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CACCIOPPOLI SETS

ABSTRACT. — The story of the theory of Caccioppoli sets is presented, together with some information about Renato Caccioppoli's life. The fundamental contributions of Ennio De Giorgi to the theory of Caccioppoli sets are sketched. A list of applications of Cacciopoli sets to the calculus of variations is finally included.

KEY WORDS: Perimeter of Lebesgue measurable sets; Caccioppoli inequality; Elliptic equations and systems; Harnack inequality; Theory of capillarity.

1. Renato Caccioppoli participated in the IV Congress of the Italian mathematical union, that took place in Taormina, Sicily in October 1951, see [1].

At that time Caccioppoli was 47 years old and a well known mathematician, in Italy and abroad. On February 15, 1947 he had been coopted as a member of the Accademia dei Lincei.

Maria Bakunin, professor of Chemistry at the University of Naples, was coopted by the Lincei together with him. Maria was the younger sister of Renato's mother and the daughter of Michail Aleksandrovich Bakunin, the famous anarchist, see [2]. Maria admired, loved and protected Renato, whom she considered as gifted as her father.

Mauro Picone was another person who admired and protected Renato. They met in Naples in 1925, and their friendship and cooperation lasted till the untimely death of Renato. In 1926 Picone founded his National Institute for Applications of Calculus, of which Caccioppoli was the most brilliant consultant for many years. In Taormina, Picone declared:

«The INAC is the place where Functional Topology and Numerical Calculus celebrate their marriage» (see [3]).

2. In Taormina, Renato called the attention of his audience to the integral

$$\int_E \operatorname{div} \phi(x)\, dx,$$

where E is a Lebesgue measurable subset of R^n and $\phi \in [C_0^1(R^n)]^n$.

He suggested (see [1]) to study the geometrical properties of the set E, whenever a positive real number K exists, with

$$\int_E div\, \phi(x)\, dx \leqslant K \max_x |\phi(x)|, \qquad \forall \phi \in [C_0^1(R^n)]^n. \tag{1}$$

Caccioppoli was aware that (1) is equivalent to the existence of a vector finite Radon measure μ, with

$$\int_E div\,\phi(x)\,dx = \int_{\partial E} \phi \cdot d\mu, \qquad \forall \phi \in [C_0^1(R^n)]^n; \tag{2}$$

and that the total variation $|\mu|$ of μ satisfies

$$|\mu|(R^n) \leq K, \tag{3}$$

$\forall K$ satisfying (1).

Moreover, μ is unique and, by denoting with $d\mu/d|\mu|$, the derivative of μ with respect to its total variation and

$$\partial^* E = \left\{ x \in \partial E \;\middle|\; \frac{d\mu}{d|\mu|}(x) = \nu(x),\ |\nu(x)| = 1 \right\},$$

one can write (2) in the form

$$\int_E div\,\phi(x)\,dx = \int_{\partial^* E} \phi \cdot \nu d|\mu|. \tag{4}$$

Caccioppoli added the following easy remark:

If, for a given E, there exists a number K and a sequence of elementary sets $\{S_j\}$ with

$$\begin{cases} S_j \to E \text{ in } L^1_{loc}(R^n) \\ H^{n-1}(\partial S_j) \leq K, \ \forall j \end{cases} \tag{5}$$

then K satisfies (1).

At that point Caccioppoli was able to make more precise the invitation to his audience:

Study the geometry of $\partial^* E$, whenever (5) is satisfied.

He also thought to simplify the problem, by suggesting to assume

$$H^n(\partial E) = 0\,. \tag{6}$$

Namely, he was preoccupied by the existence of sets E satisfying (5) and

$$H^n(\partial E) = +\infty\,. \tag{7}$$

3. Mauro Picone, in 1952 presented to the Lincei five articles by Caccioppoli, containing more details about Taormina's Program.

Those papers did not add too much, as long as definitions, statements and proofs were concerned. L.C. Young, who reviewed them for the M.R., did not hide his strong criticism and disappointment.

In September 1952, at the Salzburg Congress of the Austrian Mathematical Society, Ennio De Giorgi presented some results, about sets of finite perimeter.

Independently of Caccioppoli, De Giorgi proved that (1) and (5) are equivalent and added another characterization for them, by proving the identity

$$|\mu| = \lim_{h \to +\infty} \int_{R^n} |D(\chi_E * G_h)|\, dx\,, \tag{8}$$

where μ is the Radon vector measure satisfying (2), χ_E the characteristic function of E and

$$G_h(x) = h^n G(hx)$$

with G the Gauss function.

De Giorgi called perimeter of E the real number

$$P(E) = |\mu|$$

whenever μ exists and is finite. The sets E, in such cases, are called sets of finite perimeter. Caccioppoli used to call them: $(n-1)$ dimensionally oriented sets.

After Caccioppoli's death in 1959, De Giorgi called them Caccioppoli sets.

Still in Salzburg, De Giorgi said that the inequality

$$H^n(E) \wedge H^n(R^n - E) \leqslant [P(E)]^{\frac{n}{n-1}}\,, \tag{9}$$

is satisfied for any measurable set E and any $n > 1$. He also conjectured that

$$inf\,\{P(E)\,|\,H^n(E) = 1\}$$

is equal to the perimeter of the balls of measure one.

This conjecture was proven by De Giorgi himself in 1958 [4].

4. In Winter 1953, De Giorgi met Caccioppoli for the first time. Caccioppoli was enthusiastically shocked by De Giorgi's results, and asked to present them to the Lincei.

The first paper by De Giorgi, about sets of finite perimeter was published in 1954 [5].

In August 1955, De Giorgi sent to Caccioppoli a second paper about the geometry of $\partial^* E$, where the proofs are independent of (6).

Caccioppoli appreciated the paper very much, since it corroborated his conjectures, and gave the paper to Ricerche di Matematica for publication [6].

At the same time, De Giorgi learned from Guido Stampacchia the existence of the XIX Hilbert Problem, about the regularity of weak solutions of elliptic equations.

De Giorgi quickly realized that he could work at the Problem, by putting together the isoperimetric properties of the balls with the so called «Caccioppoli inequality».

In October 1955, De Giorgi announced his solution of the XIX Problem, at the V UMI Congress in Pavia, see [7].

5. In November 1959, De Giorgi arrived to the Scuola Normale of Pisa, where he remained till the end of his life.

At that time De Giorgi was working hard at the proof of regularity of $\partial^* E$, whenever the Caccioppoli set E minimizes the perimeter. He was able to publish this result in December 1961 (see [8]), with the further remark

$$H^{n-1}(\partial E - \partial^* E) = 0 .$$

For seven more years, De Giorgi was convinced that

$$\partial E - \partial^* E = \emptyset .$$

What reinforced his expectation, was the fact of having been able to reduce the problem to the case of Caccioppoli cones, analytic outside the vertex.

This conjecture was proven true by James Simons [9] in 1967 for $n \leqslant 7$. Simons argument was stopped in R^8 by the cone

$$\{(x, y) \,|\, x \in R^4, \, y \in R^4, \, x^2 > y^2\} ,$$

satisfying Simons' assumptions and singular at 0.

Simons strongly believed in his argument and was convinced that such a cone disproved De Giorgi's conjecture for $n \geqslant 8$.

6. Meanwhile, something new was happening around the problem of regularity of weak solutions of elliptic systems.

In August 1966, at the IMU Congress in Moscow, Fred J. Almgren, jr. presented his regularity almost everywhere result for varifolds minimizing the mass. Almgren method was the extension to all codimensions of De Giorgi's method. Charles B. Morrey, jr., at the end of Almgren talk affirmed to see the possibility of doing the same work for proving the regularity almost everywhere for the solutions of elliptic systems.

Actually Morrey did that in 1967 [10].

Back to Pisa, from Moscow, I told De Giorgi about Morrey's remark. De Giorgi was convinced that Morrey could be successful, and that Morrey's conjecture was the best possible.

Then, De Giorgi published a simple example of a singular solution for an elliptic system [11].

Nevertheless, De Giorgi remained convinced that Simons was wrong, his cone could not be a minimum.

7. In Spring 1968, Enrico Bombieri jumped on the stage. Firstly, he pushed De Giorgi and myself to work at a direct proof of the gradient estimate for the solutions of the minimal surface equation. And we were successful [12].

Secondly, in Fall 1968, he wanted De Giorgi to share his belief in Simons' conjecture.

Finally, De Giorgi was convinced by Bombieri's reasons, and they worked together to the proof of Simons' conjecture [13].

Later on, Bombieri together with Enrico Giusti [14] were able to prove a Harnack-type inequality for solutions of elliptic equations over boundaries of Caccioppoli minimal sets.

De Giorgi results extended by Umberto Massari [15] to boundaries of Caccioppoli sets, with prescribed non-zero mean curvature, were used for a mathematical theory of capillarity surfaces [16, 17].

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