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A plane problem of incompressible magnetohydro-dynamics with viscosity and resistivity depending on the temperature

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Magnetofluidodinamica. — *A plane problem of incompressible magnetohydrodynamics with viscosity and resistivity depending on the temperature.* Nota di Giovanni Cimatti, presentata (*) dal Socio P. Villaggio.

Abstract. — The plane flow of a fluid obeying the equations of magnetohydrodynamics is studied under the assumption that both the viscosity and the resistivity depend on the temperature. Some results of existence, non-existence, and uniqueness of solution are proved.

Key words: Incompressible magnetohydrodynamics; Thermistor problem; Existence of solutions.

Riassunto. — *Un problema piano di magnetoidrodinamica incomprimibile con viscosità e resistività dipendenti dalla temperatura.* Si studia il moto piano di un fluido retto dalle equazioni della magnetoidrodinamica supponendo che tanto la viscosità quanto la resistività dipendano dalla temperatura. Si dimostrano alcuni risultati di esistenza, nonesistenza e unicità di soluzioni.

1. Introduction

The equations of magnetohydrodynamics, for a viscous, incompressible and resistive fluid, read in the stationary case and in non-dimensional form:

(1.1) $$\nabla\cdot \boldsymbol{H} = 0, \quad \nabla\cdot \boldsymbol{v} = 0,$$

(1.2) $$\nabla \times (\varrho\nabla \times \boldsymbol{H}) + (\boldsymbol{H}\cdot\nabla)\, \boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\, \boldsymbol{H} = 0,$$

(1.3) $$(\boldsymbol{v}\cdot\nabla) = -\nabla p + S(\boldsymbol{H}\cdot\nabla)\, \boldsymbol{H} + \nabla\cdot(\nu\nabla \boldsymbol{v}),$$

where $\boldsymbol{H}$ is the magnetic field and $\boldsymbol{v}$ the velocity. The non-dimensional quantity ν and ϱ, related to the viscosity and electric resistivity, are usually supposed constant. Now in a real situation (*e.g.* liquid metals like sodium or mercury) they depend strongly on the temperature u. In this paper we suppose ν, ϱ, and also the thermal conductivity κ, to be given continuous functions of the temperature:

(1.4) $$\nu = \nu(u), \quad \varrho = \varrho(u), \quad \kappa = \kappa(u).$$

In addition we have the energy equation:

(1.5) $$-\nabla\cdot(\kappa\nabla u) = \varrho(\nabla \times \boldsymbol{H})^2 + \nu \sum_{i,\,k=1}^{3} (v_{i,\,k} + v_{k,\,i})\, v_{i,\,k}.$$

We treat the plane case of an indefinite cylinder of cross section Ω, an open and bounded subset of $\boldsymbol{R}^2$ with a regular boundary Γ composed of two disjoint parts Γ_1, Γ_2. The velocity and the magnetic field are accordingly assumed to be of the form

(1.6) $$\boldsymbol{H} = h(x_1, x_2)\, \boldsymbol{i}_3, \quad \boldsymbol{v} = v(x_1, x_2)\, \boldsymbol{i}_3,$$

(*) Nella seduta del 12 dicembre 2003.

where $\boldsymbol{i}_3$ is the unit vector coinciding with the axis of the cylinder. Since in the present situation $\nabla \times \boldsymbol{H} = -\nabla h \times \boldsymbol{i}_3$, it is easily verified that eqns. (1.1), (1.2), and (1.6) reduce to

$$\nabla \cdot (\varrho(u)\, \nabla h) = 0, \tag{1.7}$$

$$\nabla \cdot (\nu(u)\, \nabla v) = 0, \tag{1.8}$$

$$-\nabla \cdot (\kappa(u)\, \nabla u) = \varrho(u)\, |\nabla h|^2 + \nu(u)\, |\nabla v|^2. \tag{1.9}$$

The first term in the R.H.S. of (1.9) corresponds to the heating in the fluid due to the Joule effect and the second term to the viscous production of heat. In the dielectric medium outside the cylinder the current density $\boldsymbol{J}$ vanishes; thus we have there, by the Maxwell's equations, $\nabla h = 0$ (*i.e.* the magnetic field is constant), and the appropriate boundary conditions are:

$$h = h_1 \text{ on } \Gamma_1, \quad h = h_2 \text{ on } \Gamma_2, \tag{1.10}$$

where h_1 and h_2 are given constants, and

$$v = \hat{v}, \quad u = \hat{u} \text{ on } \Gamma, \tag{1.11}$$

with $\hat{v}$ and $\hat{u}$ given functions. The boundary value problem (1.7)-(1.11) is similar to the thermistor problem (see [1, 2, and references therein]).

In Section 2 we prove that, under the assumption $\varrho(u) \in L^\infty(\boldsymbol{R}^1)$, $\nu(u) \in L^\infty(\boldsymbol{R}^1)$, $\kappa(u) \in L^\infty(\boldsymbol{R}^1)$ and

$$\varrho_M \geq \varrho(u) \geq \varrho_m > 0, \quad \nu_M \geq \nu(u) \geq \nu_m > 0, \quad \kappa_M \geq \kappa(u) \geq \kappa_m > 0, \tag{1.12}$$

for all $u \in \boldsymbol{R}^1$, there exists at least one solution to problem (1.7)-(1.11). In Section 3 we examine a special case in which uniqueness can be proved but, depending on the value of the integral

$$\int_0^\infty \frac{\kappa(t)}{\varrho(t)}\, dt,$$

the solution may not exist.

2. Existence of weak solutions

Using eqns. (1.7) and (1.8), the energy eqn. (1.9) can be rewritten as follows:

$$-\nabla \cdot (\kappa(u)\, \nabla u) = \nabla \cdot (h\varrho(u)\, \nabla h) + \nabla \cdot (v\nu(u)\, \nabla v). \tag{2.1}$$

Let $\tilde{u}$, $\tilde{v}$ and $\tilde{h}$ be functions such that

$$\tilde{h} \in H^2(\Omega), \quad \tilde{v} \in H^2(\Omega), \quad \tilde{u} \in H^2(\Omega), \tag{2.2}$$

and

$$\tilde{h} = h_1 \text{ on } \Gamma_1, \quad \tilde{h} = h_2 \text{ on } \Gamma_2, \quad \tilde{v} = \hat{v} \text{ on } \Gamma, \quad \tilde{u} = \hat{u} \text{ on } \Gamma. \tag{2.3}$$

Defining $U = u - \tilde{u}$ in terms of U, eqn. (2.1) becomes

$$-\nabla \cdot (\kappa(u)\, \nabla U) = \nabla \cdot (\kappa(u)\, \nabla \tilde{u}) + \nabla \cdot (h\varrho(u)\, \nabla h) + \nabla \cdot (v\nu(u)\, \nabla v), \tag{2.4}$$

with the boundary condition

$$U = 0 \text{ on } \Gamma. \tag{2.5}$$

We formulate problem (1.7), (1.8), (2.2), (2.4) and (2.5) in the following weak form: to find the functions $h \in H^1(\Omega) \cap L^\infty(\Omega)$, $v \in H^1(\Omega) \cap L^\infty(\Omega)$ and $U \in H_0^1(\Omega)$ such that the equations

$$h - \tilde{h} \in H_0^1(\Omega), \quad \int_\Omega \varrho(U + \tilde{u})\, \nabla h \cdot \nabla \xi dx = 0, \quad \forall \xi \in H_0^1(\Omega), \tag{2.6}$$

$$v - \tilde{v} \in H_0^1(\Omega), \quad \int_\Omega \nu(U + \tilde{u})\, \nabla v \cdot \nabla \xi dx = 0, \quad \forall \xi \in H_0^1(\Omega), \tag{2.7}$$

$$\begin{aligned} U \in H_0^1(\Omega), \quad \int_\Omega \kappa(U + \tilde{u})\, \nabla U \cdot \nabla \xi dx &= -\int_\Omega \kappa(U + \tilde{u})\, \nabla \tilde{u} \cdot \nabla \xi dx - \\ &- \int_\Omega h\varrho(U + \tilde{u})\, \nabla h \cdot \nabla \xi dx - \int_\Omega v\nu(U + \tilde{u})\, \nabla v \cdot \nabla \xi dx, \quad \forall \xi \in H_0^1(\Omega). \end{aligned} \tag{2.8}$$

are satisfied.

THEOREM 2.1. *If* (1.12) *and* (2.2) *hold, then there exists at least one solution to problem* (2.6)-(2.8).

PROOF. Let $\{w_k\}_{k=1}^\infty$ be a regular basis of $H_0^1(\Omega)$. Define

$$U_m(x) = \sum_{k=1}^m c_k w_k(x), \quad m \in \mathbf{N} \tag{2.9}$$

where $(c_1, \ldots, c_m) \in \mathbf{R}^m$. Using standard results of existence, uniqueness and regularity of the linear elliptic theory we can solve the problems:

$$\nabla \cdot (\varrho(U_m + \tilde{u})\, \nabla h_m) = 0 \text{ in } \Omega, \quad h_m = \tilde{h} \text{ on } \Gamma, \tag{2.10}$$

$$\nabla \cdot (\nu(U_m + \tilde{u})\, \nabla v_m) = 0 \text{ in } \Omega, \quad v_m = \tilde{v} \text{ on } \Gamma. \tag{2.11}$$

We have $h_m, v_m \in H^2(\Omega)$ and, by the weak maximum principle,

$$\|h_m\|_{L^\infty(\Omega)} \leq \tilde{h}_M, \quad \|v_m\|_{L^\infty(\Omega)} \leq \tilde{v}_M, \tag{2.12}$$

where $\tilde{h}_M = \sup\{\tilde{h}(x), x \in \Gamma\}$ and $\tilde{v}_M = \sup\{\tilde{v}(x), x \in \Gamma\}$. Multiplying (2.10) by $h - \tilde{h}$ and integrating by parts over Ω, we have

$$\begin{aligned} \int_\Omega \varrho(U_m + \tilde{u})\, |\nabla h_m|^2 dx &= \int_\Omega \varrho(U_m + \tilde{u})\, \nabla h_m \cdot \tilde{\nabla} h dx \leq \\ &\leq \left(\int_\Omega \varrho(U_m + \tilde{u})\, |\nabla h_m|^2 dx \right)^{1/2} \left(\int_\Omega \varrho(U_m + \tilde{u})\, |\nabla \tilde{h}|^2 dx \right)^{1/2}. \end{aligned}$$

This implies the estimate

$$\|\nabla h\ m\| \leq C_1. \tag{2.13}$$

In the same way we obtain from (2.11)

$$\|\nabla v_m\| \leq C_2, \tag{2.14}$$

where the constants C_1 and C_2 do not depend on m.

Let us now consider the following nonlinear system of m equations in the unknown $(c_1, \ldots, c_m) \in \boldsymbol{R}^m$

$$\int_\Omega \kappa(U_m + \tilde{u})\, \nabla U_m \cdot \nabla w_k\, dx = -\int_\Omega \kappa(U_m + \tilde{u})\, \nabla \tilde{u} \cdot \nabla w_k\, dx - \int_\Omega h_m \varrho(U_m + \tilde{u})\, \nabla h_m \cdot \nabla w_k\, dx - \int_\Omega v_m \nu(U_m + \tilde{u})\, \nabla v_m \cdot \nabla w_k\, dx, \quad k = 1, \ldots, m. \tag{2.15}$$

To prove that (2.15) has at least one solution, we use the Browder fixed point theorem. Define the map $\mathcal{F}\colon \boldsymbol{R}^m \to \boldsymbol{R}^m$ by $a_k = \mathcal{F}_k(c_1, \ldots, c_m)$, $k = 1, \ldots, m$, where

$$a_k = -\int_\Omega [\nabla \cdot (\kappa(U_m + \tilde{u})\, \nabla U_m) + \nabla \cdot (\kappa(U_m + \tilde{u})) + \nabla \cdot (h_m \varrho(U_m + \tilde{u})\, \nabla h_m) + \nabla \cdot (v_m \nu(U_m + \tilde{u})\, \nabla v_m)]\, w_k\, dx,$$

where U_m is given by (2.9) and h_m, v_m by (2.10) and (2.11). We have

$$\sum_{k=1}^m a_k c_k = \int_\Omega [\kappa(U_m + \tilde{u})\, |\nabla U_m|^2 + \kappa(U_m + \tilde{u})\, \nabla \tilde{u} \cdot \nabla U_m + h_m \varrho(U_m + \tilde{u})\, \nabla h_m \cdot \nabla U_m + v_m \nu(U_m + \tilde{u})\, \nabla v_m \cdot \nabla U_m]\, dx.$$

Moreover:

$$\left| \int_\Omega \kappa(U_m + \tilde{u})\, \nabla \tilde{u} \cdot \nabla U_m\, dx \right| \leq \kappa_M \|\nabla \tilde{u}\| \|\nabla U_m\| \leq C_3 \|\nabla U_m\|,$$

$$\left| \int_\Omega h_m \varrho(U_m + \tilde{u})\, \nabla h_m \cdot \nabla U_m\, dx \right| \leq \tilde{h}_M \varrho_M \|\nabla h_m\| \|\nabla U_m\| \leq C_4 \|\nabla U_m\|,$$

$$\left| \int_\Omega v_m \nu(U_m + \tilde{u})\, \nabla v_m \cdot \nabla U_m\, dx \right| \leq \tilde{v}_M \nu_M \|\nabla v_m\| \|\nabla U_M\| \leq C_5 \|\nabla U_m\|.$$

Thus, from (2.16) we obtain

$$\sum_{k=1}^m a_k c_k \geq \kappa_u \|\nabla U_m\|^2 - (C_3 + C_4 + C_5) \|\nabla U_m\|.$$

Hence there exists a positive constant K such that if $\sum_{k=1}^m c_k^2 = K$ we have $\sum_{k=1}^m a_k c_k \geq 0$. This implies (see [4, p. 53]) that (2.15) has at least one solution $(\bar{c}_1, \ldots \bar{c}_m)$.

Multiplying (2.15) by $\bar{c}_k$ and adding over $k = 1, \ldots, m$ we obtain

$$\|\nabla U_m\| \leq C_6. \tag{2.16}$$

By (2.13), (2.14) and (2.16) we can extract from $\{h_m\}$, $\{v_m\}$ and $\{u_m\}$ subsequences (not relabelled) such that

$$(2.17)\qquad (u_m, h_m, v_m) \to (u, h, v) \text{ weakly in } H^1(\Omega)$$

$$(u_m, h_m, v_m) \to (u, h, v) \text{ strongly in } L^p(\Omega), \quad 1 \leqslant p < \infty \text{ and a.e. in } \Omega.$$

This permits to pass to the limit for $m \to \infty$ in the equations

$$\int_\Omega \varrho(u_m)\, \nabla h_m \cdot \nabla \xi dx = 0, \quad \forall \xi \in H_0^1(\Omega),$$

and

$$(2.18)\qquad v_m - \tilde{v} \in H_0^1(\Omega), \quad \int_\Omega \nu(u_m)\, \nabla v_m \cdot \nabla \xi dx = 0, \quad \forall \xi \in H_0^1(\Omega),$$

which hold for $m \geqslant \overline{m}(\xi)$, obtaining (2.6) and (2.7). Let us fix $m \in \mathbf{N}$. Since $\{w_k\}_{k=1}^\infty$ is a basis of $H_0^1(\Omega)$, we have from (2.15)

$$(2.19)\qquad \int_\Omega \kappa(U_m + \tilde{u})\, \nabla U_m \cdot \nabla \xi dx + \int_\Omega \kappa(U_m + \tilde{u})\, \nabla \tilde{u} \cdot \nabla \xi dx =$$

$$= -\int_\Omega h_m \varrho(U_m + \tilde{u})\, \nabla h_m \cdot \xi dx - \int_\Omega v_m \nu(U_m + \tilde{u})\, \nabla v_m \cdot \nabla \xi dx, \quad k = 1, \dots, m,$$

for all $\xi \in C_0^\infty(\Omega)$. The limit $m \to \infty$ in the first two terms of (2.19) presents no difficulties. On the other hand, we have the estimates

$$(2.20)\qquad \left| \int_\Omega h_m \varrho(U_m + \tilde{u})\, \nabla h_m \cdot \nabla \xi dx - \int_\Omega h \varrho(U + \tilde{u})\, \nabla h \cdot \nabla \xi dx \right| \leqslant$$

$$\leqslant \|h_m - h\| \varrho_M \|\nabla h_m\| \|\nabla \xi\|_{L^\infty} + \tilde{h}_M \|\varrho(U\, m + \tilde{u}) - \varrho(U + \tilde{u})\| \|\nabla h_m\| \|\nabla \xi\|_{L^\infty} +$$

$$+ \left| \int_\Omega h \varrho(U + \tilde{u})(\nabla h_m - \nabla h) \cdot \nabla \xi dx \right|.$$

By (2.17) the L.H.S. tends to zero. Proceeding in a similar way with the last term in the R.H.S. of eqn. (2.18) we obtain, letting $m \to \infty$,

$$(2.21)\qquad \int_\Omega \kappa(U + \tilde{u})\, \nabla U \cdot \nabla \xi dx + \int_\Omega \kappa(U + \tilde{u})\, \nabla \widehat{u} \cdot \nabla \xi dx =$$

$$= -\int_\Omega h \varrho(U + \tilde{u})\, \nabla h \cdot \nabla \xi dx - \int_\Omega v \nu(U + \tilde{u})\, \nabla v \cdot \nabla \xi dx$$

for all $\xi \in C_0^\infty(\Omega)$ and, by density, we obtain also (2.8). □

REMARK 2.1. If (h, v, u) is a solution to the weak problem we have, by a result of regularity of N.G. Meyers [6], $\nabla h \in L^p(\Omega)$, $\nabla v \in L^p(\Omega)$, $p > 2$. Therefore, in view of (2.12), we get

$$h \varrho(u)\, \nabla h, \quad v \nu(u)\, \nabla v \in L^p(\Omega), \quad p > 2.$$

Hence, again by [6], $\nabla u \in L^p(\Omega)$, $p > 2$. If the data are regular, this remark permits to bootstrap and to obtain a classical solution.

3. Uniqueness and non-existence of solutions

In this section we examine a special case in which, quite surprisingly, problem (1.7)-(1.11) can be reduced to the Dirichlet's problem for the Laplacian and thus, in a certain sense, can be completely integrated.

The method works under the following assumptions: (*i*) $\kappa(u) > 0$, $\varrho(u) > 0$; (*ii*) v and u are constant on Γ; and (*iii*) the relation $\nu(u) = K\varrho(u)$, $K > 0$ holds. These hypotheses permit a detailed discussion upon the uniqueness and non-existence of solutions. However, (*iii*) is clearly physically restrictive.

The system thus becomes

$$\nabla \cdot (\varrho(u)\, \nabla v) = 0, \tag{3.1}$$

$$\nabla \cdot (\varrho(u)\, \nabla h) = 0, \tag{3.2}$$

$$-\nabla \cdot (\kappa(u)\, \nabla u) = \varrho(u)\, |\nabla h|^2 + K\varrho(u)\, |\nabla v|^2, \tag{3.3}$$

with the boundary conditions

$$v = V \text{ on } \Gamma_2, \quad v = 0 \text{ on } \Gamma_1 \tag{3.4}$$

$$h = H \text{ on } \Gamma_2, \quad h = 0 \text{ on } \Gamma_1 \tag{3.5}$$

$$u = 0 \text{ on } \Gamma_2, \quad u = 0 \text{ on } \Gamma_1 \tag{3.6}$$

where V and H are given constants. The special boundary conditions suggest the existence of a functional relation between v, h and u. This remark is the key to the following

THEOREM 3.1. *Suppose* (*i*), (*ii*) *and* (*iii*) *to hold, and*

$$\int_0^\infty \frac{\kappa(t)}{\varrho(t)}\, dt = \ell < \infty\,, \tag{3.7}$$

with

$$\ell > \left(1 + \frac{KV^2}{H^2}\right) \frac{H^2}{8}\,. \tag{3.8}$$

Then problem (3.1)-(3.6) *has one and only one solution, while, if*

$$\ell \leqslant \left(1 + \frac{KV^2}{H^2}\right) \frac{H^2}{8}\,, \tag{3.9}$$

the problem has no solution. If

$$\int_0^\infty \frac{\kappa(t)}{\varrho(t)}\, dt = \infty\,, \tag{3.10}$$

then there exists one and only one solution.

PROOF. Suppose (3.7) and (3.8) to hold. Define the map $\mathcal{F}\colon [0, \infty) \to [0, \ell)$

$$\mathcal{F}(u) = \int_0^u \frac{\kappa(t)}{\varrho(t)}\, dt, \tag{3.11}$$

and the other map

$$\theta = \frac{1}{2} h^2 + \frac{K}{2} v^2 + \mathcal{F}(u), \tag{3.12}$$

and consider the function $\mathcal{H}\colon [0, H] \to \left[0, \left(1 + \frac{KV^2}{H^2}\right) \frac{H^2}{8}\right]$ defined as follows:

$$\mathcal{H}(h) = -\frac{1}{2} h^2 + \frac{1}{2}\left(H + \frac{K}{H} V^2\right) h - \frac{KV^2}{2H^2} h^2. \tag{3.13}$$

By (3.7) and (3.8), the function

$$\mathcal{G} = \mathcal{F}^{-1}(\mathcal{H}(h)), \quad h \in [0, H] \tag{3.14}$$

is well-defined. Consider now the Dirichlet's problem

$$\Delta\psi = 0 \text{ in } \Omega, \tag{3.15}$$

$$\psi = 0 \text{ on } \Gamma_1, \quad \psi = \psi_2 \text{ on } \Gamma_2, \tag{3.16}$$

where

$$\psi_2 = \int_0^H \varrho(\mathcal{G}(h))\, dh. \tag{3.17}$$

By the maximum principle, the only solution $\psi(x)$ of (3.15)-(3.17) satisfies the inequality

$$0 \leqslant \psi(x) \leqslant \psi_2 \text{ in } \overline{\Omega}. \tag{3.18}$$

Define now the one-to-one mapping of $[0, H]$ onto $[0, \psi_2]$ given by

$$\mathcal{L}(h) = \int_0^h \varrho(\mathcal{G}(t))\, dt \tag{3.19}$$

where, by (3.18), the function

$$h(x) = \mathcal{L}^{-1}(\psi(x)), \quad x \in \overline{\Omega} \tag{3.20}$$

is well-defined. Putting

$$v(x) = \frac{V}{H} h(x) \tag{3.21}$$

and

$$u(x) = \mathcal{G}(h(x)), \tag{3.22}$$

the triplet $(h(x), v(x), u(x))$ is a solution to problem (3.1)-(3.6), since $\nabla\psi = \varrho(u)\, \nabla h$, and, by (3.15), we have

$$\nabla \cdot (\varrho(u)\, \nabla h) = 0, \quad \nabla \cdot (\varrho(u)\, \nabla v) = 0. \tag{3.23}$$

Moreover, $h(x)$ and $v(x)$ satisfy the boundary conditions

$$h = 0, \quad v = 0 \text{ on } \Gamma_1, \quad h = H, \quad v = V \text{ on } \Gamma_2. \tag{3.24}$$

By (3.14) we have the functional relation

$$\mathcal{H}(h(x)) = \mathcal{F}(\mathcal{G}(h(x))). \tag{3.25}$$

If $h(x)$, $v(x)$ and $u(x)$ are defined respectively by (3.20), (3.21) and (3.22), the corresponding function given by (3.12), *i.e.*

$$\theta(x) = \frac{1}{2}h^2(x) + \frac{K}{2}v^2(x) + \mathcal{F}(u(x)) = \frac{1}{2}\left(H + \frac{KV^2}{H}\right)h(x) \tag{3.26}$$

satisfies, by (3.23), the same equation as $h(x)$ and $v(x)$, *i.e.*:

$$\nabla\cdot(\varrho(u)\,\nabla\theta) = 0, \tag{3.27}$$

and the boundary conditions

$$\theta = 0 \text{ on } \Gamma_1, \quad \theta = \frac{1}{2}\left(H + \frac{KV^2}{H}\right)H \text{ on } \Gamma_2. \tag{3.28}$$

On the other hand, we have, by (3.26),

$$\varrho(u)\,\nabla\theta = h\varrho(u)\,\nabla h + K\varrho(u)\,v\nabla v + \kappa(u)\,\nabla u$$

and, in view of (3.27), (3.23) and (*ii*), we obtain

$$-\nabla\cdot(\kappa(u)\,\nabla u) = \varrho(u)\,|\nabla h|^2 + \nu(u)\,|\nabla v|^2$$

as required. Finally, we have

$$u = \mathcal{G}(h) = \mathcal{G}(0) = \mathcal{F}^{-1}(\mathcal{H}(0)) = 0 \text{ on } \Gamma_1,$$

$$u = \mathcal{G}(h) = \mathcal{G}(H) = \mathcal{F}^{-1}(\mathcal{H}(H)) = \mathcal{F}^{-1}(0) = 0 \text{ on } \Gamma_2.$$

Thus $(h(x), v(x), u(x))$ is a solution. We claim that this solution is unique. By contradiction, let $(\hat{h}, \hat{v}, \hat{u})$ be a second solution. The corresponding auxiliary function

$$\hat{\theta} = \frac{1}{2}\hat{h}^2 + \frac{k}{2}{}^2\hat{v}^2 + \mathcal{F}(\hat{u}) \tag{3.29}$$

satisfies the equation

$$\nabla\cdot(\varrho(\hat{u})\,\nabla\hat{\theta}) = 0 \tag{3.30}$$

and the boundary conditions

$$\hat{\theta} = \frac{1}{2}H^2 + \frac{K}{2}V^2, \text{ on } \Gamma_2, \quad \hat{\theta} = 0 \text{ on } \Gamma_1. \tag{3.31}$$

On the other hand, the function

$$\hat{\Phi} = \frac{1}{2}\left(H + \frac{KV^2}{H}\right)\hat{h}$$

is a solution of the equation

$$\nabla\cdot(\varrho(\hat{u})\,\nabla\hat{\Phi}) = 0 \tag{3.32}$$

and satisfies the same boundary condition (3.31) of $\hat{\theta}$. It follows $\hat{\theta}(x) = \hat{\Phi}(x)$. This implies that $\hat{u}$ and $\hat{h}$ are related by the same functional relation as u and h. That is

$$\mathcal{F}(\hat{u}) = \mathcal{H}(\hat{h}). \tag{3.33}$$

Recalling (3.7) and (3.8) we can solve (3.33) with respect to $\widehat{u}$, obtaining

$$\widehat{u} = \mathcal{F}^{-1}(\mathcal{H}(\widehat{h})) = \mathcal{G}(\widehat{h}).$$

Now

$$\widehat{\psi}(x) = \int_0^{\widehat{h}(x)} \varrho(\mathcal{G}(t))\, dt$$

is a solution to problem (3.15) and (3.16). However, this solution is unique. Thus $\psi(x) = \widehat{\psi}(x)$. Since

$$\mathcal{L}^{-1}(\widehat{\psi}(x)) = \mathcal{L}^{-1}(\psi(x)),$$

we also have $\mathcal{G}(h(x)) = \mathcal{G}(\widehat{h}(x))$ and we conclude that

$$(h(x),\, v(x),\, u(x)) = (\widehat{h}(x),\, \widehat{v}(x),\, \widehat{u}(x)).$$

Assume now (3.7) and (3.9). We claim that under these hypotheses problem (3.1)-(3.6) has no solution. By contradiction, let $(h(x), v(x), u(x))$ be a solution. We have again the functional relation

$$\mathcal{F}(u(x)) = \mathcal{H}(h(x)), \qquad x \in \overline{\Omega},$$

and, by the maximum principle,

$$0 \leqslant h(x) \leqslant H\,.$$

Thus there exists $\overline{x} \in \Omega$ such that

$$h(\overline{x}) = \frac{H}{2}\,.$$

Therefore we have

$$\mathcal{F}(u(\overline{x})) = \mathcal{H}(h(\overline{x})) = \mathcal{H}\left(\frac{H}{2}\right) = \left(1 + \frac{KV^2}{H^2}\right)\frac{H^2}{8} \geqslant \ell\,.$$

On the other hand, by (3.7) we obtain

$$\mathcal{F}(u(\overline{x})) = \int_0^{u(\overline{x})} \frac{\kappa(t)}{\varrho(t)}\, dt < \ell\,.$$

To prove existence and uniqueness when (3.10) holds we proceed in the same way, noticing that the functional relation $\mathcal{F}(u) = \mathcal{H}(h)$ in this case is always solvable with respect to u. Indeed, $\mathcal{F}(u)$ is now a strictly increasing and diverging function as $u \to \infty$. □

NOTE. With slightly more complicated calculations it is possible to treat the more general (but still constant) boundary conditions

$$h = H_1, \quad v = V_1, \quad u = u_1 \text{ on } \Gamma_1; \quad h = H_2, \quad v = V_2, \quad u = u_2 \text{ on } \Gamma_2.$$

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