Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Giuseppe Dattoli

Derivation of the Hille-Hardy type formulae and operational methods

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **14** (2003), n.2, p. 85–90.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2003_9_14_2_85_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2003.

Rend. Mat. Acc. Lincei
s. 9, v. 14:85-90 (2003)

Matematica. — *Derivation of the Hille-Hardy type formulae and operational methods.* Nota di GIUSEPPE DATTOLI, presentata (*) dal Socio C. De Concini.

ABSTRACT. — The Hille-Hardy formula is a bilinear generating function, involving products of Laguerre polynomials. We use the point of view, developed in previous publications, to propose an operational method which allows a fairly direct derivation of this kind of formulae.

KEY WORDS: Operational methods; Hermite polynomials; Laguerre polynomials; Bilinear generating functions.

RIASSUNTO. — *Derivazione delle formule di Hille-Hardy e metodi operazionali.* La formula di Hille-Hardy è una funzione generatrice bilineare relativa a prodotti di polinomi di Hermite. In questo lavoro si utilizza il punto di vista sviluppato in precedenti pubblicazioni, per proporre una derivazione diretta di tale tipo di formula.

1. INTRODUCTION

The Hermite and Laguerre polynomials can be respectively derived from the operational rules [1]

$$
\begin{aligned}
H_n(x, y) &= n! \sum_{r=0}^{[n/2]} \frac{y^r x^{n-2r}}{r!(n-2r)!} = \exp\left(y \frac{\partial^2}{\partial x^2}\right) x^n \\
L_n(x, y) &= n! \sum_{r=0}^{n} \frac{(-1)^r y^{n-r} x^r}{(r!)^2 (n-r)!} = \exp\left(-y \frac{\partial}{\partial x} x \frac{\partial}{\partial x}\right)\left(\frac{(-x)^n}{n!}\right).
\end{aligned}
\tag{1}
$$

Both polynomial forms have been written by means of two variables x, y even though they can always be reduced to the ordinary one variable form, namely

$$
H_n(x, y) = i^n y^{n/2} H_n\left(\frac{x}{2i\sqrt{y}}\right) \qquad L_n(x, y) = y^n L_n\left(\frac{x}{y}\right),
\tag{2}
$$

where $H_n(x)$ and $L_n(x)$ denotes Hermite and Laguerre polynomials in the canonical forms [2].

The use of an extra variable is however of noticeable importance. The $H_n(x, y)$ and $L_n(x, y)$ can be, indeed, viewed as the solutions of the following type of partial differential equations

$$
\frac{\partial}{\partial y} L_n(x, y) = -\frac{\partial}{\partial x} x \frac{\partial}{\partial x} L_n(x, y) \qquad L_n(x, 0) = \frac{(-x)^n}{n!}
\tag{3}
$$

and

$$
\frac{\partial}{\partial y} H_n(x, y) = \frac{\partial^2}{\partial x^2} H_n(x, y) \qquad H_n(x, 0) = x^n.
\tag{4}
$$

It is evident that the operational rules given in eq. (1) are a consequence of eqs.

(*) Nella seduta dell'11 aprile 2003.

(3, 4). Having clarified these points, let us note that the exponential operator

$$\widehat{S} = \exp\left(y \frac{\partial^2}{\partial x^2}\right) \tag{5}$$

allows the formal solution of the heat equation

$$\frac{\partial}{\partial y} F(x, y) = \frac{\partial^2}{\partial x^2} F(x, y) \qquad F(x, 0) = g(x). \tag{6}$$

Accordingly the operational solution

$$F(x, y) = \widehat{S} g(x) \tag{7}$$

can be understood in terms of a Gauss type Transform [3].

In the case in which $g(x) = \exp(-x^2)$, the following relation

$$\widehat{S} \exp(-x^2) = \frac{1}{\sqrt{1+4y}} \exp\left(-\frac{x^2}{1+4y}\right), \tag{8}$$

known as Glaisher operational rule, holds [3, 4].

Eq. (8) has been a unique tool to derive bilinear generating functions involving Hermite polynomials (see *e.g.* [4, 5]). In this paper we will take advantage from the properties of the operator

$$\widehat{T} = \exp\left(-y \frac{\partial}{\partial x} x \frac{\partial}{\partial x}\right) \tag{9}$$

to derive bilinear generating functions of the Hille-Hardy type [4].

2. Operational calculus and Laguerre derivatives

The operator

$$_L\widehat{\mathcal{D}}_x = -\frac{\partial}{\partial x} x \frac{\partial}{\partial x} \tag{10}$$

plays a central role in the theory of Laguerre polynomials and is known as Laguerre derivative [1].

The $\widehat{T}$ operator, defined in the previous section, is therefore the exponential of the Laguerre derivative. One of its must important properties is the following operational identity [1]

$$\widehat{T} \exp(-\alpha x) = \frac{1}{1-y\alpha} \exp\left(-\frac{\alpha x}{1-y\alpha}\right) \tag{11}$$

which can also be viewed as a different form of the Glaisher rule.

The function

$$C_0(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{(r!)^2} \tag{12}$$

is an eigenfunction of the Laguerre derivative [1], so that

$$\widehat{T} C_0(\gamma x) = \exp(y\gamma x)\, C_0(\gamma x). \tag{13}$$

In passing we note that $C_0(x)$ is the 0^{th} order of the so called Tricomi functions

$$C_n(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!(n+r)!} = x^{-n/2} J_n(2\sqrt{x}), \tag{14}$$

where $J_n(x)$ is the cylindrical Bessel function of first kind.

By combining eqs. (12) and (13), we find

$$\hat{T}[\exp(-\beta x)\, C_0(\gamma x)] = \frac{1}{1-y\beta} \exp\left(-\frac{\beta x - y\gamma}{1-y\beta}\right) C_0\left(\frac{\gamma x}{(1-y\beta)^2}\right) \qquad |y\beta|<1, \tag{15}$$

or, what is the same,

$$\hat{T}[\exp(-\beta x)\, J_0(2\sqrt{\gamma x})] = \frac{1}{1-y\beta} \exp\left(-\frac{\beta x - y\gamma}{1-y\beta}\right) J_0\left(\frac{2\sqrt{\gamma x}}{(1-y\beta)}\right) \qquad |y\beta|<1. \tag{16}$$

In the following section we will see how the previous results can be exploited to treat bilateral generating functions involving Laguerre polynomials.

3. Operational calculus and Laguerre polynomials

We will introduce the problem of studying generating functions of Laguerre polynomials by using a fairly direct example. Let us indeed consider the case

$$G(z, w|t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(z, w), \tag{17}$$

the use of eqs. (1), (12) and (13), allows the following conclusions

$$G(z, w|t) = \hat{T} \sum_{n=0}^{\infty} \frac{(-twz)^n}{(n!)^2} = \hat{T} C_0(zt) = \exp(wt)\, C_0(zt). \tag{18}$$

We consider the generating function

$$G(x, y; z, w|t) = \sum_{n=0}^{\infty} t^n L_n(x, y)\, L_n(z, w) \tag{19}$$

which can be recast as

$$G(x, y; z, w|t) = \hat{T}[\exp(-wtx)\, C_0(-ztx)]. \tag{20}$$

Therefore, according to eq. (15), we find ($|yt| < 1$)

$$G(x, y; z, w|t) = \frac{1}{1-ywt} \exp\left(-\frac{(wx+yz)\,t}{1-wyt}\right) C_0\left(-\frac{zxt}{(1-wyt)^2}\right). \tag{21}$$

Assuming $wt = \beta$, $-zt = \gamma$, the above relation reduces to (15), which is essentially, for $y = w = 1$, the Hille-Hardy formula.

The associated Laguerre polynomials can be defined by means of an extension of eq. (2), which yields

$$L_n^{(\alpha)}(x, y) = \exp\left[-y\left(x \frac{\partial^2}{\partial x^2} + (\alpha+1) \frac{\partial}{\partial x}\right)\right]\left[\frac{(-1)^n x^n}{n!}\right]. \tag{22}$$

By denoting the exponential operator on the r.h.s. of eq. (22) by $\widehat{T}_\alpha$ we can write the identity

$$\widehat{T}_\alpha \exp(-xt) = \frac{1}{(1-yt)^{\alpha+1}} \exp\left(-\frac{xt}{1-yt}\right) \qquad |yt| < 1 . \tag{23}$$

and

$$\widehat{T}_\alpha C_\alpha(\gamma x) = \exp(\gamma y)\, C_\alpha(\gamma x) . \tag{24}$$

The generalization of eq. (21) is therefore achieved by noting that

$$\widehat{T}_\alpha[\exp(-\beta x)\, C_\alpha(\gamma x)] = \frac{1}{(1-\beta y)^{\alpha+1}} \exp\left(-\frac{\beta x - \gamma y}{1-\beta y}\right) C_\alpha\left(\frac{\gamma x}{(1-\beta y)^2}\right). \tag{25}$$

The use of the above formula allows to recover the complete Hille-Hardy formula. By considering, indeed, the generating function

$$G_\alpha(x, y; z, w|t) = \sum_{n=0}^{\infty} \frac{n!\, t^n}{\Gamma(n+\alpha+1)} L_n^{(\alpha)}(x, y)\, L_n^{(\alpha)}(z, w) \tag{26}$$

we find ($|wyt| < 1$)

$$G_\alpha(x, y; z, w|t) = \widehat{T}_\alpha \sum_{n=0}^{\infty} \frac{(-xt)^n}{\Gamma(n+\alpha+1)} L_n^{(\alpha)}(z, w) = \tag{27}$$

$$= \widehat{T}_\alpha \exp(-wxt)\, C_\alpha(-zxt) = \frac{1}{(1-ywt)^{\alpha+1}} \exp\left(-\frac{(wx+yz)\,t}{1-wyt}\right) C_\alpha\left(-\frac{zxt}{(1-wyt)^2}\right).$$

The last term of the above equality is just the Hille-Hardy formula.

4. Concluding remarks

In the previous sections and in [5] we have shown that operational methods may provide us with a unifying tool for the derivation of bilateral generating functions.

Just to complete the scenario, we note that the use of the first eq. (1), along with the, already mentioned, identification of the action of the operator $\widehat{S}$ in terms of the Gauss transform, namely

$$\widehat{S} f(x) = \frac{1}{2\sqrt{\pi y}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(x-\xi)^2}{4y}\right) f(\xi)\, d\xi \tag{28}$$

allows the proof of the identity

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y)\, L_n(z, w) = \exp(w^2 t^2 y + wxt) \left({}_H C_0\left(2zt\left(\frac{x}{2} + wyt\right), z^2 y t^2\right)\right), \tag{29}$$

where

$$ {}_{H}C_n(x, y) = \sum_{r=0}^{\infty} \frac{(-1)^r H_r(x, y)}{r!(n+r)!} \tag{30} $$

is the Tricomi Bessel function [1] of the n^{th} order.

It is evident that the method we have discussed is fairly powerful and that it may be extended to get further results in a direct way.

The reason of interest for this type of formalism is that distributions of the type

$$ P(x, y; \alpha) = \frac{1}{1-\beta} e^{-\frac{x}{1-\beta}} e^{-\frac{y}{1-\beta}} C_0\left(-\beta \frac{xy}{(1-\beta)^2}\right) \tag{31} $$

appear in problems associated with the statistical properties of chaotic light and they have been recently exploited to study the statistical aspects of the radiation Spiking in high gain Free Electron Lasers [6].

The relevant mathematical properties have been recently discussed in [7], where it has been shown that multidimensional extensions of (31) are possible, but they require the introduction of more general forms of Laguerre and Tricomi functions, respectively defined as $(m > 2)$

$$ \begin{aligned} L_n(x, y \mid m) &= n! \sum_{r=0}^{n} \frac{(-1)^r y^{n-r} x^r}{(n-r)!\,(r!)^m} \\ C_{\{n_1, \ldots, n_{m-1}\}}(x \mid m) &= \sum_{r=0}^{\infty} \frac{(-x)^r}{(r!)(n_1+r)! \ldots (n_{m-1}+r)!}\,. \end{aligned} \tag{32} $$

The above families of polynomials and functions satisfy interesting operational relations, which can be usefully exploited to generalize the Hille-Hardy formula to the more general family of polynomials $L_n(x, y \mid m)$ as it will be shown in a forthcoming investigation.

Acknowledgements

The author expresses his sincere appreciation to Prof. P.E. Ricci for clarifying discussions and for suggestions improving the presentation of the paper.

References

[1] G. Dattoli, *Hermite-Bessel and Laguerre Bessel Functions, a by-product of the monomiality principle*. In: D. Cocolicchio - G. Dattoli - H.M. Srivastava (eds.), Proceedings of the Workshop on *Special Functions and Applications in Mathematics and Physics* (Melfi, 9-12 May 1999). Aracne Editrice, Roma 2000.

[2] L.C. Andrews, *Special Functions for Applied Mathematicians and Engineers*. Mac Millan, New York 1985.

[3] G. Dattoli, *Generalized Polynomials, Operational Identities and their Applications*. J. Comput. Appl. Math., 118, 2000, 111-123.

[4] H.M. SRIVASTAVA - H.L. MANOCHA, *A Treatise on Generating Functions*. J. Wiley, New York 1984. For earlier derivations by Miller and Lebedeff, see *e.g.* A. ERDÉLYI *et al.*, *Bateman Manuscript project*, Vol. 2.

[5] G. DATTOLI, *Bilateral Generating Functions and Operational Methods*. J. Math. Anal. and Appl., to appear.

[6] S. KRINSKY - R.L. GLUCKSTERN, *Analysis of statistical correlations and intensity spiking in the self-amplified spontaneous-emission free-electron laser*. Phys. Rev. ST Accel. Beams, 6(5), 2003, 050701-050710.

[7] G. DATTOLI - P.E. RICCI, *Multi-index Polynomials and Applications to Statistical Problems*. To appear.

Pervenuta il 18 ottobre 2002,
in forma definitiva l'8 aprile 2003.

ENEA, Unità Tecnico Scientifica Tecnologie Fisiche Avanzate
Centro Ricerche Frascati
C.P. 65 - 00044 FRASCATI RM
dattoli@frascati.enea.it