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A model of seismic excitation

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Meccanica dei solidi. — *A model of seismic excitation.* Nota (*) del Socio PIERO VILLAGGIO.

ABSTRACT. — The influence of a seismic wave on a building is customarily described as a force, a function of the time, whose explicit expression is prescribed. We here suggest a one-dimensional model able to relate this force to the sudden onset of a fault in the rock layer on which the building is built.

KEY WORDS: Seismic waves; Brittle fracture; Impact.

RIASSUNTO. — *Un modello di eccitazione sismica*. L'influenza di un'onda sismica su un edificio è abitualmente riguardata come una data forza, funzione del tempo, la cui espressione esplicita è assegnata in base a conoscenze statistiche. Si propone qui un modello monodimensionale capace di mettere in relazione questa forza con la formazione improvvisa di una faglia nello strato roccioso su cui è costruito l'edificio.

Engineers concerned with the construction of structures subject to seismic disturbances are not particularly interested in the geological causes of earthquaques but rather in the precise knowledge of the forces acting at the fundation of a building. Specifically, when it is invested by the front of a an elastic wave emanating from a given source. But, in order to simplify calculations, engineers have introduced an even simple method consisting in the substitution of the horizontal seismic action with a horizontal static force. This procedure is known as the «method of the equivalent force» (cf. *e.g.* Wakabayashi [7]).

On the other hand, seismologists, interested in the explanation of the causes of earthquaques and in the description of their propagation through a stratified medium, predict the kinematic features of the phenomenon, the global transport of energy, but cannot estimate the local mechanical effect of a strong seismic perturbation on the small region where a town is built.

We here suggest a simple elastic model for establishing a deterministic relationship between the brittle onset of a fault in an elastic layer, modelled as a rectangular panel, the propagation velocity of the ensuing elastic waves, and the displacement caused by the front of these waves as they impact the fundation of a house, considered as a simple elastic oscillator.

The model we have in mind is the following. The rocky layer is modeled as an elastic rectangular plate of length l, height h and thickness b as sketched in fig. 1. The plate is either confined between two rigid walls (fig. 1*a*) or clamped at its left end and subject to a uniform distribution of tensile stresses, say p, at the right end, which can shift horizontally (fig. 1*b*). In the case (*a*) the rod is gradually uniformly strained by a displacement δ of its right clamp, while, in the case (*b*), the uniform strain is induced by the tensions p, whose resultant is a force P. According to a definition introduced by Ericksen [4], we shall call these two conditions of loading «hard» and «soft», re-

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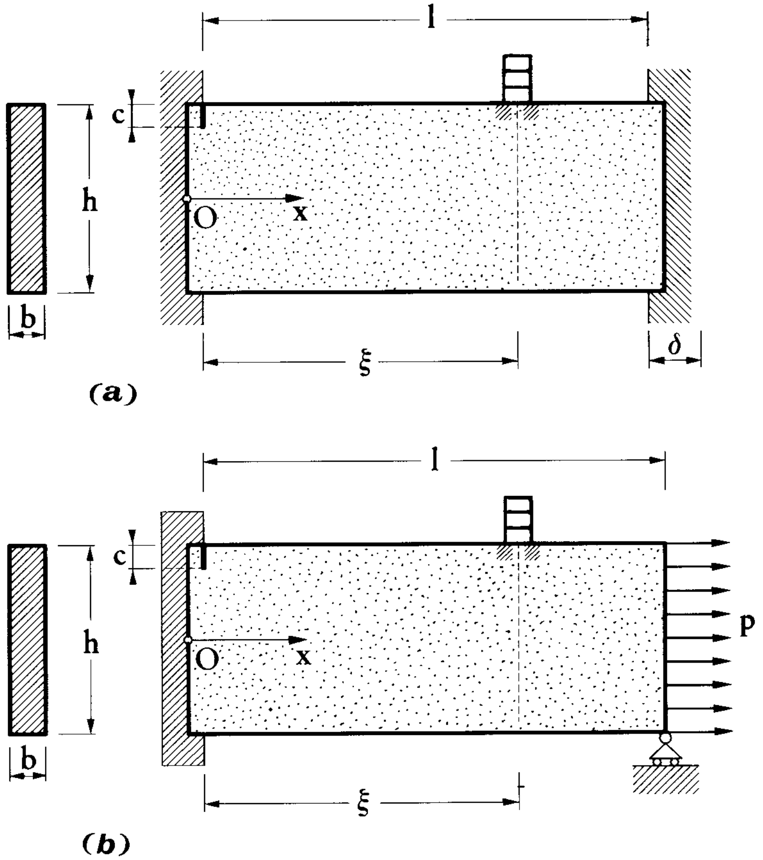

Fig. 1. – Hard and soft device of loading.

spectively. Let us denote the distance of a generic cross section from the left clamp by x, so that the right end occupies the position $x = l$ before the deformation. The linear elastic behaviour of the rock is characterized by Young's modulus E and a Poisson ratio ν.

As long as the stress state in the layer is low it will undergo a homogeneous extensional strain, increasing with the load. But, at a certain level of loading, it may happen that a small fault, modelled as an initially microscopic cut situated at the clamped end as we may always obtain after suitable choice of the origin (fig. 1), instantaneously propagates into the interior of the panel for a length c. If the layer is long the modeling of a fault by such a vertical cut if perfectly plausible (cf. Mandtl [5]). The sudden creation of a vertical fracture releases a longitudinal wave travelling in the positive x-direction. This wave is eventually reflected by the end $x = l$, but, if this point is sufficiently far from the source, $x = 0$, the reflected wave will spend a long time before reaching the source again.

This assumption permit the treatment of the problem through a simplified quasi-static theory proposed by Cox [3] more than 150 years ago, but still very effective in catching some essential features of the solution.

The procedure suggested is energetic. Let us first consider the case of the hard device sketched in fig. 1*a*. The panel is clamped at its end $x = 0$ while the other end $x = l$ is subject to a slowly increasing displacement $\delta(t)$, where t denotes the time. At each instant t, the bar transmits an axial tensile force $N(t) = E\,h\,b\,\delta/l$ and stores the strain energy

$$W_0 = \frac{1}{2} E\,h\,b \frac{\delta^2}{l}\,. \tag{1}$$

As long as this energy is small, the bar will undergo a slow static elongation without causing cracks. But, as soon as W_0 attains a certain critical value, a further possible increase of the strain energy, due to an increment of δ, is converted into an equivalent amount of fracture energy generated by the propagation of the small crack situated at the clamped section $x = 0$ of the panel. This is just an application of Griffith's criterion of brittle fracture.

In order to determine the critical value of δ, let us assume, according to the so called linear theory of fracture, that an advancement c of the fracture localized at the clamp requires a dissipation $U_1 = \gamma c b$ of energy. At the same time the strain energy, as a consequence of the reduction of the height of the left terminal cross-section from h to $(h - c)$, assume the value

$$W_1 = \frac{1}{2} E(h - c)\, b \frac{\delta^2}{l}\,. \tag{2}$$

The equilibrium becomes critical as soon as the equation

$$\frac{\partial}{\partial c}(W_1 + U_1) = 0\,, \tag{3}$$

is satisfied. Equation (3) yields the value δ_{cr} defining the onset of propagation of the crack, namely $\delta_{cr} = \sqrt{\dfrac{2\gamma l}{E\,b\,h}}$.

Let us now consider the case of «catastrofic» detachment [2], in which the initially microscopic fracture istantaneously spreads from the upper to the lower chord of the cross-section as soon as δ attains the critical value δ_{cr}. In this case the initial strain energy W_0, is suddendly converted into kinetic energy T_1 according to the conservation equation

$$W_0 = T_1\,. \tag{4}$$

The kinetic energy, aquired after the detachment, is $T_1 = \frac{1}{2}\int_0^l \varrho\,h\,b\,v^2(x)\,dx$, where ϱ is the density and $v(x)$ the longitudinal velocity of a typical cross-section after the detachment. In equation (4) all terms are known except the function $v(x)$. In order to determine $v(x)$ we apply the approximate Cox' method, according to which dynamic axial displacements are proportional to static displacements [6, §24]. Since the latter are linear, we take

$$v(x) = v_0\left(1 - \frac{x}{l}\right), \tag{5}$$

where v_0 is a constant. With this expression for $v(x)$ we calculate

$$T_1 = \frac{1}{2}\int_0^l \varrho\, h\, b\, v_0^2\left(1 - \frac{x}{l}\right)^2 dx = \frac{1}{2}\varrho\, h\, b\,\frac{l}{3}v_0^2 . \tag{6}$$

Then, from equation (4), we obtain the value

$$v_0^2 = \frac{3E}{\varrho}\,\frac{\delta_{cr}^2}{l^2} = \frac{6\gamma}{\varrho l}, \tag{7}$$

and hence, from formula (5), we can estimate the initial velocity of the ground at the foot of the building as soon as complete detachment has occured:

$$v(\xi) = \sqrt{\frac{6\gamma}{\varrho l}}\left(1 - \frac{\xi}{l}\right). \tag{8}$$

We now examine the case of soft device, shown in fig. 1*b*. Here the energy criterion must be modified since we must account for the work of the exterior increasing force $P(t)$. The total energy stored by the rod, has the form

$$E_0 = W_0 - L_0, \tag{9}$$

where W_0 $\left(W_0 = \frac{1}{2}\,\frac{P^2 l}{E h b}\right)$ is the strain energy and L_0 $\left(L_0 = \frac{P^2 l}{E h b} = 2W_0\right)$ the exterior work. After the propagation of a fracture at the clamp for an extent c the total energy becomes

$$E_1 = W_1 - L_1 + U_1 = -\frac{1}{2}\,\frac{P^2 l}{E(h-c)\,b} + \gamma\, b\, c . \tag{10}$$

Propagation occurs as soon as $\frac{\partial E_1}{\partial c}$ vanishes, that is when P^2 attains the values $P_{cr}^2 = 2\,\frac{E\gamma b^2}{l}(h-c)^2$. But, since we have supposed that the initial crack is very small with respect to h, we obtain the value

$$P_{cr} = \sqrt{\frac{2E\gamma b^2 h^2}{l}} . \tag{11}$$

Assume again that, immediately after reaching the critical state, the crack at the clamp instantaneously traverses the entire cross-section so that the initial energy $E_0 = W_0 - L_0$ is suddendly converted into the energy $E_1 = W_1 - L_1 + T_1$, where $W_1 = 0$, $L_1 = L_0$, and T_1 is the kinetic energy, equal to $\frac{1}{2}\int_0^l \varrho\, h\, b\, v^2(x)\, dx$.

The choice of the function $v(x)$ can be made by applying again Cox' method, and the simplest form of $v(x)$ compatible with the end conditions is $v = v_0 = const$, which implies $T_1 = \frac{1}{2} \varrho\, h\, b\, l\, v_0^2$. Thus, from the equation $W_0 = T_1$, we find

$$v_0^2 = \frac{P_{cr}^2}{\varrho\, E\, h^2\, b^2} = \frac{2\gamma}{\varrho l}\,. \tag{12}$$

All sections of the rod undergo the same velocity $v_0 = \sqrt{\frac{2\gamma}{\varrho l}}$.

Once the istantaneous velocities of the ground are determined either by (8) or (12), the subsequent motion of the building can be exactly predicted by the laws of elementary mechanics, and so are the horizontal forces acting on each store.

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