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Sets of finite perimeter associated with vector fields and polyhedral approximation

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Calcolo delle variazioni. — *Sets of finite perimeter associated with vector fields and polyhedral approximation.* Nota (*) di Francescopaolo Montefalcone, presentata dal Socio F. Ricci.

Abstract. — Let $X = (X_1, \dots, X_m)$ be a family of bounded Lipschitz continuous vector fields on $\mathbb{R}^n$. In this paper we prove that if E is a set of finite X-perimeter then his X-perimeter is the limit of the X-perimeters of a sequence of euclidean polyhedra approximating E in L^1-norm. This extends to Carnot-Carathéodory geometry a classical theorem of E. De Giorgi.

Key words: Carnot-Carathéodory metric; Perimeter; Polyhedra.

Riassunto. — *Insiemi di perimetro finito rispetto ad una famiglia di campi vettoriali e approssimazione poliedrale.* Sia data in $\mathbb{R}^n$ una m-upla $X = (X_1, \dots, X_m)$ di campi vettoriali lipschitziani e limitati. In questo lavoro dimostriamo che se E è un insieme di X-perimetro finito allora l'X-perimetro di E è il limite degli X-perimetri di una successione di poliedrali euclidee approssimanti E in norma L^1. Questo risultato estende alle geometrie di tipo Carnot-Carathéodory un classico teorema di E. De Giorgi.

1. Introduction

In the last few years, a Geometric Measure Theory in metric spaces of very general type has been developed. This project, already hidden in Federer's book [13], has been explicitly formulated and carried on by several authors. We only mention some of them: De Giorgi [11, 12], Gromov [20], Preiss and Tisěr [23], David and Semmes [9], Cheeger [8], Ambrosio and Kirchheim [3, 4], Garofalo and Nhieu [18] and Franchi, Serapioni and Serra Cassano [16, 17]. In particular, in some papers the progress in this direction is somehow connected with the development of the theory of anisotropic Sobolev spaces and that of degenerated elliptic operators of the form *sum of squares* (see, for instance, [5, 7, 14, 15, 18]). In connection with this project has been introduced the notion of function of bounded variation with respect to a family of vector fields and that related of perimeter (see [16, 18]). To give a notion of function of bounded variation with respect to a family of vector fields, there are several approaches, all reminding of the euclidean case, which actually turn out to be equivalent as proved in [16]. More precisely, let $X = (X_1, \dots, X_m)$ be a family of Lipschitz continuous vector fields in $\mathbb{R}^n$. We naturally identify these vector fields with first order differential operators, *i.e.* $X_j(x) = (c_{1j}(x), \dots, c_{nj}(x)) = \sum_{i=1}^{n} c_{ij}(x)\,\partial_i$, $j = 1, \dots, m$, where $c_{ij}(x)$ are Lipschitz continuous functions. If Ω is an open set of $\mathbb{R}^n$, the space $BV_X(\Omega)$ can be defined equivalently in several ways. If $u \in L^1(\Omega)$ we say that $u \in BV_X(\Omega)$ if one of the following holds (see [5, 16, 18, 21]):

$(\mathcal{D}_1)$ *the distributional gradient Xu is an $\mathbb{R}^m$-valued Radon measure*;

(*) Pervenuta in forma definitiva all'Accademia il 4 luglio 2003.

($\mathcal{D}_2$) $|Xu|(\Omega) < \infty$ *where the total variation* $|Xu|(\Omega)$ *is defined to be*

$$|Xu|(\Omega) = \sup\left\{ \int_\Omega u \operatorname{div}_X(\psi)\, dx : \psi \in C_0^1(\Omega, \mathbb{R}^m),\ |\psi|_m \leq 1 \right\},$$

where $\operatorname{div}_X(\psi) = -\sum_{j=1}^{m} X_j^* \psi_j$ *and* X_j^* *is the operator formally adjoint to* X_j *in* $L^2(\mathbb{R}^n)$;

($\mathcal{D}_3$) *u belongs to the domain of finiteness of the* L^1*-relaxed of the functional*

$$F(v) := \begin{cases} \int_\Omega \sqrt{\sum_j (X_j v)^2}\, dx & \text{if } v \in C^1(\Omega) \cap L^1(\Omega), \\ +\infty & \text{otherwise}. \end{cases}$$

Consequently we may define the X-perimeter in Ω of a measurable set E as *the total* X*-variation in* Ω *of the characteristic function of* E.

We stress that the last assertion ($\mathcal{D}_3$) is a particular case of a general definition given in [21] (see also [1] for the case of Alfhors regular metric measure spaces).

Moreover let us mention that in this framework has been recently proved in [22] a variational approximation of the perimeter associated with a family of Carnot-Carathéodory vector fields.

We should remind that some of the most useful results of the euclidean case as the Anzelotti-Giaquinta Theorem and the coarea formula of Fleming-Rischel, can be naturally extended (see Section 1.1) to BV_X with no further restriction on the nature of vector fields [16, 18].

With reference to the «further restriction» mentioned above we should make a short comment. Indeed we know that if the family of vector fields $X = (X_1, \dots, X_m)$ satisfies suitable geometric properties as for instance the Hörmander condition (*i.e.* the rank of the Lie algebra generated by $X_1, \dots, X_m \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ equals n at any point of $\mathbb{R}^n$), we can canonically associate with X a metric d_X, called Carnot-Carathéodory metric, defined as *the minimum time necessary to connect two points of the space by curves everywhere tangent to the linear space spanned by the vector fields.* Carnot-Carathéodory metrics arise naturally in studying isoperimetric inequalities or those of Sobolev-Poincaré type, because the intrinsec dimension of the geometry associated with a family X of vector fields is given by the asymptotic behavior of the measure of spheres in the metric d_X. Now, instead of a general philosophy which suggests to adapt the classical tools for the Carnot-Carathéodory geometries, the density theorem and the coarea formula are independent of the geometry itself, since totally degenerated situations are possible.

In this paper we want to show that, preserving in this context the classical definition of polyhedron (see Definition 2.1 below), a satisfying notion of polyhedral approximation can be given. The result obtained extends a well known theorem of E. De Giorgi which states (see [10]): *the perimeter of a set E is the lower limit of the perimeters of the polyhedra approximating E in* L^1*-norm.* Indeed we show that the following holds

THEOREM [Polyhedral Approximation Theorem]. *Let Ω be an open subset of $\mathbb{R}^n$ and denote by $\mathcal{P}^n$ the class of n-dimensional polyhedra of $\mathbb{R}^n$. Let $X = (X_1, \dots, X_m)$ be a family of bounded Lipschitz continuous vector fields in $\mathbb{R}^n$. Finally let $E \subset \mathbb{R}^n$ be a set of finite X-perimeter in Ω such that $\mathbf{1}_E \in L^1(\mathbb{R}^n)$. Then there exists a sequence $\{\Sigma_i\}_{i \in \mathbb{N}} \subset \mathcal{P}^n$ such that*

(1) $\lim_{i \to \infty} \|\mathbf{1}_{\Sigma_i} - \mathbf{1}_E\|_{L^1(\Omega)} = 0$,

(2) $\lim_{i \to \infty} |X\mathbf{1}_{\Sigma_i}|(\Omega) = |X\mathbf{1}_E|(\Omega)$.

We point out that the boundedness assumptions on the family of vector fields can be dropped if the Carnot-Carathéodory distance associated with $X_1, \dots, X_m$ is finite, namely every two points of $\mathbb{R}^n$ are accessible from one another, as in the case of Hörmander vector fields. We point out also that the assumption $\mathbf{1}_E \in L^1(\mathbb{R}^n)$ can be dropped in many situations. Indeed, if for instance the vector fields satisfy the Hörmander condition and the open set Ω is suitably regular, then by the relative isoperimetric inequality (see [18]) we can obtain that either $\mathbf{1}_E$ or $\mathbf{1}_{E^c}$ belong to $L^1(\mathbb{R}^n)$.

In accordance with the definition of perimeter associated with a family of vector fields we may also give the notion of *partial X_j-perimeter*, $j = 1, \dots, m$. More precisely we say that E has finite X_j-perimeter in Ω if $|X_j\mathbf{1}_E|(\Omega) < \infty$ where

$$|X_j\mathbf{1}_E|(\Omega) = \sup\left\{ \int_\Omega \mathbf{1}_E X_j^*(\psi)\, dx : \psi \in C_0^1(\Omega),\ |\psi| \leq 1 \right\}.$$

This definition allows us to give immediately a further characterization of sets of finite X-perimeter which generalizes the first one given by Caccioppoli [6, 10].

COROLLARY. *Let $X = (X_1, \dots, X_m)$ be a family of bounded Lipschitz continuous vector fields in $\mathbb{R}^n$ and let $\Omega \subseteq \mathbb{R}^n$ be open. Let $E \subset \mathbb{R}^n$ be a measurable set such that $\mathbf{1}_E \in L^1(\Omega)$. If for any $j = 1, \dots, m$, there exists a sequence $\{\Sigma_i^j\}_{i \in \mathbb{N}} \subset \mathcal{P}^n$ such that*

(1) $\lim_{i \to \infty} \|\mathbf{1}_{\Sigma_i^j} - \mathbf{1}_E\|_{L^1(\Omega)} = 0$,

(2) *there exists $C_j \in \mathbb{R}_+$ such that* $\sup_i |X_j\mathbf{1}_{\Sigma_i^j}|(\Omega) \leq C_j$,

then E has finite X-perimeter in Ω and there exists a sequence $\{\Sigma_i\}_{i \in \mathbb{N}} \subset \mathcal{P}^n$ such that

(3) $\lim_{i \to \infty} \|\mathbf{1}_{\Sigma_i} - \mathbf{1}_E\|_{L^1(\Omega)} = 0$;

(4) $\lim_{i \to \infty} |X_j\mathbf{1}_{\Sigma_i}|(\Omega) = |X_j\mathbf{1}_E|(\Omega)$.

This results are analogous to the corresponding theorems of the euclidean setting. Nevertheless we should notice that this is not a priori obvious. Indeed one could imagine to have to handle with an intrinsec notion of polyhedra, shaped to the vector fields but actually, what is obtained agrees with the role played in this context by the classical Friedrichs' mollifiers, which turn out to be independent of the geometry of the vector fields [16, 18].

1.1. *Functions with bounded X-variation and sets of finite X-perimeter.*

Throughout this paper $\Omega \subseteq \mathbb{R}^n$ is an open set. If $x, y \in \mathbb{R}^n$, we denote by $|x|_n$ and $\langle x, y \rangle$ the euclidean norm and the scalar product, respectively. $B_r(x)$ denotes the open euclidean ball of radius r centered at x. If $A \subset \mathbb{R}^n$, $\mathbf{1}_A$ denotes the characteristic function of A, $|A|_n$ its n-dimensional Lebesgue measure and $\mathcal{H}^{n-1}(A)$ its $(n-1)$-dimensional Hausdorff measure. We denote by $C^k(\Omega, \mathbb{R}^m)$ the space of $\mathbb{R}^m$-valued functions k times continuously differentiable, by $\mathrm{Lip}(\Omega, \mathbb{R}^m)$ the space of $\mathbb{R}^m$-valued Lipschitz continuous functions and by $C_0^k(\Omega, \mathbb{R}^m)$ that of $\mathbb{R}^m$-valued functions k times continuously differentiable with compact support contained in Ω. We will use spherically symmetric mollifiers J_ε defined by $J_\varepsilon(x) = \varepsilon^{-n} J\left(\frac{x}{\varepsilon}\right)$, where $J \in C_0^\infty(\mathbb{R}^n)$, $J \geqslant 0$, $\mathrm{spt}\, J \subset B_1(0)$ and $\int_{\mathbb{R}^n} J(x)\, dx = 1$.

From now on we assume that $X_1, \dots, X_m$ are locally Lipschitz continuous vector fields on $\mathbb{R}^n$, where $X_j = (c_{1j}, \dots, c_{nj})$, $j = 1 \dots, m$. We shall identify these vector fields with first order differential operators, *i.e.* $X_j(x) = (c_{1j}(x), \dots, c_{nj}(x)) = \sum_{i=1}^n c_{ij}(x)\, \partial_i, j = 1, \dots, m$, where $c_{ij}(x)$ are Lipschitz continuous functions and we set $C = \mathrm{row}\,[\mathrm{X}_1, \dots, \mathrm{X}_\mathrm{m}]$.

Given $Y \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ we denote by Y^* the operator formally adjoint in $L^2(\mathbb{R}^n)$, *i.e.* the operator which for all $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} \varphi Y \psi \, dx = \int_{\mathbb{R}^n} \psi Y^* \varphi \, dx.$$

Moreover we define the *X-divergence* of $\varphi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^m)$ as follows

$$\mathrm{div}_X(\varphi) := -\sum_{j=1}^m X_j^* \varphi_j.$$

Finally we introduce the following test functions

$$F(\Omega, \mathbb{R}^m) := \{\varphi \in C_0^1(\Omega, \mathbb{R}^m) : |\varphi|_m \leqslant 1\}.$$

DEFINITION 1.1. *We say that $u \in L^1_{\mathrm{loc}}(\Omega)$ has bounded X-variation in Ω if*

$$|Xu|(\Omega) := \sup_{\varphi \in F(\Omega, \mathbb{R}^m)} \int_\Omega u \, \mathrm{div}_X(\varphi)\, dx < \infty,$$

and we put

$$BV_X(\Omega) = \{u \in L^1(\Omega) : |Xu|(\Omega) < \infty\}.$$

REMARK 1.2. *The space $BV_X(\Omega)$ can be equivalently defined as the set of all $u \in L^1(\Omega)$ such that there exists an m-vector valued Radon measure $\mu = (\mu_1, \dots, \mu_m)$*

on Ω such that for all $\varphi \in C_0^1(\Omega, \mathbb{R}^m)$

$$\int_\Omega u \operatorname{div}_X(\varphi)\, dx = -\int_\Omega \langle \varphi, d\mu \rangle$$

where $\langle \varphi, d\mu \rangle = \sum_{j=1}^m \varphi_j d\mu_j$. Notice that by Riesz Theorem it follows that $|Xu|(\Omega) = |\mu|(\Omega)$.

REMARK 1.3 [16]. *Let u and Xu be in $L^1(\Omega)$ and set $Xu = (X_1 u, \dots, X_m u)$. Then we have*

$$|Xu|(\Omega) = \int_\Omega |Xu|_m\, dx\,.$$

DEFINITION 1.4. *We say that a measurable subset E of $\mathbb{R}^n$ has finite X-perimeter in Ω if $|Xu|(\Omega)$ is finite and we call X-perimeter of E in Ω the quantity*

$$|\partial E|_X(\Omega) := |X\mathbf{1}_E|(\Omega)\,.$$

If E is a set of finite X-perimeter in Ω then by Remark 1.2 it follows that the distributional derivative $\mu = X_{\mathbf{1}_E}$ is a m-vector valued Radon measure and $|\mu|(U) = |X_{\mathbf{1}_E}|(U) = |\partial E|_X(U)$, for any open $U \Subset \Omega$. Moreover, by the polar decomposition theorem (see [2]) there exists a $|\mu|$-measurable function $\nu_E \colon \Omega \to \mathbb{R}^m$ such that $\mu = \nu_E |\mu|$ and also that $|\nu_E| = 1$ $|\mu|$-a.e.; we call ν_E the *X-generalized inner normal* of E.

REMARK 1.5 [7]. *Suppose that the boundary of $E \subset \mathbb{R}^n$ is an $(n-1)$-dimensional C^2-manifold in Ω, then*

$$|\partial E|_X(\Omega) = \int_{\Omega \cap \partial E} |C(x)\, \nu(x)|_m\, d\mathcal{H}^{n-1}(x)\,,$$

where ν denotes the inner unit normal of ∂E. In this case we have that the X-generalized inner normal of E is

$$\nu_E = \frac{C(x)\, \nu(x)}{|C(x)\, \nu(x)|_m}\,,$$

whenever $C(x)\nu(x) \neq 0$, and 0 otherwise, for $|\partial E|_X$-a.e. $x \in \Omega$. Now if $X = (\partial_1, \dots, \partial_n)$, then the above identity reduces to the well known formula

$$|\partial E|(\Omega) = \mathcal{H}^{n-1}(\Omega \cap \partial E)\,,$$

where the left-hand side is the classical perimeter of E in Ω (see [19] for the euclidean case).

Now let us remind some properties of bounded X-variation functions that we will used later on (see also [16, 18, 21]).

PROPOSITION 16 [Lower semicontinuity]. *Let* $u, u_h \in L^1(\Omega)$, $h \in \mathbb{N}$, *such that* $u_k \to u$ *in* $L^1_{\mathrm{loc}}(\Omega)$. *Then*

$$|Xu|(\Omega) \leq \liminf_{h \to \infty} |Xu_h|(\Omega).$$

As in classical case smooth functions are dense in BV_X in the following weak sense

THEOREM 1.7 [Density for BV_X functions]. *Let* $X = (X_1, \dots, X_m)$ *as above and let* $\Omega \subset \mathbb{R}^n$ *be open. If* $u \in BV_X(\Omega)$ *there exists a sequence* $\{u_h\}_{h \in \mathbb{N}} \subset C^\infty(\Omega) \cap BV_X(\Omega)$ *such that*

$$\lim_{h \to \infty} \|u_h - u\|_{L^1(\Omega)} = 0 \qquad \textit{and} \qquad \lim_{h \to \infty} \int_\Omega |Xu_h|_m \, dx = |Xu|(\Omega).$$

Finally we can prove the following generalized coarea formula.

THEOREM 1.8 [Coarea formula for BV_X functions]. *Let* $X = (X_1, \dots, X_m)$ *as above and let* $\Omega \subset \mathbb{R}^n$ *be open. Let* $u \in BV_X(\Omega)$ *and set* $E_t = \{x \in \Omega : u(x) > t\}$. *Then*

(1) E_t *has finite X-perimeter for almost every* $t \in \mathbb{R}$;

(2) $|Xu|(\Omega) = \int_{-\infty}^{+\infty} |\partial E_t|_X(\Omega)\, dt$.

Conversely, if $u \in L^1(\Omega)$ *and* $\int_{-\infty}^{+\infty} |\partial E_t|_X(\Omega)\, dt < \infty$, *then* $u \in BV_X(\Omega)$ *and* (2) *holds.*

2. POLYHEDRAL APPROXIMATION

DEFINITION 2.1. *Let denote by* H^n *the family of all affine hyperplanes of* $\mathbb{R}^n$. *We say that* Σ *is a n-dimensional polyhedron of* $\mathbb{R}^n$ *if there exists* $m_\Sigma \in \mathbb{N}$ *and* $\{H_i\}_{i=1}^{m_\Sigma} \subset \mathsf{H}^n$ *such that*

$$\partial \Sigma \subseteq \bigcup_{i=1}^{m_\Sigma} H_i.$$

By $\mathcal{P}^n$ *we denote the family of n-dimensional polyhedra of* $\mathbb{R}^n$.

We state now the main result of this paper and some first consequences.

THEOREM 2.2. *Let* $X = (X_1, \dots, X_m)$ *be a family of bounded Lipschitz continuous vector fields in* $\mathbb{R}^n$ *and let* $\Omega \subseteq \mathbb{R}^n$ *be open. Let* $E \subset \mathbb{R}^n$ *be a set of finite X-perimeter in* Ω *such that* $\mathbf{1}_E \in L^1(\mathbb{R}^n)$. *Then there exists a sequence* $\{\Sigma_i\}_{i \in \mathbb{N}} \subset \mathcal{P}^n$ *such that*

(1) $\lim_{i \to \infty} \|\mathbf{1}_{\Sigma_i} - \mathbf{1}_E\|_{L^1(\Omega)} = 0$,

(2) $\lim_{i \to \infty} |\partial \Sigma_i|_X(\Omega) = |\partial E|_X(\Omega)$.

The proof of this theorem will be given in Section 2.2. We stress that the boundedness assumptions on the family of vector fields are not necessary if the Carnot-Carathéodory distance associated with $X_1, \dots, X_m$ is finite, *i.e.* every two points of $\mathbb{R}^n$

are accessible from one another, as for instance in the case of Hörmander vector fields (see also Remark 2.11 below).

REMARK 2.3. *If Σ is a n-dimensional polyhedron, then by definition we have*

$$\partial\Sigma = \bigcup_{h=1}^{m_\Sigma} \Sigma_h^{n-1},$$

where $\{\Sigma_h^{n-1}: h = 1, \dots, m_\Sigma\}$ is a finite set of $n-1$-dimensional polyhedra of $\mathbb{R}^n$ with mutually disjoint relative interiors and from Remark 1.5 we get

$$|\partial\Sigma|_X(\Omega) = \sum_{h=1}^{m_\Sigma} \int_{\Omega\cap\partial E} |C(x)\,\nu_h|_m\, d\mathcal{H}^{n-1}(x),$$

where ν_h is the inner unit normal vector of Σ_h^{n-1}, which turns out to be constant.

From Theorem 2.2 this formula yields a concrete expression for a numerical approximation of the X-perimeter of any subset of $\mathbb{R}^n$. Together with the definition of X-perimeter we may give that of *partial perimeter along a vector field.*

DEFINITION 2.4. *Let $X = (X_1, \dots, X_m)$ be a family of bounded Lipschitz continuous vector fields in $\mathbb{R}^n$ and let $\Omega \subseteq \mathbb{R}^n$ be open. We say that E has finite X_j-perimeter in Ω if $\mathbf{1}_E \in L^1(\Omega)$ and $|X_j\mathbf{1}_E|(\Omega) < \infty$, where*

$$|X_j\mathbf{1}_E|(\Omega) = \sup\left\{\int_\Omega \mathbf{1}_E X_j^*(\psi)\,dx : \psi \in C_0^1(\Omega),\ |\psi| \leqslant 1\right\}.$$

In this case we call partial X_j-perimeter of E in Ω the quantity

$$|\partial E|_{X_j}(\Omega) := |X_j\mathbf{1}_E|(\Omega).$$

This definition agrees with Definition 1.3 and it is the same if the family $X = (X_1, \dots, X_m)$ reduces to a unique vector field. Therefore the partial perimeter along a vector field enjoys the same properties of the X-perimeter established in Section 1.1.

REMARK 2.5. *Let Ω be open and let $E \subset \mathbb{R}^n$ such that $\mathbf{1}_E \in L^1(\Omega)$. Then*

$$|\partial E|_X(\Omega) \leqslant \sum_{j=1}^m |\partial E|_{X_j}(\Omega).$$

This easily follows by definitions. Indeed if $\varphi = (\varphi_1, \dots, \varphi_m) \in F(\Omega, \mathbb{R}^m)$ we get

$$\int_{\Omega\cap E} \mathrm{div}_X(\varphi)\,dx =$$

$$= -\int_{\Omega\cap E} \sum_{j=1}^m X^*(\varphi_j)\,dx \leqslant \sum_{j=1}^m \sup\left\{\int_{\Omega\cap E} X_j^*(\psi_j)\,dx : \psi_h \in C_0^1(\Omega),\ |\psi_j| \leqslant 1\right\}.$$

Therefore, taking the supremum with respect to $\varphi \in F(\Omega, \mathbb{R}^m)$, we get

$$|\partial E|_X(\Omega) \leqslant \sum_{j=1}^{m} \sup \left\{ \int_{\Omega \cap E} X_j^*(\psi_j)\, dx : \psi_j \in C_0^1(\Omega),\ |\psi_j| \leqslant 1 \right\} = \sum_{j=1}^{m} |\partial E|_{X_j}(\Omega).$$

From what proven we can easily deduce the following simple characterization for sets of finite X-perimeter (see also [6, 9]).

COROLLARY 2.6. *Let $X = (X_1, \dots, X_m)$ be a family of bounded Lipschitz continuous vector fields in $\mathbb{R}^n$ and let $\Omega \subseteq \mathbb{R}^n$ be open. Let E be a measurable subset of $\mathbb{R}^n$ such that $\mathbf{1}_E \in L^1(\Omega)$. If for any $j \in \{1, \dots, m\}$ there exists $\{\Sigma_i^j\}_{i \in \mathbb{N}} \subset \mathcal{P}^n$ such that*

(1) $\lim_{i \to \infty} \|\mathbf{1}_{\Sigma_i^j} - \mathbf{1}_E\|_{\mathcal{L}^1(\Omega)} = 0$;

(2) *there exists $C_j > 0$ such that* $\sup_{i \in \mathbb{N}} |\partial \Sigma_i^j|_{X_j}(\Omega) < C_j$;

then E has finite X-perimeter in Ω and there exists $\{\Sigma_i\}_{i \in \mathbb{N}} \subset \mathcal{P}^n$ such that

(3) $\lim_{i \to \infty} \|\mathbf{1}_{\Sigma_i} - \mathbf{1}_E\|_{\mathcal{L}^1(\Omega)} = 0$;

(4) $\lim_{i \to \infty} |\partial \Sigma_i|_X(\Omega) = |\partial E|_X(\Omega)$.

PROOF. The first part of the statement follows from the above remark observing that

$$|\partial E|_X(\Omega) \leqslant \sum_{j=1}^{m} |\partial E|_{X_j}(\Omega) \leqslant \sum_{j=1}^{m} \liminf_{i \to \infty} |\partial \Sigma_i^j|_{X_j}(\Omega) < \sum_{j=1}^{m} C_j < \infty,$$

whereas the second part is just a reformulation of Theorem 2.2. □

2.1. *An approximation lemma.*

Let us fix now some notations. Let $\mathsf{A}_n(\mathbb{R})$ denote the affine group of $\mathbb{R}^n$ *i.e.* the group of trasformations of $\mathbb{R}^n$ onto itself, represented by the equations $\xi(x) = Ax + b$, $\det A \neq 0$, where A and b are $n \times n$ and $n \times 1$ matrices, respectively. By $\mathsf{O}_n(\mathbb{R})$ we denote the orthogonal group, *i.e.* the group of all $n \times n$ real matrices A such that $AA^T = I$, where I denotes the $n \times n$ unit matrix. Let us denote by $\mathsf{M}_n(\mathbb{R})$ the group of the motions of $\mathbb{R}^n$, *i.e.* the subgroup of the affine group, defined by the equations $\xi(x) = Ax + b$, $A \in \mathsf{O}_n(\mathbb{R})$. Let $\mathrm{conv}(A)$ denote the convex hull of $A \subseteq \mathbb{R}^n$. If $v, w \in \mathbb{R}^n$, $[v, w]$ denotes the close line segment joining them. If $r \in \mathbb{R}$ and $t \in \mathbb{R}^n$, we denote by δ_r the dilation with center the origin and ratio r and by τ_t the translation defined by t. We set $\mathcal{Q}^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n : \max_i |x_i| \leqslant 1\}$ that is the unit n-cube of $\mathbb{R}^n$ and if $\alpha > 0$ we put $\mathcal{Q}^n_\alpha := \delta_\alpha \mathcal{Q}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \max_i |x_i| \leqslant \alpha\}$.

If $a \in \mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0,\ i = 1, \dots, n\}$, we set

$$\mathcal{T}^n_a := \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \frac{x_i}{a_i} \leqslant 1,\ x_i \geqslant 0,\ i = 1, \dots, n \right\},$$

and we call *standard n-tetrahedron of $\mathbb{R}^n$ relative to the vector a* any subset of $\mathbb{R}^n$ isometric to $\mathcal{T}^n_a$. Finally we denote by V the set of vertices of $\mathcal{T}^n_a$, that is $V_a = \{0\} \cup \{v_i =$

$= a_i e_i \colon i = 1, \ldots, n\}$, where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. Clearly one has $\mathcal{T}_a^n = \operatorname{conv}(V_a)$. The following elementary lemma will be used in the proof of Lemma 2.8 and it shows that the unit n-cube of $\mathbb{R}^n$ can be covered by means of a fixed number of isometric standard n-tetrahedra.

LEMMA 2.7. *Let us fix $n \geq 2$ and set $I_n = \{1, \ldots, 2^{n-1} n!\}$. Then there exist $a \in \mathbb{R}_+^n$ and a family of motions $\Gamma = \{\gamma_i\}_{i \in I_n} \subset \mathsf{M}_n(\mathbb{R})$ satisfying*

(1) $\mathcal{Q}^n = \bigcup_{i \in I_n} \gamma_i(\mathcal{T}_a^n)$;

(2) $\operatorname{int} \gamma_i(\mathcal{T}_a^n) \cap \operatorname{int} \gamma_j(\mathcal{T}_a^n) = \emptyset$ *for all* $i, j \in I_n$, $i \neq j$;

(3) $\operatorname{conv}(\gamma_i(V_a) \cap \gamma_j(V_a)) = \gamma_i(\mathcal{T}_a^n) \cap \gamma_j(\mathcal{T}_a^n)$ *for all* $i, j \in I_n$, $i \neq j$.

Explicitly $\gamma_i(x) = A_i x + b_i$, $x \in \mathbb{R}^n$, *where* $A_i \in \mathsf{O}_n(\mathbb{R})$ *and* $b_i \in \mathbb{R}^n$, *for* $i \in I_n$.

PROOF. The proof is straightforward when $n = 2$. In this case we simply triangulate the unit square by tracing its diagonals. Then we can use an inductive argument to find such triangulations in the following way: after triangulating the faces of the unit n-cube we get the required n-tetrahedra by connecting the center of the n-cube to all the vertices of the $(n-1)$-tethraedra forming the triangulation of the faces. Now if we choose a tetrahedron, then all the others can be obtained by means of a motion of this one. □

We state now one of the main ingredients in the proof of our Polyhedral Approximation Theorem.

LEMMA 2.8. *Let $X = (X_1, \ldots, X_m)$ be a family of Lipschitz continous vector fields in $\mathbb{R}^n$. Fix $\alpha > 0$ and let $\mathcal{Q}_\alpha^n$ be as above. Let $\varphi \in C^\infty(\mathcal{Q}_\alpha^n)$, $\varphi \geq 0$. Then for any ε_1, ε_2, $\varepsilon_3 > 0$ there exists a piecewise linear function $\psi \colon \mathcal{Q}_\alpha^n \to \mathbb{R}$, $\psi \geq 0$, $\psi = \psi(\varphi, \alpha, \varepsilon_1, \varepsilon_2, \varepsilon_3)$, satisfying*

(1) $0 < \psi(x) - \varphi(x) < \varepsilon_1$ *for all* $x \in \mathcal{Q}_\alpha^n$;

(2) $\|\psi - \varphi\|_{L^1(\mathcal{Q}_\alpha^n)} < \varepsilon_2$;

(3) $\int_{\mathcal{Q}_\alpha^n} |X\psi|_m \, dx \leq \int_{\mathcal{Q}_\alpha^n} |X\varphi|_m \, dx + \varepsilon_3$.

REMARK 2.9. *By setting*

$$\Psi = \left\{(x_1, \ldots, x_{n+1}) \in \mathcal{Q}_\alpha^n \times \mathbb{R} : 0 \leq x_{n+1} \leq \psi(x),\ x = (x_1, \ldots, x_n)\right\},$$

we have that $\Psi \in \mathcal{P}^{n+1}$, i.e. *$\Psi$ is a polyhedral set of $\mathbb{R}^{n+1}$.*

PROOF OF LEMMA 2.8. Choose a positive $k \in \mathbb{N}$ and set

$$\frac{\alpha}{k}\mathbb{Z}^n := \left\{t \in \mathbb{R}^n \colon t = \frac{\alpha}{k} z,\ z \in \mathbb{Z}^n\right\} \quad \text{and} \quad \Delta := \left\{t \in \frac{\alpha}{k}\mathbb{Z}^n \colon \bigcup_t \tau_t \circ \delta_{\alpha/k}(\mathcal{Q}^n) = \mathcal{Q}_\alpha^n\right\}.$$

Let $\Gamma = \{\gamma_i\}_{i \in I_n} \subset \mathsf{M}_n(\mathbb{R})$ the family of motions of Lemma 2.7, where $\gamma_i(x) = A_i x + b_i$. Now we define a family of functions $\xi_{i,k,t} \colon \mathbb{R}^n \to \mathbb{R}$ depending on $i \in I_n$, $k > 0$ and

$t \in \Delta$, by setting

$$\xi_{i,k,t}(y) := \frac{\alpha}{k} A_i y + \frac{\alpha}{k} b_i + t, \qquad y \in \mathbb{R}^n.$$

Clearly $\xi_{i,k,t} \in \mathsf{A}_n(\mathbb{R})$ and we have

$$\mathcal{Q}^n_\alpha = \bigcup_{t\in\Delta} \bigcup_{i \in I_n} \xi_{i,k,t}(\mathcal{C}^n_a),$$

where $\mathcal{C}^n_a$ is defined in Lemma 2.7. Let $V_a = \{0\} \cup \{v_i : i = 1, \dots, n\}$ be the set of the vertices of $\mathcal{C}^n_a$. From now on we set $v_0 = 0$ and thus we have

$$v_i = \begin{cases} 0 & \text{if } i = 0, \\ a_i e_i & \text{if } i = 1, \dots, n. \end{cases}$$

Moreover we set $\Lambda = \{\lambda_i \in L(\mathbb{R}^n, \mathbb{R}) : \lambda_i(v_j) = \delta_{i,j},\ i, j = 0, 1, \dots, n\}$, where $L(\mathbb{R}^n, \mathbb{R})$ denotes the space of linear functionals on $\mathbb{R}^n$. Then

$$\lambda_0(y) = 1 - \sum_{i=1}^n a_i^{-1} y_i \quad \text{ and } \quad \lambda_i = a_i^{-1} y_i,\ i = 1, \dots, n.$$

From these expressions it follows that $\nabla\lambda_0 = (-a_1^{-1}, \dots, -a_n^{-1})$ and $\nabla\lambda_i = a_i^{-1} e_i$, for any $i = 1, \dots, n$. Fix now k and let us define the function $\widetilde{\psi}_{i,t}\colon \mathbb{R}^n \to \mathbb{R}$, as follows

$$\widetilde{\psi}_{i,t}(x) := \sum_{h=0}^n (\varphi \circ \xi_{i,k,t})(v_h)(\lambda_h \circ \xi^{-1}_{i,k,t})(x), \qquad x \in \mathbb{R}^n.$$

The graph of this function is an affine hyperplane of $\mathbb{R}^{n+1}$ which interpolates the points of the graph of φ representing the images of the n-tetrahedron $\xi_{i,k,t}(\mathcal{C}^n_a)$. Starting from the family of functions $\{\widetilde{\psi}_{i,t}\}_{i,t}$ where $i \in I_n$ and $t \in \Delta$, we can construct a unique map $\widetilde{\psi} : \mathcal{Q}^n_\alpha \to \mathbb{R}$, continuous and piecewise linear, $\widetilde{\psi} = \widetilde{\psi}(k, \alpha, \varphi)$, such that

$$\widetilde{\psi}\,|_{\xi_{i,k,t}(\mathcal{C}^n_a)} = \widetilde{\psi}_{i,t}\,|_{\xi_{i,k,t}(\mathcal{C}^n_a)}.$$

STEP 1. There exists a constant $\mathcal{C} > 0$, $\mathcal{C} = \mathcal{C}(\varphi, \alpha)$, such that

$$|X\widetilde{\psi}|_m \leq |X\varphi|_m + k^{-1}\mathcal{C} \ \text{ for a.e. } \ x \in \mathcal{Q}^n_\alpha. \tag{1}$$

PROOF OF STEP 1. Let us observe that the function $\widetilde{\psi}$, by Rademacher's theorem, is differentiable for a.e. $x \in \mathcal{Q}^n_\alpha$ and that $X\widetilde{\psi} = C\nabla\widetilde{\psi}$, where $C = \mathrm{row}\,[X_1, \dots, X_m]$ is the matrix of the coefficients of the vector fields. Thus we can proceed directly, looking for a local expression of $\nabla\widetilde{\psi}$. Then by definition we have

$$\widetilde{\psi}(x) = \sum_{h=0}^n (\varphi \circ \xi_{i,k,t})(v_h)(\lambda_h \circ \xi^{-1}_{i,k,t})(x) \ \text{ for } \ x \in \operatorname{int} \xi_{i,k,t}(\mathcal{C}^n_a).$$

Now since

$$\frac{\partial}{\partial x_j}(\lambda_h \circ \xi^{-1}_{i,k,t})(x) = \left\langle \nabla\lambda_h(\xi^{-1}_{i,k,t}(x)),\ \frac{k}{\alpha} A_i^{-1} e_j \right\rangle \quad (j = 1, \dots, n)$$

if $x \in \operatorname{int} \xi_{i,k,t}(\mathcal{C}_a^n)$ we get

$$\nabla \widetilde{\psi}(x) = \frac{k}{\alpha} A_i \sum_{h=0}^{n} (\varphi \circ \xi_{i,k,t})(v_h)\, \nabla \lambda_h(\xi_{i,k,t}^{-1}(x)) =$$

$$= \frac{k}{\alpha} A_i \left((\varphi \circ \xi_{i,k,t})(v_0)(-a_1^{-1}, -a_2^{-1}, \dots, -a_n^{-1}) + \sum_{h=1}^{n} a_h^{-1} (\varphi \circ \xi_{i,k,t})(v_h)\, e_h \right) =$$

$$= \frac{k}{\alpha} A_i \sum_{h=1}^{n} a_h^{-1} ((\varphi \circ \xi_{i,k,t})(v_h) - (\varphi \circ \xi_{i,k,t})(v_0))\, e_h .$$

By Taylor's formula, for $h \in \{1, \dots, n\}$ there exists $z_h \in \left[\frac{\alpha}{k} b_i + t, \frac{\alpha}{k} b_i + t + \frac{\alpha}{k} A_i v_h \right] \subset \operatorname{int} \xi_{i,k,t}(\mathcal{C}_a^n)$ such that

$$\varphi \circ \xi_{i,k,t}(v_h) - \varphi \circ \xi_{i,k,t}(v_0) = \varphi\left(\frac{\alpha}{k} A_i v_h + \frac{\alpha}{k} b_i + t \right) - \varphi\left(\frac{\alpha}{k} b_i + t \right) =$$

$$= \left\langle \nabla \varphi\left(\frac{\alpha}{k} b_i + t \right), \frac{\alpha}{k} A_i v_h \right\rangle + \frac{\alpha^2}{2k^2} \langle \mathcal{H}_\varphi(z_h)\, A_i v_h, A_i v_h \rangle .$$

Since $v_h = a_h e_h (h = 1, \dots, n)$ we get

$$\nabla \widetilde{\psi}(x) = \frac{k}{\alpha} A_i \sum_{h=1}^{n} a_h^{-1} \left(a_h \frac{\alpha}{k} \left\langle A_i^{-1} \nabla \varphi\left(\frac{\alpha}{k} b_i + t \right), e_h \right\rangle + \frac{\alpha^2}{2k^2} \langle \mathcal{H}_\varphi(z_h)\, A_i v_h, A_i v_h \rangle \right) e_h .$$

Therefore

$$\nabla \widetilde{\psi}(x) = \nabla \varphi\left(\frac{\alpha}{k} b_i + t \right) + \frac{\alpha}{2k} \sum_{h=1}^{n} \langle \mathcal{H}_\varphi(z_h)\, A_i v_h, A_i e_h \rangle\, A_i e_h .$$

Applying Taylor's formula again, we have

$$\nabla \varphi\left(\frac{\alpha}{k} b_i + t \right) = \nabla \varphi(x) - \sum_{h=1}^{n} \left\langle \mathcal{H}_\varphi(\tilde{z}_h)\, e_h, x - \frac{\alpha}{k} b_i - t \right\rangle e_h ,$$

where $\tilde{z}_h \in \left[\frac{\alpha}{k} b_i + t, x \right]$, for $h = 1, \dots, n$. Setting $y = \xi_{i,k,t}^{-1}(x)$ and substituting we find

$$\nabla \widetilde{\psi}(x) = \nabla \varphi(x) + \frac{\alpha}{k} \sum_{h=1}^{n} \left\{ \frac{1}{2} \langle \mathcal{H}_\varphi(z_h)\, A_i v_h, A_i e_h \rangle\, A_i e_h - \langle \mathcal{H}_\varphi(\tilde{z}_h)\, e_h, A_i y \rangle\, e_h \right\} .$$

Therefore by definition of X-gradient it follows that for any $x \in \operatorname{int} \xi_{i,k,t}(\mathcal{C}_a^n)$

$$X \widetilde{\psi}(x) = X \varphi(x) + \frac{\alpha}{2k} \sum_{h=1}^{n} \left\{ C(x) \left(\langle \mathcal{H}_\varphi(z_h) A_i v_h, A_i e_h \rangle A_i e_h - 2 \langle \mathcal{H}_\varphi(\tilde{z}_h)\, e_h, A_i y \rangle\, e_h \right) \right\} .$$

For $i \in I_n$, $t \in \Delta$, we set

$$\mathcal{F}_{i,k,t}(x) := \frac{\alpha}{2} \sum_{h=1}^{n} \left\{ C(x) \left(\langle \mathcal{H}_\varphi(z_h) A_i v_h, A_i e_h \rangle A_i e_h - 2 \langle \mathcal{H}_\varphi(\tilde{z}_h)\, e_h, A_i \rangle\, e_h \right) \right\} ,$$

and also

$$\mathcal{F}(x) := \mathcal{F}_{i,k,t}(x) \quad \text{for } x \in \operatorname{int} \xi_{i,k,t}(\mathcal{C}_a^n) .$$

Then

$$X\widetilde{\psi}(x) = X\varphi(x) + \frac{1}{k}\,\mathcal{F}(x) \quad \text{for a.e.} \quad x \in \mathcal{Q}^n_\alpha,$$

and we have

$$\max_{x\in\mathcal{Q}^n_\alpha} |\mathcal{F}(x)|_m \leqslant \frac{\alpha}{2}\max_{x\in\mathcal{Q}^n_\alpha}\|C(x)\|\|\mathcal{H}_\varphi(x)\|\sum_{h=1}^{n}\left(|A_i v_h|_n\,|A_i e_h| + 2\,|A_i y|_n\right) \leqslant$$

$$\leqslant 2n\alpha \max_i |a_i| \max_{x\in\mathcal{Q}^n_\alpha}\|C(x)\|\|\mathcal{H}_\varphi(x)\| = \mathcal{C} = \mathcal{C}(\varphi,\alpha) < \infty\,,$$

and (1) of Step 1 follows. □

Let now $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\,|\mathcal{Q}^n_\alpha|_n^{-1}\}$ and $\tilde{\varepsilon} > 0$ be such that $2\,\tilde{\varepsilon} < \varepsilon$. Then by the uniform continuity of φ on $\mathcal{Q}^n_\alpha$ and since $\operatorname{diam}\left(\tau_t \circ \delta_{\alpha/k}(\mathcal{Q}^n)\right) = 2\,\frac{\alpha}{k}\sqrt{n}$, we may choose $\tilde{k} = k(\tilde{\varepsilon}, \alpha, \varphi) \in \mathbb{N}$ such that, if $k > \tilde{k}$, then

$$\max_{t\in\Delta} \operatorname{osc}\left(\varphi;\, \tau_t \circ \delta_{\alpha/k}(\mathcal{Q}^n)\right) < \tilde{\varepsilon}\,.$$

By taking $k \geqslant \max\{\tilde{k}, \varepsilon_3^{-1}\mathcal{C}\,|\mathcal{Q}^n_\alpha|_n\}$, $k \in \mathbb{N}$, and setting

$$\psi(x) := \widetilde{\psi}(x) + \tilde{\varepsilon}\,,$$

we find that ψ satisfies (1), (2) and (3). Indeed if $x \in \xi_{i,k,t}(\mathcal{T}^n_a)$, remembering that $\sum_{h=0}^{n} \lambda_h = 1$, we have

$$|\varphi(x) - \widetilde{\psi}(x)| \leqslant \max_{t\in\Delta} \operatorname{osc}(\varphi;\, \tau_t \circ \delta_{\alpha/k}(\mathcal{Q}^n)) < \tilde{\varepsilon}\,,$$

and thus $\psi - \varphi = \tilde{\varepsilon} + \widetilde{\psi} - \varphi > 0$ and we get (1), whereas (2) follows by observing that

$$\|\psi - \varphi\|_{L^1(\mathcal{Q}^n_\alpha)} \leqslant \varepsilon\,|\mathcal{Q}^n_\alpha|_n < \varepsilon_2.$$

Finally, since $k \geqslant \varepsilon_3^{-1}\mathcal{C}\,|\mathcal{Q}^n_\alpha|_n$, we get (3) by integrating on $\mathcal{Q}^n_\alpha$ both sides of the inequality (1) of Step 1. □

REMARK 2.10. *Let $\Omega \subseteq \mathbb{R}^n$ be open. We stress that the previous proof of Lemma* 2.8 *show how, with the same choice of ψ, one also has*

$$\int_{\Omega\cap\mathcal{Q}^n_\alpha} |X\psi|_m\,dx \leqslant \int_{\Omega\cap\mathcal{Q}^n_\alpha} |X\varphi|_m\,dx + \varepsilon_3.$$

2.2. *Proof of Theorem* 2.2.

This proof will be divided in some steps. We begin with the following

STEP 1. *There exists a sequence $\{\varphi_i\}_{i\in\mathbb{N}} \subset C^\infty(\Omega) \cap BV_X(\Omega)$ such that*

(1.1) *for all $\varepsilon > 0$ there exists $i_\varepsilon \in \mathbb{N}$ such that if $i \geqslant i_\varepsilon$ then*

$$\|\varphi_i - \mathbf{1}_E\|_{L^1(\Omega)} \leqslant \varepsilon\,;$$

(1.2) *for all* $\eta \in \left]0, \frac{1}{4}\right[$ *there exists* $i_\eta \in \mathbb{N}$ *such that if* $i \geq i_\eta$ *then*

$$\int_\Omega |X\varphi_i(x)|\,dx \leq |\partial E|_X(\Omega) + \frac{1}{2}\eta\ ;$$

(1.3) *for all* $i \in \mathbb{N}$ *there exists a constant* $\mathcal{R}_i > 0$ *such that* $\varphi_i(x) = 0$ *if* $|x|_n > \mathcal{R}_i$.

Proof of Step 1. Since $\mathbf{1}_E \in BV_X(\Omega)$ from Theorem 1.7 we immediately get that there exists a sequence $\{\tilde{\varphi}_i\}_{i\in\mathbb{N}} \subset C^\infty(\Omega) \cap BV_X(\Omega)$ satisfying (1.1) and (1.2) with $\frac{1}{2}\eta$ replaced by $\frac{1}{4}\eta$. Actually to prove (1.3) it is enough to show that there exists a sequence $\{\psi_h\}_{h\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ such that for $i \in \mathbb{N}$ we have

$$\lim_{h\to\infty} \|\tilde{\varphi}_i - \tilde{\varphi}_i\psi_h\|_{L^1(\Omega)} = 0\ , \tag{2}$$

and that there exists $h_{i,\eta} \in \mathbb{N}$ such that if $h \geq h_{i,\eta}$ then

$$\int_\Omega |X(\tilde{\varphi}_i\psi_h)|_m\,dx \leq \int_\Omega |X\tilde{\varphi}_i|_m\,dx + \frac{1}{4}\eta\ . \tag{3}$$

Indeed if (3) holds, then to prove (1.2) it will be enough to put $\varphi_i = \tilde{\varphi}_i\psi_{h_{i,\eta}}$. Let now ϑ_h be a smooth function, $\vartheta_h\colon [0, \infty[\to [0, 1]$, such that $\vartheta_h = 1$ on $[0, h]$, $\vartheta_h = 0$ on $[h+1, \infty[$, $|\vartheta_h'| \leq 2$ and put $\psi_h(x) = \vartheta_h(|x|_n)$. Clearly (2) follows by dominate convergence theorem. On the other hand

$$\int_\Omega |X(\tilde{\varphi}_i\psi_h)|_m\,dx \leq \int_\Omega \psi_h\,|X\tilde{\varphi}_i|_m\,dx + \int_\Omega |\tilde{\varphi}_i\|X\psi_h|_m\,dx \leq$$

$$\leq \int_\Omega |X\tilde{\varphi}_i|_m\,dx + \int_\Omega \mathbf{1}_{\{h<|x|_n\leq h+1\}}\,|\tilde{\varphi}_i|\,|X\psi_h|_m\,dx\ ,$$

and (3) follows since $X\psi_h \to 0$ as $h \to \infty$, for any $x \in \mathbb{R}^n$ (remember that $\psi_h = 1$ on $B_h(0)$, $\tilde{\varphi}_h \in L^1(\Omega)$ and the supremum of $|X\psi_h(x)|_m$ is finite since the vector fields have bounded coefficients). □

Remark 2.11. *The boundedness assumptions on the family of vector fields are not needed if the Carnot-Carathéodory distance associated with X is finite. Indeed in this case we can replace* $\vartheta_h(|x|_n)$ *by a smooth approximation of* $\psi_h(d_X(x, 0))$ *of the form* $J_\varepsilon * \psi_h(d_X(x, 0))$, *where* d_X *denotes the Carnot-Carathéodory distance.*

By setting

$$\Phi_{i,\eta} = \left\{x \in \Omega : \varphi_i(x) \geq \eta,\ \eta \in \left]0, \frac{1}{4}\right[,\ i \in \mathbb{N}\right\},$$

and applying (1.3) of Step 1 it follows that $\Phi_{i,\eta}$ is bounded for any $\eta \in \left]0, \frac{1}{4}\right[$ and $i \in \mathbb{N}$. Since $\mathbf{1}_E \in L^1(\Omega)$ and by the boundedness of $\Phi_{i,\eta}$ we get that if $\varepsilon > 0$,

$\eta \in \left]0, \frac{1}{4}\right[$ and $i \in \mathbb{N}$ there exists a positive constant $\alpha = \alpha(\varepsilon, \eta, i)$ such that

(4) $$\int_{(\mathbb{R}^n \setminus \mathcal{Q}^n_\alpha) \cap \Omega} \mathbf{1}_E \, dx < \varepsilon \quad \text{and} \quad \varphi_i|_{\mathbb{R}^n \setminus \mathcal{Q}^n_\alpha} < \eta\,.$$

STEP 2. *Let $\varepsilon > 0$, $\eta \in \left]0, \frac{1}{4}\right[$ and $i \in \mathbb{N}$. Let $\alpha = \alpha(\varepsilon, \eta, i)$ be such that* (4) *holds. Then there exists a piecewise linear function $\psi : \mathcal{Q}^n_\alpha \to \mathbb{R}$, $\psi \geqslant 0$, $\psi = \psi(\varepsilon, \eta, i, \alpha)$, satisfying*

(2.1) $\Psi^{n+1} = \left\{(x_1, \dots, x_{n+1}) \in \mathcal{Q}^n_\alpha \times \mathbb{R} : 0 \leqslant x_{n+1} \leqslant \psi(x_1, \dots, x_n)\right\} \in \mathcal{P}^{n+1}$;

(2.2) $0 < \psi(x) - \varphi_i(x) < \frac{1}{2}\eta$ *for all* $x \in \mathcal{Q}^n_\alpha$;

(2.3) $\|\psi - \varphi_i\|_{\mathcal{L}^1(\mathcal{Q}^n_\alpha)} < \varepsilon$;

(2.4) $\int_{\mathcal{Q}^n_\alpha \cap \Omega} |X\psi|_m \, dx \leqslant \int_{\mathcal{Q}^n_\alpha \cap \Omega} |X\varphi_i|_m \, dx + \frac{1}{2}\eta.$

Observing that $\varphi_i|_{\mathcal{Q}^n_\alpha} \in C^\infty(\mathcal{Q}^n_\alpha)$ and $\varphi_i \geqslant 0$, this is just the content of Lemma 2.8 combined with Remark 2.9 and Remark 2.10. Moreover from (1.2) of Step 1 and (2.4) of Step 4 the following statement holds:

STEP 3. *Let $\eta \in \left]0, \frac{1}{4}\right[$ and i_η be as in* (1.2) *of Step* 1 *and $i \in \mathbb{N}$, $i \geqslant i_\eta$. Let α, ψ be as in Step* 2*. Then we have*

$$\int_{\mathcal{Q}^n_\alpha \cap \Omega} |X\psi|_m \, dx \leqslant |\partial E|_X(\Omega) + \eta\,.$$

Furthermore from (4) and (2.2) of Step 2 we get that

(5) $$\psi(x) < 2\eta \quad \text{for all } x \in \partial \mathcal{Q}^n_\alpha.$$

Let $\{H_t\}_{t \in \mathbb{R}}$ be the family of affine hyperplanes of $\mathbb{R}^{n+1}$ that are parallel to the subspace spanned by the first n vectors $e_1, \dots, e_n$ of the standard basis of $\mathbb{R}^{n+1}$, *i.e.* for $t \in \mathbb{R}$,

$$H_t = \left\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = t\right\}.$$

Let us denote $\Psi_t = \left\{x \in \operatorname{int}(\mathcal{Q}^n_\alpha) : \psi(x) > t\right\}$. If we choose $t \in]2\eta, 1[$, then by (5) we get $\overline{\Psi_t} \subset \operatorname{int} \mathcal{Q}^n_\alpha$. Indeed whenever $x \in \overline{\Psi_t}$, we have

$$\psi(x) \geqslant t > 2\eta > \max_{\partial \mathcal{Q}^n_\alpha} \psi\,.$$

If t it is not a local maximum for ψ, then

(6) $$\Sigma_t := \pi_{n+1}(H_t \cap \Psi^{n+1}) = \overline{\Psi_t},$$

where π_{n+1} denotes the orthogonal projection on the hyperplane H_0. Notice that the number of local maxima of ψ is finite. Denote by I_ψ this set.

STEP 4. *For any $t \in]2\eta, 1[\setminus I_\psi$ we have*

$$|\partial \Sigma_t|_X(\Omega) = |\partial \Sigma_t|_X(\operatorname{int}(\mathcal{Q}^n_\alpha) \cap \Omega) = |\partial \Psi_t|_X(\operatorname{int}(\mathcal{Q}^n_\alpha) \cap \Omega).$$

The first equality above follows by observing that $\Sigma_t \subset \operatorname{int}(\mathcal{Q}^n_\alpha)$ whereas the second one follows from the definition of Σ_t by observing that $|\Psi_t \triangle \Sigma_t|_n = 0$ (see [19] for the classical case and [21] for the generalized statement).

Step 5. *Let $\varepsilon > 0$, $\eta \in \left]0, \frac{1}{4}\right[$ and i_η be as in Step 1. Let $i \in \mathbb{N}$, $i \geq i_\eta$. Finally let $\alpha > 0$ be as in Step 2. Then there exists $\tilde{t} \in]2\eta, 1-\eta[$, $\tilde{t} = \tilde{t}(\varepsilon, \eta, i)$, such that*

$$(1-3\eta)\,|\partial\Sigma_{\tilde{t}}|_X(\Omega) \leq |\partial E|_X(\Omega) + \eta\,. \tag{7}$$

Proof of Step 5. By applying Theorem 1.8 and the above Step 4 we get

$$\int_{\operatorname{int}(\mathcal{Q}^n_\alpha)\cap\Omega} |X\psi|_m\,dx = \int_{-\infty}^{+\infty} |\partial\Psi_t|_X(\operatorname{int}(\mathcal{Q}^n_\alpha)\cap\Omega)\,dt \geq$$

$$\geq \int_{2\eta}^{1-\eta} |\partial\Psi_t|_X(\operatorname{int}(\mathcal{Q}^n_\alpha)\cap\Omega)\,dt = \int_{2\eta}^{1-\eta} |\partial\Sigma_t|_X(\Omega)\,dt\,.$$

By virtue of Step 3, being $i \geq i_\eta$, we obtain

$$\int_{2\eta}^{1-\eta} |\partial\Sigma_t|_X(\Omega)\,dt < |\partial E|_X(\Omega) + \eta\,.$$

Then, let us show that there exists $\tilde{t} \in]2\eta, 1-\eta[$ such that (7) holds. To this end it is enough to show that

$$\int_{2\eta}^{1-\eta} |\partial\Sigma_t|_X(\Omega)\,dt \geq |\partial E|_X(\Omega) + \eta\,.$$

Suppose by contradiction there is an $\eta \in \left]0, \frac{1}{4}\right[$ such that, whenever $s \in]2\eta, 1-\eta[$,

$$\int_{2\eta}^{1-\eta} |\partial\Sigma_t|_X(\Omega)\,dt < (1-3\eta)\,|\partial\Sigma_s|_X(\Omega)\,.$$

By integrating both sides of this equation for $s \in]2\eta, 1-\eta[$ we get a contradiction. □

Step 6. *Let $\varepsilon > 0$, $\eta \in \left]0, \frac{1}{4}\right[$ and $i \geq \max\{i_\varepsilon, i_\eta\}$, where i_ε and i_η are determined as in Step 1. Let $\tilde{t} \in]2\eta, 1-\eta[$, $\tilde{t} = \tilde{t}(\varepsilon, \eta, i)$, be as in Step 5. Then*

$$|(E\cap\Omega) \triangle \Sigma_{\tilde{t}}|_n < \varepsilon\left(1 + \frac{2}{\eta}\right). \tag{8}$$

Proof of Step 6. By definition of Σ_t it follows that for any $t \in]2\eta, 1-\eta[$

$$\begin{cases} \psi(x) \geq t, & x \in \Sigma_t, \\ \psi(x) < t, & x \in \mathcal{Q}^n_\alpha \setminus \Sigma_t. \end{cases}$$

Hence

$$\begin{cases} \psi(x) - \mathbf{1}_E(x) \geq \tilde{t} > 2\eta, & x \in \Sigma_{\tilde{t}} \setminus (E \cap \Omega \cap \Sigma_{\tilde{t}}), \\ \mathbf{1}_E(x) - \psi(x) \geq 1 - \tilde{t} > \eta, & x \in (E \cap \Omega \cap \mathcal{Q}^n_{,\alpha}) \setminus (E \cap \Omega \cap \Sigma_{\tilde{t}}). \end{cases}$$

From (1.1) of Step 1 and (2.3) of Step 2 we have $\|\psi - \mathbf{1}_E\|_{L^1(\Omega)} \leq 2\varepsilon$. Therefore

$$\eta |((E \cap \Omega \cap \mathcal{Q}^n_{,\alpha}) \cup \Sigma_{\tilde{t}}) \setminus (E \cap \Omega \cap \Sigma_{\tilde{t}})|_n < \|\psi - \mathbf{1}_E\|_{L^1(\Omega)} \leq 2\varepsilon .$$

Finally, by the first statement of (4) the claim follows. □

At this point we may achieve the proof as follows. Let $\{\varepsilon_i\}_{i \in \mathbb{N}}$ be a vanishing sequence and let $\{\eta_i\}_{i \in \mathbb{N}}$ be the sequence obtained from the previous one by setting $\eta_i = \sqrt{\varepsilon_i}$. From what was proven above, for any $i \in \mathbb{N}$ there exists a sequence $\{t_i\}_{i \in \mathbb{N}} \subset]2\eta_i, 1 - \eta_i[$ which defines, according to (6), a sequence of polyhedra $\{\Sigma_i\}_{i \in \mathbb{N}} \subset \mathcal{P}^n$, where we have set $\Sigma_i = \Sigma_{t_i}$, satisfying (1) and (2) of Theorem 2.2. In fact for (1) it is enough to observe that (8) implies

$$\lim_{i \to \infty} |(E \cap \Omega) \triangle \Sigma_{t_i}|_n = \lim_{i \to \infty} \|\mathbf{1}_{\Sigma_i} - \mathbf{1}_E\|_{L^1(\Omega)} = 0 .$$

Finally from Step 5 it follows that

$$\limsup_{i \to \infty} |\partial \Sigma_i|_X(\Omega) \leq |\partial E|_X(\Omega),$$

which together with Proposition 1.6 proves (2). □

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