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SUB-LAPLACIANS OF HOLOMORPHIC L^p-TYPE ON EXPONENTIAL LIE GROUPS

ABSTRACT. — In this survey article, I shall give an overview on some recent developments concerning the L^p-functional calculus for sub-Laplacians on exponential solvable Lie groups. In particular, I shall give an outline on some recent joint work with W. Hebisch and J. Ludwig on sub-Laplacians which are of holomorphic L^p-type, in the sense that every L^p-spectral multiplier for $p \neq 2$ will be holomorphic in some domain.

KEY WORDS: Solvable Lie group; Sub-Laplacian; Lebesgue space; Spectral multiplier.

1. INTRODUCTION

Let T be a self-adjoint linear operator on a Hilbert space $L^2(X, d\mu)$, and denote by $T = \int_{\mathbb{R}} \lambda dE_\lambda$ its spectral resolution.

If m is a bounded Borel function on $\mathbb{R}$, then we call m an *L^p-multiplier for* T $(1 \leq p < \infty)$, if $m(T) := \int_{\mathbb{R}} m(\lambda) dE_\lambda$ extends from $L^p \cap L^2(X, d\mu)$ to a bounded operator on $L^p(X, d\mu)$. We shall denote by $\mathcal{M}_p(T)$ the space of all L^p-multipliers for T, and by $\sigma_p(T)$ the L^p-spectrum of T.

In this survey article, the operators of interest will be Laplacians or sub-Laplacians on manifolds, mostly even Lie groups, but the setting will apply to larger classes of operators as well, for instance to «Laplacians» on homogeneous trees (see *e.g.* [17]).

We say that T admits a *differentiable L^p-functional calculus*, if there exists some $k \in \mathbb{N}$, such that $C_0^k(\mathbb{R}) \subset \mathcal{M}_p(T)$. A typical example for this type of behaviour is the classical Laplacian $T = -\Delta = -\sum_j \frac{\partial^2}{\partial x_j{}^2}$ on Euclidean space $\mathbb{R}^d$. In this case, $m(-\Delta)$ is the convolution operator $f \mapsto f \star K_m$, where the convolution kernel K_m is the inverse Fourier transform of the function $\xi \mapsto m(|\xi|^2)$ on $\mathbb{R}^d$. More generally, it is known that every left-invariant Laplacian or sub-Laplacian on a connected Lie group G of polynomial growth admits a differentiable L^p-functional calculus, for $1 \leq p < \infty$. There are even multiplier theorems of Marcinkiewicz-Mikhlin-Hörmander type known for such operators, see *e.g.* [1, 6, 11, 12, 14, 23-26].

On the other hand, we shall say that T is of *holomorphic L^p-type*, if there exist some non-isolated point λ_0 in the L^2-spectrum $\sigma_2(T)$ and an open complex neighborhood W of λ_0 in $\mathbb{C}$, such that every $m \in \mathcal{M}_p(T) \cap C_\infty(\mathbb{R})$ extends holomorphically to W. Here, $C_\infty(\mathbb{R})$ denotes the space of all continuous functions on $\mathbb{R}$ vanishing at infinity.

It is this rather opposite behaviour on which I shall concentrate.

REMARK 1.1. If T is of holomorphic L^p-type, and if in addition T admits a joint core in every space $L^q(X)$, then

$$\overline{W} \subset \sigma_p(T)\,. \tag{1.1}$$

In particular,

$$\sigma_2(T) \subsetneq \sigma_p(T)\,. \tag{1.2}$$

A fundamental class of operators with such type of behaviour are Laplace-Beltrami operators on Riemannian symmetric spaces of the non-compact type. Such an operator is of holomorphic L^p-type for every $p \in [1\,,\infty[\,, p \neq 2$. Even the maximal domain $W = W_p$ to which all L^p-multipliers will extend holomorphically is known in this case [8, 3, 31, 2]. The «bad» behaviour of these operators with respect to L^p-functional calculi is closely linked to the exponential volume growth of the underlying Riemannian manifolds.

2. EXPONENTIAL LIE GROUPS

In the sequel, I shall restrict myself to the case where $X = G$ is an *exponential* Lie group, which means that the exponential mapping $\exp : \mathfrak{g} \to G$ is a diffeomorphism from the Lie algebra $\mathfrak{g}$ of G onto G. Such groups are known to be solvable.

We fix a *left-invariant* Haar measure dg on G, and shall identify an element X of the Lie algebra with the *right-invariant* vector field on G, which is given by

$$Xf(g) := \lim_{t \to 0} \frac{1}{t}\left[f((\exp tX)g) - f(g)\right].$$

Choose right invariant vector fields $X_1\,, \ldots\,, X_k$ in $\mathfrak{g}$ generating $\mathfrak{g}$ as a Lie algebra, and form the *sub-Laplacian*

$$L = -\sum_{j=1}^{k} X_j^2\,.$$

By [29, 19] L is hypoelliptic and essentially self-adjoint as an operator on $L^2(G, dg)$ with domain $\mathcal{D}(G)$. We denote its closure again by L. Since G is amenable, one has

$$\sigma_2(L) = [0\,, \infty[\,. \tag{2.1}$$

QUESTION. Which type of L^p-functional calculi does $T := L$ admit?

In view of what has been said before, in trying to answer this question we may concentrate on the case where the group G has exponential volume growth. But then the example of Laplace-Beltrami operators on symmetric spaces rather suggests that L should be of holomorphic L^p-type for $p \neq 2$. It therefore rather came as a surprise when the following fact was discoverd:

- Certain (sub-) Laplacians on certain exponential Lie groups of exponential volume growth, notably so-called AN-groups, do admit differentiable L^p-functional calculi, see *e.g.* [13, 9, 4, 27, 15, 28].

A typical example is the $ax + b$-group, whose Lie algebra has a basis T, X satisfying the commutation relations $[T, X] = X$.

- On the other hand, it turned out that there are exponential Lie groups which do admit sub-Laplacians of holomorphic L^p-type:
  Denote by $B = \exp(\mathfrak{b})$ *Boidol's group*, *i.e.* the Lie group whose Lie algebra $\mathfrak{b}$ is spanned by elements T, X, Y, U, satisfying the following commutation relations:
  $$[T, X] = X,\ \ [T, Y] = -Y,\ \ [X, Y] = U,\ \text{ where } U \text{ is central}.$$
  Then the sub-Laplacian $L := -(T^2 + X^2 + Y^2)$ is of holomorphic L^p-type for every $p \neq 2$, see [7].

These results indicate that the question above can have no easy answer, and it is indeed still widely open. Nevertheless, interesting progress has been made in extending the results in [7] to wider classes of groups and operators L, for instance in [22] and furthermore in [16], which lead to a certain conjecture which I am going to describe next.

Let me only mention that important progress has recently been made also on the question whether those sub-Laplacians, which do admit differentiable L^p-functional calculi, even allow for multipliers of Mikhlin-Hörmander type (see [17]).

In order to formulate our conjecture, I have to briefly recall some basic facts from the representation theory of exponential Lie groups, see *e.g.* [5, 22].

2.1. *Unitary representations.*

If $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation of G on the Hilbert space $\mathcal{H} = \mathcal{H}_\pi$, then we denote the integrated representation of $L^1(G) = L^1(G, dg)$ again by π, *i.e.* $\pi(f)\xi := \int_G f(g)\pi(g)\xi\, dg$ for every $f \in L^1(G)$, $\xi \in \mathcal{H}$. For $X \in \mathfrak{g}$, we denote by $d\pi(X)$ the infinitesimal generator of the one-parameter group of unitary operators $t \mapsto \pi(\exp tX)$.

For a given function f on G, we write

$$\left[\lambda(g)f\right](x) := f(g^{-1}x), \quad g, x \in G,$$

for the *left-regular* action of G.

Recall that the *modular function* Δ_G on G is defined by the equation

$$\int_G f(xg)\, dx = \Delta_G(g)^{-1} \int_G f(x)\, dx, \quad g \in G.$$

We put

$$f^*(g) := \Delta_G^{-1}(g)\overline{f(g^{-1})}.$$

Then $f \mapsto f^*$ is an isometric involution on $L^1(G)$, and for any unitary representation π of G, we have

$$\pi(f)^* = \pi(f^*).$$

The group G is said to be *symmetric*, if the associated group algebra $L^1(G)$ is symmetric, *i.e.* if every element $f \in L^1(G)$ with $f^* = f$ has a real spectrum with respect to the involutive Banach algebra $L^1(G)$.

Recall that the unitary dual $\hat{G}$ of G, *i.e.* the space of all equivalence classes of irreducible unitary representations of G, can be constructed by means of the *orbit method.* More precisely, if $\ell \in \mathfrak{g}^*$ is a linear form on $\mathfrak{g}$, one can always find a so-called Vergne-polarization $\mathfrak{p} = \mathfrak{p}(\ell)$ for ℓ. This is a suitable subalgebra of $\mathfrak{g}$ which is isotropic with respect to the skew form $B_\ell(X, Y) := \ell([X, Y])$ on $\mathfrak{g}$, *i.e.* $\ell([X, Y]) = 0 \quad \forall X, Y \in \mathfrak{p}$, and of maximal possible dimension among such isotropic subalgebras. Then

$$\chi_\ell(\exp X) := e^{i\ell(X)}, \quad X \in \mathfrak{p}$$

defines a unitary character of the subgroup $P := \exp(\mathfrak{p})$, and one can show that the induced representation

$$\pi_\ell := \mathrm{ind}_P^G \chi_\ell$$

acts irreducibly on its representation space $\mathcal{H}_\ell$. Moreover, two such representations π_ℓ and $\pi_{\ell'}$ are equivalent if and only if there exists some $g \in G$ such that $\ell' = \mathrm{Ad}^*(g)\ell$, where Ad^* denotes the co-adjoint representation of G on $\mathfrak{g}^*$. Finally, one can show that every irreducible unitary representation of G is equivalent to one of these induced representation, *i.e.* if we denote by

$$\Omega(\ell) := \mathrm{Ad}^*(G)\ell$$

the *coadjoint orbit* of ℓ, then the so-called *Kirillov-map*

$$K : \mathfrak{g}^*/\mathrm{Ad}^*(G) \to \hat{G}, \quad \Omega(\ell) \mapsto [\pi_\ell]$$

is a bijection. Even more is true. Both spaces $\mathfrak{g}^*/\mathrm{Ad}^*(G)$ and $\hat{G}$ can be endowed with natural topologies, the first with the quotient topology induced from the Euclidean topology of $\mathfrak{g}^*$, and the second with the Jacobson topology, pulled back from the primitive ideal space of the C^*-hull of $L^1(G)$. Then the following deep theorem holds true

THEOREM 2.1 [21]. *The Kirillov map K is a homeomorphism.*

2.2. *The conjecture.*

The following theorem links our problem to the symmetry question for G.

THEOREM 2.2 (see [20]). *If G is symmetric, then for every sub-Laplacian L on G one has $\sigma_p(L) = \sigma_2(L) \quad \forall\, 1 \le p < \infty$.*

In view of (1.1), this implies in particular that G is non-symmetric, if there exists a sub-Laplacian on G which is of holomorphic L^p-type for some $p \neq 2$.

The symmetry question has been studied for exponential Lie groups by various authors, with fundamental contributions by H. Leptin, and a complete answer had eventually been given by D. Poguntke in terms of the following condition, which had originally been introduced by J. Boidol in a different context.

If $\ell \in \mathfrak{g}^*$, denote by $\mathfrak{g}(\ell) := \ker \mathrm{ad}^*(\ell) = \{X \in \mathfrak{g} : \ell([X, Y]) = 0 \quad \forall\ Y \in \mathfrak{g}\}$ the *stabilizer* of ℓ under the coadjoint action ad^*. Moreover, if $\mathfrak{m}$ is any Lie algebra, consider the descending central series

$$\mathfrak{m} = \mathfrak{m}^1 \supset \mathfrak{m}^2 \supset \dots ,$$

i.e. $\mathfrak{m}^2 = [\mathfrak{m}, \mathfrak{m}]$, and $\mathfrak{m}^{k+1} = [\mathfrak{m}, \mathfrak{m}^k]$, and denote by $\mathfrak{m}^\infty$ its «bottom», *i.e.*

$$\mathfrak{m}^\infty = \bigcap_k \mathfrak{m}^k .$$

$\mathfrak{m}^\infty$ is the smallest ideal $\mathfrak{k}$ in $\mathfrak{m}$ such that $\mathfrak{m}/\mathfrak{k}$ is nilpotent. Put

$$\mathfrak{m}(\ell) := \mathfrak{g}(\ell) + [\mathfrak{g}, \mathfrak{g}] .$$

Then we say that ℓ respectively the associated coadjoint orbit $\Omega(\ell)$ satisfies *Boidol's condition* (B), if

$$\text{(B)} \qquad \ell\,|_{\mathfrak{m}(\ell)^\infty} \neq 0 .$$

THEOREM 2.3 [30]. *The exponential Lie group G is non-symmetric if and only if there exists a coadjoint orbit satisfying Boidol's condition.*

Boidol's group is, by the way, the lowest dimensional example of a non-symmetric exponential Lie group.

Non-symmetry of G alone, however, does not necessarily force sub-Laplacians on G to be of holomorphic L^p-type, as is shown by the study of some distinguished Laplacian on the AN-group arizing in the Iwasawa-decomposition of $SL(3, \mathbb{R})$. This fact, as well as the proof in the main theorem in [22], suggests the following

CONJECTURE. *There exists a sub-Laplacian on G which is of holomorphic L^p-type for some $p \neq 2$ if and only if there exists some coadjoint orbit which is closed and satisfies Boidol's condition* (B).

3. THE MAIN RESULT

If Ω is a coadjoint orbit, and if $\mathfrak{n}$ denotes the nilradical of $\mathfrak{g}$, then

$$\Omega|_{\mathfrak{n}} := \{\ell|_{\mathfrak{n}} : \ell \in \Omega\} \subset \mathfrak{n}^*$$

will denote the restriction of Ω to $\mathfrak{n}$.

The following result has been proved recently in joint work with W. Hebisch und J. Ludwig.

THEOREM 3.1 [16]. *Let G be an exponential solvable Lie group, and assume that there exists a coadjoint orbit $\Omega(\ell)$ satisfying Boidol's condition* (B), *whose restriction to the nilradical $\mathfrak{n}$ is closed. Then every sub-Laplacian on G is of holomorphic L^p-type, for $1 \leq p < \infty$, $p \neq 2$.*

REMARKS. (*a*) The condition that the restriction of a coadjoint orbit to the nilradical be closed is a stronger condition than the closedness of the orbit itself.

(*b*) Under the hypotheses of the theorem, we obtain in particular that

$$\sigma_2(L) \subsetneq \sigma_p(L) \quad \text{for} \quad p \neq 2 .$$

This results has been proved independently by D. Poguntke (Poguntke, oral communication).

(*c*) What we really use in the proof is the following property of the orbit Ω:

Ω is closed, and for every real character ν of $\mathfrak{g}$ which does not vanish on $\mathfrak{g}(\ell)$, there exists a sequence $\{\tau_n\}_n$ of real numbers such that $\lim_{n\to\infty} \Omega + \tau_n\nu = \infty$ *in the orbit space.*

Here, a *character* means an element $\nu \in \mathfrak{g}^*_{\mathbb{C}}$, such that $\nu([\mathfrak{g}, \mathfrak{g}]) = \{0\}$.

This property is a consequence of the closedness of $\Omega|_{\mathfrak{n}}$. There are, however, many examples where the condition above is satisfied, so that the conclusion of the theorem still holds, even though the restriction of Ω to the nilradical is not closed. It is an open problem whether the condition above automatically holds whenever the orbit Ω is closed.

In the remaining part of the article, I shall try to explain the meaning of our conditions in Theorem 3.1 and sktech some of the main ideas in its proof. For full proofs and further details, the interested reader is referred to the articles [22, 16].

3.1. *On the meaning of the conditions in Theorem 3.1.*

Let us denote by $C^*(G)$ the C^*-hull of the group algebra $L^1(G)$, und recall that λ denotes the left-regular representation of G. If we consider $\lambda(L^1(G))$ as a subspace of the space of bounded operators $\mathcal{B}(L^2(G))$ on $L^2(G)$ then, since G is amenable,

$$C^*(G) = \overline{\lambda\big(L^1(G)\big)} \subset \mathcal{B}\big(L^2(G)\big) .$$

Of course, every unitary representation of $L^1(G)$ extends to $C^*(G)$. The following result translates the topological condition that an orbit be closed into an analytic condition; it is a consequence of Theorem 2.1.

Proposition 3.2. *Suppose G is an exponential solvable Lie group, and let $\ell \in \mathfrak{g}^*$. If the orbit $\Omega(\ell)$ is closed, then $\pi_\ell(C^*(G))$ is the algebra of all compact operators on $\mathcal{H}_\ell$. In particular, $\pi_\ell(f)$ is compact for every $f \in L^1(G)$.*

The second result is a kind of «Riemann-Lebesgue lemma».

Proposition 3.3. *Suppose G is an exponential solvable Lie group, and let $\ell \in \mathfrak{g}^*$ with coadjoint orbit $\Omega := \Omega(\ell)$. Assume that the restriction of Ω to the nilradical $\mathfrak{n}$ of the Lie algebra $\mathfrak{g}$ is closed. Then the orbit Ω is closed itself, and for any real character ν of $\mathfrak{g}$ which does not vanish on the stabilizer $\mathfrak{g}(\ell)$ of ℓ, we have that*

$$\lim_{|\tau|\to\infty} \Omega + \tau\nu = \lim_{|\tau|\to\infty} \Omega(\ell + \tau\nu) = \infty \tag{3.1}$$

in the orbit space. In particular,

$$\lim_{|\tau|\to\infty} \|\pi_{\ell+\tau\nu}(f)\| = 0 \tag{3.2}$$

for every $f \in L^1(G)$.

3.2. *Representations on mixed L^p-spaces.*

Another important ingredient in the proof of Theorem 3.1 is the construction of certain bounded representations on mixed L^p-spaces. For related constructions in the context of semi-simple Lie groups, see [10] (compare also [18] for an earlier appearance of mixed L^p-spaces on groups).

A careful analysis of «roots» on $\mathfrak{g}$ reveals that, given any $p \in [1, \infty[$, one may find Euclidean spaces X, Y and a real character $\Delta_{\underline{p}} : G \to \mathbb{R}$, a kind of «modulus for L^p», such that

$$\pi_\ell^{\underline{p}}(g) := \Delta_{\underline{p}}(g)\lambda(g), \quad g \in G \tag{3.3}$$

defines an isometric representation of G on the mixed L^p-space $L^{\underline{p}} := L^p(X, L^2(Y))$, endowed with the norm

$$\|f\|_{\underline{p}} := \left(\int_X \left(\int_Y |f(x, y)|^2 \, dy \right)^{p/2} dx \right)^{1/p} .$$

More precisely, λ and hence $\pi_\ell^{\underline{p}}$ act on functions $f : G \to \mathbb{C}$, satisfying the covariance condition

$$f(xp) = \overline{\chi}_\ell(p) f(x) \quad \forall\, x \in G, p \in P ,$$

with χ_ℓ and P as in Section 2.1, and the measure space $X \times Y$ is identified with the quotient space G/P.

This can be done in such a way that

$$\pi_\ell^{\underline{2}} \simeq \pi_\ell . \tag{3.4}$$

3.3. *A holomorphic family of operators.*

In the sequel, we shall always make the following

ASSUMPTION. $\ell \in \mathfrak{g}^*$ satisfies Boidol's condition, and $\Omega(\ell)|_{\mathfrak{n}}$ is closed.

By means of Boidol's condition, one can then set up the construction in such a way that

$$\Delta_{\underline{p}} \Delta_{\underline{2}}^{-1}\big(\exp(X)\big) = e^{(\frac{1}{2} - \frac{1}{p})\nu(X)}, \quad X \in \mathfrak{g} , \tag{3.5}$$

where $\nu \in \mathfrak{g}^*$ is a real character of $\mathfrak{g}$ satisfying

$$\nu|_{\mathfrak{g}(\ell)} \neq 0 . \tag{3.6}$$

For any complex number z in the strip

$$\Sigma := \big\{ \zeta \in \mathbb{C} : |\mathrm{Im}\, \zeta| < 1/2 \big\} ,$$

let Δ_z be the complex character of G given by

$$\Delta_z\big(\exp(X) \big) := e^{-iz\nu(X)}, \quad X \in \mathfrak{g} ,$$

and χ_z the unitary character

$$\chi_z(\exp(X)) := e^{-i\mathrm{Re}\,(z)\nu(X)}\,, \quad X \in \mathfrak{g}\,.$$

Since, by (3.5),

$$\Delta_z = \chi_z \Delta_{\underline{p(z)}} \Delta_{\underline{2}}^{-1}\,,$$

if we define $p(z) \in]1\,,\infty[$ by the equation

$$\mathrm{Im}\,(z) = 1/2 - 1/p(z)\,, \tag{3.7}$$

we see that the representation π_ℓ^z, given by

$$\pi_\ell^z(x) := \Delta_z(x)\pi_\ell(x) = \chi_z(x)\pi_\ell^{\underline{p(z)}}(x)\,, \quad x \in G\,, \tag{3.8}$$

is an isometric representation on the space $L^{\underline{p(z)}}$.

Next, observe that for every $t > 0$, e^{-tL} is a convolution operator

$$e^{-tL}f = h_t \star f\,, \tag{3.9}$$

where the $\{h_t\}_{t>0}$ form a 1-parameter semigroup of smooth probability measures in $L^1(G)$, the *heat-semigroup.* By means of Gaussian-type estimates for these heat kernels, one can then conclude that

$$\pi_\ell^z(h_1) = \pi_\ell^{\underline{q}}(\Delta_z \Delta_{\underline{q}}^{-1} h_1) \in \mathcal{B}(L^{\underline{q}})\,, \quad 1 \leq q < \infty\,. \tag{3.10}$$

Moreover, from Proposition 3.2 one obtains:

$$\pi_\ell^z(h_1) = \pi_\ell(\Delta_z h_1) \quad \text{is compact on} \quad L^{\underline{2}}\,. \tag{3.11}$$

Let us define a family of operators by setting

$$T(z) := \pi_\ell^z(h_1)\,, \quad z \in \Sigma\,.$$

More precisely, we write $T_q(z)$ in place of $T(z)$, if we consider $T(z)$ as a bounded operator on $L^{\underline{q}}$. The spectrum of $T_q(z)$ will be denoted by $\sigma_q(z)$. Adapting a classical interpolation theorem of M.A. Krasnoselskii, one deduces from (3.10) and (3.11) the following

Proposition 3.4. (*a*) *The mapping* $\Sigma \ni z \mapsto T_q(z)$ *is an analytic family of compact operators on* $L^{\underline{q}}$ *in the sense of Kato, for every* $1 < q < \infty$.
(*b*) *If* $z \in \mathbb{R}$, *then the following additional properties hold true:*

- (*i*) $T_2(z)$ *is self-adjoint on* $L^{\underline{2}}$.
- (*ii*) $\sigma_q(z) = \sigma_2(z) \subset \mathbb{R} \quad \forall\, 1 < q < \infty$.
- (*iii*) *If* λ *is an eigenvalue of* $T_2(z)$, *then* $\lambda \in \sigma_p(z)$, *and if* ξ *is any associated eigenfunction in* $L^{\underline{p}}$, *then* ξ *lies in every* $L^{\underline{q}}$, *for* $1 < q < \infty$.

Notice in particular that $\sigma_q(z)$ is discrete away from the origin, if $1 < q < \infty$. This allows for the application of analytic perturbation theory, which eventually leads to the following

PROPOSITION 3.5. *Let $1 \leq p_0 < 2$. There exist an open neighborhood U of a point $z_0 \in \mathbb{R}$ in the complex strip Σ and holomorphic mappings*

$$\lambda : U \to \mathbb{C}$$

and

$$\xi : U \to \bigcap_{p_0 \leq p \leq p_0'} L^p_- ,$$

such that $\xi(z) \neq 0$ and

$$T(z)\xi(z) = \lambda(z)\xi(z) \quad \textit{for every} \quad z \in U . \tag{3.12}$$

Moreover, shrinking U, if necessary, one can find a constant $C > 0$ such that

$$\|\xi(z)\|_{L^p_-} \leq C \quad \textit{for every} \quad z \in U,\ p \in [p_0\, ,\, p_0'] . \tag{3.13}$$

3.4. *Application to spectral multipliers.*

Assume now that $F \in \mathcal{M}_p(L) \cap C_\infty(\mathbb{R})$.

Then, applying a duality argument, one may conclude that also $F \in \mathcal{M}_{p'}(L)$, and so, by interpolation, that $F \in \mathcal{M}_q(L)$, and

$$\|F(L)\|_{L^q \to L^q} \leq C, \quad \text{for every} \quad q \in [p\, ,\, p'] . \tag{3.14}$$

(here, we assume w.r. that $p \leq 2$.)

Now, recall the following result by R. Coifman and G. Weiss.

THEOREM 3.6 (Transference). *Suppose G is an amenable group, and let $K \in C_0(G)$. Denote by $\lambda(K)$ the convolution operator $\lambda(K) : f \mapsto K \star f$ on $L^q(G)$. Then, if π is any isometric representation of G on a Lebesgue space $L^q(Z)$, one has the estimate*

$$\|\pi(K)\|_{L^q(Z) \to L^q(Z)} \leq \|\lambda(K)\|_{L^q(G) \to L^q(G)} \tag{3.15}$$

The idea is then to apply this to the operator $F(L)$, but there are certain obstacles to be overcome.

Of course, by the Schwartz' kernel theorem, $F(L)$ is of the form $F(L)f = K \star f$, where K is a suitable distribution on G, but we cannot expect K to lie in $C_0(G)$. By replacing the multiplier $F(\lambda)$ by $e^{-\lambda}F(\lambda)$, which amounts to replacing the kernel K by the smooth kernel $h_1 \star K$, we may at least assume that $K \in C^\infty(G)$.

Next, in order to force the support of K to be compact, one may devise a sequence of Herz-Schur-multipliers ϕ_n on G of compact support and tending to 1 uniformly on compact sets. Such a sequence does exist, since G is amenable. One can then approximate K in a certain way by the kernels $K_n := K\phi_n$, which have compact support.

Grossly oversimplifying (and somewhat cheating), let us therefore henceforth assume for simplicity that $K \in C_0(G)$. Then there still remains the problem that we have to adapt Theorem 3.5 to the setting of our mixed L^p-spaces. However, this is no serious obstacle, since one can embed $L^{p(z)} = L^{p(z)}(X, L^2(Y))$ into some space $L^{p(z)}(Z)$, for instance by means of Rademacher functions and Khintchin's inequality.

Eventually one can then apply the transference method and obtains from (3.14) that

$$\|\pi_\ell^z(K)\|_{L^{p(z)} \to L^{p(z)}} \leq C \quad \forall\, z \in \Sigma \quad \text{such that} \quad p(z) \in [p\,, p'] \,. \tag{3.16}$$

We choose $\psi \in C_0(X \times Y)$ such that $\langle \xi(z_0)\,, \psi \rangle \neq 0$, where $z_0 \in \mathbb{R}$ and $\xi(z)$ are as in Proposition 3.5. We then obtain a holomorphic mapping

$$h : U \to \mathbb{C}\,, \quad z \mapsto \langle \pi_\ell^z(K)\xi(z)\,, \psi \rangle$$

on a suitable neighborhood U of z_0 in the complex plane, since, by (3.13) and (3.15),

$$|h(z)| \leq \|\pi_\ell^z(K)\|_{L^{p(z)} \to L^{p(z)}} \|\xi(z)\|_{L^{p(z)}} \|\psi\|_{L^{p(z)'}}$$

is uniformly bounded on U.

- Consider the case $\operatorname{Im} z = 0$.

If we define $\mu(z)$ by $\lambda(z) = e^{-\mu(z)}$, then we have

$$\pi_\ell^z(h_1)\xi(z) = T(z)\xi(z) = \lambda(z)\xi(z)\ ,$$

hence

$$d\pi_\ell^z(L)\xi(z) = \mu(z)\xi(z)\,.$$

By means of spectral theory on Hilbert spaces, one concludes that

$$h(z) = \langle \pi_\ell^z(F(L))\xi(z)\,, \psi \rangle = \langle F(d\pi_\ell^z(L))\xi(z)\,, \psi \rangle = F(\mu(z))\, \langle \xi(z)\,, \psi \rangle\ ,$$

i.e.

$$F \circ \mu(z) = \frac{h(z)}{\langle \xi(z)\,, \psi \rangle}\,, \quad \forall z \in U \cap \mathbb{R}\,. \tag{3.17}$$

Here, we assume that we have chosen U so small that the denominator of (3.17) does not vanish.

Clearly the right-hand side of (3.17) extends holomorphically to U, and thus $F \circ \mu$ extends to a holomophic function on U. It thus only remains to show that the function μ cannot be constant, in order to conclude that the multiplier F is holomorphic near $\lambda_0 := \mu(z_0)$. It is here where the «Riemann-Lebesgue-Lemma» Proposition 3.3 comes into play.

Indeed, if $z = \tau \in \mathbb{R}$, then $\pi_\ell^z \simeq \pi_{\ell+\tau\nu}$, hence, by (3.2), $\lim_{\tau\to\infty} \|\pi_\ell^\tau(h_1)\| = 0$. This implies $\lambda(\tau) \to 0$, hence $\mu(\tau) \to \infty$ as $\tau \to \infty$. Thus μ is not constant and so, modifying $z_0 \in \mathbb{R} \cap U$ slightly, if necessary, we may assume that $\mu'(z_0) \neq 0$. But then μ is a local bi-holomorphism, and so F is holomorphic near λ_0.

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