Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Alberto Bressan

Some remarks on multidimensional systems of conservation laws

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **15** (2004), n.3-4, p. 225–233.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2004_9_15_3-4_225_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2004.

Rend. Mat. Acc. Lincei
s. 9, v. 15:225-233 (2004)

ALBERTO BRESSAN

SOME REMARKS ON MULTIDIMENSIONAL SYSTEMS OF CONSERVATION LAWS

ABSTRACT. — This note is concerned with the Cauchy problem for hyperbolic systems of conservation laws in several space dimensions. We first discuss an example of ill-posedness, for a special system having a radial symmetry property. Some conjectures are formulated, on the compactness of the set of flow maps generated by vector fields with bounded variation.

KEY WORDS: Hyperbolic system; Conservation laws; Several space dimensions.

1. HYPERBOLIC SYSTEMS WITH THE COMMUTATIVE PROPERTY

A nonlinear system of conservation laws in several space dimensions takes the form

$$\frac{\partial}{\partial t} u + \sum_{\alpha=1}^{m} \frac{\partial}{\partial x_\alpha} F^\alpha(u) = 0. \tag{1.1}$$

Here $u = (u_1, \ldots, u_n)$ is the vector of conserved quantities and $x = (x_1, \ldots, x_m)$ is the space variable. By the chain rule, every smooth solution of (1.1) satisfies the quasilinear system

$$u_t + \sum_{\alpha=1}^{m} A^\alpha(u) u_{x_\alpha} = 0, \qquad A^\alpha \doteq D_u F^\alpha. \tag{1.2}$$

For every $u \in \mathbb{R}^n$, we assume that the system satisfies the following

HYPERBOLICITY CONDITION. For every unit vector $\xi = (\xi_1, \ldots, \xi_m)$, the matrix

$$A(u, \xi) \doteq \sum_\alpha \xi_\alpha A^\alpha(u)$$

has n real eigenvalues $\lambda_1(u, \xi), \ldots, \lambda_n(u, \xi)$ and a basis of real eigenvectors $r_1(u, \xi), \ldots, r_n(u, \xi)$. Because of the strong nonlinearity of the equations, smooth initial data can develop shocks in finite time. One can hope to find global solutions only within a space of discontinuous functions. The conservation equations must then be interpreted in a distributional sense.

Scalar conservation laws in several space dimensions [11], as well as hyperbolic systems in one space dimension [17, 3] are now fairly well understood. On the contrary, developing a mathematical theory of multi-dimensional systems of conservation laws remains a challenging task. Indeed, at the present time not even the global existence of solutions is known, in any meaningful generality. The existing literature is mainly concerned with

1. Local existence of smooth solutions in a Sobolev space H^s, where s is sufficiently large to allow the embedding $H^s \subset \mathcal{C}^1$. Existence and uniqueness of solutions are obtained up to the first time where one of the first derivatives blows up [9, 16]. A special case where solutions remain smooth for all times was studied in [15].

2. Special solutions, having a particularly simple structure. For example, piecewise smooth solutions having one shock across a moving hypersurface [12], or solutions of two-dimensional Riemann problems [18], where the initial data are constant over sectors of the (x_1, x_2)-plane.

However, having solved these special cases we are still a long way from a full understanding of the general solution. In particular, unlike the one-dimensional case, it is not clear whether one can recover the solution of the general Cauchy problem by approximations in terms of several Riemann problems. In our view, a more promising approach is to look first at special classes of hyperbolic systems, possessing additional properties that may simplify their analysis. In each case, one should study the whole set of solutions, checking whether the evolution equations determine a flow depending continuously on the initial data.

A special class of multi-dimensional hyperbolic systems, which can be naturally singled out, consists of those systems of the form (1.2) where all matrices A^α commute. We observe that

$$A^\alpha A^\beta = A^\beta A^\alpha \tag{1.3}$$

provided that the $n \times n$ hyperbolic matrices A^α, A^β admit a common basis of eigenvectors $\{r_1, \dots, r_n\}$. The relevance of this assumption can be easily appreciated in connection with the Cauchy problem for a linear system with constant coefficients:

$$u_t + \sum_\alpha A^\alpha u_{x_\alpha} = 0, \qquad u(0, x) = \bar{u}(x). \tag{1.4}$$

Let $\{l_1, \dots, l_n\}$ be a common basis of left eigenvectors of the matrices A^α, normalized so that

$$l_i \cdot r_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The general solution can now be explicitly written as the superposition of n travelling waves:

$$u(t, x) = \sum_i \phi_i(x - \boldsymbol{c}_i t) r_i, \qquad \phi_i(s) = l_i \cdot \bar{u}(s). \tag{1.5}$$

Here the speed of the i-th wave is $\boldsymbol{c}_i = (c_{i,1}, \dots, c_{i,m})$. For $i = 1, \dots, n$ and $\alpha = 1, \dots, m$, the component $c_{i,\alpha}$ is the eigenvalue of the matrix A^α corresponding to the eigenvector r_i.

From the explicit formula (1.5), it clear that the solution operator $\bar{u} \mapsto S_t \bar{u} \doteq u(t)$ is continuous as a map from $\boldsymbol{L}^p \mapsto \boldsymbol{L}^p$, for all $p \in [0, \infty[$. We remark that, writing the solution by means of a Fourier transform, one always obtains the continuity of the solution operator S_t as a map from $\boldsymbol{L}^2$ into itself [16]. Indeed, the $\boldsymbol{L}^2$ norm is preserved under Fourier transform. On the other hand, a deep result of P. Brenner [2] shows that, for $p \neq 2$, the solution operator S_t is continuous as a linear transformation within $\boldsymbol{L}^p$ only if all matrices A^α commute. Based on this result, J. Rauch proved that, for a non-commutative system of conservation laws, the total variation of a solution can be amplified by an arbitrarily large factor [14]. In particular, no a priori BV bounds can be available. This is in sharp contrast with the one dimensional case, where the basic existence and uniqueness theory heavily rely on estimates on the total variation [10, 6, 3]. In order to obtain existence results for multidimensional systems, one should therefore search for new compactness theorems, not based on a priori bounds on the total variation.

2. A simple commutative system

Among systems that enjoy the commutativity property, we can further specialize to those of the special form

$$\frac{\partial}{\partial t} u_i + \sum_{\alpha=1}^{m} \frac{\partial}{\partial x_\alpha} \big(f_\alpha(|u|)\, u_i \big) = 0. \tag{2.1}$$

Such systems have been used as simplified models for magneto-hydrodynamics, and are discussed in [7, 16]. See also [13] for a detailed study of the one-dimensional case.

The solution of the Cauchy problem for (2.1) with initial data

$$u(0, x) = \bar{u}(x) \tag{2.2}$$

can be constructed in two steps.

1. The norm $\varrho \doteq |u|$ is obtained by solving the Cauchy problem for a scalar conservation law

$$\varrho_t + \sum_{\alpha=1}^{m} \big(f_\alpha(\varrho)\,\varrho\big)_{x_\alpha} = 0 \qquad \varrho(0, x) = |\bar{u}(x)|. \tag{2.3}$$

2. To compute the angular component $\theta \doteq u/|u|$, we observe that θ is constant along trajectories of the discontinuous O.D.E.

$$\dot{x} = f\big(\varrho(t, x)\big), \tag{2.4}$$

where $f = (f_1, \dots, f_m)$. Calling $t \mapsto x(t) \doteq \Phi_t(y)$ the trajectory starting at $x(0) = y$, we thus have

$$\theta\big(t, \Phi_t(y)\big) = \theta(0, y) = \bar{u}(y)/|\bar{u}(y)|.$$

Given $t > 0$ and $x \in \mathbb{R}^m$, call $y \doteq \Phi^{-t} x$ be the (unique) point such that $\Phi_t y = x$. If the measurable maps Φ^{-t} are well defined (up to sets of zero measure), the angular component of the solution of (2.1)-(2.2) is then obtained as

$$\theta(t, x) = \bar{\theta}\big(\Phi^{-t} x\big), \tag{2.5}$$

where $\bar{\theta} \doteq \bar{u}/|\bar{u}|$.

Assuming $\bar{u} \in \boldsymbol{L}^\infty$, by the fundamental result of Kruzhkov [11] one can find a unique solution $\varrho = \varrho(t, x)$ of the scalar conservation law (2.3), continuously depending on the initial data in the $\boldsymbol{L}^1$ norm. On the other hand, when this solution ϱ develops shocks, the O.D.E. (2.4) has a discontinuous right hand side. In the literature, no existence-uniqueness theorem is yet available which covers this case.

The recent analysis in [4] shows that, even with bounded initial data, in two space dimensions the Cauchy problem can be ill posed in $\boldsymbol{L}^1$.

Proposition 1. *There exists a flux function $f = (f_1, f_2)$ and a sequence of initial data $\bar{u}_\nu$, $\nu \geq 1$, such that the following holds.*

- *f is Lipschitz continuous, piecewise affine.*
- *Each u_ν is piecewise constant, namely $\bar{u}_\nu(x) \in \{2, 3, 4\}$.*
- *As $\nu \to \infty$ one has the convergence of the initial data $\bar{u}_\nu \to \bar{u}$ in $\boldsymbol{L}^1$.*
- *The corresponding solutions $u_\nu(t, \cdot) = S_t \bar{u}_\nu$ are well defined for all $t > 0$.*
- *There exists $\tau > 0$ such that the sequence $u_\nu(\tau, \cdot)$ does not converge in $\boldsymbol{L}^1$. It admits a weak limit, which is NOT a weak solution of the nonlinear system* (2.1).

A counterexample with the above properties was constructed in [4]. For each $\nu \geq 1$, one can always solve (2.2) and obtain the corresponding density function $\varrho_\nu = \varrho_\nu(t, x)$. By the results in [11], the $\boldsymbol{L}^1$ convergence $\varrho_\nu \to \varrho$ holds. However, as shown in figure 1, for this limit function ϱ the corresponding O.D.E. (2.4) now has a mixing property. Points which initially lie on the gray region Ω_1 are transported by the flow onto the region Ω_2, then onto Ω_3, etc. After a finite amount of time, the flow $\boldsymbol{\Phi}_t$ is no longer invertible and the representation (2.5) fails.

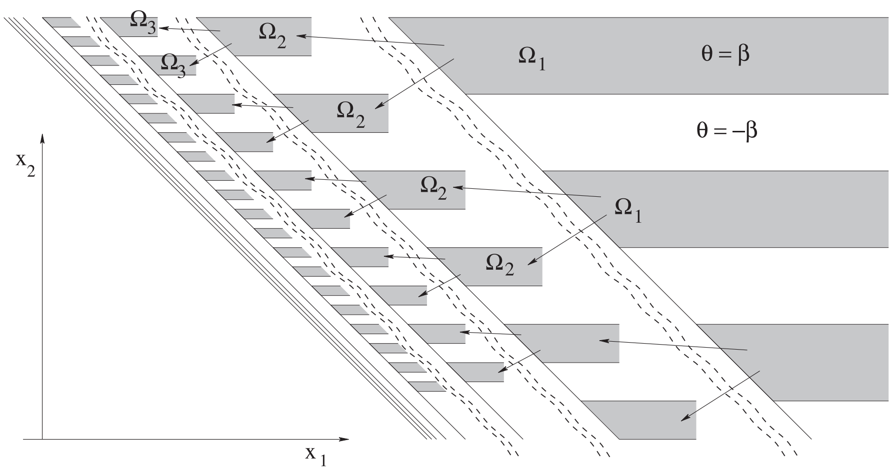

Fig. 1.

3. Flows of BV vector fields

In the previous example, compactness was lost because of oscillations. We remark that the conservation equations imply

$$\int_{\Phi_t(\Omega)} \varrho(t, x)\, dx = \int_{\Omega} \varrho(0, x)\, dx \tag{3.1}$$

for every $t \geq 0$ and every measurable set $\Omega \subset \mathbb{R}^m$. Here $\Phi_t : x(0) \mapsto x(t)$ is the flow map generated by (2.4). If the initial density satisfies $0 < a \leq \varrho(0, x) \leq b$, a standard comparison argument yields

$$a \leq \varrho(t, x) \leq b. \tag{3.2}$$

Setting $\kappa \doteq b/a$, from (3.1)-(3.2) we deduce

$$\frac{1}{\kappa}\,\mathrm{meas}\,(\Omega) \leq \mathrm{meas}\,\big(\Phi_t(\Omega)\big) \leq \kappa\,\mathrm{meas}\,(\Omega) \tag{3.3}$$

for every $t \geq 0$ and $\Omega \subset \mathbb{R}^m$. In one space dimension, the estimate (3.3) is sufficient to guarantee the Lipschitz continuity of the flow. Indeed, for any two trajectories x_1, x_2, the identity

$$\int_{x_1(t)}^{x_2(t)} \varrho(t, x)\, dx = \int_{x_1(0)}^{x_2(0)} \varrho(0, x)\, dx$$

and the assumption $\varrho \in [a, b]$ imply

$$x_2(t) - x_1(t) \leq \frac{b}{a}\big(x_2(0) - x_1(0)\big).$$

In the multidimensional case, for initial data in $\boldsymbol{L}^\infty$ the counterexample in [4] shows that an infinite amount of mixing can take place. This determines a loss of compactness and hence the ill posedness of the Cauchy problem. An interesting open question is whether the system (2.1) can be well posed within the class of functions such that

$$\varrho \doteq |u| \in BV, \qquad \theta \doteq \frac{u}{|u|} \in \boldsymbol{L}^\infty .$$

In this case, one has to solve a possibly discontinuous O.D.E. of the form

$$\dot{x} = g(t, x) \tag{3.4}$$

where the right hand side $g(t, x) \doteq f\big(\varrho(t, x)\big)$ is a time-dependent vector field with bounded variation. For discontinuous vector fields with divergence is in $\boldsymbol{L}^\infty$ or in $\boldsymbol{L}^1$, existence and uniqueness of the corresponding flow has been studied by DiPerna and Lions [8] and very recently by Ambrosio [1]. However, in the presence of shocks, the divergence of g becomes a measure which is not absolutely continuous w.r.t. Lebesgue measure. The existence of a measurable flow for (3.4) remains an open problem (*). A basic step in this direction would involve a proof of the following conjecture.

(*) Added in proof: see however the recent paper by L. Ambrosio - C. De Lellis, *Hyperbolic conservation laws in several space dimensions*. International Mathematics Research Notices, 41, 2003, 2205-2220.

Consider a sequence of time-dependent, smooth vector fields $g_\nu : [0, T] \times \mathbb{R}^m \mapsto \mathbb{R}^m$. Assume that they all satisfy the uniform bounds

$$\|g_\nu\|_{L^\infty} \leq C_1, \tag{3.5}$$

$$\|g_\nu\|_{BV} \doteq \int_0^T \int_{\mathbb{R}^m} \left| \frac{\partial}{\partial t} g_\nu \right| + \sum_{\alpha=1}^m \left| \frac{\partial}{\partial x_\alpha} g_\nu \right| dx dt \leq C_2 . \tag{3.6}$$

Call $t \mapsto x(t) \doteq \Phi_t^\nu(y)$ the solution of

$$\dot{x} = g_\nu(t, x), \qquad x(0) = y .$$

Moreover, assume that the fluxes Φ_t^ν are all nearly incompressible, so that, for every bounded set $\Omega \subset \mathbb{R}^m$,

$$\frac{1}{C_3} \text{meas}(\Omega) \leq \text{meas}\big(\Phi_t^\nu(\Omega)\big) \leq C_3 \,\text{meas}(\Omega), \tag{3.7}$$

for some constant C_3 and all $t \in [0, T]$, $\nu \geq 1$.

CONJECTURE 1. With the above assumptions, by possibly extracting a subsequence one has the convergence

$$\Phi_t^\nu \to \Phi_t \qquad \text{in } \boldsymbol{L}^1_{\text{loc}} \tag{3.8}$$

for some measurable flow Φ_t, also satisfying (3.7).

We remark that, by a standard compactness theorem [19], one can always extract a subsequence $f_{\nu'}$ converging to some BV vector field f in $\boldsymbol{L}^1_{\text{loc}}$. However, one cannot use the same argument here, to prove compactness of the family of flows, because the functions Φ_t^ν may have arbitrarily large total variation.

4. MIXING FLOWS

Because of the assumption (3.7), loss of compactness can only be due to an unbounded amount of oscillations in the flows Φ^ν. It would thus be interesting to provide a quantitative estimate on the amount of mixing that can be generated by a BV vector field. A convenient setting for this problem can be the following. Consider the m-dimensional torus $\boldsymbol{T}^m \doteq \mathbb{R}^2/\mathbb{Z}^2$. Let $f = f(t, x)$ be a smooth, time dependent vector field on $\boldsymbol{T}^m$, and call $y \mapsto \Phi_t(y)$ the corresponding flow map. In other words, $\Phi_t(y) \doteq x(t, y)$, where $t \mapsto x(t, y)$ is the solution of the Cauchy problem

$$\dot{x} = f(t, x) \qquad x(0) = y . \tag{4.1}$$

Fix any $\Omega \subset \boldsymbol{T}^m$. We say that the map Φ_t *mixes* Ω *up to scale* ε if, for every ball B_ε of radius ε one has (fig. 2)

$$\frac{\text{meas}(\Omega)}{2} \cdot \text{meas}(B_\varepsilon) \leq \text{meas}\big(B_\varepsilon \cap \Phi_t(\Omega)\big) \leq 2\, \text{meas}(\Omega) \cdot \text{meas}(B_\varepsilon). \tag{4.2}$$

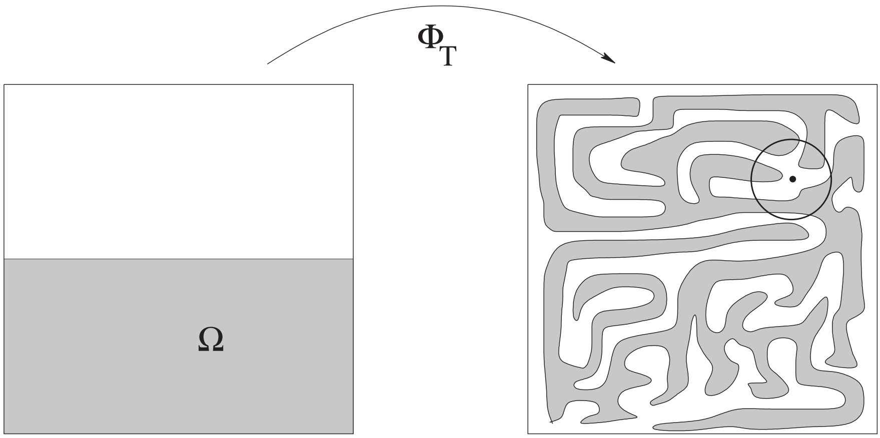

Fig. 2.

It is reasonable to expect that vector fields satisfying a uniform BV bound can mix a set Ω only up to a finite scale.

CONJECTURE 2. For each set $\Omega \subset \boldsymbol{T}^m$ there exists a constant C_Ω such that the following holds. Let $f = f(t, x)$ is a smooth vector field whose flow Φ satisfies

$$\frac{1}{2}\,\text{meas}\,(A) \leq \text{meas}\left(\Phi_t^f(A)\right) \leq 2\ \text{meas}\,(A) \tag{4.3}$$

for every $t \geq 0$ and $A \subseteq \boldsymbol{T}^m$, and such that Φ_T mixes Ω up to scale ε. Then

$$\sum_{\alpha=1}^{m} \int_0^T \int_{\boldsymbol{T}^m} \left| \frac{\partial}{\partial x_\alpha} f \right| dxdt \geq C_\Omega \cdot |\ln \varepsilon|\,. \tag{4.4}$$

A seemingly equivalent conjecture is the following. Consider two sets $\Omega,\ \Omega' \subset \mathbb{R}^m$, with meas (Ω) = meas $(\Omega') = 1$. Assume that there exists a measure-preserving transformation $\psi : \Omega \mapsto \Omega'$ such that

$$|x - \psi(x)| \leq \varepsilon \qquad \text{for all } x \in \Omega\,. \tag{4.5}$$

Let $f : [0,\ T] \times \mathbb{R}^m \mapsto \mathbb{R}^m$ be a smooth vector field whose flow Φ^f is almost incompressible, so that (4.3) holds for every $A \subset \mathbb{R}^m$. Moreover, assume that the flow Φ_T separates the two sets $\Omega,\ \Omega'$, namely

$$|\Phi_T(x) - \Phi_T(x')| \geq 1 \qquad \text{for all } x \in \Omega,\ x' \in \Omega'\,. \tag{4.6}$$

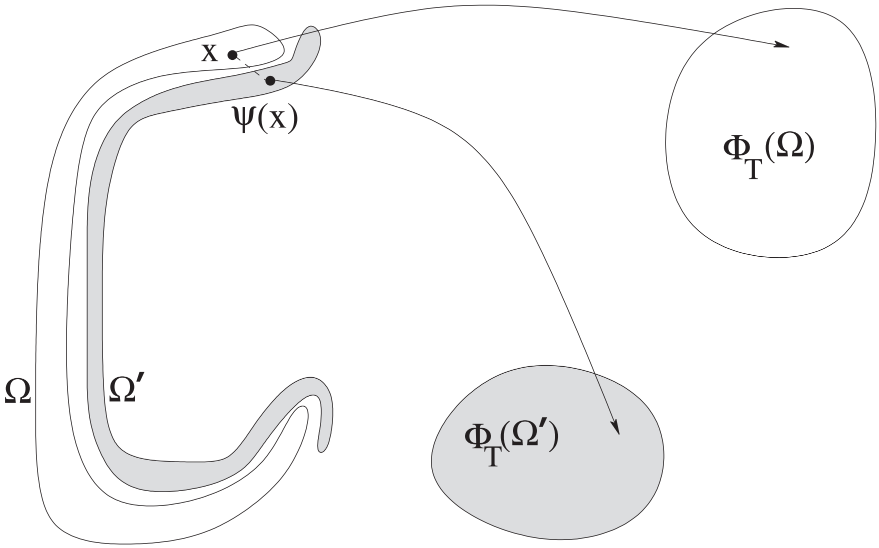

Fig. 3.

CONJECTURE 3. There exists a constant C, independent of f, ε, Ω, Ω', such that the above assumptions (4.3), (4.5) and (4.6) imply

$$\sum_{\alpha=1}^{m} \int_0^T \int_{\mathbb{R}^m} \left| \frac{\partial}{\partial x_\alpha} f \right| dx dt \geq C \cdot |\ln \varepsilon| . \tag{4.7}$$

Intuitively, if the flow Φ_T maps couples of nearby points to quite different locations (fig. 3), the vector field f must be far from constant. A quantitative estimate in this direction would be provided by (4.7). A much simplified one-dimensional analogue of this estimate was proved in [5].

REFERENCES

[1] L. AMBROSIO, *Transport equation and the Cauchy problem for BV vector fields*. Preprint 2003.

[2] P. BRENNER, *The Cauchy problem for the symmetric hyperbolic systems in L_p*. Math. Scand., 19, 1966, 27-37.

[3] A. BRESSAN, *Hyperbolic systems of conservation laws. The one dimensional Cauchy problem*. Oxford University Press, 2000.

[4] A. BRESSAN, *An ill posed Cauchy problem for a hyperbolic system in two space dimensions*. Rend. Sem. Mat. Univ. Padova, 110, 2003, 103-117.

[5] A. BRESSAN, *A lemma and a conjecture on the cost of rearrangements*. Rend. Sem. Mat. Univ. Padova, 110, 2003, 97-102.

[6] A. BRESSAN - T.P. LIU - T. YANG, *L^1 stability estimates for $n \times n$ conservation laws*. Arch. Rational Mech. Anal., 149, 1999, 1-22.

[7] C. DAFERMOS, *Hyperbolic conservation laws in continuum physics*. Springer-Verlag, Berlin 1999.

[8] R. DiPerna - P.L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*. Invent. Math., 98, 1989, 511-517.
[9] L. Gärding, *Problèmes de Cauchy pour les systèmes quasi-linéaires d'ordre un strictement hyperboliques*. In: *Les E.D.P's*, vol. 117, Paris 1963, Colloques Internationaux du CNRS, 33-40.
[10] J. Glimm, *Solutions in the large for nonlinear hyperbolic systems of equations*. Comm. Pure Appl. Math., 18, 1965, 697-715.
[11] S. Kruzhkov, *First-order quasilinear equations with several space variables*. Math. USSR Sbornik, 10, 1970, 217-273.
[12] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*. Springer-Verlag, New York 1984.
[13] E.Y. Panov, *On the theory of generalized entropy solutions of the Cauchy problem for a class of non-strictly hyperbolic systems of conservation laws*. Sbornik: Mathematics, 191, 2000, 121-150.
[14] J. Rauch, *BV estimates fail for most quasilinear hyperbolic systems in dimensions greater than one*. Comm. Math. Phys., 106, 1986, 481-484.
[15] D. Serre, *Solutions classiques globales des équations d'Euler pour un fluide parfait compressible*. Ann. Inst. Fourier, 47, 1997, 139-153.
[16] D. Serre, *Systems of Conservation Laws I*. Cambridge University Press, 2000.
[17] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*. Springer-Verlag, New York 1983.
[18] Y. Zheng, *Systems of conservation laws: two-dimensional Riemann problems*. Birkhäuser, 2001.
[19] W.P. Ziemer, *Weakly Differentiable Functions*. Springer-Verlag, New York 1989.

S.I.S.S.A.
Via Beirut, 4 - 34014 Trieste
bressan@sissa.it