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## Unique determination of local $CR$-maps by their jets: a survey

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# UNIQUE DETERMINATION OF LOCAL CR-MAPS BY THEIR JETS: A SURVEY

Abstract. — We survey results on unique determination of local CR-automorphisms of smooth CR-manifolds and of local biholomorphisms of real-analytic CR-submanifolds of complex spaces by their jets of finite order at a given point. Examples generalizing [28] are given showing that the required jet order may be arbitrarily high.

Key words: Jets of biholomorphic maps; Real hypersurfaces of finite type; CR-manifolds.

## 1. The scope of the problem

For many geometric structures, their automorphisms are known to be uniquely determined by finitely many derivatives at a given point. For example, an isometry $\varphi\colon M \to M$ of a connected Riemannian manifold $M$ is uniquely determined by its value $\varphi(p)$ and its differential $d\varphi(p)$ at any fixed point $p \in M$. In this case, we say that $\varphi$ is uniquely determined by its 1-jet $j_p^1\varphi$. Another example is given by complex projective transformations of $\mathbb{P}^n$ that are uniquely determined by their 2-jets but not by their 1-jets at a fixed point. In Complex Analysis an analogous statement to the first example is H. Cartan's uniqueness theorem [16] saying that biholomorphic automorphisms of a bounded domain in $\mathbb{C}^n$ are uniquely determined by their 1-jets at any fixed point of the domain. An analogous statement to the second example is the well-known fact that the biholomorphic automorphisms of the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ are uniquely determined by their 2-jets (but not by their 1-jets) at a fixed boundary point $p \in \partial\mathbb{B}^n$. An interesting feature of this unique determination by 2-jets is that it also holds for *local biholomorphic maps* of $\mathbb{C}^n$ leaving an open piece of the boundary $\partial\mathbb{B}^n$ invariant and that, furthermore, $\partial\mathbb{B}^n$ can be replaced by any real-analytic hypersurface $M \subset \mathbb{C}^n$ with nondegenerate Levi form:

Theorem 1.1. *Let $M$ be a real-analytic hypersurface through a point $p$ in $\mathbb{C}^n$ with nondegenerate Levi form and let $H_j\colon U_j \to V_j$, $j = 1, 2$, be two biholomorphic maps, where $U_j$, $V_j$ are open subsets in $\mathbb{C}^n$, $p \in U_j$ and $H_j(M \cap U_j) \subset M$ for any $j$. Then, if $H_1$ and $H_2$ have the same 2-jets at $p$, they coincide in a neighborhood of $p$.*

Theorem 1.1 follows from the classical results of E. Cartan [15], N. Tanaka [35], S.-S. Chern and J.K. Moser [17]. The first natural question, whether the condition on the Levi form can be removed, has obviously negative answer: the *Levi-flat hypersurface* given by $\operatorname{Im} w = 0$ in some local coordinates $z = (z, w) \in \mathbb{C}^2$ is invariant under all biholomorphic maps of $\mathbb{C}^2$ of the form $H(z, w) = (z + F(w), w)$, where $F$ is any entire holomorphic function on $\mathbb{C}$. Clearly the above maps are not determined by their jets at 0 of any finite order.

If $M \subset \mathbb{C}^n$ is a merely smooth Levi-nondegenerate real hypersurface rather than real-analytic, a statement analogous to Theorem 1.1 holds for local CR-diffeomorphisms. (The reader is referred to the Appendix for this and other basic notions used here.)

The illustrated phenomenon suggests the nontrivial and, in general, open problem of finding conditions on real submanifolds $M \subset \mathbb{C}^n$ that imply the conclusion similar to that of Theorem 1.1 with 2-jets possibly replaced by $k$-jets for some finite number $k$. If this conclusion holds for some $k$, we say that local biholomorphisms of $M$ at $p$ are uniquely determined by their $k$-jets (at $p$). Here a *local biholomorphisms of $M$ at $p$* is always assumed to be defined in a neighborhood of $p$ in $\mathbb{C}^n$, to fix $p$ and to send a neighborhood of $p$ in $M$ into $M$.

## 2. The existence of a unique jet determination

For real-analytic hypersurfaces in $\mathbb{C}^2$, necessary and sufficient conditions for unique jet determination were recently obtained by the author jointly with P. Ebenfelt and B. Lamel [20]:

Theorem 2.1. *Local biholomorphisms of a real-analytic hypersurface $M \subset \mathbb{C}^2$ at a point $p \in M$ are uniquely determined by their $k$-jets for some $k$ if and only if $M$ is not Levi-flat (near $p$).*

Thus, in $\mathbb{C}^2$, the Levi-flat hypersurface is the only one, for which its local biholomorphisms are not uniquely determined by their jets of any order. In the case when $M$ is of *finite type* at $p$, *i.e.* does not contain a complex hypersurface through $p$, the statement of Theorem 2.1 follows from previous results of M.S. Baouendi, P. Ebenfelt and L.P. Rothschild [6, Corollary 2.7] and from a more recent stronger result of M.S. Baouendi, N. Mir and L.P. Rothschild [7, Corollary 1.8]. In fact, it shown in [7] that unique $k$-jet determination for some $k$ holds for local biholomorphisms (and even for finite holomorphic maps) of CR-submanifolds of any codimension that are of finite type and holomorphically nondegenerate (see below).

In the case when $M$ is of infinite type at $p$, the proof of Theorem 2.1 consists of three steps. The first step is to establish, for any $k$, unique determination of $k$-jets of local biholomorphisms along the complex hypersurface $S \subset M$ through $p$ by their $(k+1)$-jets at the origin (such $S$ exists and its germ at $p$ is unique since $M$ is assumed to be of infinite type). The second step is based on the method of singular complete systems due to P. Ebenfelt [19] used to obtain singular ODE's along real curves transverse to $S$ satisfied by all local biholomorphisms. Finally, the third step is to show that holomorphic solutions of the ODE's are uniquely determined by their jets at the reference point.

The statement of Theorem 2.1 and its generalization for formal invertible maps was independently obtained by R.T. Kowalski [29] for so-called hypersurfaces $M \subset \mathbb{C}^2$ of 1-*infinite type* as defined in [19].

In dimension higher than 2, the situation is different: the hypersurface given by $\operatorname{Im} w = |z_1|^2$ in local coordinates $(z_1, \dots, z_{n-1}, w) \in \mathbb{C}^n$, $n \geq 3$, is not Levi-flat but has no unique determination property as in Theorem 2.1. Here, as well as for the Levi-flat hypersurface, the lack of any unique jet determination is due to the invariance of the hypersurface $M \subset \mathbb{C}^n$ under translations in some coordinate direction. Such translations form a complex one-parameter family of transformations. We are led to the following definition originally due to N. Stanton [33].

DEFINITION 2.2. A real-analytic submanifold $M \subset \mathbb{C}^n$ is called *holomorphically degenerate* at a point $p \in M$ if one of the following equivalent conditions holds:

($i$) There exists a nontrivial complex one-parameter family of local biholomorphisms leaving $M$ invariant, *i.e.* a holomorphic map $\varphi \colon \Delta \times U \to V$, where $\Delta \subset \mathbb{C}$, $U, V \subset \mathbb{C}^n$ are open connected neighborhoods of 0 and $p$ respectively, such that $\varphi(0, p) = p$, $\partial\varphi/\partial t \not\equiv 0$ and each $\varphi(t, \cdot)$ is biholomorphic and sends $M \cap U$ into $M$.

($ii$) There exists a nontrivial holomorphic vector field on $\mathbb{C}^n$ in a neighborhood $U$ of $p$ whose both real and imaginary parts are tangent to $M \cap U$.

Otherwise $M$ is called *holomorphically nondegenerate* at $p$.

It is easy to see that that both conditions ($i$) and ($ii$) are equivalent. A less trivial fact is that, for a connected real-analytic submanifold $M$, the property to be holomorphically nondegenerate is independent of the point $p \in M$ (see [1, Proposition 1.2.1]). In particular, it follows that conditions ($i$) and ($ii$) in Definition 2.2 are also equivalent to

($iii$) $M$ can be straightened at some point $q \in M$ in the connected component of $p$, *i.e.* there exist local holomorphic coordinates $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$, in a neighborhood $U$ of $q$ such that $M \cap U = \{(z, w) : z \in M_0\}$, where $M_0 \subset \mathbb{C}^{n-1}$ is some real-analytic submanifold.

We also note that, for $n \geq 3$, a holomorphically nondegenerate hypersurface may be everywhere Levi-degenerate, the simplest example is the tube over the «light cone» given by $(\operatorname{Re} z_1)^2 + \cdots + (\operatorname{Re} z_{n-1})^2 = (\operatorname{Re} w)^2$ in the coordinates $(z_1, \dots, z_{n-1}, w) \in \mathbb{C}^n$. The reader is referred to Proposition 3.3 for other characterizations of holomorphic nondegeneracy that can be often easier to check.

Using the family $\varphi \colon \Delta \times U \to V$ in ($i$), it is not difficult to see that, for any holomorphic function $F \colon U \to \Delta$ with $F(p) = 0$, $dF(p) = 0$, the map $Z \mapsto \varphi(F(Z), Z)$ defines local biholomorphism of $M$ at $p$. Hence, if $M$ is holomorphically degenerate at $p$, its local biholomorphisms are not uniquely determined by their jets of any order. For hypersurfaces in $\mathbb{C}^2$, Theorem 2.1 shows that the opposite statement holds, since in this case holomorphic degeneracy is equivalent to Levi-flatness (see definition ($iii$)). The corresponding question for hypersurfaces of higher dimension is open:

CONJECTURE 2.3. *Local biholomorphisms of a real-analytic hypersurface $M \subset \mathbb{C}^n$ $(n \geq 3)$ at a point $p \in M$ are uniquely determined by their $k$-jets for some $k$ if and only if $M$ is holomorphically nondegenerate at $p$.*

The statement of Conjecture 2.3 obviously fails if $M$ is taken to be a CR-submanifold of higher codimension rather than a hypersurface, *e.g.* any product $M = \mathbb{R} \times \times M_0 \subset \mathbb{C} \times \mathbb{C}^n$ where $M_0 \subset \mathbb{C}^n$ a Levi-nondegenerate hypersurface, is a counterexample.

In the important case when $M$ is of *finite type* at $p$, Conjecture 2.3 has been recently solved [7, Corollary 1.8]. Previous results in the direction of Conjecture 2.3 are due to [12, 2, 3, 38, 5, 6]. Conjecture 2.3 is open for any holomorphically nondegenerate hypersurface $M$ of infinite type in $\mathbb{C}^n$ with $n \geq 3$. The simplest example of such a hypersurface is given by $\operatorname{Im} w = (\operatorname{Re} w)(|z_1|^2 + |z_2|^2)$ in the coordinates $(z_1, z_2, w) \in \mathbb{C}^3$.

In [34] N. Stanton showed that $M$ is holomorphically nondegenerate at $p$ if and only if the Lie algebra $\operatorname{aut}(M, p)$ of germs at $p$ of its infinitesimal automorphisms (*i.e.* of holomorphic vector fields in $\mathbb{C}^n$ whose real parts are tangent to $M$), vanishing at $p$, is finite-dimensional. Using local flows of vector fields, it is not difficult to see that unique jet determination for local biholomorphisms implies that infinitesimal automorphisms in $\operatorname{aut}(M, p)$ are also determined by their jets at $p$ of the same order and, hence, form a finite-dimensional Lie algebra. The converse implication is not known and is equivalent to Conjecture 2.3. Motivated by the mentioned result of N. Stanton, Conjecture 2.3 can be extended to submanifolds of any codimension as follows.

CONJECTURE 2.4. *Local biholomorphisms of a generic real-analytic submanifold $M \subset \mathbb{C}^n$ are uniquely determined by their $k$-jets for some $k$ (at a point $p \in M$) if and only if the Lie algebra* $\operatorname{aut}(M, p)$ *is finite-dimensional.*

One direction is easy to obtain: if $\operatorname{aut}(M, p)$ is infinite-dimensional, the elements of $\operatorname{aut}(M, p)$ are not uniquely determined by their $k$-jets at $p$ for any $k$ and, therefore, no unique jet determination is possible for local biholomorphisms. Note that infinitesimal automorphisms are often easier to handle than local biholomorphisms since they are local solutions of a linear PDE system.

## 3. THE PROBLEM OF FINDING THE OPTIMAL JET ORDER

In previous paragraph we were interested in finding some integer $k$, for which we have unique determination by $k$-jets. It is natural to ask what $k$ is the minimal possible. It is well-known that local biholomorphisms of the quadric

$$Q := \left\{(z, w) \in \mathbb{C}^2 : \operatorname{Im} w = |z|^2\right\} \tag{1}$$

at $p = 0$ are uniquely determined by their 2-jets at 0 but not by their 1-jets because they are given by

$$(z, w) \mapsto \frac{\left(c(z + aw),\, |c|^2 w\right)}{1 - 2i\bar{a}\; z - (r + i|a|^2)w}\,, \quad c \in \mathbb{C}^*,\ a \in \mathbb{C},\ r \in \mathbb{R}\,. \tag{2}$$

As mentioned in Theorem 1.1, the 2-jet determination holds for all real-analytic Levi-nondegenerate hypersurfaces $M$. More precisely, it follows from the theory of S.S. Chern and J.K. Moser [17] that to every local biholomorphism of $M$ at $p$, parameters $c$, $a$ and $r$ as in (2) can be associated and the biholomorphism is uniquely determined by them.

The parameters $c$ and $a$ are in fact determined by the 1-jets of the biholomorphism (2). Moreover, V.K. Beloshapka [11, Lemma 10] proved that, if $(M, p)$ is *nonspherical* (*i.e.* not locally equivalent to any quadric given by $\operatorname{Im} w = q(z, \bar{z})$ where $q: \mathbb{C}^{n-1} \times \mathbb{C}^{n-1} \to \mathbb{C}$ is a hermitian form), then $r$ is uniquely determined by the other parameters $c$ and $a$. Hence, in this case, local biholomorphisms of $(M, p)$ are uniquely determined by their 1-jets. This result was refined by A.V. Loboda [31] who showed that, in this case, the biholomorphisms are in fact uniquely determined by the restrictions of their differentials to $T_p^c M$. We mention also the result of V.K. Beloshapka [12] showing unique determination of local biholomorphisms by their 2-jets for any generic Levi-nondegenerate submanifold $M \subset \mathbb{C}^n$ of higher codimension:

Theorem 3.1. *Let $M \subset \mathbb{C}^n$ be a generic real-analytic submanifold of any codimension with nondegenerate Levi form at $p$ (see below). Then its local biholomorphisms are uniquely determined by their 2-jets at $p$.*

More recently, the Levi-degenerate case has been studied. Here the following higher order nondegeneracy condition appears to be relevant:

Definition 3.2. A smooth CR-submanifold $M \subset \mathbb{C}^N$ is called *finitely nondegenerate* at $p$ (see [4, §11.1]) if, for some integer $l \geq 1$,

$$\text{(3)}\quad \operatorname{span}_{\mathbb{C}}\{\mathcal{T}_{L_s}(\ldots\mathcal{T}_{L_2}(\mathcal{T}_{L_1}\theta)\ldots)(p) : 0 \leq s \leq l,\, L_j \in \Gamma(T^{0,1}M),\, \theta \in \Gamma(T^{*0}M)\} = T_p^{*1,0}M,$$

where $T^{*0}M$ and $T^{*1,0}M$ denote the bundles of complex 1-forms that vanish on $T^cM \times \mathbb{C}$ and on $T^{0,1}M$ respectively and $\mathcal{T}_L$ is the Lie derivative along $L$, given by $\mathcal{T}_L\omega = \mathrm{i}_L d\omega$, where $\mathrm{i}_L$ denotes the contraction.

The reader is referred to [9, 26] for equivalent definitions and further discussion of this notion. Here we mention that the Levi-nondegeneracy condition of V.K. Beloshapka [12] consists of two parts, the first one is equivalent to 1-nondegeneracy in the sense of Definition 3.2 and the second one says that the CR-submanifold $M$ is of type 2 at $p$ (*i.e.* $\mathbb{C} \oplus T_pM$ is spanned by $(1, 0)$ and $(0, 1)$ vector fields and their Lie brakets of length 2). If $M$ is a real hypersurface, the second condition follows from the first one. In higher codimension the conditions are independent.

The condition of finite nondegeneracy is closely related to that of holomorphic nondegeneracy (see [1, 4]):

Proposition 3.3. *A real-analytic CR-submanifold $M \subset \mathbb{C}^n$ is holomorphically nondegenerate at $p \in M$ if and only if the following equivalent conditions hold*:

(*i*) *There exists a point of the connected component of $p$ in $M$ where $M$ is finitely nondegenerate;*
(*ii*) *There exists a point of the connected component of $p$ in $M$ where $M$ is $l$-nondegenerate for some $l$ that does not exceed the CR-dimension of $M$.*

For arbitrary generic finitely nondegenerate submanifolds of finite type, unique jet determination with estimates on the required jet order has been obtained by M.S. Baouendi, P. Ebenfelt and L.P. Rothschild [3] when $M \subset \mathbb{C}^n$ is a real-analytic generic submanifold, by P. Ebenfelt [18] when $M \subset \mathbb{C}^n$ is a smooth hypersurface and by the author jointly with S.-Y. Kim [26] when $M$ is a smooth abstract CR-manifold of any CR-codimension:

THEOREM 3.4. *Let $M$ be a smooth CR-manifold of any codimension $d$ which is $l$-nondegenerate and of finite type at a point $p$. Then local CR-automorphisms of $M$ at $p$ are uniquely determined by their $k$-jets with $k = (d+1)l$. If $M$ is in addition a real-analytic generic submanifold in $\mathbb{C}^n$, its local biholomorphisms at $p$ are also uniquely determined by their $k$ with the same $k$.*

In the proof of Theorem 3.4 in [26], a method of «approximate Segre sets» is applied to obtain a complete differential PDE system that is satisfied by all local CR-diffeomorphisms and that expresses their jets of order $2k+1$ (with $k$ as in Theorem 3.4) through lower order jets. Such a system is essentially reduced to ODE's along real curves and unique determination by $2k$-jets follows from the determination of the solutions of the ODE's. Unique determination by $k$-jets as stated in Theorem 3.4 follows then from a result of M.S. Baouendi, P. Ebenfelt and L.P. Rothschild [5, Theorem 2] that implies under the assumptions of Theorem 3.4 that $2k$-jets of local CR-automorphisms are uniquely determined by their $k$-jets at the same point.

Theorem 3.4 contains, in particular, Theorem 1.1 and gives the right estimate $k=2$, since $l=d=1$ in that case. It also treats some Levi-degenerate hypersurfaces, *e.g.* the tube over the light cone mentioned above which is everywhere 2-nondegenerate. The fact that one needs higher order derivatives in Theorem 3.4 as the degeneracy $l$ grows is common for results about Levi-degenerate hypersurfaces. Therefore, the following result of [20] seems to be unexpected:

THEOREM 3.5. *If $M \subset \mathbb{C}^2$ is a real-analytic hypersurface of finite type at $p \in M$, then local biholomorphisms of $M$ at $p$ are always uniquely determined by their 2-jets.*

Note that any finitely nondegenerate hypersurface $M \subset \mathbb{C}^n$ is automatically of finite type, whereas the example $M = \{\operatorname{Im} w = |z|^{2m}\} \subset \mathbb{C}^2$ with $m \geq 2$ shows that the converse is not true. The example of the quadric (1) shows that the order $k=2$ given by Theorem 3.5 cannot be reduced anymore. The uniform estimate on the jet order in Theorem 3.5 motivates the following

CONJECTURE 3.6. *For any $n \geq 3$, there exists an integer $k = k(n)$ such that, for any holomorphically nondegenerate real-analytic hypersurface $M \subset \mathbb{C}^n$ of finite type, its local biholomorphisms at a point $p \in M$ are uniquely determined by their $k$-jets.*

Examples in the following section show that the assumption that $M$ is of finite type cannot be removed.

## 4. EXAMPLES OF JET DETERMINATION OF ARBITRARILY HIGH ORDER

Very recently R.T. Kowalski [28] gave the first example of a real hypersurface in $\mathbb{C}^n$, for which $k=3$ is needed, *i.e.* local biholomorphisms of which (at a given point) are uniquely determined by their 3-jets but not by their 2-jets. The following lemma gives a generalization of his example showing that, for any $k$, there exists a hypersurface $M_k \subset \mathbb{C}^2$ through 0 (of infinite type at 0) whose local biholomorphisms are uniquely determined by their $(k+1)$-jets at 0 but not by their $r$-jets for any $r \leq k$. In particular, the condition that $M$ is of finite type in Theorem 3.5 cannot be removed.

Lemma 4.1. *Let $Q$ be given by* (1) *and, for every $k \geq 2$, consider the holomorphic map $\varphi_k\colon \mathbb{C}^2 \to \mathbb{C}^2$, $(z, w) \mapsto (zw^{k-1}, w^k)$. Then the preimage $\varphi_k^{-1}(Q)$ contains a real-analytic hypersurface $M_k \subset \mathbb{C}^2$ through* 0 *such that the local biholomorphisms of $M_k$ at* 0 *are precisely those of the form*

$$(z, w) \mapsto \frac{\left(c(z+aw)\,,\ |c|^{2/(1-k)}w\right)}{\left(1 - 2i\bar{a}\ zw^{k-1} - (r + i|a|^2)w^k\right)^{1/k}}\ , \tag{4}$$

*where the data $c$, $a$ and $r$ are as in* (2) *and the principal branch of the $k$th root near* 0 *is chosen.*

Proof. We write $w = u + iv \in \mathbb{C}$. Then $\varphi_k^{-1}(Q)$ is given by

$$\operatorname{Im}(u+iv)^k = |z|^2(u+iv)^{k-1}(u-iv)^{k-1}$$

that can be rewritten as

$$u^{k-1}v + \sum_{j=2}^{k} b_j u^{k-j}v^j = |z|^2 \sum_{l=0}^{2k-2} c_l u^{2k-2-l}v^l$$

for suitably chosen real coefficients $b_2, \ldots, b_k$, $c_0, \ldots, c_{2k-2}$. We assume $u \neq 0$, divide both sides by $u^k$ and set $t = v/u$:

$$t + \sum_{j=2}^{k} b_j t^j = |z|^2 u^{k-2} \sum_{l=0}^{2k-2} c_l t^l\,. \tag{5}$$

By the implicit function theorem, (5) can be solved for $t$ near $(z, u, t) = 0$ in the form $t = \Phi(|z|^2, u)$, where $\Phi$ is a real-analytic function defined in a neighborhood of 0 with $\Phi(0) = 0$. Going back to the original variables $(z, w = u + iv)$ we conclude that $\varphi_k^{-1}(Q)$ contains the real-analytic hypersurface

$$M_k := \left\{(z, w) : \operatorname{Im} w = (\operatorname{Re} w)\Phi(|z|^2, \operatorname{Re} w)\right\}$$

defined in a neighborhood of 0 and passing through 0.

Any biholomorphism $H$ from (4) satisfies $\varphi_k \circ H = H_0 \circ \varphi_k$, where $H_0$ is a local biholomorphism of $Q$ of the form (2) with the same parameters $a$ and $r$ (the parameter $c$ changes to its power). Hence $H$ leaves the germ at 0 of $\varphi_k^{-1}(Q)$ invariant. Moreover, $H$ fixes both 0 and the tangent space $T_0M_k = \{dv = 0\}$. Hence, if $u \neq 0$ and both $z$ and $\operatorname{Im} w/\operatorname{Re} w$ are sufficiently small, then $\operatorname{Re} G(z, w) \neq 0$ and both $F(z, w)$ and $\operatorname{Im} G(z, w)/\operatorname{Re} G(z, w)$ are also sufficiently small, where we write $H = (F, G)$. By the choice of $\Phi$, it follows that $H(z, w) \in M_k$. Thus $H$ leaves the germ $(M_k, 0)$ invariant.

On the other hand, suppose that $H$ is a local biholomorphism of $(M_k, 0)$ defined in a neighborhood $V$ of 0 in $\mathbb{C}^2$. Then $\varphi_k$ is biholomorphic (onto its image) in a neighborhood $U \subset V$ of the interval $\{0\} \times (0, \varepsilon] \subset M_k$ for $\varepsilon > 0$ sufficiently small and maps $M_k \cap U$ into the quadric $Q$. We conclude that $H_0 := \varphi_k \circ H \circ \varphi_k^{-1}$ is a local biholomorphism sending an open piece of $Q$ with 0 in the closure into $Q$ such that $H_0(0, x_n) \to 0$ for any sequence $x_n \in (0, \varepsilon^{1/k}]$, $x_n \to 0$ as $n \to \infty$. Then $H_0$ extends as a birational self-map of $\mathbb{C}^2$ sending an open piece of $Q$ into $Q$ having no poles in 0 and defines a local biholomorphism of $Q$ at 0 which is known to be of the form (2). It is now straightforward to show that the only local biholomorphism $H$ satisfying $\varphi_k \circ H = H_0 \circ \varphi_k$ and fixing $T_0 M_k$ is of the form (4).

For $k = 2$, it can be shown that the hypersurface $M_k$ given by Lemma 4.1 coincides with the hypersurface in [28]. Observe that $M_k$ in Lemma 4.1 is always «generically spherical», *i.e.* is locally biholomorphically equivalent to the quadric $Q$ at every point $p \in M_k$ outside a proper subvariety. This remark motivates the following

Problem 4.2. Does there exist a real-analytic hypersurface $M \subset \mathbb{C}^2$ which is not locally biholomorphically equivalent to $Q$ at any point and whose local biholomorphisms at a point $p \in M$ are not uniquely determined by their 2-jets?

## 5. Other related results

Unique jet determination has been also established for various classes of smooth CR-maps (more general than CR-automorphisms) between smooth CR-manifolds as well as for local holomorphic maps sending one real-analytic generic submanifolds $M \subset \mathbb{C}^n$ and $M' \subset \mathbb{C}^{n'}$. The reader is referred to [38, 5] for local holomorphic maps sending $M$ into $M'$ and $T_p^c M$ surjectively onto $T_{f(p)}^c M'$, to [8] for local biholomorphisms sending $M$ into itself «up to a finite order», to [22, 25, 30] for so-called finitely-nondegenerate local holomorphic maps and to [26] for smooth finitely-nondegenerate CR-maps.

The results on unique jet determination can be refined to give a description of the actual dependence of the maps on their jets. The reader is referred to [2, 38, 22, 25, 8, 18, 30, 20, 26, 29] for related results.

We conclude by mentioning a result of W. Kaup and H. Upmeier [24] on unique determination of complexified infinitesimal automorphisms of arbitrary bounded domains by their 2-jets at a fixed point of the domain and results of D.M. Burns and S.G. Krantz [14], of X. Huang [23], of G. Gentili and S. Migliorini [21] and of the author jointly with L. Baracco and G. Zampieri [10] on unique determination of biholomorphic automorphisms of certain bounded domains by their «higher order jets» at a fixed point on the boundary.

## 6. Appendix: notation and definitions

A smooth ($\mathcal{C}^\infty$) real submanifold $M \subset \mathbb{C}^n$ is called a *CR-submanifold* if its complex tangent space $T_x^c M := T_x M \cap i T_x M$ has constant dimension. If also $T_x M + i T_x M = T_x \mathbb{C}^n$, $M$ is called *generic*. The restriction $J$ of the standard complex structure of $\mathbb{C}^n$ defines a bundle automorphism $J: T^c M \to T^c M$. The triple $(M, T^c M, J)$ is called the *CR-structure* of $M$. The complex dimension of $T_x^c M$ and the real codimension of $T_x^c M$ in $T_x M$ are called *CR-dimension* and *CR-codimension* of $M$ respectively. A $(1, 0)$

(resp. $(0\,,1)$) vector field on $M$ is a complex vector field (in $\mathbb{C}\otimes TM$) of the form $X - iJX$ (resp. $X + iJX$), where $X_x \in T^c_x M$. Every CR-submanifold $M \subset \mathbb{C}^n$ satisfies the *integrability condition* saying that the Lie bracket of two $(1\,,0)$ vector fields is again a $(1\,,0)$ vector field (the same holds also for $(0\,,1)$ vector fields). A CR-structure $(M\,,T^cM\,,J)$ can be also specified abstractly on $M$. Then $T^cM \subset TM$ is a subbundle, $J\colon T^cM \to T^cM$ is a bundle automorphism satisfying $J^2 = -\mathrm{id}$ and the integrability condition is postulated. In this case $M$ is often called an *abstract CR-manifold*.

A smooth map $f\colon M \to M'$ between two (abstract) CR-manifolds is called CR, if its differential $f_*$ sends $T^cM$ into $T^cM'$ and is $J$-linear there. In the special case $M = M'$ and $f$ a diffeomorphism in a neighborhood of a point $p$ fixing $p$ we say that $f$ is a CR-automorphism of $M$ at $p$. If $M$ is a real-analytic CR-submanifolds in $\mathbb{C}^n$, the relation with local biholomorphisms is given by a theorem of G. Tomassini [36]: a real-analytic diffeomorphism of $M$ is CR if and only if it is locally a restriction of a local biholomorphism of $M$ in $\mathbb{C}^n$.

The *finite type* condition here is always understood in the sense of J.J. Kohn [27] and T. Bloom and I. Graham [13] and means that the Lie algebra generated by $(1\,,0)$ and $(0\,,1)$ vector fields on $M$ spans the complexified tangent space of $M$ at $p$. If $M$ is real-analytic, this condition is equivalent to *minimality* of $M$ in the sense of A.E. Tumanov [37] by a theorem of T. Nagano [32] (see also [4]).

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