Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Augusto Visintin

Quasilinear hyperbolic equations with hysteresis

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **15** (2004), n.3-4, p. 235–247.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2004_9_15_3-4_235_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2004.

Rend. Mat. Acc. Lincei
s. 9, v. 15:235-247 (2004)

AUGUSTO VISINTIN

QUASILINEAR HYPERBOLIC EQUATIONS WITH HYSTERESIS

ABSTRACT. — Hysteresis operators are illustrated, and a weak formulation is studied for an initial- and boundary-value problem associated to the equation $(\partial^2/\partial t^2)[u + \mathcal{F}(u)] + Au = f$; here $\mathcal{F}$ is a (possibly discontinuous) hysteresis operator, A is a second order elliptic operator, f is a known function. Problems of this sort arise in plasticity, ferromagnetism, ferroelectricity, and so on. In particular an existence result is outlined.

KEY WORDS: Hysteresis; Hysteresis operator; Quasilinear hyperbolic equations; Existence of weak solutions.

1. HYSTERESIS OPERATORS

The word *hysteresis* originates from an ancient Greek term, meaning *lag in arrival.* It seems that it was first used in connection with ferromagnetism in 1882. Hysteresis occurs in several physical phenomena: plasticity, ferromagnetism, ferroelectricity, porous media filtration, phase transitions, superconductivity, the recently developed theory of materials with *shape memory*, and many others. In spite of the obvious importance of these phenomena, the investigation of the properties of hysteresis in the framework of function spaces only began less than 40 years ago, cf. [2]. In the 1970s M.A. Krasnosel'skiĭ and a group of Russian mathematicians systematically studied the concept of *hysteresis operator*, acting in spaces of time dependent functions; see Krasnosel'skiĭ and Pokrovskiĭ's monograph [8]. Since the beginning of the 1980s other mathematicians have also been studying hysteresis phenomena, especially in connection with P.D.E.s and applicative problems; see *e.g.* the monographs [4, 9, 16], the proceedings [15], and the survey [17].

In order to outline a simplified picture of hysteresis, let us consider a system characterized by two scalar variables, u and w, and assume that at any instant $t \in [0, T]$, $w(t)$ depends on the previous evolution of u (memory effect) and on the initial state; that is,

$$w(t) = [\mathcal{F}(u, w^0)](t) \qquad \forall t \in [0, T]. \tag{1.1}$$

For any fixed w^0, $\mathcal{F}(\cdot, w^0)$ represents an operator acting in some Banach space of time-dependent functions. This *memory operator* must also be causal, *i.e.*,

$$\forall (u_1, w^0), (u_2, w^0) \in \mathrm{Dom}(\mathcal{F}), \ \ \forall t \in [0, T], \tag{1.2}$$
$$\text{if } u_1 = u_2 \text{ in } [0, t], \text{ then } [\mathcal{F}(u_1, w^0)](t) = [\mathcal{F}(u_2, w^0)](t).$$

In typical examples, the state (u, w) is confined to a set $\mathcal{L} \subset \boldsymbol{R}^2$, cf. fig. 1; we then assume that

$$\text{if } (u(0), w^0) \in \mathcal{L} \text{ then } [\mathcal{F}(u, w^0)](0) = w^0. \tag{1.3}$$

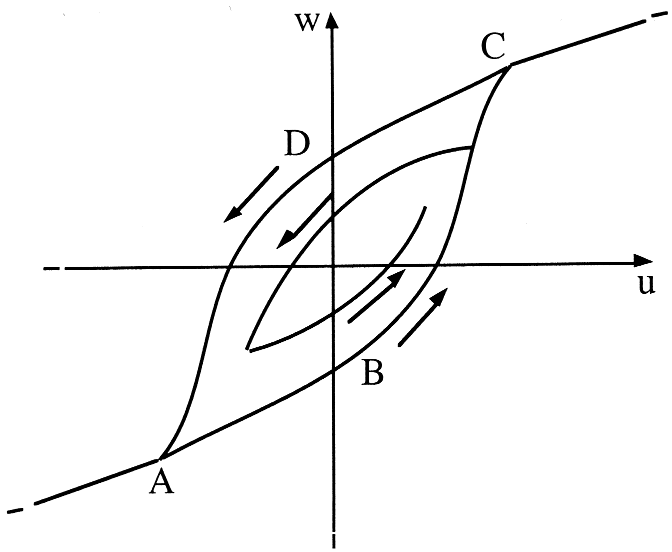

Fig. 1. – Example of hysteresis loop. The exterior loop and an (incomplete) internal loop are outlined.

Rate-Independence. We name *hysteresis operator* any causal memory operator $\mathcal{F}: u \mapsto w$ such that the path of the pair $(u(t), w(t))$ is invariant w.r.t. any increasing diffeomorphism $\varphi : [0, T] \to [0, T]$, that is,

$$\mathcal{F}(u \circ \varphi, w^0) = \mathcal{F}(u, w^0) \circ \varphi \quad \text{in } [0, T]; \tag{1.4}$$

in other terms, if $u \mapsto w$ then $u \circ \varphi \mapsto w \circ \varphi$. This means that at any instant t, $w(t)$ only depends on $u([0, t])$, and on the order in which values have been attained. This property also allows one to draw hysteresis curves, without the need of relating them to any specific time-law of the input curve $u(t)$. In particular, if the input function u is periodic, the w vs. u relation shall not depend on the frequency.

It is easy to see that rate-independence excludes any viscous-type memory, like that represented by time-convolution. In reality, even in most typical hysteresis phenomena, memory is not purely rate-independent, as rate-dependent effects are superposed to hysteresis.

Although so far we assumed that at any instant t the state of the system is characterized by the pair $(u(t), w(t))$, several hysteresis operators can also account for internal variables.

P.D.E.s with Hysteresis. Some partial differential equations including hysteresis operators have been studied. In particular results have been derived for quasi-linear or semi-linear parabolic equations of the form

$$\frac{\partial}{\partial t}[u + \mathcal{F}(u, w^0)] + Au = f, \tag{1.5}$$

$$\frac{\partial u}{\partial t} + Au + \mathcal{F}(u, w^0) = f, \tag{1.6}$$

and for quasi-linear hyperbolic equations like

$$\frac{\partial^2}{\partial t^2}[u + \mathcal{F}(u, w^0)] + Au = f, \tag{1.7}$$

$$\frac{\partial}{\partial t}[u + \mathcal{F}(u, w^0)] + v \cdot \nabla u = f. \tag{1.8}$$

Here $\mathcal{F}$ is a (possibly discontinuous) scalar hysteresis operator, A is an elliptic operator, f and v are given functions. Equations of this sort arise in ferromagnetism, ferroelectricity, elasto-plasticity, and so on. Initial- and boundary-value problems associated either with (1.5), or with (1.6), or with (1.8) are well-posed.

In Sections 2 and 3 of this survey we review some simple examples of continuous and discontinuous scalar hysteresis operators. In Section 4 we provide a weak formulation for an initial- and boundary-value problem associated to equation (1.7), for a large class of (possibly discontinuous) scalar hysteresis operators, and outline an existence result.

Finally, a remark about the classification of P.D.E.s with hysteresis is in order. Any scalar hysteresis operator, $\mathcal{F}$, is reduced to a superposition operator on any time interval in which the input function is monotone (either increasing or decreasing). Let us denote by $S_{\mathcal{F}}$ this class of superposition operators; in typical examples, they are associated to (possibly multivalued) nondecreasing functions. We then say that an equation which includes $\mathcal{F}$ is parabolic (hyperbolic, resp.) whenever it would be so if the operator $\mathcal{F}$ were replaced by any element of $S_{\mathcal{F}}$. As these equations are nonlinear, we also extend the usual denomination of semi-linearity, quasi-linearity and full nonlinearity by the same criterion. This classification can also be extended to some vector hysteresis operators.

2. Examples of continuous hysteresis operators

Hysteresis operators acting in Banach spaces of time dependent functions, *e.g.* $C^0([0, T])$ or $W^{1,1}(0, T)$, are typically constructed via the following procedure:

(*i*) first the operator is defined for all piecewise monotone inputs;

(*ii*) uniform continuity is derived w.r.t. the strong topology of the Banach space;

(*iii*) the operator is then extended by continuity to the whole space.

Here we review some especially simple classes of hysteresis operators.

The Duhem Model [6]. Let $g_1, g_2 \in C^1(\mathbf{R}^2)$. For any $w^0 \in \mathbf{R}$ and any $u \in C^1([0, T])$, this model is defined via the Cauchy problem

$$\begin{cases} \dfrac{dw}{dt} = g_1(u, w)\left(\dfrac{du}{dt}\right)^+ - g_2(u, w)\left(\dfrac{du}{dt}\right)^- & \text{in }]0, T[, \\ w(0) = w^0. \end{cases} \tag{2.1}$$

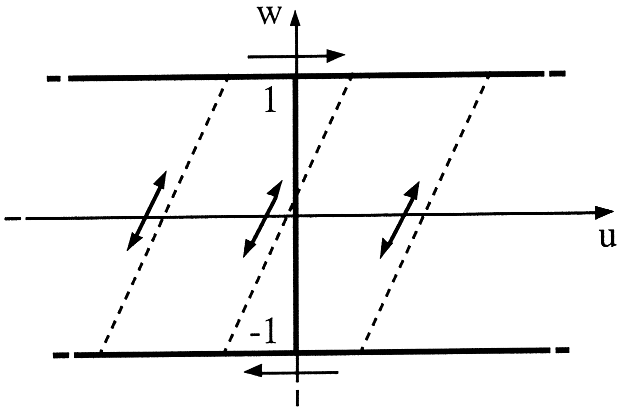

Fig. 2. – Stop.

It is easy to see that this system defines a rate-independent operator $C^1([0, T]) \to$ $\to C^1([0, T]): u \mapsto w$. Consistently with the irreversibility of hysteresis phenomena, here we assume that t is nondecreasing, whence $dt \geq 0$. The differential equation $(2.1)_1$ is then formally equivalent to

$$\text{(2.2)} \qquad \begin{cases} \dfrac{dw}{du} = g_1(u, w) & \text{if } u \text{ is increasing,} \\ \dfrac{dw}{du} = g_2(u, w) & \text{if } u \text{ is decreasing.} \end{cases}$$

Under suitable assumptions on the functions g_1 and g_2, the operator $(u, w^0) \mapsto w$ can be extended by continuity to $W^{1,1}(0, T)$, see *e.g.* [16, Chap. V]. This formulation can easily be modified, to confine (u, w) to a subset of $\boldsymbol{R}^2$.

Stop: Prandtl's Model of Elasto-Plasticity [12]. This model can be represented by a simple rheological model, which consists in a linear spring coupled *in series* with a friction element, cf. fig. 2.

Let us set

$$\mathrm{sign}(\xi) := \{-1\} \quad \forall \xi < 0, \qquad \mathrm{sign}(0) := [0, 1], \qquad \mathrm{sign}(\xi) := \{1\} \quad \forall \xi > 0.$$

Assuming that all mechanical coefficients are normalized, denoting the strain by u and the stress by w, and denoting the time derivative by $'$, we represent the operator $u \mapsto w$ by the system

$$\text{(2.3)} \qquad \begin{cases} w' + \mathrm{sign}^{-1}(w) \ni u' & \text{in }]0, T[, \\ w(0) = w^0. \end{cases}$$

The rate-independence is obvious. This differential inclusion is equivalent to a variational inequality:

$$\text{(2.4)} \qquad w \in [-1, 1], \qquad (w' - u')(w - v) \leq 0 \quad \forall v \in [-1, 1].$$

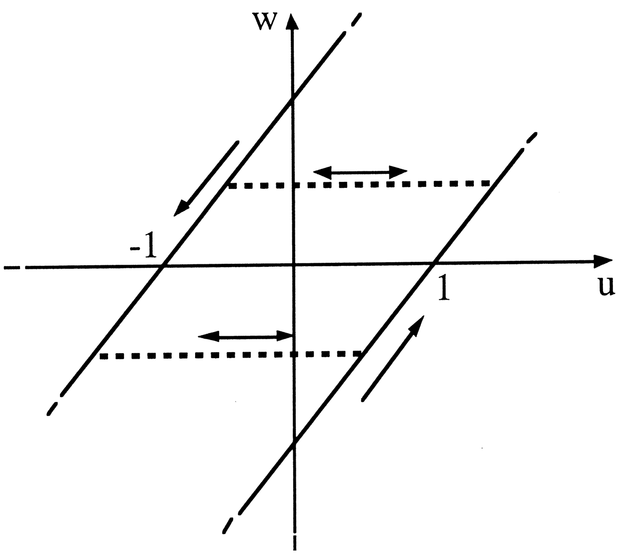

Fig. 3. – Play.

Play: Model of Plasticity with Strain-Hardening. This operator can also be represented by a rheological model: a linear spring coupled *in parallel* with a friction element. Still assuming that all mechanical coefficients are normalized, and (at variance with above) denoting the strain by w and the stress by u, here we have

$$(2.5)\qquad \begin{cases} \operatorname{sign}(w') + w \ni u & \text{in }]0, T[, \\ w(0) = w^0, \end{cases}$$

cf. fig. 3. The rate-independence is obvious here, too. This differential inclusion is also equivalent to a variational inequality:

$$(2.6)\qquad (w-u)(w'-v) + |w'| - |v| \leq 0 \qquad \forall v \in \boldsymbol{R}.$$

The play operator $\mathcal{F}: u \mapsto w$ can be extended by continuity to an operator acting in $C^0([0, T])$. One can also show a regularizing property: $\mathcal{F}\big(C^0([0, T])\big) \subset BV(0, T)$. This result is easily understood: by the uniformly continuity of the input function u, the pair $(u, \mathcal{F}(u))$ can move from the *ascending* branch $\{(u, w) \in \boldsymbol{R}^2 : w = u - 1\}$ to the *descending* branch $\{(u, w) \in \boldsymbol{R}^2 : w = u + 1\}$ just a finite number of times, if any. Therefore $\mathcal{F}(u)$ is a piecewise monotone function of time, hence its total variation is finite.

Prandtl-Ishlinskiĭ Models [7, 13]. The above formulation of the stop and play operators via differential inclusions and variational inequalities allows for an automatic extension to tensors. Suitable series and parallel arrangements of the corresponding rheological models yield more complex constructions (*Prandtl-Ishlinskiĭ models*), which can be represented by systems of variational inequalities, cf. *e.g.* [16, Chap. III].

These models also account for occurrence of internal variables, and are widely used in elasto-plasticity. Coupled with the laws of dynamics or statics, they yield problems for which several results have been established in the framework of Sobolev spaces, cf. *e.g.* [16, Chap. VII].

3. Discontinuous hysteresis

Relay Operator. Let us denote by $C_r^0([0, T[)$ the space of functions that are continuous on the right in $[0, T[$. For any pair $\varrho := (\varrho_1, \varrho_2) \in \boldsymbol{R}^2$ $(\varrho_1 < \varrho_2)$ we introduce the (*delayed*) *relay operator*

$$h_\varrho : C^0([0, T]) \times \{-1, 1\} \to BV(0, T) \cap C_r^0([0, T[).$$

For any $u \in C^0([0, T])$ and any $\xi = -1$ or 1, first let us set $X_t := \{\tau \in]0, t] : u(\tau) = \varrho_1 \text{ or } \varrho_2\}$ for any $t \in]0, T]$, and define the function $w = h_\varrho(u, \xi) : [0, T] \to \{-1, 1\}$ as follows:

$$w(0) := \begin{cases} -1 & \text{if } u(0) \leqslant \varrho_1, \\ \xi & \text{if } \varrho_1 < u(0) < \varrho_2, \\ 1 & \text{if } u(0) \geqslant \varrho_2, \end{cases} \tag{3.1}$$

$$w(t) := \begin{cases} w(0) & \text{if } X_t = \emptyset \\ -1 & \text{if } X_t \neq \emptyset \text{ and } u(\max X_t) = \varrho_1 \\ 1 & \text{if } X_t \neq \emptyset \text{ and } u(\max X_t) = \varrho_2 \end{cases} \quad \forall t \in]0, T]. \tag{3.2}$$

These conditions define w uniquely in $[0, T]$. For instance let $u(0) < \varrho_1$; then $w(0) = -1$, and $w(t) = -1$ as long as $u(t) < \varrho_2$; if at some instant u reaches ϱ_2 then w jumps up to 1, where it remains as long as $u(t) > \varrho_1$; if later u reaches ϱ_1, w jumps down to -1; and so on, cf. fig. 4.

For any function $u \in C^0([0, T])$ the number of oscillations of u between ϱ_1 and ϱ_2 is necessarily finite, because of uniform continuity; hence w can just have a finite number of jumps between -1 and 1 if any. Therefore w is piecewise constant and its total variation in $[0, T]$ is finite. It is easy to check that w is also continuous on the right in $[0, T[$ and that $h_\varrho(\cdot, w^0) : u \mapsto w$ is rate-independent, piecewise monotone (*i.e.*, if u is either nondecreasing or nonincreasing in an interval $[t_1, t_2] \subset [0, T]$, then w has the same monotony in that interval), and order preserving (*i.e.*, if $u_1 \leqslant u_2$ then $w_1 \leqslant w_2$).

Extension of the Relay Operator. When dealing with P.D.E.s, one is induced to extend the graph of the relay operator, by allowing w to attain intermediate values between -1 and 1 as follows, in order to get a closed operator in suitable function spaces. For any $u \in C^0([0, T])$ and any $\xi \in [-1, 1]$, let us set $w \in k_\varrho(u, \xi)$ if and only

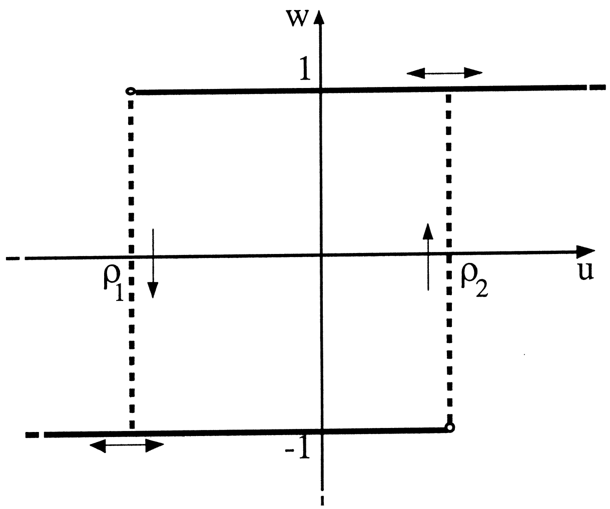

Fig. 4. – (Delayed) relay. If $u(t) \neq \varrho_1, \varrho_2$, then w is constant in a neighbourhood of t. Moreover w can jump from -1 to 1 only for $u = \varrho_2$, and from 1 to -1 only for $u = \varrho_1$.

if w is measurable in $]0, T[$ and

$$(3.3) \qquad w(0) \in \begin{cases} \{-1\} & \text{if } u(0) < \varrho_1, \\ [-1, \xi] & \text{if } u(0) = \varrho_1, \\ \{\xi\} & \text{if } \varrho_1 < u(0) < \varrho_2, \\ [\xi, 1] & \text{if } u(0) = \varrho_2, \\ \{1\} & \text{if } u(0) > \varrho_2, \end{cases}$$

$$(3.4) \qquad w(t) \in \begin{cases} \{-1\} & \text{if } u(t) < \varrho_1, \\ [-1, 1] & \text{if } \varrho_1 \leq u(t) \leq \varrho_2, \\ \{1\} & \text{if } u(t) > \varrho_2, \end{cases}$$

$$(3.5) \qquad \begin{cases} \text{if } u(t) \neq \varrho_1, \varrho_2, & \text{then } w \text{ is constant in a neighbourhood of } t, \\ \text{if } u(t) = \varrho_1, & \text{then } w \text{ is nonincreasing in a neighbourhood of } t, \\ \text{if } u(t) = \varrho_2, & \text{then } w \text{ is nondecreasing in a neighbourhood of } t. \end{cases}$$

Such a function w exists and belongs to $BV(0, T)$, because of the same argument we used for h_ϱ. Thus

$$(3.6) \qquad k_\varrho : C^0([0, T]) \times [-1, 1] \to \mathcal{P}(BV(0, T)).$$

The behaviour of k_ϱ is outlined in fig. 5. Note that its graph in the (u, w)-plane includes the whole rectangle $[\varrho_1, \varrho_2] \times [-1, 1]$. k_ϱ will be named *completed relay operator.* This completion procedure is somehow analogous to that of replacing the sign_0 function (with $\text{sign}_0(0) := 0$) by the sign graph ($\text{sign}(0) := [-1, 1]$).

If the space variable is inserted as a parameter in the input and output functions, u

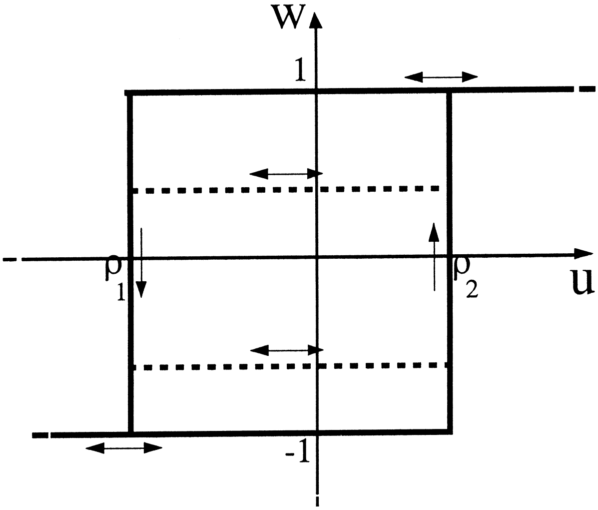

Fig. 5. – Completed relay. If $u(t) \neq \varrho_1, \varrho_2$, then w is constant in a neighbourhood of t. If $u(t) = \varrho_1$ ($u(t) = \varrho_2$, resp.), then w is nonincreasing (nondecreasing, resp.) in a neighbourhood of t. The pair (u, w) can attain any value in the rectangle $[\varrho_1, \varrho_2] \times [-1, 1]$.

and w, one can show that the multivalued operator k_ϱ is closed w.r.t. the strong topology of $L^2(\Omega; C^0([0, T]))$ for u and the weak star topology of $L^2_{w^*}(\Omega; BV(0, T))$ for w. This is especially convenient for the study of P.D.E.s that contain the relay operator

Weak Formulation of the Relay Operator.

The operator k_ϱ can be represented by the following system.

(*i*) *Confinement condition*:

$$\begin{cases} |w| \leqslant 1 \\ (w-1)(u-\varrho_2) \geqslant 0 \qquad \text{a.e. in }]0, T[; \\ (w+1)(u-\varrho_1) \geqslant 0 \end{cases} \tag{3.7}$$

this constrains the states that are accessible to the pair (u, w).

(*ii*) *Dissipation condition*:

$$\int_0^t u\, dw \geqslant \int_0^t [\varrho_2 (dw)^+ - \varrho_1 (dw)^-] = \tag{3.8}$$

$$= \frac{\varrho_2 - \varrho_1}{2} \int_0^t |dw| + \frac{\varrho_2 + \varrho_1}{2} [w(t) - w(0)] =: \Psi_\varrho(w, t) \qquad \forall t \in]0, T];$$

this accounts for the (dissipative) dynamics of the pair (u, w). Both conditions are illustrated in fig. 5 above.

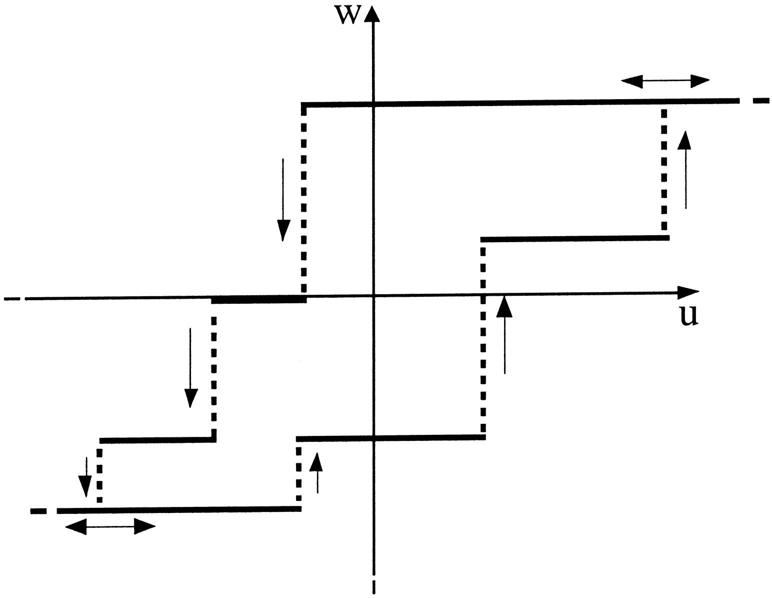

Fig. 6. – Preisach model obtained by assemblying three relays.

The Preisach Model [14]. The pairs of admissible thresholds of relay operators form the so called *Preisach (half-)plane*

(3.9) $$\mathcal{P} := \{\varrho = (\varrho_1, \varrho_2) \in \boldsymbol{R}^2 : \varrho_1 < \varrho_2\}.$$

Let us denote by $\mathcal{R}$ the family of Borel measurable functions $\mathcal{P} \to \{-1, 1\}$, and by $\{\xi_\varrho\}$, or more briefly ξ, a generic element of $\mathcal{R}$. Let us fix any finite positive measure μ over $\mathcal{P}$, and define the corresponding *Preisach operator*

(3.10) $$\begin{cases} \mathcal{H}_\mu : C^0([0, T]) \times \mathcal{R} \to L^\infty(0, T) \cap C_r^0([0, T[), \\ [\mathcal{H}_\mu(u, \xi)](t) := \int_{\mathcal{P}} [h_\varrho(u, \xi_\varrho)](t)\, d\mu(\varrho) \quad \forall t \in [0, T]. \end{cases}$$

This operator is rate-independent, piecewise monotone and order preserving.

In general $\mathcal{H}_\mu(u, \xi)$ maps continuous functions to discontinuous ones, cf. fig. 6. It is then convenient to replace the integrand h_ϱ by k_ϱ. However, under simple hypotheses on the measure μ, $\mathcal{H}_\mu$ operates and is continuous in $C^0([0, T])$.

4. Quasilinear hyperbolic equations with hysteresis

Let Ω be a regular domain of $\boldsymbol{R}^3$ and set $Q := \Omega \times]0, T[$. Let us assume that we are given a (measurable) field $\varrho : \Omega \to (\mathcal{P}, \mu)$, and the functions

$$(4.1) \qquad \begin{cases} u^0, w^0 \in L^2(\Omega), \quad F \in L^2\big(0, T; H^{-1}(\Omega)\big); \\ |w^0| \leqslant 1, \ \ w^0 = -1 \ \text{ if } u^0 < \varrho_1, \ \ w^0 = 1 \ \text{ if } u^0 > \varrho_2 \ \text{ a.e. in } \Omega. \end{cases}$$

We formulate an initial- and boundary-value problem in the framework of Sobolev spaces.

Problem 4.1. *To find* $U \in H^1(Q)$ *and* $w \in L^\infty(Q)$ *such that*

$$(4.2) \qquad \begin{cases} \gamma_0 U = 0 \ \textit{a.e. in} \ (\Omega \times \{0\}) \cup (\partial\Omega \times]0, T[), \\ |w| \leqslant 1 \ \textit{a.e. in} \ Q, \quad \dfrac{\partial w}{\partial t} \in C^0(\overline{Q})', \end{cases}$$

$$(4.3) \qquad \frac{\partial}{\partial t}(u + w) - \Delta U = F \quad \textit{in} \ H^{-1}(Q)\left(u := \frac{\partial U}{\partial t}\right),$$

$$(4.4) \qquad \begin{cases} (w-1)(u-\varrho_2) \geqslant 0 \\ (w+1)(u-\varrho_1) \geqslant 0 \end{cases} \quad \textit{a.e. in} \ Q,$$

$$(4.5) \qquad \frac{1}{2}\int_\Omega [u(x,t)^2 + |\nabla U(x,t)|^2 - u^0(x)^2 + \Psi_\varrho(w(x,\cdot),t)]\,dx \leqslant$$

$$\leqslant \int_0^t \langle F, u\rangle d\tau \quad \textit{for a.a.} \ \ t \in]0, T[,$$

$$(4.6) \qquad (u+w)\big|_{t=0} = u^0 + w^0 \qquad \textit{in} \ H^{-1}(\Omega).$$

Interpretation. By differentiating (4.3) in time and setting $f := \partial F/\partial t$, we get the equation

$$(4.7) \qquad \frac{\partial^2}{\partial t^2}(u+w) - \Delta u = f \quad \text{in} \ \mathcal{D}'(Q).$$

As $|w| \leqslant 1$, (4.4) is equivalent to the confinement condition (3.7) a.e. in Q. If $u \in L^2\big(0, T; H_0^1(\Omega)\big)$, we can multiply (4.7) by $u = \dfrac{\partial U}{\partial t}$, and integrate in space and time. It is then easy to see (4.5) is *formally* equivalent to

$$(4.8) \qquad \int_0^t {}_{H^{-1}(\Omega)}\left\langle \frac{\partial}{\partial \tau}(u+w), u\right\rangle_{H_0^1(\Omega)} d\tau \geqslant$$

$$\geqslant \frac{1}{2}\int_\Omega [u(x,t)^2 - u^0(x)^2]\,dx + \int_{\overline{\Omega}} \Psi(w,t) \qquad \text{for a.a.} \ \ t \in]0, T[.$$

This derivation is rigorous whenever the left-hand side has a meaning as a duality pairing over $\Omega \times]0, t[$. (4.5) may then be regarded as a weak formulation of the dissipation condition (3.8) a.e. in Ω. The latter, (3.7) and the initial condition «$w(\cdot, 0) = w^0$

a.e. in Ω» (which is implicit in the above equations) represent the hysteresis relation

(4.9) $$w \in k_\varrho(u, w^0) \quad \text{a.e. in } \Omega.$$

Therefore Problem 4.1 is a weak formulation of an initial- and boundary-value problem associated to the system (4.7), (3.7), (3.8), namely, the equation (4.7) coupled with the hysteresis relation (4.9).

By the discontinuity of the w vs. u relation, the equation (4.7) accounts for the occurrence of moving fronts which separate regions characterized by different values of w. The location of these fronts is a priori unknown, namely, they are *free boundaries.*

For $N = 1$, Problem 4.1 can represent processes in a univariate insulating ferrimagnetic material. The equation (4.7) can be derived from the Maxwell equations, assuming that (with standard notation) the field D is proportional to E, that $J = 0$, and denoting the fields H and M by u and w, respectively. The same problem can also represent evolution in a univariate insulating ferroelectric material; in this case u and w stay for the fields E and P, respectively.

THEOREM 4.1 [18]. *If* (4.1) *is fulfilled and*

(4.10) $$F \in L^1\big(0, T; L^2(\Omega)\big) + W^{1,1}\big(0, T; H^{-1}(\Omega)\big),$$

then Problem 1 *has a solution* (U, w) *such that*

(4.11) $$U \in W^{1,\infty}\big(0, T; L^2(\Omega)\big) \cap L^\infty\big(0, T; H_0^1(\Omega)\big).$$

The argument is based upon

(*i*) approximation via implicit time-discretization,

(*ii*) derivation of a priori estimates,

(*iii*) passage to the limit by compactness and lower semicontinuity.

In particular, denoting the solution of the time-discretized problem by (u_m, w_m) ($m \in \mathbf{N}$), the dissipation condition (3.8) yields an estimate on $\int_\Omega dx \int_0^T |dw_m| = \|\partial w_m / \partial t\|_{C^0(\overline{Q})'}$. Hysteresis thus provides additional time-regularity. The argument also uses the following compactness result, which is based on Banach-space interpolation and on the compactness of Sobolev injections.

LEMMA 4.2. *If the sequences* $\{z_m\}, \{w_m\}$ *are such that*

$$z_m \to z \quad \text{weakly in } L^2(Q) \cap H^{-1}\big(0, T; H^1(\Omega)\big),$$

$$w_m \to w \quad \text{weakly star in } L^\infty(Q),$$

$$\left\| \frac{\partial w_m}{\partial t} \right\|_{L^1(Q)} \leqslant \text{Constant},$$

then

$$\iint_Q w_m z_m \, dxdt \to \iint_Q wz \, dxdt \, .$$

The above existence result is based on the dissipative character of hysteresis, and has no analog for quasi-linear hyperbolic equations without hysteresis. Thus the equation (4.7) turns out to be one of the few known examples in which analysis is made easier by occurrence of hysteresis.

Uniqueness of the solution of Problem 4.1 is an open question.

EXTENSIONS. Problem 4.1 and Theorem 4.1 can be extended in two main directions:

(*i*) the relay operator can be replaced by the Preisach model, cf. [18];

(*ii*) one can deal with the vector setting, *i.e.*, with u and w ranging in $\boldsymbol{R}^3$.

As we mentioned above, processes in an insulating ferrimagnetic material can the represented by the Maxwell equations; assuming that the field $\vec{D}$ is proportional to $\vec{E}$, one gets an equation of the form

$$\frac{\partial^2}{\partial t^2}(\vec{H} + \vec{M}) + \mathrm{curl}^2 \vec{H} = \vec{f}\,, \tag{4.12}$$

(here written with normalized coefficients). The relation between $\vec{M}$ and $\vec{H}$ can be represented by a vector extension of the relay operator, cf. [5, 10, 11]; an initial- and boundary-value problem can be formulated in the framework of Sobolev spaces, and existence of a solution can be proved. For this and other models of evolution in magnetic materials with hysteresis, we refer to [19].

REFERENCES

[1] G. BERTOTTI, *Hysteresis in Magnetism.* Academic Press, Boston 1998.

[2] R. BOUC, *Solution périodique de l'équation de la ferrorésonance avec hystérésis.* C.R. Acad. Sci. Paris, Série A, 263, 1966, 497-499.

[3] M. BROKATE *et al.*, *Phase Transitions and Hysteresis.* Lecture Notes in Mathematics, vol. 1584 (A. Visintin, ed.). Springer-Verlag, Berlin 1994.

[4] M. BROKATE - J. SPREKELS, *Hysteresis and Phase Transitions.* Springer, Berlin 1996.

[5] A. DAMLAMIAN - A. VISINTIN, *Une généralisation vectorielle du modèle de Preisach pour l'hystérésis.* C.R. Ac. Sci. Paris I, 297, 1983, 437-440.

[6] P. DUHEM, *The evolution of mechanics.* Sijthoff and Noordhoff, Alphen aan den Rijn, 1980. Original edition: *L'évolution de la méchanique*. Joanin, Paris 1903.

[7] A.Y. ISHLINSKĬ, *Some applications of statistical methods to describing deformations of bodies.* Izv. Akad. Nauk S.S.S.R., Techn. Ser., 9, 1944, 580-590 (in Russian).

[8] M.A. KRASNOSEL'SKIĬ - A.V. POKROVSKIĬ, *Systems with Hysteresis.* Springer, Berlin 1989 (Russian ed. Nauka, Moscow 1983).

[9] P. KREJČÍ, *Convexity, Hysteresis and Dissipation in Hyperbolic Equations.* Gakkotosho, Tokyo 1997.

[10] I.D. MAYERGOYZ, *Vector Preisach model of hysteresis.* J. Appl. Phys., 63, 1988, 2995-3000.

[11] I.D. MAYERGOYZ, *Mathematical Models of Hysteresis.* Springer, New York 1991.

[12] L. PRANDTL, *Spannungsverteilung in plastischen Körpern.* In: C.B. BIEZENO - J.M. BURGERS (eds.), *Proceedings of the first International Congress for Applied Mechanics* (Delft, 22-26 April 1924). J. Waltman Jr., Delft 1925, 43-54.
[13] L. PRANDTL, *Ein Gedankenmodell zur kinetischen Theorie der festen Körper.* Z. Angew. Math. Mech., 8, 1928, 85-106.
[14] F. PREISACH, *Über die magnetische Nachwirkung.* Z. Physik, 94, 1935, 277-302.
[15] A. VISINTIN (ed.), *Models of Hysteresis.* Longman, Harlow 1993.
[16] A. VISINTIN, *Differential Models of Hysteresis.* Springer, Berlin 1994.
[17] A. VISINTIN, *Six talks on hysteresis.* C.R.M. Proceedings and Lecture Notes, 13, 1998, 207-236.
[18] A. VISINTIN, *Quasi-linear hyperbolic equations with hysteresis.* Ann. Inst. H. Poincaré, Nonlinear Analysis, 19, 2002, 451-476.
[19] A. VISINTIN, *Maxwell's equations with vector hysteresis.* Arch. Rat. Mech. Anal., in press.

Dipartimento di Matematica
Università degli Studi di Trento
Via Sommarive, 14 - 38050 POVO DI TRENTO TN
Visintin@science.unitn.it