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Fisica matematica. — *Asymptotic behaviour* ($t \to +0$) *of the interface for the critical case of undercooled Stefan problem.* Nota di IVAN G. GÖTZ, MARIO PRIMICERIO e JUAN J.L. VELÁZQUEZ, presentata (*) dal Socio M. Primicerio.

ABSTRACT. — The critical case of solvability of a two-phase Stefan problem with supercooled liquid phase is considered. Asymptotic analysis is performed of the behaviour of the free boundary in the vicinity of the initial time.

KEY WORDS: Stefan problem; Supercooling; Asymptotics.

RIASSUNTO. — *Comportamento asintotico dell'interfase per il caso critico del problema di Stefan con sovraraffreddamento.* Si considera il caso critico di solubilità di un problema di Stefan a due fasi in presenza di sovraraffreddamento della fase liquida. Viene condotta l'analisi asintotica del comportamento del contorno libero nell'intorno dell'istante iniziale.

Let us consider the one-dimensional Stefan problem:

(1) $\theta_t - \theta_{xx} = 0\,,$ in $Q_T\,,$

(2) $L\dot{s}(t) = \theta_x(s(t)-0\,,\,t) - \theta_x(s(t)+0\,,\,t)\,,$ $t > 0\,,$

(3) $\theta(s(t)\,,\,t) = 0\,,$ $t > 0\,,$

(4) $\theta(x, 0) = \theta_0(x)\,,$ $x \in [-1\,,\,1]\,,$ $s(0) = 0\,,$

(5) $\theta(\pm 1\,,\,t) = \theta^{\pm}(t)\,,$ $t > 0\,,$

where θ is the temperature, L is the dimensionless latent heat and the function s describes the phase-change boundary. The curve $x = s(t)$ divides the domain $Q_T = (-1\,,\,1) \times (0\,,\,T)$ into the solid phase $Q_T^- = Q_T \cup \{x < s(t)\}$ and the liquid one $Q_T^+ = Q_T \cup \{x > s(t)\}$. The problem (1)-(5) is called one phase problem, if

(6)
$$\theta_0(x) = 0\,, \quad x \in (-1\,,\,0)\,,$$
$$\theta^-(t) = 0\,, \quad t > 0\,,$$

otherwise it is called two-phase problem.

We are interested in studying the critical case for solvability of the problem (1)-(5) (see [1, 2]), namely:

(7) $$-L < \theta_0(x) \leq 0\,, \qquad 0 < x < 1\,, \quad \theta^+(t) \leq 0 \quad t > 0\,,$$

(8) $$\lim_{x \to +0} \theta_0(x) = -L.$$

These conditions describe the undercooled liquid ($\theta < 0$), which is deeply undercooled ($\theta \leq -L$) at one point $x = 0$. The following conditions on the initial and boundary

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temperature in Q_T^- ensure that the solid phase is not superheated:

$$\theta_0(x) \leq 0\,, \qquad -1 < x < 0\,, \quad \theta^-(t) \leq 0 \quad t > 0. \tag{9}$$

The property (8) is critical in the following sense. If $\theta_0(x) > -L + \epsilon$, $\epsilon > 0$ for $x \in (0\,,1)$, and $\theta_0(0+0) < 0$, then the function s has the well-known self-similar asymptotic $s(t) = Ct^{\frac{1}{2}} + o(t^{\frac{1}{2}})$, see *e.g.* [5]. If otherwise $\theta_0(x) \leq -L$ for $x \in (0\,,\sigma)$, $\sigma > 0$, then no classical solution exists such that $\lim_{t\to+0} s(t) = 0$. For a discussion of possible regularization see [3].

On the other hand the conditions (7), (8) are not completely artificial. If we consider the two-phase Stefan problem, then the free boundary may jump over some interval, occupied by deeply undercooled liquid at some instant $t = \bar{t}$ see [4]. After this event we have a classical solution, and the initial data at $t = \bar{t} + 0$ satisfy conditions (7), (8).

The following existence result is valid:

THEOREM 1. *Let us suppose, under conditions* (7) , (9), *that the functions* θ_0 , $\theta^\pm$ *are bounded and piecewise continuous. Then there exists a classical solution of the problem* (1)-(5) *on some time interval* (0 , T), *such that*:

$$\theta \in L_2(0\,,T; H^1(0\,,1)) \cup L_\infty(Q_T)\,, \tag{10}$$

(11) *s is a continuous nondecreasing function,*

$$\text{either} \quad T = +\infty \quad \text{or} \quad s(T) = 1. \tag{12}$$

The proof of this result for one-phase problems is given in [2] and for two phase problems in [4]. Now, we state and prove the main result of the paper:

THEOREM 2. *Let us assume, in addition to conditions of Theorem* 1:

$$\theta_0(x) = -L + Ax^k + o(x^k)\,, \qquad \text{for} \quad x > 0\,, \tag{13}$$

$$\theta_0(x) > Bx\,, \qquad \text{for} \quad x < 0\,, \quad \theta^-(t) > -B\,, \qquad \text{for} \quad t > 0\,, \tag{14}$$

where A , B and k are some positive constants. Then the function s has the following asymptotic behaviour:

$$s(t) = \left(\frac{L}{A}t\right)^{\frac{1}{\beta}} + o(t^{\frac{1}{\beta}})\,, \qquad \text{for} \quad t > 0\,, \tag{15}$$

where $\beta = k + 2$.

PROOF OF THEOREM 2. We begin with a Baiocchi type transformation:

$$u(x\,,t) = \begin{cases} \displaystyle\int_{s(t)}^x dy \int_{s(t)}^y d\xi(\theta(\xi\,,t) + L)\,, & x > s(t)\,, \\ 0\,, & \text{otherwise.} \end{cases} \tag{16}$$

The function u satisfies the following equations:

$$\begin{aligned} &u_t - u_{xx} = -H(u)(L + (x - s(t))\theta_x(s(t) - 0\,,t))\,, \ \text{in} \quad Q_T\,, \\ &u(x,0) = u_0(x)\,, \quad -1 < x < 1\,, \end{aligned} \tag{17}$$

where

$$u_0(x) = \begin{cases} \int_0^x dy \int_0^y d\xi(\theta_0(\xi) + L), & x > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

Using assumption (14) we can easily construct a barrier function for θ in Q_T^-:

$$\theta(x, t) > B(x - s(t)), \quad \text{in} \quad Q_T^-, \tag{19}$$

hence

$$0 < \theta_x(s(t) - 0, t) < B, \qquad t > 0. \tag{20}$$

Due to this property we can build a subsolution for the problem (17):

$$w(x, t) = \begin{cases} (\underline{A}x^\beta - (L + Bx)t)_+, & x > 0 \\ 0, & \text{otherwise}, \end{cases}$$

where $\underline{A} \in (0, A)$ and

$$(y)_+ = \begin{cases} y, & \text{if} \quad y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let us take $x_{\underline{A}} \in (0, 1)$, $t_{\underline{A}} > 0$ such that the function $v = u - w$ is nonnegative on the parabolic boundary of the domain $Q_{\underline{A}} = (0, x_{\underline{A}}) \times (0, t_{\underline{A}})$. This is possible due to condition (13). Since

$$w_t - w_{xx} \leq -H(w)(L + Bx + \beta(\beta - 1)\underline{A}x^{\beta-2}),$$

hence

$$\begin{aligned} v_t - v_{xx} \geq & -(H(u) - H(w))(L + (x - s(t))\theta_x(s(t) - 0, t)) - \\ & - H(w)(x(\theta_x(s(t) - 0, t) - B) - s(t)\theta_x(s(t) - 0, t) - \beta(\beta - 1)\underline{A}x^{\beta-2}) \\ & \leq \left[-\frac{(H(u) - H(w))}{v}(L + (x - s(t))\theta_x(s(t) - 0, t)) \right] v. \end{aligned}$$

The coefficient in front of v is nonpositive, therefore by the maximum principle we obtain:

$$v = u - w \geq 0 \qquad \text{in} \quad Q_{\underline{A}}. \tag{21}$$

Since

$$u(x, t) \equiv 0 \quad \text{for} \quad x < s(t),$$

$$w(x, t) \equiv 0 \quad \text{for} \quad t > \frac{\underline{A}x^\beta}{L + Bx},$$

hence

$$s^{-1}(x) \geq \frac{\underline{A}x^\beta}{L + Bx} \quad x \in (0, x_{\underline{A}}). \tag{22}$$

Since we can take an arbitrary $\underline{A} \in (0, A)$, therefore the inequality (22) gives

$$s(t) \leq \left(\frac{L}{A}t\right)^{\frac{1}{\beta}} + o(t^{\frac{1}{\beta}}), \qquad \text{for} \quad t > 0. \tag{23}$$

In order to get the opposite inequality we use the following comparison result:

LEMMA 1. *Let $\widetilde{u}$, $\widetilde{s}$ be a Baiocchi transformed one-phase solution of the problem* (1)-(5), *so that $\widetilde{u} \geq u$ on the parabolic boundary of some domain Ω, then*

$$\begin{aligned} \widetilde{u} &\geq u \qquad && in \quad \Omega\,, \\ \widetilde{s}(t) &\leq s(t)\,, && t \in (0\,,\,T). \end{aligned} \tag{24}$$

This statement becomes obvious if we compare the equation (17) with the equation for $\widetilde{u}$:

$$\widetilde{u}_t - \widetilde{u}_{xx} = -LH(\widetilde{u}) \qquad \text{in} \quad Q_T.$$

Now we need to prove the opposite to inequality (23) for the function $\widetilde{s}$. For this purpose we construct a supersolution of the one-phase problem:

$$v = \widetilde{w} + z \qquad \text{in} \quad Q_T\,,$$

where

$$\widetilde{w}(x\,,\,t) = \begin{cases} (\underline{A}x^{\beta} - Lt)_+\,, & \text{for} \quad x > 0\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

and the function z satisfies

$$z_t - z_{xx} = \delta(x - \lambda(t))\beta\bar{A}\left(\frac{L}{\bar{A}}t\right)^{\frac{\beta-1}{\beta}} \qquad \text{in} \quad Q_T\,,$$

$$z(x\,,0) = 0\,, \quad -1 < x < 1\,, \quad z(\pm 1\,,\,t) = 0\,, \quad t > 0\,,$$

where $\lambda(t) = \left(\frac{L}{\bar{A}}t\right)^{\frac{1}{\beta}}$, $\bar{A} > A$. The function v satisfies the following problem:

$$\begin{aligned} v_t - v_{xx} &= -LH(x - \lambda(t))\,, \qquad \text{in} \quad Q_T\,, \\ v = \widetilde{w}\,, &\quad \text{on the parabolic boundary of} \quad Q_T. \end{aligned}$$

Let us take $\widetilde{u} = \widetilde{w}$ on the parabolic boundary of Q_T, then (21), (22) yield

$$\begin{aligned} \widetilde{u} &\geq \widetilde{w} \qquad && \text{in} \quad Q_T\,, \\ \widetilde{s}(t) &\leq \lambda(t)\,, && t \in (0\,,\,T)\,, \end{aligned} \tag{25}$$

since $B = 0$ for the one-phase problem. Thus we have

$$v_t - v_{xx} = -LH(x - \lambda(t)) \geq -LH(x - \widetilde{s}(t)) = \widetilde{u}_t - \widetilde{u}_{xx}\,,$$

which gives the inequality

$$v \geq \widetilde{u} \quad \text{in} \quad Q_T. \tag{26}$$

For $x < \lambda(t)$ we have $v(x\,,\,t) = z(x\,,\,t)$, and

$$\begin{aligned} z(x\,,\,t) &= \int_0^t \frac{d\tau}{(4\pi(t-\tau))^{1/2}} \int_{\mathbb{R}} \exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right) \beta\bar{A}\left(\frac{L}{\bar{A}}\tau\right)^{\frac{\beta-1}{\beta}} \delta(x - \lambda(\tau))d\xi \\ &= \int_0^t \exp\left(-\frac{(x-\lambda(\tau))^2}{4(t-\tau)}\right) \beta\bar{A}\left(\frac{L}{\bar{A}}\tau\right)^{\frac{\beta-1}{\beta}} \frac{d\tau}{(4\pi(t-\tau))^{1/2}}. \end{aligned}$$

Then

$$z(x, t) < C_{\beta, \bar{A}} \int_0^t \frac{\tau^{1-1/\beta} d\tau}{(t-\tau)^{1/2}} = \widetilde{C}_{\beta, \bar{A}} t^{1+1/2-1/\beta}, \quad \text{in} \quad Q_T,$$

hence (26) yields

$$\widetilde{u}(x, t) \le C t^{1+1/2-1/\beta}, \quad \text{for} \quad x < \lambda(t). \tag{27}$$

On the other hand the following result holds:

LEMMA 2. *Let σ satisfy*

$$\sigma_t - \sigma_{xx} \le -H(\sigma) \quad \text{in} \quad D_K = (-K, K) \times (-K^2, 0),$$
$$0 \le \sigma(x, t) \le \frac{1}{4} K^2 \quad \text{on the parabolic boundary of} \quad D_k,$$

then $\sigma(0, 0) = 0$.

This result is rather classical and it follows from the supersolution:

$$\overline{\sigma}(x, t) = -\frac{1}{2} t + \frac{1}{4} x^2.$$

Then

$$\overline{\sigma}(\pm K, t) \ge \frac{1}{4} K^2, \qquad \overline{\sigma}(x, -K^2) \ge \frac{1}{2} K^2.$$

Therefore, by comparison we deduce $0 < \sigma(0, 0) \le \overline{\sigma}(0, 0) = 0$.

We will use this result in order to estimate the position of the free boundary $\widetilde{s}(t)$. Let us pick

$$x \ge (1-\delta) \left(\frac{L}{\bar{A}} t \right)^{\frac{1}{\beta}}, \qquad \delta > 0,$$

for δ and t small enough. Given $\eta > 0$ small enough and independent on t we find that points $(\overline{x}, \overline{t})$ such that

$$|\overline{x} - x| \le \sqrt{\eta t}, \quad |\overline{t} - t| \le \eta t, \tag{28}$$

satisfy the inequality:

$$\overline{x} \le \left(1 - \frac{\delta}{2} \right) \left(\frac{L}{\bar{A}} t \right)^{\frac{1}{\beta}}.$$

Then

$$\widetilde{u}(\overline{x}, \overline{t}) \le z(\overline{x}, \overline{t}) \le C \overline{t}^{1+1/2-1/\beta} = \eta t \left(\frac{\overline{t}}{t} \frac{C}{\eta} \overline{t}^{\frac{\beta-2}{2\beta}} \right) \le \frac{1}{4}$$

for all $\overline{t}$ satysfying (28). Then using Lemma 2 with $K = \sqrt{\eta t}$ we obtain

$$\widetilde{u}(x, t) = 0,$$

hence

$$\widetilde{s}(t) \ge (1-\delta) \left(\frac{L}{\bar{A}} t \right)^{\frac{1}{\beta}}, \quad \text{for } t \text{ small enough.}$$

Taking into account, that $\delta > 0$ is arbitrarily small, we obtain

(29) $$\liminf_{t\to+0} \frac{\widetilde{s}(t)}{t^{\frac{1}{\beta}}} \geq \left(\frac{L}{\bar{A}}t\right)^{\frac{1}{\beta}} .$$

Now we can use Lemma 1 in a domain $\Omega = (-x', x') \times (0, t')$, where x' and t' are chosen so that $\widetilde{u} \geq u$ on the parabolic boundary of the domain Ω. This is possible since $\bar{A}$ is chosen arbitrarily, so that $\bar{A} > A$. Therefore using (24) and (29) we get

(30) $$\liminf_{t\to+0} \frac{s(t)}{t^{\frac{1}{\beta}}} \geq \left(\frac{L}{A}t\right)^{\frac{1}{\beta}} ,$$

which together with (23) gives the desired asymptotic behaviour (15).

REMARK 1. Theorem 2 gives informations on the behaviour of free boundary in the vicinity of $t = 0$. For this reason, it is clear that values taken by the data «far» from $x = 0, t = 0$ will be irrelevant. It is straightforward to release the assumptions (7), (14).

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