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Shoshana Kamin, Philip Rosenau

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Shoshana Kamin - Philip Rosenau

CONVERGENCE TO THE TRAVELLING WAVE SOLUTION FOR A NONLINEAR REACTION-DIFFUSION EQUATION

Abstract. — We study the behaviour of the solutions of the Cauchy problem

$$u_t = (u^m)_{xx} + u(1 - u^{m-1}), \quad x \in \mathbb{R}, \quad t > 0 \quad u(0, x) = u_0(x), \quad u_0(x) \geqslant 0,$$

and prove that if initial data $u_0(x)$ decay fast enough at infinity then the solution of the Cauchy problem approaches the travelling wave solution spreading either to the right or to the left, or two travelling waves moving in opposite directions. Certain generalizations are also mentioned.

Key words: Asymptotic behaviour of solutions; Nonlinear diffusion; Reaction-diffusion equation; Travelling waves.

1. Introduction

In this *Note* we study the behaviour of the solutions of the Cauchy problem

$$u_t = (u^m)_{xx} + u(1 - u^{m-1}), \quad x \in \mathbb{R}, \quad t > 0 \tag{1.1}$$

$$u(0, x) = u_0(x), \quad u_0(x) \geqslant 0, \quad x \in \mathbb{R} \tag{1.2}$$

where $m > 1$. We prove that if initial data $u_0(x)$ decay fast enough at infinity then the solution of (1.1), (1.2) approaches the travelling wave solution. It may be a travelling wave (TW) spreading to the right, a TW spreading to the left, or two TW's moving in opposite directions.

The first paper in which convergence to the TW solution was demonstrated, is the celebrated work by Kolmogorov, Petrovsky and Piskunov [11] where the prototypical reaction - linear diffusion equation

$$u_t = u_{xx} + f(u), \tag{1.3}$$

was considered. Thereafter, a plethora of results were obtained with a variety of reaction functions $f(u)$ (see [23, 24, and the references therein]).

Equation (1.1) studied in the present paper addresses the impact of nonlinear diffusion on these processes. Curiously enough, though the importance of nonlinear diffusion is universally acknowledged, very rarely it is included. Depending on the interpretation of u, the nonlinear diffusion may represent a nonlinear conduction in plasma transport model or, in a biological context, the dispersion of a crowd avoiding specie. For instance, the diffusion of lubricating bacteria such as Paenibacillus dendriformitis is described by diffusion which is proportional to the bacterial density.

Equation (1.1) degenerates at the points where $u = 0$ and thus the definition of a weak solution is required [14, 21]. The main feature of the solutions of (1.1) is the finite speed of propagation of disturbances. In particular, the function

$$U(t, x) = \left\{ \left[1 - e^{\frac{m-1}{m}(x - t - x_0)} \right]_+ \right\}^{\frac{1}{m-1}} \tag{1.4}$$

is the TW solution of (1.1) and has support $x \leqslant t + x_0$, bounded by the straight line $x = t + x_0$ [1, 13, 17].

We note the existence of other TW's wherein the wave speed > 1. However, these solutions do not have a sharp front.

In the next section we present the properties of the solutions of equation (1.1) which are essential for the proof of convergence. Two of them, the «weighted» conservation of mass (Lemma 2.2) and the contraction principle (Lemma 2.3), are analogous to the corresponding features of the viscous conservation laws. They allow us to employ the approach used in the works on stability of shock waves in viscous conservation laws [6, 15, 19] and a variant of an idea presented in [4, 16].

Detailed proofs of Lemmas 2.2 and 2.3 of Section 2 will be given in our forthcoming paper [9] where we study convergence to the TW solutions of the equation

$$u_t = (u^m)_{xx} + a(u^m)_x + u - u^m, \quad a = \text{const}. \tag{1.5}$$

In [9] we also present some conclusions for the equations

$$u_t = (u^m)_{xx} + f(u). \tag{1.6}$$

The existence of a TW for (1.6) is studied in [12, 18]. The most complete analysis is performed in [7]. In Section 3 we prove the convergence to a TW with a sharp front for equation (1.1).

Our last comment is

REMARK. Let $u(t, x)$ be a solution of (1.5) with $a = 0$, $m = 2$, and let

$$u(t, x) = 4(\tau + 1) r^{-1/4} z(\tau, r), \qquad r = e^{2x}, \qquad \tau = e^t - 1.$$

Then $z(\tau, r)$ solves the equation

$$\frac{1}{r^{7/4}} z_\tau = (z^2)_{rr}, \quad r > 0, \quad t > 0, \tag{1.7}$$

and the TW solution (1.4) transforms to

$$z = \frac{4 r^{1/4}}{\tau + 1} \left(1 - b \frac{r^{1/4}}{\sqrt{\tau + 1}} \right)_+, \qquad r > 0. \tag{1.8}$$

Therefore the convergence of the solution of (1.1) to the TW solution (1.4) is equivalent to the convergence of the solution of (1.7) to the solution (1.8) which happens to be a dipole solution of (1.7). A similar connection holds for any $m > 1$ (see [17]).

2. Some properties of the solutions of (1.1)

We first recall some known results. We consider here weak solutions of equation (1.1). Let $S = \mathbb{R}^+ \times \mathbb{R}$ and $S_T = (0, T] \times \mathbb{R}$.

DEFINITION. By a solution of the Cauchy problem (1.1), (1.2) we mean a nonnegative function $u \in L^\infty(S_T)$ which satisfies the identity

$$\iint_{S_T} [\zeta_t u + \zeta_{xx} u^m + \zeta(u - u^m)]\, dx dt + \int_{\mathbb{R}} \zeta(0, x) u_0(x)\, dx = 0$$

for any $\zeta \in C^{2,1}(S_T)$ which vanishes for large $|x|$ and at $t = T$.

Suppose that $u_0(x)$ is a nonnegative continuous function, $u_0 \in L^\infty(\mathbb{R})$. It is known that under such an assumption there exists a unique bounded weak solution of the Cauchy problem (1.1), (1.2). This solution is continuous at the points where $u(t, x) > 0$ and

(2.1) $$u(t_0, x_0) > 0 \Rightarrow u(t, x_0) > 0 \qquad \forall t > t_0.$$

Moreover for every x_0 there exists $T = T(x_0)$ such that

(2.2) $$u(t, x_0) > 0 \quad \text{for every} \quad t \geq T(x_0).$$

In what follows we use a comparison principle: if u_1 and u_2 are weak solutions of (1.1) then

(2.3) $$u_1(0, x) \leq u_2(0, x) \Rightarrow u_1(t, x) \leq u_2(t, x), \qquad x \in \mathbb{R}.$$

Also, every weak solution may be obtained as a limit of the classical solutions u_ε,

(2.4) $$u_\varepsilon \to u \quad \text{uniformly on any bounded set}.$$

For the proof of the aforementioned results we refer the reader to [10, 14, 21] and to the references given in [8]. We use below two special solutions:

1. The TW solution (1.4) which we denote by U_q, where

(2.5) $$q = \int_{-\infty}^{\infty} U_q(t, x) e^{x-t}\, dx = \int_{-\infty}^{\infty} U_q(0, x) e^x\, dx.$$

The reason for such notation will be clarified shortly. By computation $q = q(x_0) = e^{x_0 \frac{m-1}{2m-1}}$.

2. A compact expanding wave, [2, 13, 17, 22],

(2.6) $$W(t, x) = A(t) \left\{ \left[1 - D(t) \cosh \frac{m-1}{m} x \right]_+ \right\}^{\frac{1}{m-1}}.$$

The functions $A(t)$ and $D(t)$ are solutions of the system of two, first order, ordinary differential equations.

LEMMA 2.1. *Assume $u_0(x) = 0$ for $x > x_1$. Then there exists a large enough A such that*

$$u(t, x) = 0 \quad \textit{for} \quad x \geq t + A.$$

LEMMA 2.2 (Conservation law). *Let* $u(t, x)$ *be a weak solution of* (1.1), (1.2) *and suppose that*

$$\int_{-\infty}^{\infty} u_0(x)\, e^x\, dx = q < \infty .$$

Then for all $T \geq 0$

$$\int_{-\infty}^{\infty} u(T, x)\, e^{x-T}\, dx = q. \tag{2.7}$$

LEMMA 2.3 (Contraction principle). *Let* u *and* v *be weak solutions of* (1.1), (1.2) *and*

$$\int_{-\infty}^{\infty} u_0\, e^x\, dx < \infty , \qquad \int_{-\infty}^{\infty} v_0(x)\, e^x\, dx < \infty . \tag{2.8}$$

Then

$$\int_{-\infty}^{\infty} |u(T, x) - v(T, x)|\, e^{x-T}\, dx \leq \int_{-\infty}^{\infty} |u(\tau, x) - v(\tau, x)|\, e^{x-\tau}\, dx \tag{2.9}$$

for $0 \leq \tau \leq T$.

REMARK. Changing x to $-x$ we obviously get the corresponding conservation of mass under the assumption that

$$\int_{-\infty}^{\infty} u_0\, e^{-x}\, dx < \infty$$

and the contraction principle under the assumptions $\int_{-\infty}^{\infty} u_0\, e^{-x}\, dx < \infty$, $\int_{-\infty}^{\infty} v_0\, e^{-x}\, dx < \infty$.

3. THE MAIN RESULTS

THEOREM 3.1. *Suppose that* $u_0(x) \not\equiv 0$,

$$\int_{-\infty}^{\infty} u_0\, e^x\, dx = q < \infty \tag{3.1}$$

and U_q *is the TW solution* (1.4)

$$\int_{-\infty}^{\infty} U_q(t, x)\, e^{x-t}\, dx = q . \tag{3.2}$$

Then

$$\int_{-\infty}^{\infty} |u(t, x) - U_q(t, x)| e^{x-t} dx \to 0 \tag{3.3}$$

as $t \to \infty$. *Moreover* $u(t,x) - U_q(t,x) \to 0$ *uniformly inside any strip* $\alpha \leqslant x - t \leqslant \beta$.

PROOF. We prove Theorem 3.1, first under the assumption that $u_0(x)$ has compact support on the right. Then, by Lemma 2.1, there exists a constant A such that

$$u(t, x) = 0 \quad \text{for} \quad x \geqslant t + A. \tag{3.4}$$

Let $u_h(t, x) = u(t+h, x+h)$, $h \geqslant 0$, $h \to \infty$. By (3.1) and Lemma 2.2

$$\int_{-\infty}^{\infty} u_h(t, x) e^{x-t} dx = \int_{-\infty}^{\infty} u(t+h, y) e^{y-t-h} dy = q. \tag{3.5}$$

Sequence $\{u_h(t, x)\}$ is uniformly bounded, and thus, is equicontinuous on any bounded set in $\mathbb{R}^+ \times \mathbb{R}$ [5]. Therefore there exists a subsequence $h_i \to \infty$ such that

$$u_{h_i}(t, x) \to w(t, x), \tag{3.6}$$

and the convergence is uniform on any bounded set. The limit function w is defined for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ and is a weak solution of (1.1). It follows from (3.4) that for all $h \geqslant 0$, $u_h(t, x) = 0$ for $x \geqslant t + A$. Therefore

$$w(t, x) = 0 \quad \text{for} \quad x \geqslant t + A, \tag{3.7}$$

$$\int |u_h(t, x) - w(t, x)| e^{x-t} dx \to 0 \quad \text{as} \quad h \to \infty \tag{3.8}$$

and

$$\int w(t, x) e^{x-t} dx = q. \tag{3.9}$$

We will prove below that

$$w = U_q. \tag{3.10}$$

Define

$$I^h(t) = \int |u_h(t, x) - U_q(t, x)| e^{x-t} dx. \tag{3.11}$$

Obviously

$$I^h(t) = \int |u(t+h, x+h) - U_q(t+h, x+h)| e^{x-t} dx =$$

$$= \int |u(t+h, y) - U_q(t+h, y)| e^{y-(t+h)} dy = I^0(t+h).$$

By the contraction principle, $I^0(t+h)$ is a nonincreasing function of t and h, and therefore, there exists

$$\lim_{h \to \infty} I^h(t) = I^\infty \geqslant 0 \quad \text{for all} \quad t. \tag{3.12}$$

It follows from (3.8), (3.11) and (3.12) that for all $t \geqslant 0$

$$\int_{-\infty}^{\infty} |w(t, x) - U_q(t, x)| e^{x-t} dx = I^{\infty}. \tag{3.13}$$

To prove (3.10) it is enough to show that $I^{\infty} = 0$. Define $\overline{u}(t, x)(\underline{u}(t, x))$ as a solution of (1.1) with $\overline{u}(0, x) = \max\{w(0, x), U_q(0, x)\}$ ($\underline{u}(0, x) = \min\{w(0, x), U_q(0, x)\}$). By the comparison principle

$$\overline{u}(t, x) \geqslant \max\{w(t, x), U_q(t, x)\}, \qquad \underline{u}(t, x) \leqslant \min\{w(t, x), U_q(t, x)\}.$$

Suppose $I^{\infty} > 0$. We prove below that such an assumption leads to a contradiction. For the proof we use the strong maximum principle which is valid in the region where the equation is not degenerate.

By (3.2) and (3.9) for all $t \geqslant 0$

$$\int w e^{x-t} dx = \int U_q e^{x-t} dx. \tag{3.14}$$

By the definition of U_q given in (1.4) and (2.5) we have

$$U_q(\tau, x+\tau) > 0 \quad \text{for} \quad x < x_0, \qquad U_q(\tau, x+\tau) = 0 \quad \text{for} \quad x \geqslant x_0.$$

Since

$$w(0, x_1) \not\equiv U_q(0, x) \quad \text{and} \quad \int_{-\infty}^{\infty} w(0, x) e^x dx = \int_{-\infty}^{\infty} U_q(0, x) dx \tag{3.15}$$

there exists $\tilde{x}$ such that

$$w(0, \tilde{x}) > U_q(0, \tilde{x}). \tag{3.16}$$

There are two possibilities:

(*i*) $\tilde{x}$ may be chosen such that $U_q(0, \tilde{x}) > 0$, which means $\tilde{x} < x_0$.

(*ii*) (3.16) holds only for $\tilde{x} \geqslant x_0$.

First assume (*i*). In this case there exists some interval (c, d) such that the difference $w(0, x) - U_q(0, x)$ changes sign on (c, d) at least once and

$$w(0, x) > 0, \qquad U_q(0, x) > 0 \quad \text{for} \quad x \in [c, d]. \tag{3.17}$$

Then we have for $x \in [c, d]$

$$\overline{u}(0, x) - w(0, x) \not\equiv 0, \qquad \overline{u}(0, x) - U_q(0, x) \not\equiv 0. \tag{3.18}$$

Because of (3.17) the difference $\overline{u} - w$ is a solution of some nondegenerate parabolic equation with smooth coefficients in the cylinder $(c, d) \times (0, 1)$. By the strong maximum principle using (3.18) we obtain that $\overline{u}(1, x) - w(1, x) > 0$, $\overline{u}(1, x) - U_q(1, x) > 0$ on (c, d) and hence

$$\overline{u}(1, x) > \max\{w(1, x), U_q(1, x)\}$$

for all $x \in (c, d)$.

Therefore

$$\int_{\mathbb{R}} \left[\overline{u}(1, x) - \underline{u}(1, x)\right] e^{x-1}\, dx >$$

$$> \int \left[\max\left\{w(1, x), U_q(1, x)\right\} - \min\left\{w(1, x), U_q(1, x)\right\}\right] e^{x-1}\, dx =$$

$$= \int_{\mathbb{R}} |w(1, x) - U_q(1, x)|\, e^{x-1}\, dx = I^{\infty}.$$

On the other hand, by the contraction principle

$$\int_{\mathbb{R}} \left[\overline{u}(1, x) - \underline{u}(1, x)\right] e^{x-1}\, dx \leqslant \int_{\mathbb{R}} \left[\overline{u}(0, x) - \underline{u}(0, x)\right] e^{x}\, dx =$$

$$= \int_{\mathbb{R}} |w(0, x) - U_q(0, x)|\, e^{x}\, dx = I^{\infty}$$

and we have a contradiction. Thus the assumption $I^{\infty} > 0$ is false. Consequently $I^{\infty} = 0$ and as a result, $w \equiv U_q$.

Second case: (*ii*). We cannot use the same proof as in case (*i*) because the parabolic equation for the difference $\overline{u} - w$ may be degenerate. Therefore one needs to use the positivity property (2.1), (2.2) to conclude that there exist $\tau > 0$ and $\tilde{x} < x_0 + \tau$ such that $w(\tau, \tilde{x}) > U_q(\tau, \tilde{x}) > 0$. Now, changing t to $t - \tau$, we return to the previous case and obtain that $w \equiv U_q$.

Finally we obtain that $\lim\limits_{h_i \to \infty} u_{h_i}$ does not depend on the subsequence, and the whole sequence u_h converges to U_q. The convergence is in the «weighted» L_1 norm as defined by (3.3) and, as follows from the proof, it is uniform on any compact set. Thus we have

$$\int |u(t+h, x+h) - U_q(t+h, x+h)|\, e^{x-t}\, dx = \int |u(\tau, y) - U_q(\tau, y)|\, e^{y-\tau}\, dy \to 0$$

as $\tau \to \infty$ and

$$|u(\tau, y) - U_q(\tau, y)| \to 0$$

uniformly on every set

$$\alpha < y - \tau < \beta$$

for any fixed α, β. Thus Theorem 3.1 is proved for the case where $u_0(x)$ has compact support on the right. Now suppose that only (3.1) holds. Let $u_{0,\varepsilon}(0, x)$ be a sequence of functions, each compactly supported from the right, and

$$\int |u_{0,\varepsilon}(0, x) - u_0(x)|\, e^{x}\, dx \leqslant \varepsilon. \tag{3.19}$$

The sequence $u_{0,\varepsilon}$ may be chosen such that

$$\int u_{0,\varepsilon}(x)\, e^{x}\, dx = q.$$

Let $u_\varepsilon(t, x)$ be the solution of (1.1) with initial data

$$u_\varepsilon(0, x) = u_{0,\varepsilon}(x).$$

By (3.19) and the contraction principle

$$\int |u_\varepsilon(t, x) - u(t, x)| e^{x-t} dx \leqslant \varepsilon. \tag{3.20}$$

Moreover, as we proved above,

$$\int |u_\varepsilon(t, x) - U_q(t, x)| e^{x-t} dx \to 0 \tag{3.21}$$

as $t \to \infty$. Because ε is arbitrarily small it follows from (3.20) and (3.21) that (3.3) holds. □

As a corollary, note that changing x to $-x$, leads to the convergence to a TW that moves to the left.

THEOREM 3.2. *Suppose that* $u_0(x) \not\equiv 0$,

$$\int_{\mathbb{R}} u_0 e^{-x} dx = q^* < \infty \tag{3.22}$$

and U_{q^*} *is the TW solution* (1.4),

$$\int_{\mathbb{R}} U_{q^*}(t, x) e^{x-t} dx = q^*.$$

Then

$$\int_{\mathbb{R}} |u(t, x) - U_{q^*}(t, -x)| e^{-x-t} dx \to 0$$

as $t \to \infty$. *Moreover,*

$$|u(t, x) - U_{q^*}(t, -x)| \to 0$$

uniformly inside any strip $\alpha \leqslant x + t \leqslant \beta$.

REMARK. Suppose that both conditions (3.1) and (3.22) are satisfied. In particular, if the initial function $u_0(x)$ has a compact support, then we can apply Theorems 3.1 and 3.2 and conclude that as $t \to \infty$ $u(t, x) \to U_q(t, x)$ for $\alpha + t < x < \beta + t$ and $u(t, x) \to U_{q^*}(t, x)$ for $\alpha_1 - t < x < \beta_1 - t$, where $\alpha, \beta, \alpha_1, \beta_1$ are arbitrary given constants and the convergence is uniform in the corresponding strips.

A typical question is then: what is the behaviour of the solution in the $\beta_1 - t < x < \alpha + t$ domain? It is natural to expect that in this region the solution is close to 1. This, indeed, is proved in [9].

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School of Mathematical Sciences
Tel Aviv University
TEL AVIV 69978 (Israele)
kamin@post.tau.ac.il
rosenau@post.tau.ac.il