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TANGENTIAL CAUCHY-RIEMANN EQUATIONS ON QUADRATIC CR MANIFOLDS

Abstract. — We study the tangential Cauchy-Riemann equations $\overline{\partial}_b u = \omega$ for $(0, q)$-forms on quadratic CR manifolds. We discuss solvability for data ω in the Schwartz class and describe the range of the tangential Cauchy-Riemann operator in terms of the signatures of the scalar components of the Levi form.

Key words: Tangential Cauchy-Riemann complex; Kohn Laplacian; CR manifolds; Global solvability; Hypoellipticity.

1. Introduction

Let V be an n-dimensional complex vector space, W an m-dimensional real vector space, $W^{\mathbb{C}}$ the complexification of W, and

$$\Phi : V \times V \longrightarrow W^{\mathbb{C}}$$

a Hermitean map (*i.e.* $\Phi(z, z') = \overline{\Phi(z', z)}$ for every $z, z' \in V$, where complex conjugation in $W^{\mathbb{C}}$ is referred to the real form W).

We consider the associated *quadratic manifold*

$$S = \{(z, t + iu) \in V \times W^{\mathbb{C}} : u = \Phi(z, z)\} \tag{1}$$

in $n + m$ complex dimensions. Then S is a CR manifold of CR-dimension n and real codimension m.

We consider the $\overline{\partial}_b$-complex on S, mapping $(0, q)$-forms on S into $(0, q + 1)$-forms, for $0 \leq q \leq n$.

We shall consistently use the parameters $(z, t) \in V \times W$ to denote the element $(z, t + i\Phi(z, z)) \in S$. A natural Lie group structure can be introduced on $V \times W$ (as described in Section 1); this group will be denoted by G_Φ.

The fiber of the vector bundle $\Lambda^{0,q}(T^* S)$ over each point of S can be identified in the trivial way with the exterior product $\Lambda_q = \Lambda^{0,q}(V^*)$. Through the identification of S with $V \times W = G_\Phi$, we then regard $(0, q)$-forms on S as vector valued functions on G_Φ with values in Λ_q.

Depending on the integrability or regularity conditions imposed on the forms under consideration, we shall denote the different spaces of $(0, q)$-forms as $L^2(G_\Phi) \otimes \Lambda_q$, $\mathcal{S}(G_\Phi) \otimes \Lambda_q$, $\mathcal{S}'(G_\Phi) \otimes \Lambda_q$, etc.

We shall also need other linear bundles over G_Φ, with fibers $\mathrm{End}\,(\Lambda_q)$, $\mathrm{Hom}\,(\Lambda_q, \Lambda_{q+1})$, etc. The corresponding spaces of sections will be denoted in a similar way.

We address the following problem: determine under which assumptions on the Hermitean form Φ, $q \geq 1$ and the $\overline{\partial}_b$-closed form $\omega \in \mathcal{S}(G_\Phi) \otimes \Lambda_q$, the equation $\overline{\partial}_b u = \omega$ has a solution $u \in \mathcal{S}'(G_\Phi) \otimes \Lambda_q$.

Our approach is based on the results in [7] concerning the Kohn Laplacian

$$\Box_b^{(q)} = \overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b$$

acting on $(0\,,q)$-forms on S. The operator $\overline{\partial}_b^*$ is the adjoint of the L^2-closure of $\overline{\partial}_b$ w.r. to the Haar measure $dz\,dt$ on G_Φ and to a fixed inner product on V.

The answer depends on the signatures of the scalar-valued forms

$$\Phi^\lambda(z\,,z') = \lambda\big(\Phi(z\,,z')\big)\ ,$$

depending on $\lambda \in W^*$.

In general, W^* decomposes as the union of an open regular set, consisting of those λ for which Φ^λ is non-degenerate, and its complement, the singular set. We further decompose the regular set as the union of the open sets Ω_q, defined by the condition that Φ^λ has q positive and $n-q$ negative eigenvalues. Some Ω_q may be empty, and it may well happen that all of them are empty, *i.e.* that Φ^λ is degenerate for every λ.

In general, we denote by $\Omega \subset W^*$ the set of those λ for which Φ^λ has maximum rank. If there are non-degenerate Φ^λ, then Ω is the regular set.

In [7] we proved the following theorem.

THEOREM 1.1. *The following are equivalent*:

(*i*) Ω_q *is non-empty*;

(*ii*) $\Box_b^{(q)}$ *is locally solvable,* i.e. *given any smooth* $(0\,,q)$*-form* ω*, the equation* $\Box_b^{(q)} u = \omega$ *has a solution in a fixed neighborhood of the origin*;

(*iii*) $\Box_b^{(q)}$ *has a tempered fundamental solution,* i.e. $K_q \in \mathcal{S}'(G_\Phi) \otimes \mathrm{End}(\Lambda_q)$ *such that* $\Box_b^{(q)}(\omega * *K_q) = \omega$ *for every* $\omega \in \mathcal{S}(G_\Phi) \otimes \Lambda_q$;

(*iv*) *the* L^2*-null-space of* $\Box_b^{(q)}$ *is trivial.*

If Ω_q *is non-empty, then* $\Box_b^{(q)}$ *has a relative fundamental solution* $K_{q,rel} \in \mathcal{S}'(G_\Phi) \otimes \otimes \mathrm{End}\,(\Lambda_q)$, i.e. *such that* $\Box_b^{(q)}(\omega * K_{q,rel}) = (I - \mathcal{C}_q)\omega$ *for every* $\omega \in \mathcal{S}(G_\Phi) \otimes \Lambda_q$*, where* $\mathcal{C}_q$ *is the orthogonal projection of* $L^2(G_\Phi) \otimes \Lambda_q$ *onto the null-space of* $\Box_b^{(q)}$.

The notation we have used is such that, if f and g are functions on G_Φ with values in Λ_q and in $\mathrm{End}\,(\Lambda_q)$ respectively, then

$$f * g(z\,,t) = \int_{G_\Phi} g\big((w\,,u)^{-1}(z\,,t)\big) f(w\,,u)\,dw\,du$$

takes values in Λ_q.

We derive from this the following result.

THEOREM 1.2. *Let $q \geq 1$. The equation $\overline{\partial}_b u = \omega$ has a solution $u \in \mathcal{S}'(G_\Phi) \otimes \Lambda_{q-1}$ for a given $\overline{\partial}_b$-closed form $\omega \in \mathcal{S}(G_\Phi) \otimes \Lambda_q$ if and only if $C_q \omega = 0$. In particular, the equation has a solution $u \in \mathcal{S}'(G_\Phi) \otimes \Lambda_{q-1}$ for every $\omega \in \mathcal{S}(G_\Phi) \otimes \Lambda_q$ such that $\overline{\partial}_b \omega = 0$ if and only if Ω_q is empty.*

The proof will require a precise description of the L^2-null-space of $\Box_b^{(q)}$. This will be done in Section 3.

The question of solvability for the Cauchy-Riemann complex has drawn a great deal of interest since Lewy's celebrated example of a non-solvable differential operator [6]. Such question is also of interest for extension phenomena, such as Bochner's theorem, see [1-5] for historical background and references.

The relations between solvability of the $\overline{\partial}_b$-complex and signatures of the scalar components of the Levi form for general CR manifolds have been investigated by several authors [9-11].

In the case of a quadratic CR manifold S, Rossi and Vergne [8] showed that if Φ is non-degenerate, *i.e.* is there exists $\lambda \in W^*$ such that Φ^λ is non-degenerate, then condition (*i*) in Theorem 1.1 is necessary and sufficient for the solvability of the $\overline{\partial}_b$-equation. The degenerate case is not included in their analysis. On the other hand, it is as an immediate consequence of our Theorem 1.2 that if Φ is degenerate, then Ω_q is empty and the $\overline{\partial}_b$-equation $\overline{\partial}_b u = \omega$ is solvable for all $\overline{\partial}_b$-closed, $(0, q)$-forms ω in the Schwartz class, for all $q \geq 1$.

2. THE LIE GROUP ASSOCIATED TO S AND ITS REPRESENTATIONS

We define the following product between two elements $(z, t), (z', t') \in V \times W$:

$$(z, t)(z', t') = \left(z + z', t + t' + 2\operatorname{Im}\Phi(z, z')\right),$$

which induces a step-two nilpotent Lie group structure on $V \times W$. We call G_Φ this group and $\mathfrak{g}_\Phi$ its Lie algebra.

For $v \in V$, let $\partial_v f$ denote the directional derivative of a function f in the direction v. The left-invariant vector field X_v on G_Φ that coincides with ∂_v at the origin is given by

$$X_v f(z, t) = \partial_v f(z, t) + 2\operatorname{Im}\Phi(z, v) \cdot \nabla_t f(z, t).$$

If J denotes the complex structure on V, we define $Z_v, \overline{Z}_v \in {\mathfrak{g}_\Phi}^{\mathbb{C}}$ as

$$Z_v = \tfrac{1}{2}(X_v - iX_{Jv}) = \tfrac{1}{2}(\partial_v - i\partial_{Jv}) + i\overline{\Phi(z, v)} \cdot \nabla_t,$$

$$\overline{Z}_v = \tfrac{1}{2}(X_v + iX_{Jv}) = \tfrac{1}{2}(\partial_v + i\partial_{Jv}) - i\Phi(z, v) \cdot \nabla_t.$$

The relevance of the group G_Φ in our context is justified by the fact that the operators $\overline{Z}_v$ coincide with the tangential Cauchy-Riemann operators on S.

The following commutation rules hold:

$$
\begin{aligned}
&[Z_v, Z_{v'}] = [\overline{Z}_v, \overline{Z}_{v'}] = 0 , \\
&[Z_v, \overline{Z}_{v'}] = -2i\Phi(v, v') \cdot \nabla_t .
\end{aligned}
\tag{2}
$$

Hence G_Φ is step-two nilpotent.

It can be shown that the Lie algebras that arise in this way can be characterized as follows.

Proposition 2.1. *A real Lie algebra $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathfrak{g}_\Phi$ associated to a quadratic* CR *manifold if and only if $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{w}$, with $[\mathfrak{v}, \mathfrak{v}] \subseteq \mathfrak{w}$, $[\mathfrak{g}, \mathfrak{w}] = 0$ and there is a complex structure J on $\mathfrak{v}$ such that $[Jv, Jv'] = [v, v']$ for every $v, v' \in \mathfrak{v}$.*

We summarize the description of the irreducible unitary representations of G_Φ.

For $\lambda \in W^*$, let Φ^λ be the scalar-valued form $\Phi^\lambda(v, v') = \lambda(\Phi(v, v'))$. Denote by V_0^λ the radical of Φ^λ, *i.e.* the subspace of the v such that $\Phi^\lambda(v, v') = 0$ for every $v' \in V$.

Let V_1^λ be the orthogonal complement of V_0^λ in V, w.r. to the fixed inner product. Let also V_r^λ be a real form of V_1^λ on which Φ^λ is real. For $z \in V$, we set $z = z' + z''$, with $z' \in V_1^\lambda$, $z'' \in V_0^\lambda$, and $z' = x' + iy'$ with $x', y' \in V_r^\lambda$.

Lemma 2.2. *Let $\lambda \in W^*$, and let η be a linear functional on V_0^λ. Define the representation $\pi_{\lambda,\eta}$ of G_Φ on $L^2(V_r^\lambda)$ as*

$$
\big(\pi_{\lambda,\eta}(z, t)\phi\big)(\xi) = e^{i(\lambda(t)+2\mathrm{Re}\,\eta(z''))} e^{-2i\Phi^\lambda(y', \xi+x')} \phi(\xi + 2x') .
\tag{3}
$$

Then $\pi_{\lambda,\eta}$ is an irreducible unitary representation of G_Φ. Conversely, any irreducible unitary representation of G_Φ is equivalent to one and only one $\pi_{\lambda,\eta}$.

It is convenient to diagonalize Φ^λ with respect to an orthonormal basis $\{v_1^\lambda, \dots, v_n^\lambda\}$ of V, in such a way that $v_j^\lambda \in V_r^\lambda$ for $j \le \nu(\lambda)$ and $v_j^\lambda \in V_0^\lambda$ for $j > \nu(\lambda)$, where $0 \le \nu(\lambda) = \mathrm{rank}\,\Phi^\lambda = \dim V_1^\lambda \le n$. We set

$$
\mu_j = \mu_j(\lambda) = \Phi^\lambda(v_j^\lambda, v_j^\lambda) .
\tag{4}
$$

Calling

$$
Z_j^\lambda = \tfrac{1}{2}(X_{v_j^\lambda} - iX_{Jv_j^\lambda}) , \quad \overline{Z}_j^\lambda = \tfrac{1}{2}(X_{v_j^\lambda} + iX_{Jv_j^\lambda}) ,
$$

a standard computation gives that

$$
\begin{aligned}
d\pi_{\lambda,\eta}(Z_k^\lambda) &= \begin{cases} \partial_{\xi_j} - \mu_k \xi_k & \text{if } k \le \nu \\ i\overline{\eta}_{k-\nu(\lambda)} & \text{if } k > \nu \end{cases} \\
d\pi_{\lambda,\eta}(\overline{Z}_k^\lambda) &= \begin{cases} \partial_{\xi_j} + \mu_k \xi_k & \text{if } k \le \nu \\ i\eta_{k-\nu(\lambda)} & \text{if } k > \nu , \end{cases}
\end{aligned}
\tag{5}
$$

with $\eta_{k-\nu(\lambda)} = \eta(v_k^\lambda) \in \mathbb{C}$.

For a function f on G_Φ, we define

$$\pi_{\lambda,\eta}(f) = \int f(z,t)\pi_{\lambda,\eta}(z,t)^{-1}\, dzdt\,. \tag{6}$$

This definition has the effect that $\pi_{\lambda,\eta}(f*g) = \pi_{\lambda,\eta}(g)\pi_{\lambda,\eta}(f)$, and that

$$\pi_{\lambda,\eta}(\mathcal{L}f) = d\pi_{\lambda,\eta}(\mathcal{L})\pi_{\lambda,\eta}(f)\ ,$$

for any left-invariant differential operator $\mathcal{L}$.

We denote by h_j the j-th Hermite function on the real line:

$$h_j(t) = (2^j\sqrt{\pi}j!)^{-1/2}(-1)^j e^{t^2/2}\frac{d^j}{dt^j}e^{-t^2}\ , \tag{7}$$

and, for a given a multi-index $m \in \mathbb{N}^{\nu(\lambda)}$, we set

$$h_m^\lambda(\xi) = \prod_{j=1}^{\nu(\lambda)} |\mu_j|^{1/4} h_{m_j}\big(|\mu_j|^{1/2}\xi_j\big)\,. \tag{8}$$

By (7) and (8) above we have that $(d_t - t)h_j = \sqrt{2(j+1)}h_{j+1}$ and $(d_t + t)h_j = -\sqrt{2j}h_{j-1}$. From these it follows that

$$(\partial_{\xi_k} + \mu_k\xi_k)h_m^\lambda = (-\operatorname{sgn}\mu_k)\big((2m_k+1)|\mu_k| - \mu_k\big)^{1/2} h^\lambda_{m-(\operatorname{sgn}\mu_k)e_k}\ ,$$

where $e_k = (0,\dots,1,\dots,0)$ denotes the k-th element of the standard basis. Hence, for any unitary irreducible representation $\pi_{\lambda,\eta}$ of G_Φ,

$$d\pi_{\lambda,\eta}(\overline{Z}_k^\lambda)h_m^\lambda = \begin{cases} (-\operatorname{sgn}\mu_k)\big((2m_k+1)|\mu_k| - \mu_k\big)^{1/2} h^\lambda_{m-(\operatorname{sgn}\mu_k)e_k} & \text{if } k \le \nu(\lambda) \\ i\eta_{k-\nu(\lambda)}h_m^\lambda & \text{if } k > \nu(\lambda)\,. \end{cases}$$

Analogously, one gets

$$d\pi_{\lambda,\eta}(Z_k^\lambda)h_m^\lambda = \begin{cases} \operatorname{sgn}\mu_k\big((2m_k+1)|\mu_k| + \mu_k\big)^{1/2} h^\lambda_{m+(\operatorname{sgn}\mu_k)e_k} & \text{if } k \le \nu(\lambda) \\ i\overline{\eta}_{k-\nu(\lambda)}h_m^\lambda & \text{if } k > \nu(\lambda)\,. \end{cases}$$

The matrix coefficient $\langle \pi_{\lambda,\eta}(f)h_m^\lambda, h_{m'}^\lambda\rangle$ will be denoted as $\widehat{f}(\lambda,\eta;m,m')$. For a $(0,q)$-form ω, the notation $\widehat{\omega}(\lambda,\eta;m,m') \in \Lambda_q$ will be used with the same meaning. If Φ^λ is non-degenerate, so that $V_0^\lambda = \{0\}$, we drop the parameter η.

Define

$$D(\lambda) = \prod_{j=1}^{\nu(\lambda)} |\mu_j|\,.$$

LEMMA 2.3. *The function $D(\lambda)$ is smooth on Ω, the subset of W^* where $\nu(\lambda)$ is maximum. The Plancherel formula for G_Φ is*

$$\|f\|_2^2 = \int_\Omega \int_{(V_0^\lambda)^*} \|\pi_{\lambda,\eta}(f)\|_{HS}^2\, d\eta\, D(\lambda)\, d\lambda\ , \tag{9}$$

where $d\lambda$ is an appropriately normalized Lebesgue measure on W^* and $d\eta$ is the volume element on $(V_0^\lambda)^*$ induced by the inner product on V.

Observe that the domain of integration in (9) has a natural differentiable structure, as it can be identified with $\Omega \times \mathbb{C}^{n-\nu}$, with $\nu = \max \nu(\lambda)$.

3. The $\overline{\partial}_b$-complex on G_Φ and the null space of $\Box_b^{(q)}$

Let $\{v_1, \dots, v_n\}$ be any orthonormal basis of V with respect to the given inner product. Let $(z_1, \dots, z_n)$ denote the coordinates on V with respect to this basis. We denote by $d\overline{z}^I$ the $(0, q)$-form $d\overline{z}_{i_1} \wedge \cdots \wedge d\overline{z}_{i_q}$, where $I = (i_1, \dots, i_q)$ is a strictly increasing multi-index. Given a $(0, q)$-form $\phi = \sum_{|I|=q} \phi_I d\overline{z}^I$ with smooth coefficients, we have

$$\overline{\partial}_b \phi = \sum_{|I|=q} \sum_{k=1}^{n} \overline{Z}_k(\phi_I) d\overline{z}_k \wedge d\overline{z}^I = \sum_{|J|=q+1} \sum_{k,|I|=q} \epsilon_{kI}^J \overline{Z}_k(\phi_I) d\overline{z}^J , \tag{10}$$

where

$$Z_j = \tfrac{1}{2}(X_{v_j} - iX_{Jv_j}) , \quad \overline{Z}_j = \tfrac{1}{2}(X_{v_j} + iX_{Jv_j}) , \quad j = 1, \dots, n ,$$

and $\epsilon_{kI}^J = 0$ if $J \neq \{k\} \cup I$ as sets, and it equals the parity of the permutation that rearranges $(k, i_1, \dots, i_q)$ in increasing order if $J = \{k\} \cup I$.

Then $\overline{\partial}_b^*$ can be easily computed to yield that

$$\overline{\partial}_b^* \left(\sum_{|I|=q} \phi_I d\overline{z}^I \right) = \sum_{|J|=q-1} \left(- \sum_{k,|I|=q} \epsilon_{kJ}^I Z_k \phi_I \right) d\overline{z}^J . \tag{11}$$

The Kohn Laplacian is defined as $\Box_b^{(q)} = \overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b$. This is explicitly computed in [7, Proposition 2.1].

We assume now that Ω_q is non-empty. Then also $\Omega_{n-q} = -\Omega_q$ is non-empty and it contains a Zariski-open subset Ω'_{n-q} where the number of distinct eigenvalues of Φ^λ is maximum. It is shown in [7] that locally on Ω'_{n-q} the eigenvalues $\mu_j(\lambda)$ and the basis elements v_j^λ in (4) are well-defined real-analytic functions of λ.

We can therefore find a locally finite open covering $\{U_j\}$ of Ω'_{n-q} such that for each j there is an orthonormal coordinate system $(z_1^\lambda, \dots, z_n^\lambda)$ on V that varies smoothly with $\lambda \in U_j$ and diagonalizes Φ^λ as $\Phi^\lambda(z, z) = \sum_{k=1}^n \mu_k |z_k^\lambda|^2$.

For a multi-index L of length q with entries $\ell_1 < \ell_2 < \cdots < \ell_q$, we denote by ω_L^λ the form $d\overline{z}_{\ell_1}^\lambda \wedge \cdots \wedge d\overline{z}_{\ell_q}^\lambda$.

Let $\overline{L} = \overline{L}_j$ the multi-index of length q containing those k for which $\mu_k < 0$. Let also $\{\rho_j\}$ be a smooth partition of unity on Ω'_{n-q} subordinated to the given covering.

The following result is proven in [7, Lemma 5.1].

Lemma 3.1. *Let* $\omega \in L^2(G_\Phi) \otimes \Lambda_q$. *The following are equivalent*:

(*i*) ω *is in the null space of* $\Box_b^{(q)}$;

(*ii*) $\pi_\lambda(\omega) = 0$ *a.e. outside of* Ω_{n-q} *and, a.e. on each* U_j, $\pi_\lambda(\omega) = T^\lambda \otimes \omega_{\overline{L}}^\lambda$, *where* T^λ *is a Hilbert-Schmidt operator on* $L^2(V_r^\lambda)$, *with range in the linear span of* h_0^λ.

Corollary 3.2. *The subspace* $(\mathcal{S}(G_\Phi) \otimes \Lambda_q) \cap \ker \Box_b^{(q)}$ *is dense in* $(L^2(G_\Phi) \otimes \Lambda_q) \cap \ker \Box_b^{(q)}$ *in the* L^2*-topology. Moreover, if* $\omega \in (L^2(G_\Phi) \otimes \Lambda_q) \cap \ker \Box_b^{(q)}$ *then* $\overline{\partial}_b\omega = \overline{\partial}_b^*\omega = 0$ *in the sense of distributions.*

Proof. By Lemma 5.1 in [7] it follows that $\omega \in L^2(G_\Phi) \otimes \Lambda_q$ lies in $\ker \Box_b^{(q)}$ if and only if $\widehat{\omega}(\lambda; k, \ell) \neq 0$ implies that $\lambda \in \Omega_{n-q}$, $\ell = 0$ and $\widehat{\omega}(\lambda; k, 0) = c(\lambda, k)\omega_{\overline{L}}^\lambda$ is such that

$$\|\omega\|_{L^2}^2 = \int_{\Omega_{n-q}} \sum_k |c(\lambda, k)|^2 D(\lambda)\, d\lambda .$$

Therefore, for any $\varepsilon > 0$ it is possible to find a positive integer k_0 and Schwartz functions ψ_k with support in Ω_{n-q}, identically zero for $k > k_0$, and such that

$$\int_{\Omega_{n-q}} \sum_k |c(\lambda, k) - \psi_k(\lambda)|^2 D(\lambda)\, d\lambda < \varepsilon .$$

By Lemmas 3.1 and 5.2 in [7] there exists $\psi \in \mathcal{S}(G_\Phi) \otimes \Lambda_q$ such that $\widehat{\psi}(\lambda; k, \ell) = \delta_{0\ell}\psi_k(\lambda)\omega_{\overline{L}}^\lambda$. Hence $\psi \in \ker \Box_b^{(q)}$ and $\|\omega - \psi\| < \varepsilon$.

The second assertion is clear for Schwartz forms and follows from the density above for an L^2-form. $\Box$

4. Proof of Theorem 1.2

The proof is based on the following lemma.

Lemma 4.1. *There is a family* $\{K_q\}_{0 \le q \le n}$, *with* $K_q \in \mathcal{S}'(G_\Phi) \otimes \mathrm{End}(\Lambda_q)$, *satisfying the following properties*

(*i*) K_q *is a fundamental* (resp. *a relative fundamental*) *solution of* $\Box_b^{(q)}$ *if* Ω_q *is empty* (resp. *non-empty*);

(*ii*) *the following identity holds*

$$\overline{\partial}_b(\omega * K_q) = (\overline{\partial}_b\omega) * K_{q+1} , \tag{12}$$

for all $\omega \in \mathcal{S}(G_\Phi) \otimes \Lambda_q$.

Assuming the validity of the lemma, we prove Theorem 1.2.

Proof of Theorem 1.2. Given ω as in the statement, it suffices to define $u = \overline{\partial}_b^*(\omega * K_q)$. Since $\overline{\partial}_b\omega = 0$, we have

$$\overline{\partial}_b^*\overline{\partial}_b(\omega * K_q) = \overline{\partial}_b^*(\overline{\partial}_b\omega * K_{q+1}) = 0 ,$$

by (12). Then

$$\overline{\partial}_b u = \overline{\partial}_b \overline{\partial}_b^*(\omega * K_q) = \Box_b^{(q)}(\omega * K_q) = (I - \mathcal{C}_q)\omega\,.$$

On the other hand, notice that if $\omega \in \mathcal{S}(G_\Phi)\otimes\Lambda_q$ is such that $\mathcal{C}_q\omega \neq 0$, then the equation $\overline{\partial}_b u = \omega$ cannot be solved. Indeed, using Corollary 3.2, let $\omega_k \in (\mathcal{S}(G_\Phi)\otimes\Lambda_q)\cap\ker\Box_b^{(q)}$ be a sequence converging to $\mathcal{C}_q\omega$ in $L^2(G_\Phi)\otimes\Lambda_q$. Then,

$$\|\mathcal{C}_q\omega\|_{L^2}^2 = \langle\omega\,,\mathcal{C}_q\omega\rangle = \lim_{k\to+\infty}\langle\overline{\partial}_b u\,,\omega_k\rangle = \lim_{k\to+\infty}\langle u\,,\overline{\partial}_b^*\omega_k\rangle = 0\ ,$$

a contradiction. □

Proof of Lemma 4.1. Given $\lambda\in\Omega$ and (in case $\nu = \operatorname{rank}\Phi^\lambda < n$) $\eta\in(V_0^\lambda)^*$, define the operator $A_q^{\lambda,\eta}$ on $L^2(V_r^\lambda)\otimes\Lambda_q$ as

$$A_q^{\lambda,\eta}(h_m^\lambda\otimes\omega_L^\lambda) = \begin{cases} 0 & \text{if } \nu = n,\ \lambda\in\Omega_q \\ & m=0 \text{ and } L=\overline{L} \\ \dfrac{1}{\alpha_L^\lambda + |\eta|^2 + \sum_{j=1}^n (2m_j+1)|\mu_j|}(h_m^\lambda\otimes\omega_L^\lambda) & \text{otherwise}\,. \end{cases}$$

When $\nu = n$, we simply drop η from this formula altogether. Due to its diagonal form, it is easy to check that $A_q^{\lambda,\eta}$ is a bounded operator.

The (relative) fundamental solutions K_q constructed in [7] are such that for any pair of Schwartz $(0\,,q)$-forms $\omega\,,\sigma$ on G_Φ,

$$\langle\omega * K_q\,,\sigma\rangle = -\int_\Omega\int_{(V_0^\lambda)^*}\langle\!\langle A_q^{\lambda,\eta}\pi_{\lambda,\eta}(\omega)\,,\pi_{\lambda,\eta}(\sigma)\rangle\!\rangle\,d\eta\,D(\lambda)\,d\lambda\,. \tag{13}$$

The inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ is the ordinary Hilbert-Schmidt inner product for operators on $L^2(V_r^\lambda)\otimes\Lambda_q$. If $\nu = n-1$, the integral in $d\eta$ in (13) may not be absolutely convergent for certain values of λ, and it must be taken in a principal value sense.

We shall show below that

$$d\pi_{\lambda,\eta}(\overline{\partial}_b)\circ A_q^{\lambda,\eta} = A_{q+1}^{\lambda,\eta}\circ d\pi_{\lambda,\eta}(\overline{\partial}_b)\,. \tag{14}$$

This implies that, if $\omega\in\mathcal{S}(G_\Phi)\otimes\Lambda_q$ and $\sigma\in\mathcal{S}(G_\Phi)\otimes\Lambda_{q+1}$, then

$$\langle\!\langle A_q^{\lambda,\eta}\pi_{\lambda,\eta}(\omega)\,,\pi_{\lambda,\eta}(\overline{\partial}_b^*\sigma)\rangle\!\rangle = \langle\!\langle A_{q+1}^{\lambda,\eta}\pi_{\lambda,\eta}(\overline{\partial}_b\omega)\,,\pi_{\lambda,\eta}(\sigma)\rangle\!\rangle\,. \tag{15}$$

Hence

$$\langle\omega * K_q\,,\overline{\partial}_b^*\sigma\rangle = \langle(\overline{\partial}_b\omega) * K_q\,,\sigma\rangle\ , \tag{16}$$

which implies (12). When $\nu = n-1$, the derivation of (16) from (15) requires some more care, but we leave the details to the interested reader.

The proof of (14) is very easy when $\nu < n$ and $\eta \neq 0$, or when $\nu = n$ and $\lambda\notin\Omega_q\cup\Omega_{q+1}$. In both cases, in fact, $A_q^{\lambda,\eta} = -d\pi_{\lambda,\eta}(\Box_b^{(q)})^{-1}$ and $A_{q+1}^{\lambda,\eta} = -d\pi_{\lambda,\eta}(\Box_b^{(q+1)})^{-1}$. It is the sufficient to apply $d\pi_{\lambda,\eta}$ to both sides of the identity $\overline{\partial}_b\Box_b^{(q)} = \Box_b^{(q+1)}\overline{\partial}_b$.

Assume now that $\nu = n$ and $\lambda \in \Omega_q$. We recall that

$$d\pi_\lambda(\overline{\partial}_b)(h_m^\lambda \otimes \omega_L^\lambda) = \sum_{|J|=q+1} \left(\sum_k \epsilon_{kL}^J d\pi_\lambda(\overline{Z}_k^\lambda) h_m^\lambda \right) \otimes \omega_J^\lambda \ ,$$

and that

$$d\pi_{\lambda,\eta}(\overline{Z}_k^\lambda) = \sum_{|J|=q+1} \left(\sum_k \epsilon_{kL}^J (-\operatorname{sgn}\mu_k) \big((2m_k+1)|\mu_k| - \mu_k\big)^{1/2} h^\lambda_{m-(\operatorname{sgn}\mu_k)e_k} \right) \otimes \omega_J^\lambda \, .$$

Hence,

$$d\pi_\lambda(\overline{\partial}_b) \circ A_q^\lambda (h_m^\lambda \otimes \omega_L^\lambda) =$$

$$= \sum_{|J|=q+1} \sum_k \epsilon_{kL}^J \frac{(-\operatorname{sgn}\mu_k)\big(|\mu_k|(2m_k+1) - \mu_k\big)^{1/2}}{\alpha_L^\lambda + \sum_{j=1}^n (2m_j+1)|\mu_j|} (h^\lambda_{m-(\operatorname{sgn}\mu_k)e_k} \otimes \omega_J^\lambda) \tag{17}$$

if $m \neq 0$ or $L \neq \overline{L}$ and $d\pi_\lambda(\overline{\partial}_b) \circ A_q^{\lambda,\eta}(h_m^\lambda \otimes \omega_L^\lambda) = 0$ otherwise.

On the other hand,

$$d\pi_\lambda(\overline{\partial}_b)(h_m^\lambda \otimes \omega_L^\lambda) = \sum_{|J|=q+1} \sum_k \epsilon_{kL}^J (-\operatorname{sgn}\mu_k)\big(|\mu_k|(2m_k+1) - \mu_k\big)^{1/2} (h^\lambda_{m-(\operatorname{sgn}\mu_k)e_k} \otimes \omega_J^\lambda) \ ,$$

so that

$$A_{q+1}^{\lambda,\eta} \circ d\pi_\lambda(\overline{\partial}_b)(h_m^\lambda \otimes \omega_L^\lambda) =$$

$$= \sum_{|J|=q+1} \sum_k \epsilon_{kL}^J \frac{(-\operatorname{sgn}\mu_k)\big(|\mu_k|(2m_k+1) - \mu_k\big)^{1/2}}{\alpha_J^\lambda - 2\mu_k + \sum_{j=1}^n (2m_j+1)|\mu_j|} (h^\lambda_{m-(\operatorname{sgn}\mu_k)e_k} \otimes \omega_J^\lambda) \, . \tag{18}$$

Notice that $\lambda \in \Omega_q$ implies that $\lambda \notin \Omega_{q+1}$. If $J = \{k\} \cup L$ as sets, then it is easy to check that $\alpha_L^\lambda = \alpha_J^\lambda - 2\mu_k$. When $L = \overline{L}$ and $k \notin \overline{L}$ then $\mu_k > 0$ which implies that

$$d\pi_\lambda(K_{q+1}) \circ d\pi_\lambda(\overline{\partial}_b)\big(h_0^\lambda \otimes \omega_{\overline{L}}^\lambda\big) = 0 \, .$$

Thus, (17) and (18) prove equality (14) for $\lambda \in \Omega_q$.

The argument for $\lambda \in \Omega_{q+1}$ is similar to the case $\lambda \in \Omega_q$ and we omit it. □

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