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LIOUVILLE TYPE THEOREMS FOR SOME CONFORMALLY INVARIANT FULLY NONLINEAR EQUATIONS

Abstract. — This is a report on some joint work with Aobing Li on Liouville type thorems for some conformally invariant fully nonlinear equations.

Key words: Liouville type theorems; Conformally invariant equations.

This is a report on some joint work with Aobing Li, the full proof of which, as well as some of our other results on conformally invariant fully nonlinear equations, can be found in [8-12].

For $n \geqslant 3$, consider

$$-\Delta u = \frac{n-2}{2} u^{\frac{n+2}{n-2}}, \qquad \text{on } \mathbb{R}^n. \tag{1}$$

It was proved by Obata [15] and Gidas, Ni and Nirenberg [6] that any positive C^2 solution of (1) which is regular at infinity must be of the form

$$u(x) = (2n)^{\frac{n-2}{4}} \left(\frac{a}{1 + a^2 |x - \bar{x}|^2} \right)^{\frac{n-2}{2}},$$

where $a > 0$ and $\bar{x} \in \mathbb{R}^n$. A function u is said to be regular at infinity if $|x|^{2-n} u\left(\frac{x}{|x|^2}\right)$ can be extended as a C^2 function near $x = 0$. The hypothesis that u is regular at infinity was removed by Caffarelli, Gidas and Spruck [1]; this is important for applications. The method in [6] is completely different from that of [15]. The method used in our proof of the Liouville type theorems on conformally invariant fully nonlinear equations (Theorem 1 and Theorem 3) is in the spirit of [6] rather than that of [15]. As in [1], the superharmonicity of the solution has played an important role in our proof of Theorem 3, see Lemma 1 below. On the other hand, under some additional hypothesis on the solution near infinity, the superharmonicity of the solution is not needed, see Theorem 1 and Remark 2 below.

Somewhat different proofs of the result of Caffarelli, Gidas and Spruck were given in [5, 14] and [13]. A related result of Gidas and Spruck in [7] states that there is no positive solution to the equation $-\Delta u = u^p$ in $\mathbb{R}^n$ when $1 < p < \frac{n+2}{n-2}$. An extension of this result to fully nonlinear equations is given in [11] (see also [12] for a somewhat more detailed version).

Let ψ be a Möbius transformation and let u be any positive C^2 function u on $\mathbb{R}^n$, we have the identity

$$\left(u_\psi^{-\frac{n+2}{n-2}} \Delta u_\psi\right) = \left(u^{-\frac{n+2}{n-2}} \Delta u\right) \circ \psi\,, \qquad \text{on } \mathbb{R}^n,$$

where $u_\psi := |J_\psi|^{\frac{n-2}{2n}} (u \circ \psi)$ and J_ψ denotes the Jacobian of ψ. A Möbius transformation is a finite composition of translations, multiplications by non-zero constants and the inversion $x \to x/|x|^2$.

Let $\mathcal{S}^{n\times n}$ be the set of $n \times n$ real symmetric matrices, $\mathcal{S}_+^{n\times n} \subset \mathcal{S}^{n\times n}$ be the set of positive definite matrices, and $O(n)$ be the set of $n \times n$ real orthogonal matrices.

For $H \in C^0(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^{n\times n})$, we say, as in [9], that $H(\cdot, u, \nabla u, \nabla^2 u)$ is conformally invariant on $\mathbb{R}^n$ if for any Möbius transformation ψ and any positive function $u \in C^2(\mathbb{R}^n)$, we have

$$H(\cdot, u_\psi, \nabla u_\psi, \nabla^2 u_\psi) \equiv H(\cdot, u, \nabla u, \nabla^2 u) \circ \psi \qquad \text{on } \mathbb{R}^n.$$

It was proved in [9] that H is conformally invariant if and only if it is of the form

$$H(\cdot, u, \nabla u, \nabla^2 u) \equiv F(A^u)\,,$$

where

$$A^u := -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \nabla^2 u + \frac{2n}{(n-2)^2} u^{-\frac{2n}{n-2}} \nabla u \otimes \nabla u - \frac{2}{(n-2)^2} u^{-\frac{2n}{n-2}} |\nabla u|^2 I\,,$$

I is the $n \times n$ identity matrix and F is invariant under orthogonal conjugation, *i.e.*

$$F(O^{-1} M O) = F(M) \quad \forall A \in \mathcal{S}^{n\times n},\ O \in O(n)\,.$$

Let $U \subset \mathcal{S}^{n\times n}$ be an open set satisfying

$$(2) \qquad O^{-1} U O = U, \qquad \forall\, O \in O(n)\,,$$

and

$$(3) \qquad U \cap \{M + tN \mid 0 < t < \infty\} \text{ convex} \qquad \forall\, M \in \mathcal{S}^{n\times n},\ N \in \mathcal{S}_+^{n\times n},$$

and let $F \in C^1(U)$ satisfy

$$(4) \qquad F(O^{-1} M O) = F(M), \qquad \forall\, M \in U,\ O \in O(n)\,,$$

and

$$(5) \qquad \left(\frac{\partial F}{\partial M_{ij}}(M)\right) > 0\,, \qquad \forall\, M \in U\,.$$

The following theorem extends the above mentioned result of Obata and Gidas, Ni and Nirenberg to general conformally invariant operators of elliptic type.

Theorem 1 [9]. *For $n \geq 3$, let $U \subset \mathcal{S}^{n\times n}$ be open and satisfy* (2) *and* (3), *and let $F \in C^1(U)$ satisfy* (4) *and* (5). *Assume that $u \in C^2(\mathbb{R}^n)$ satisfies*

$$F(A^u) = 1\,, \qquad u > 0\,, \qquad A^u \in U\,, \qquad \text{on } \mathbb{R}^n,$$

and

(6) $\qquad u_{0,1}$ *can be extended to apositive C^2 function near the origin*,

where $u_{0,1}(x) := |x|^{2-n}u(x/|x|^2)$. *Then for some* $\bar{x} \in \mathbb{R}^n$, *and some positive constants* a *and* b *satisfying* $2b^2a^{-2}I \in U$ *and* $F(2b^2a^{-2}I) = 1$,

$$u(x) \equiv \left(\frac{a}{1 + b^2|x - \bar{x}|^2}\right)^{\frac{n-2}{2}}, \qquad \forall x \in \mathbb{R}^n.$$

Taking

$$F(M) = \mathrm{Trace}\,(M) \quad \text{and} \quad U = \mathcal{S}^{n\times n} \ \ (\text{or } U = \{M \in \mathcal{S}^{n\times n} \,|\, \mathrm{Trace}\,(M) > 0\}),$$

we have

$$F(A^u) = -\frac{2}{n-2}\frac{\Delta u}{u^{\frac{n+2}{n-2}}},$$

and Theorem 1 in this case is the result of Obata and Gidas, Ni and Nirenberg.

For $1 \leqslant k \leqslant n$, let

$$\sigma_k(\lambda) = \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \lambda_{i_1} \ldots \lambda_{i_k}, \qquad \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n,$$

denote the k-th symmetric function, and let

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \,|\, \sigma_1(\lambda) > 0, \ldots, \lambda_k(\lambda) > 0\},$$

$$U_k = \{M \in \mathcal{S}^{n\times n} \,|\, \lambda(M) \in \Gamma_k\},$$

and

$$F_k(M) = \sigma_k(\lambda(M))^{\frac{1}{k}}, \qquad M \in U_k.$$

Here, and in the following, $\lambda(M)$ denotes the eigenvalues of M. For $1 \leqslant k \leqslant n$, it is known that (F_k, U_k) satisfies the hypothesis of Theorem 1 (see *e.g.* [2]), and the result in this case is due to Viaclovsky [16, 17]).

Theorem 1 requires the strong hypothesis (6) on u near infinity. On the other hand, the conclusion still holds under the following weaker hypothesis (see [9]):

(7) $\quad u_{0,1}$ can be extended to a positive continuous function near the origin,

and $u_{0,1}$ satisfies

$$\limsup_{x\to 0}(x \cdot \nabla u_{0,1}(x)) < \frac{n-2}{2}u_{0,1}(0), \tag{8}$$

and

$$\lim_{x\to 0}(|x|^2|\nabla u_{0,1}(x)|) = 0. \tag{9}$$

For applications, as mentioned earlier, it is of importance to establish such results without imposing any hypothesis on u near infinity. For this we first have

Theorem 2 [9]. *For* $n \geqslant 3$ *and* $1 \leqslant k \leqslant n$, *let* $u \in C^2(\mathbb{R}^n)$ *satisfy*

$$F_k(\lambda(A^u)) = 1, \qquad u > 0, \qquad A^u \in U_k, \quad \text{on } \mathbb{R}^n.$$

Then for some $a > 0$ *and* $\bar{x} \in \mathbb{R}^n$,

$$u(x) = c(n, k) \left(\frac{a}{1 + a^2 |x - \bar{x}|^2} \right)^{\frac{n-2}{2}}, \qquad \forall x \in \mathbb{R}^n,$$

where $c(n, k) = 2^{(n-2)/4} \binom{n}{k}^{(n-2)/4k}$.

For $n \geqslant 3$ and $k = 1$, Theorem 2 is the above mentioned result of Caffarelli, Gidas and Spruck. For $n = 4$ and $k = 2$, the result is due to Chang, Gursky and Yang [3]. For $n = 5$ and $k = 2$, as well as for $n \geqslant 6$, $k = 2$ and under the assumption $\int_{\mathbb{R}^n} u^{\frac{2n}{n-2}} < \infty$, the result was independently obtained by Chang, Gursky and Yang [4].

More recently we have obtained the following general Liouville type theorem which extends all the above mentioned results except for Theorem 1.

Theorem 3 [10-12]. *For* $n \geqslant 3$, *let* $U \subset \mathcal{S}^{n \times n}$ *be open and satisfy* (2) *and* (3), *and let* $F \in C^1(U)$ *satisfy* (4) *and* (5). *Assume that* $u \in C^2(\mathbb{R}^n)$ *satisfies*

$$F(A^u) = 1, \qquad u > 0, \qquad A^u \in U, \qquad on\ \mathbb{R}^n, \tag{10}$$

and

$$\Delta u \leqslant 0 \qquad on\ \mathbb{R}^n. \tag{11}$$

Then, for some $\bar{x} \in \mathbb{R}^n$ *and some constants* $a > 0$ *and* $b \geqslant 0$ *satisfying* $2b^2 a^{-2} I \in U$ *and* $F(2b^2 a^{-2} I) = 1$, *we have*

$$u(x) \equiv \left(\frac{a}{1 + b^2 |x - \bar{x}|^2} \right)^{\frac{n-2}{2}}, \qquad \forall x \in \mathbb{R}^n. \tag{12}$$

Remark 1. *We proved in* [9] *that* (11) *is not needed if we impose some hypothesis on u near infinity* (i.e. (7), (8) *and* (9)).

Remark 2. *If U satisfies*

$$\mathrm{Trace}\,(M) \geqslant 0 \qquad \forall M \in U, \tag{13}$$

then (11) *is automatically satisfied* (*since* $A^u \in U$). (13) *is satisfied by* $U = U_k$ *for all* $1 \leqslant k \leqslant n$.

Question 1. *Can the hypothesis* (11) *be removed in Theorem* 3?

In the rest of this note we outline the proof of Theorem 3 in [11], see also [12] for a somewhat more detailed version.

We first give a lemma. A slightly weaker version of the lemma was established in [10] (which is enough in the proof of Theorem 3).

Lemma 1 [12]. *For* $n \geqslant 2$, *let* B_1 *be the unit ball (centered at the origin) in* $\mathbb{R}^n$. *Let* $a \in \mathbb{R}$, $p, q \in \mathbb{R}^n$, $p \neq q$, *and let* $u \in L^1_{\mathrm{loc}}(B_1 \setminus \{0\})$ *satisfy*

$$\Delta u \leqslant 0 \quad in\ B_1 \setminus \{0\}\ in\ the\ distribution\ sense,$$

$$\liminf_{|x| \to 0} [u(x) - (a + p \cdot x)] \, |x|^{-1} \geqslant 0 ,$$

and

$$\liminf_{|x| \to 0} [u(x) - (a + q \cdot x)] \, |x|^{-1} \geqslant 0 .$$

Then

$$\liminf_{|x| \to 0} u(x) > a .$$

OUTLINE OF THE PROOF OF THEOREM 3. By the superharmonicity and the positivity of u,

$$\liminf_{|x| \to \infty} (|x|^{n-2} u(x)) > 0 . \tag{14}$$

As in the proof of Lemma 2.1 in [13], we know that for any $x \in \mathbb{R}^n$, there exists $\lambda_0(x) > 0$ such that

$$u_{x,\lambda}(y) := \left(\frac{\lambda}{|y-x|} \right)^{n-2} u \left(x + \frac{\lambda^2 (y-x)}{|y-x|^2} \right) \leqslant u(y) , \quad \forall \ |y-x| \geqslant \lambda, \ 0 < \lambda < \lambda_0(x) .$$

Define, for any $x \in \mathbb{R}^n$,

$$\bar{\lambda}(x) := \sup \{ \mu \,|\, u_{x,\lambda}(y) \leqslant u(y), \ \forall |y-x| \geqslant \lambda, \ 0 < \lambda < \mu \} .$$

Let

$$\alpha := \liminf_{|x| \to \infty} (|x|^{n-2} u(x)) . \tag{15}$$

We know from (14) that $0 < \alpha \leqslant \infty$.

If $\alpha = \infty$, the moving sphere procedure can never stop (no touching of u and $u_{x,\lambda}$ at infinity may occur) and therefore $\bar{\lambda}(x) = \infty$ for any $x \in \mathbb{R}^n$. This follows from arguments in [13] and [9]. The strong maximum principle and the Hopf lemma are used in the arguments to rule out the touching of u and $u_{x,\lambda}$ in $\{y \,|\, |y-x| > \lambda\}$ or u and $u_{x,\lambda}$ become tangential to each other on $\{y \,|\, |y-x| = \lambda\}$. All these are based on the conformal invariance of the equation: For a solution u of (10) and for any $x \in \mathbb{R}^n$ and any $\lambda > 0$, $u_{x,\lambda}$ is still a solution, *i.e.*,

$$F(A^{u_{x,\lambda}}) = 1 , \quad A^{u_{x,\lambda}} \in U, \qquad \text{in } \mathbb{R}^n \setminus B_\lambda(x) ,$$

where $B_\lambda(x)$ denotes the ball of radius λ and centered at x.

Once we know that $\bar{\lambda}(x) = \infty$ for all $x \in \mathbb{R}^n$, we have, by the definition of $\bar{\lambda}(x)$,

$$u_{x,\lambda}(y) \leqslant u(y) , \qquad \forall \ |y-x| \geqslant \lambda > 0 .$$

This implies (see *e.g.*, Lemma 11.2 in [13]) that $u \equiv$ *constant*, and Theorem 3 is proved in this case.

From now on, we assume that $0 < \alpha < \infty$. Since the moving sphere procedure stops at $\bar{\lambda}(x)$, we must have, by using the above mentioned arguments in [13] and [9],

$$\liminf_{|y| \to \infty} (u(y) - u_{x,\bar{\lambda}(x)}(y)) \, |y|^{n-2} = 0 , \tag{16}$$

i.e.,

(17) $$\alpha = \bar{\lambda}(x)^{n-2} u(x), \qquad \forall\, x \in \mathbb{R}^n.$$

For a Möbius transformation ϕ, we use notation

$$u_\phi := |J_\phi|^{\frac{n-2}{2n}} (u \circ \phi),$$

where J_ϕ denotes the Jacobian of ϕ.

For $x \in \mathbb{R}^n$, let

$$\phi^{(x)}(y) := x + \frac{\bar{\lambda}(x)^2 (y-x)}{|y-x|^2},$$

and we know that $u_{\phi(x)} = u_{x, \bar{\lambda}(x)}$.

For $\psi(y) := \dfrac{y}{|y|^2}$, $x \in \mathbb{R}^n$, define

$$w^{(x)} := (u_{\phi^{(x)}})_\psi = u_{\phi^{(x)} \circ \psi}.$$

By (16) and (17),

(18) $$w^{(x)}(0) = \alpha, \qquad \forall\, x \in \mathbb{R}^n,$$

and

$$\liminf_{y \to 0} u_\psi(y) = \alpha.$$

By the conformal invariance of the equation satisfied by u, we know that $u_\psi \in C^2(\mathbb{R}^n \setminus \{0\})$ and $\Delta u_\psi \leqslant 0$ in $\mathbb{R}^n \setminus \{0\}$.

By the definition of $\bar{\lambda}(x)$ and by the construction of $w^{(x)}$, we have, for some $\delta(x) > 0$,

$$w^{(x)} \in C^2(B_{\delta(x)}), \qquad \forall\, x \in \mathbb{R}^n,$$

$$u_\psi \geqslant w^{(x)} \quad \text{in } B_{\delta(x)} \setminus \{0\}, \qquad \forall\, x \in \mathbb{R}^n,$$

An application of Lemma 1 yields

$$\nabla w^{(x)}(0) = \nabla w^{(0)}(0), \; \textit{i.e.} \;\; \nabla w^{(x)}(0) \quad \text{is independent of } x \in \mathbb{R}^n.$$

A calculation gives, using (17), that

$$\nabla w^{(x)}(0) = (n-2)\, \bar{\lambda}(x)^{n-2} u(x)\, x + \bar{\lambda}(x)^n \nabla u(x) = (n-2)\, \alpha x + \alpha^{\frac{n}{n-2}} u(x)^{\frac{n}{2-n}} \nabla u(x).$$

Since $\vec{V} := \nabla w^{(x)}(0)$ is independent of x, we have

$$\nabla_x \left(\frac{n-2}{2} \alpha^{\frac{n}{n-2}} u(x)^{-\frac{2}{n-2}} - \frac{(n-2)\,\alpha}{2} |x|^2 + \vec{V} \cdot x \right) \equiv 0,$$

from which we deduce that u is of the form (12) with $a, b > 0$. Theorem 3 is established.

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