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Meccanica dei fluidi. — *Start-up of channel-flow of a Bingham fluid initially at rest.* Nota di Irene Daprà e Giambattista Scarpi, presentata (*) dal Socio E. Marchi.

Abstract. — We present an analytical solution of plane motion for a Bingham fluid initially at rest subjected to a suddenly applied constant pressure gradient. Using the Laplace transform we obtain expressions which allow a direct easy calculation of the velocity, of the plug thickness and of the rate of flow as function of time.

Key words: Bingham fluid; Unsteady flow; Channel-flow.

Riassunto. — *Inizio del moto piano per un fluido di Bingham.* In questa *Nota* viene presentata una soluzione analitica per il moto piano laminare di un fluido di Bingham inizialmente in quiete sottoposto ad un gradiente di pressione costante. Si ottengono espressioni per il calcolo della velocità e dell'ampiezza del nucleo solido centrale in funzione del tempo.

1. Introduction

The beginning of the motion in a Bingham fluid has been studied in the past, either for numerical calculation purposes (*e.g.* by Duggins [1], Mitra [2], Hammad [3], Ly and Bellet [4], Al Khatib and Wilson [5], Makarov, Zhdanova and Polozova [6]) or analytically. A fundamental paper of Safronchik [7] gives a general expression for the velocity which requires the solution of a nonlinear integral equation; for the case we are studying Safronchik gives an approximated solution valid only for small value of time. Analitical papers of Glowinski [8], Huilgol and Mena [9] do not give an explicit solution in terms of velocity. A paper of Atabek [10] for axisymmetrical flow contains an error in the setting up of the boundary conditions (see [6]), and that of Amadei and Savage [11] for plane flow contains the same error: the equation of motion in the fluid domain is usable in the region between the wall and the solid plug, which are the boundaries where the non-slip condition and the yield stress respectively have to be imposed; in [11] the condition on the stress is imposed at the layer axis, which is out of the region of validity of the related equation. This has several consequencies on the final results: *e.g.*, the expression for the velocity in the fluid region does not depend on the thickness of the solid core. An extensive reference is in the review article by Bird, Dai and Yarusso [12].

The present work gives an analytical solution for the velocity, the plug thickness, the shear stress and the rate of flow as a function of time that is easily suitable for numerical calculations.

(*) Nella seduta del 9 gennaio 2004.

2. Problem statement

We consider a plane horizontal layer of constant thickness $2h$ filled with a Bingham fluid, initially at rest; at time $t = 0$ we apply a constant pressure gradient, strong enough to start a laminar motion. For a rectilinear flow, taking into account the continuity equation, the equation of motion can be written:

$$-\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} = \varrho \frac{\partial u}{\partial t} \tag{1}$$

were x is the direction of the motion, y the normal to the layer $(-h \leqslant y \leqslant h)$, p the pressure, τ the shear stress (τ_{xy}), u the velocity, ϱ the fluid density and t the time. The relation between shear stress and shear rate for a Bingham fluid in plane horizontal laminar flow is

$$\tau = \tau_0 \operatorname{sign}\left(\frac{\partial u}{\partial y}\right) + \mu \frac{\partial u}{\partial y} \tag{2}$$

where τ_0 is the yield stress and μ is the viscosity.

Putting $-\dfrac{\partial p}{\partial x} = P_0 > 0$ and introducing the following dimensionless quantities: $\xi = x/h$, $\eta = y/h$, $\nu = u\mu/h^2 P_0$, $\theta = \tau/hP_0$, $\theta_0 = \tau_0/hP_0$, $T = t\mu/\varrho h^2$, (1) and (2) become:

$$\mathrm{H}(T) + \frac{\partial \theta}{\partial \eta} = \frac{\partial \nu}{\partial T} \tag{3}$$

where $\mathrm{H}(T)$ is the Heaviside unit-step function, and

$$\theta = \theta_0 \operatorname{sign}\left(\frac{\partial \nu}{\partial \eta}\right) + \frac{\partial \nu}{\partial \eta}\,. \tag{4}$$

Initial and boundary conditions are respectively:

$$\nu(\eta,\, T < 0) = 0 \tag{5}$$

and

$$\nu(\eta = \pm 1,\, T) = 0 \tag{6}$$

(no-slip condition at the wall).

We suppose $P_0 > \tau_0/h$ so that the fluid moves in the positive ξ direction; because of symmetry, we may study the problem only in the region defined by $0 \leqslant \eta \leqslant 1$ where $\partial\nu/\partial\eta$ and θ assume negative values.

If we call η_0 the local semi-amplitude of the plug, which obviously depends on T, we obtain from (4) $\theta = -\theta_0$ for $\eta = \eta_0(T)$ and thus another boundary condition:

$$\left.\frac{\partial \nu}{\partial \eta}\right|_{\eta = \eta_0(T)} = 0\,. \tag{7}$$

Writing the equation of motion for the solid plug it is easy to verify that the stress depends linearly on the distance from the ξ axis:

$$\frac{\partial \theta}{\partial \eta} = -\frac{\theta_0}{\eta_0}\,. \tag{8}$$

Deriving (4) in respect to η we obtain

$$\frac{\partial\theta}{\partial\eta} = \frac{\partial^2 \nu}{\partial\eta^2}\,. \tag{9}$$

Equation (3) becomes then

$$\mathrm{H}(T) + \frac{\partial^2 \nu}{\partial\eta^2} = \frac{\partial\nu}{\partial T}\,. \tag{10}$$

3. Solution

We take the Laplace transform of (10) with respect to T; putting

$$\widehat{\nu}(\eta, s) = \mathscr{L}[\nu(\eta, T)] = \int_0^\infty \nu(\eta, T)\, e^{-sT}\,\mathrm{d}T$$

and recalling (5) we obtain

$$\frac{\partial^2 \widehat{\nu}}{\partial\eta^2} - s\widehat{\nu} = -\frac{1}{s}\,. \tag{11}$$

The general solution is then

$$\widehat{\nu}(\eta, s) = \frac{1}{s^2} + F\,\mathrm{Sinh}\,(\eta\sqrt{s}) + G\,\mathrm{Cosh}\,(\eta\sqrt{s}) \tag{12}$$

where F and G are any function of s.

Equation (12), which is applicable only where the fluid's behaviour is Newtonian ($\eta_0 \leqslant \eta \leqslant 1$), must satisfy the no-slip condition at the wall

$$\widehat{\nu}(1, s) = 0 \tag{13}$$

and the condition $\theta = -\theta_0$ as $\eta = \eta_0(T)$, *i.e.*

$$\left.\frac{\partial\widehat{\nu}}{\partial\eta}\right|_{\eta = \eta_0(T)} = 0\,. \tag{14}$$

Equation (10) with conditions (5) (6) and (7) may be considered as the governing law of a linear system with time-varying parameters (see *e.g.* [13]), where the parameter is $\eta_0(T)$, and thus $\widehat{\nu}(\eta, s)$ is the correspondent system-function.

Conditions (13) and (14) give

$$\begin{cases} F\,\mathrm{Sinh}\,(\sqrt{s}) + G\,\mathrm{Cosh}\,(\sqrt{s}) = -\dfrac{1}{s^2} \\ F\,\mathrm{Cosh}\,(\eta_0\sqrt{s}) + G\,\mathrm{Sinh}\,(\eta_0\sqrt{s}) = 0 \end{cases} \tag{15}$$

and then

$$F = -\frac{1}{s^2}\,\frac{\mathrm{Sinh}\,(\eta_0\sqrt{s})}{\varDelta} \qquad G = \frac{1}{s^2}\,\frac{\mathrm{Cosh}\,(\eta_0\sqrt{s})}{\varDelta}$$

where

$$\Delta = -\operatorname{Cosh}[(1-\eta_0)\sqrt{s}]\,. \tag{16}$$

Solution (12) becomes then

$$\widehat{v}(\eta, s) = \frac{1}{s^2}\,\frac{\operatorname{Cosh}[(1-\eta_0)\sqrt{s}] - \operatorname{Cosh}[(\eta-\eta_0)\sqrt{s}]}{\operatorname{Cosh}[(1-\eta_0)\sqrt{s}]}\,. \tag{17}$$

We can retrieve $v(\eta, T)$ by inverting the Laplace transform by means of the integral

$$v(\eta, T) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sT}\,\widehat{v}(\eta, s)\,\mathrm{d}s\,. \tag{18}$$

Expression (17) has a first order pole in $s=0$ and a countable infinity of first order poles when

$$\operatorname{Cosh}[(1-\eta_0)\sqrt{s}] = 0\,;$$

thus when

$$s = s_k = -\frac{\pi^2}{4}\left(\frac{2k+1}{1-\eta_0}\right)^2 \tag{19}$$

where $k = 0, 1, 2, \ldots$.

Applying to (18) the residue theorem we have

$$v(\eta, T) = \sum \operatorname{Res}[e^{sT}\,\widehat{v}(\eta, s)]\,.$$

For $s=0$ we obtain

$$\operatorname{Res}[e^{sT}\,\widehat{v}(\eta, s)]_{s=0} = \lim_{s\to 0} s\widehat{v}(\eta, s)\,e^{sT} = \frac{1}{2}[(1-\eta_0)^2 - (\eta-\eta_0)^2]\,,$$

and for $s = s_k$

$$\operatorname{Res}[e^{sT}\,\widehat{v}(\eta, s)]_{s=s_k} = \lim_{s\to s_k}(s-s_k)\,\widehat{v}(\eta, s)\,e^{sT} =$$

$$= -(-1)^k\,\frac{16(1-\eta_0)^2}{\pi^3(2k+1)^3}\exp\left[-\frac{\pi^2}{4}\left(\frac{2k+1}{1-\eta_0}\right)^2 T\right]\cos\left[\frac{\eta-\eta_0}{1-\eta_0}\,\frac{\pi}{2}(2k+1)\right].$$

Consequently

$$v(\eta, T) = \frac{1}{2}[(1-\eta_0)^2 - (\eta-\eta_0)^2] - \tag{20}$$

$$-\frac{16(1-\eta_0)^2}{\pi^3}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}\exp\left[-\frac{\pi^2}{4}\left(\frac{2k+1}{1-\eta_0}\right)^2 T\right]\cos\left[\frac{\eta-\eta_0}{1-\eta_0}\,\frac{\pi}{2}(2k+1)\right];$$

equation (20) allows to calculate $v(\eta, T)$ at time T when the plug amplitude is $\eta_0(T)$.

To obtain $\eta_0(T)$ we observe that at the interface, *i.e.* as $\eta = \eta_0$, the acceleration, and thus $\partial\theta/\partial\eta$ must be continuous; then, recalling (8) and (9):

$$\frac{\partial^2 v}{\partial \eta^2}\bigg|_{\eta=\eta_0^+} = \frac{\partial \theta}{\partial \eta}\bigg|_{\eta=\eta_0^+} = \frac{\partial \theta}{\partial \eta}\bigg|_{\eta=\eta_0^-} = -\frac{\theta_0}{\eta_0}\,. \tag{21}$$

From (20) we have

$$\frac{\partial^2 v(\eta, T)}{\partial \eta^2} = -1 + \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\exp\left[-\frac{\pi^2}{4}\left(\frac{2k+1}{1-\eta_0}\right)^2 T\right]\cos\left[\frac{\eta-\eta_0}{1-\eta_0}\,\frac{\pi}{2}(2k+1)\right]$$

which in $\eta = \eta_0$ gives

$$\frac{\partial^2 v(\eta, T)}{\partial \eta^2}\bigg|_{\eta_0^+} = -1 + \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\exp\left[-\frac{\pi^2}{4}\left(\frac{2k+1}{1-\eta_0}\right)^2 T\right]. \tag{22}$$

Inserting (22) in (21), we have

$$\eta_0(T) = \frac{-\theta_0}{\dfrac{\partial^2 v}{\partial \eta^2}\bigg|_{\eta_0}} = \theta_0\left\{1 - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)}\exp\left[-\frac{\pi^2}{4}\left(\frac{2k+1}{1-\eta_0}\right)^2 T\right]\right\}^{-1}; \tag{23}$$

we may either assign a value to T and then solve (23) numerically in respect to $\eta_0(T)$ or assign a value to $\eta_0(T)$ and solve it in respect to T; as $T \to \infty$ $\eta_0 \to \eta_\infty = \theta_0$.

If $T \to 0^+$ and thus $\eta_0 \to 1$ (21) gives

$$\frac{\partial^2 v}{\partial \eta^2}\bigg|_{\eta_0} = -\theta_0$$

and thus we obtain

$$\lim_{T\to 0}\frac{\partial^2 v(\eta, T)}{\partial \eta^2}\bigg|_{\eta_0} = -1 + \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\exp\left[-\frac{\pi^2(2k+1)^2}{4}\,\frac{T}{(1-\eta_0)^2}\right] = -\theta_0. \tag{24}$$

The term $T/(1-\eta_0)^2$ tends to a limit, which depends on θ_0, as $T \to 0$ and $\eta_0 \to 1$; it is easy to verify that for small values of time T $\eta_0 \simeq 1 - C\sqrt{T}$, as Safronchik demonstrated in his paper [7]. Calculating numerically from (24) the value of $z = \lim_{T\to 0}[T/(1-\eta_0)^2]$ as a function of θ_0 we obtain the plot of fig. 1: if $\theta_0 = 0$ (Newtonian fluid) $z = 0$, and if $\theta_0 \to 1$ (no motion is possible, because the fluid behaves as a solid) $z \to \infty$.

As $T \to \infty$ (20) gives the known asymptotic velocity distribution for $\eta \geq \eta_\infty$, where η_∞ is the asymptotic value of η_0

$$\lim_{T\to\infty} v(\eta, T) = v_\infty(\eta) = \frac{1}{2}[(1-\eta_\infty)^2 - (\eta-\eta_\infty)^2]\,. \tag{25}$$

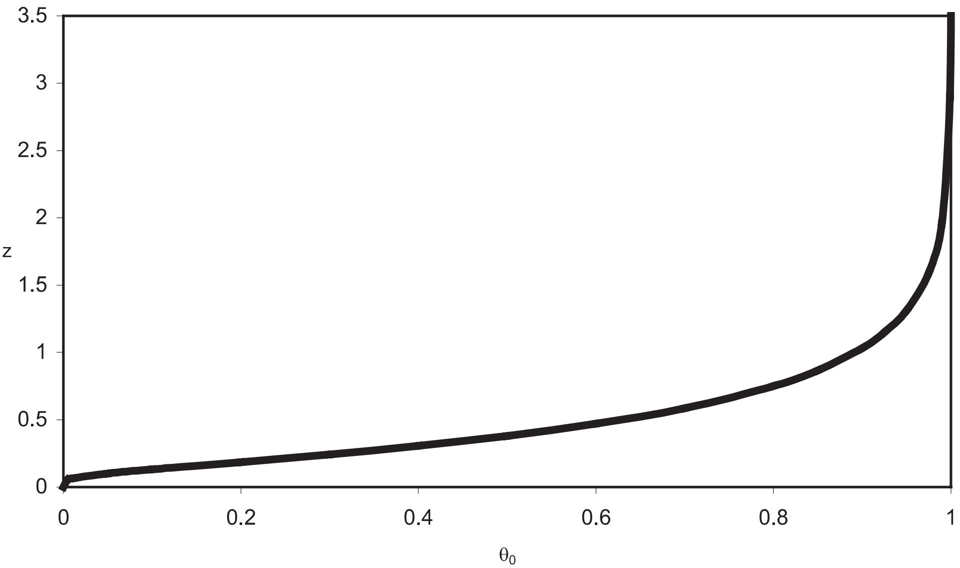

Fig. 1. – $z = \lim_{T \to 0} [T/(1-\eta_0)^2]$ versus θ_0.

Analogously, if we let $T \to 0$ in equation (20) and recall that it is usable only in the region $\eta \geqslant \eta_0 \to 1$, we obtain $v = 0$; using (20) and (23) we may easily calculate the velocity profile for $\eta \geqslant \eta_0$.

We evaluate the rate of flow Q, which is made up of the contribution Q_1 of the solid plug (of amplitude $2\eta_0$):

$$Q_1(T) = 2\eta_0 v(\eta_0, T) =$$

$$= \eta_0 \left\{ (1-\eta_0)^2 - \frac{32(1-\eta_0)^2}{\pi^3} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} \exp\left[-\frac{\pi^2}{4} \left(\frac{2k+1}{1-\eta_0} \right)^2 T \right] \right\}$$

and of the contribution Q_2 of the fluid region:

$$Q_2(T) = 2 \int_{\eta_0}^{1} v(\eta, T)\, d\eta =$$

$$= \frac{2}{3}(1-\eta_0)^3 - \frac{64(1-\eta_0)^3}{\pi^4} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} \exp\left[-\frac{\pi^2}{4} \left(\frac{2k+1}{1-\eta_0} \right)^2 T \right];$$

and thus

$$(26) \qquad Q(T) = Q_1(T) + Q_2(T) = (1-\eta_0)^2 \left(\frac{2+\eta_0}{3} \right) -$$

$$- \frac{32(1-\eta_0)^2}{\pi^3} \sum_{k=0}^{\infty} \left\{ \left[\eta_0(-1)^k + \frac{2(1-\eta_0)}{(2k+1)\pi} \right] \frac{1}{(2k+1)^3} \exp\left[-\frac{\pi^2}{4} \left(\frac{2k+1}{1-\eta_0} \right)^2 T \right] \right\};$$

as $T \to \infty$ (26) gives the asymptotic value Q_∞ of the rate of flow:

$$Q_\infty = \lim_{T\to\infty} Q(T) = (1-\eta_\infty)^2\left(\frac{2+\eta_\infty}{3}\right).$$

Using (4) we obtain the value of the shear stress in the Newtonian region:

$$\theta = -\theta_0 - \left\{(\eta-\eta_0) - \frac{8}{\pi^2}(1-\eta_0)\cdot \right.$$
$$\left. \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \exp\left[-\frac{\pi^2}{4}\left(\frac{2k+1}{1-\eta_0}\right)^2 T\right] \sin\left[\frac{\eta-\eta_0}{1-\eta_0}\frac{\pi}{2}(2k+1)\right]\right\}. \tag{27}$$

If $\theta_0 = 0$ *i.e.* if the fluid is a Newtonian one, equations (20), (26) and (27) give the known results for Newtonian fluids.

4. Final Remarks

Equations (20) and (23) which gives the analytical solution of the problem, allow an easy calculation of the velocity profile and of the thickness of the plug: equations (26) and (27) allow the calculation of the rate of flow in the whole layer, and of the shear stress in the Newtonian region respectively. As an example, we have drawn the

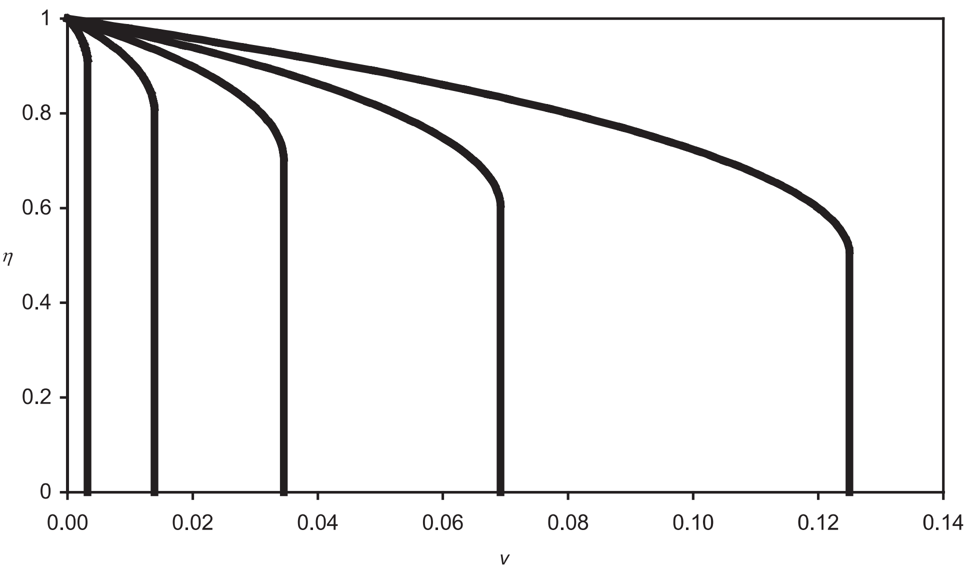

Fig. 2. – Velocity profile: yield stress $\theta_0 = 0.5$, asymptotic plug semi-amplitude $\eta_\infty = 0.5$ (dimensionless variables); from left to right: $\eta_0 = 0.9$ $T = 0.00426$, $\eta_0 = 0.8$ $T = 0.0198$, $\eta_0 = 0.7$ $T = 0.0545$, $\eta_0 = 0.6$ $T = 0.0132$, $\eta_0 = 0.5$ $T = \infty$.

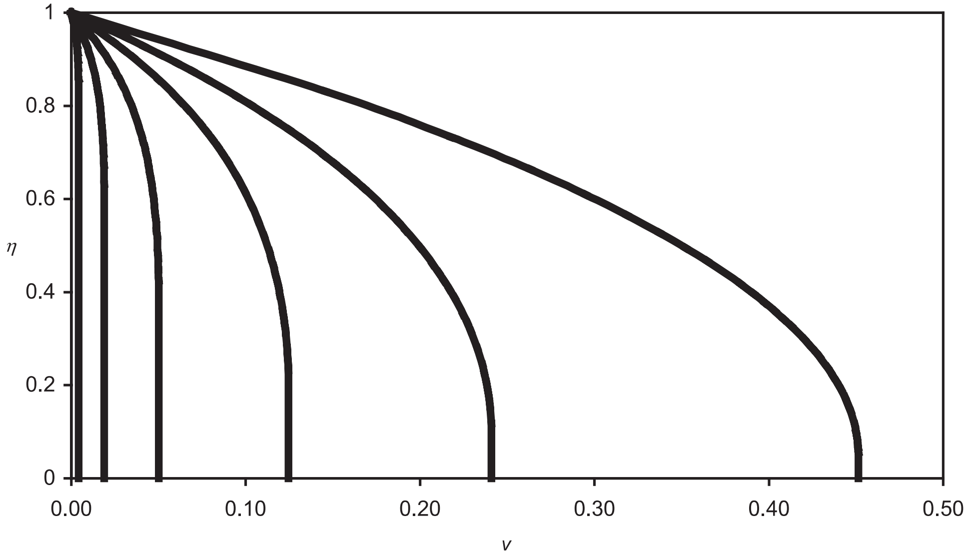

Fig. 3. – Velocity profile: yield stress $\theta_0 = 0.05$, asymptotic plug semi-amplitude $\eta_\infty = 0.05$ (dimensionless variables); from left to right: $\eta_0 = 0.8$ $T = 0.00431$, $\eta_0 = 0.6$ $T = 0.0193$, $\eta_0 = 0.4$ $T = 0.0519$, $\eta_0 = 0.2$ $T = 0.0136$, $\eta_0 = 0.1$ $T = 0.307$, $\eta_0 = 0.05$ $T = \infty$.

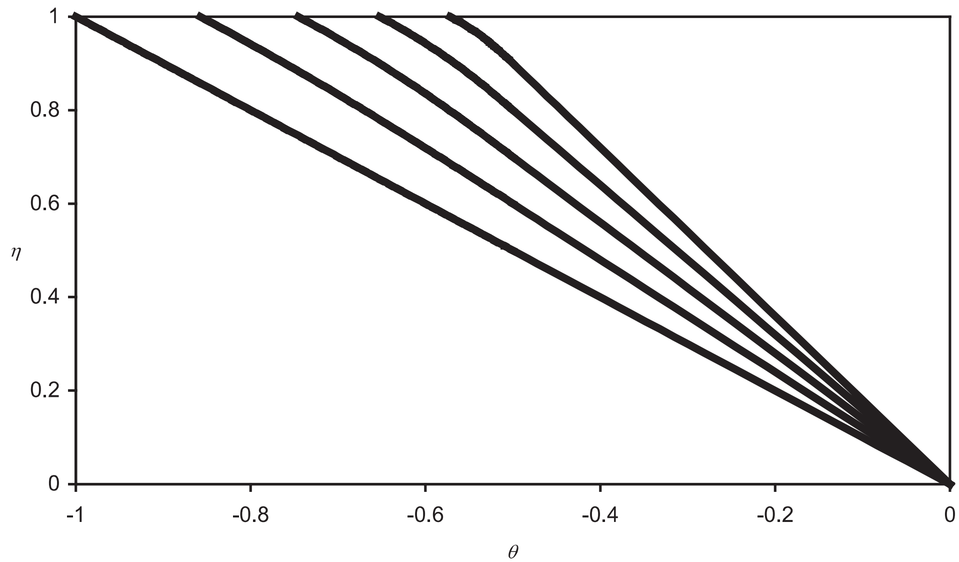

Fig. 4. – Shear stress profile; yield stress $\theta_0 = 0.5$, asymptotic plug semi-amplitude $\eta_\infty = 0.5$ (dimensionless variables); from right to left: $\eta_0 = 0.9$ $T = 0.00426$, $\eta_0 = 0.8$ $T = 0.0198$, $\eta_0 = 0.7$ $T = 0.0545$, $\eta_0 = 0.6$ $T = 0.0132$, $\eta_0 = 0.05$ $T = \infty$.

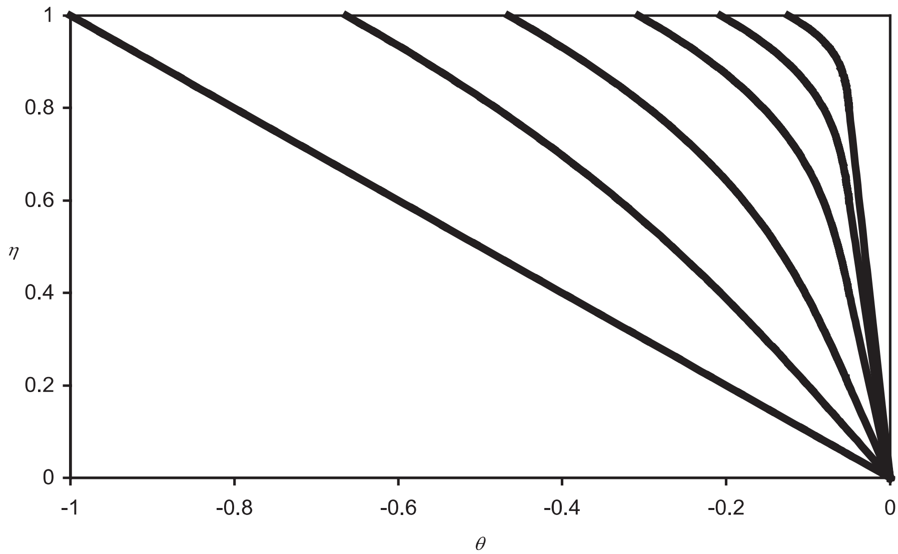

Fig. 5. – Shear stress profile: yield stress $\theta_0 = 0.05$, asymptotic plug semi-amplitude $\eta_\infty = 0.05$ (dimensionless variables); from right to left: $\eta_0 = 0.8$ $T = 0.00431$, $\eta_0 = 0.6$ $T = 0.0193$, $\eta_0 = 0.4$ $T = 0.0519$, $\eta_0 = 0.2$ $T = 0.0136$, $\eta_0 = 0.01$ $T = 0.307$, $\eta_0 = 0.05$ $T = \infty$.

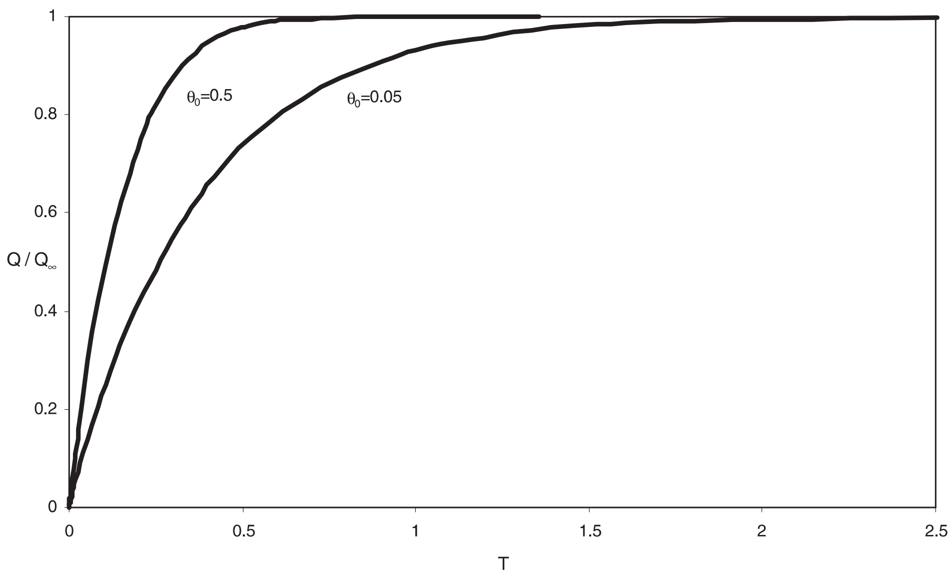

Fig. 6. – Semi-amplitude of η_∞/η_0 versus time T for two values of the yield stress: $\theta_0 = 0.5$ and $\theta_0 = 0.05$ (dimensionless variables).

velocity (figs. 2, 3) and stress profiles (figs. 4, 5) for two values of the dimensionless yield stress $\theta_0 = 0.5$ and $\theta_0 = 0.05$. In fig. 6 and in fig. 7 we have plotted the values of η_∞/η_0 and of Q/Q_∞ respectively, as function of T for the same values of θ_0.

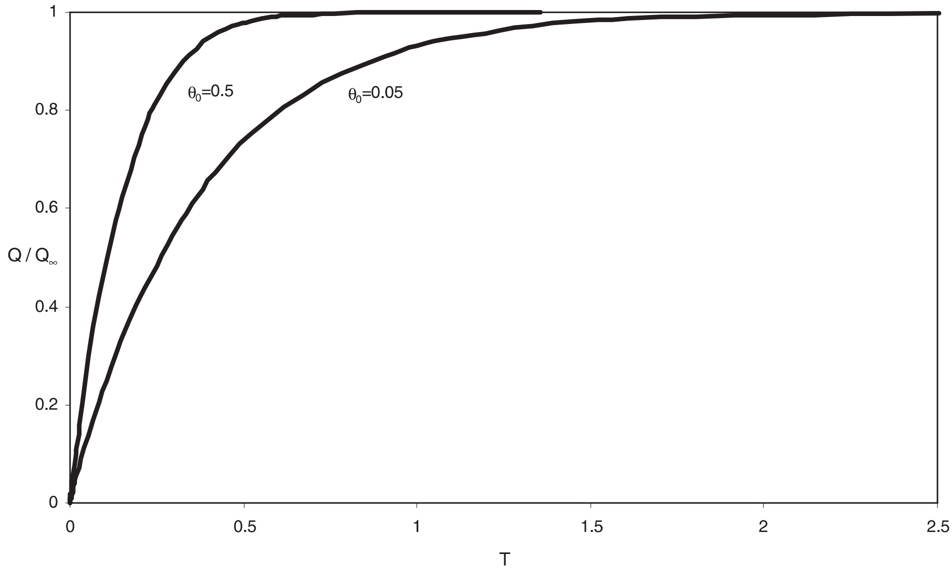

Fig. 7. – Rate of flow Q/Q_∞ versus time T for two values of the yield stress: $\theta_0 = 0.5$ and $\theta_0 = 0.05$ (dimensionless variables).

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