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DIFFUSION AND CROSS-DIFFUSION IN PATTERN FORMATION

ABSTRACT. — We discuss the stability and instability properties of steady state solutions to single equations, shadow systems, as well as 2×2 systems. Our basic observation is that the more complicated the pattern are, the more unstable they tend to be.

KEY WORDS: Diffusion; Cross-diffusion; Shadow systems; Steady states; Stability.

0. INTRODUCTION

The purpose of this expository paper is to discuss the roles of diffusion and cross-diffusion in modeling pattern formation in various branches of science. To focus our attention more on diffusion and/or cross-diffusion, we shall limit our considerations to *autonomous* equations and systems with reaction terms involving only the unknowns but *not* their gradients. From the point of view of pattern formation, we shall mainly concentrate on the stability properties of steady states of those equations/systems under the homogeneous Neumann boundary condition (*i.e.* no flux boundary condition).

Using diffusion to model pattern formation has a long history. For instance, the well known concept of «Turing patterns» goes back to the celebrated paper of Turing [36] in 1952, in which an attempt was made to model the remarkable regeneration phenomenon of *hydra*. However, mathematical progress in understanding those patterns has only been made relatively recently.

It turns out to be a general principle that the stability properties of a steady state are closely related to the «shape» of the steady state. Roughly speaking, the more complicated the shape of the steady state, the less stable the steady state is.

For example, in Section 1 below we will show that for a solution of a single equation with homogeneous Neumann boundary condition to be stable, it must be a constant if the domain is convex – a nice result due to Matano [18]. This may be regarded as «stability implies triviality» for single equations. In Section 3 we will show that, in one space dimension and still under homogeneous Neumann boundary condition, for a (time-dependent) solution of a «shadow system» (*i.e.* a reaction-diffusion equation coupled with a non-local ordinary differential equation) to be stable, it must be (eventually) monotone in space. In short, «stability implies monotonicity» holds for shadow systems – a recent result due to [21]. For 2×2 systems, the situation can be very complicated and will be illustrated via an example in Section 4. Section 2 is devoted to the discussion of stability properties of single reaction-diffusion equations under homogeneous Dirichlet boundary condition.

In certain models, diffusion seems simply inadequate from either the modeling or mathematical points of view. For instance, in the classical Lotka-Volterra competition-diffusion system in population dynamics, no nonconstant steady state is possible when the competition is weak. Furthermore, it is also not entirely reasonable to assume that individuals in the competition system move around in a completely random fashion. Thus, an attempt was made in 1979 [31] to model segregation phenomena in competing species by taking inter-specific population pressure into account. This results in a cross-diffusion system which will be discussed in Section 5.

Finally, we remark that the article [20] appeared in 1998 is closely related to the present one. This current expository paper can be considered as an updated and more detailed version of [20].

1. Single equations with Neumann boundary conditions

We start our discussion on the stability analysis of solutions to single equations with homogeneous Neumann boundary conditions

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $f \in C^1(\boldsymbol{R})$, Ω is a bounded smooth domain in $\boldsymbol{R}^n$, ν is the unit outer normal to $\partial\Omega$. In order to discuss the notion of stability in an intuitive way, it is best to introduce the corresponding parabolic initial-boundary problem

$$\begin{cases} v_t = \Delta v + f(v) & \text{in } \Omega \times \boldsymbol{R}_+, \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times \boldsymbol{R}_+, \\ v(x, 0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{1.2}$$

A solution of (1.1) is said to be a *steady state* of (1.2), and a solution of u of (1.1) is said to be *stable* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|v(\cdot, t) - u(\cdot)\|_{L^\infty(\Omega)} < \varepsilon$ for all $t > 0$ provided that $\|v_0 - u\|_{L^\infty(\Omega)} < \delta$. A steady state u is said to be *asymptotically stable* if there exists $\delta > 0$ such that $\|v(\cdot, t) - u(\cdot)\|_{L^\infty(\Omega)} \to 0$ as $t \to \infty$ provided that $\|v_0 - u\|_{L^\infty(\Omega)} < \delta$. Naturally we say that u is unstable if it is not stable. It is also possible to discuss the stability of a solution u to (1.1) without going into its parabolic counterpart (1.2). This may be done via the «linearized stability». Standard arguments show that (see *e.g.* [18, Theorem 3.3, p. 423]) *if u is stable, then*

$$\mathcal{H}(\varphi) = \int_\Omega [\,|D\varphi|^2 - f'(u)\varphi^2] \geqslant 0 \tag{1.3}$$

for all $\varphi \in H^1(\Omega)$. Putting this in a different way, we look at the linearized problem of (1.1) at this particular solution u

$$\begin{cases} \Delta\varphi + f'(u)\varphi + \lambda\varphi = 0 & \text{in } \Omega, \\ \dfrac{\partial \varphi}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Denoting the first eigenvalue by λ_1, we have

$$\lambda_1 = \mathrm{Min}\,\{\mathcal{H}(\varphi)\,|\,\varphi \in H^1(\Omega),\ \|\varphi\|_{L^2(\Omega)} = 1\} \tag{1.5}$$

and, the assertion (1.3) follows from the following result which can be proved easily.

PROPOSITION 1.1. *If* $\lambda_1 < 0$, *then* u *is unstable.*

It is generally believed that the diffusion process is a «smoothing» and «trivializing» process. Thus in a closed system it seems reasonable to expect that *the only stable steady states are constants* (i.e. *spatially homogeneous*). It turns out that this is indeed the case *for single equations* (1.1) or (1.2) *provided that the domain* Ω *is nice*, e.g. *convex* (for systems of equations with different diffusion coefficients, this is generally not true and we shall discuss this later). This result was proved by Matano [18] in 1979 (see [5] also). Matano also showed that this result also holds for other domains such as annuli $\{x \in \boldsymbol{R}^n\,|\,a < |x| < b\}$, and gave a counterexample showing that *for certain non-convex domains, nontrivial stable steady states of* (1.1) *or* (1.2) *do exist.* Following Matano's proof, we see that the role of convexity is contained in the following.

LEMMA 1.2. *Let* Ω *be a bounded smooth convex domain in* $\boldsymbol{R}^n$. *Suppose that* $v \in C^3(\overline{\Omega})$ *with* $\dfrac{\partial v}{\partial \nu} = 0$ *on* $\partial\Omega$. *Then*

$$\frac{\partial}{\partial \nu}\,|Dv|^2 \leqslant 0 \quad \text{on } \partial\Omega .$$

The main result in this section may be stated as follows.

THEOREM 1.3. *If* Ω *is convex, then the only stable solutions of* (1.1) *are constants.*

The approach is to show that if u is a non-constant solution of (1.1), then λ_1 (given by (1.5)) must be negative. This is achieved by choosing appropriate test functions in (1.3). For a detailed proof we refer the interested readers to [18]. However, it seems natural to question *a priori* whether this approach would work. For, it seems that if $f' < 0$ on $\boldsymbol{R}$, then $\mathcal{H}(\varphi)$ is always positive for all $\varphi \not\equiv 0$ in $H^1(\Omega)$. It turns out that *if* $f' < 0$ *on* $\boldsymbol{R}$, *then* (1.1) has no non-constant solutions. To prove this, we let u be a solution of (1.1). Integrating the equation yields $\int_{0\Omega} f(u(x))\,dx = 0$ and thus there exists a unique a such that $f(a) = 0$ (since f is monotonically decreasing). Without loss of generality, we may assume that $a = 0$, *i.e.* $f(0) = 0$ (for, we may set $v \equiv u - a$, then $\Delta v + \tilde{f}(v) = 0$ and $\dfrac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$, where $\tilde{f}(v) = f(v + a)$. Thus $\tilde{f}(0) = f(a) = 0$). Assume $u \not\equiv 0$, then $\{x \in \Omega\,|\,u(x) > 0\}$ and $\{x \in \Omega\,|\,u(x) < 0\}$ are both non-empty. Let $u(P) = \max_{\overline{\Omega}} u$. Then $u(P) > 0$ and we have two cases:

(*i*) $P \in \Omega$. Since $f(u(P)) < 0$ ($f < 0$ on $\boldsymbol{R}_+$) we have $\Delta u(P) > 0$. On the other hand, u assumes its maximum at P, so $\Delta u(P) \leqslant 0$, a contradiction.

(*ii*) $P \in \partial\Omega$. Choose a ball $B \subsetneq \Omega$ which is tangent to $\partial\Omega$ at P with $u > 0$ on $\overline{B}$. Then $f(u(x)) < 0$ on $\overline{B}$, and $\Delta u(x) > 0$ on B with $u(P) = \max_{\overline{B}} u$. By Hopf's boundary point lemma, $\dfrac{\partial u}{\partial \nu} > 0$ at P, which contradicts the boundary condition $\dfrac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$.

In [18], an example is given to illustrate the importance of the convexity of Ω in the above theorem; namely, a *stable nonconstant solution* for (1.1) on a *dumbell-shaped domain* Ω was constructed with a *bistable* nonlinearity $f(u)$. Further research in this direction has been conducted by many authors, see [9, 11] and the references therein.

2. Single equations with Dirichlet boundary conditions

It is clear that we can define the notions of stability, asymptotic stability, linearized stability and instability for solutions to single equations under homogeneous Dirichlet boundary conditions

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

in a similar fashion as we did in § 1. Attempts have been made to obtain the counterpart of Theorem 1.3 in § 1. However, the situation here is more complicated. In [13] (see [33]) the following result was established:

Proposition 2.1. *Let Ω be a ball or an annulus. Then a stable solution of* (2.1) *must not change sign in Ω.*

We ought to remark that, in general, even if Ω is *convex*, a *stable* solution of (2.1) is *not* necessarily of one sign. Such an example was constructed in (H. Matano, private communication) and [33].

To prove Proposition 2.1, we proceed as follows. Note that an interesting intermediate step in the proof is that *stability implies radial symmetry*.

Let u be a stable steady state of

$$\begin{cases} v_t = \Delta v + f(v) & \text{in } \Omega \times \boldsymbol{R}_+, \\ v = 0 & \text{on } \partial\Omega \times \boldsymbol{R}_+, \\ v(x, 0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{2.2}$$

As the first step, we claim that *u must be radial*. To this end, we set

$$T_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \qquad i, j = 1, \dots, n$$

where $x = (x_1, \dots, x_n) \in \boldsymbol{R}^n$. A stragihtforward computation shows that

$$\Delta T_{ij} = T_{ij} \Delta .$$

Applying T_{ij} to the equation in (2.1), we have

$$\begin{cases} \Delta(T_{ij}u) + f'(u)T_{ij}u = 0 & \text{in } \Omega, \\ T_{ij}u = 0 & \text{on } \partial\Omega. \end{cases}$$

(recall that Ω has rotational symmetry). Since (1.3) holds for all $\varphi \in H_0^1(\Omega)$, $T_{ij}u$ is the first eigenfunction of the Schrödinger operator $\Delta + f'(u)$ if $T_{ij}u \not\equiv 0$. $T_{ij}u$ must then have only one sign in Ω, which is impossible. Hence $T_{ij}u \equiv 0$ for all $1 \leqslant i, j \leqslant n$ and our assertion is proved.

We now divide the rest of the proof into two cases.

CASE 1. $\Omega = B_b$ (*i.e.* Ω is the ball of radius b centered at the origin).

Since u is radial, it satisfies

$$\begin{cases} u_{rr} + \dfrac{n-1}{r} u_r + f(u) = 0 \text{ in } 0 \leqslant r \leqslant b, \\ u(b) = 0. \end{cases}$$

Suppose that u changes sign in $(0, b)$. Then there exists an $r_0 \in (0, b)$ such that $u_r(r_0) = 0$. Differentiating the above equation with respect to r, we obtain

$$(u_r)_{rr} + \frac{n-1}{r}(u_r)_r + f'(u)u_r - \frac{n-1}{r^2}u_r = 0.$$

Multiplying the above equation by $r^{n-1}u_r$ and integrating over $(0, r_0)$, we have by (1.3), that

$$-(n-1)\int_{B_{r_0}} \frac{u_r^2}{r^2}dx = \int_{B_{r_0}} |\nabla u_r|^2 dx - \int_{B_{r_0}} f'(u)|u_r|^2 dx \geqslant 0$$

since the function

$$\varphi(r) = \begin{cases} u_r(r) & \text{if } r \leqslant r_0, \\ 0 & \text{if } r_0 \leqslant r \leqslant b \end{cases}$$

belongs to $H_0^1(\Omega)$. Therefore, $u_r \equiv 0$ in $(0, r_0)$ which implies that u is a constant in B_{r_0} and thus in Ω, which is a contradiction.

CASE 2. $\Omega = \{x \in \boldsymbol{R}^n \mid a < |x| < b\}$ where $0 < a < b < \infty$. Now u satisfies

$$\begin{cases} u_{rr} + \dfrac{n-1}{r} u_r + f(u) = 0 \text{ in } (a, b), \\ u(a) = u(b) = 0. \end{cases}$$

Suppose that u changes sign, there exist r_0, r_1 such that $a < r_0 < r_1 < b$ and $u_r(r_0) = u_r(r_1) = 0$. Differentiating the above equation and repeating the same arguments as in Case 1, we obtain that $u_r \equiv 0$ in (r_0, r_1) (in the present case, the «test function» is chosen to be

$$\varphi(r) = \begin{cases} u_r(r) & \text{if } r_0 \leqslant r \leqslant r_1 \\ 0 & \text{if } r \geqslant r_1 \text{ or } r \leqslant r_0, \end{cases}$$

which clearly belongs to $H_0^1(\Omega)$). Thus u is a constant in (r_0, r_1) which again implies u is a constant in Ω, a contradiction, and the proof of Proposition 2.1 is complete.

For general nonlinearity $f(u)$, even positive solutions of (2.1) are often unstable. To guarantee stability for positive solutions, we need to restrict ourselves to special classes of nonlinearities.

PROPOSITION 2.2. *Let u be a positive solution of* (2.1) *where f satisfies the following condition*:

$$\frac{f(z)}{z} \text{ is decreasing in } z > 0. \tag{2.3}$$

Then u must be the only positive solution of (2.1) *and is stable.*

Well known examples include the case $f(u) = e^{-u}$. The sublinear case $f(u) = u^\tau$, $0 < \tau < 1$, although not C^1 in $\boldsymbol{R}$, can be handled by exactly the same arguments.

The proof of Proposition 2.2 makes use of the well known «monotone iteration method»; or, the «upper- and lower-solutions method», which is standard by now, and is therefore omitted here. We refer the interested readers to [19] for a detailed proof (for the monotone iteration method, see [30, 3]).

3. SHADOW SYSTEMS

From Theorem 1.3 in Section 1, it seems clear that single equations with homogeneous Neumann boundary conditions are simply inadequate in modeling nontrivial pattern in reality. Therefore we need to go to systems, and 2×2 systems already admit many stable steady state solutions with highly nontrivial patterns. As a first step in understanding 2×2 systems, we shall first study the shadow systems which, in some sense, lie between single equations and 2×2 systems.

For a 2×2 system

$$\begin{cases} u_t = d_1 \Delta u + f(u, v) & \text{in } \Omega \times [0, T), \\ v_t = d_2 \Delta v + g(u, v) & \text{in } \Omega \times [0, T), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times [0, T), \end{cases} \tag{3.1}$$

it has been known for quite some time that when both the diffusion coefficients d_1, d_2 are large, the dynamics of (3.1) is essentially determined by the corresponding system of ordinary differential equations, at least in many important cases. It has also been understood that when one of the diffusion coefficients, say, d_2 is large, the dynamics of

(3.1) is essentially determined by the following *shadow system*

$$(3.2)\qquad \begin{cases} u_t = d_1 \Delta u + f(u, \xi) & \text{in } \Omega \times [0, T), \\ \xi_t = |\Omega|^{-1} \int_\Omega g(u, \xi)\, dx & \text{in } [0, T), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times [0, T), \end{cases}$$

again in many important cases (see [8]). Note that the equation for v in (3.1) is replaced by an ordinary differential equation for ξ with *nonlocal* effects.

In [21] it is established that *any bounded (not necessarily stationary) stable solution of* (3.2) *in* $n = 1$ *must be either asymptotically homogeneous or eventually monotone in* x. In particular, the fact that «stability implies monotonicity» for the shadow system (3.2) we discussed at the beginning of this paper holds. To make the basic idea transparent we first treat the steady state case.

PROPOSITION 3.1 [21]. *Suppose that $f(u, v)$ and $g(u, v)$ are of class C^1. Then any spatially inhomogeneous non-monotone steady state of*

$$(3.3)\qquad \begin{cases} u_t = u_{xx} + f(u, \xi) \ \ \text{in } (0, 1) \times [0, \infty), \\ u_x(0, t) = 0 = u_x(1, t), \quad t > 0, \\ \xi_t = \int_0^1 g(u, \xi)\, dx, \quad t > 0, \end{cases}$$

is unstable.

The proof relies heavily on symmetry properties of the domain $\Omega = (0, 1)$ and thus is strictly one-dimensional.

We begin with the notion of k-symmetry. We say that a function $u(x)$ is *k-symmetric* in $[0, 1]$, $k \geq 2$, if the restriction $u(x)$, $x \in \left[\dfrac{i-1}{k}, \dfrac{i+1}{k}\right]$, is even symmetric with respect to the point $x = i/k$ for all $i = 1, 2, \ldots, k-1$, that is,

$$u(x) = u(2i/k - x) \ \text{ for all } x \in \left[\frac{i-1}{k}, \frac{i+1}{k}\right].$$

We call a solution (u, ξ) of (3.3) k-symmetric if $u(x, t)$ is k-symmetric for every t.

Let $(u(x), \xi)$ be a stationary solution of (3.3), that is, $(u(x), \xi)$ satisfies

$$(3.4)\qquad \begin{cases} u'' + f(u, \xi) = 0, \quad x \in (0, 1), \\ u'(0) = 0 = u'(1), \\ \int_0^1 g(u(x), \xi)\, dx = 0. \end{cases}$$

Clearly, if $(u(x), \xi)$ is a nonconstant non-monotone solution of (3.4), then $u(x)$ is k-symmetric with some $k \geq 2$ and monotone in $[0, 1/k]$.

Let us consider the following eigenvalue problem associated with the linearized operator around $u(x)$:

$$\tag{3.5} \begin{cases} \ell\,\varphi(x) = \varphi \ll (x) + f_u(u(x), \xi)\,\varphi(x), & x \in (0, 1), \\ \varphi'(0) = 0 = \varphi'(1). \end{cases}$$

According to the Sturm-Liouville theory, the eigenvalues of (3.5) are real numbers $\ell_0 > \ell_1 > \ell_2 > \ldots \to -\infty$, and the corresponding eigenfuncitons $\varphi_0, \varphi_1, \varphi_2, \ldots$, are characterized by the property that φ_j has exactly j zeros in $(0, 1)$. We assume that these eigenfunctions are normalized in $L^2(0, 1)$.

Next, let us consider the eigenvalue problem

$$\tag{3.6} \begin{cases} \tilde{\ell}\,\tilde{\varphi}(x) = \tilde{\varphi}''(x) + f_u(u(x), \xi)\,\tilde{\varphi}(x), & x \in (0, 1/k), \\ \tilde{\varphi}'(0) = 0 = \tilde{\varphi}'(1/k). \end{cases}$$

We denote by $\tilde{\ell}_j$ and $\tilde{\varphi}_j$ the jth eigenvalue and corresponding eigenfunction of (3.6), respectively. We assume that the eigenfunctions are normalized in $L^2(0, 1/k)$. Since $\tilde{\varphi}_j$ has exactly j zeros in $(0, 1/k)$, it follows from reflection and the number of zeros that

$$\tilde{\ell}_j = \ell_{jk}, \quad \tilde{\varphi}_j(x) \equiv \sqrt{k}\varphi_{jk}(x) \ \text{ on } [0, 1/k],$$

for all $j = 0, 1, 2, \ldots$.

LEMMA 3.2. *Let $w(x)$ be any k-symmetric function on $[0, 1]$. Then*

$$\int_0^1 w(x)\,\varphi_j(x)\,dx = 0, \quad \text{for all } j \neq 0, k, 2k, \ldots.$$

PROOF. Let $\langle \cdot, \cdot \rangle_{L^2(a,b)}$ denote the L^2-inner product on (a, b). By reflection, we have for $x \in (0, 1/k)$

$$\begin{aligned} w &= \sum_{j=0}^{\infty} \langle w, \tilde{\varphi}_j \rangle_{L^2(0,1/k)} \tilde{\varphi}_j \\ &= \sum_{j=0}^{\infty} k \langle w, \varphi_{jk} \rangle_{L^2(0,1/k)} \varphi_{jk} \\ &= \sum_{j=0}^{\infty} \langle w, \varphi_{jk} \rangle_{L^2(0,1)} \varphi_{jk}. \end{aligned}$$

Hence, again by reflection, we obtain

$$w = \sum_{j=0}^{\infty} \langle w, \varphi_{jk} \rangle_{L^2(0,1)} \varphi_{jk} \ \text{ on } [0, 1].$$

On the other hand, we can expand w as

$$w = \sum_{j=0}^{\infty} \langle w, \varphi \rangle_{L^2(0,1)} \varphi_j \ \text{ on } [0, 1].$$

Comparing these two expansions termwise, we obtain the conclusion.

LEMMA 3.3. *If $u(x)$ is k-symmetric, then the eigenvalues of* (3.5) *satisfy* $\ell_0 > \ell_1 > \ldots > \ell_{k-1} > 0$.

PROOF. Differentiating (3.4) by x, we obtain

$$\{u'(x)\}'' + f_u(u(x), \xi)\, u'(x) = 0, \quad x \in (0, 1).$$

We also have $u'(0) = u'(1) = 0$. Clearly $u'(x)$ has $k-1$ zeros in (0,1) and $\varphi_j(x)$ has exactly j zeros in (0,1). Then it follows from the Sturm comparison theorem that $\ell_{k-1} > 0$.

We now give a proof of Proposition 3.1.

PROOF OF PROPOSITION 3.1. Let $(u(x), \xi)$ be any spatially inhomogeneous non-monotone solution of (3.12), and consider the eigenvalue problem

$$\begin{cases} \lambda \Phi(x) = \Phi''(x) + f_u(u(x), \xi)\, \Phi(x) + f_v(u(x), \xi)\, \eta, \quad x \in (0, 1), \\ \lambda \eta = \int_0^1 \{ g_u(u(x), \xi)\, \Phi(x) + g_v(u(x), \xi)\, \eta \}\, dx, \\ \Phi'(0) = 0 = \Phi'(1). \end{cases} \tag{3.7}$$

Since $g_u(u(x), \xi)$ is k-symmetric with some $k \geqslant 2$, it follows from Lemma 3.2 that

$$\int_0^1 g_u(u(x), \xi)\, \varphi_j(x)\, dx = 0 \quad \text{for } j \neq 0, k, 2k, \ldots$$

Hence $(\lambda, \Phi, \eta) = (\ell_j, \varphi_j, 0)$ satisfies (3.7) if $j \neq 0, k, 2k, \ldots$ This implies that $(\widetilde{\Phi}, \tilde{\eta}) = (e^{\ell_j t} \varphi_j(x), 0)$ satisfies the linearized system for (3.3)

$$\begin{cases} \widetilde{\Phi}_t = \widetilde{\Phi}_{xx} + f_u(u, \xi)\, \widetilde{\Phi} + f_v(u, \xi)\, \tilde{\eta}, \quad 0 < x < 1, \quad t > 0, \\ \tilde{\eta}_t = \int_0^1 [g_u(u, \xi)\, \widetilde{\Phi} + g_v(u, \xi)\, \tilde{\eta}]\, dx, \; t > 0, \\ \widetilde{\Phi}_x(0, t) = 0 = \widetilde{\Phi}_x(1, t), \quad t > 0 \end{cases}$$

if $j \neq 0, k, 2k, \ldots$ Since $\ell_j > 0$ for $j = 1, 2, \ldots, k-1$ by Lemma 3.3, the steady state (u, ξ) is unstable.

The proof of the «parabolic» version of Proposition 3.1 is more involved, and we refer the interested readers to [21] for details.

Among major problems left open concerning (3.2) is perhaps the multi-dimensional analogue of Proposition 3.1 for, say, convex domains. Very little is known so far in this generality.

On the other hand, with more specific shadow systems, there are stability and instability results for domains in multi-dimensions. See *e.g.* [27].

4. Diffusion systems

Stability properties for diffusion systems have been studied for many important models. However, there seems to be no general results. In particular, the counterpart for the property that «stability implies triviality» for single equations, or that «stability implies monotonicity» for shadow systems, has not been established even for 2×2 diffusion systems. The situation here seems quite complicated, and remains as a major research direction.

Therefore, in this section, instead of surveying stability and instability results for various systems, we shall use an activator-inhibitor system to illustrate the role of diffusions in pattern formation.

The regeneration phenomenon of *hydra*, first discovered by A. Trembley [35] in 1744, is one of the earliest examples in morphogenesis. *Hydra*, an animal of a few millimeters in length, is made up of approximately 100,000 cells of about 15 different types. It consists of a «head» region located at one end along its length. Typical experiments on *hydra* involve removing part of its «head» region and transplanting it to other parts of the body column. Then, a new «head» will form if and only if the transplanted area is sufficiently far from the (old) «head». These observations have led to the assumption of the existence of two chemical substances – a slowly diffusing (short-range) activator and a rapidly diffusing (long-range) inhibitor. In 1952, A. Turing [36] argued, although diffusion is a smoothing and trivializing process in a system of a single chemical, for systems of two or more chemicals, different diffusion rates could force the uniform steady states to become unstable and lead to nonhomogeneous distributions of such reactants. This is now known as the «diffusion-driven instability». Exploring this idea further, in 1972, Gierer and Meinhardt [7] proposed the following activator-inhibitor system (already normalized) to model the above regeneration phenomenon of *hydra*:

$$\begin{cases} U_t = d_1 \Delta U - U + \dfrac{U^p}{V^q} & \text{in } \Omega \times [0, T), \\ \tau V_t = d_2 \Delta V - V + \dfrac{U^r}{V^s} & \text{in } \Omega \times [0, T), \\ \dfrac{\partial U}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega \times [0, T) \end{cases} \tag{4.1}$$

where, as before, Δ is the usual Laplacian, Ω is a bounded smooth domain in $\boldsymbol{R}^n$, ν denotes the outward unit normal to $\partial\Omega$, $T \leqslant \infty$, and the constants τ, d_1, d_2, p, q, r are all positive, $s \geqslant 0$ and

$$0 < \frac{p-1}{q} < \frac{r}{s+1}. \tag{4.2}$$

Here U represents the density of the slowly diffusing activator which activates both U and V, and V represents the density of the rapidly diffusing inhibitor which suppresses both U and V. Therefore, both U and V are positive, and

d_1 is very small while d_2 is very large. The parameter τ here reflects the *response* rate of V versus the change of U.

Condition (4.2) is mathematical, under which one can prove easily that the equilibrium $\overline{U} \equiv 1$ and $\overline{V} \equiv 1$ of the corresponding kinetic system (for ordinary differential equation)

$$\begin{cases} \overline{U}_t = -\overline{U} + \dfrac{\overline{U}^p}{\overline{V}^q} \\ \tau \overline{V}_t = -\overline{V} + \dfrac{\overline{U}^r}{\overline{V}^s} \end{cases}$$

is *stable if* $\tau < (s+1)/(p-1)$. However, once the diffusion terms are introduced with d_1 small and d_2 large in (4.1), linearized analysis of (4.1) shows that the equilibrium $U \equiv 1$ and $V \equiv 1$ becomes unstable and bifurcations occur, thus the «diffusion-driven instability» takes place.

Note that (4.1) does not have variational structure. One way to solve (4.1) is using the «shadow system» approach. More precisely, since d_2 is large, we divide the second equation in (4.1) by d_2 and let d_2 tend to ∞. It seems reasonable to expect that, for each fixed t, V tends to a (spatially) harmonic function that must be a constant by the boundary condition. That is, as $d_2 \to \infty$, V tends to a spatially homogeneous function $\xi(t)$. Thus, integrating the second equation in (4.1) over Ω, we reduce (4.1) to the following «shadow system»

$$\begin{cases} U_t = d_1 \Delta U - U + \dfrac{U^p}{\xi^q} & \text{in } \Omega \times [0, T), \\ \tau \xi_t = -\xi + \dfrac{1}{|\Omega|} \xi^{-s} \displaystyle\int_\Omega U^r(x, t)\, dx & \text{in } [0, T), \\ \dfrac{\partial U}{\partial \nu} = 0 & \text{on } \partial\Omega \times [0, T), \end{cases} \tag{4.3}$$

where $|\Omega|$ denotes the measure of Ω. Although the above reduction can be verified rigorously in some cases [22], we must point out that it is more important to solve (4.1) via solutions of (4.3). It turns out that the steady states of (4.3) and their stability properties are closely related to that of the original system (4.1) and that the study of the steady states of (4.3) essentially reduces to that of the following single equation (by a suitable scaling argument):

$$\begin{cases} \varepsilon^2 \Delta u - u + u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

In the case $n = 1$, a lot of work has been done by I. Takagi [34]. For $n \geqslant 2$, the situation becomes far more interesting. The pioneering work [22-24] have produced a single-peak spike-layer solution u_ε of (4.4) in 1993. Furthermore, steady states of the shadow system (4.3) as well as the original system (4.1) have been constructed from u_ε

– at least for small d_1 and large d_2, and their stability properties have been investigated [25-27].

5. A cross-diffusion system

Although diffusion is generally regarded as a trivializing process in single equations (see Section I above), we have seen how *different diffusion rates* could produce patterns strikingly different from trivial ones for 2×2 systems. However, for that to happen, *reaction terms are essential as well*: for some systems, no matter what the diffusion rates are, no nonconstant steady state could possibly exist. For example, the classical Lotka-Volterra competition-diffusion system takes the following form:

$$(5.1) \qquad \begin{cases} u_t = d_1 \Delta u + u(a_1 - b_1 u - c_1 v) & \text{in } \Omega \times (0, \infty), \\ v_t = d_2 \Delta v + v(a_2 - b_2 u - c_2 v) & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial \nu} = 0 = \dfrac{\partial v}{\partial \nu} & \text{on } \partial\Omega \times (0, \infty) \end{cases}$$

where all the constants a_i, b_i, c_i, d_i, $i = 1, 2$, are positive, and u, v are nonnegative. Here, as is explained in [37], u and v represent the population densities of two competing species (a nice and thorough reference for (2.11) is the recent monograph by Cantrell and Cosner [4]). For convenience, we set $A = \dfrac{a_1}{a_2}$, $B = \dfrac{b_1}{b_2}$, and $C = \dfrac{c_1}{c_2}$. It is well known that in the «weak competition» case, *i.e.*

$$(5.2) \qquad B > A > C,$$

the constant steady state $(u_*, v_*) \equiv \left(\dfrac{a_1 c_2 - a_2 c_1}{b_1 c_2 - b_2 c_1}, \dfrac{b_1 a_2 - b_2 a_1}{b_1 c_2 - b_2 c_1} \right)$ is *globally asymptotically stable regardless of the diffusion rates d_1 and d_2*. This implies, in particular, that *no nonconstant steady state can exist for any diffusion rates d_1, d_2*.

On the other hand, it seems not entirely reasonable to add just diffusions to models in population dynamics, since individuals do *not* move around completely randomly. In particular, while modeling segregation phenomena for two competing species one must take into account the population pressures created by the competitors. In an attempt to model segregation pheonomena between two competing species, Shigesada, Kawasaki and Teramoto [31] proposed in 1979 the following cross-diffusion model

$$(5.3) \qquad \begin{cases} u_t = \Delta[(d_1 + \varrho_{12} v) u] + u(a_1 - b_1 u - c_1 v) & \text{in } \Omega \times (0, T), \\ v_t = \Delta[(d_2 + \varrho_{21} u) v] + v(a_1 - b_2 u - c_2 v) & \text{in } \Omega \times (0, T), \\ \dfrac{\partial u}{\partial \nu} = 0 = \dfrac{\partial v}{\partial \nu} & \text{on } \partial\Omega \times (0, T), \end{cases}$$

where ϱ_{12} and ϱ_{21} represent the cross-diffusion pressures and are nonnegative (in fact, the model in [31] also includes «self-diffusion» pressures that turn out to be not so different from the usual diffusion as is shown in [14]. Here, for simplicity, we shall discuss only (5.3)).

We first focus on the effect of cross-diffusion on steady states. To illustrate the significance of cross-diffusions, we again go to the weak competition case (*i.e.* $B > A > C$) since in this case (5.3) has no nonconstant steady states if both $\varrho_{12} = \varrho_{21} = 0$ (we refer to [10] for some interesting disccusions on the ecological significance of coexistence, «competition-exclusion», and weak/strong competitions. One point of view is that whether « competition-exclusion» holds in nature is a matter of interpretation. See [37]). Recent work of Lou and myself [14, 15] show that, indeed, *if one of the two cross-diffusion rates, say* ϱ_{12}, *is large, then* (5.3) *will have nonconstant steady states provided that* d_2 *belongs to a proper range*. On the other hand, *if both* ϱ_{12} *and* ϱ_{21} *are small, then* (5.3) *will have no nonconstant steady states under the condition* (5.2). This shows the introduction of cross-diffusion does seem to help create patterns.

In the «strong competition» case, *i.e.* $B < A < C$, even the situation of steady states solutions of (5.1) becomes more interesting and complicated, and is *not* completely understood. Nonetheless, cross-diffusions still have similar effects in help creating nontrivial patterns of (5.3). We refer the interested readers to [14, 15] for details.

So far in this section, we have only touched upon the existence and nonexistence of nonconstant steady states. It seems natural and important to ask if we can derive any qualitative properties (such as the spike-layers in the previous section) of those steady states. Our first step in this direciton is to classify *all the possible (limiting) steady states* as one of the cross-diffusion pressures tends to infinity.

THEOREM 5.1 [15]. *Suppose for simplicity that* $\varrho_{21} = 0$. *Suppose further that* $B \neq A \neq C$, $n \leqslant 3$, *and* $\dfrac{a_2}{d_2} \neq \lambda_k$ *for all* k, *where* λ_k *is the kth eigenvalue of* $-\Delta$ *on* Ω *with zero Neumann boundary data. Let* (u_j, v_j) *be a nonconstant steady state solution of* (5.3) *with* $\varrho_{12} = \varrho_{12,j}$. *Then by passing to a subsequence if necessary, either* (*i*) *or* (*ii*) *holds as* $\varrho_{12,j} \to \infty$, *where*

(*i*) $\left(u_j, \dfrac{\varrho_{12,j}}{d_1} v_j\right) \to (u, v)$ *uniformly,* $u > 0$, $v > 0$, *and*

$$\text{(5.4)} \qquad \begin{cases} d_1 \Delta[(1+v)u] + u(a_1 - b_1 u) = 0 & \text{in } \Omega, \\ d_2 \Delta v + v(a_2 - b_2 u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 = \dfrac{\partial v}{\partial \nu} & \text{on } \partial\Omega; \end{cases}$$

and

(*ii*) $(u_j, v_j) \to \left(\dfrac{\zeta}{w}, w\right)$ *uniformly,* ζ *is a positive constant,* $w > 0$, *and*

$$\text{(5.5)} \qquad \begin{cases} d_2 \Delta w + w(a_2 - c_2 w) - b_2 \zeta = 0 & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \frac{1}{w}\left(a_1 - \frac{b_2 \zeta}{w} - c_1 w\right) = 0. \end{cases}$$

The proof is quite lengthy. The most important step in the proof is to obtain *a priori* bounds on steady states of (5.3) that are independent of ϱ_{12}.

We ought to remark that both systems (5.4) and (5.5) possess spike-layer solutions. For instance, using a suitable change of variables, the equation in (5.5) may be transformed into (4.4) with $p = 2$. Thus our results in [22-24] apply. Perhaps we ought to point out that in fact, what is important is the ratio of cross-diffusion versus diffusion ϱ_{12}/d_1 in which d_1 can also vary. A deeper classification result is obtained in [15] as $\varrho_{12} \to \infty$ in (5.3) in terms of various possibilities of ϱ_{12}/d_1 and d_1.

To see how (4.4) turns up in (5.5), at least heuristically, we proceed as follows. Formally, setting

$$\zeta = u_* v_* \quad \text{and} \quad w = v^* - \varphi, \tag{5.6}$$

we have

$$d_2 \Delta\varphi - (c_2 v_* - b_2 u_*)\varphi + c_2 \varphi^2 = 0. \tag{5.7}$$

Rescaling (5.7) we obtain (4.4) provided that

$$c_2 v_* - b_2 u_* > 0, \tag{5.8}$$

which is equivalent to

$$\begin{cases} \frac{1}{2}(B + C) > A \ \text{ if } \ B > A > C, \\ \frac{1}{2}(B + C) < A \ \text{ if } \ B < A < C. \end{cases} \tag{5.9}$$

Note that in (5.6) we need $w > 0$, or, $v^* > \varphi$. In $n = 1$ this is guaranteed by

$$A > \frac{1}{4}(B + 3C). \tag{5.10}$$

Under these conditions, our results in [22-24] imply that (2.17) has spike-layer solutions for d_2 small. Observe that those solutions tend to 0 as $d_2 \to 0$ except at isolated points. Let φ be *e.g.* a solution of (4.4) guaranteed by [22-24]. Then the pair $(w, u_* v_*)$ satisfies the differential equation with the homogeneous Neumann boundary condition in (5.5), and it *almost* satisfies the integral constraint in (5.5) since w is close to v_* a.e. for d_2 small. It is then not hard to find a solution, for d_2 small, near the pair $(w, u_* v_*)$ by the Implicit Function Theorem, as was done in [15].

Although (5.4) is still an elliptic system, it is a bit easier to analyze than the original one. We refer the interested reader to [15] for details.

It turns out that both alternatives (*i*) and (*ii*) in Theorem 5.1 occur under suitable conditions. Therefore, to understand the steady states of (5.3) a good model would be (5.4) or (5.5), at least when ϱ_{12} is large. In the recent work of Lou, Yotsutani and myself [17], we were able to achieve an almost *complete* understanding of the «shadow» system (5.5) for $n = 1$ (and Ω is an interval, say, (0,1)). To illustrate our results, we include the following

THEOREM 5.2. *Suppose $B < C$. Then* (5.5) *does not have any nonconstant solution if either one of the following two conditions holds*:

(*i*) $d_2 \geq a_2/\pi^2$

(*ii*) $A \leq B$

THEOREM 5.3. *Suppose $B < C$. Then* (5.5) *has a nonconstant solution if $d_2 \leq a_2/\pi^2$ and $A \geq (B + C)/2$.*

The case $d_2 \leq a_2/\pi^2$ and $B < A < (B + C)/2$ is more delicate – existence holds for d_2 closer to a_2/π^2 while nonexistence holds when d_2 is near 0.

The behavior of solutions is also obtained for d_2 close to one of the two endpoints, 0 or a_2/π^2.

THEOREM 5.4. (*i*) *As $d_2 \to a_2/\pi^2$, $(w, \zeta) \to (0, 0)$ in such a way that*

$$\frac{\zeta}{w} \to \frac{a_2(1+\mu)}{2[\mu + (1-\mu)\sin^2(\pi x/2)]}$$

uniformly on $[0, 1]$ *where* $\mu = (2A/B) - 1 - 2\sqrt{(A/B)^2 - (A/B)} \in (0, 1]$.

(*ii*) *As $d_2 \to 0$ we have*

(α) *if* $A < \dfrac{B+3C}{4}$, *then*

$$\zeta \to \frac{a_2^2}{b_2 c_2} \cdot \frac{(B-A)(A-C)}{(B-C)^2}$$

$$w(0) \to 2\frac{a_2}{c_2} \cdot \frac{A - (B+3C)/4}{B-C}$$

$$w(\cdot) \to \frac{a_2}{c_2} \cdot \frac{B-A}{B-C} \quad \text{on } (0, 1],$$

(β) *if* $A \geq \dfrac{B+3C}{4}$, *then* $\zeta \to \dfrac{3}{16} \cdot \dfrac{a_2^2}{b_2 c_2}$, $w(0) \to 0$, *and* $w \to \dfrac{3a_2}{4c_2}$ *on* $(0, 1]$.

It seems interesting to note that the limits in (β) above are independent of a_1, b_1, c_1.

Our method of proof here is a bit unusual: we convert the problem of solving (w, ζ) of (5.5) to a problem of solving its «representation» in a different parameter space. This is done first *without* the integral constraint in (5.5). Then we use the integral constraint to find the «solution cuve» in the new parameter space as the diffusion rate d_2 varies. This method turns out to be very powerful as it gives fairly precise information about the solution.

Of course, our ultimate goal is to be able to obtain the steady state of (5.3) from our knowledge of the simpler limiting systems (5.4) or (5.5). This turns out to be possible, at least in the one-dimensional case $\Omega = [0, 1]$, as the next two results show (for simplicity, we shall assume that $\varrho_{21} = 0$ in the next two theorems).

THEOREM 5.5 [15]. *Suppose that $A > B$. There exists a small $d^* > 0$ such that for any $d_2 \in (0, d^*)$, we can find a large $\tilde{d} > 0$ such that if $d_1 > \tilde{d}$ is fixed, then there exists* *a large* $\alpha > 0$ *such that if* $\varrho_{12} > \alpha$, (5.3) *has a non-constant positive steady state* (u, v), *with* $(u, \varrho_{12} v) \to (\bar{u}, \bar{v})$ *uniformly in* $[0, 1]$ *as* $\varrho_{12} \to \infty$, *where* $(\bar{u}, \bar{v})$ *is a non-constant positive solution of* (5.4).

THEOREM 5.6 [15]. *Suppose that* $d_1 > 0$ *is fixed and that either* $A \in \left(\frac{1}{2}(B+C), \left(\frac{1}{4}B + \frac{3}{4}C\right)\right)$ *or* $A \in \left(\left(\frac{1}{4}B + \frac{3}{4}C\right), \frac{1}{2}(B+C)\right)$. *There exists a small* $d^* > 0$ *such that for* $d_2 \in (0, d^*)$ *we can find a large* $\alpha > 0$ *such that if* $\varrho_{12} > \alpha$, *has a non-constant positive steady state* (u, v) *with* $(u, v) \to \left(\frac{\zeta}{w}, w\right)$ *as* $\varrho_{12} \to \infty$ *where* $w > 0$, *non-constant and* (w, ζ) *is a solution of* (5.5).

The proofs of Theorems 5.5 *and* 5.6 involve careful analysis of the linearized systems of (5.4) and (5.5) at their non-constant positive solutions.

The stability properties of the various steady states obtained in the results above are yet to be studied. It is worth noting that even the local existence question for (5.3) is highly nontrivial and was resolved in a series of long papers by H. Amann [1, 2] in the early 1990s. The global existence question for (5.3) remains open, although progress has been made under various assumptions on the smallness of the cross-diffusion coefficients ϱ_{12}, ϱ_{21}, the dimension n, or the initial values $u(x, 0)$, $v(x, 0)$. Interesting related results with different but similar reaction terms have also been obtained (see *e.g.* [6, 12, 16, 28, 29, 32, 38], and the references therein).

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