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Totally bounded differential polynomial systems in $\mathbb{R}^2$

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Equazioni differenziali ordinarie. — *Totally bounded differential polynomial systems in* $\mathbb{R}^2$. Nota di ROBERTO CONTI e MARCELLO GALEOTTI, presentata (*) dal Socio R. Conti.

ABSTRACT. — Totally bounded differential systems in $\mathbb{R}^2$ are defined as having all trajectories bounded. By Dulac's finiteness theorem it is proved that totally bounded polynomial systems exhibit an unbounded «annulus» of cycles. The portrait of the remaining trajectories is examined in the case the system has, in $\mathbb{R}^2$, a unique singular point. Work is in progress concerning the study of totally bounded polynomial systems with two singular points.

KEY WORDS: Polynomial systems; Total boundedness; Petals.

RIASSUNTO. — *Sistemi differenziali totalmente limitati in* $\mathbb{R}^2$. Si definiscono i sistemi differenziali totalmente limitati in $\mathbb{R}^2$ come quelli di cui tutte le traiettorie sono limitate. Applicando il teorema di finitezza di Dulac, si dimostra che i sistemi polinomiali totalmente limitati sono caratterizzati dall'esistenza di un «anello» illimitato di cicli. La configurazione delle restanti traiettorie viene studiata nel caso che il sistema possieda, al finito, un unico punto singolare. Ricerche in corso riguardano lo studio di sistemi polinomiali totalmente limitati con due punti singolari.

1. DEFINITIONS

A polynomial differential system in $\mathbb{R}^2$ is a pair of ordinary differential equations

$$\dot{x} = X(x, y), \qquad \dot{y} = Y(x, y) \tag{1.1}$$

where $\dot{x} = dx/dt$, $\dot{y} = dy/dt$, $t \in \mathbb{R}$ and $X(x, y)$, $Y(x, y)$ are polynomials of $(x, y) \in \in \mathbb{R}^2$ with real coefficients.

Usually, t is referred to as time.

The system is of degree n if n is the maximum of the degrees of $X(x, y)$, $Y(x, y)$.

We shall study the class of polynomial differential systems satisfying

DEFINITION 1. *We say that* (1.1) *is* totally bounded *if each trajectory is bounded.*

2. DEGREE AND TOTAL BOUNDEDNESS

An important contribution to the study of polynomial systems comes from a result of M. Galeotti and M. Villarini [3]. They proved in fact that, if n is even, the polynomial system (1.1) has one unbounded trajectory at least.

It follows

THEOREM 1. *The degree of a totally unbounded polynomial system is odd.*

(*) Nella seduta dell'8 febbraio 2002.

3. Singular points

If (1.1) has no singular point, then by Bendixson theorem it has no cycle either, so that each trajectory is open, with empty limit sets, hence unbounded.

Therefore we have

Proposition 1. *A totally bounded polynomial system has one singular point at least.*

Actually, the singular points may be infinitely many.

As it is wellknown, it took more than sixty years to prove Dulac's finiteness theorem asserting that polynomial differential systems in $\mathbb{R}^2$ have at most a finite number of limit cycles: see, for instance, [2, 4].

As an immediate consequence of finiteness theorem, an *isolated* singular point of a polynomial system, totally bounded or not, must be a center, a focus, or a tangential limit point, *i.e.*, the limit point S of trajectories γ whose tangent line at $P \in \gamma$ tends to a line through S as P tends to S along γ.

4. Annular region

As a consequence of finiteness theorem on totally bounded systems, we have

Theorem 2. *If* (1.1) *is totally bounded, a limit cycle cannot surround all the singular points.*

Proof. Let γ be a limit cycle and let the singular points all belong to the region interior to γ. By the analyticity of Poincaré return map there must exist a neighborhood of γ whose exterior part is covered by open trajectories which spiral around γ towards γ (away from γ).

Such trajectories, bounded by assumption, must be contained in the region interior to a limit cycle γ' surrounding γ and spiral away from γ' (towards γ').

This argument can be repeated starting from γ' and gives rise to an infinite sequence of (expanding) limit cycles, against the finiteness theorem. □

Totally bounded polynomial systems are characterized by

Theorem 3. *A polynomial system* (1.1) *is totally bounded if and only if there exists a family of cycles covering an annular region whose «outer» boundary in* $\mathbb{R}^2$ *is empty.*

Proof. The «if» part is obvious.

To prove the «only if» part let there exist $r > 0$ such that the circle $\Gamma_r : x^2 + y^2 = r^2$ contains in its interior all the trajectories which are closed, *i.e.*, all the cycles and all the singular points. Γ_r cannot be invariant.

Let P be a point exterior to Γ_r. Then the trajectory γ_P is open.

If one of its limit sets is empty then γ_P is unbounded. If both the limit sets of γ_P are non empty they are interior to Γ_r, so both the trajectories γ_P^-, γ_P^+ must cross Γ_r. Since (1.1) is a polynomial system, the crossing points with the circle Γ_r are finite in number.

Let $P_k = (0\,,\,k_r)$ and let Q_k be the intersection of $\gamma^+_{P_k}$ (or, indifferently, of $\gamma^-_{P_k}$) with Γ_r nearest to P_k. As $k \to \infty$ the sequence $\{Q_k\}$ has a limit Q_∞ and the trajectory γ_{Q_∞} is unbounded. □

From now on the annular region covered by cycles of a totally bounded polynomial system will be referred to as the *annulus* $\mathcal{A}$ of the system. We shall denote by $\partial\mathcal{A}$ the «inner» boundary of $\mathcal{A}$.

Clearly $\partial\mathcal{A}$ is a finite union of singular points and open trajectories, homoclinic and/or heteroclinic, oriented concordantly with the cycles of $\mathcal{A}$. Let Σ denote the Poincaré sphere in $\mathbb{R}^3$ associated to (1.1) and let Σ_∞ denote the equator of Σ.

If the system is totally bounded Σ_∞ is the «outer» boundary on Σ of the *annulus* $\mathcal{A}$. Σ_∞ may be a cycle on Σ or contain singular points at infinity.

PROBLEM 1. *If* (1.1) *is totally bounded, may* Σ_∞ *consist uniquely of singular points at infinity?*

REMARK 1. Let (1.1) be totally bounded. Any cycle from $\mathcal{A}$ surrounds all the singular points. Therefore, if they are finite in number, the sum of their indices is equal to *one*.

5. PETALS AND FLOWERS

Independent of total boundedness it is convenient for polynomial systems to introduce the following notion.

DEFINITION 2. *Let* γ *be an open trajectory of the polynomial system* (1.1) *homoclinic at a singular point S and such that the Jordan region* $\mathcal{J}$ *interior to* $\gamma \cup \{S\}$ *does not contain singular points.*

If there is no trajectory surrounding γ *with the same properties as* γ *we shall say that the closure of* $\mathcal{J}$ *is a petal of* (1.1) *at S.*

In other words a petal at S is the closure of a maximal elliptic region at S.

THEOREM 4. *Petals of a polynomial system are finite in number.*

PROOF. Let S be a singular point with petals. For simplicity let $S = (0\,,0)$. Since (1.1) is a polynomial system of degree n, a circle $C_r : x^2 + y^2 = r^2$ is invariant or it has at most $2(n+1)$ contacts with (1.1), *i.e.*, points $(x\,,y)$ satisfying

$$x^2 + y^2 = r^2\,, \qquad xX(x\,,y) + yY(x\,,y) = 0$$

If there existed infinitely many petals at O the circle C_r, for r sufficiently close to zero, would have more than $2(n+1)$ contacts, so it would be invariant, which is impossible. □

DEFINITION 3. *The union of all petals at S will be called a* flower *at S.*

6. A unique singular point

We shall now consider in some detail the simplest class of totally bounded polynomial systems, namely that of totally bounded systems with a unique singular point.

Recall first that a center S is, by definition, a *global center* if all the trajectories $\neq \{S\}$ are cycles.

Using Section 3 it is easy to verify

THEOREM 5. *Let* (1.1) *be a totally bounded polynomial differential system with a unique singular point S.*

Then:

a) *there are no limit cycles*;

b) *S is not a focus nor a saddle point*;

c) *if S is a center, S is a global center*;

d) *all the open trajectories, if any, are homoclinic at S*;

e) *the inner boundary $\partial\mathcal{A}$ of the annulus $\mathcal{A}$ is $\{S\}$ or it is the union of $\{S\}$ with a finite number of open trajectories.*

Using Definitions 2 and 3 we have

THEOREM 6. *Under the assumptions of Theorem* 5 *the compact set $\mathbb{R}^2 \backslash \mathcal{A}$ either reduces to $\{S\}$ and S is a global center, or $\mathbb{R}^2 \backslash \mathcal{A}$ is a flower at S.*

7. Example of petal

In the linear case ($n = 1$) the system is totally bounded if and only if O is a global center. When $n = 3, 5\ldots$ there are other possibilities.

For instance let us consider

$$\left\{\begin{aligned} \dot{x} &= x(ax + by) + (x^2 + y^2)y \\ \dot{y} &= (ax + by)y - x(x^2 + y^2). \end{aligned}\right. \tag{7.1}$$

Using polar coordinates ρ, θ, (7.1) becomes $\dot{\rho} = (a\cos\theta + b\sin\theta)\rho^2$, $\dot{\theta} = -\rho^2$, so the trajectories are represented by graphs of functions $\theta \mapsto \rho(\theta)$

$$\left\{\begin{aligned} \rho(\theta) &= -a\sin\theta + b\cos\theta + r_0 \\ r_0 &= \rho(\theta_0) + a\sin\theta_0 - b\cos\theta_0 \end{aligned}\right. \tag{7.2}$$

and θ_0 satisfies

$$a\cos\theta_0 + b\sin\theta_0 = 0.$$

Returning to x, y coordinates we see from (7.2) that the trajectories are the bicircular (hence bounded) quartics (lemniscates)

$$(x^2 + y^2)^2 + 2(ay - bx)(x^2 + y^2) + \left[(ay - bx)^2 - r_0^2(x^2 + y^2)\right] = 0 \tag{7.3}$$

depending on the parameter r_0^2, orthogonally symmetrical with respect to the line $ax + by = 0$.

O is a double point of (7.3) with tangents $(b^2 - r_0^2)x^2 - 2abxy + (a^2 - r_0^2)y^2 = 0$.

The trajectories are homoclinic at O for $r_0^2 < a^2 + b^2$, cycles for $a^2 + b^2 < r_0^2$, and $\partial\mathcal{A}$ corresponds to $a^2 + b^2 = r_0^2$.

Example 1. $a = 1,\ b = -1$

$$\begin{cases} \dot{x} = x^2 - xy + x^2y + y^3 \\ \dot{y} = xy - y^2 - x^3 - xy^2. \end{cases}$$

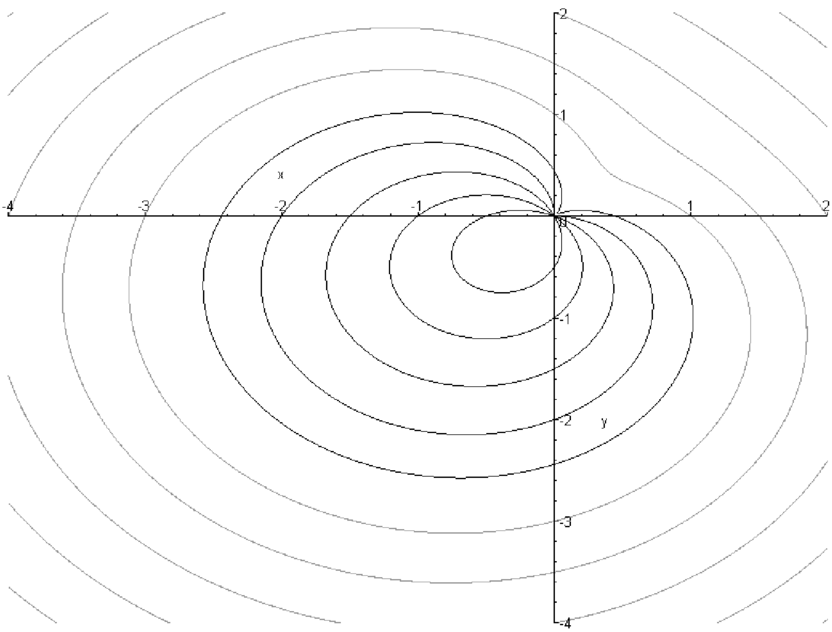

Fig. 1.

8. Number of petals

Theorem 6 is refined by

Theorem 7. *Under the assumptions of Theorem* 5 *the number of petals can be* $1, 2, \ldots, n-2$.

Proof. Let us consider the system of degree $2m + 1$

$$\begin{cases} \dot{x} = xH_h(x, y) + (x^2 + y^2)^m y \\ \dot{y} = H_h(x, y)y - x(x^2 + y^2)^m \end{cases} \tag{8.1}$$

where h, m are integers satisfying

$$m = 1, 2 \ldots; \qquad 1 \le h \le 2m - 1 \tag{8.2}$$

and $H_h(x, y)$ is a homogeneous polynomial of degree h

$$H_h(\lambda x, \lambda y) = \lambda^h H_h(x, y), \qquad \lambda \in R \tag{8.3}$$

non divisible by $x^2 + y^2$.

O is the unique singular point of (8.1). In fact

$$\dot{x}^2 + \dot{y}^2 = \rho^{2h+2}\left[H_h^2(\cos\theta, \sin\theta) + \rho^{4m-2h}\right] = 0 \Leftrightarrow \rho = 0.$$

In polar coordinates (8.1) can be written

$$\dot{\rho} = \rho^{h+1} H_h(\cos\theta, \sin\theta).$$

Hence the trajectories of (8.1) are represented by graphs of functions $\theta \mapsto \rho(\theta) > 0$ satisfying

$$\frac{d}{d\theta}\rho^{2m-h} = -(2m-h)H_h(\cos\theta, \sin\theta), \tag{8.4}$$

i.e.,

$$\rho^{2m-h} = -(2m-h)\int_0^\theta H_h(\cos\varphi, \sin\varphi)d\varphi + r^{2m-h}. \tag{8.5}$$

Then (8.1) is totally bounded if and only if

$$\int_0^{2\pi} H_h(\cos\varphi, \sin\varphi)d\varphi = 0. \tag{8.6}$$

Therefore if (8.1) is totally bounded the equation

$$H_h(\cos\varphi, \sin\varphi)d\varphi = 0 \tag{8.7}$$

has at least one root φ_0.

Actually, if φ_0 is a root of (8.7), also $\varphi_0 + \pi$ is a root, because of (8.3). According to (8.4) the pair of roots φ_0, $\varphi_0 + \pi$ corresponds to a pair of limit tangent lines at O oppositely oriented to each other, hence it corresponds to a petal at O.

The assertion follows from (8.2). □

Figures 2, 3, 4 represent examples of phase portraits of (8.1) with $m = 2$ ($n = 5$).

Theorem 7 is completed by

THEOREM 8. *Under the assumptions of Theorem* 5 *the number* k *of petals satisfies the inequality*

$$k \leq n - 2 \tag{8.8}$$

PROOF. Write

$$X(x, y) = X_h(x, y) + \text{higher order terms}$$
$$Y(x, y) = Y_h(x, y) + \text{higher order terms}$$

with $X_h(x, y)$, $Y_h(x, y)$ homogeneous polynomials of degree h and

$$X_h^2(x, y) + Y_h^2(x, y) \not\equiv 0.$$

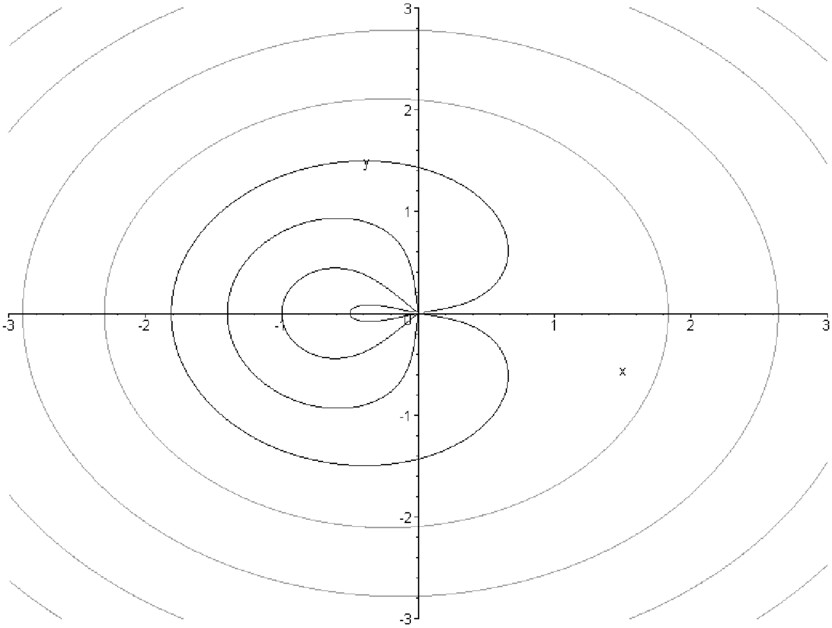

Fig. 2. – $\begin{cases} \dot{x} = -xy + x^4y + 2x^2y^3 + y^5 \\ \dot{y} = -y^2 - x^5 - 2x^3y^2 - xy^4 \end{cases}$ $(H_h(x, y) = -y)$.

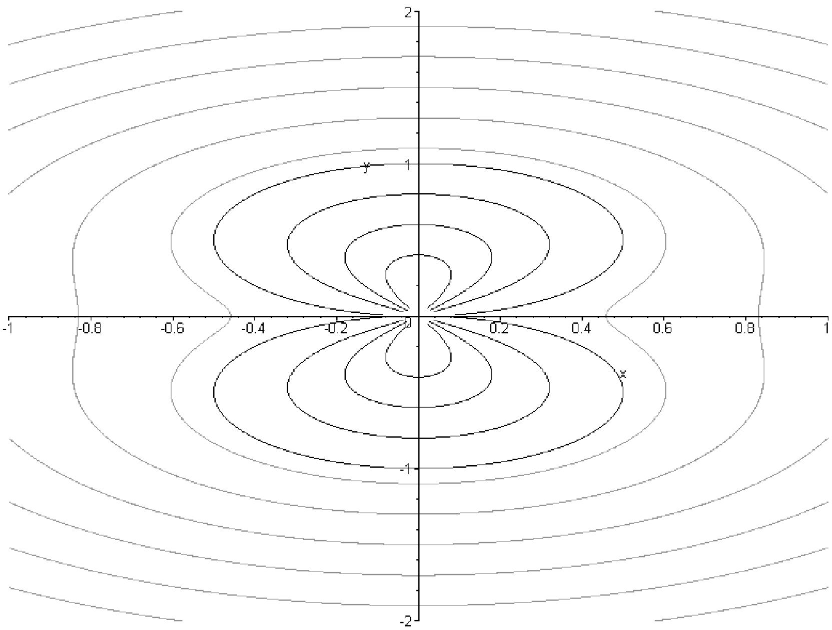

Fig. 3. – $\begin{cases} \dot{x} = -x^2y + x^4y + 2x^2y^3 + y^5 \\ \dot{y} = -xy^2 - x^5 - 2x^3y^2 - xy^4 \end{cases}$ $(H_h(x, y) = -xy)$.

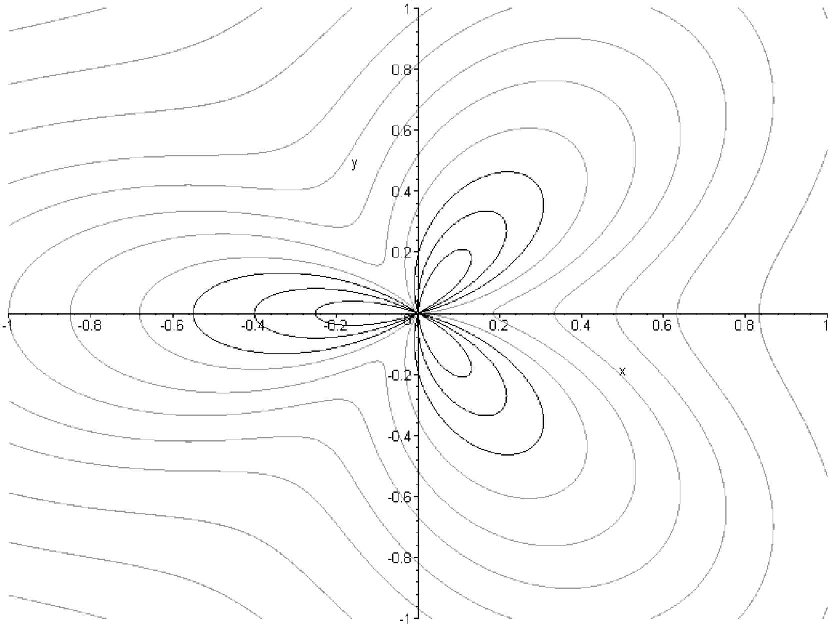

Fig. 4. – $\begin{cases} x = -3x^3y + xy^3 + x^4y + 2x^2y^3 + y^5 \\ \dot{y} = -3x^2y^2 + y^4 - x^5 - 2x^3y^2 - xy^4 \end{cases}$ $\quad (H_h(x, y) = -3x^2y)$.

It is clear that (1.1) cannot be homogeneous, therefore $n > h$.

On the other hand, a result of A.N. Berlinskii [1] states that, if S is a singular point of an analytical system having n_e elliptic sectors and n_h hyperbolic sectors, then $n_e + n_h \leq 2h$.

In our case $n_e = n_h = k$ and thus $h \geq k$.

Suppose first

$$yX_h(x, y) - xY_h(x, y) \equiv 0.$$

Then

$$X_h(x, y) = xZ_{h-1}(x, y)\,, \qquad Y_h(x, y) = yZ_{h-1}(x, y)$$

with $Z_{h-1}(x, y)$ homogeneous polynomial of degree $h-1$.

Any circumference $x^2 + y^2 = \varepsilon^2$, with ε sufficiently small, has at least $2k$ contacts with the vector field $(\dot{x}, \dot{y})$. This implies $h - 1 \geq k$, hence $n \geq k + 2$, *i.e.* (8.8).

Suppose now

$$yX_h(x, y) - xY_h(x, y) \not\equiv 0.$$

The limit tangent lines at S are finite in number. If $ax + by = 0$ is not one of them, it is easily seen to have, in a sufficiently small disc centered at S, an odd number of contacts with (1.1).

Then h is an odd number.

If k is even, $n > h > k$ implies $n \geq k + 2$ (in fact $n > k + 2$, as n is odd). If k is odd then, possibly, $h = k$. However, as n is odd and greater than h, it holds $n \geq k + 2$ again and the proof is complete. □

REMARK 2. A far more general result, of which Theorem 8 is a particular case, can be proved, namely

THEOREM 9. *Let* (1.1) *be a polynomial system of degree n with a singular point S having $k > 1$ equally oriented elliptic sectors. Then, except possibly for* (1.1) *homogeneous of degree* 3, $k \leq n - 2$, i.e. (8.8).

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