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Analisi matematica. — *Ground States of Nonlinear Schrödinger Equations with potentials vanishing at infinity.* Nota di ANTONIO AMBROSETTI, VERONICA FELLI e ANDREA MALCHIODI, presentata (*) dal Socio A. Ambrosetti.

ABSTRACT. — In this preliminary *Note* we outline the results of the forthcoming paper [2] dealing with a class on nonlinear Schrödinger equations with potentials vanishing at infinity. Working in weighted Sobolev spaces, the existence of a ground state is proved. Furthermore, the behaviour of such a solution, as the Planck constant tends to zero (semiclassical limit), is studied proving that it concentrates at a point.

KEY WORDS: Nonlinear Schrödinger equations; Weighted Sobolev spaces; Critical point theory.

RIASSUNTO. — *Stati fondamentali per equazioni di Schrödinger nonlineari con potenziali che si annullano all'infinito.* In questa *Nota* preliminare presentiamo i risultati del lavoro [2] dove studiamo una classe di equazioni di Schrödinger nonlineari con potenziali che tendono a zero all'infinito. Lavorando in spazi di Sobolev con peso, dimostriamo l'esistenza di una soluzione fondamentale. Di tale soluzione è anche studiato il comportamento quando la costante di Planck tende a zero (limite semiclassico) dimostrando che essa si concentra in un punto.

1. INTRODUCTION

We consider, for $N \geq 3$, the stationary Nonlinear Schrödinger Equations

$$(1) \qquad \begin{cases} -\varepsilon^2 \Delta v + V(x) v = K(x) v^p, & x \in \mathbb{R}^N, \\ v \in W^{1,2}(\mathbb{R}^N), \quad v(x) > 0, & \lim_{|x| \to \infty} v(x) = 0, \end{cases}$$

where $1 < p < \frac{N+2}{N-2}$. We address here two problems: (*i*) the existence, for $\varepsilon > 0$ fixed, of a solution v_ε of (1) with minimal energy (ground state); (*ii*) the behavior (concentration) of v_ε as $\varepsilon \to 0$. The main novelty with respect to most of the (broad) literature dealing with (1) is that we assume that the potentials V and K decay to zero as $|x| \to \infty$. Precisely, we suppose

(V) $\quad V: \mathbb{R}^N \to \mathbb{R}$ is smooth and $\exists\, \alpha, a_1, a_2 > 0$ such that

$$\frac{a_1}{1 + |x|^\alpha} \leq V(x) \leq a_2,$$

and, respectively

(K) $\quad K: \mathbb{R}^N \to \mathbb{R}$ is smooth and $\exists\, \beta, a_3 > 0$:

$$0 < K(x) \leq \frac{a_3}{1 + |x|^\beta}.$$

If $V \sim |x|^{-\alpha}$ as $|x| \to \infty$, with $\alpha > 0$, the spectrum of the linear operator $-\Delta + V$ is

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$[0, +\infty)$, and this prevents the use of perturbation methods as in [3, 4]. On the other hand, we cannot even apply critical point theory working in $W^{1,2}(\mathbb{R}^N)$. For these reasons, we consider the weighted space L_K^q of measurable $u : \mathbb{R}^N \to \mathbb{R}$ such that

$$|u|_{q,K} = \left[\int_{\mathbb{R}^N} K(x)\, |u(x)|^q \, dx \right]^{\frac{1}{q}} < \infty ,$$

as well as the weighted Sobolev spaces $\mathcal{H}_\varepsilon$ defined by setting

$$\mathcal{H}_\varepsilon = \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} [\varepsilon^2 |\nabla u(x)|^2 + V(x) u^2(x)] \, dx < \infty \right\}.$$

$\mathcal{H}_\varepsilon$ is a Hilbert space with scalar product

$$(u|v)_\varepsilon = \int_{\mathbb{R}^N} [\varepsilon^2 \nabla u(x) \cdot \nabla v(x) + V(x) u(x) v(x)] \, dx$$

and norm $\|u\|_\varepsilon^2 = (u|u)_\varepsilon$. Let

$$\sigma = \sigma_{N,\alpha,\beta} = \begin{cases} \dfrac{N+2}{N-2} - \dfrac{4\beta}{\alpha(N-2)}, & \text{if } 0 < \beta < \alpha \\ 1 \text{ otherwise}. \end{cases}$$

The above weighted spaces have been introduced in [6], where the following result is proved.

THEOREM 1. *Let $N \geqslant 3$ and suppose that $(V), (K)$ hold with $\alpha \in (0, 2]$ and $\beta > 0$, respectively. Then for all $\varepsilon > 0$, $\mathcal{H}_\varepsilon \hookrightarrow L_K^{p+1}$ provided*

$$\sigma \leqslant p \leqslant \frac{N+2}{N-2}.$$

Furthermore, the embedding of $\mathcal{H}_\varepsilon$ into L_K^q is compact provided

$$\sigma < p < \frac{N+2}{N-2}. \tag{2}$$

2. AN EXISTENCE RESULT

The preceding Theorem implies that the functional I_ε

$$I_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left[\varepsilon^2 |\nabla u(x)|^2 + V(x) u^2(x) \right] dx - \frac{1}{p+1} \int_{\mathbb{R}^N} K(x) |u(x)|^{p+1} dx, \quad u \in \mathcal{H}_\varepsilon,$$

is well defined and $I_\varepsilon \in C^1(\mathcal{H}_\varepsilon, \mathbb{R})$. It is easy to check that I_ε has the Mountain Pass geometry. Furthermore, the Palais-Smale condition is satisfied if (2) holds, since in such a case $\mathcal{H}_\varepsilon$ is compactly embedded into L_K^q. This immediately implies:

LEMMA 2. *Let* (V), (K) *hold with* $0 < \alpha \leqslant 2$, $\beta > 0$, *respectively, and suppose that* p *satisfies* (2). *Then*

$$b_\varepsilon = \inf_{u \in \mathcal{H}_\varepsilon \setminus \{0\}} \max_{t \geqslant 0} I_\varepsilon(tu)$$

is a critical level of I_ε *and carries a critical point* $v_\varepsilon \in \mathcal{H}_\varepsilon$ *of* I_ε.

Here v_ε is a critical point in the sense that $(I'_\varepsilon(v_\varepsilon) \,|\, u) = 0$ for all $u \in \mathcal{H}_\varepsilon$. By local elliptic regularity, it follows that v_ε is for all $\varepsilon > 0$ a positive (classical) solution of the equation

$$-\varepsilon^2 \Delta v + V(x) v = K(x) v^p, \quad x \in \mathbb{R}^N. \tag{3}$$

REMARKS. *a*) Lemma 2 also follows from [7, Thm. 3.1] combined with Theorem 1. Let us point out that the case in which $p = \sigma$ or $p = \dfrac{N+2}{N-2}$ is also studied in [7, Thm. 3.2], under some further restriction on V and K. Lemma 2 is also somewhat related to the results of [5].

b) If V is «bounded away from zero and infinity», namely $0 < \inf_{\mathbb{R}^N} V \leqslant \sup_{\mathbb{R}^N} V < +\infty$, we can directly work in $W^{1,2}(\mathbb{R}^N)$. In such a case we recover the compactness assuming that (K) holds. If $V \sim |x|^{-\alpha}$ as $|x| \to \infty$, with $\alpha \in (0, 2]$, while K is bounded away from zero and infinity, one could show that $b_\varepsilon = 0$ and hence there are no Mountain Pass solution. The same remark holds if $p < \sigma$. On the other hand, solutions with higher energy might exist. For example, if both V and K are bounded away from zero and infinity, the existence of a solution to (1) is proved *e.g.* in [3] provided ε is *sufficiently small* and $V^\theta K^{-2/(p-1)}$, with $\theta = \dfrac{p+1}{p-1} - \dfrac{N}{2}$, has a «stable» stationary point, like *e.g.* a maximum or a minimum. □

Next, we will show that the Mountain-Pass solution $v_\varepsilon \in \mathcal{H}_\varepsilon$ is indeed a ground state, provided $0 < \alpha < 2$. In order to prove this fact, some sharp decay estimates are carried out, leading to the following integral estimate (in the lemma below, we have highlighted the dependence on ε because it will be used in the next section, dealing with the concentration phenomenon).

LEMMA 3. *Let* (V), (K) *hold with* $0 < \alpha < 2$, $\beta > 0$, *respectively, and suppose that* p *satisfies* (2). *Moreover, let* v_ε *be solutions of* (3) *and suppose there exists* $\Gamma > 0$ *such that*

$$\|v_\varepsilon\|_{\mathcal{H}_\varepsilon} \leqslant \Gamma \varepsilon^n. \tag{4}$$

Then there exist $R_\Gamma > 0$, *and constants* $C_{1,2} > 0$, *depending only on* Γ, *such that, for all* $R \geqslant R_\Gamma$ *there holds*

$$\int_{|x| > R} [\varepsilon^2 |\nabla v_\varepsilon|^2 + V(x) v_\varepsilon^2]\, dx \leqslant C_1 \varepsilon^N \exp\left\{ -C_2 \varepsilon^{-1} R^{\frac{2-\alpha}{2}} \right\}. \tag{5}$$

The preceding lemma is used to show that $v_\varepsilon \in L^2(\mathbb{R}^N)$. Roughly, if $y \in \mathbb{R}^N$ with $|y| \gg 1$, one gets:

$$\int_{|x-y|<1} v_\varepsilon^2\, dx = \int_{|x-y|<1} V(x)\, v_\varepsilon^2 \cdot \frac{1}{V(x)}\, dx \leqslant c_1\, |y|^\alpha \int_{|x-y|<1} V(x)\, v_\varepsilon^2\, dx .$$

Then (5) implies

$$\int_{|x-y|<1} v_\varepsilon^2\, dx \leqslant c_2\, |y|^\alpha \exp\{-c_3\, |y|^{1-\alpha/2}\},$$

and from this it follows that $v_\varepsilon \in L^2(\mathbb{R}^N)$. Then one gets that $v_\varepsilon \in W^{1,2}(\mathbb{R}^N)$ as well as that $\lim_{|x|\to\infty} v(x) = 0$, proving our main existence result:

THEOREM 4. *Let* (V), (K) *hold with* $0 < \alpha < 2$, $\beta > 0$, *respectively, and suppose that* p *satisfies* (2). *Then the Mountain-Pass solution* v_ε *found in Lemma* 2 *is such that* $v_\varepsilon \in W^{1,2}(\mathbb{R}^N)$, $v_\varepsilon \in C^2(\mathbb{R}^N)$, $v_\varepsilon(x) > 0$ *and* $\lim_{|x|\to\infty} v_\varepsilon(x) = 0$ *and thus is a ground state of* (1).

3. CONCENTRATION AS $\varepsilon \to 0$

Concerning the behavior of the Mountain-Pass solution v_ε, our main result is Theorem 5 below. Let

$$\mathcal{A}(x) := [V(x)]^\theta [K(x)]^{-2/(p-1)}, \qquad \theta = \frac{p+1}{p-1} - \frac{N}{2}.$$

The auxiliary potential $\mathcal{A}$ has been previously introduced dealing with potentials V and K bounded away from zero and infinity, see *e.g.* [3], where it is proved that concentration occurs at the «stable» stationary points of $\mathcal{A}$. Moreover, let us point out that $\mathcal{A}$ has a global minimum since $\lim_{|x|\to\infty} \mathcal{A}(x) = +\infty$ provided (2) holds.

THEOREM 5. *Let the same assumptions as in Theorem* 4 *hold. Then, the preceding ground state* v_ε *concentrates at a global minimum* x^* *of* $\mathcal{A}$. *More precisely,* v_ε *has a unique maximum* x_ε *such that* $x_\varepsilon \to x^*$ *as* $\varepsilon \to 0$, *and*

$$v_\varepsilon(x) \sim U^*\left(\frac{x - x_\varepsilon}{\varepsilon}\right), \quad \textit{as } \varepsilon \to 0,$$

where U^* *is the unique positive radial solution of*

$$-\Delta U^* + V(x^*)\, U^* = K(x^*)(U^*)^p .$$

The proof of Theorem 5 is based upon the following Lemmas. The first one provides an uniform bound on the Mountain-Pass critical level of I_ε.

LEMMA 6. *There exists* $\Gamma > 0$ *such that* $b_\varepsilon \leqslant \Gamma \varepsilon^N$, *for all* $\varepsilon > 0$ *small.*

To see this, we introduce the functional J_ε: $W^{1,2}(\mathbb{R}^N) \mapsto \mathbb{R}$, by setting

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left[\varepsilon^2 |\nabla u|^2 + a_2 u^2\right]dx - \frac{1}{p+1}\int_{\mathbb{R}^N} K(x)\,|u|^{p+1}\,dx,$$

where, according to assumption (V), $\sup V \leqslant a_2$. Since $W^{1,2}(\mathbb{R}^N) \subset \mathcal{H}_\varepsilon$, we infer

$$b_\varepsilon \leqslant \tilde{b}_\varepsilon := \inf_{u \in W^{1,2}(\mathbb{R}^N)\setminus\{0\}} \max_{t \geqslant 0} J_\varepsilon(tu). \tag{6}$$

Furthermore, letting

$$\mathfrak{J}_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left[\,|\nabla u|^2 + a_2 u^2\right]dx - \frac{1}{p+1}\int_{\mathbb{R}^N} K(\varepsilon x)\,|u|^{p+1}\,dx.$$

One finds that $u_\varepsilon(x)$ is a critical point of $\mathfrak{J}_\varepsilon$ iff $\tilde{u}_\varepsilon(x) := u_\varepsilon(x/\varepsilon)$ is a critical point of J_ε. Using the perturbation method introduced in [1], we can look for critical points of $\mathfrak{J}_\varepsilon$ near those of the *unperturbed* functional $\mathfrak{J}_0 \equiv \mathfrak{J}_{\varepsilon=0}$. Up to translation, we can assume that $K(0) = \max K$. Then $\mathfrak{J}_\varepsilon$ has, for $\varepsilon > 0$ small, a critical point u_ε such that $u_\varepsilon \to U$ as $\varepsilon \to 0$, where U is the unique positive radial solution of

$$-\Delta U + a_2 U = K(0)\,U^p, \qquad U \in W^{1,2}(\mathbb{R}^N).$$

Furthermore, U is a Mountain Pass critical point of $\mathfrak{J}_0$ and the same holds true for u_ε. The preceding information, together with $\mathfrak{J}_\varepsilon(u_\varepsilon) \to \mathfrak{J}_0(U)$ as $\varepsilon \to 0$ and the equality $J_\varepsilon(\tilde{u}_\varepsilon) = \varepsilon^N \mathfrak{J}_\varepsilon(u_\varepsilon)$, readily imply that $\tilde{b}_\varepsilon \leqslant \Gamma\varepsilon^N$, and the lemma follows using (6).

Lemma 6 implies that (4) holds. Then Lemma 3 applies and this allows us to prove a pointwise uniform exponential decay for the solutions v_ε

LEMMA 7. *There exist* C_1, $C_2 > 0$ *and* $d > 0$, *depending only on* Γ, p, N, α *and* β, *such that*

$$|v_\varepsilon(x)| \leqslant C_1\,|x|^d \varepsilon^{-d} \exp\left\{-C_2\,\varepsilon^{-1}\,|x|^{\frac{2-\alpha}{2}}\right\}; \qquad for\,|x| \gg 1. \tag{7}$$

Let x_ε denote a point of maximum for v_ε. Thus, $\Delta v_\varepsilon(x_\varepsilon) \leqslant 0$ whence $V(x_\varepsilon)K^{-1}(x_\varepsilon) \leqslant v_\varepsilon^{p-1}(x_\varepsilon)$. Then, using (7), we deduce that

$$V(x_\varepsilon)K^{-1}(x_\varepsilon) \leqslant C_1\,|x_\varepsilon|^{d(p-1)}\,\varepsilon^{-d(p-1)} \exp\left\{-C_2\,\varepsilon^{-1}\,|x_\varepsilon|^{\frac{2-\alpha}{2}}\right\}.$$

Since from $(V)-(K)$ we know that $V(x)K^{-1}(x) \sim |x|^{\beta-\alpha}$ as $|x| \to \infty$, there exists a constant $C' > 0$ such that

$$|x_\varepsilon| \leqslant C', \qquad \forall\, \varepsilon \sim 0. \tag{8}$$

Finally one also shows that there exists a constant $C'' > 0$

$$\|v_\varepsilon\|_{L^\infty} \geqslant C''. \tag{9}$$

Equations (8), (9) and the preceding Lemmas allow us to carry over standard arguments which lead to prove Theorem 5.

The complete proofs of the results sketched in the present *Note*, are contained in the forthcoming paper [2].

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References

[1] A. Ambrosetti - M. Badiale, *Homoclinics: Poincaré-Melnikov type results via a variational approach*. Ann. Inst. H. Poincaré Analyse Non Linéaire, 15, 1998, 233-252.

[2] A. Ambrosetti - V. Felli - A. Malchiodi, *Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity*. Preprint S.I.S.S.A., n. 16/2004/M, Febr. 2004; to appear on Journal Europ. Math. Soc.

[3] A. Ambrosetti - A. Malchiodi - S. Secchi, *Multiplicity results for some nonlinear singularly perturbed elliptic problems on* $\mathbb{R}^n$. Arch. Rat. Mech. Anal., 159, 2001, 253-271.

[4] Y.Y. Li, *On a singularly perturbed elliptic equation*. Adv. Diff. Eqns., 2, 1997, 955-980.

[5] E.S. Noussair - C.A. Swanson, *Decaying solutions of semilinear elliptic equations in* $\mathbb{R}^N$. SIAM J. Math. Anal., 20, 6, 1989, 1336-1343.

[6] B. Opic - A. Kufner, *Hardy-type inequalities*. Pitman Res. Notes in Math. Series 219, Longman Scientific & Technical, Harlow 1990.

[7] M. Schneider, *Entire solutions of semilinear elliptic problems with indefinite nonlinearities*. PhD Thesis, Universität Mainz, 2001.

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