L)$, let $\mathcal{L} \to X \to \operatorname{Spec} \mathbb{C}[\varepsilon]$ be the corresponding infinitesimal deformation of $L \to C$. We have a natural cup product

$$H^1(C, \Sigma_L) \otimes H^0(C, L) \to H^1(C, L) \tag{1}$$

and an exact sequence

$$0 \to \mathcal{O}_C \to \Sigma_L \to \Theta_C \to 0$$

where Θ_C is the tangent sheaf on C. We recall also the fundamental applications

$$\mu_0 : W \otimes H^0(C, K - L) \longrightarrow H^0(C, K)$$

where $W \subseteq H^0(C, L)$ is a $(r+1)$-dimensional subspace, and

$$\mu : W \otimes H^0(C, K - L) \longrightarrow H^0(C, K \otimes \Sigma_L^*)$$

defined by duality from (1). The linear map

$$\mu_1 : \operatorname{Ker} \mu_0 \longrightarrow H^0(C, 2K)$$

is defined by the following commutative diagram:

$$\begin{array}{ccc} & & 0 \\ & & \uparrow \\ W \otimes H^0(C, K-L) & \xrightarrow{\mu_0} & H^0(C,K) \\ \| & & \uparrow \\ W \otimes H^0(C, K-L) & \xrightarrow{\mu} & H^0(C, K \otimes \Sigma_L^*) \\ \uparrow & & \uparrow \\ \operatorname{Ker}\mu_0 & \xrightarrow{\mu_1} & H^0(C, 2K) \\ & & \uparrow \\ & & 0 \end{array}$$

(see [1, p. 18]); locally we have $\mu_1(\sigma \otimes \tau) = d\sigma \otimes \tau = \tau d\sigma$. As in [2], let us denote by $\mathfrak{M}^1_{g,k}$ the subvariety of $\mathfrak{M}_g$ whose points correspond to curves admitting a linear series of degree k and dimension one. There is a natural identification:

$$T_C(\mathfrak{M}^1_{g,k}) = (\operatorname{Coker}\mu_1)^* \tag{2}$$

(see [3, 8.22]). Moreover, if $W = < s, t > \subseteq H^0(C, L)$, the Base-Point-Free Pencil Trick (see [4, p. 126]) provides an explicit isomorphism:

$$\begin{array}{rcl} H^0(C, K-2L) & \xrightarrow{\simeq} & \operatorname{Ker}\mu_0 \\ \rho & \longmapsto & s \otimes \rho t - t \otimes \rho s \end{array}$$

and the map μ_1 may be rewritten as follows:

$$\begin{array}{rcl} \mu_1 : H^0(C, K-2L) & \longrightarrow & H^0(C, 2K) \\ \rho & \longmapsto & \rho t^2 d(\frac{s}{t}) \,. \end{array} \tag{3}$$

3. Tangent spaces to higher theta loci

This section is entirely devoted to the proof of Theorem 3. The argument naturally splits into three steps. In the first and the third one we closely follow the proof of Theorem 1 in [7]; in the second step, instead, we develop a completely independent treatment, which seems to be more direct and geometric in nature.

3.1. *First step.*

Here we fix the set-up. Chosen an affine covering $\{U_1, U_2\}$ of C such that

$$\widetilde{L}(U_1) \cong \mathcal{O}_C(U_1)e_1$$

$$\widetilde{L}(U_2) \cong \mathcal{O}_C(U_2)e_2$$

we have

$$K(U_1) \cong \mathcal{O}_C(U_1)hda \ \cong \mathcal{O}_C(U_1)e_1^{2m}$$
$$K(U_2) \cong \mathcal{O}_C(U_2)h'da' \cong \mathcal{O}_C(U_2)e_2^{2m}$$

where hda and $h'da'$ are 1-forms which locally generate the canonical sheaf K. Moreover, if $\alpha_{12} \in \mathcal{O}_C(U_1 \cap U_2)^*$ is the transition function of $\widetilde{L}$, then α_{12}^{2m} is the transition function of K.

If $D \in H^1(C, T_C) \cong H^0(C, 2K)^*$, D defines a derivation $D : \mathcal{O}_C(U_1 \cap U_2) \to \to \mathcal{O}_C(U_1 \cap U_2)$ and so an infinitesimal deformation $C[\varepsilon]$ of C, obtained by gluing together $\mathrm{Spec}(\mathcal{O}_C(U_1) \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2))$ and $\mathrm{Spec}(\mathcal{O}_C(U_2) \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2))$ along $\mathrm{Spec}(\mathcal{O}_C(U_1 \cap \cap U_2) \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2))$ via the function $f \mapsto f + \varepsilon D(f)$.

Lemma 1. *Let $K[\varepsilon]$ be the canonical bundle of $C[\varepsilon]$. Then the transition function of $K[\varepsilon]$ is given by*

$$\alpha_{12}^{2m}\left(1 + \varepsilon\left(\frac{d(D(a))}{da} + \frac{D(h)}{h}\right)\right).$$

Proof. This is just a straightforward verification; if β_{12} is the sought-for cocycle, then

$$\begin{aligned}\beta_{12}h'da' = (h + \varepsilon D(h))d(a + \varepsilon D(a)) = hda + \varepsilon D(h)da + \varepsilon d(D(a))h = \\ = hda\left(1 + \varepsilon\left(\frac{d(D(a))}{da} + \frac{D(h)}{h}\right)\right)\end{aligned}$$

and the thesis follows. □

Corollary 1. *Let $\widetilde{L}[\varepsilon]$ be the deformation of $\widetilde{L}$ on $C[\varepsilon]$ induced by the deformation of C. Then the transition function of $\widetilde{L}[\varepsilon]$ is given by*

$$\alpha_{12}\left(1 + \frac{\varepsilon}{2m}\left(\frac{d(D(a)}{da} + \frac{D(h)}{h}\right)\right).$$

3.2. *Second step.*

Now we discuss the crucial point of the argument. The following simple lemma is probably well-known, but we restate and reprove it here for completeness' sake.

Lemma 2. *The section $s \in H^0(C, \widetilde{L})$ extends to a section of $\widetilde{L}[\varepsilon]$ when the curve C deforms up to the first order in the direction $D \in H^1(C, T_C)$ if and only if $D(s) = 0$ in $H^1(C, \widetilde{L})$.*

Proof. Let s_1, s_2 be local sections of $H^0(C, \widetilde{L})$ such that $\alpha_{12}s_2 = s_1$. Their first order deformations $s_1 + \varepsilon t_1$, $s_2 + \varepsilon t_2$ patch together to give a global section of $\widetilde{L}[\varepsilon]$ if and only if

$$\gamma_{12}(s_2 + \varepsilon t_2) = s_1 + \varepsilon t_1 \tag{4}$$

where

$$\gamma_{12} = \alpha_{12}\left(1 + \frac{\varepsilon}{2m}\left(\frac{d(D(a))}{da} + \frac{D(h)}{h}\right)\right)$$

is the transition function of $\widetilde{L}[\varepsilon]$. Since $\gamma_{12}(s_2) = s_1 + \varepsilon D(s_1)$, condition (4) becomes $D(s_1) = t_1 - \gamma_{12}t_2$, which represents a coboundary and as such is 0 in $H^1(C, \widetilde{L})$. □

To apply Lemma 2 in our context an explicit formulation for $D(fe_1)$ is needed. This is provided by the following

LEMMA 3. *Let $fe_1 \in \widetilde{L}(U_1)$ be a local section of $\widetilde{L}$. Then*

$$D(fe_1) = \left(\frac{f}{2m}\left(\frac{d(D(a))}{da} + \frac{D(h)}{h}\right) + D(f)\right) e_1.$$

PROOF. Since

$$\alpha_{12}e_2 = e_1$$

and

$$\alpha_{12}\left(1 + \frac{\varepsilon}{2m}\left(\frac{d(D(a))}{da} + \frac{D(h)}{h}\right)\right) e_2 = e_1 + \varepsilon D(e_1)\,,$$

it follows that

$$D(e_1) = \frac{1}{2m}\left(\frac{d(D(a))}{da} + \frac{D(h)}{h}\right) e_1$$

and

$$D(fe_1) = f\,D(e_1) + D(f)e_1 = \left(\frac{f}{2m}\left(\frac{d(D(a))}{da} + \frac{D(h)}{h}\right) + D(f)\right) e_1\,,$$

as claimed. □

COROLLARY 2. *Define*

$$\widetilde{\nu} : H^1(C, T_C) \longrightarrow \mathrm{Hom}(H^0(C, \widetilde{L})\,, H^1(C, \widetilde{L}))$$

$$D \longmapsto \left(fe_1 \mapsto \left(\frac{f}{2m}\left(\frac{d(D(a))}{da} + \frac{D(h)}{h}\right) + D(f)\right) e_1\right).$$

Then $T_C\left(\mathfrak{Th}^r_{g,m}\right) = \mathrm{Ker}\,\widetilde{\nu}$.

3.3. *Third step.*

We will obtain the thesis from Corollary 2, as soon as we show that $\widetilde{\nu} = (\widetilde{\mu})^*$ (up to a scalar multiple). Chosen $\sigma = fe_1$ and $\tau = ge_1^{2m-1}$, in order to prove the duality between $\widetilde{\nu}$ and $\widetilde{\mu}$ we have to verify that $\widetilde{\nu}(D)(\sigma)\cdot\tau = D\cdot\widetilde{\mu}(\sigma\otimes\tau)$ in $H^1(C, K)$, or that $\mathrm{res}(\widetilde{\nu}(D)(\sigma)\cdot\tau) = \mathrm{res}(D\cdot\widetilde{\mu}(\sigma\otimes\tau))$, where $\mathrm{res} : H^1(C, K) \longrightarrow H^0(C, \mathcal{O}_C) \cong \mathbb{C}$ is the duality homomorphism.

On one hand,

$$\begin{aligned}\widetilde{\nu}(D)(\sigma)\cdot\tau &= \left(\frac{f}{2m}\left(\frac{d(D(a))}{da}+\frac{D(h)}{h}\right)+D(f)\right)e_1 g e_1^{2m-1} = \\ &= \left(\frac{fg}{2m}\left(\frac{d(D(a))}{da}+\frac{D(h)}{h}\right)+D(f)g\right)e_1^{2m} = \\ &= \frac{fgh}{2m}d(D(a))+\frac{fgD(h)}{2m}da + D(f)ghda = \\ &= \frac{fgh}{2m}d(D(a))+\frac{fgD(a)}{2m}dh + hD(a)gdf = \\ &= \frac{1}{2m}d(fghD(a)) - \frac{1}{2m}hD(a)fdg - \frac{1}{2m}hD(a)gdf + hD(a)gdf = \\ &= \frac{hD(a)}{2m}\big((2m-1)gdf - fdg\big) + \frac{1}{2m}d(fghD(a)).\end{aligned}$$

Hence for every $p \in C \setminus U_1$ we have

$$\mathrm{res}_p(\widetilde{\nu}(D)(\sigma)\cdot\tau) = \mathrm{res}_p\left(\frac{hD(a)}{2m}\big((2m-1)gdf - fdg\big)\right).$$

On the other side,

$$D\cdot\widetilde{\mu}(\sigma\otimes\tau) = D\cdot\Big(\big((2m-1)gdf - fdg\big)\,hda\Big) = \big((2m-1)gdf - fdg\big)\,hD(a).$$

Hence we deduce

$$\mathrm{res}(D\cdot\widetilde{\mu}(\sigma\otimes\tau)) = \sum_p \mathrm{res}_p\Big(\big((2m-1)gdf - fdg\big)\,hD(a)\Big) = \mathrm{res}(2m\widetilde{\nu}(D)(\sigma)\cdot\tau)$$

where the sum runs over $p \in C \setminus U_1$.

Theorem 3 is now completely proved. □

4. Dimension of theta loci

In this section we turn to the proof of Theorem 2. Fix a general point P in an irreducible component of $\mathfrak{Th}_g^r$; P corresponds to a smooth curve C of genus g with a theta-characteristic L such that $h^0(C, L) = r + 1$. Consider the natural map $\mu : \bigwedge^2 H^0(C, L) \to H^0(C, 2K)$ defined by $\mu(\sigma\wedge\tau) = \sigma d\tau - \tau d\sigma$. The elements in $\bigwedge^2 H^0(C, L)$ of the form $\sigma\wedge\tau$ are called *decomposable* (see [9, Definition 2.9]); the projectivization of the set of decomposable elements is indeed a Grassmannian $\mathcal{G}$ of lines in $\mathbb{P}^r$. We denote by $\mathcal{S}(\mathcal{G})$ the secant variety of $\mathcal{G}$ and we recall that $\dim\mathcal{S}(\mathcal{G}) = 4r-7$.

If $\mathbb{P}\,\mathrm{Ker}\,\mu \cap \mathcal{S}(\mathcal{G}) = \emptyset$, then $\dim\mathrm{Ker}\,\mu + \dim\mathcal{S}(\mathcal{G}) + 1 \leq \dim\bigwedge^2 H^0(C, L) = \dim\mathrm{Ker}\,\mu + \dim\mathrm{Im}\,\mu$, so $\dim\mathcal{S}(\mathcal{G}) + 1 \leq \dim\mathrm{Im}\,\mu$. By applying Nagaraj's theorem (*i.e.* Theorem 3 with $m = 1$) we get:

$$\dim\mathfrak{Th}_g^r \leq \dim T_P(\mathfrak{Th}_g^r) = 3g-3-\dim\mathrm{Im}\,\mu \leq 3g-3-\dim\mathcal{S}(\mathcal{G})-1 = 3g-4r+3$$

i.e. the thesis.

Now assume that $\mathbb{P}\,\mathrm{Ker}\,\mu \cap \mathcal{S}(\mathcal{G}) \neq \emptyset$. In this case we have $\mu(s \wedge t - u \wedge v) = 0$ for some s, t, u, v. We take the two (distinct) unidimensional linear subseries of the half-canonical g^r_{g-1} defined by $< s, t >$ and $< u, v >$ and we focus our attention on the morphism:

$$\psi = (f_1, f_2) : C \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

obtained as the product of the two morphisms associated to the above linear subseries. It is a direct consequence of the theory of the normal sheaf to a morphism as developed in [1, 2], that ψ cannot be birational to the image (in [9, p. 110] the interested reader will find all the details). Hence ψ factors through another curve Γ:

$$\psi : C \longrightarrow \Gamma \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$

If $g(\Gamma) > 0$ then C is a degree $\delta \geq 2$ covering of a genus $\gamma \geq 1$ curve, so it depends only on $2g - 2 - (2\delta - 3)(\gamma - 1)$ moduli.
If instead $g(\Gamma) = 0$, then the half-canonical series is composed with a pencil, *i.e.* there exists a $g^1_k \subseteq \mathbb{P}H^0(C, \widetilde{L})$ with $mg^1_k = g^r_{g-1}$.
If $m = 1$ we have

$$\psi = (f_1, f_2) : C \xrightarrow{k} \mathbb{P}^1 \xrightarrow{(1,1)} \mathbb{P}^1 \times \mathbb{P}^1$$

so ψ should be composed with the diagonal morphism and $f_1 = f_2$, which is excluded. So we may assume $m \geq 2$ and $k = \frac{g-1}{m} \leq \frac{g-1}{2} < \frac{g}{2} + 1$. Hence $\dim \mathfrak{M}^1_{g,k} = 2g + 2k - 5$ (see [2, eq. (2.3), p. 346]) and using (2) we obtain

$$2g + 2k - 5 = \dim \mathfrak{M}^1_{g,k} \leq \dim T_P(\mathfrak{M}^1_{g,k}) = 3g - 3 - \dim \mathrm{Im}\,\mu_1\,. \tag{5}$$

If $m = 2$ just notice that $2\widetilde{L} = L$, so $K - 2\widetilde{L} \sim L$ and $h^0(C, K - 2\widetilde{L}) = r + 1$. From (3) it follows that $\dim \mathrm{Im}\,\mu_1 = r + 1$, so (5) gives

$$2g + 2\frac{g-1}{2} - 5 \leq 3g - 3 - r - 1$$

i.e. $r \leq 2$, which is excluded by hypothesis.

LEMMA 4. *Let*

$$\widetilde{\mu} : H^0(C, \widetilde{L}) \otimes H^0(C, K - \widetilde{L}) \longrightarrow H^0(C, 2K)$$

$$\sigma \otimes \tau \longmapsto \frac{\tau^2}{\sigma^{2m-2}} d\left(\frac{\sigma^{2m-1}}{\tau}\right).$$

Then $\dim \mathrm{Im}\,\widetilde{\mu} \geq g - k + 1$.

PROOF. Fix $\sigma \in H^0(C, \widetilde{L})$, $\sigma \neq 0$, so that $\sigma^{2m-1} \in H^0(C, K - \widetilde{L})$. Complete σ^{2m-1} to a basis of $H^0(C, K - \widetilde{L})$: $(\sigma^{2m-1}, \tau_1, \ldots, \tau_s)$, where $s = h^0(C, K - \widetilde{L}) - 1$. We claim that $\widetilde{\mu}(\sigma \otimes \cdot)$ embeds $< \tau_1, \ldots, \tau_s >$ into $H^0(C, 2K)$. In fact, assume on the contrary that $\widetilde{\mu}(\sigma \otimes \sum_{i=1}^s a_i\tau_i) = 0$ with some $a_i \neq 0$. Then locally it should be $d\left(\frac{\sigma^{2m-1}}{\sum_{i=1}^s a_i\tau_i}\right) = 0$ and so $\sum_{i=1}^s a_i\tau_i - c\sigma^{2m-1} = 0$, whence $a_1 = \ldots = a_s = 0$. By the claim we may conclude just applying Riemann-Roch. $\square$

At last we are in position to prove Theorem 2 also for $m \geq 3$. In this case, we have $k \leq \frac{g-1}{3}$, so Lemma 4 yields the following estimate:

$$\dim \operatorname{Im} \widetilde{\mu} \geq \frac{2g+4}{3}.$$

From Theorem 3 we deduce:

$$\dim \mathfrak{Th}^r_g \leq \dim \mathfrak{Th}^r_{g,m} \leq \dim T_P(\mathfrak{Th}^r_{g,m}) = 3g-3-\dim \operatorname{Im} \widetilde{\mu} \leq 3g-3-\frac{2g+4}{3}.$$

To get the thesis, it will be sufficient to check that

$$3g-3-\frac{2g+4}{3} \leq 3g-4r+3,$$

but such an inequality is a straightforward consequence of the numerical hypothesis $g \geq 6r-11$. So the proof is over. □

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