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Collisions and fractures: a model in *SBD*

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Calcolo delle variazioni. — *Collisions and fractures: a model in SBD.* Nota (*) di
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ABSTRACT. — We investigate collisions (assumed to be instantaneous) and fractures of three-dimensional solids. Equations of motion and constitutive laws provide a set of partial differential equations, whose corresponding variational problem may be solved in the space of special functions with bounded deformations (*SBD*), exploiting the direct method of calculus of variations.

KEY WORDS: Collisions; Fractures; Velocities of bounded deformations.

RIASSUNTO. — *Collisioni e fratture: un modello in SBD.* Studiamo il fenomeno delle collisioni (assunte istantanee) e della formazione di fratture in solidi nel caso tridimensionale. Le equazioni di moto e le leggi costitutive danno origine ad un sistema di equazioni alle derivate parziali, il cui corrispondente problema variazionale viene risolto nello spazio delle funzioni speciali a deformazione limitata, utilizzando il metodo diretto del calcolo delle variazioni.

1. INTRODUCTION

In this paper, we investigate collisions and fractures of solids. Consider, for instance, a plate falling on the floor and breaking, or a rock avalanching from a mountain on a concrete protecting wall (depending on the circumstances either both the rock and the concrete wall break, or only one breaks, or none of them breaks). We address this problem at the engineering macroscopic level and derive a model by discontinuum mechanics theory. Hence, our aim is to investigate this subject both from mechanical and analytical point of view. Let us observe that in our approach collisions are assumed to be instantaneous, as they are very short when compared to the flight time of the solids (cf. [11]). Then, after observing that the state quantities at collision time t are constant, we recall that a collision is characterized by a time discontinuity of the velocity field. By $\boldsymbol{u}^-$ we denote the smooth velocity field before the collision and by $\boldsymbol{u}^+$ the velocity field after the collision. As a consequence of the collision, fractures may appear in the solid (cf. [8]). We characterize a fracture resulting from the collision by a spatial discontinuity of the velocity field $\boldsymbol{u}^+$. The equations of motion are derived by the principle of virtual work at time t, which includes interior percussions accounting for the very large stresses and forces related to the cinematic incompatibilities (cf. [10]). More precisely, we have interior volume percussion stresses and interior surface percussions. The latter appear on the contact surface as well on the fracture surfaces. The constitutive laws are derived by use of dissipative potentials (cf. [13]) in coherence with the second law of thermodynamics, *i.e.* to satisfy the Clausius-Duhem inequality.

Let us now briefly describe the mechanical phenomenon. We consider a solid, located in a smooth domain $\Omega \subset \boldsymbol{R}^3$, colliding on a part of its boundary $\partial\Omega_1$ with a fixed

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obstacle. The remaining part $\partial\Omega_2$ of the boundary is free. The system we consider is made of the solid and the obstacle, whose velocity is assumed to remain equal to 0, as the obstacle is taken very massive. This model has been introduced in [7] to describe fractures caused by collisions and percussions in a one-dimensional system.

The PDE's system resulting from the equations of motion and from the constitutive laws is investigated in a variational framework.

2. The model

2.1. *The principle of virtual work.*

The equations of motion are derived by the principle of virtual work at collision time (a work is a duality pairing in mathematical terms), in which surface and volume percussions are considered. The virtual work of the interior percussions is defined by (Γ denotes the fracture)

$$\mathcal{T}_{\text{int}}(\boldsymbol{v}^+, \boldsymbol{v}^-) = -\int_{\Omega\setminus\Gamma} \Sigma : \mathcal{E}\left(\frac{\boldsymbol{v}^+ + \boldsymbol{v}^-}{2}\right) d\Omega + \int_{\Gamma} \boldsymbol{R}\cdot\left[\frac{\boldsymbol{v}^+ + \boldsymbol{v}^-}{2}\right] d\mathcal{H}^2 - \int_{\partial\Omega_1} \boldsymbol{R}\left(\frac{\boldsymbol{v}^+ + \boldsymbol{v}^-}{2}\right) d\mathcal{H}^2, \tag{2.1}$$

where $d\mathcal{H}^2$ stands for the two-dimensional Hausdorff measure (from a mechanical point of view it corresponds to the surface measure), $\mathcal{E}(\boldsymbol{v}) = (1/2(v_{i,j} + v_{j,i}))$ is the classical symmetric strain rate, and $\boldsymbol{v}^+$, $\boldsymbol{v}^-$ are virtual velocity fields. The fracture Γ is oriented ($\boldsymbol{n}$ in the following of this paragraph stands for the normal vector). Thus, we are allowed to consider a «left» part ($\boldsymbol{v}_l$) and a «right» part ($\boldsymbol{v}_r$) of the velocity field w.r.t. the orientation of the fracture Γ (see [3] for details). The spatial velocity discontinuity is indicated by $[\boldsymbol{v}] := \boldsymbol{v}_r - \boldsymbol{v}_l$. Note that in $\mathcal{T}_{\text{int}}$ we have introduced the volume percussion stress Σ and the surface percussion $\boldsymbol{R}$ defined on the unknown fracture Γ and the contact surface $\partial\Omega_1$. The virtual work of acceleration forces is

$$\mathcal{T}_{\text{acc}}(\boldsymbol{v}^+, \boldsymbol{v}^-) = \int_{\Omega} \varrho(\boldsymbol{u}^+ - \boldsymbol{u}^-)\cdot\frac{\boldsymbol{v}^+ + \boldsymbol{v}^-}{2}\, d\Omega, \tag{2.2}$$

where ϱ is the density of the solid and $\varrho(\boldsymbol{u}^+ - \boldsymbol{u}^-)$ is the collision inertial percussion. There is no exterior action, thus the exterior virtual work is zero. The equations of motion

$$\begin{aligned} &\varrho(\boldsymbol{u}^+ - \boldsymbol{u}^-) - \operatorname{div}\Sigma = 0 \text{ in } \Omega\setminus\Gamma, \\ &[\Sigma]\,\boldsymbol{n} = 0, \quad \Sigma\boldsymbol{n} = \boldsymbol{R} \text{ on } \Gamma, \\ &\Sigma\boldsymbol{n} + \boldsymbol{R} = 0 \text{ on } \partial\Omega_1, \\ &\Sigma\boldsymbol{n} = 0 \text{ on } \partial\Omega_2, \end{aligned} \tag{2.3}$$

result from the principle of virtual work

$$\forall\, \boldsymbol{v}^+, \boldsymbol{v}^- \quad \mathcal{T}_{\text{acc}}(\boldsymbol{v}^+, \boldsymbol{v}^-) = \mathcal{T}_{\text{int}}(\boldsymbol{v}^+, \boldsymbol{v}^-). \tag{2.4}$$

Note that in (2.3), we have used the same notation for the normals to the fractures and to the boundary.

2.2. *The constitutive laws and the resulting equations.*

We assume that the velocity $\boldsymbol{u}^-$ before the collision is a datum of our problem, and let the unknown be the velocity $\boldsymbol{u}^+$ after the collision. The constitutive laws for Σ and $\boldsymbol{R}$ which have to satisfy the Clausius-Duhem inequalities (cf. [7, 10]) are defined by three functions: the volume dissipative function Φ and the surface dissipative functions Φ_Γ, $\Phi_{\partial\Omega_1}$. We choose the volume dissipative function as

$$\Phi(\mathcal{E}(\boldsymbol{u}^+ + \boldsymbol{u}^-)) = k_0 \,|\, \mathcal{E}(\boldsymbol{u}^+ + \boldsymbol{u}^-)\,| + \frac{k_1}{2}\,|\, \mathcal{E}(\boldsymbol{u}^+ + \boldsymbol{u}^-)\,|^2. \tag{2.5}$$

For simplicity's sake, we use the same symbol to denote the length of a vector $|\,\boldsymbol{x}\,| = \sqrt{x_i x_i}$ and the norm of a symmetric tensor $|A| = \sqrt{A_{ij}A_{ij}}$. Hence, the fracture dissipative function on Γ is addressed as follows

$$\Phi_\Gamma([\boldsymbol{u}^+ + \boldsymbol{u}^-]) = 2k_2\sqrt{|[\boldsymbol{u}^+ + \boldsymbol{u}^-]|} + k_3\,|[\boldsymbol{u}^+ + \boldsymbol{u}^-]| + I_+([\boldsymbol{u}^+ + \boldsymbol{u}^-]\cdot\boldsymbol{n}). \tag{2.6}$$

The constants k_i, $i = 0, \dots, 3$, are chosen to be strictly positive. Let us comment on our choices regarding the mechanical behaviour of the body. The volume dissipative function (2.5) ensures a classical behaviour away from the fractures. The effect of the function $2k_2\sqrt{|[\boldsymbol{u}^+ + \boldsymbol{u}^-]|}$ is to avoid having many fractures with small discontinuities. The indicator function I_+ of $\boldsymbol{R}^+$ takes into account the impenetrability condition on the interior fractures

$$[\boldsymbol{u}^+]\cdot\boldsymbol{n} \geqslant 0. \tag{2.7}$$

Thus, taking into account the natural orientation of the boundary, the unilateral boundary condition on $\partial\Omega_1$ may be described by a dissipative function depending on $(\boldsymbol{u}^+ + \boldsymbol{u}^-)$ (recall that the velocity of the obstacle is 0)

$$\Phi_{\partial\Omega_1}(\boldsymbol{u}^+ + \boldsymbol{u}^-) = I_-(\boldsymbol{u}^+\cdot\boldsymbol{n}) = I_+((\boldsymbol{u}^- - (\boldsymbol{u}^+ + \boldsymbol{u}^-))\cdot\boldsymbol{n}), \tag{2.8}$$

where I_- is the indicator function of $\boldsymbol{R}^-$. We point out that in (2.8) we have implicitly considered an exterior velocity of Ω corresponding to the velocity of the obstacle (equal to 0) and dealt with the unilateral boundary condition on $\partial\Omega_1$ including the obstacle in the domain of the model. Hence, we observe that the dissipative functions Φ and $\Phi_{\partial\Omega_1}$ are pseudo-potentials of dissipation, *i.e.* positive convex functions attaining their minimum 0 at the origin [13] (cf. Remark 2.1). On the contrary, Φ_Γ is not because it is not a convex function of $[\boldsymbol{u}^+ + \boldsymbol{u}^-]$. Nevertheless, Φ_Γ may be split into a convex part, Φ_Γ^c, and a non convex part, Φ_Γ^{nc}. Its generalised subdifferential set $\bar{\partial}\Phi_\Gamma$ is the sum of the subdifferential set of the convex part and of the extended derivative of the non convex part (cf. [7] for a similar assumption). We address the following constitutive laws

$$\begin{gathered}\Sigma \in \partial\Phi(\mathcal{E}(\boldsymbol{u}^+ + \boldsymbol{u}^-)) \ \text{ in } \Omega\backslash\Gamma, \quad -\boldsymbol{R} \in \bar{\partial}\Phi_\Gamma([\boldsymbol{u}^+ + \boldsymbol{u}^-]) \ \text{ on } \Gamma,\\ \boldsymbol{R} \in \partial\Phi_{\partial\Omega_1}(\boldsymbol{u}^+ + \boldsymbol{u}^-) \ \text{ on } \partial\Omega_1.\end{gathered} \tag{2.9}$$

It is easy to prove that, owing to the above prescriptions, the Clausius-Duhem inequality is satisfied. This follows by standard properties of the pseudo-potentials of dissipation and the results presented in [7].

The equations for the velocity $\boldsymbol{u}^+$ are derived by the principle of virtual work (2.4), combined with the constitutive relations (2.9), and the expression of dissipative potentials (2.6) and (2.8). They are written in the smooth bounded domain $\Omega \subset \boldsymbol{R}^3$ with boundary $\partial\Omega = \overline{\partial\Omega_1} \cup \overline{\partial\Omega_2}$ ($\partial\Omega_1$ has a strictly positive measure and such that it is $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$). We get

$$
\begin{aligned}
&\varrho\boldsymbol{u}^+ - \operatorname{div}\Sigma = \varrho\boldsymbol{u}^- \quad \text{in } \Omega\backslash\Gamma,\\
&\Sigma \in \partial\Phi(\mathcal{E}(\boldsymbol{u}^+ + \boldsymbol{u}^-)) \quad \text{in } \Omega\backslash\Gamma,\\
&\Sigma\boldsymbol{n} \in \overline{\partial}\Phi_\Gamma[\boldsymbol{u}^+ + \boldsymbol{u}^-],\ [\Sigma]\ \boldsymbol{n} = 0 \quad \text{on } \Gamma,\\
&\Sigma\boldsymbol{n} + \partial\Phi_{\partial\Omega_1}(\boldsymbol{u}^+ + \boldsymbol{u}^-) \ni 0 \quad \text{on } \partial\Omega_1,\\
&\Sigma\boldsymbol{n} = 0 \quad \text{on } \partial\Omega_2.
\end{aligned}
\tag{2.10}
$$

In the following, we deal with a variational formulation of the above problem, written in a suitable space of discontinuous and sufficiently smooth velocities.

REMARK 2.1. *Because the solid is falling on the obstacle, we have*

$$\boldsymbol{u}^- \cdot \boldsymbol{n} \geqslant 0 \quad \textit{in } \partial\Omega_1. \tag{2.11}$$

This condition ensures that $\Phi_{\partial\Omega_1}(\boldsymbol{u}^+ + \boldsymbol{u}^-)$ is zero at the origin (i.e. *if $\boldsymbol{u}^+ + \boldsymbol{u}^- = 0$*), *which has to be verified if $\Phi_{\partial\Omega_1}$ is a pseudo-potential of dissipation* (cf. (2.8)). *From now on we assume this condition is satisfied.*

3. Variational formulation

We look for $\boldsymbol{u}^+$ minimizing a functional whose Euler equations correspond to (2.10). To this aim, we consider the weak formulation of a minimization problem associated to the following functional

$$
\mathcal{F}(\boldsymbol{v}) = \int_{\Omega\backslash\Gamma} \Phi(\mathcal{E}(\boldsymbol{v} + \boldsymbol{u}^-))\, d\Omega + \int_\Omega \left(\frac{\varrho}{2}(\boldsymbol{v})^2 - \varrho\boldsymbol{v}\cdot\boldsymbol{u}^-\right) d\Omega +
+ \int_\Gamma \Phi_\Gamma([\boldsymbol{v} + \boldsymbol{u}^-])\, d\mathcal{H}^2 + \int_{\partial\Omega_1} \Phi_{\partial\Omega_1}(\boldsymbol{v} + \boldsymbol{u}^-)\, d\mathcal{H}^2.
\tag{3.1}
$$

We address the corresponding weak problem in the space $SBD(\Omega)$ of special functions with bounded deformations (cf. [2]). Before proceeding, let us specify some useful notation used in the following. We recall that a function $\boldsymbol{u}$ is a function of bounded deformation (cf. [14, 12, 15]), *i.e.* $\boldsymbol{u} \in BD(\Omega)$, if $\boldsymbol{u} \in L^1(\Omega)$ and its distributional symmetric gradient $E\boldsymbol{u}$ is a bounded Radon measure, *i.e.* $E\boldsymbol{u} \in \mathcal{M}_b(\Omega)$. For every $\boldsymbol{u} \in BD(\Omega)$, we let the jump set $J_{\boldsymbol{u}}$ to be defined as the set of points $x \in \Omega$ where $\boldsymbol{u}$ has two different one sided Lebesgue limits, $\boldsymbol{u}_r(x) \neq \boldsymbol{u}_l(x)$ ($[\boldsymbol{u}] \neq 0$), w.r.t. the direction $\boldsymbol{n}_{\boldsymbol{u}}$ normal to the approximate tangent space to $J_{\boldsymbol{u}}$. We recall that $J_{\boldsymbol{u}}$ is a countably rectifiable set. Hence, $E\boldsymbol{u}$ can be split in the sum of an absolutely continuous part w.r.t. the Lebesgue measure $d\Omega$, $E^a\boldsymbol{u} = \mathcal{E}\boldsymbol{u}\, d\Omega$ (here $\mathcal{E}\boldsymbol{u}$ denotes the density of the measure $E^a\boldsymbol{u}$), and a singular part, $E^s\boldsymbol{u} = E^j\boldsymbol{u} + E^c\boldsymbol{u}$, where $E^j\boldsymbol{u}$ stands for the jump part (restriction of $E^s\boldsymbol{u}$ to $J_{\boldsymbol{u}}$) and $E^c\boldsymbol{u}$ the Cantorian part (restriction of $E^s\boldsymbol{u}$ to $\Omega\backslash J_{\boldsymbol{u}}$). A function $\boldsymbol{u} \in BD(\Omega)$ belongs to $SBD(\Omega)$ if and only if $E^c\boldsymbol{u} = 0$. Moreover, it has been proved in [2] that

$$E^j\boldsymbol{u} = [\boldsymbol{u}] \odot \boldsymbol{n}_{\boldsymbol{u}}\, d\mathcal{H}^2 \lfloor J_{\boldsymbol{u}}, \tag{3.2}$$

where $\odot$ is the symmetric tensor product and $d\mathcal{H}^2 \lfloor J_{\boldsymbol{u}}$ the 2-Hausdorff measure restricted to $J_{\boldsymbol{u}}$. In the sequel $\boldsymbol{n}$ will denote the normal to the boundary. Towards the aim of including the fixed obstacle in the system (whose velocity $\boldsymbol{v}_{\text{ext}}$ is 0), we extend the domain Ω outside the part $\partial\Omega_1$. In particular, we aim to deal on $\partial\Omega_1$ with the discontinuity $\boldsymbol{v}_{\text{ext}} - \boldsymbol{v}$ ($\boldsymbol{v}$ is here the interior trace on $\partial\Omega_1$ of the velocity $\boldsymbol{v}$ defined in Ω). The extended domain $\Omega' \supset \Omega$ is smooth and such that a part of its boundary is $\partial\Omega_2$ ($\partial\Omega \cap \partial\Omega' = \overline{\partial\Omega_2}$). The extension is in contact with the initial domain on boundary $\partial\Omega_1$, *i.e.* $\overline{\Omega} \cap (\overline{\Omega' \backslash \Omega}) = \overline{\partial\Omega_1}$. Then, we can provide a unified weak formulation of the impenetrability condition on the fractures and on the boundary $\partial\Omega_1$. We require that (cf. [7])

$$\int_{J_{\boldsymbol{v}}} \phi[\boldsymbol{v}]\cdot\boldsymbol{n}_{\boldsymbol{v}}\, d\mathcal{H}^2 + \int_{\partial\Omega_1} \phi(-\boldsymbol{v})\cdot\boldsymbol{n}\, d\mathcal{H}^2 \geqslant 0, \tag{3.3}$$

for any $\phi \in C_c^0(\Omega')$ with $\phi \geqslant 0$. Condition (3.3) is equivalent to

$$[\boldsymbol{v}]\cdot\boldsymbol{n}_{\boldsymbol{v}}\, d\mathcal{H}^2 \lfloor J_{\boldsymbol{v}} + ((-\boldsymbol{v})\cdot\boldsymbol{n})\, d\mathcal{H}^2 \lfloor \partial\Omega_1 \geqslant 0, \tag{3.4}$$

in the sense of measures in Ω' (in the following we use the notation $\mathcal{M}(\Omega')$).

We assume, as it is usual, that before the collision the solid moves with a rigid body velocity, *i.e.* $E\boldsymbol{u}^- = 0$. Then, we can make precise the functional we aim to minimize. To simplify notation, but without loss of generality, we put the physical constants equal to 1 in (3.1), and deal with

$$\mathcal{G}(\boldsymbol{v}) = \frac{1}{2}\int_{\Omega} |\mathcal{E}(\boldsymbol{v})|^2\, d\Omega + |E\boldsymbol{v}|(\Omega) + \int_{\Omega}\left(\frac{1}{2}(\boldsymbol{v})^2 - \boldsymbol{v}\cdot\boldsymbol{u}^-\right) d\Omega + \int_{J_{\boldsymbol{v}}} |[\boldsymbol{v}]|^{1/2}\, d\mathcal{H}^2, \tag{3.5}$$

where $|E\boldsymbol{v}|(\Omega)$ is the total variation of the measure $E\boldsymbol{v}$. We address the problem in the following convex subset of smooth kinematically admissible velocities $SCV(\Omega, \partial\Omega_1)$ (accounting for (3.4))

$$\begin{aligned} &SCV(\Omega, \partial\Omega_1) := \{\boldsymbol{v} \in SBD(\Omega) \cap L^2(\Omega);\\ &[\boldsymbol{v}]\cdot\boldsymbol{n}_{\boldsymbol{v}}\, d\mathcal{H}^2 \lfloor J_{\boldsymbol{v}} + ((-\boldsymbol{v})\cdot\boldsymbol{n})\, d\mathcal{H}^2 \lfloor \partial\Omega_1 \geqslant 0 \text{ in } \mathcal{M}(\Omega')\}. \end{aligned} \tag{3.6}$$

The problem we aim to solve reads

$$\inf\{\mathcal{G}(\boldsymbol{v}) : \boldsymbol{v} \in SCV(\Omega, \partial\Omega_1)\}. \tag{3.7}$$

The following theorem states existence of a solution for (3.7).

THEOREM 3.1. *Let* $\boldsymbol{u}^-$ *be a rigid body velocity satisfying* (2.11). *Then, the minimization problem* (3.7) *admits a solution.*

To prove Theorem 3.1 we use the direct method of calculus of variations, which is based on compactness and lower semicontinuity arguments. We consider a minimizing sequence $\boldsymbol{v}_n$ of $\mathcal{G}$ in $SCV(\Omega, \partial\Omega_1)$ (in particular we have $\boldsymbol{v}_n \in SBD(\Omega)$). The first step is to show that $\mathcal{G}$ is coercive in *SBD*. By (3.5), recalling in particular the definition of norm in *BD*, *i.e.* $\|\boldsymbol{u}\|_{BD(\Omega)} = \|\boldsymbol{u}\|_{L^1(\Omega)} + |E\boldsymbol{u}|(\Omega)$, and applying the Young inequality, we are allowed to deduce

$$\|\boldsymbol{v}_n\|_{BD(\Omega)} + \int_{J_{v_n}} |[\boldsymbol{v}_n]|^{1/2}\, d\mathcal{H}^2 + \|\mathcal{E}\boldsymbol{v}_n\|_{L^2(\Omega)} + \|\boldsymbol{v}_n\|_{L^2(\Omega)} \leqslant c, \tag{3.8}$$

for a positive constant c independent of n. In the sequel, to simplify notation, we use the same symbol c for possibly different positive constants, in particular not depending on n. Next, we prove a weak compactness result holding in *SBD* and allowing us to pass to the limit $\boldsymbol{v}_n \in SCV \to \boldsymbol{v} \in SCV$ (in a suitable weak sense), in such a way that $\mathcal{G}$ turns out to be lower semicontinuous, *i.e.* $\liminf_{n\to+\infty} \mathcal{G}(\boldsymbol{v}_n) \geqslant \mathcal{G}(\boldsymbol{v})$. The compactness (Section 3.1) and lower semicontinuity (Section 3.2) results exploit two theorems we have proved for rather general dissipative potentials.

3.1. *A weak compactness and convergence theorem in SBD.*

We prove the following compactness theorem holding in $SBD(\Omega)$, which apply to the sequence $\boldsymbol{v}_n$ owing to (3.8).

Theorem 3.2. *Let $\boldsymbol{u}_n \in SBD(\Omega)$ a sequence of functions fulfilling*

- *$\boldsymbol{u}_n$ is uniformly bounded in $BD(\Omega)$;*
- *$\mathcal{E}\boldsymbol{u}_n$ are equi-integrable;*
- *there exists a function $\Psi : [0, +\infty) \to [0, +\infty)$, non decreasing and such that $\Psi(t)/t \to +\infty$ as $t \searrow 0$, satisfying*

$$\sup_n \int_{J_{u_n}} \Psi(|[\boldsymbol{u}_n]|)\, d\mathcal{H}^2 < +\infty. \tag{3.9}$$

Then, there exists a subsequence still denoted by $\boldsymbol{u}_n$, which converges in $L^1(\Omega)$ to some $\boldsymbol{u} \in SBD(\Omega)$. Moreover, the Lebesgue part and the jump part of $E\boldsymbol{u}_n$ converge weakly separately, i.e. *the Lebesgue part converges weakly in $L^1(\Omega)$ and the jump part weakly in the sense of measures.*

Remark 3.3. *The above theorem generalizes an analogous compactness and lower semicontinuity result holding in SBV* (cf. [3, 1, 4]). *Nonetheless, we recall that we cannot directly apply the result in SBD, as the Korn inequality does not still hold in this space* (cf. [12]) *and a bound for the antisymmetric part of the gradient does not follow from the assumptions of the Theorem 3.2.*

The demonstration of Theorem 3.2 is based on similar arguments as that exploited in [6] (cf. also [5]). Nonetheless, our result is fairly different as we introduce the function Ψ applied to the jump part of $E\boldsymbol{u}_n$ and we do not require that the measure of the fracture (*i.e.* the set of discontinuities) is uniformly bounded w.r.t. n. Thus, for the sake of brevity, we point out only the part of the proof concerning the bound and the lower semicontinuity of Ψ while we refer to [6] for the remaining details. The main idea is to reduce to one-dimensional sections of *SBD* functions (*i.e.* investigate the velocity field restricted to fixed directions in $\boldsymbol{R}^3$), then apply the results known for *SBV* in $1D$ domains, and finally integrate these informations on all directions. This method is based on the fact that we can characterize *SBD* functions through their one-dimensional sections exploiting fine properties of this space (cf. [2]). It has been proved that $\boldsymbol{u} \in SBD(\Omega)$ if and only if for any $\xi = \xi_i + \xi_j$ (ξ_i, $i = 1, 2, 3$ being a basis in $\boldsymbol{R}^3$) and for $\mathcal{H}^2$ a.e. $y \in \Omega^\xi$, one has $u_y^\xi \in SBV(\Omega_y^\xi)$, where,

$$u_y^\xi(t) = u^\xi(y + t\xi) = \boldsymbol{u}(y + t\xi)\cdot\xi \tag{3.10}$$

$$\Omega^\xi = \{y \in \boldsymbol{R}^3 : y\cdot\xi = 0,\ \Omega_y^\xi \neq \emptyset\} \tag{3.11}$$

$$\Omega_y^\xi = \{t \in \boldsymbol{R} : y + t\xi \in \Omega\}. \tag{3.12}$$

Now, owing to the boundedness of $\boldsymbol{u}_n$ (cf. (3.8)) and weak* compactness results in $BD(\Omega)$ (which can be identified with the dual of a Banach space), we find a suitable subsequence $\boldsymbol{u}_n$ weakly* converging in $BD(\Omega)$ to some $\boldsymbol{u} \in BD(\Omega)$, *i.e.* $\boldsymbol{u}_n \to \boldsymbol{u}$ strongly in $L^1(\Omega)$ and $E\boldsymbol{u}_n \to E\boldsymbol{u}$ in the sense of measures in Ω (in the following we still denote any subsequences by $\boldsymbol{u}_n$ to simplify notation). Thus, the first step is to prove that $\boldsymbol{u} \in SBD(\Omega)$, showing that $u_y^\xi \in SBV(\Omega_y^\xi)$ for any ξ and a.e. y. To this aim, let us consider the sequence $u_{ny}^\xi \in SBV(\Omega_y^\xi)$, defined as by (3.10). Applying the results of [2] and proceeding as in [6], we can prove for a constant c not depending on ξ or n

$$\int_{\Omega^\xi}\left(\int_{\Omega_y^\xi} |(u_{ny}^\xi)'(t)|\,dt + \int_{J_{u_{ny}^\xi}} |[u_{ny}^\xi]|\,d\mathcal{H}^0\right)dy \leqslant |E\boldsymbol{u}_n|(\Omega) \leqslant c, \tag{3.13}$$

where $(u_{ny}^\xi)'$ denotes the classical derivative w.r.t. t. Analogously proceeding, we infer that

$$\int_{\Omega^\xi}\int_{J_{u_{ny}^\xi}} \Psi(|[u_{ny}^\xi]|)\,d\mathcal{H}^0\,dy = \int_{J_{\boldsymbol{u}_n}} \Psi(|[\boldsymbol{u}_n]\cdot\xi|)\,d\mathcal{H}^2 \leqslant \int_{J_{\boldsymbol{u}_n}} \Psi(|[\boldsymbol{u}_n]|)\,d\mathcal{H}^2 \leqslant c. \tag{3.14}$$

Then, applying a compactness theorem in $SBV(\Omega_y^\xi)$ (cf. *e.g.* [1]), we claim that there exists a subsequence such that $u_{ny}^\xi \to \eta$ in $L^1(\Omega_y^\xi)$, with $\eta \in SBV(\Omega_y^\xi)$, and eventually identify $\eta = u_y^\xi$ by the convergence of $\boldsymbol{u}_n$. Thus, we get $\boldsymbol{u} \in SBD(\Omega)$. Moreover, we can infer that $(u_{ny}^\xi)'$ converges weakly to $(u_y^\xi)'$ in $L^1(\Omega_y^\xi)$ (and also that $[u_{ny}^\xi] \to [u_y^\xi]$ in the sense of measures on Ω_y^ξ). Now, we aim to discuss the asymptotic behaviour of $E\boldsymbol{u}_n$. Owing to the above convergences stated for one-dimensional sections, exploiting semicontinuity arguments and applying the Fatou lemma (mainly to get results for a subsequence of $\boldsymbol{u}_n$ not depending on ξ or $y \in \Omega^\xi$), we proceed as in [6], and show that there exists a subsequence of $\boldsymbol{u}_n$ such that for any $w \in L^1(\Omega)$, there holds $\int_\Omega |(\mathcal{E}\boldsymbol{u})\,\xi\cdot\xi - w| \leqslant \liminf_{n\to+\infty}\int_\Omega |(\mathcal{E}\boldsymbol{u}_n)\,\xi\cdot\xi - w|$. In particular, it follows that (cf. [3]) $(\mathcal{E}\boldsymbol{u}_n)\,\xi\cdot\xi$ converges weakly in $L^1(\Omega)$ (actually in our applications in $L^2(\Omega)$) to

$(\mathcal{E}\boldsymbol{u})\,\xi\cdot\xi$, for any $\xi=\xi_i+\xi_j$, $i,j=1,2,3$. Then, due to the symmetry of $\mathcal{E}\boldsymbol{u}_n$, we eventually get $\mathcal{E}\boldsymbol{u}_n\to\mathcal{E}\boldsymbol{u}$ weakly in $L^1(\Omega)$. Finally, the convergence in the sense of measures of $E^j\boldsymbol{u}_n$ easily follows combining the convergence in the sense of measures of the strain rates $E\boldsymbol{u}_n$ and the convergence of the regular parts $\mathcal{E}\boldsymbol{u}_n$ (to $E\boldsymbol{u}$ and $\mathcal{E}\boldsymbol{u}$, respectively).

Now, we come back to the proof of Theorem 3.1. Owing to (3.8), we apply standard weak compactness results in L^2 and Theorem 3.2 to eventually get

$$\boldsymbol{v}_n\to\boldsymbol{v} \text{ strongly in } L^1(\Omega), \text{ weakly in } L^2(\Omega), \tag{3.15}$$

$$\mathcal{E}\boldsymbol{v}_n\to\mathcal{E}\boldsymbol{v} \text{ weakly in } L^2(\Omega), \tag{3.16}$$

$$E^j(\boldsymbol{v}_n)\to E^j(\boldsymbol{v}) \text{ in the sense of measures in } \Omega, \tag{3.17}$$

for a suitable subsequence and $\boldsymbol{v}\in SBD(\Omega)$. It remains to prove that

$$\mathcal{G}(\boldsymbol{v})\leqslant\liminf_{n\to+\infty}\mathcal{G}(\boldsymbol{v}_n), \tag{3.18}$$

and

$$\boldsymbol{v}\in SCV(\Omega,\partial\Omega_1). \tag{3.19}$$

3.2. *A lower semicontinuity result in SBD.*

Towards the aim of proving (3.18), we prove the following lower semincontinuity result holding in $SBD(\Omega)$.

THEOREM 3.4. *Let $\boldsymbol{v}_n\in SBD(\Omega)$ fulfill the assumptions of Theorem* 3.2. *In addition, we require the function Ψ to be lower semicontinuous, subadditive*, i.e.

$$\Psi(a+b)\leqslant\Psi(a)+\Psi(b), \tag{3.20}$$

and satisfying

$$\Psi(ab)=\Psi(a)\,\Psi(b) \tag{3.21}$$

for any $a,b\geqslant 0$. Then, there exists a subsequence fulfilling the thesis of Theorem 3.2 *s.t.*

$$\liminf_{n\to+\infty}\int_{J_{\boldsymbol{u}_n}}\Psi(|[\boldsymbol{u}_n]|)\,d\mathcal{H}^2\geqslant\int_{J_{\boldsymbol{u}}}\Psi(|[\boldsymbol{u}]|)\,d\mathcal{H}^2. \tag{3.22}$$

We proceed as in the proof of Theorem 3.2. Thus, for ξ and y fixed, we consider the one-dimensional sections u^{ξ}_{ny}, and apply the lower semicontinuity result result stated by [9, Theorem 2.10] and holding *e.g.* in $SBV(\Omega^{\xi}_{y})$. By similarly arguing as above and in [6], we can eventually infer that for any direction ξ there holds

$$\liminf_{n\to+\infty}\int_{J_{\boldsymbol{u}_n}}\Psi(|[\boldsymbol{u}_n]\cdot\xi|)\,d\mathcal{H}^2\geqslant\int_{J_{\boldsymbol{u}}}\Psi(|[\boldsymbol{u}]\cdot\xi|)\,d\mathcal{H}^2. \tag{3.23}$$

To get (3.22), we integrate (3.23) on all directions $\xi\in S^2:=\{\xi\in\boldsymbol{R}^3:\ |\xi|=1\}$. For any $\boldsymbol{v}\in\boldsymbol{R}^3$ and $\xi\in S^2$, we can write $|\boldsymbol{v}\cdot\xi|=|\boldsymbol{v}|\,|\cos(\theta(\boldsymbol{v},\xi))|$, where $\theta(\boldsymbol{v},\xi)$ is the angle between $\boldsymbol{v}$ and ξ. Using Fatou's lemma, the Fubini-Tonelli theorem, and strongly exploiting (3.21), we can infer that

$$(3.24)\qquad \int_{J_u} \Psi(|[u]|)\int_{S^2} \Psi(|\cos\theta(u,\xi)|) = \int_{S^2}\int_{J_u} \Psi(|[u]\cdot\xi|) \leqslant$$

$$\leqslant \liminf_{n\to+\infty} \int_{S^2}\int_{J_{u_n}} \Psi(|[u_n]\cdot\xi|) = \liminf_{n\to+\infty} \int_{J_{u_n}} \Psi(|[u_n]|)\int_{S^2} \Psi(|\cos\theta(u_n,\xi)|)\,.$$

Thus, after observing that, as we integrate over all the directions ξ, there holds

$$(3.25)\qquad \int_{S^2} \Psi(|\cos\theta(u,\xi)|)\,d\xi = \int_{S^2} \Psi(|\cos\theta(u_n,\xi)|)\,d\xi = \hat{c},$$

we can divide (3.24) by $\hat{c}$ and eventually get (3.22).

Hence, (3.18) follows by Theorem 3.4 and well-known weak lower semicontinuity of norms.

REMARK 3.5. *To prove* (3.22), *we have strongly exploited the properties* (3.21), *which is not required for an analogous result in SBV* (*also in the higher dimensional case, cf.* [9, Theorem 2.12]). *Thus, one could think to extend the result of Theorem* 3.4 *to a more general function* Ψ, *with weaker assumptions than* (3.21). *Nonetheless, to our knowledge this remains as an open question.*

Now, to complete the proof of the existence result, we have to prove (3.19), *i.e.* show that v satisfies the impenetrability condition on the extended domain Ω'. More precisely, we have to verify that (3.4) is fulfilled. We observe that by Theorem 3.2 we get a convergence in the sense of measures for $E^j v_n$, but an analogous result does not hold for the traces on $\partial\Omega_1$ of v_n. Indeed, it is known that the trace operator $BD(\Omega)\to$ $\to L^1(\partial\Omega)$ is not continuous if the two spaces are endowed with the weak topology (cf. [14, 3]), which is our case. Thus, we aim to find a convergence in the sense of measures also for the traces, which we consider as jumps between the velocity of Ω and the velocity of the obstacle ($\Omega'\setminus\Omega$). We point out that this is the same idea we have exploited in modelling the impenetrability condition by the dissipative function $\Phi_{\partial\Omega_1}$ in (2.8) (cf. also (3.6)). We consider a field of velocity which describes the velocity of the whole system located in Ω', given by the velocity of the body and the velocity of the fixed obstacle. In particular, we will consider the unilateral boundary condition on $\partial\Omega_1$ as an impenetrability condition prescribed on a jump of the extended velocity field. We let z_n defined in Ω' be equal to v_n in Ω (velocity of the solid) and 0 in $\Omega'\setminus\overline{\Omega}$ (velocity of the obstacle). We note that $z_n\in SBV(\Omega')$ and $J_{z_n}=J_{v_n}\cup\partial\Omega_1$. Moreover, we infer that

$$(3.26)\qquad [z_n]\odot n_{z_n}\,d\mathcal{H}^2\lfloor J_{z_n} = [v_n]\odot n_{v_n}\,d\mathcal{H}^2\lfloor J_{v_n} + ((-v_n)\odot n)\,d\mathcal{H}^2\lfloor\partial\Omega_1\,.$$

It is a standard matter to verify that (cf. (3.8))

$$(3.27)\qquad \|z_n\|_{BD(\Omega')} + \|z_n\|_{L^2(\Omega')} + \|\mathcal{E}(z_n)\|_{L^2(\Omega')} \leqslant c,$$

independently of n. Hence, after recalling the continuity of the trace operator $BD(\Omega) \to L^1(\partial\Omega)$, we get

$$(3.28) \qquad \int_{J_{z_n}} |[\boldsymbol{z}_n]|^{1/2} = \int_{J_{v_n}} |[\boldsymbol{v}_n]|^{1/2} + \int_{\partial\Omega_1} |[\boldsymbol{v}_n]|^{1/2} \leqslant$$

$$\leqslant c + \int_{\partial\Omega_1} |\boldsymbol{v}_n|^{1/2} \leqslant c\left(1 + \int_{\partial\Omega_1} (1 + |\boldsymbol{v}_n|)\right) \leqslant c(1 + \|\boldsymbol{v}_n\|_{BD(\Omega)}) \leqslant c.$$

Thus, the sequence $\boldsymbol{z}_n$ satisfies the assumptions of Theorem 3.2. In particular, we can find a subsequence converging in $SBD(\Omega')$ to $\boldsymbol{z}$, where $\boldsymbol{z} = \boldsymbol{v}$ in Ω and $\boldsymbol{z} = 0$ in $\Omega' \setminus \overline{\Omega}$. Indeed, $\boldsymbol{z}_n$ converges strongly in $L^1(\Omega')$ to $\boldsymbol{z}$, and consequently a.e. in Ω', as well $\boldsymbol{v}_n$ converges strongly in $L^1(\Omega)$ to $\boldsymbol{v}$, and consequently a.e. in Ω. By the above arguments and Theorem 3.2, it follows in particular

$$(3.29) \qquad [\boldsymbol{v}_n] \odot \boldsymbol{n}_{\boldsymbol{v}_n} d\mathcal{H}^2 \lfloor J_{\boldsymbol{v}_n} + ((-\boldsymbol{v}_n) \odot \boldsymbol{n})\, d\mathcal{H}^2 \lfloor \partial\Omega_1 = [\boldsymbol{z}_n] \odot \boldsymbol{n}_{\boldsymbol{z}_n} d\mathcal{H}^2 \lfloor J_{\boldsymbol{z}_n} \to$$

$$\to [\boldsymbol{z}] \odot \boldsymbol{n}_{\boldsymbol{z}} d\mathcal{H}^2 \lfloor J_{\boldsymbol{z}} = [\boldsymbol{v}] \odot \boldsymbol{n}_{\boldsymbol{v}} d\mathcal{H}^2 \lfloor J_{\boldsymbol{v}} + ((-\boldsymbol{v}) \odot \boldsymbol{n})\, d\mathcal{H}^2 \lfloor \partial\Omega_1,$$

in the sense of measures in Ω'. We observe that (3.29) means that for any $\sigma_{ij} \in C_c^0(\Omega')$ with $\sigma_{ij} = \sigma_{ji}$ there holds ($[\cdot]_i$ means the i-component of $[\cdot]$)

$$(3.30) \qquad \int_{J_{v_n}} [\boldsymbol{v}_n]_i \sigma_{ij} n_{\boldsymbol{v}_n j} d\mathcal{H}^2 + \int_{\partial\Omega_1} (-\boldsymbol{v}_n)_i \sigma_{ij} n_j d\mathcal{H}^2 \to \int_{J_v} [\boldsymbol{v}]_i \sigma_{ij} n_{\boldsymbol{v}j} d\mathcal{H}^2 + \int_{\partial\Omega_1} (-\boldsymbol{v})_i \sigma_{ij} n_j d\mathcal{H}^2.$$

Let us choose $\sigma = (\xi_k \otimes \xi_k)\,\phi$, $\sigma_{ij} = (\xi_k)_i (\xi_k)_j \phi$, with no summation w.r.t. k, where ξ_k are the unit vectors in $\boldsymbol{R}^3$ (whose components are $(\xi_k)_i$ with $i = 1, 2, 3$) and $\phi \in C_c^0(\Omega')$. We get for $k = 1, 2, 3$

$$(3.31) \qquad \int_{J_{v_n}} [\boldsymbol{v}_n]_k n_{\boldsymbol{v}_n k} \phi d\mathcal{H}^2 + \int_{\partial\Omega_1} (-\boldsymbol{v}_n)_k n_k \phi d\mathcal{H}^2 \to \int_{J_v} [\boldsymbol{v}]_k n_{\boldsymbol{v}k} \phi d\mathcal{H}^2 + \int_{\partial\Omega_1} (-\boldsymbol{v})_k n_k \phi d\mathcal{H}^2.$$

Now, we are in the position of deducing (3.4). Indeed, after summing up (3.31) w.r.t. k, we have that the normal discontinuities converge in the sense of measures and the weak impenetrability condition holding for $\boldsymbol{v}_n$ pass to the limit as $n \to +\infty$.

REMARK 3.6. *Let us point out that even though from the analytical point of view one could expect to treat boundary conditions* (*as the impenetrability condition*) *in a different way than the impenetrability conditions on the interior traces*, i.e., *using the notion of interior traces of the velocities, the problem shows that it is not the correct way, as the mechanical application suggests. Indeed, we need to extend the domain and to deal with the whole set of impenetrability conditions* (*on the interior fractures and on the boundary*), *to get significant mathematical results. This fact agrees with the mechanical point of view which includes the obstacle in the system, so that the trace of the velocity of the body is a jump of the global velocity* (i.e., *the difference between the velocity of the obstacle and the velocity of the solid which is colliding*). *The same ideas apply for more sophisticated constitutive laws on colliding boundary* $\partial\Omega_1$.

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