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Classical, viscosity and average solutions for PDE's with nonnegative characteristic form

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Analisi matematica. — *Classical, viscosity and average solutions for PDE's with nonnegative characteristic form.* Nota (*) di CRISTIAN E. GUTIÉRREZ e ERMANNO LANCONELLI, presentata dal Socio A. Tesei.

ABSTRACT. — We compare several definitions of weak solutions to second order partial differential equations with nonnegative characteristic form.

KEY WORDS: Weak solutions; Viscosity solutions; Second order PDE's with nonnegative characteristic form.

RIASSUNTO. — *Soluzioni classiche, viscose ed in media per equazioni differenziali alle derivate parziali con forma caratteristica non negativa.* In questa *Nota* confrontiamo alcune nozioni di soluzione per equazioni alle derivate parziali del secondo ordine con forma caratteristica semidefinita positiva.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be open and $u \colon \Omega \to \mathbb{R}$ a continuous function. For $x \in \Omega$ and $r > 0$ denote

$$\mathcal{M}_r u(x) = \fint_{B_r(x)} u(y)\, dy,$$

where $B_r(x)$ is the Euclidean ball centered at x with radius r. Then, by the Gauss-Koebe Theorem, u is harmonic in Ω if and only if $u(x) = \mathcal{M}_r u(x)$ for all x and r such that $B_r(x) \Subset \Omega$. This classical and very well known result has been generalized in countless directions. Here we are interested in the following asymptotic version due to Brelot [2]: if $u : \Omega \to \mathbb{R}$ is continuous, then u is harmonic in Ω if and only if

$$\lim_{r \to 0} \frac{\mathcal{M}_r u(x) - u(x)}{r^2} = 0, \qquad \forall x \in \Omega. \tag{1.1}$$

The only if part of this theorem is a trivial consequences of Gauss Theorem. In the case where u is C^2 the same statement also follows from the classical Pizzetti's formula for C^2 functions:

$$\frac{1}{2(N+2)} \Delta u(x) = \lim_{r \to 0} \frac{\mathcal{M}_r u(x) - u(x)}{r^2}, \qquad \forall x \in \Omega, \tag{1.2}$$

where Δ denotes the Laplacian in $\mathbb{R}^N$. An enlightening step towards the proof of Brelot's Theorem is the following remark: a continuous function u satisfying (1.1) solves the equation $\Delta u = 0$ in Ω in the viscosity sense of Crandall, Ishii and Lions, see Theorem 3.3 below. From the general regularity theory for viscosity solutions, it follows that u is smooth and therefore harmonic. We notice that Ramaswamy [14] was able to prove that viscosity solutions to Laplace's equation are harmonic by using basic tools from potential theory.

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The main purpose of this paper is to compare several definitions of weak solutions to second order linear pde's with nonnegative characteristic form. Given that the regularity theory of viscosity solutions for these operators is not available, we follow an approach closer to the one considered in [14].

The paper is organized as follows. In Section 2, we prove the equivalence of classical and viscosity solutions to (2.1) assuming Conditions 2.1 and 2.2, extending the results of [14]. In Section 3, we introduce our notion of asymptotic-average solution assuming a general asymptotic representation formula for C^2 functions u involving a sort of integral averages of u and Lu. We then prove that these solutions are viscosity solutions. These results combined with the ones in Section 2 give that the notions of classical, viscosity and average solutions are equivalent if Conditions 2.1 and 2.2 hold.

Finally, in Section 4 we show several noteworthy examples of operators to which our results apply. In particular, 4.2 extends to a class of hypoelliptic operators the Theorems of Gauss-Koebe, Brelot and Ramaswamy.

Closing this introduction we would like to mention the recent papers [7] and [15] containing results related to ours.

2. $\boldsymbol{H}$-SOLUTIONS AND VISCOSITY SOLUTIONS

Let us consider the linear second order operator:

$$L := \sum_{i,j=1}^{N} a_{ij}(x)\, \partial_{x_i x_j} + \sum_{i=1}^{N} b_i(x)\, \partial_{x_i}, \qquad x \in X, \tag{2.1}$$

where X is an open subset of $\mathbb{R}^N$. Throughout the paper, we assume without further comments that the matrix $A(x) = (a_{ij}(x))_{1 \leq i,j \leq N}$ is symmetric and non-negative definite at any $x \in X$, and the functions a_{ij} and b_i are continuous. In this section, we suppose that L satisfies the following condition:

CONDITION 2.1. *For every bounded open set V such that $\overline{V} \subset X$ there exists a function $h \in C^2(V)$ such that $Lh < 0$ and $h \geq 0$ in V.*

It is well known that this condition implies the following maximum principle: if $u \in C^2(V)$ and

$$\begin{aligned} Lu \geq 0 \quad & \text{in } V \\ \limsup_{x \to y} u(x) \leq 0, \quad & \forall y \in \partial V, \end{aligned}$$

then $u \leq 0$ in V.

Given $\Omega \subset X$ open, we define

$$\boldsymbol{H}(\Omega) = \{u \in C^2(\Omega) : Lu = 0\}.$$

The bounded open set V such that $\overline{V} \subset X$ will be called $\boldsymbol{H}$-regular if for any $\phi \in C(\partial V)$ there exists a unique function $H_\phi^V \in \boldsymbol{H}(V) \cap C(\overline{V})$ such that $H_\phi^V = \phi$ on ∂V.

In this section, we also assume that the following property holds:

CONDITION 2.2. *There exists a class $\boldsymbol{B}$ of $\boldsymbol{H}$-regular open sets that form a basis of the Euclidean topology of X.*

DEFINITION 2.3 ($\boldsymbol{H}$-subsolutions). *Let $\Omega \subset X$ be open. The function $u : \Omega \to \mathbb{R}$ is an $\boldsymbol{H}$-subsolution in Ω if u is upper semicontinuous and if for each $V \in \boldsymbol{B}$, $\overline{V} \subset \Omega$, and $\phi \in C(\partial V)$ with $\phi \geqslant u$ on ∂V, we have*

$$H_\phi^V \geqslant u \quad \textit{in } V.$$

A function u is a $\boldsymbol{H}$-supersolution if $-u$ is a $\boldsymbol{H}$-subsolution.

We shall denote by $\boldsymbol{H}_*(\Omega)$ ($\boldsymbol{H}^*(\Omega)$) the set of the $\boldsymbol{H}$-subsolutions (supersolutions) on Ω. By using the maximum principle stated before, one easily recognize that if $u \in C^2(\Omega)$ and $Lu \geqslant 0$ in Ω, then $u \in \boldsymbol{H}_*(\Omega)$. We also have $\boldsymbol{H}(\Omega) = \boldsymbol{H}_*(\Omega) \cap \boldsymbol{H}^*(\Omega)$, for any open set $\Omega \subset X$. In what follows we call every member of $\boldsymbol{H}(\Omega)$ a $\boldsymbol{H}$-solution in Ω.

DEFINITION 2.4 (Viscosity solutions). *The upper semicontinuous function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution to $Lu = 0$ in Ω if whenever $\phi \in C^2(\Omega)$ and $x_0 \in \Omega$ are such that $(u - \phi)(x) \leqslant (u - \phi)(x_0)$ for all x in a neighborhood of x_0, then we must have*

$$L\phi(x_0) \geqslant 0.$$

A function $u : \Omega \to \mathbb{R}$ is a viscosity supersolution if $-u$ is a viscosity subsolution. A real function u is a viscosity solution to $Lu = 0$ if it is both a viscosity subsolution and supersolution.

Note that in order to check that u is a viscosity subsolution (supersolution) it is enough to use test functions $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a strict local max (min) at x_0. Because if for example $u - \varphi$ has a maximum at x_0 and we set $\overline{\varphi}(x) = \varphi(x) + \varepsilon |x - x_0|^2$, then $u - \overline{\varphi}$ has a strict maximum at x_0. Since $L\overline{\varphi}(x_0) = L\varphi(x_0) + 2\varepsilon \operatorname{trace} A(x_0)$, letting $\varepsilon \to 0$ we get $L\varphi(x_0) \geqslant 0$.

PROPOSITION 2.5 (Maximum Principle for viscosity subsolutions). *Let V be a bounded open set such that $\overline{V} \subset X$. Let $u : V \to \mathbb{R}$ be a viscosity subsolution such that*

$$\limsup_{x \to y} u(x) \leqslant 0 \qquad \forall y \in \partial V. \tag{2.2}$$

Then $u \leqslant 0$ in V.

PROOF. Let $h \in C^2(V)$ be such that $Lh < 0$ and $h \geqslant 0$ in V. For any fixed $\varepsilon > 0$ define $v_\varepsilon := u - \varepsilon h$. Let $x_0 \in \overline{V}$ be such that

$$\sup_V v_\varepsilon = \sup_{V \cap B_r(x_0)} v_\varepsilon \qquad \forall r > 0. \tag{2.3}$$

Such a point must exist, since $\overline{V}$ is compact. Suppose $x_0 \in V$. Since u is upper semicontinuous, we get

$$\sup_V v_\varepsilon = \lim_{r \to 0}\left(\sup_{V \cap B_r(x_0)} v_\varepsilon\right) = \limsup_{x \to x_0} v_\varepsilon \leqslant v_\varepsilon(x_0)\,.$$

It follows that x_0 is a maximum point for v_ε. As a consequence, since u is a viscosity subsolution, $L(\varepsilon h)(x_0) \geqslant 0$. This contradicts the condition $Lh < 0$ in V. Then, the point x_0 must belong to ∂V. This, together with (2.2) and condition $h \leqslant 0$ in V, imply

$$\sup_V v_\varepsilon = \lim_{r \to 0}\left(\sup_{V \cap B_r(x_0)} v_\varepsilon\right) = \limsup_{x \to x_0} (u(x) - \varepsilon h(x)) \leqslant \limsup_{x \to x_0} u(x) \leqslant 0\,.$$

Thus, $v_\varepsilon = u - \varepsilon h \leqslant 0$ in V for every $\varepsilon > 0$. Letting ε go to zero, we get $u \leqslant 0$ in V. □

We compare the notion of viscosity solution with that of $\boldsymbol{H}$-solution.

THEOREM 2.6. *Let Ω be an open subset of X and $u : \Omega \to \mathbb{R}$ an upper semicontinuous function. The function u is an $\boldsymbol{H}$-subsolution if and only if u is a viscosity subsolution to $Lu = 0$.*

PROOF. We first prove the «only if» part. Suppose by contradiction that u is an $\boldsymbol{H}$-subsolution which is not a viscosity subsolution. Then there exists $h \in C^2(\Omega)$ and $x_0 \in \Omega$ such that $(u - h)(x) < (u - h)(x_0)$ for all $x \in B_\delta(x_0) \setminus \{x_0\}$, (strict maximum) with

$$Lh(x_0) < 0\,.$$

By continuity

$$Lh(x) < 0 \qquad \text{for all } x \in B_\mu(x_0).$$

We have that h is an $\boldsymbol{H}$-supersolution to $Lh = 0$ in $B_R(x_0)$ with $R = \min\{\delta, \mu\}$. Recall that the maximum of $u - h$ at x_0 is strict. We may assume that $(u - h)(x_0) = 0$. By Condition 2.2, let $V \in \boldsymbol{B}$, $\overline{V} \subset B_R(x_0)$ with $x_0 \in V$ and

$$M = \max_{\partial V} (u(x) - h(x)) < 0\,,$$

and set $h^*(x) = h(x) + M$. We have $h^*(x) \geqslant u(x)$ on ∂V, and h^* is an $\boldsymbol{H}$-supersolution in $B_R(x_0)$.

We claim that $h^* \geqslant u$ in V. Since u is an $\boldsymbol{H}$-subsolution, it follows taking $\phi = h^*$ on ∂V and letting H_R^* being the $\boldsymbol{H}$-solution with $H_R^* = h^*$ in ∂V, that $H_R^* \geqslant u$ in V. On the other hand, since h^* is an $\boldsymbol{H}$-supersolution in $B_R(x_0)$, we get that $H_R^* \leqslant h^*$ in V. Therefore $u \leqslant H_R^* \leqslant h^*$ in V, and the claim is proved. Hence $h^*(x_0) = h(x_0) + M \geqslant u(x_0)$, and since $u(x_0) = h(x_0)$, we get $M \geqslant 0$, a contradiction.

We are left with the «if» part. Suppose u is a viscosity subsolution and take an $\boldsymbol{H}$-

regular open set $V \subset \overline{V} \subset \Omega$. Let $\phi \in C(\partial V)$ be such that $\phi \geqslant u|_{\partial V}$. Since H_ϕ^V is a C^2 solution to $Lv = 0$, it follows from the definition of viscosity subsolution that $u - H_\phi^V$ is a viscosity subsolution in V. Moreover,

$$\limsup_{V \ni x \to y} (u(x) - H_\phi^V(x)) \leqslant u(y) - \phi(y) \leqslant 0 \qquad \forall y \in \partial V.$$

Then, by Proposition 2.5 we get $u \leqslant H_\phi^V$ in V and u is an $\boldsymbol{H}$-subsolution. This completes the proof. □

3. Average solutions

In this section, we assume that there exist linear operators $\mathcal{M}_r$, $\mathcal{N}_r$ such that $\mathcal{M}_r$ is defined for all semicontinuous functions in X, $\mathcal{N}_r$ is defined for all continuous functions, and both have values in the class of functions on X. In addition, we assume that if $u \geqslant 0$ in a neighborhood of x, then $\mathcal{M}_r u$, $\mathcal{N}_r u \geqslant 0$, for all r sufficiently small, and the following representation formula holds:

$$u(x) = \mathcal{M}_r u(x) - \mathcal{N}_r(Lu)(x) + o(Q_r(x)), \tag{3.1}$$

as $r \to 0$, for all $x \in \Omega \subset X$, for all $u \in C^2(\Omega)$, where $Q_r(x)$ is nonnegative and satisfies

$$\lim_{r \to 0} \frac{\mathcal{N}_r(w(x))}{Q_r(x)} = w(x), \tag{3.2}$$

for all $w \in C(\Omega)$ and for all $x \in \Omega$.

Definition 3.1 (Asymptotically L-harmonic). *Let $u \in C(\Omega)$, $\Omega \subset \mathbb{R}^n$ open. u is asymptotically L-harmonic in Ω*, i.e., AL*-harmonic, if*

$$\lim_{r \to 0} \frac{\mathcal{M}_r u(x) - u(x)}{Q_r(x)} = 0,$$

for each $x \in \Omega$.

Obviously, if $u \in C^2(\Omega)$ and $Lu = 0$ in Ω, then u is AL-harmonic in Ω. We will show that AL-harmonic solutions are viscosity solutions to $Lu = 0$. To this end we introduce the notions of sub and super AL-harmonicity.

Definition 3.2. *An upper semicontinuous function $u : \Omega \to \mathbb{R}$ is AL-subharmonic in Ω if*

$$\liminf_{r \to 0} \frac{\mathcal{M}_r u(x) - u(x)}{Q_r(x)} \geqslant 0,$$

for each $x \in \Omega$. The function u is AL-superharmonic in Ω if $-u$ is AL-subharmonic in Ω.

From the previous definitions, we straightforwardly obtain the following properties:

1. If u is sub and super AL-harmonic, then u is AL-harmonic in Ω.
2. If $u \in C^2(\Omega)$ and $Lu \geqslant 0$ in Ω, then u is AL-subharmonic in Ω.

THEOREM 3.3. *If u is AL-subharmonic in Ω, then u is a viscosity subsolution to $Lu = 0$ in Ω.*

PROOF. Let $\phi \in C^2(\Omega)$ and $x_0 \in \Omega$ be such that $u - \phi \leqslant (u - \phi)(x_0)$ in a neighborhood of x_0. From (3.1) we have that $\mathcal{M}_r 1 = 1 + o(Q_r(x))$ and consequently

$$\frac{\mathcal{M}_r(u-\phi)(x_0) - (u-\phi)(x_0)}{Q_r(x_0)} = \frac{\mathcal{M}_r((u-\phi) - (u-\phi)(x_0))(x_0)}{Q_r(x_0)} + o(1) \leqslant o(1)$$

for every $r > 0$ sufficiently small. Since u is AL-subharmonic then

$$0 \leqslant \liminf_{r\to 0} \frac{\mathcal{M}_r u(x_0) - u(x_0)}{Q_r(x_0)} \leqslant \lim_{r\to 0} \frac{\mathcal{M}_r \phi(x_0) - \phi(x_0)}{Q_r(x_0)} = L\phi(x_0)$$

and the theorem is proved. □

From this theorem we immediately get the following corollary.

COROLLARY 3.4. *Suppose L satisfies Conditions* 2.1 *and* 2.2. *If u is AL-subharmonic in Ω, then u is an **H**-subsolution in Ω. In particular, if u is AL-harmonic in Ω, then $u \in C^2(\Omega)$ and $Lu = 0$.*

PROOF. If u is AL-subharmonic, then by Theorem 3.3, u is a viscosity subsolution to $Lu = 0$ in Ω, so from Theorem 2.6 u is an $\boldsymbol{H}$-subsolution in Ω. □

We close this section with the following.

REMARK 3.5. If u and v are sub and super AL-harmonic functions respectively in a bounded open set Ω, and

$$\limsup_{x\to y} u(x) \leqslant \liminf_{x\to y} v(x), \qquad \forall y \in \partial\Omega,$$

then $u \leqslant v$ in Ω. Indeed, it is quite easy to show that the function $w = u - v$ is AL-subharmonic in Ω so that, by Theorem 3.3, it is a viscosity subsolution to $Lu = 0$ in Ω. Moreover, w satisfies the boundary condition $\limsup_{x\to y} w(x) \leqslant 0$ for every $y \in \Omega$. Then, by the Maximum Principle in Proposition 2.5, we get $w \leqslant 0$ in Ω, and the assertion is proved.

In order to emphasize the result just proved, we would like to recall that an important open problem in the general setting of second order PDE's with nonnegative characteristic form is to know if a comparison principle for viscosity solutions holds.

4. Examples

4.1. Example 1. Suppose L has the from in (2.1) with $b_1 = \cdots = b_N = 0$. By [4, Theorem, p. 514] we have

$$u(x) = \mathcal{M}_r u(x) - C_N r^2 Lu(x) + o(r^2)$$

where

$$\mathcal{M}_r u(x) = \fint_{|y|=r} u(x + B(x)\, y)\, d\sigma(y),$$

$B(x)$ is the unique positive square root of $A(x) = (a_{ij}(x))$, and C_N is a positive constant depending only on N. Then (3.1) holds with

$$\mathcal{N}_r w(x) = C_N r^2 w(x), \qquad \text{and} \qquad Q_r(x) = r^2.$$

This type of representation formulas were used by Pucci and Talenti [13] in the elliptic, and by Pagani [11] in the parabolic and the elliptic degenerate cases.

4.2. Example 2. Suppose L is in divergence form, $L = \sum_{i,j=1}^{N} \partial_i(a_{ij}\partial_j) + \sum_{i=1}^{N} b_i \partial_i$, where the coefficients are smooth and $\operatorname{div}(b_1, \dots, b_N) = 0$. We assume $X = \mathbb{R}^N$ and the operator L has a global fundamental solution $\Gamma(x, y)$ which is smooth outside of the diagonal and is such that

1. $\Gamma(\cdot, y), \Gamma(x, \cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ for all $x, y \in \mathbb{R}^n$;
2. $\Gamma(x, y) \geq 0$ and $\Gamma(x, y) \to 0$ as $|y| \to \infty$ for each $x \in \mathbb{R}^N$; and
3. $\limsup_{y \to x} \Gamma(x, y) = +\infty$ for each $y \in \mathbb{R}^N$.

Given $x \in \mathbb{R}^n$ and $r > 0$, we define the L-ball of center x and radius r by

$$\Omega_r(x) = \{y \in \mathbb{R}^N : \Gamma(x, y) > 1/r\}.$$

These L-balls have the following properties:

1. $\Omega_r(x) \subset \Omega_R(x)$ for $r \leq R$;
2. $\Omega_r(x) \neq \emptyset$ for each $r > 0$;
3. for each $\delta > 0$ there exists $\bar{r} = \bar{r}(\delta) > 0$ such that $\Omega_{\bar{r}}(x) \subset B_\delta(x)$;
4. by Sard's lemma, the set $\partial\Omega_r(x) = \{y : \Gamma(x, y) = 1/r\}$ is a smooth manifold of dimension $N - 1$ for almost every $r > 0$.
5. $\frac{1}{r}|\Omega_r(x)| \to 0$ as $r \to 0$, where $|\cdot|$ denotes Lebesgue measure.

We assume that the following Green's representation formula holds for every $r > 0$

$$u(x) = \int_{\partial\Omega_r(x)} u(y) \langle A(y)\, D_y \Gamma(x, y), D_y \Gamma(x, y)\rangle \frac{1}{|D_y \Gamma(x, y)|}\, d\sigma(y) - \int_{\Omega_r(x)} \left(\Gamma(x, y) - \frac{1}{r}\right) Lu(y)\, dy, \tag{4.1}$$

and for all $u \in C^2$, where $A(y) = (a_{ij}(y))$. Then (3.1) holds with the obvious meaning for $\mathcal{M}_r$ and $\mathcal{N}_r$, and with $o(Q_r(x))$ replaced by 0. Moreover (3.2) holds with

$$Q_r(x) = \int_{\Omega_r(x)} \left(\Gamma(x, y) - \frac{1}{r}\right) dy. \tag{4.2}$$

An easy computation shows that

$$Q_r(x) = \int_0^r \frac{|\Omega_s(x)|\, s^2}{ds}. \tag{4.3}$$

If $L = \Delta$, then (4.1) is the classical Gauss-Poisson-Jensen formula. If $L = \Delta - \partial_t$ and $Lu = 0$, then (4.1) was proved by B. Pini [12] and W. Fulks [5], (see also [16]). When L is a sum of squares of vector fields satisfying the hypoellipticity Hörmander's condition, formula (4.1) appears in [3]. For Kolmogorov-Fokker-Planck operators, (4.1) was proved in [8, 6, 9]. Finally, for a class of hypoelliptic parabolic operators (4.1) was proved in [10].

From (4.1) we can easily obtain solid representation formulas. Multiplying (4.1) by r^α with $\alpha > -1$ and integrating in r, we then get for each $u \in C^2(\Omega)$ the following formula:

$$u(x) = \mathcal{M}_r^{(\alpha)} u(x) - \mathcal{N}_r^{(\alpha)}(Lu)(x), \tag{4.4}$$

where

$$\mathcal{M}_r^{(\alpha)} u(x) = \int_{\Omega_r(x)} u(y)\, K_r^{(\alpha)}(x, y)\, dy, \tag{4.5}$$

$$K_r^{(\alpha)}(x, y) = \frac{\alpha + 1}{r^{\alpha + 1}} \frac{\langle A(y)\, D_y \Gamma(x, y), D_y \Gamma(x, y)\rangle}{\Gamma(x, y)^{2 + \alpha}}, \tag{4.6}$$

and

$$\mathcal{N}_r^{(\alpha)} w(x) = \frac{\alpha + 1}{r^{\alpha + 1}} \int_0^r \varrho^\alpha \int_{\Omega_\varrho(x)} w\left(\Gamma(x, y) - \frac{1}{\varrho}\right) dy\, d\varrho. \tag{4.7}$$

If we set

$$Q_r^{(\alpha)}(x) = \mathcal{N}_r^{(\alpha)}(1)(x), \tag{4.8}$$

then

$$\lim_{r \to 0} \frac{\mathcal{N}_r^{(\alpha)} w(x)}{Q_r^{(\alpha)}(x)} = w(x), \tag{4.9}$$

for any $w \in C(\Omega)$ and each $x \in \Omega$. Then (3.1) holds with $\mathcal{M}_r$, $\mathcal{N}_r$ and Q_r given by (4.5), (4.7), (4.8) respectively. We notice that using (4.2) and (4.3) one gets

$$Q_r^{(\alpha)}(x) = \int_0^r \frac{|\Omega_s(x)|}{s^2}(1-(s/r)^{\alpha+1})\,ds. \tag{4.10}$$

We explicitly remark that Corollary 3.4 can be applied to the operators considered in the present example if we assume that for each compact set $K \subset \mathbb{R}^N$ there exists $1 \leqslant j \leqslant$ $\leqslant N$ such that $\inf_K a_{jj} > 0$. In fact, this assumption trivially implies Condition 2.1, while Condition 2.2 follows as in [1].

4.3. Example 3. In $\mathbb{R}^{N+1}$ we consider the differential operator

$$Lu = \sum_{i,j=1}^{p} a_{ij}(z)\,\partial_{x_i x_j} u + \langle x, BDu\rangle - \partial_t, \qquad D = (D_1, \ldots, D_N), \tag{4.11}$$

where $z = (x, t) = (x_1, \ldots, x_N, t)$ is a point in $\mathbb{R}^{N+1}$ and $1 \leqslant p \leqslant N$. We assume the coefficients $a_{ij} = a_{ji}$ are continuous in $\mathbb{R}^{N+1}$ and such that, for a suitable constant $\mu > 0$,

$$\mu^{-1}\sum_{j=1}^{p} \xi_j^2 \leqslant \sum_{i,j=1}^{p} a_{ij}(z)\,\xi_i \xi_j \leqslant \mu \sum_{j=1}^{p} \xi_j^2,$$

for any $(\xi_1, \ldots, \xi_p) \in \mathbb{R}^p$ and any $z \in \mathbb{R}^{N+1}$. We also assume that B is an $N \times N$ constant matrix satisfying the condition

$$\operatorname{rank} \operatorname{Lie}\{\partial_{x_1}, \ldots, \partial_{x_p}, Y\}(z) = N+1, \qquad \forall z \in \mathbb{R}^{n+1}, \tag{4.12}$$

where $Y = \langle x, BD\rangle - \partial_t$. We stress that these conditions imply the hypoellipticity of the frozen operators

$$L_{z_0} = \sum_{i,j=1}^{p} a_{ij}(z_0)\,\partial_{x_i x_j} + Y, \qquad z_0 \in \mathbb{R}^{N+1} \text{ fixed}.$$

The Kolmogorov-Fokker-Plank operators in $\mathbb{R}^{2n+1}$

$$L = \sum_{i,j=1}^{n} a_{ij}(z)\,\partial_{x_i x_j} + \sum_{i=1}^{n}(x_i\,\partial_{x_{n+i}} + x_{n+i}\,\partial_{x_i}) - \partial_t,$$

and

$$L = \sum_{i,j=1}^{n} a_{ij}(z)\,\partial_{x_i x_j} + \sum_{i=1}^{n} x_i\,\partial_{x_{n+i}} - \partial_t,$$

satisfy all the previous conditions. They correspond with the case $N = 2n$, $p = n$ and B given by

$$\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix}, \qquad I_n = n \times n \text{ identity matrix},$$

respectively.

Another remarkable example is given by

$$L = a_{11}(z)\,\partial_{x_1}^2 + x_1\,\partial_{x_2} + \ldots + x_{x_{N-1}}\,\partial_{x_N} - \partial_t,$$

corresponding to the case $p = 1$ and

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot\cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Let us now come back to the general case. Denote by $A = A(z)$ the following $N \times \times N$ block matrix

$$A(z) = \begin{bmatrix} A_0(z) & 0 \\ 0 & 0 \end{bmatrix},$$

where $A_0(z) = (a_{ij}(z_0))_{i,j=1,\ldots,p}$. We also define

$$C_{z_0}(t) = \int_0^t E^T(s)\, A(z_0)\, E(s)\, ds, \qquad E(s) = \exp(-sB).$$

It follows from (4.12) that $C_{z_0}(t) > 0$ for any $z \in \mathbb{R}^{N+1}$, and any $t > 0$. Then the frozen operator L_{z_0} has a fundamental solution $\Gamma_{z_0}(z, \zeta)$ given by

$$\Gamma_{z_0}(z, \zeta) = \Gamma_{z_0}(\zeta^{-1} \circ z), \tag{4.13}$$

where $\Gamma_{z_0}(x, t) = 0$ if $t \leqslant 0$, and

$$\Gamma_{z_0}(x, t) = (4\pi)^{-N/2} e^{-t \operatorname{tr} B} \sqrt{\det C_{z_0}(t)^{-1}} \exp\left(-\frac{1}{4}\langle C_{z_0}(t)^{-1} x, x\rangle\right),$$

if $t > 0$, see [9]. We have denoted by $\circ$ the following composition law in $\mathbb{R}^{N+1}$

$$(x, t) \circ (y, \tau) = (y + E(\tau)\, x, t + \tau).$$

$(R^{N+1}, \circ)$ is a Lie group and L_{z_0} is invariant with respect to the left translations in the group. In (4.13), ζ^{-1} denotes the opposite of ζ with respect to $\circ$. For every $z = (x, t) \in \mathbb{R}^{N+1}$ and $r > 0$ we define

$$\Omega_r(z) = \{\zeta \in \mathbb{R}^{N+1} : \Gamma_z(z, \zeta) > 1/r\}.$$

Then, for any function $u \in C^2(\mathbb{R}^{N+1})$ we have the following representation formula

$$u(z) = \mathcal{M}_r u(z) - \mathcal{N}_r(L_z u)(z), \tag{4.14}$$

where

$$\mathcal{M}_r u(z) = \frac{1}{r} \int_{\Omega_r(z)} u(\zeta)\, \frac{\langle A(z)\, D_\xi \Gamma_z(z, \zeta), D_\xi \Gamma_z(z, \zeta)\rangle}{\Gamma_z^2(z, \zeta)}\, d\zeta, \qquad \zeta = (\xi, \tau), \tag{4.15}$$

and

$$\mathcal{N}_r(w)(z) = \frac{1}{r} \int_0^r \int_{\Omega_\varrho(z)} \left(\Gamma_z(z, \zeta) - \frac{1}{\varrho}\right) w(\zeta)\, d\zeta\, d\varrho, \tag{4.16}$$

see [9]. We remark that $\mathcal{M}_r$ and $\mathcal{N}_r$ are linear increasing operators since their kernels are nonnegative. Moreover, if we define $Q_r(z) = \mathcal{N}_r(1)(z)$, then

$$\lim_{r\to 0} \frac{\mathcal{N}_r(w)(z)}{Q_r(z)} = w(z),$$

for any $w \in C(\mathbb{R}^{N+1})$ and $z \in \mathbb{R}^{N+1}$. From (4.8), (4.10) with $\alpha = 0$, we obtain

$$Q_r(z) = \int_0^r \frac{|\Omega_s(z)|}{s^2}\left(1 - \frac{s}{r}\right) ds. \tag{4.17}$$

This implies

$$\lim_{r\to 0} \frac{\mathcal{N}_r(L_z u)(z)}{Q_r(z)} = L_z u(z) = Lu(z).$$

Then, for every $u \in C^2(\mathbb{R}^{N+1})$, and since $L_z u(\zeta) = Lu(\zeta) + o(1)$ as $\zeta \to z$, we have

$$\mathcal{N}_r(L_z(u))(z) = \mathcal{N}_r(Lu)(z) + o(Q_r(z)),$$

where we have used the fact that $\Omega_r(z)$ shrinks to z as $r \to 0$. Replacing this identity in (4.14) we obtain (3.1) with $\mathcal{M}_r$, $\mathcal{N}_r$ and Q_r given by (4.15), (4.16), and (4.17) respectively.

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