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EXISTENCE AND APPROXIMATION RESULTS FOR GRADIENT FLOWS

ABSTRACT. — This note addresses the Cauchy problem for the gradient flow equation in a Hilbert space H

$$\begin{cases} u'(t) + \partial\phi(u(t)) \ni 0 & \text{a.e. in } (0, T), \\ u(0) = u_0, \end{cases}$$

where $\phi : \mathrm{H} \to (-\infty, +\infty]$ is a proper, lower semicontinuous functional which is not supposed to be a (smooth perturbation of a) convex functional and $\partial\phi$ is (a suitable limiting version of) its subdifferential. The interest for this kind of equations is motivated by a number of examples, which show that several mathematical models describing phase transitions phenomena and leading to systems of evolutionary PDEs have a *common gradient flow structure*. In particular, when quasi-stationary models are considered, highly non-convex functionals naturally arise. We will present some existence results for the solution of the gradient flow equation by exploiting a variational *approximation* technique, featuring some ideas from the theory of *Minimizing Movements*.

KEY WORDS: Phase transitions; Evolution problems; Gradient flows; Minimizing Movements.

1. THE GRADIENT FLOW STRUCTURE OF SOME PHASE FIELD MODELS

In this section we first recall some basic examples, which will motivate our further abstract discussion.

EXAMPLE 1 (Phase relaxation). In [28], A. Visintin proposed a relaxed formulation of the classical *Stefan problem*, which consists of the system

$$\partial_t e - \Delta(e - \chi) = 0 \qquad \text{in } \Omega \times (0, T), \tag{1.1}$$

$$\varepsilon\partial_t \chi + (\chi - e) + V'(\chi) \ni 0 \qquad \text{in } \Omega \times (0, T), \tag{1.2}$$

with suitable initial and lateral boundary conditions on (e, χ). Here Ω is an open bounded subset of $\mathbb{R}^d$, e can be interpreted as the enthalpy density of a physical system subject to, *e.g.*, a solid-liquid phase transition, χ is the phase variable taking values in $[0, 1]$ and representing the local proportion of the two phases, $\varepsilon > 0$ is a relaxation parameter, $V : \mathbb{R} \to [0, +\infty]$ is a convex potential which is finite only in $[0, 1]$ and confines the values of χ. The simplest choice for V is represented by the indicator function of $[0, 1]$

$$V(x) = I_{[0,1]}(x) := \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant 1, \\ +\infty & \text{otherwise}. \end{cases} \tag{1.3}$$

The «derivative» (in fact, the subdifferential) of V is given by the inverse H^{-1} of the

Heaviside graph

$$(1.4) \qquad V'(x) = H^{-1}(x) := \begin{cases} (-\infty, 0] & \text{if } x = 0, \\ 0 & \text{if } 0 < x < 1, \\ [0, +\infty) & \text{if } x = 1, \end{cases} \qquad D(V') = [0, 1].$$

As it stands, this model has been proposed to account for supercooling and superheating effects in phase transition phenomena, and its well-posedness has been proved in [28]; the classical *Stefan problem* can be recovered as $\varepsilon \downarrow 0$.

As we will show later on, this system is naturally associated to the *convex* Lyapunov functional

$$(1.5) \qquad \Phi_1(e, \chi) := \int_\Omega \left(\frac{1}{2} |e - \chi|^2 + V(\chi) \right) dx,$$

which is decreasing in time along the solutions of (1.1-1.2).

EXAMPLE 2 (Phase field). An alternative model for melting and freezing processes is the classical phase field system proposed by G. Caginalp [9, 10], in which the energy balance equation (1.1) is coupled to

$$(1.6) \qquad \varepsilon \partial_t \chi + (\chi - e) + \frac{1}{\eta} W'(\chi) - \eta \Delta \chi = 0 \quad \text{in } \Omega \times (0, T),$$

W' being the derivative of a double well potential function W vanishing exactly at 0 and 1, *e.g.*

$$(1.7) \qquad W(x) = \frac{1}{4} x^2 (x - 1)^2,$$

while η is a (positive) relaxation parameter. We will show that, as for the previous example, further insight may be gained into the nature of the system (1.1, 1.6) by taking into account the *non convex* Lyapunov functional

$$(1.8) \qquad \Phi_{2,\eta}(e, \chi) := \int_\Omega \left(\frac{1}{2} |e - \chi|^2 + \frac{1}{\eta} W(\chi) + \frac{\eta}{2} |\nabla \chi|^2 \right) dx.$$

Phase transitions models as gradient flows for the «entropy» functional. In both the previous examples, the evolution of the physical phenomena under interest can be modelled by a system of partial differential equations in the variables e and χ. Such a system is somehow naturally related to a (not necessarily convex) functional Φ, (as in (1.5), (1.8)),

$$\Phi(e, \chi) = -S(e, \chi, \nabla \chi, \ldots),$$

which is the opposite of the (mathematical) entropy functional S and contains all the specific information of the model. Within this approach, the equilibrium states of the system correspond to the minima of the functional Φ, which in particular satisfy the resulting system of Euler equations

$$\begin{cases} \dfrac{\delta\Phi}{\delta e} = 0, \\ \dfrac{\delta\Phi}{\delta\chi} = 0. \end{cases}$$

Accordingly, the evolution dynamics of the system is yielded by the system

$$\begin{cases} \partial_t e - \operatorname{div}\left(\nabla \dfrac{\delta\Phi}{\delta e}\right) = 0 & \text{in } \Omega \times (0, T), \\ \varepsilon\partial_t \chi + \dfrac{\delta\Phi}{\delta\chi} = 0 & \text{in } \Omega \times (0, T). \end{cases} \tag{1.9}$$

As a matter of fact, (1.9) has the same form of the systems (1.1-1.2) and (1.1, 1.6) (for $\Phi = \Phi_1$, $\Phi = \Phi_{2,\eta}$, respectively). On the other hand, let us note that (1.9) has a *gradient flow structure* which can be better understood if we reformulate (1.9) by inverting the Laplace operator in the first equation (*e.g.*, supposing for simplicity that homogeneous Dirichlet boundary conditions on $\dfrac{\delta\Phi}{\delta e}$ are imposed), thus obtaining

$$\begin{cases} (-\Delta)^{-1} \partial_t e + \dfrac{\delta\Phi}{\delta e} = 0 & \text{in } \Omega \times (0, T), \\ \varepsilon\partial_t \chi + \dfrac{\delta\Phi}{\delta\chi} = 0 & \text{in } \Omega \times (0, T). \end{cases} \tag{1.10}$$

The above system can be further rephrased in terms of the vector $u := (e, \chi)$: if $\boldsymbol{J}_\varepsilon$ is the duality map between $\mathrm{H}_\varepsilon := H^{-1}(\Omega) \times L^2(\Omega)$ and $H_0^1(\Omega) \times L^2(\Omega)$, induced by the norm

$$\|u\|_\varepsilon^2 := \langle -\Delta^{-1} e, e\rangle + \varepsilon \int_\Omega |\chi|^2\, dx, \quad \text{so that} \quad \boldsymbol{J}_\varepsilon(e, \chi) := (-\Delta^{-1} e, \varepsilon\chi),$$

then (1.10) becomes

$$\boldsymbol{J}_\varepsilon(\partial_t u) + \frac{\delta\Phi}{\delta u} = 0 \quad \text{in } (0, T), \tag{1.11}$$

which is the *gradient flow equation* for the functional Φ in the Hilbert space H_ε.

The variational approximation scheme. One of the standard ways to approximate (1.11) (starting from an initial condition $u(0) = u_0$) is to fix a time step $\tau = T/N$, $N \in \mathbb{N}$, inducing a uniform partition

$$\mathrm{P}_\tau := \{t_0 = 0,\ t_1 = \tau, \dots, t_n = n\tau, \dots, t_N = T\} \quad \text{of the interval } (0, T),$$

and to find a discrete approximation $U^n \approx u(t_n)$ of the values of the solution u at the nodes t_n by solving recursively the *implicit Euler scheme*

$$\boldsymbol{J}_\varepsilon\left(\frac{U^n - U^{n-1}}{\tau}\right) + \frac{\delta\Phi}{\delta u}(U^n) = 0 \quad n = 1, \dots, N; \quad U^0 := u_0. \tag{1.12}$$

Note that (1.12) can be recast in terms of the variables e and χ, yielding for $U^n = (E^n, \chi^n)$

$$(1.13)\qquad \begin{cases} (-\Delta)^{-1}\left(\dfrac{E^n - E^{n-1}}{\tau}\right) + \dfrac{\delta\Phi}{\delta e}(E^n) = 0, \\ \varepsilon\dfrac{\chi^n - \chi^{n-1}}{\tau} + \dfrac{\delta\Phi}{\delta\chi}(\chi^n) = 0, \end{cases}$$

which is exactly the time implicit approximation scheme associated to (1.10). On the other hand, (1.12) is the Euler equation for the variational problem

$$(1.14)\qquad \begin{cases} \text{find } U^n \in \mathrm{H}_\varepsilon \text{ which minimizes} \\ \mathrm{F}_\varepsilon(\tau, U^{n-1}; U) := \dfrac{1}{2\tau}\|U - U^{n-1}\|_\varepsilon^2 + \Phi(U) \text{ among all } U \in \mathrm{H}_\varepsilon. \end{cases}$$

This minimization problem is solvable under suitable lower-semicontinuity and coercivity assumptions on Φ. If $\{U^n\}_{n=1}^N$ is a sequence of solutions, we can consider the piecewise constant interpolants

$$(1.15)\qquad \overline{U}_\tau(t) := U^n \quad \text{if } t \in ((n-1)\tau, n\tau],$$

and we can try to recover the solution u as the (uniform) limit in H_ε of $\overline{U}_\tau$ as $\tau \downarrow 0$.

In the previous examples, the convergence of this approximation procedure is guaranteed by the convexity of Φ_1 (ex. 1) and of (a quadratic perturbation of) Φ_2 (ex. 2). Nevertheless, let us also notice that the quadratic perturbation needed to «convexify» Φ_2 is of order ε^{-1} and blows up when ε vanishes. This fact reflects a crucial aspect of the family of «quasi-stationary models» which we are going to discuss, and explains the higher intrinsic difficulty of their analysis.

Quasi-stationary problems. The *quasi-stationary* models can be formally obtained from the general phase field system (1.9) by taking the limit as $\varepsilon \downarrow 0$:

$$(1.16)\qquad \begin{cases} \partial_t e - \operatorname{div}\left(\nabla\dfrac{\delta\Phi}{\delta e}\right) = 0 & \text{in } \Omega \times (0, T), \\ \dfrac{\delta\Phi}{\delta\chi} = 0 & \text{in } \Omega \times (0, T). \end{cases}$$

Apart from the case of a convex functional Φ, where convergence can be rigorously proved, the passage to the limit can be justified only at the discrete level of the variational steps (1.14), thus assuming that the parameter ε vanishes much more quickly than the step size τ of the approximations. If this is the case, it is easy to see by standard Γ-convergence results that, under fairly mild lower semicontinuity and coercivity assumptions on Φ, the Γ-limit of the functionals $U \mapsto \mathrm{F}_\varepsilon(\tau, U^{n-1}; U)$ is

$$(1.17)\qquad \mathrm{F}_0(\tau, U^{n-1}; U) := \frac{1}{2\tau}\|U - U^n\|_0^2 + \Phi(U) = \\ = \frac{1}{2\tau}\|E - E^{n-1}\|_{H^{-1}(\Omega)}^2 + \Phi(E, \chi),$$

and therefore the solutions of (1.14) converge as $\varepsilon \downarrow 0$ (up to subsequences) to solutions of the recursive scheme

$$U^n \in \operatorname*{argmin}_U \mathrm{F}_0(\tau, U^{n-1}; U) \tag{1.18}$$

or, equivalently,

$$(E^n, \chi^n) \in \operatorname*{argmin}_{(E,\chi)} \left\{ \frac{1}{2\tau} \|E - E^{n-1}\|^2_{H^{-1}(\Omega)} + \Phi(E, \chi) \right\}. \tag{1.19}$$

We immediately check that the first variation of (1.19) yields

$$\begin{cases} (-\Delta)^{-1}\left(\dfrac{E^n - E^{n-1}}{\tau}\right) + \dfrac{\delta\Phi}{\delta e}(E^n) = 0, \\ \dfrac{\delta\Phi}{\delta\chi}(\chi^n) = 0, \end{cases} \tag{1.20}$$

which is just the implicit Euler discretization of (1.16). Let us quickly review the new problems associated to the previous examples after this asymptotic procedure.

EXAMPLE 3 (Quasi-stationary phase relaxation). Choosing $\Phi = \Phi_1$ in (1.16), we obtain the quasi-stationary version of (1.1-1.2)

$$\begin{cases} \partial_t e - \Delta(e - \chi) = 0 & \text{in } \Omega \times (0, T), \\ \chi + V'(\chi) \ni e & \text{in } \Omega \times (0, T), \end{cases} \tag{1.21}$$

which, in the case (1.4), corresponds to the *weak formulation* of the Stefan problem (see *e.g.* [29]). Note that the second equation may be rewritten as

$$e - \chi = \beta(e), \quad \text{with} \quad \beta(e) := \begin{cases} e & \text{if } e \leq 0, \\ 0 & \text{if } 0 < e < 1, \\ e - 1 & \text{if } e \geq 1. \end{cases} \tag{1.22}$$

EXAMPLE 4 (*Quasi-stationary phase field model*). Selecting the functional $\Phi = \Phi_{2,\eta}$ given by (1.8), (1.16) yields the *quasistationary phase field system*

$$\begin{cases} \partial_t e - \Delta(e - \chi) = 0 & \text{in } \Omega \times (0, T), \\ \chi - e + \dfrac{1}{\eta} W'(\chi) - \eta \Delta \chi = 0 & \text{in } \Omega \times (0, T). \end{cases} \tag{1.23}$$

P. Plotnikov and V.N. Starovoitov [23] and R. Schätzle [27] proved the existence of a variational solution of (1.23) in the (technically quite different) cases of homogeneous Dirichlet and of Neumann boundary conditions for $e - \chi$, respectively. In their approach, the proof of the convergence of a time-discrete approximation relies on Holmgren uniqueness theorem or on refined spectral analysis arguments, which essentially depend on the particular shape and regularity of the double well potential (1.7) and of the elliptic operator.

EXAMPLE 5 (Stefan-Gibbs-Thomson Problem). Another interesting example can be obtained by taking the limit of (1.23) as $\eta \downarrow 0$. Here the variational formulation

(1.19) with $\Phi := \Phi_{2,\eta}$ is particularly convenient, since it is well known that the Γ-limit of (1.8) is

$$\Phi_{2,0}(e,\chi) := \int_\Omega \left(\frac{1}{2} |e-\chi|^2 + I_{\{0,1\}}(\chi) \right) dx + \gamma \int_\Omega |D\chi|, \tag{1.24}$$

where $I_{\{0,1\}}$ is the (highly non convex!) indicator function of the two pointed set $\{0,1\}$ ($I_{\{0,1\}}$ being defined in the same way as the functional (1.3) for the interval $[0,1]$), $\int_\Omega |D\chi|$ is the total variation of χ, extended to $+\infty$ outside of $BV(\Omega)$ (see *e.g.* [29, XI.1]), and γ is a positive coefficient related to the shape of W. The corresponding evolution problem has to be formulated as a coupling between a diffusion equation and a variational inequality, and takes the form

$$\begin{cases} \partial_t e - \Delta(e-\chi) = 0 & \text{in } \Omega \times (0,T), \\ \Phi_{2,0}(e(t),\chi(t)) \leq \Phi_{2,0}(e(t),v) & \forall v \in BV(\Omega), \text{ a.e. in } (0,T). \end{cases} \tag{1.25}$$

Note that (1.25) provides a variational formulation of the Stefan problem with the Gibbs-Thomson law for the evolving interface (which can be inferred directly from the quasi-stationary variational condition) and has been proposed by A. Visintin; the existence of a (so called Lyapunov) solution to (1.25) with homogeneous Dirichlet conditions for $e-\chi$ has been proved by S. Luckhaus in [18] (whose proof is also presented in [29, Ch. VIII]).

Quasi-stationary phase field models as gradient flows of the reduced entropy.

The main feature of the variational scheme (1.18) $\sim$ (1.19) is its degeneration in the metric quadratic part of the functional

$$\frac{1}{2\tau}\|U-U^{n-1}\|_0^2 = \frac{1}{2\tau}\|E-E^{n-1}\|_{H^{-1}(\Omega)}^2, \qquad U=(E,\chi) \in H^{-1}(\Omega) \times L^2(\Omega), \tag{1.26}$$

which is the square of a seminorm and in fact controls only the E-component of the couple $U=(E,\chi) \in H^{-1}(\Omega) \times L^2(\Omega)$. This fact suggests splitting (1.19) into two iterated minimization problems, thus writing

$$\min_{E,\chi} \left\{ \frac{1}{2\tau}\|E-E^{n-1}\|_{H^{-1}(\Omega)}^2 + \Phi(E,\chi) \right\} =$$

$$= \min_E \left\{ \frac{1}{2\tau}\|E-E^{n-1}\|_{H^{-1}(\Omega)}^2 + \left(\min_\chi \Phi(E,\chi) \right) \right\}.$$

Therefore, it seems natural to work in the reduced Hilbert space $\mathrm{H}_0 := H^{-1}(\Omega)$ by introducing the *reduced functional*

$$\phi(e) := \inf_\chi \Phi(e,\chi), \qquad \forall e \in \mathrm{H}, \tag{1.27}$$

and by writing the recursive minimization algorithm (1.19) as

$$\begin{cases} \text{find } E^n, \chi^n \text{ which minimize respectively} \\ \dfrac{1}{2\tau}\|E - E^{n-1}\|^2_{H^{-1}(\Omega)} + \phi(E) \text{ among all } E \in H^{-1}(\Omega), \\ \Phi(E^n, \chi) \text{ among all } \chi \in L^2(\Omega). \end{cases} \tag{1.28}$$

In principle, this splitting allows to write an independent evolution equation for e, which we interpret as the *gradient flow for* ϕ in $\mathrm{H}_0 := H^{-1}(\Omega)$,

$$(-\Delta)^{-1}\partial_t e + \frac{\delta\phi}{\delta e} = 0, \tag{1.29}$$

and to recover $\chi = \chi(t)$ at each time t as the solution of

$$\Phi\big(e(t), \chi(t)\big) = \phi\big(e(t)\big) \leqslant \Phi\big(e(t), v\big) \qquad \forall v \in L^2(\Omega), \text{ a.e. in } (0, T). \tag{1.30}$$

Lack of convexity. Let us rewrite the reduced entropy for the Examples 3, 4, and 5 we detailed in the previous section: in the first case, we have

$$\phi_1(e) := \int_\Omega j(e)\, dx, \quad j(e) = \min_\chi \left(\frac{1}{2}(e-\chi)^2 + V(\chi)\right). \tag{1.31}$$

In the particular case (1.3), (1.31) gives

$$j(e) = \begin{cases} \dfrac{1}{2}e^2 & e < 0, \\ 0 & \text{if } 0 \leqslant e \leqslant 1, \\ \dfrac{1}{2}(e-1)^2 & \text{if } e > 1, \end{cases} \tag{1.32}$$

which was introduced by [6] in his gradient flow formulation of the Stefan problem; in particular $\delta\phi_1/\delta e = \beta(e)$, with β as in (1.22). The functionals corresponding to Examples 4 and 5 are

$$\phi_{2,\eta}(e) := \inf_\chi \int_\Omega \left(\frac{1}{2}|e-\chi|^2 + \frac{1}{\eta}W(\chi) + \frac{\eta}{2}|\nabla\chi|^2\right) dx, \tag{1.33}$$

$$\phi_{2,0}(e) := \inf_\chi \int_\Omega \left(\frac{1}{2}|e-\chi|^2 + I_{\{0,1\}}(\chi)\, dx\right) dx + \gamma \int_\Omega |D\chi|. \tag{1.34}$$

Note that, while ϕ_1 is convex, $\phi_{2,\eta}$ and $\phi_{2,0}$ are neither convex nor C^1 perturbations of a convex functional. The lack of convexity properties is indeed somehow latent in the analytical difficulties of the latter two models and can be better understood by highlighting a further characteristic of the above functionals.

Concave perturbations. Let us denote by

$$
(1.35)\qquad \begin{cases} \psi_\eta(\chi) := \displaystyle\int_\Omega \left(\frac{1}{2} |\chi|^2 + \frac{1}{\eta} W(\chi) + \frac{\eta}{2} |\nabla \chi|^2 \right) dx, \\[2ex] \psi_0(\chi) := \displaystyle\int_\Omega \left(\frac{1}{2} |\chi|^2 + I_{\{0,1\}}(\chi) \right) dx + \gamma \int_\Omega |D\chi|, \end{cases}
$$

and observe that for $\eta \geq 0$

$$
(1.36)\qquad \phi_{2,\eta}(e) = \inf_\chi \left(\frac{1}{2} \int_\Omega e^2\, dx + \psi_\eta(\chi) - \int_\Omega e\chi\, dx \right) =
$$

$$
= \frac{1}{2} \int_\Omega e^2\, dx - \sup_\chi \left(\langle e, \chi \rangle_{L^2(\Omega)} - \psi_\eta(\chi) \right) = \frac{1}{2} \int_\Omega e^2\, dx - \psi_\eta^*(e),
$$

where ψ_η^* is the conjugate function of ψ_η w.r.t. the $L^2(\Omega)$ scalar product.

Being ψ_η^* always convex, independently of the convexity of ψ_η, we have thus represented ϕ_η as a *concave* perturbation of the convex square L^2-norm. We know that ψ_η^* is C^1 only if ψ_η^{**} is (at least locally) uniformly convex, which is surely not verified for small values of η; moreover, we have also to take into account that our underlying space is $H^{-1}(\Omega)$ and not $L^2(\Omega)$.

Nonetheless, the gradient flow structure of these problems, and the particular decomposition (1.36) as well, will be sufficient to derive the existence of their solution by applying a general variational result, which allows for a great flexibility in the choice of the functionals and of the ambient spaces and, at least we hope, can be applied to several other different situations.

Inspired by this discussion, we will devote the following sections to the analysis of gradient flow equations in arbitrary Hilbert spaces, first of all handling carefully the matter of defining a *suitable* notion of subdifferential in this *non-convex* setting.

2. Abstract gradient flows in Hilbert spaces

Let H be a *separable Hilbert* space with scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$, and let

$$
(2.1)\qquad \phi : \mathrm{H} \to (-\infty, +\infty] \text{ a proper, } \textit{lower semicontinuous} \text{ function,}
$$

with non empty proper domain $D(\phi) := \{v \in \mathrm{H}: \phi(v) < +\infty\}$, such that for some $\tau_* > 0$

$$
(2.2)\qquad \text{the function } v \mapsto \frac{1}{2\tau_*} |v|^2 + \phi(v) \quad \text{has strongly compact sublevels.}
$$

For the moment, we denote by $\nabla\phi : \mathrm{H} \to 2^{\mathrm{H}}$, with proper domain $D(\nabla\phi) \subset D(\phi)$, a suitable choice for the subdifferential of ϕ, which we will make precise in the sequel. The evolution problem associated to the gradient flow equation for ϕ is

PROBLEM 2.1. *Given the initial datum $u_0 \in D(\phi)$, find a function $u : (0, T) \to \mathrm{H}$* such that

$$u'(t) + \nabla\phi(u(t)) \ni 0 \ \textit{a.e. in } (0, T), \ \textit{and } u(0) = u_0. \tag{2.3}$$

As we suggested in the previous section, we are interested in solutions of (2.3) which result from the approximating variational scheme

$$\begin{cases} U^0 = u_0, \quad U^n \in \operatorname*{argmin}_{V \in \mathrm{H}} \mathrm{F}(\tau, U^{n-1}; V), \quad n = 1, \dots, N, \\ \mathrm{F}(\tau, U^{n-1}; V) := \dfrac{|V - U^{n-1}|^2}{2\tau} + \phi(V), \end{cases} \tag{2.4}$$

as a limit (up to the possible extraction of a subsequence) for $\tau \downarrow 0$ of the piecewise constant interpolants $\overline{U}_\tau$ defined by (1.15).

This variational approach has been used by several authors in many different contexts (see *e.g.* [7, 13, 14, 18, 29, 1, 17, 15, 3]); in our framework, the existence of at least one *discrete solution* $\overline{U}_\tau$ associated to a sequence $\{U^n\}_{n=1}^N$ solving (2.4) is an immediate consequence of (2.1) and (2.2), at least for $\tau < \tau_*$.

The choice of $\nabla\phi$. A reasonable definition of $\nabla\phi$ should imply that any discrete solution $\{U^n\}_{n=1}^N$ of (2.4) solves as well the associated Euler equation

$$U_0 := u_0, \quad \frac{U^n - U^{n-1}}{\tau} + \nabla\phi(U^n) \ni 0 \quad n = 1, \dots, N. \tag{2.5}$$

Dealing with a minimization problem, a preliminary choice for $\nabla\phi$ could be the so called *Fréchet subdifferential* defined by

$$\xi \in \partial_F\phi(v) \ \Leftrightarrow \ v \in D(\phi), \liminf_{w \to v} \frac{\phi(w) - \phi(v) - \langle \xi, w - v \rangle}{|w - v|} \geqslant 0. \tag{2.6}$$

It is easy to see that a vector ξ is the Fréchet differential of ϕ at a point v if and only if it belongs to the Fréchet subdifferential of ϕ and $-\phi$. Moreover, for any *convex* functional ϕ the Fréchet subdifferential coincides with the usual one.

Among the general properties which are satisfied by $\partial_F\phi$, we mention the *convexity* of its values

$$\partial_F\phi(v) \quad \text{is a closed convex subset of H}, \quad \forall v \in \mathrm{H} \tag{2.7}$$

and the *chain rule*

$$\begin{cases} \text{if } t \mapsto v(t), \ t \mapsto \phi(v(t)) \ \text{are differentiable at } t_0, \\ \xi \in \partial_F\phi(v(t_0)), \ \text{then } \dfrac{d}{dt}\phi(v(t))\,|_{t=t_0} = \langle \xi, v'(t_0) \rangle. \end{cases} \tag{2.8}$$

Unfortunately, simple finite dimensional examples as $\phi(x) := -|x|$, $x \in \mathbb{R}$, show that for general Lipschitz concave functions ϕ (or concave perturbations of convex functions) the graph of $\partial_F\phi$ is not closed in the product space $\mathrm{H} \times \mathrm{H}$: therefore, it is natural to consider a version of its closure. At least two choices are possible:

DEFINITION 2.2 (*Limiting subdifferentials*). *We say that* ξ *belongs to the limiting subdifferential* $\partial_e \phi$ *of* ϕ *if*

$$\exists\, \xi_n,\, v_n \in \mathrm{H}:\ \xi_n \in \partial_F \phi(v_n),\ v_n \to v,\ \xi_n \rightharpoonup \xi,\ \limsup_{n\uparrow+\infty} \phi(v_n) < +\infty\,. \tag{2.9}$$

REMARK 2.3. The basic assumption of the theory developed by [19], as pointed out in [2], is that $\partial\phi \equiv \partial_e \phi$, *i.e.* the graph of $\partial_F \phi$ is strongly-weakly closed. As we mentioned before, this assumption seems too strong in view of applications to quasi-stationary phase field models. On the other hand, in general $\partial_e \phi$ does not satisfy (2.7) and (2.8) any more; we shall see that a general existence result can be proved if it satisfies at least one of these two properties.

Existence results for «Lyapunov» and «energy» solutions. Corresponding to (2.9), we can introduce two notions of solution for Problem 2.1:

DEFINITION 2.4. *We say that* $u \in H^1(0, T; \mathrm{H})$ *is a* Lyapunov solution *of Problem* 2.1 *if it satisfies the differential equation*

$$u(0) = u_0, \quad u'(t) + \partial_e \phi(u(t)) \ni 0 \ \textit{for a.e.}\ \ t \in (0, T), \tag{2.10}$$

and the energy inequality

$$\phi(u(t)) + \int_0^t |u'(\lambda)|^2 \, d\lambda \leq \phi(u_0) \quad \forall\, t \in (0, T). \tag{2.11}$$

We say that u is an energy solution *of Problem* 2.1 *if it satisfies the differential equation* (2.11) *and the energy identity*

$$\phi(u(t)) + \int_0^t |u'(\lambda)|^2 \, d\lambda = \phi(u_0) \quad \forall t \in (0, T). \tag{2.12}$$

It is not difficult to show that the family $\{\overline{U}_\tau\}_{\tau>0}$ is relatively compact with respect to the strong uniform convergence. Following [14], we call $GMM(u_0, \mathrm{F})$ the set of all limit points u as $\tau \downarrow 0$ such that

$$u \in GMM(u_0, \mathrm{F}) \ \Leftrightarrow\ \exists\, \tau_k \downarrow 0 : \overline{U}_{\tau_k}(t) \to u(t) \quad \forall t \in (0, T). \tag{2.13}$$

REMARK 2.5. In the case of a (quadratic perturbation of a) *convex* functional ϕ, we have

$$\partial_F \phi = \partial_e \phi\,;$$

existence, uniqueness, and regularity of the solution of (2.10) follow from the well known theory developed by Komura [16], Crandall-Pazy [12], Brézis [7]: cf. the monograph [8]. Further, the whole family $\overline{U}_\tau$ is converging and no further extraction of subsequences are needed; this property follows from *a priori* [11, 4, 25, 26] and *a posteriori* [22] estimates of the error $|u(t) - \overline{U}_\tau(t)|$; those estimates are independent of the compactness assumption (2.2), which therefore can be avoided. Up to now, no general error estimates are known in the non convex case.

The following results provide a description of $GMM(u_0, \mathrm{F})$ under some extra assumption on ϕ and its gradients.

THEOREM 1 (Lyapunov solutions). *Assume ϕ fulfills* (2.1, 2.2), *and suppose that*

$$\partial_e \phi(v) \text{ is a singleton for every } v \in D(\partial_e \phi). \tag{2.14}$$

Then, any function $u \in GMM(u_0, \mathrm{F})$ is a Lyapunov solution *of Problem* 2.1. *Moreover, if the additional* continuity assumption holds

$$v_n \to v, \quad \exists\ \xi_n \in \partial_F \phi(v_n) : \sup_n \{|\xi_n|, \phi(v_n)\} < +\infty \Rightarrow \phi(v_n) \to \phi(v), \tag{2.15}$$

then the following refined energy inequality *holds*

$$\int_s^t |u'(\sigma)|^2 d\sigma + \phi(u(t)) \leq \phi(u(s)), \quad \textit{for a.e. } s, t \in (0, T), \quad \textit{with } s \leq t. \tag{2.16}$$

THEOREM 2 (Energy solutions). *Assume ϕ fulfills* (2.1, 2.2) *and suppose that the following* chain rule inequality *holds*

$$\begin{cases} v \in H^1(0, T; \mathrm{H}), \ell \in L^2(0, T; \mathrm{H}) \textit{ and } \ell(t) \in \partial_e \phi(v(t)) \textit{ a.e. in } (0, T), \\ |\phi(v(t)) - \phi(v(s))| \leq \displaystyle\int_s^t |\ell(\sigma)||v'(\sigma)| d\sigma \ \forall 0 \leq s \leq t \leq T. \end{cases} \tag{2.17}$$

Then any function $u \in GMM(u_0, \mathrm{F})$ is an energy solution *of Problem* 2.1 *and $-u'(t)$ coincides with the projection of the origin on the* closed affine envelope *of $\partial_e \phi(u(t))$ for a.e. $t \in (0, T)$; in particular, it coincides with its element of minimal norm.*

REMARK 2.6. A particular case in which the chain rule (2.17) is satisfied is provided by *dominated concave perturbations* (cf. (1.36)) of convex functionals, *e.g.* when there exists a convex l.s.c. function $\psi : \mathrm{H} \to [0, +\infty]$ and positive constants α_ϱ, C_ϱ depending on $\varrho \in (0, +\infty)$ such that the sum $\phi + \psi$ is convex and

$$\begin{cases} \text{if } v \in D(\partial_F \phi), \theta \in \partial_F \phi(v) \text{ with } \max\left(\phi(v), |v|^2\right) \leq \varrho, \text{ then} \\ \partial\psi(v) \neq \emptyset, \quad |\xi| \leq \alpha_\varrho |\theta| + C_\varrho \quad \forall \xi \in \partial\psi(v). \end{cases} \tag{2.18}$$

Observe that in the case of a finite dimensional Hilbert space H, any globally defined concave functional ϕ satisfies (2.18); thus Theorem 2 can be considered as an extension to infinite dimensional Hilbert spaces of [5], where gradient flows of concave functionals in the euclidean spaces are considered.

REMARK 2.7. Theorems 1 and 2 are particular cases of more general results discussed in the forthcoming paper [24]; we refer the interested reader to this paper for the related proofs, which combine ideas from the Minimizing Movement approach and tools from the theory of the so called «Young measures» in spaces of infinite dimension.

3. Applications

The quasi-stationary phase field model. Recalling the discussion of Example 3 and the related «reduced entropy» formulation (1.27, 1.29, 1.33) in the first Section, we have the following result.

Theorem 3.1. *In the Hilbert space* $\mathrm{H} := H^{-1}(\Omega)$, *let us choose an initial datum* $e_0 \in L^2(\Omega)$ *and an element* $e \in GMM(e_0, \mathrm{F})$ *for the functional* F *defined as in* (2.4) *and with* $\phi = \phi_{2,\eta}$ *given by* (1.33). *Then* e *is an* energy solution *of Problem* 2.1, *and setting*

$$\theta \in L^2(0, T; H_0^1(\Omega)), \quad -\Delta\theta = -\partial_t e, \quad \chi := e - \theta, \tag{3.1}$$

the couple (e, χ) *is a solution of* (1.23) *with*

$$e, \chi \in L^2(0, T; H^1(\Omega)), \quad e - \chi \in L^2(0, T; H_0^1(\Omega)), \quad \partial_t e \in L^2(0, T; H^{-1}(\Omega)).$$

Remark 3.2. More generally, we can replace $\Phi_{2,\eta}$ of (1.8) by a functional of the type

$$\widetilde{\Phi} := \int_\Omega \frac{1}{2}|e - \chi|^2\, dx + F(\chi), \quad \widetilde{\phi}(e) := \inf_\chi \widetilde{\Phi}(e, \chi), \tag{3.2}$$

where $F : H^1(\Omega) \to (-\infty, +\infty]$ is a weakly lower semicontinuous, coercive functional, *i.e.*

$$\lim_{\|v\|_{H^1(\Omega)} \to +\infty} F(v) = +\infty .$$

We can thus consider the *generalized* phase-field system given by (1.1), coupled with the variational inequality

$$\chi(t) \in H^1(\Omega), \quad \widetilde{\Phi}(e(t), \chi(t)) \leq \widetilde{\Phi}(e(t), v) \quad \forall v \in H^1(\Omega), \text{ a.e. in } (0, T). \tag{3.3}$$

For such a system, a result analogous to Theorem 3.1 can be proved, by solving the gradient flow equation (2.10) for the functional $\widetilde{\phi}$ in $\mathrm{H} := H^{-1}(\Omega)$ (in the case of homogeneous Dirichlet boundary conditions on $e - \chi$) or in $\mathrm{H} := \big(H^1(\Omega)\big)'$ (in the case of homogeneous Neumann boundary conditions on $e - \chi$).

Remark 3.3. Keeping the notation of the previous Remark, a crucial property for the limiting subdifferentials of $\widetilde{\phi}$, which enlightens the link between the abstract formulation (2.10) and the system (1.23), is provided by the following formula

$$\xi \in \partial_e \widetilde{\phi}(e) \;\Rightarrow\; \xi = -\Delta(e - \chi), \quad \text{for some } \chi \in M(e), \tag{3.4}$$

where

$$M(e) := \mathrm{argmin}_\chi \widetilde{\Phi}(e, \chi). \tag{3.5}$$

The Stefan-Gibbs-Thomson problem. As it can be deduced from the preliminary discussion of Example 5, we can obtain another proof of the existence result of [18] for the Stefan-Gibbs-Thomson problem (1.25) by solving in $\mathrm{H} := H^{-1}(\Omega)$ the gradient flow equation (2.10) for the functional $\phi_{2,0}$ previously introduced in (1.34). In this case, taking into account Remark 3.3, the crucial property is that the set

$$M(e) := \operatorname{argmin}_{\chi \in BV(\Omega)} \Phi_{2,0}(e, \chi), \tag{3.6}$$

which is non-empty for every $e \in D(\phi_{2,0})$, satisfies

$$e \in D(\partial_e \phi_{2,0}) \Rightarrow \begin{cases} M(e) = \{\chi(e)\} \text{ is a singleton}, \\ \partial_e \phi_{2,0}(e) = -\Delta(e - \chi(e)). \end{cases} \tag{3.7}$$

So, $\partial_e \phi_{2,0}$ is trivially convex, and Theorem 1 yields the following result:

THEOREM 3.4. *In the Hilbert space* $\mathrm{H} := H^{-1}(\Omega)$, *let us choose an initial datum* $e_0 \in L^2(\Omega)$ *and an element* $e \in GMM(e_0, \mathrm{F})$ *for the functional* F *defined as in* (2.4) *and* $\phi = \phi_{2,0}$ *given by* (1.34). *Then e is a* Lyapunov solution *of Problem* 2.1 *and, defining* θ, χ *as in* (3.1), *the couple* (e, χ) *is a solution of* (1.25) *with*

$$e, \chi \in L^2(0, T; H^1(\Omega)), \quad e - \chi \in L^2(0, T; H_0^1(\Omega)), \quad \partial_t e \in L^2(0, T; H^{-1}(\Omega)).$$

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