Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **15** (2004), n.3-4, p. 169–181.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2004_9_15_3-4_169_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2004.

Rend. Mat. Acc. Lincei
s. 9, v. 15:169-181 (2004)

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ON A PHASE TRANSITION MODEL OF PENROSE-FIFE TYPE

ABSTRACT. — We deal with a Penrose-Fife type model for phase transition. We assume a rather general constitutive low for the heat flux and treat the Dirichlet and Neumann boundary condition for the temperature. Some of our proofs apply to different types of boundary conditions as well and improve some results existing in the literature.

KEY WORDS: Phase transition; Penrose-Fife model; Heat flux law; Boundary conditions; Well-posedness.

1. INTRODUCTION

A wide literature deals with the so-called phase field models for phase transitions. Such models introduce a new physical variable, the phase parameter φ (which is a scalar in the simplest cases), besides the temperature ϑ, and their mathematical description consists in a system of partial differential equations that rules the pair (ϑ, φ).

To give an idea of such a framework, we start with the well-known two phase Stefan problem. If ϑ and φ denote the relative temperature (around the transition temperature $\vartheta = 0$) and the proportion of the solid phase, respectively, the Stefan problem can be described by the energy balance

$$\partial_t(\vartheta + \lambda\varphi) + \operatorname{div} \boldsymbol{q} = g \tag{1.1}$$

and by a pointwise relationship between ϑ and φ, namely

$$\varphi \in \mathcal{H}(\vartheta). \tag{1.2}$$

Here, the positive constant λ, the vector field $\boldsymbol{q}$, and the function g are the latent heat, the heat flux, and the heat source, respectively, while $\mathcal{H}$ is the Heaviside graph in the euclidean plane defined by

$$\mathcal{H}(\vartheta) = 0,\ [0, 1],\ 1 \quad \text{according to} \quad \vartheta < 0, \quad \vartheta = 0, \quad \vartheta > 0.$$

In equation (1.1), the heat flux is related to ϑ by the Fourier law

$$\boldsymbol{q} = -k\nabla\vartheta \tag{1.3}$$

where k is a positive constant, at least in the simplest case.

In such a formulation, the solid and the liquid regions are the sets where $\varphi = 0$ and $\varphi = 1$, respectively, and their complement, where φ takes values between 0 and 1, is the so-called mushy region. When a classical solution to the Stefan problem exists, the mushy region is empty and the solid and liquid regions are separated by a moving smooth surface.

Phase field model replace such a sharp interface with a thin transition layer. From the mathematical point of view, this corresponds to replace (1.2) by a differential inclusion that rules the phase dynamics, namely

$$\mu\partial_t\varphi - \nu\Delta\varphi + \mathcal{H}^{-1}(\varphi) \ni \lambda\vartheta \tag{1.4}$$

where μ and ν are small positive constants. Note that the Stefan problem is formally obtained by setting $\mu = \nu = 0$ in (1.4). A different phase dynamics is given by the Allen-Cahn equation

$$\mu\partial_t\varphi - \nu\Delta\varphi + W'(\varphi) = \lambda\vartheta \tag{1.5}$$

where W is a double well potential, *e.g.*, $W(\varphi) = \varphi^2(1-\varphi)^2$. For the wide literature on phase field models we directly refer, *e.g.*, to the books [1] and [17].

Assuming $\lambda = \mu = \nu = 1$ to simplify the notation, we observe that both (1.4) and (1.5) are particular cases of the differential inclusion

$$\partial_t\varphi - \Delta\varphi + \partial j(\varphi) + \sigma'(\varphi) \ni \vartheta \tag{1.6}$$

and that (1.6) is the gradient flow governed by the free energy functional

$$\mathcal{F}_\vartheta(\varphi) = \frac{1}{2}\int_\Omega |\nabla\varphi|^2 + \int_\Omega \big(j(\varphi) + \sigma(\varphi) - \vartheta\varphi\big).$$

In (1.6), ∂j is the subdifferential of j, where

(1.7) $\quad j : (-\infty, +\infty) \to [0, +\infty]$ is convex, proper, and lower semicontinuous

and σ is a smooth function. More precisely, we can assume that

(1.8) $\quad \sigma \in C^1(-\infty, +\infty)$ and σ' is Lipschitz continuous.

Indeed, in (1.4) j is the indicator function of $[0, 1]$ and $\sigma = 0$, while in (1.5) j and σ are the convex and concave parts of the double well potential W, respectively.

More recently, Penrose and Fife derived in [15] a new model for phase transitions which uses the absolute temperature rather then the relative temperature and guarantees thermodynamical consistency, namely, the absolute temperature is positive and the second principle is satisfied in the form of the Clausius-Duhem inequality. From the mathematical point of view, the main difference with respect to the previous phase field models is that the inverse $1/\vartheta$ of the absolute temperature ϑ enters the phase dynamics, directly. A rather general version of the related system of partial differential equations reads

$$\partial_t(\vartheta + \varphi) + \operatorname{div}\boldsymbol{q} = g \qquad \partial_t\varphi - \Delta\varphi + \partial j(\varphi) + \sigma'(\varphi) \ni \frac{1}{\vartheta_c} - \frac{1}{\vartheta}$$

where ϑ_c is the critical value of the absolute temperature ϑ and $\boldsymbol{q}$ is related to ϑ by a suitable constitutive law.

Initial and boundary value problem for similar systems have been already studied from several viewpoints and a number of results is known. As we are interested mainly in well-posedness, we quote the papers [3-5, 7-14, 16, 18] and just mention that there are results also in different directions, like, *e.g.*, long-time behaviour, numerical approximation, and additional memory effects. However, different variants of Penrose-Fife type systems require different techniques, in general, even when dealing just with well-posedness.

Here, we discuss a rather general class of constitutive laws for the heat flux and treat the Dirichlet and Neumann problems for the temperature (more precisely, for a function of the temperature). We note that such boundary conditions are considered very seldom and that the third boundary value problem is mainly studied. Our starting point is the couple of papers [3] and [4], where the constitutive law for the heat flux is settled in the form

$$\boldsymbol{q} = -\nabla\alpha(\vartheta). \tag{1.9}$$

In (1.9), α is a given smooth function on $(0, +\infty)$. The main assumption on it is that

$$\alpha \text{ is concave and strictly increasing}. \tag{1.10}$$

Moreover, α must have a suitable behaviour as its argument tends to 0 and to $+\infty$. In the present exposition, we deal with the following case

$$r^2\alpha'(r) = 1 + o(1) \qquad \text{as } r \to 0^+ \tag{1.11}$$

$$r^d\alpha'(r) = c_\infty + o(1) \qquad \text{as } r \to +\infty \qquad \text{with } 0 \leqslant d \leqslant 1 \tag{1.12}$$

where c_∞ is a positive constant. We note that the case $d = 0$ corresponds to a perturbation of the linear behaviour of $\alpha(\vartheta)$ for high values of the temperature, so that the parameter d can be seen as a measure of the distance from the Fourier law (1.3) with $k = c_\infty$. In the worst case ($d = 1$) we can deal with, α has a logarithmic behaviour at infinity.

Coming back to the papers we have just quoted, we remark that an existence result is given in [3] under much more general assumptions on α near 0 and with a latent heat that can depend on φ. On the contrary, the regularity and uniqueness results contained in [4] are proved just in the particular case $d = 0$. Finally, both papers consider the third boundary value problem for $\alpha(\vartheta)$, namely

$$\partial_n\alpha(\vartheta) + c\alpha(\vartheta) = h \tag{1.13}$$

where ∂_n is the outward normal derivative, c is a positive constant, and h is a given function on the boundary of the domain.

Here, we present the results contained in [6] and in [2] with some simplification. On one side, we assume that the latent heat is constant as above, while a nonconstant latent heat could be considered. On the other side, some of the conditions we assume to make the exposition simpler are not necessary. Finally, some of the results are just mentioned without any precise statement, for brevity.

2. Main results

In order to simplify the notation, we set

$$V := H^1(\Omega), \qquad H := L^2(\Omega), \qquad W_d := W^{1,4/(d+2)}(\Omega)$$

$$H_n^2 := \{v \in H^2(\Omega) : \partial_n v = 0 \ \text{ on } \Gamma\}$$

where Ω is a bounded and connected open set in the 3-dimensional euclidean space with smooth boundary Γ. Moreover, we identify H to a subspace of V' in the usual way and denote the duality pairing between V' and V by $\langle\cdot,\cdot\rangle$. In the case of the Dirichlet problem, we replace V' by $H^{-1}(\Omega)$ in such an identification and use the same notation $\langle\cdot,\cdot\rangle$ for the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Moreover, we introduce two more state variables, u and ξ, related to the previous ones by the conditions

$$u = \alpha(\vartheta) \quad \text{and} \quad \xi \in \partial j(\varphi). \tag{2.1}$$

Such relations must hold a.e. in $\Omega \times (0,T)$, where $T > 0$ is a given final time. In all the problems we are going to consider, we look for four functions, ϑ, φ, u, and ξ, satisfying the regularity condition (R) stated below and relationships (2.1).

REGULARITY (R). *We require that*

$$\begin{aligned}
&\vartheta \in L^\infty(0,T;H) \cap L^2(0,T;W_d)\\
&\varphi \in L^2(0,T;H_n^2) \cap H^1(0,T;H)\\
&u \in L^2(0,T;V), \quad \xi \in L^2(\Omega \times (0,T))\\
&\vartheta > 0 \quad \textit{a.e. in } \Omega \times (0,T) \quad \textit{and} \quad 1/\vartheta \in L^2(0,T;V).
\end{aligned}$$

We point out that (R) contains the homogeneous Neumann condition for φ as well. Moreover, we require an additional regularity property, namely

$$\text{either } \partial_t\vartheta \in L^2(0,T;H^{-1}(\Omega)) \quad \text{or} \quad \partial_t\vartheta \in L^2(0,T;V') \tag{2.2}$$

according to whether we are dealing with the Dirichlet condition or with other types of boundary conditions for u. In any case, the phase dynamics is described by the equation

$$\partial_t\varphi - \Delta\varphi + \xi + \sigma'(\varphi) = -\frac{1}{\vartheta} \tag{2.3}$$

with a new σ (obtained by subtracting a linear term with slope $1/\vartheta_c$ to the previous one). On the contrary, the energy balance and the boundary condition for u are stated in different forms according to the boundary condition we want to consider. In the case of the Dirichlet condition, we ask that

$$\langle \partial_t(\vartheta(t)+\varphi(t)), v\rangle + \int_\Omega \nabla u(t)\cdot\nabla v = \int_\Omega g(t)v \tag{2.4}$$

$$\text{for every } v \in H_0^1(\Omega) \text{ and for a.a. } t \in (0,T)$$

$$u = u_\Gamma \quad \text{on } \Gamma \times (0,T) \tag{2.5}$$

where u_Γ is given. In the case of the Neumann problem, instead, we insert the boundary condition in the variational formulation as usual, *i.e.*,

$$\langle \partial_t(\vartheta(t)+\varphi(t)), v\rangle + \int_\Omega \nabla u(t)\cdot\nabla v = \int_\Omega g(t)v + \int_{\Gamma\times(0,T)} h(t)v \tag{2.6}$$

$$\text{for every } v \in V \text{ and for a.a. } t \in (0,T)$$

and the same we do for the third type condition (1.13), namely

$$\langle \partial_t(\vartheta(t) + \varphi(t)), v\rangle + \int_\Omega \nabla u(t) \cdot \nabla v + c \int_\Gamma uv = \int_\Omega g(t) v + \int_{\Gamma \times (0,T)} h(t) v \tag{2.7}$$

for every $v \in V$ and for a.a. $t \in (0, T)$.

As in the introduction, h is a given function and c is a positive constant. Finally, we give the initial conditions

$$\vartheta(0) = \vartheta_0 \quad \text{and} \quad \varphi(0) = \varphi_0 \tag{2.8}$$

with suitable initial data.

For the sake of convenience, we specify the regularity of the given data later on. Now, we summarize the assumptions on the structure and state the three problems corresponding to the above boundary conditions.

Assumption (H). *We assume that the functions j and σ satisfy* (1.7) *and* (1.8). *Moreover $\alpha \in C^2(0, +\infty)$ satisfies* (1.10), (1.11), *and* (1.12).

Problem (P_1). *Find $(\vartheta, \varphi, u, \xi)$ satisfying* (R), *the first condition* (2.2), (2.1), *and solving the initial-boundary value problem* (2.3), (2.4), (2.5), *and* (2.8).

Problem (P_2). *Find $(\vartheta, \varphi, u, \xi)$ satisfying* (R), *the second condition* (2.2), (2.1), *and solving the initial-boundary value problem* (2.3), (2.6), *and* (2.8).

Problem (P_3). *Find $(\vartheta, \varphi, u, \xi)$ satisfying* (R), *the second condition* (2.2), (2.1), *and solving the initial-boundary value problem* (2.3), (2.7), *and* (2.8).

A problem like (P_3) is solved in [3] and [4] under general assumptions on the data. More precisely, [3] gives an existence result under assumptions on the structure that are much more general than (H) as far as the behaviour of α near 0 is concerned. On the contrary, [4] deals just with the case $d = 0$ since the Lipschitz continuity of α^{-1} is used there. Moreover, the Lipschitz continuity of the function

$$\ell(r) := \alpha(r) + \frac{1}{r}, \qquad r \in (0, +\infty) \tag{2.9}$$

is assumed as well. In [4] the authors prove a uniqueness result assuming the further regularity condition on the solution

$$\vartheta \in L^\infty(0, T; V). \tag{2.10}$$

In the same paper, a regularity result ensuring (2.10) is proved under suitable assumptions on the data.

Remark 2.1. We note that α^{-1} is globally Lipschitz continuous if and only if $d = 0$ and that it has an exponential behaviour at infinity in the worst case $d = 1$. Moreover, the function ℓ is Lipschitz continuous on $(\delta, +\infty)$ for any $\delta > 0$, even in the case $0 \leqslant d \leqslant 1$, as a consequence of (1.12) and of the smoothness of α. However, its global Lipschitz continuity cannot be deduced from (H).

Now we present the results of [6] and [2] regarding problems (P_1) and (P_2), respectively. Some ideas on their proofs are given in the next section.

As far as (P_1) is concerned, we think of an extension of the Dirichlet datum u_Γ to the whole domain and still term u_Γ such an extension.

THEOREM 2.2. *Assume* (H). *Assume moreover*

$$g \in L^2(0, T; H), \quad u_\Gamma \in L^2(0, T; V) \cap H^1(0, T; H) \cap L^\infty(\Omega \times (0, T))$$

$$\vartheta_0 \in H, \quad \vartheta_0 > 0 \ \ a.e.\ in\ \Omega, \quad 1/\vartheta_0 \in H, \quad \varphi_0 \in V, \quad j(\varphi_0) \in L^1(\Omega).$$

Then problem (P_1) *has at least a solution.*

THEOREM 2.3. *Assume* (H) *and let* $(\vartheta, \varphi, u, \xi)$ *be a solution to problem* (P_1) *satisfying also* (2.10). *Then such a solution is unique.*

REMARK 2.4. More precisely, in [6] it is proved that the solution given by Theorem 2.2 can be obtained as the limit of the solutions to problems of type (P_3) depending on a parameter $\varepsilon > 0$ that one lets tend to zero.

Moreover, the same paper contains a regularity result ensuring (2.10). Such a result is analogous to the one given in [4] as far as the regularity of the data and of the solution is concerned, but it holds essentially just under assumptions (H). In particular, it holds for all $d \in [0, 1]$. Indeed, the precise further assumption needed is just a reinforcement of (1.11), namely

$$r^2 \alpha'(r) = 1 + O(r) \qquad \text{as } r \to 0^+ . \tag{2.11}$$

Note that (2.11) is stronger than (1.11) and weaker than the Lipschiz continuity of ℓ.

Now, we deal with problem (P_2) and present the main results proved in [2]. The existence of a solution depends on a further assumption on j which seems to be just technical. We assume that

$$|s| \leq c_0(1 + j(r)) < +\infty \quad \text{for any } r \in (-\infty, +\infty) \text{ and } s \in j(r) \tag{2.12}$$

for some $c_0 > 0$. Hence, ∂j can still be multivalued. However, it cannot grow more than exponentially at infinity and the second condition (2.1) cannot include constraints on φ.

THEOREM 2.5. *Assume* (H) *and* (2.12). *Assume moreover*

$$g \in L^2(0, T; H), \quad h \in L^\infty(\Gamma \times (0, T))$$

$$\vartheta_0 \in H, \quad \vartheta_0 > 0 \ \ a.e.\ in\ \Omega, \quad 1/\vartheta_0 \in H, \quad \varphi_0 \in V, \quad j(\varphi_0) \in L^1(\Omega).$$

Then problem (P_2) *has at least a solution.*

THEOREM 2.6. *Assume in addition that the function* ℓ *defined in* (2.9) *is Lipschitz continuous. Then the solution given by Theorem* 2.5 *is unique.*

REMARK 2.7. More precisely, in [2] it is proved that the solution given by Theorem 2.5 can be obtained as the limit of the solutions to problems of type (P_3) depending on a parameter $\varepsilon > 0$ that one lets tend to zero. Moreover, the assumption on h could be replaced by a weaker assumption depending on d. Finally, we note that the technique used in [2] to prove Theorem 2.6 applies to problems (P_1) and (P_3) as well, *i.e.*, the uniqueness of the solution to such problems is guaranteed without any further regularity assumption on the solution itself, provided that ℓ is Lipschitz continuous.

3. Outline of the proofs

As far as Theorem 2.2 is concerned, we refer to [6], where the solution to (P_1) is obtained letting ε tend to zero in a family of problems (P_3). More precisely, one takes

$$c = \frac{1}{\varepsilon} \quad \text{and} \quad h = \frac{1}{\varepsilon} u_\Gamma \tag{3.1}$$

in (P_3) and treats the obtained problem like an approximating problem. Hence, one uses compactness and monotonicity methods and shows that its solution $(\vartheta_\varepsilon, \varphi_\varepsilon, u_\varepsilon, \xi_\varepsilon)$ tends to a solution to (P_1) as $\varepsilon \to 0^+$ in a suitable topology, at least for a subsequence.

The basic a priori estimate can be formally obtained this way. Setting $\vartheta_\Gamma := := \alpha^{-1}(u_\Gamma)$, we test equation (2.7) (where we have replaced c and h according to (3.1)) with $v = \vartheta_\varepsilon + u_\varepsilon - \vartheta_\Gamma - u_\Gamma$ and integrate over $(0, t)$; next, we multiply (2.3) by $\partial_t \varphi_\varepsilon$ and integrate over $\Omega \times (0, t)$; then we sum the obtained equalities to each other. We refer to [6] for the treatement (partially inspired by [3]) of all the integrals and just note two things about the formula we get. First, the left hand side contains the sum

$$\int_\Omega A(\vartheta_\varepsilon(t)) + \int_{\Omega \times (0, t)} |\nabla u_\varepsilon|^2 + \int_{\Omega \times (0, t)} |\partial_t \varphi_\varepsilon|^2 \tag{3.2}$$

where A is the unique primitive of α with $\min A = 0$. On the other hand, the integrals containing $u_\varepsilon \partial_t \varphi$ and $\partial_t \varphi / \vartheta_\varepsilon$ partially cancel and just the term

$$\int_{\Omega \times (0, t)} \ell(\vartheta_\varepsilon) \partial_t \varphi_\varepsilon$$

survives in their sum. Such a term is treated with the help of an elementary inequality which follows from (H) (see [6, Lemma 3.2]), namely

$$\ell^2(r) \leq \delta \alpha^2(r) + c_\delta \big(1 + A(r)\big) \qquad \text{for any } r \in (0, +\infty)$$

where c_δ depends only on α and on the parameter $\delta > 0$. Clearly, such an inequality helps a lot, compared with (3.2) and the possibility of applying the Gronwall lemma. The final basic estimate we obtain is

$$\begin{aligned}
&\|A(\vartheta_\varepsilon)\|_{L^\infty(0, T; L^1(\Omega))} + \|j(\varphi_\varepsilon)\|_{L^\infty(0, T; L^1(\Omega))} + \|\vartheta_\varepsilon\|_{L^\infty(0, T; H)} + \\
&\qquad + \|u_\varepsilon\|_{L^2(0, T; V)} + \|\varphi_\varepsilon\|_{L^\infty(0, T; V)} + \|\partial_t \varphi_\varepsilon\|_{L^2(0, T; H)} \\
&\qquad + \varepsilon^{-1/2} \|u_\varepsilon - u_\Gamma\|_{L^2(\Gamma \times (0, T))} + \|\nabla \vartheta_\varepsilon^{1 - d/2}\|_{L^2(0, T; H)} \leq c
\end{aligned}$$

where c does not depend on ε. In particular is already clear that the Dirichlet condition for u will be satisfied in the limit.

However, the above test function v is not admissible and the correct procedure used in [6] replaces v with a V-valued approximation of it.

In order to apply weak and strong compactness results and overcome the difficulties related to the limits of the nonlinear terms, more estimates are proved in [6]. In particular, one shows that the limit satisfies the regularity conditions (R).

As far as Theorem 2.2 is concerned, we refer to [6] and just note that the proof applies to problems (P_2) and (P_3) as well, and the same remark holds for the technique used to prove the regularity result for (P_1) just mentioned in the previous section. However, it must be pointed out that the outline of the proofs of both the uniqueness and the regularity results of [6] is strongly inspired by [4]. The modification relies essentially in one point: [6] uses the properties stated below instead of the Lipschitz continuity of α^{-1} and ℓ

$$\text{the function } \ell \circ \alpha^{-1} \text{ is globally Lipschitz continuous}$$

$$|(\ell \circ \alpha^{-1})'(s)| \leq c\sqrt{(\alpha^{-1})'(s)} \quad \text{for any real } s$$

where c is a positive constant. The corresponding proofs are given in [6, Lemma 3.3] and in [6, Lemma 3.4], respectively, and are completely elementary. The first property is true whenever assumptions (H) hold. On the contrary, the second one needs the additional assumption (2.11).

The same outline we have sketched to prove Theorem 2.2 is used as far as Theorem 2.5 is concerned. In this case, one simply takes $c = \varepsilon$ in (P_3) and starts estimating suitable norms of the approximate solution $(\vartheta_\varepsilon, \varphi_\varepsilon, u_\varepsilon, \xi_\varepsilon)$. The basic estimate should follow similarly as before. Here, assuming that the point where $\alpha(r)$ vanishes is $r = 1$, one should choose (a V-valued approximation of) $v = \vartheta_\varepsilon + u_\varepsilon - 1$ in (2.7) (with $c = \varepsilon$) and multiply (2.3) by $\partial_t \varphi$. After integrating and summing up, we obtain what we could call the basic equality. We do not write it for brevity and just observe that its left hand side contains the sum

$$\frac{1}{2}\int_\Omega |\vartheta_\varepsilon(t)|^2 + \int_{\Omega \times (0,t)} |\nabla u_\varepsilon|^2 + \int_\Omega j(\varphi_\varepsilon(t)) + \int_{\Omega \times (0,t)} |\partial_t \varphi_\varepsilon|^2. \tag{3.3}$$

A partial cancellation helps as before. On the contrary, one immediately finds some trouble related with the source terms.

This difficulty is overcome with the help of the further assumption (2.12). The procedure used in [2] splits the basic estimate in two parts and uses just assumptions (H) and (2.12). Here, we assume ℓ to be Lipschitz continuous (see also Remark 2.1) and proceede formally (*i.e.*, we choose exactly the above v) in order to make the exposition simple and transparent. The source terms are

$$\int_{\Gamma \times (0,t)} h\vartheta_\varepsilon \quad \text{and} \quad \int_0^t \langle f(s), u_\varepsilon(s) \rangle \, ds \tag{3.4}$$

where we have introduced $f \in L^2(0, T; V')$ by setting

$$\langle f(s), v \rangle = \int_\Omega g(s) v + \int_\Gamma h(s) v \qquad \text{for } v \in V \text{ and } s \in (0, T).$$

We can deal with the first term (3.4) owing to the inequality

$$\|v\|_{L^{d_\bullet}(\Gamma)} \leq \delta \|\nabla \alpha(v)\|_H^2 + c_\delta (1 + \|v\|_H^2)$$

where $d_\bullet \geq 1$ is given by $d_\bullet := 4/(1 + 3d)$. Such an inequality holds for any $\delta > 0$, some $c_\delta = c(\delta, \Omega, \alpha)$, and for any positive v that makes the right hand side finite. Its proof is given in [2, Lemma 3.1] and uses just sharp trace theorems and standard inequalities. A simple application and an integration over $(0, t)$ yield

$$\int_{\Gamma \times (0, t)} h \vartheta_\varepsilon \leq \delta \int_{\Omega \times (0, t)} |\nabla u_\varepsilon|^2 + c_\delta \int_{\Omega \times (0, t)} |\vartheta_\varepsilon|^2 + c_\delta$$

(with a new c_δ), provided that h is bounded (more generally, h could belong to a suitable space depending on d), and the last integrals can be controlled using the Gronwall lemma, since the left hand side of our basic equality contains the sum (3.3). As far as the second (3.4) is concerned, we can estimate it like

$$\int_0^t \langle f(s), u_\varepsilon(s) \rangle ds \leq c(g, h) \|u_\varepsilon\|_{L^2(0, t; V)}$$

with full V-norm, while the left hand side of our basic equality contains just the integral

$$\int_{\Omega \times (0, t)} |\nabla u_\varepsilon|^2.$$

On the other hand, the Poincaré inequality

$$\|u_\varepsilon(t) - m_\varepsilon(t)\|_V^2 \leq c(\Omega) \int_\Omega |\nabla u_\varepsilon(t)|^2 \qquad \text{where } m_\varepsilon(t) := \frac{1}{|\Omega|} \int_\Omega u_\varepsilon(t)$$

does not solve immediately our problem. So, some work has to be done. We add and subtract the mean value and are led to estimate the integral

$$\int_0^t \langle f(s), m_\varepsilon(s) \rangle ds = \int_0^t m_\varepsilon(s) \langle f(s), 1 \rangle ds \leq \|f\|_{L^\infty(0, T; V')} \|1\|_V \int_0^t |m_\varepsilon(s)| ds$$

where we are assuming $f \in L^\infty(0, T; V')$ just for the sake of simplicity. Now, we can treat the last integral this way

$$\int_0^t |m_\varepsilon(s)| ds \leq \frac{1}{|\Omega|} \int_{\Omega \times (0, t)} |\alpha(\vartheta_\varepsilon)| \leq \frac{1}{|\Omega|} \int_{\Omega \times (0, t)} \frac{1}{\vartheta_\varepsilon} + \frac{1}{|\Omega|} \int_{\Omega \times (0, t)} |\ell(\vartheta_\varepsilon)|.$$

The ℓ-term can be easily controlled using the Lipschitz continuity of ℓ we are assuming for the sake of simplicity. Instead, the mean value of $1/\vartheta_\varepsilon$ has to be treated carefully. However, it can be read in (2.3) and estimated. Noting that $\Delta\varphi_\varepsilon$ has zero mean value (thanks to the Neumann condition for φ_ε contained in (R)) and owing to the Lipschitz continuity of σ', we have

$$\int_{\Omega\times(0,t)} \frac{1}{\vartheta_\varepsilon} \leq c \int_{\Omega\times(0,t)} \big(|\partial_t\varphi_\varepsilon| + |\xi_\varepsilon| + |\varphi_\varepsilon| + 1\big).$$

Now, remembering that the left hand side of our basic equality contains the sum (3.3), we see that assumption (2.12) allows us to overcome the difficulty.

We conclude by sketching the proof of Theorem 2.6. In order to simplify the notation, we use the same symbol c for constants that might be different from each other, even in the same line of our estimates.

Let $(\vartheta_i, \varphi_i, u_i, \xi_i)$, $i=1,2$, be two solution to (P_2). Following [4], we write (2.6) for such solutions, take the difference, and integrate the obtained equation over $(0,t)$. Next, we write (2.3) for the two solution and take the difference. We obtain the system

$$\int_\Omega \big(\vartheta(t) + \varphi(t)\big)v + \int_\Omega \nabla\Big(\int_0^t u\Big)\cdot\nabla v = 0 \qquad \text{for every } v\in V \text{ and for a.a. } t\in(0,T)$$

$$\partial_t\varphi - \Delta\varphi + \xi + \sigma'(\varphi_1) - \sigma'(\varphi_2) = \zeta \qquad \text{a.e. in } \Omega\times(0,T)$$

where we have set for convenience

$$\vartheta := \vartheta_1 - \vartheta_2, \quad u := u_1 - u_2, \quad \varphi := \varphi_1 - \varphi_2$$

$$\xi := \xi_1 - \xi_2, \quad \zeta := \zeta_1 - \zeta_2, \quad \zeta_i := -\frac{1}{\vartheta_i} \quad \text{for } i = 1, 2.$$

Now, we write the first equation for $t=s$, choose $v=u(s)$, and integrate over $(0,t)$. Then, we multiply the second equation by φ and integrate over $\Omega\times(0,t)$ owing to the Neumann condition for φ. Finally, we sum the obtained equalities to each other and obtain

$$\int_{\Omega\times(0,t)} \vartheta u + \frac{1}{2}\int_\Omega \Big|\nabla\int_0^t u\Big|^2 + \frac{1}{2}\int_\Omega |\varphi(t)|^2 + \int_{\Omega\times(0,t)} |\nabla\varphi|^2 + \int_{\Omega\times(0,t)} \xi\varphi =$$

$$= \int_{\Omega\times(0,t)} (\zeta - u)\varphi + \int_{\Omega\times(0,t)} \big(\sigma'(\varphi_2) - \sigma'(\varphi_1)\big)\varphi.$$

Essentially at this point, [4] and [6] use u_i and replace ϑ_i with $\alpha^{-1}(u_i)$. Here, instead, we use ϑ_i rather that u_i and replace $\zeta_i - u_i$ by $-\ell(\vartheta_i)$. Forgetting some positive terms and taking advantage of the Lipschitz continuity of both ℓ and σ', we deduce that

$$\int_{\Omega\times(0,t)} \vartheta u + \frac{1}{2}\int_\Omega |\varphi(t)|^2 + \int_{\Omega\times(0,t)} |\nabla\varphi|^2 \leq c\int_{\Omega\times(0,t)} |\vartheta||\varphi| + c\int_{\Omega\times(0,t)} \varphi^2.$$

Clearly, the main point is compensating the second last integral with the left hand side since the last term can be controlled using the Gronwall lemma. To this aim, we estimate the first integral on the left hand side from below owing to the following inequality

$$(r_1 - r_2)\big(\alpha(r_1) - \alpha(r_2)\big) \geq \delta_0 \frac{(r_1 - r_2)^2}{1 + r_1^d + r_2^d}$$

which holds for some constant $\delta_0 > 0$ and any $r_1, r_2 > 0$ and is an easy consequence of the mean value theorem and of our assumptions (H). Moreover, we estimate the troubling integral from above accordingly. We obtain

$$\delta_0 \int_{\Omega \times (0,t)} \frac{\vartheta^2}{1 + \vartheta_1^d + \vartheta_2^d} + \frac{1}{2}\|\varphi(t)\|_H^2 + \int_{\Omega \times (0,t)} |\nabla \varphi|^2 \leq$$

$$\leq \frac{\delta_0}{2} \int_{\Omega \times (0,t)} \frac{\vartheta^2}{1 + \vartheta_1^d + \vartheta_2^d} + c \int_{\Omega \times (0,t)} (1 + \vartheta_1^d + \vartheta_2^d)\varphi^2$$

and note that we could already apply the Gronwall lemma if $d = 0$. If $d > 0$, we estimate the non trivial contributions to the last integral owing to the Hölder inequality with suitable $p, q \geq 1$ satisfying $(1/p) + (2/q) = 1$. We choose

$$p := \frac{2}{d} \quad \text{and} \quad q := \frac{4}{2-d}$$

and note that $q \leq 4$. We obtain for $i = 1, 2$

$$\int_{\Omega \times (0,t)} \vartheta_i^d \varphi^2 \leq \int_0^t \|\vartheta_i^d\|_{L^p(\Omega)} \|\varphi\|_{L^q(\Omega)}^2 \, ds = \int_0^t \|\vartheta_i\|_H^d \|\varphi\|_{L^q(\Omega)}^2 \, ds \leq c \int_0^t \|\varphi\|_{L^4(\Omega)}^2 \, ds$$

where c accounts also for the norm of ϑ_i in $L^\infty(0, T; H)$. Now, the compact embedding $V \subset L^4(\Omega)$ yields

$$\|v\|_{L^4(\Omega)}^2 \leq \delta \|\nabla v\|_H^2 + c_\delta \|v\|_H^2$$

for any $\delta > 0$ and any $v \in V$ and for some constant c_δ. Hence, we deduce

$$\int_{\Omega \times (0,t)} \vartheta_i^d \varphi^2 \leq \frac{1}{4} \int_{\Omega \times (0,t)} |\nabla \varphi|^2 + c \int_0^t \|\varphi(s)\|_H^2 \, ds .$$

Therefore, we can apply the Gronwall lemma and conclude that ϑ and φ vanish identically, *i.e.*, $\vartheta_1 = \vartheta_2$ and $\varphi_1 = \varphi_2$, whence also $u_1 = u_2$, obviously. Finally, we deduce that $\xi_1 = \xi_2$ by comparison in (2.3), and the proof is complete.

REMARK 3.1. One could replace equation (2.3) by the system

$$\partial_t \varphi - \Delta w = 0, \qquad w \in -\Delta \varphi + \partial j(\varphi) + \sigma'(\varphi) + \frac{1}{\vartheta}$$

which is equivalent to a fourth order equation of Cahn-Hilliard type for φ. One uses to add the homogeneous Neumann condition $\partial_n w = 0$ on the boundary. If we still keep the energy balance equation and the boundary and initial conditions we have dealt with before, we obtain a Penrose-Fife type model with conserved order parameter. Indeed, the integral of $\varphi(t)$ over Ω does not depend on time.

From the mathematical point of view, the main difference with respect to the second order system is that the time derivative $\partial_t \varphi$ is estimated in $L^2(0, T; V')$ rather than in $L^2(0, T; H)$. Nevertheless, one can prove similar results. This is done in a joint work with A. Marson, where we are dealing with well-posedness, essentially under the same assumptions (H). Indeed, just the case $d = 1$ is missing. Some results on the third type problem for u have been already established. However, our work is still in progress.

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