Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Stanislav Nikolaevich Antontsev, Jesús Ildefonso Díaz, Hermenegildo Borges de Oliveira

Stopping a viscous fluid by a feedback dissipative field: II. The stationary Navier-Stokes problem

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **15** (2004), n.3-4, p. 257–270.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2004_9_15_3-4_257_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2004.

Rend. Mat. Acc. Lincei
s. 9, v. 15:257-270 (2004)

Stanislav Nikolaevich Antontsev - Jesús Ildefonso Díaz -
Hermenegildo Borges de Oliveira

STOPPING A VISCOUS FLUID BY A FEEDBACK DISSIPATIVE FIELD: II. THE STATIONARY NAVIER-STOKES PROBLEM

Abstract. — We consider a planar stationary flow of an incompressible viscous fluid in a semi-infinite strip governed by the Navier-Stokes system with a feed-back body forces field which depends on the velocity field. Since the presence of this type of non-linear terms is not standard in the fluid mechanics literature, we start by establishing some results about existence and uniqueness of weak solutions. Then, we prove how this fluid can be stopped at a finite distance of the semi-infinite strip entrance by means of this body forces field which depends in a sub-linear way on the velocity field. This localization effect is proved by reducing the problem to a fourth order non-linear one for which the localization of solutions is obtained by means of a suitable energy method.

Key words: Navier-Stokes system; Body forces field; Non-linear fourth order equation; Energy method; Localization effect.

1. Introduction

We study the planar stationary flow of an incompressible viscous fluid in a semi-infinite strip $\Omega = (0, \infty) \times (0, L)$, $L > 0$, given by the following system of equations

$$-\nu\Delta \boldsymbol{u} + (u\cdot\nabla)\boldsymbol{u} = \boldsymbol{f} - \nabla p \quad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{u}(0, y) = \boldsymbol{u}_*(y), \quad y \in (0, L) \tag{1.3}$$

$$\boldsymbol{u}(x, 0) = \boldsymbol{u}(x, L) = \boldsymbol{0}, \quad x \in (0, \infty) \tag{1.4}$$

$$|\boldsymbol{u}(x, y)| \to 0, \quad \text{as } x \to \infty \text{ and } y \in (0, L), \tag{1.5}$$

where $\boldsymbol{x} = (x, y) \in \mathbb{R}^2$, $\boldsymbol{u}(\boldsymbol{x}) = (u(\boldsymbol{x}), v(\boldsymbol{x}))$ is the velocity vector field, $p = p(\boldsymbol{x})$ stands here for the hydrostatic pressure divided by the constant density of the fluid and ν is the kinematics viscosity coefficient.

We assume that the possible non-zero velocity at the strip entrance, $\boldsymbol{u}_*(y) = (u_*(y), v_*(y))$, satisfies the compatibility conditions

$$\boldsymbol{u}_*(0) = \boldsymbol{u}_*(L) = \boldsymbol{0}, \quad \int_0^L u_*(s)\,ds = 0. \tag{1.6}$$

The body forces are given in a feedback form, $\boldsymbol{f}: \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$, $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) = (f_1(\boldsymbol{x}, \boldsymbol{u}), f_2(\boldsymbol{x}, \boldsymbol{u}))$, and such that, for every $\boldsymbol{u} \in \mathbb{R}^2$, $\boldsymbol{u} = (u(\boldsymbol{x}), v(\boldsymbol{x}))$, and for almost all $\boldsymbol{x} \in \Omega$,

$$-\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})\cdot\boldsymbol{u} \geqslant \delta\chi_f(\boldsymbol{x})\,|u|^{1+\sigma} - g(\boldsymbol{x}) \tag{1.7}$$

for some $\delta > 0$, $0 < \sigma < 1$ and

$$g \in L^1(\Omega^{x_g}), \quad g \geqslant 0, \quad g(\boldsymbol{x}) = 0 \quad \text{a.e. in} \quad \Omega_{x_g} \tag{1.8}$$

for some x_f, x_g with $0 \leqslant x_g < x_f \leqslant \infty$ and x_f large enough, where $\Omega^{x_g} = (0, x_g) \times \times (0, L)$ and $\Omega_{x_g} = (x_g, \infty) \times (0, L)$. The function χ_f denotes the characteristic function of the interval $(0, x_f)$, *i.e.*, $\chi_f(\boldsymbol{x}) = 1$, if $x \in (0, x_f)$ and $\chi_f(\boldsymbol{x}) = 0$, if $x \notin (0, x_f)$.

In this paper we prove that the fluid can be stopped at a finite distance of the semi-infinite strip entrance once we assume the body forces field depending in a sublinear way on the velocity field. This localization effect is proved by reducing the problem to a fourth order non-linear one for which the localization of solutions is obtained by means of a suitable energy method. The results we present here, are an extension to the Navier-Stokes system of our previous works [1-3] dealing with the Stokes system. In addition, we present here a new phenomenon related to the case of non homegeneous body forces (see Section 4). We show that under suitable conditions we have a stronger lack of propagation since a *stagnation line* for the forces field (in the sense that $|f_2(\boldsymbol{x}, \boldsymbol{u})| \leqslant C(x_s - x)_+^{\zeta}$ for some suitable C, $\zeta > 0$ and $x_s > 0$) remains being a *stagnation line* for the velocity field (*i.e.* $\boldsymbol{u}(x, y) = \boldsymbol{0}$ for $x > x_s$ and any $y \in (0, L)$). This property have some resemblances with the so called waiting time property for parabolic problems and the non-diffusion of the support property for some scalar elliptic problems (see [4]).

2. Existence and uniqueness results

The presence of non-linear terms defined by $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$, and to the best of our knowledge, is new in fluid mechanics setting. Thus, we collect in this section some results about existence and uniqueness of problem (1.1)-(1.8).

We shall search solutions such that $\int_\Omega |\nabla \boldsymbol{u}|^2 d\boldsymbol{x} < \infty$. Moreover, due to the fact that the Poincaré inequality

$$\int_0^L |u|^p \, dy \leqslant \left(\frac{L}{\pi}\right)^p \int_0^L |u'|^p \, dy, \tag{2.9}$$

holds for every $u \in W_0^{1,p}(0, L)$ and $1 \leqslant p < \infty$ (see, *e.g.*, [8]), our searched solution will be an element of the Sobolev space $\boldsymbol{H}^1(\Omega)$ simplifying, in this way, the functional framework needed for other unbounded domains.

To define the notion of a weak solution, we introduce the functional spaces

$$\widetilde{\boldsymbol{H}}(\Omega) = \\ = \left\{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) : \operatorname{div} \boldsymbol{u} = 0, \ \boldsymbol{u}(0, .) = \boldsymbol{u}_*(.), \boldsymbol{u}(., 0) = \boldsymbol{u}(., \mathrm{L}) = \boldsymbol{0}, \lim_{x \to \infty} |\boldsymbol{u}| = 0\right\},$$

$$\widetilde{\boldsymbol{H}}_0(\Omega) = \\ = \left\{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) : \operatorname{div} \boldsymbol{u} = 0, \ \boldsymbol{u}(0, .) = \boldsymbol{0}, \ \boldsymbol{u}(., 0) = \boldsymbol{u}(., \mathrm{L}) = \boldsymbol{0}, \lim_{x \to \infty} |\boldsymbol{u}| = 0\right\}.$$

DEFINITION 2.1. *A vector function* $\boldsymbol{u}$ *is a weak solution of problem* (1.1)-(1.8), *if:*
(*i*) $\boldsymbol{u} \in \widetilde{\boldsymbol{H}}(\Omega)$, $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \in \boldsymbol{L}^1_{loc}(\Omega)$;
(*ii*) *For every* $\varphi \in \widetilde{\boldsymbol{H}}_0(\Omega) \cap \boldsymbol{L}^\infty(\Omega)$ *with compact support,*

$$\nu \int_\Omega \nabla \boldsymbol{u} : \nabla \varphi \, d\boldsymbol{x} + \int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \varphi \, d\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \varphi \, d\boldsymbol{x}.$$

In this section, we shall assume that $\boldsymbol{f}: \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$, with $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) = (f_1(\boldsymbol{x}, \boldsymbol{u}), f_2(\boldsymbol{x}, \boldsymbol{u}))$ and $\boldsymbol{u} = (u(\boldsymbol{x}), v(\boldsymbol{x}))$, is given by

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) = -\delta \chi_f(\boldsymbol{x})(|u(\boldsymbol{x})|^{\sigma-1} u(\boldsymbol{x}), 0) - \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}), \tag{2.10}$$

for some $\delta > 0$, $0 < x_f \leq \infty$ and $0 < \sigma < 1$. Here, $\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u})$ is a Carathéodory function such that

$$\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}) \cdot \boldsymbol{u} \geq -g(\boldsymbol{x}), \quad \text{for every } \boldsymbol{u} \in \mathbb{R}^2 \text{ and a.e. } \boldsymbol{x} \in \Omega, \tag{2.11}$$

for some

$$g \in \mathrm{L}^1(\Omega^{x_g}), \quad g \geq 0, \quad g(\boldsymbol{x}) = 0 \quad \text{a.e. in} \quad \Omega_{x_g}, \quad 0 \leq x_g < x_f \tag{2.12}$$

and

$$H_M \in \mathrm{L}^1(\Omega^{x_f}), \quad \text{for all} \quad M > 0, \quad H_M(\boldsymbol{x}) = \sup_{|\boldsymbol{u}| \leq M} |\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u})|. \tag{2.13}$$

THEOREM 2.1. *Let us assume* $\boldsymbol{u}_* \in \boldsymbol{H}^{1/2}(0, L)$, $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ *satisfies* (2.10)-(2.13) *and the following growth condition holds: there exist some positive constants* M, C, *a function* $G \in \mathrm{L}^p(\Omega)$, *for some* $p > 1$ *and* $s \in (0, 2)$, *such that*

$$|\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u})| \leq C|\boldsymbol{u}|^s + G(\boldsymbol{x}), \tag{2.14}$$

for every $|\boldsymbol{u}| > M$ *and a.e. in* Ω. *In addition, we assume that the problem* (1.1)-(1.6) *with* $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{0})$ *has a unique weak solution in* $\Omega^R = (0, R) \times (0, L)$, *for every* $R > 0$. *Then, there exists, at least, one weak solution* $\boldsymbol{u}$ *of problem* (1.1)-(1.6). *Moreover,* $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \cdot \boldsymbol{u}$ *lies in* $\mathrm{L}^1(\Omega)$ *and* $\boldsymbol{u}$ *satisfies to the energy estimate*

$$\int_\Omega (|\nabla \boldsymbol{u}|^2 + \chi_f |u|^{1+\sigma}) \, d\boldsymbol{x} \leq C_1^2, \tag{2.15}$$

where $C_1 = C_1(L, \delta, s, p, \nu, \sigma, \|g\|_{\mathrm{L}^1(\Omega^{x_g})}, \|G\|_{\mathrm{L}^p(\Omega)}, \|\boldsymbol{u}_*\|_{\boldsymbol{H}^{1/2}(0, L)})$.

PROOF. *First step.* We start by considering the auxiliary problem, in $\Omega^N = (0, N) \times (0, L)$, with $N \in \mathbb{N}$ given,

$$-\nu \Delta \boldsymbol{u}^N + (\boldsymbol{u}^N \cdot \nabla) \boldsymbol{u}^N = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}^N) - \nabla p^N \quad \text{in } \Omega^N, \tag{2.16}$$

$$\operatorname{div} \boldsymbol{u}^N = 0 \quad \text{in } \Omega^N, \tag{2.17}$$

$$\boldsymbol{u} = \boldsymbol{u}_*(y), \quad \text{for } x = 0 \tag{2.18}$$

$$\boldsymbol{u} = \boldsymbol{0}, \quad \text{for } x = N \text{ and } y = 0, L. \tag{2.19}$$

With no lost of generality, we assume $N > 1$ and let $\boldsymbol{U}^1$ be an extension of $\boldsymbol{u}_*$ to $\Omega^1 = (0, 1) \times (0, L)$ such that: (*i*) $\boldsymbol{U}^1 \in \boldsymbol{H}^1(\Omega^1)$; (*ii*) $\operatorname{div} \boldsymbol{U}^1 = 0$ in Ω^1; (*iii*) $\boldsymbol{U}^1 = \boldsymbol{u}_*$ on $x = 0$, $\boldsymbol{U}^1 = \boldsymbol{0}$ on $x = 1$, $y = 0$ and on $y = L$, in the trace sense. One can prove that for any $\alpha > 0$, there exists an extension $\boldsymbol{U}^1$ satisfying (*i*)-(*iii*) above and verifying

$$\left| \int_{\Omega^1} \boldsymbol{v}\cdot\nabla U^1\cdot\boldsymbol{v}\,d\boldsymbol{x} \right| \leqslant \alpha\|\nabla v\|^2_{L^2(\Omega^1)}, \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}^1(\Omega^1).$$

Moreover,

$$\|\boldsymbol{U}^1\|_{\boldsymbol{H}^1(\Omega^1)} \leqslant C_2(L)\|\boldsymbol{u}_*\|_{\boldsymbol{H}^{1/2}(0,\,L)}$$

(see, *e.g.*, [7]). Now, we consider the extension $\boldsymbol{U}^N$ to Ω^N such that $\boldsymbol{U}^N = \boldsymbol{U}^1$ if $x < 1$ and $\boldsymbol{U}^N = \boldsymbol{0}$ if $x \geqslant 1$. From what we have said above, $\boldsymbol{U}^N \in \boldsymbol{H}^1(\Omega^N)$,

$$\|\boldsymbol{U}^N\|_{\boldsymbol{H}^1(\Omega^N)} \leqslant C_3(L)\|\boldsymbol{u}_*\|_{\boldsymbol{H}^{1/2}(0,\,L)} \tag{2.20}$$

and

$$\left| \int_{\Omega^N} \boldsymbol{v}\cdot\nabla U^N\cdot\boldsymbol{v}\,d\boldsymbol{x} \right| \leqslant \alpha\|\nabla v\|^2_{L^2(\Omega^N)}, \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}^1(\Omega^N). \tag{2.21}$$

Second step. We look for solutions $\boldsymbol{u}^N$ of the form $\boldsymbol{u}^N = \boldsymbol{w}^N + \boldsymbol{U}^N$, where $\boldsymbol{U}^N$ is the extension given in the First step and $\boldsymbol{w}^N$ solves the problem

$$-\nu\Delta\boldsymbol{w}^N + (\boldsymbol{w}^N\cdot\nabla)\,\boldsymbol{w}^N = \boldsymbol{g}(\boldsymbol{x},\boldsymbol{w}^N) - (\boldsymbol{w}^N\cdot\nabla)\,\boldsymbol{U}^N - (\boldsymbol{U}^N\cdot\nabla)\,\boldsymbol{w}^N - \nabla p^N \quad \text{in } \Omega^N, \tag{2.22}$$

$$\operatorname{div}\boldsymbol{w}^N = 0 \quad \text{in } \Omega^N, \tag{2.23}$$

$$\boldsymbol{w}^N = \boldsymbol{0} \quad \text{at } \partial\Omega^N, \tag{2.24}$$

where $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{w}^N) = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{w}^N + \boldsymbol{U}^N) + \nu\Delta\boldsymbol{U}^N - (\boldsymbol{U}^N\cdot\nabla)\,\boldsymbol{U}^N$. Given $\boldsymbol{v}^N \in \boldsymbol{L}^2(\Omega^N)$, $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{v}^N) \in \boldsymbol{L}^q(\Omega^N)$, with $q = \min(2/s,\,p)$, and there exists a unique weak solution $\boldsymbol{w}^N \in \boldsymbol{H}^1_0(\Omega^N)$ of problem (2.22)-(2.24) with the body forces given by $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{v}^N)$ (see, *e.g.*, [7]). Thus, we can define a non-linear operator $\Lambda : \boldsymbol{L}^2(\Omega^N)\times[0,\,1] \to \boldsymbol{L}^2(\Omega^N)$, by setting

$$\Lambda(\boldsymbol{v}^N,\,\lambda) = \boldsymbol{w}^N, \tag{2.25}$$

associated to the problem (2.22)-(2.24) with the body forces given by $\lambda\boldsymbol{g}(\boldsymbol{x},\boldsymbol{v}^N)$. Multiplying (2.22: with $\lambda\boldsymbol{g}(\boldsymbol{x},\boldsymbol{v}^N)$) by $\boldsymbol{w}^N$, integrating by parts over Ω^N, using (2.21), (2.23)-(2.24) and the Sobolev embedding

$$\boldsymbol{H}^1(\Omega^N) \to \boldsymbol{L}^q(\Omega^N), \quad 1 \leqslant q < \infty, \tag{2.26}$$

we obtain the estimate

$$\|\Lambda(\boldsymbol{v}^N,\,\lambda)\|_{\boldsymbol{L}^2(\Omega^N)} = \|\boldsymbol{w}^N\|_{\boldsymbol{H}^1_0(\Omega^N)} < C_4(\alpha,\,L,\,p,\,s,\,\nu,\,R,\,\|\boldsymbol{v}^N\|_{\boldsymbol{L}^2(\Omega^N)}), \tag{2.27}$$

where R is taken to be $R > \max(\|G\|_{L^p(\Omega)},\,\|\boldsymbol{u}_*\|_{\boldsymbol{H}^{1/2}(0,\,L)},\,1)$. Then, from (2.27), the operator (2.25) maps $\boldsymbol{L}^2(\Omega^N)\times[0,\,1]$ into a bounded subset of $\boldsymbol{H}^1_0(\Omega^N)$ and from the Sobolev compact embedding $\boldsymbol{H}^1_0(\Omega^N) \to \boldsymbol{L}^2(\Omega^N)$, it is a completely continuous operator. Moreover $\Lambda(\boldsymbol{v}^N,\,0) = \boldsymbol{0}$ and from the Leray-Shauder Fixed Point Theorem, $\Lambda(\cdot,\,1)$ has a fixed point, $\Lambda(\boldsymbol{w}^N,\,1) = \boldsymbol{w}^N$. This proves the existence of, at least, one weak solution $\boldsymbol{w}^N \in \boldsymbol{H}^1_0(\Omega^N)$ of the problem (2.22)-(2.24), with $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{w}^N)$. Consequently the existence of, at least, one weak solution $\boldsymbol{u}^N \in \boldsymbol{H}^1(\Omega^N)$ of the problem (2.16)-(2.19) is assured.

Third step. Multiplying (2.22: with $g(x, w^N)$) by w^N, integrating by parts over Ω^N, using (1.7)-(1.8), (2.21), (2.23)-(2.24), the Sobolev embedding (2.26), Young's inequality with a suitable ε and finally replacing $w^N = u^N - U^N$, we obtain the following estimate independent of N

$$\int_{\Omega^N} (|\nabla u^N|^2 + \chi_f |u^N|^{1+\sigma})\, dx \leq C_5, \tag{2.28}$$

with $C_5 = C_5(L, \delta, s, p, \nu, \|g\|_{L^1(\Omega^{x_g})}, \|G\|_{L^p(\Omega)}, \|u_*\|_{H^{1/2}(0, L)})$.

Fourth step. Now, for each $N \in \mathbb{N}$, we consider a sequence u_k^N of weak solutions to problems (2.16)-(2.19) and thus satisfying (2.28). In consequence, using a standard diagonal process and that $f(x, u_k^N)$ is a Carathéodory function, we can choose a subsequence $u_k^{N_k}$ such that $u_k^{N_k}$ tends to u, weakly in $H^1(\Omega^R)$, as k tends to infinity, and $f(x, u_k^{N_k})$ tends to $f(x, u)$, in $L^1(\Omega^R)$, as k tends to infinity, for every $R > 0$. In addition, u satisfies to the energy estimate (2.15). Now, by the Sobolev embedding (2.26), we get that $f(x, u) \cdot u \in L^1(\Omega)$, $f(x, u) \in L^1(\Omega)$ and u is a weak solution to the nonlinear problem (1.1)-(1.6). □

In some situations, we can prove the existence result dropping the growth condition (2.14). But then, in order to control the convergence of suitable approximations, we need to assume a vectors angle condition.

THEOREM 2.2. *Theorem* 2.1 *is still valid if we replace the growth condition* (2.14) *by the following vectors angle condition: there exists* $\varepsilon > 0$ *such that*

$$|\angle(h(x, u), u)| \notin \left(\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon\right) \tag{2.29}$$

for every $|u| > M$ *and a.e. in* Ω, *where* $\angle(a, b)$ *denotes the angle formed by the vectors* a *and* b. *Here the energy estimate* (2.15) *takes the form*

$$\int_{\Omega} (|\nabla u|^2 + \chi_f |u|^{1+\sigma} + |h(x, u) \cdot u|)\, dx \leq C_1^2, \tag{2.30}$$

where, now, $C_1 = C_1(L, \delta, \nu, \sigma, \|g\|_{L^1(\Omega^{x_g})}, \|u_*\|_{H^{1/2}(0, L)})$.

PROOF. *First step.* All that is written in the First step of the Proof of Theorem 2.1 is valid here. Moreover, using Hölder inequality, one can prove

$$\int_{\Omega^N} |U^N|^p\, dx \leq C_6(L, p) \|u_*\|^p_{H^{1/2}(0, L)}, \quad \text{for } 1 \leq p < 2.$$

Second step. We consider, first, the intermediary case in which we assume, additionally,

$$|h(x, u)| \leq C\chi_f(x), \tag{2.31}$$

for some positive constant C, for all $u \in \mathbb{R}^2$ and almost all $x \in \Omega$.

If we consider the problem (2.16)-(2.19), with $f(x, u)$ replaced by $f(x)$ given arbitrarily, for instance $f \in L^2(\Omega^N)$, then we know the existence of a unique weak solution

$\boldsymbol{u}^N \in \boldsymbol{H}^1(\Omega^N)$ (see, *e.g.*, [7]), which satisfies to the energy relation

$$\nu \int_{\Omega^N} \nabla u^N : \nabla(\boldsymbol{u}^N - \boldsymbol{U}^N)\, d\boldsymbol{x} + \int_{\Omega^N} \boldsymbol{u}^N \cdot \nabla \boldsymbol{u}^N \cdot (\boldsymbol{u}^N - \boldsymbol{U}^N)\, d\boldsymbol{x} = \int_{\Omega^N} \boldsymbol{f} \cdot (\boldsymbol{u}^N - \boldsymbol{U}^N)\, d\boldsymbol{x}. \tag{2.32}$$

Proceeding, *e.g.*, as in the last reference, using (2.20) and (2.21), one can prove the following estimate

$$\|\boldsymbol{u}^N\|^2_{\boldsymbol{H}^1(\Omega^N)} \leqslant C_7(L, \nu, \|\boldsymbol{u}_*\|_{\boldsymbol{H}^{1/2}(0,L)}, \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega^N)}).$$

Using the Schauder's fixed point theorem in the same manner we did in [3], we prove the existence of, at least, one weak solution $\boldsymbol{u}^N \in \boldsymbol{H}^1(\Omega^N)$ to the problem (2.16)-(2.19), with $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}^N)$.

Third step. We point out that from assumptions (2.11) and (2.12),

$$|\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}) \cdot u| \leqslant \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}) \cdot u + 2g(\boldsymbol{x}), \tag{2.33}$$

for every $\boldsymbol{u} \in \mathbb{R}^2$ and almost all $\boldsymbol{x} \in \Omega$. In the energy relation (2.32) satisfied by $\boldsymbol{u}^N$, we use the assumption (2.10), next we add $|\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}^N) \cdot \boldsymbol{u}^N|$ to both sides of the resultant equation, we use assumptions (2.12), (2.31), (2.33) and we use the Hölder inequality. Then, we use (2.21), the Sobolev embedding (2.26) and we apply the Young inequality with a suitable $\varepsilon > 0$, to obtain the *a priori* estimate independent of N for $\boldsymbol{u}^N$,

$$\int_{\Omega^N} (|\nabla \boldsymbol{u}^N|^2 + \chi_f |u^N|^{1+\sigma} + |\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}^N) \cdot \boldsymbol{u}^N|)\, d\boldsymbol{x} \leqslant C_8,$$

where $C_8 = C_8(L, \delta, \nu, \sigma, \|\boldsymbol{u}_*\|_{\boldsymbol{H}^{1/2}(0,L)}, \|g\|_{L^1(\Omega^{x_g})})$.

Fourth step. Keeping in mind the assumption (2.31) and proceeding exactly as in [3], we prove the existence of a weak solution $\boldsymbol{u}$ to the problem (1.1)-(1.6) and which satisfies the energy estimate (2.30).

Fifth step. To proceed with the general case, *i.e.*, dropping condition (2.31), we use the same truncation and approximation argument as we did in [3]. □

REMARK 2.1. *Condition* (2.29) *does not imply any upper restriction on the growth of* $|\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})|$ *with respect to* $\boldsymbol{u}$ *and due to that, sometimes, this type terms are called* strongly non-linear.

Moreover, if we assume a non-increasing condition on $\boldsymbol{f}$, we can prove a uniqueness result.

THEOREM 2.3. *Let* $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ *be two weak solutions of* (1.1)-(1.6) *and let us assume the inequality*

$$(\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}_1) - \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}_2)) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) \leqslant 0 \tag{2.34}$$

holds for every $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{R}^2$ *and almost all* $\boldsymbol{x} \in \Omega$. *Assume the data* ν *and* $\boldsymbol{u}_*$ *such that the problem* (1.1)-(1.6), *with* $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{0})$, *has a unique weak solution in* $\widetilde{\boldsymbol{H}}(\Omega)$, *then* $\boldsymbol{u}_1 = \boldsymbol{u}_2$.

PROOF. Let $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ be two weak solutions. Then, according to Definition 2.1, $\boldsymbol{u}_1 - u_2 \in \boldsymbol{H}_0^1(\Omega)$ and

$$\nu \int_\Omega |\nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2)|^2 \, d\boldsymbol{x} + \int_\Omega (\boldsymbol{u}_1 - \boldsymbol{u}_2) \cdot \nabla \boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) \, d\boldsymbol{x} =$$

$$= \int_\Omega (\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}_1) - \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}_2)) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) \, d\boldsymbol{x}.$$

Now, proceeding as in [3], we prove

$$\nu \int_\Omega |\nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2)|^2 \, d\boldsymbol{x} + \int_\Omega (\boldsymbol{u}_1 - \boldsymbol{u}_2) \cdot \nabla \boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) \, d\boldsymbol{x} \leqslant 0 .$$

Then arguing as for the proof of the uniqueness result for $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{0})$ we get that

$$\int_\Omega |\nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2)|^2 \, d\boldsymbol{x} = 0$$

and from Poincaré inequality (2.9), we get the result. □

REMARK 2.2. *The assumption of a unique weak solution for problem* (1.1)-(1.6) *with* $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{0})$ *is fundamental to prove the uniqueness of weak solutions for general Navier-Stokes problems with prescribed forces field. This assumption is equivalent to assume*

$$-\int_\Omega \varphi \cdot \nabla \boldsymbol{u} \cdot \varphi \, d\boldsymbol{x} < C \|\nabla \varphi\|^2_{L^2(\Omega)}, \quad \textit{with } C < \nu,$$

for every $\boldsymbol{u} \in \widetilde{\boldsymbol{H}}$ *and* $\varphi \in \widetilde{\boldsymbol{H}}_0$. *Or, in our specific problem, is equivalent to assume*

$$\frac{L}{\pi} C_1 \sqrt{2} < \nu, \quad C_1 \textit{ given in } (2.15) \textit{ or } (2.30).$$

3. LOCALIZATION EFFECT

In the previous section has been established the existence of a weak solution having a finite global energy

$$E := \int_\Omega (|\nabla \boldsymbol{u}|^2 + \chi_f |u|^{1+\sigma}) \, d\boldsymbol{x}.$$

THEOREM 3.1. *Assume* $\boldsymbol{f}$ *satisfies* (1.7) *and* (1.8). *Then*:

(*i*) *if* $x_f = \infty$ (x_f *is given in* (1.8)), $\boldsymbol{u}$ *is any weak solution of* (1.1)-(1.6) *with finite energy* E, *then* $\boldsymbol{u}(x, y) = \boldsymbol{0}$ *for* $x > a'$, *where* $a' = a'(E, L, \delta, \nu, \sigma)$ *is a positive constant*;

(*ii*) *if* $x_f < \infty$, *then there exists at least one weak solution* $\boldsymbol{u}$ *of* (1.1)-(1.6) *with a finite energy* E, *such that if* $a' \leqslant x_f$, *then* $\boldsymbol{u}(x, y) = \boldsymbol{0}$ *for* $x > a'$;

(*iii*) *if, in addition, we assume* $\boldsymbol{f}$ *non-increasing, then conclusion* (*ii*) *holds for the unique solution of* (1.1)-(1.6).

We proceed as in [3], introducing the associated stream function ψ

(3.35) $$u = \psi_y \quad \text{and} \quad v = -\psi_x \quad \text{in } \Omega$$

and we reduce the study of problem (1.1)-(1.6), to the consideration of the following fourth order problem where the pressure term does not appear anymore,

(3.36) $$\nu \Delta^2 \psi + \frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial x} = \psi_y \Delta \psi_x - \psi_x \Delta \psi_y \quad \text{in } \Omega,$$

(3.37) $$\psi(x, 0) = \psi(x, L) = \frac{\partial \psi}{\partial n}(x, 0) = \frac{\partial \psi}{\partial n}(x, L) = 0 \quad \text{for } x \in (0, \infty),$$

(3.38) $$\psi(0, y) = \int_0^y u_*(s)\, ds, \quad \frac{\partial \psi}{\partial n}(0, y) = v_*(y) \quad \text{for } y \in (0, L),$$

(3.39) $$\psi(x, y), \ |\nabla \psi(x, y)| \to 0, \quad \text{as } x \to \infty \quad \text{and for } y \in (0, L).$$

Here $\boldsymbol{f} = (f_1, f_2) = (f_1(\boldsymbol{x}, \psi_y, -\psi_x), f_2(\boldsymbol{x}, \psi_y, -\psi_x))$ and the notion of weak solution is adapted to the information we have on the function $\boldsymbol{f}$.

DEFINITION 3.1. *A function ψ is a weak solution of problem* (3.36)-(3.39), *if*:

(*i*) $\psi \in \mathrm{H}^2(\Omega)$, $\boldsymbol{f}(\boldsymbol{x}, \psi_y, -\psi_x) \in \boldsymbol{L}^1_{loc}(\Omega)$;

(*ii*) $\psi(0, y) = \int_0^y u_*(s)\, ds$, $\frac{\partial \psi}{\partial n}(0, y) = v_*(y)$, $\psi(x, 0) = \psi(x, L) = \frac{\partial \psi}{\partial n}(x, 0) = \frac{\partial \psi}{\partial n}(x, L) = \psi(0, L) = 0$, *and* ψ, $|\nabla \psi| \to 0$, *when* $x \to \infty$;

(*iii*) *For every* $\phi \in \mathrm{H}^2_0(\Omega) \cap \mathrm{W}^{1,\infty}(\Omega)$ *with compact support*,

$$\nu \int_\Omega \Delta \psi \Delta \phi \, d\boldsymbol{x} - \int_\Omega (f_1 \phi_y - f_2 \phi_x)\, d\boldsymbol{x} = \int_\Omega \Delta \psi (\psi_x \phi_y - \psi_y \phi_x)\, d\boldsymbol{x}.$$

To establish the localization effect, as stated in Theorem 3.1, we proceed as in [3] and we prove the followings lemmas.

LEMMA 3.1. *If $\boldsymbol{u}$ is a weak solution of* (1.1)-(1.8) *in the sense of Definition* 2.1, *then ψ, given by* (3.35), *is a weak solution of* (3.36)-(3.39) *in the sense of Definition* 3.1.

LEMMA 3.2. *Let ψ be a weak solution of* (3.36)-(3.39) *with E finite. Assume that $\boldsymbol{f}$ satisfies* (1.7) *and* (1.8) *with $x_f = \infty$. Then, for every $a > x_g$, and every positive integer $m \geq 2$*

(3.40)
$$\int_\Omega (\nu |\mathrm{D}^2 \psi|^2 + \delta |\psi_y|^{1+\sigma})(x-a)_+^m \, d\boldsymbol{x} \leq$$
$$\leq 2m\nu \int_\Omega |\Delta \psi| |\psi_x| (x-a)_+^{m-1} d\boldsymbol{x} + 2m\nu \int_\Omega |\psi_y| |\psi_{xy}| (x-a)_+^{m-1} d\boldsymbol{x} +$$
$$+ m(m-1)\nu \int_\Omega |\Delta \psi| |\psi| (x-a)_+^{m-2} d\boldsymbol{x} + m \int_\Omega |\Delta \psi| |\psi_y| |\psi| (x-a)_+^{m-1} d\boldsymbol{x},$$

where $|\mathrm{D}^2 \psi|^2 = \psi_{xx}^2 + 2\psi_{xy}^2 + \psi_{yy}^2$.

From the term on the left-hand side of the inequality (3.40), it will arise the energy type term which depends on a

$$E_m(a) = \int_\Omega (|D^2\psi|^2 + |\psi_y|^{1+\sigma})(x-a)_+^m \, d\boldsymbol{x}.$$

We observe that $E_0(0) = E$ and $(E_m(a))^{(k)} = (-1)^k m!/(m-k)! E_{m-k}(a)$, $0 \leqslant k \leqslant m$.

Then, we prove the more difficult part expressed in the following lemma.

LEMMA 3.3. *Let ψ be a weak solution of* (3.36)-(3.39) *and let us assume $\boldsymbol{f}$ satisfies* (1.7) *and* (1.8) *with $x_f = \infty$. Then, the following differential inequality holds for $a \geqslant x_g$ (x_g is given in* (1.8)*):*

$$E_m(a) \leqslant C_9(E_{m-2}(a))^{\mu_1} + C_{10}(E_{m-2}(a))^{\mu_2}, \tag{3.41}$$

for every integer $m > 3$, where $C_i = C_i(L, m, \delta, \nu, \sigma)$, $i = 9, 10$ are positive constants and $\mu_j = \mu_j(m, \sigma) > 1$, $j = 1, 2$. Moreover, $E_2(a) < \infty$ for any $a \geqslant x_g$. In fact,

$$E_2(a) \leqslant CE_0(a) + C_{11}(E_0(a))^{\mu_1} + C_{12}(E_0(a))^{\mu_2}, \tag{3.42}$$

where $C_i = C_i(L, \delta, \nu, \sigma)$, $i = 11, 12$, are positive constants, $\mu_j = \mu_j(\sigma) > 1$, $j = 1, 2$.

PROOF. We rewrite (3.40) as

$$\int_\Omega (\nu|D^2\psi|^2 + \delta|\psi_y|^{1+\sigma})(x-a)_+^m \, d\boldsymbol{x} \leqslant 2m\nu I_1 + 2m\nu I_2 + m(m-1)\nu I_3 + mJ.$$

Applying the Cauchy inequality with $\varepsilon = \nu/(2m)$ to the term J and then taking the minimum on the left-hand side, we obtain

$$\min\left(\frac{\nu}{2}, \delta\right) E_m(a) \leqslant 2m\nu I_1 + 2m\nu I_2 + m(m-1)\nu I_3 + \frac{m^2}{\nu} J_2,$$

where

$$J_2 = \int_\Omega \psi_y^2 \psi^2 (x-a)_+^{m-2} \, d\boldsymbol{x}.$$

If we assume $m > 3$, the estimations of I_1, I_2 and I_3 obtained in [3] lead to

$$\min\left(\frac{\nu}{2}, \delta\right) E_m(a) \leqslant \varepsilon C_{13} E_m(a) + \frac{1}{\varepsilon} C_{14}(E_{m-2}(a))^{\mu} + C_{15} J_2, \tag{3.43}$$

where

$$\mu = 1 + 2\,\frac{1-\sigma}{4(1+\sigma) + (1-\sigma)m}, \tag{3.44}$$

$C_{13} = C_{13}(m, \nu)$, $C_{14} = C_{14}(L, m, \nu, \sigma)$ and $C_{15} = C_{15}(m, \nu)$. To estimate J_2, we use two fundamental one-dimensional inequalities. The first one is the Poincaré inequality (2.9). The second is the Ladyzhenskaya inequality (see [11] and [9])

$$\int_0^L u(y)^4 \, dy \leqslant 2\left(\int_0^L u(y)^2 \, dy\right)\left(\int_0^L u'(y)^2 \, dy\right),$$

valid for sufficiently regular functions u such that $u(0) = u(L) = 0$. Then, we apply

Hölder inequality to J_2, next we use the Green theorem to prove,

$$\int_0^L \phi^2\, dy = -2\int_x^\infty \int_0^L \phi\phi_x\, dydx \leqslant 2\left(\int_\Omega \phi^2\, d\boldsymbol{x}\right)^{1/2}\left(\int_\Omega \phi_x^2\, d\boldsymbol{x}\right)^{1/2}$$

for every $x \geqslant 0$ and every function ϕ with the same regularity and boundary values of our ψ or ψ_y. In a final step, we use the Cauchy inequality and from the definition of $E_0(0) \equiv E$ and $E_{m-2}(a)$, we prove

$$J_2 \leqslant 2\sqrt{C_{16}}E\left(\frac{L}{\pi}\right)^{3-\theta}(E_{m-2}(a))^{\frac{\mu+1}{2}}, \quad C_{16} = C_{16}(m, \nu), \tag{3.45}$$

where $\theta = [2(1+\sigma) + (1-\sigma)m]/[4(1+\sigma) + (1-\sigma)m]$. Then, (3.43) comes

$$\min\left(\frac{\nu}{2}, \delta\right) E_m(a) \leqslant \varepsilon C_{13} E_m(a) + \frac{1}{\varepsilon} C_{14}(E_{m-2}(a))^\mu + C_{15}(E_{m-2}(a))^{\frac{\mu+1}{2}},$$

where, now, $C_{15} = C_{15}(E, L, m, \nu, \sigma)$. Then, choosing an appropriated ε, we obtain the fractional differential inequality (3.41).

If $m = 2$, the estimates on I_1, I_2 and I_3 (with $m = 2$) obtained in [3] lead to

$$\min\left(\frac{\nu}{2}, \delta\right) E_2(a) \leqslant \varepsilon C_{17} E_2(a) + \varepsilon C_{18} E_0(a) + \frac{1}{\varepsilon} C_{19}(E_0(a))^\mu + C_{20} J_{2(m=2)},$$

where

$$\theta = \frac{2}{3+\sigma} \quad \text{and} \quad \mu = \frac{4}{3+\sigma}, \tag{3.46}$$

$C_{17} = C_{17}(\nu)$, $C_{18} = C_{18}(\nu)$ $(C_{18} = 2C_{17})$, $C_{19} = C_{19}(L, \nu, \sigma)$ and $C_{20} = C_{20}(\nu)$. Taking $m = 2$ in (3.45), $C_{21} = C_{16}$ with $m = 2$,

$$J_{2(m=2)} \leqslant 2\sqrt{C_{21}}E\left(\frac{L}{\pi}\right)^{3-\theta}(E_0(a))^{\frac{\mu+1}{2}}.$$

Finally, choosing an appropriated ε, we obtain the differential inequality (3.42). □

Proof of Theorem 3.1. We start with the case $x_f = \infty$. Taking $m = 4$ in Lemma 3.3, we have the fractional differential inequality

$$E_4(a) \leqslant C_{22}(E_2(a))^{\mu_1} + C_{23}(E_2(a))^{\mu_2}, \tag{3.47}$$

where, from (3.44),

$$\mu_1 = \mu = \frac{5-\sigma}{4} \quad \text{and} \quad \mu_2 = \frac{\mu+1}{2} = \frac{9-\sigma}{8}$$

and $C_{22} = C_{22}(L, m, \delta, \nu, \sigma)$, $C_{23} = C_{23}(L, m, \delta, \nu, \sigma)$. Using Lemma 3.3 with $m = 2$ and because of the finiteness of E, we can easily see that $E_2(a)$ is finite. Then, from Lemma 5.1, with $m = 4$, $p = 2$, $w = 2(9-\sigma)/(1-\sigma) > m$, $1 < \mu_1 = (5-\sigma)/4 < m/(m-p)$ and $1 < \mu_2 = (9-\sigma)/8$ $(\equiv \gamma) < m/(m-p)$, because $0 < \sigma < 1$, the support of $E_0(a)$ is a bounded interval $[0, a^*]$ with $a^* \leqslant a'$, where from (5.56)

$$a' = \frac{15+\sigma}{1-\sigma} C_{24}^{\frac{4}{7+\sigma}} E^{\frac{1}{2(7+\sigma)}}, \qquad C_{24} = C_{24}(E, L, \delta, \nu, \sigma).$$

Then $E_0(a) = 0$ for $a > a'$, which implies $u = 0$ and v is constant almost everywhere for $x > a'$. Finally from (1.4), $v = 0$ too in the same domain.

For the case $x_f < \infty$, the proof follows exactly as in [3]. □

REMARK 3.1. *All the remarks made in* [3, Section 3] *remain valid in this case. For instance, in the case of* $\sigma = 1$, *the above arguments lead to the inequality*

$$E_m(a) \leqslant C_{25} E_{m-2}(a), \quad \textit{for } a \geqslant x_g,$$

and, again, we can only derive an exponential decay for this case, which is optimal (*see Horgan* [9] *where the exponential decay estimate is derived using analogous arguments to those developed by Knowles* [10] *and Toupin* [12] *in their energy approach to the investigation of the Saint-Venant's Principle in classical elasticity theory*). *Moreover, the results of* [3, Section 4] *where we have considered* $\Omega = (0, \infty) \times (L_1(x), L_2(x))$, *can be extended for this case.*

REMARK 3.2. *A simple proof of the localization effect can be obtained by proving that*

$$|\psi| \leqslant C_{26}(L) \|\psi\|_{H^2(\Omega)} \tag{3.48}$$

and (3.41) *and* (3.42) *would come as in* [3], *but with the constants depending also on* $\|\psi\|_{H^2(\Omega)}$. *Another idea of a simple proof, is to assume* $E_m(a) \leqslant 1$ *or* $E_m(a) \geqslant 1$ *and again* (3.41) *and* (3.42) *would come as in* [3], *with* $\mu = \min(\mu_1, \mu_2)$ *or* $\mu = \max(\mu_1, \mu_2)$, *respectively.*

4. FORCES FIELD WITH A STAGNATION LINE

In the arguments we have considered in Section 3, one can realize that the parameter a we have chosen is such that $a \geqslant x_g$, with x_g given in (1.8). To work with $a < x_g$, we have to assume an extra condition on the second component of the forces field. In this section, we assume the body forces field satisfy:

$$-\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \cdot \boldsymbol{u} \geqslant \delta \chi_f(\boldsymbol{x}) |u|^{1+\sigma} \tag{4.49}$$

for every $\boldsymbol{u} \in \mathbb{R}^2$, for almost all $\boldsymbol{x} \in \Omega$ and for some $\delta > 0$, $0 < \sigma < 1$; and

$$|f_2(\boldsymbol{x}, \boldsymbol{u})| \leqslant C(x_s - x)_+^{\zeta} \tag{4.50}$$

for some x_f, x_s with $0 \leqslant x_s < x_f \leqslant \infty$, x_f large enough, C and ζ positive constants, with ζ to be specified later on. Because of this last condition, we say the second component of the body forces field has a *stagnation line* at $x = x_s$. The existence and uniqueness of a weak solution for this case, is guaranteed by Theorem 2.1, where $\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{0}$, and Theorem 2.3, respectively.

THEOREM 4.1. *There exists some positive constants C and ζ such that if* (4.50) *holds, then* $\boldsymbol{u} = \boldsymbol{0}$ *for* $x > x_s$ *and any* $y \in (0, L)$.

PROOF. If we consider $a \geqslant x_s$, we fall in the conditions studied in Section 3 with $g = 0$. Then we have to add to the right-hand side of (3.40) the terms $C(mK_1 + K_2)$ resulting from condition (4.50), where

$$K_1 = \int_\Omega (x_s - x)_+^\zeta \, |\psi| (x-a)_+^{m-1} \, d\boldsymbol{x} \quad \text{and} \quad K_2 = \int_\Omega (x_s - x)_+^\zeta \, |\psi_x| (x-a)_+^m \, d\boldsymbol{x}. \tag{4.51}$$

This lead us to the counterpart of (3.41),

$$E_m(a) \leqslant C_{27}(E_{m-2}(a))^\mu + C_{28} E_{m,2\zeta}(a), \tag{4.52}$$

for every integer $m > 3$, where $C_i = C_i(L, m, \delta, \nu, \sigma)$, $i = 27, 28$, μ is given by (3.44) and

$$E_{m,\zeta}(a) = \int_\Omega (x_g - x)_+^\zeta \, (x-a)_+^m \, d\boldsymbol{x}, \qquad m \geqslant 2. \tag{4.53}$$

Taking $m = 4$ in (4.52) and using an integration by parts on (4.53), we arrive at the counterpart of (3.47), where we put $E_4(a) = z(a)$,

$$z(a) \leqslant C_{29}(z''(a))^\mu + C_{30}(x_s - a)_+^{2\zeta+4}, \tag{4.54}$$

where $C_{29} = C_{29}(L, \delta, \nu, \sigma)$ and $C_{30} = C_{30}(\zeta, L, \delta, \nu, \sigma)$, The solutions of (4.54), with $\zeta = \dfrac{(2-\mu)}{\mu - 1}$ (notice that from (3.46), $\zeta > 3$), are of the form $z(a) = C(x_s - a)_+^{\frac{2\mu}{\mu-1}}$ with the positive constant C satisfying

$$C - C_{29} C^\mu \left[\frac{2\mu(\mu+1)}{(\mu-1)^2} \right]^\mu - C_{30} \leqslant 0.$$

Then $E_4(a) = C(x_s - a)_+^{\frac{2\mu}{\mu-1}}$ and consequently $\boldsymbol{u} = \boldsymbol{0}$ for $x \geqslant x_s$. □

REMARK 4.1. *From the physical point of view, this means the fluid stops at the same stagnation line as the second component of the body forces.*

5. APPENDIX

Here we prove the following result whose applications go beyond this article.

LEMMA 5.1. *Let* $f \in L^1(\mathbb{R}^+)$, $f \geqslant 0$ *a.e. in* $\mathbb{R}^+$ *and let us put*

$$E_m(a) = \int_a^\infty f(x)(x-a)^m \, dx.$$

Assume that the fractional differential inequality

$$E_m(a) \leqslant C_1(E_{m-p}(a))^{\mu_1} + C_2(E_{m-p}(a))^{\mu_2} \tag{5.55}$$

holds for all $a \geqslant 0$, *where* $0 < p < m < w = p\gamma/(\gamma - 1)$, C_1 *and* C_2 *are positive constants and* $1 < \mu_1, \mu_2 < m/(m-p)$ *and* $\gamma = \min(\mu_1, \mu_2)$. *Assume* $E_{m-p}(a)$ *is finite for any* $a \geqslant 0$. *Then, the support of* $E_0(a)$ *is a bounded interval* $[0, a_*]$, *with* $a_* \leqslant a'$, *where*

$$a' = (w - m + 1) C^{\frac{1}{(w-m)(\gamma-1)}} E^{\frac{1}{w-m}}, \qquad C = C(C_1, C_2, E, m, p, \mu_1, \mu_2). \tag{5.56}$$

Proof. For all $0 < p < m$ and all $a \geqslant 0$, we have by Hölder inequality

$$E_{m-p}(a) \leqslant (E_m(a))^{\frac{m-p}{m}} (E_0(a))^{\frac{p}{m}}. \tag{5.57}$$

Given $\gamma = \min(\mu_1, \mu_2)$, by the monotonicity of $E_{m-p}(a)$, we have for $i = 1, 2$,

$$(E_{m-p}(a))^{\mu_i} \leqslant (E_{m-p}(0))^{\mu_i - \gamma} (E_{m-p}(a))^{\gamma}$$

and, from (5.57), we have for $i = 1, 2$,

$$(E_{m-p}(a))^{\mu_i} \leqslant (E_m(0))^{\frac{m-p}{m}(\mu_i - \gamma)} (E_0(0))^{\frac{p}{m}(\mu_i-\gamma)} (E_{m-p}(a))^{\gamma}. \tag{5.58}$$

On the other side, from (5.55) and (5.57),

$$E_m(0) \leqslant \sum_{i=1}^{2} C_i (E_m(0))^{\frac{m-p}{m}\mu_i} (E_0(0))^{\frac{p}{m}\mu_i}.$$

Requiring that $\mu_i < m/(m-p)$, for $i = 1, 2$, we obtain when using Young inequality with $\varepsilon = 1/[2(C_1 + C_2)]$, $E_m(0) \leqslant K$, where

$$K := \sum_{i=1}^{2} 2C_i \left[2(C_1 + C_2) \frac{m-p}{m} \mu_i \right]^{\frac{(m-p)\mu_i}{m-(m-p)\mu_i}} \frac{m-(m-p)\mu_i}{m} (E_0(0))^{\frac{p\mu_i}{m-(m-p)\mu_i}}.$$

Then, for $i = 1, 2$, (5.58) comes

$$(E_{m-p}(a))^{\mu_i} \leqslant K^{\frac{m-p}{m}(\mu_i-\gamma)} (E_0(0))^{\frac{p}{m}(\mu_i-\gamma)} (E_{m-p}(a))^{\gamma}. \tag{5.59}$$

From (5.55), (5.57) and (5.59),

$$E_m(a) \leqslant \sum_{i=1}^{2} C_i K^{\frac{m-p}{m}(\mu_i-\gamma)} (E_0(0))^{\frac{p}{m}(\mu_i-\gamma)} (E_m(a))^{\frac{m-p}{m}\gamma} (E_0(a))^{\frac{p}{m}\gamma}.$$

Then, requiring that $\gamma < m/(m-p)$ and $w = p\gamma/(\gamma - 1) > m$,

$$E_m(a) \leqslant C^{\frac{m}{(w-m)(\gamma-1)}} (E_0(a))^{\frac{p\gamma}{(w-m)(\gamma-1)}} \tag{5.60}$$

where

$$C = \sum_{i=1}^{2} C_i \left\{ 2C_i \left[2(C_1+C_2) \frac{m-p}{m} \mu_i \right]^{\alpha_i} \frac{m-(m-p)\mu_i}{m} E^{\beta_i} \right\}^{\gamma_i} E^{\delta_i},$$

with $\alpha_i = (m-p)\mu_i/[m-(m-p)\mu_i]$, $\beta_i = p\mu_i/[m-(m-p)\mu_i]$, $\gamma_i = (m - p)(\mu_i - \gamma)/m$, $\delta_i = [p(\mu_i - \gamma)]/m$. Let us put $m - 1 = p$ in (5.57). Then, from (5.60),

$$E_1(a) \leqslant C^{\frac{1}{(w-m)(\gamma-1)}} (E_0(a))^{\frac{w-m+1}{w-m}}.$$

Since $E_1' = -E_0$, this is a first order differential inequality, whose explicit integration ends the proof. □

References

[1] S.N. Antontsev - J.I. Díaz - H.B. de Oliveira, *Stopping a viscous fluid by a feedback dissipative external field: I. The stationary Stokes equations.* Book of abstratcs of NSEC8, Euler International Mathematical Institute, St. Petersburg 2002.

[2] S.N. Antontsev - J.I. Díaz - H.B. de Oliveira, *On the confinement of a viscous fluid by means of a feedback external field.* C.R. Mécanique, 330, 2002, 797-802.

[3] S.N. Antontsev - J.I. Díaz - H.B. de Oliveira, *Stopping a viscous fluid by a feedback dissipative field: I. The stationary Stokes problem.* J. Math. Fluid Mech., to appear.

[4] S.N. Antontsev - J.I. Díaz - S.I. Shmarev, *Energy Methods for Free Boundary Problems, Applications to non-linear PDEs and fluid mechanics.* Birkhäuser, Boston 2002.

[5] F. Bernis, *Extinction of the solutions of some quasilinear elliptic problems of arbitrary order: Part 1.* Proc. Symp. Pure Math., 45, 1986, 125-132.

[6] F. Bernis, *Qualitative properties for some non-linear higher order degenerate parabolic equations.* Houston J. Math., 14, 1988, 319-352.

[7] G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Nonlinear Steady Problems*. Springer-Verlag, New York 1994.

[8] D. Gilbarg - N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, Berlin-Heidelberg 1998.

[9] C.O. Horgan, *Plane entry flows and energy estimates for the Navier-Stokes equations.* Arch. Rat. Mech. and Analysis, 68, 1978, 359-381.

[10] J.K. Knowles, *On Saint-Venant's Principle in the Two-Dimensional Linear Theory of Elasticity*. Arch. Ration. Mech. Anal., 21, 1966, 1-22.

[11] O.A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow.* Mathematics and its Applications, 2, Gordon and Breach, New York 1969.

[12] R.A. Toupin, *Saint-Venant's Principle.* Arch. Ration. Mech. Anal., 18, 1965, 83-96.

S.N. Antontsev:
Departamento de Matemática
Universidade da Beira Interior
6201-001 Covilhã (Portogallo)
anton@ubi.pt

H.B. de Oliveira:
Faculdade de Ciências e Tecnologia
Universidade do Algarve
8000-062 Faro (Portogallo)
holivei@ualg.pt

J.I. Díaz:
Facultad de Matemáticas
Universidad Complutense
28040 Madrid (Spagna)
ji_diaz@mat.ucm.es