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A strengthening of the Nyman-Beurling criterion for the Riemann hypothesis

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Teoria dei numeri. — *A strengthening of the Nyman-Beurling criterion for the Riemann hypothesis.* Nota (*) di LUIS BÁEZ-DUARTE, presentata dal Socio E. Bombieri.

ABSTRACT. — According to the well-known Nyman-Beurling criterion the Riemann hypothesis is equivalent to the possibility of approximating the characteristic function of the interval $(0, 1]$ in mean square norm by linear combinations of the dilations of the fractional parts $\{1/ax\}$ for real a greater than 1. It was conjectured and established here that the statement remains true if the dilations are restricted to those where the a's are positive integers. A constructive sequence of such approximations is given.

KEY WORDS: Riemann zeta function; Riemann hypothesis; Nyman-Beurling theorem.

RIASSUNTO. — *Un rafforzamento del criterio di Nyman-Beurling per l'ipotesi di Riemann.* Il noto criterio di Nyman-Beurling per la validità dell'ipotesi di Riemann è equivalente alla possibilità di approssimare la funzione caratteristica dell'intervallo $(0, 1]$ in media quadratica per mezzo di combinazioni lineari delle dilatazioni delle parti frazionarie $\{1/ax\}$ con a reale e maggiore di 1. In questa *Nota* si stabilisce la validità della congettura che il criterio resta vero se le dilatazioni sono ristrette a quelle dove il parametro a è un intero. Si dà inoltre una costruzione esplicita di tali approssimazioni.

1. INTRODUCTION

We denote the fractional part of x by $\varrho(x) = x - [x]$, and let χ stand for the characteristic function of the interval $(0, 1]$. μ denotes the Möbius function. We shall be working in the Hilbert space $\mathcal{H} := L_2((0, \infty), dx)$, where our main object of interest is the subspace of *Beurling functions*, which we define to be the linear hull of the family $\{\varrho_a \mid 1 \leq a \in \mathbb{R}\}$ with

$$\varrho_a(x) := \varrho(1/ax).$$

The much smaller subspace $\mathcal{B}^{nat}$ of *natural Beurling functions* is generated by $\{\varrho_a \mid a \in \mathbb{N}\}$. The Nyman-Beurling criterion [14, 6] states, in a slightly modified form [4] (the original formulation is related to $L_2((0, 1), dx)$), that the Riemann hypothesis is equivalent to the statement that $\chi \in \overline{\mathcal{B}}$, but it has recently been conjectured by several authors (see [1-5, 9-13, 17, 18]) that this condition could be substituted by $\chi \in \overline{\mathcal{B}^{nat}}$. We state this as a theorem to be proved below.

THEOREM 1. *The Riemann hypothesis is equivalent to the statement that* $\chi \in \overline{\mathcal{B}^{nat}}$.

To properly gauge the strength of this theorem note this: not only is $\mathcal{B}^{nat}$ a rather thin subspace of $\mathcal{B}$ but, as is easily seen, it is also true that $\overline{\mathcal{B}}$ is much larger than $\overline{\mathcal{B}^{nat}}$.

(*) Pervenuta in forma definitiva all'Accademia il 6 settembre 2002.

By necessity all authors have been led in one way or another to the *natural approximation*

$$F_n := \sum_{a=1}^{n} \mu(a)\, \varrho_a, \tag{1.1}$$

which tends to $-\chi$ both a.e. and in L_1 norm when restricted to $(0, 1)$ (see [1]), but which has been shown [2, 3] to diverge in $\mathcal{H}$. Other related sequences have been studied with little success (see [3, 9, 18]). This is probably connected to the fact that should they converge at all to $-\chi$ in $\mathcal{H}$ they must do so very slowly, since [4, 7] for any $F = \sum_{k=1}^{n} c_k \varrho_{a_k}$, $a_k \geqslant 1$, if $N = \max a_k$, then

$$\|F - \chi\|_{\mathcal{H}} \geqslant \frac{C}{\sqrt{\log N}}. \tag{1.2}$$

This, as well as considerations of *summability* of series, led the author to define for $\varepsilon > 0$ and $x > 0$

$$f_\varepsilon(x) := \sum_{a=1}^{\infty} \frac{\mu(a)}{a^\varepsilon} \varrho_a(x) = \frac{1}{x\zeta(\varepsilon + 1)} - \sum_{a \leqslant 1/x} \frac{\mu(a)}{a^\varepsilon} \left[\frac{1}{ax}\right]. \tag{1.3}$$

Assuming the Riemann hypothesis we shall prove that $f_\varepsilon \in \overline{\mathcal{B}^{nat}}$ for all small positive ε, which then implies, *unconditionally*, as shall be seen, that $f_\varepsilon \overset{\mathcal{H}}{\to} -\chi$ as $\varepsilon \downarrow 0$, so that $\chi \in \overline{\mathcal{B}^{nat}}$.

2. The Proof

2.1. *Two technical lemmas.*

Here $s = \sigma + i\tau$ with σ and τ real. A well-known theorem of Littlewood (see [16, Theorem 14.25 (A)]) about the convergence of the Dirichlet series for $1/\zeta(s)$ can be generalized and stated with a precise error term (see [5, Lemme 2]) as follows:

Lemma 1 (Balazard-Saias). *Let $1/2 \leqslant \alpha < 1$, $\delta > 0$, and $\varepsilon > 0$. If $\zeta(s)$ does not vanish in the half-plane $\Re(s) > \alpha$, then for $n \geqslant 2$ and $\alpha + \delta \leqslant \Re(s) \leqslant 1$ we have*

$$\sum_{a=1}^{n} \frac{\mu(a)}{a^s} = \frac{1}{\zeta(s)} + O_{\alpha,\delta,\varepsilon}\left(n^{-\delta/3}(1 + |\tau|)^\varepsilon\right). \tag{2.1}$$

It is important to note that the next lemma is independent of the Riemann or even the Lindelöf hypothesis.

Lemma 2. *For $0 \leqslant \varepsilon \leqslant \varepsilon_0 < 1/2$ there is a positive constant $C = C(\varepsilon_0)$ such that for all τ*

$$\left| \frac{\zeta\left(\frac{1}{2} - \varepsilon + i\tau\right)}{\zeta\left(\frac{1}{2} + \varepsilon + i\tau\right)} \right| \leqslant C(1 + |\tau|)^\varepsilon. \tag{2.2}$$

Proof. We bring in the functional equation of $\zeta(s)$ to bear as follows

$$\left| \frac{\zeta\left(\frac{1}{2} - \varepsilon + i\tau\right)}{\zeta\left(\frac{1}{2} + \varepsilon + i\tau\right)} \right| = \left| \frac{\zeta\left(\frac{1}{2} - \varepsilon - i\tau\right)}{\zeta\left(\frac{1}{2} + \varepsilon + i\tau\right)} \right| = \pi^{-\varepsilon} \left| \frac{\Gamma\left(\frac{1}{4} + \frac{1}{2}\varepsilon + \frac{1}{2}i\tau\right)}{\Gamma\left(\frac{1}{4} - \frac{1}{2}\varepsilon + \frac{1}{2}i\tau\right)} \right|,$$

then the conclusion follows easily from well-known asymptotic formulae for the gamma function in a vertical strip [15, (21.51), (21.52)]. □

2.2. *The Proof proper of Theorem* 1.

It is clear that we need not prove the *if* part of Theorem 1. So let us assume that the Riemann hypothesis is true. We define

$$f_{\varepsilon, n} := \sum_{a=1}^{n} \frac{\mu(a)}{a^{\varepsilon}} \varrho_a, \qquad (\varepsilon > 0).$$

It is easy to see that

$$f_{\varepsilon, n}(x) = \frac{1}{x} \sum_{a=1}^{n} \frac{\mu(a)}{a^{1+\varepsilon}} - \sum_{a=1}^{n} \frac{\mu(a)}{a^{\varepsilon}} \left[\frac{1}{ax} \right], \tag{2.3}$$

then, noting that the terms of the right-hand sum drop out when $a > 1/x$, we obtain the pointwise limit

$$f_{\varepsilon}(x) = \lim_{n \to \infty} f_{\varepsilon, n}(x) = \frac{1}{x\zeta(1+\varepsilon)} - \sum_{a \leqslant 1/x} \frac{\mu(a)}{a^{\varepsilon}} \left[\frac{1}{ax} \right]. \tag{2.4}$$

Then again for fixed $x > 0$ we have

$$\lim_{\varepsilon \downarrow 0} f_{\varepsilon}(x) = - \sum_{a \leqslant 1/x} \mu(a) \left[\frac{1}{ax} \right] = -\chi(x), \tag{2.5}$$

by the fundamental property of Möbius numbers. The task at hand now is to prove these pointwise limits are also valid in the $\mathcal{H}$-norm. To this effect we introduce a new Hilbert space

$$\mathcal{K} := L_2((-\infty, \infty), (2\pi)^{-1/2}\, dt),$$

and note, by virtue of Plancherel's theorem, that the Fourier-Mellin map $\boldsymbol{M}$ defined by

$$\boldsymbol{M}(f)(\tau) := \int_0^{\infty} x^{-\frac{1}{2} + i\tau} f(x)\, dx, \tag{2.6}$$

is an invertible isometry from $\mathcal{H}$ to $\mathcal{K}$.

A well-known identity, which is at the root of the Nyman-Beurling formulation, probably due to Titchmarsh [16, (2.1.5)], namely

$$-\frac{\zeta(s)}{s} = \int_0^\infty x^{s-1} \varrho_1(x)\, dx, \qquad (0 < \Re(s) < 1),$$

immediately yields, denoting $X_\varepsilon(x) = x^{-\varepsilon}$,

$$M(X_\varepsilon f_{2\varepsilon,n})(\tau) = -\frac{\zeta\left(\frac{1}{2} - \varepsilon + i\tau\right)}{\frac{1}{2} - \varepsilon + i\tau} \sum_{a=1}^{n} \frac{\mu(a)}{a^{\frac{1}{2}+\varepsilon+i\tau}}, \qquad (0 < \varepsilon < 1/2). \tag{2.7}$$

By Littlewood's theorem [16, 14.25 (A)] if we let $n \to \infty$ in the right-hand side of (2.7) we get the pointwise limit

$$-\frac{\zeta\left(\frac{1}{2} - \varepsilon + i\tau\right)}{\frac{1}{2} - \varepsilon + i\tau} \sum_{a=1}^{n} \frac{\mu(a)}{a^{\frac{1}{2}+\varepsilon+i\tau}} \to -\frac{\zeta\left(\frac{1}{2} - \varepsilon + i\tau\right)}{\zeta\left(\frac{1}{2} + \varepsilon + i\tau\right)} \frac{1}{\frac{1}{2} - \varepsilon + i\tau}. \tag{2.8}$$

To see that this limit also takes place in $\mathcal{H}$ we choose the parameters in Lemma 1 as $\alpha = 1/2$, $\delta = \varepsilon > 0$, $\varepsilon \leqslant 1/2$, and $n \geqslant 2$ to obtain

$$\sum_{a=1}^{n} \frac{\mu(a)}{a^{\frac{1}{2}+\varepsilon+i\tau}} = \frac{1}{\zeta\left(\frac{1}{2} + \varepsilon + i\tau\right)} + O_\varepsilon((1 + |\tau|)^\varepsilon).$$

If we now use Lemma 2 and a consequence of the Riemann hypothesis, the Lindelöf hypothesis applied to the abscissa $1/2 - \varepsilon$, we obtain a positive constant K_ε such that for all real τ

$$\left| -\frac{\zeta\left(\frac{1}{2} - \varepsilon + i\tau\right)}{\frac{1}{2} - \varepsilon + i\tau} \sum_{a=1}^{n} \frac{\mu(a)}{a^{\frac{1}{2}+\varepsilon+i\tau}} \right| \leqslant K_\varepsilon (1 + |\tau|)^{-1+2\varepsilon}.$$

It is then clear that for $0 < \varepsilon \leqslant \varepsilon_0 < 1/2$ the left-hand side of (2.8) is uniformly majorized by a function in $\mathcal{H}$. Thus the convergence does take place in $\mathcal{H}$ which implies that

$$X_\varepsilon f_{2\varepsilon,n} \xrightarrow{\mathcal{H}} X_\varepsilon f_{2\varepsilon}.$$

But $x^{-\varepsilon} > 1$ for $0 < x < 1$, and for $x > 1$

$$f_{2\varepsilon,n}(x) = \frac{1}{x} \sum_{a=1}^{n} \frac{\mu(a)}{a^{1+\varepsilon}} \ll \frac{1}{x}, \qquad (x > 1), \tag{2.9}$$

uniformly in n, which easily implies that one also has $\mathcal{H}$-convergence for $f_{2\varepsilon,n}$ when

$n \to \infty$. We have thus shown that

$$f_{\varepsilon, n} \xrightarrow{\mathcal{H}} f_\varepsilon \in \overline{\mathcal{B}^{nat}},$$

for all sufficiently small $\varepsilon > 0$. Moreover, since we have identified the pointwise limit in (2.8) we now have

$$\boldsymbol{M}(X_\varepsilon f_{2\varepsilon})(t) = -\frac{\zeta\left(\frac{1}{2} - \varepsilon + i\tau\right)}{\zeta\left(\frac{1}{2} + \varepsilon + i\tau\right)} \frac{1}{\frac{1}{2} - \varepsilon + i\tau}.$$

Now we apply Lemma 2 and obtain, at this juncture *without the assumption of the Riemann hypothesis*, that $\boldsymbol{M}(X_\varepsilon f_{2\varepsilon})$ converges in $\mathcal{K}$, thus $X_\varepsilon f_{2\varepsilon}$ converges in $\mathcal{H}$, and this means that f_ε also converges in $\mathcal{H}$ as $\varepsilon \downarrow 0$ by an argument entirely similar to that used for $f_{\varepsilon, n}$. The identification of the pointwise limit in (2.5) finally gives

$$f_\varepsilon \xrightarrow{\mathcal{H}} -\chi . \qquad \square$$

REMARK. The proof of Theorem 1 provides in turn a new proof, albeit of a stronger theorem, of the Nyman-Beurling criterion bypassing Hardy space techniques.

It should also be clear that we have shown this criterion to be true:

COROLLARY. *The Riemann hypothesis is equivalent to the $\mathcal{H}$-convergence of $f_{\varepsilon, n}$ as $n \to \infty$ for all sufficiently small $\varepsilon > 0$.*

3. A CONSTRUCTIVE APPROXIMATING SEQUENCE

In the proof of Theorem 1 we did not employ the dependence on n in the Balazard-Saias Lemma 1. After the first version of this paper M. Balazard and E. Bombieri, independently of each other, mentioned to the author that a suitable choice of ε would lead to a quantitative estimate of the error term. We owe special thanks to M. Balazard for the statement and proof of the following theorem:

THEOREM 2. *If the Riemann hypothesis is true then there is a constant $c > 0$ such that the distance in $\mathcal{H}$ between χ and*

$$-\sum_{a=1}^{n} \mu(a)\, e^{-c \frac{\log a}{\log\log n}} \varrho_a \tag{3.1}$$

is $\ll (\log\log n)^{-1/3}$.

PROOF. We shall only sketch the proof of this proposition. Applying the Fourier-Mellin map (2.6) to $f_{\varepsilon,n}+\chi$ we have from Plancherel's theorem that

$$2\pi\|f_{\varepsilon,n}+\chi\|^2_{\mathcal{H}} = \int_{\Re(z)=1/2} \left| \zeta(z)\sum_{a=1}^{n}\frac{\mu(a)}{a^{z+\varepsilon}}-1\right|^2 \frac{|dz|}{|z|^2} \leqslant$$

$$\leqslant 2\int_{\Re(z)=1/2}\left|\zeta(z)\left(\sum_{a=1}^{n}\frac{\mu(a)}{a^{z+\varepsilon}}-\frac{1}{\zeta(z+\varepsilon)}\right)\right|^2\frac{|dz|}{|z|^2}+$$

$$+2\int_{\Re(z)=1/2}\left|\frac{\zeta(z)}{\zeta(z+\varepsilon)}-1\right|^2\frac{|dz|}{|z|^2}.$$

The second integral on the right-hand side above is estimated to be $\ll\varepsilon^{2/3}$ as follows. If the distance between $z=\frac{1}{2}+it$ and the nearest zero of ζ is larger than δ, say, the upper bound

$$\left|\frac{\zeta(z)}{\zeta(z+\varepsilon)}-1\right| \ll \varepsilon\delta^{-1}(|t|+1)^{1/4}$$

follows from the classical estimate

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|\gamma-\tau|\leqslant 1}\frac{1}{s-\varrho}+O\big(\log(2+|\tau|)\big),$$

where $s=\sigma+i\tau$, $1/2\leqslant\sigma\leqslant 3/4$, $\tau\in\mathbb{R}$ and $\varrho=\beta+i\gamma$ denotes a generic zero of the ζ function, by integration and exponentiation, provided ε/δ is small enough. In the other case, one uses an estimate of Burnol [8] stating that under the Riemann hypothesis

$$\left|\frac{\zeta(z)}{\zeta(z+\varepsilon)}\right| \ll |z|^{\varepsilon/2}, \qquad \Re(z)=1/2, \qquad 0<\varepsilon\leqslant 1/2\,. \tag{3.2}$$

Integrating these two inequalities on the corresponding sets, one gets an upper bound $\ll\varepsilon^2/\delta^2+\delta$, and chooses $\delta=\varepsilon^{2/3}$. The first integral, on the other hand, succumbs to a special form of the Balazard-Saias Lemma 1, namely

$$\sum_{a=1}^{n}\frac{\mu(a)}{a^{\frac{1}{2}+\varepsilon+it}} = \frac{1}{\zeta\left(\frac{1}{2}+\varepsilon+it\right)}+O(n^{-\varepsilon/3}e^{b\mathcal{L}(t)}), \qquad c/\log\log n\leqslant\varepsilon\leqslant 1/2\,,$$

where $\mathcal{L}(t):=\log(|t|+3)/\log\log(|t|+3)$. We thus have

$$2\pi\|f_{\varepsilon,n}+\chi\|^2_{\mathcal{H}} \ll n^{-2\varepsilon/3}\int_{-\infty}^{\infty}e^{O(\mathcal{L}(t))}\frac{dt}{\frac{1}{4}+t^2}+\varepsilon^{2/3} \ll n^{-2\varepsilon/3}+\varepsilon^{2/3},$$

provided that $\varepsilon\geqslant c/\log\log n$, whereupon one chooses $\varepsilon=c/\log\log n$ to reach the conclusion. □

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