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BOUNDARY TRACE OF SOLUTIONS OF SEMILINEAR ELLIPTIC EQUALITIES AND INEQUALITIES

ABSTRACT. — The boundary trace problem for positive solutions of

$$-\Delta u + g(x, u) \geqslant 0$$

is considered for nonlinearities of absorption type, and three different methods for defining the trace are compared. The boundary trace is obtained as a generalized Borel measure. The associated Dirichlet problem with boundary data in the set of such Borel measures is studied.

KEY WORDS: Laplacian; Poisson potential; Singularities; Radon measures; Borel measures; Balayage; Convergence in measure.

1. INTRODUCTION

Let Ω be an open domain of $\mathbb{R}^N$ with a C^2 boundary and $g : \Omega \times \mathbb{R} \mapsto g(x, r)$ a continuous real-valued function. If $u \in C^1(\Omega)$ is a solution of

$$-\Delta u + g(x, u) \geqslant 0 \quad \text{in } \Omega, \tag{1.1}$$

a natural problem is to associate to this function an extended notion of boundary value called the *boundary trace* of u.

It is wellknown that if u is a positive harmonic function in Ω, there exists a Radon measure μ on $\partial\Omega$ which is the **boundary trace** of u on $\partial\Omega$ in the following natural sense:

$$\lim_{t \downarrow 0} \int_{\Sigma_t} u(\sigma, t)\, \zeta_t(\sigma, t)\, dS_t = \int_{\partial\Omega} \zeta(\sigma)\, d\mu, \tag{1.2}$$

for any $\zeta \in C_0(\partial\Omega)$, in which formula $\Sigma_t = \{x \in \Omega : \varrho(x) = t\}$ $(t > 0)$ and

$$\varrho(x) = \operatorname{dist}(x, \partial\Omega), \quad \forall x \in \Omega.$$

We denote by dS_t is the induced surface measure on Σ_t, $\sigma = \sigma(x)$ the orthogonal projection of x on $\partial\Omega$ and put $\zeta_t(\sigma, t) = \zeta(\sigma)$. The above formulation is consistent since the mapping

$$\Pi : (\sigma, t) \mapsto x$$

is a local diffeomorphism from $\partial\Omega \times (0, t_0)$ onto $\Omega_{t_0} = \{x \in \Omega : \varrho(x) < t_0\}$, for some $t_0 > 0$. Moreover, if Ω admits a Poisson kernel P^Ω (this is always the case if it is bounded) the Herglotz formula asserts that u admits an integral representation

$$u(x) = \int_{\partial\Omega} P^\Omega(x, y)\, d\mu(y) = \mathbb{P}^\Omega_\mu(x), \quad \forall x \in \Omega. \tag{1.3}$$

The extension of the above results to positive super-harmonic functions has been performed by Doob. If u is a positive super-harmonic function in Ω, it admits a boundary trace which is a nonnegative Radon measure μ, and, for any compact subset K of $\mathbb{R}^N$,

$[\Delta u] \in L^1(\Omega \cap K; \varrho dx)$. Furthermore if Ω is bounded and G^Ω is the Green kernel in Ω, there holds:

$$u(x) = \int_\Omega G^\Omega(x, y)[-\Delta u]\, dy + \int_{\partial\Omega} P^\Omega(x, y)\, d\mu(y). \tag{1.4}$$

In order to extend the linear theory to the semilinear one, we consider positive solutions of (1.1), assuming that $g(x, u) \geqslant 0$ (notice that the case $g(x, u) \leqslant 0$, is described by Doob's result). Actually, in the subharmonic framework, the nonlinear term plays a crucial role.

There are several approaches for defining the boundary trace of positive solutions u of nonlinear equations such as (1.1) and we present three of them here.

- The first method is based upon convexity and duality arguments. It was first developed for the following type of equation

$$-\Delta u + u^q = 0, \tag{1.5}$$

where $q > 1$ [13-15, 10, 11, 3, 4]. The boundary trace $Tr_{\partial\Omega}(u)$ of u exists in the class $\mathfrak{B}_+^{reg}(\partial\Omega)$ of outer regular positive Borel measures, not necessarily locally bounded. There exists a critical exponent $q_c = (N+1)/(N-1)$. If $1 < q < q_c$ the generalized Dirichlet problem

$$\begin{cases} -\Delta u + u^q = 0 & \text{in } \Omega, \\ Tr_{\partial\Omega}(u) = \nu \in \mathfrak{B}_+^{reg}(\partial\Omega), \end{cases} \tag{1.6}$$

is uniquely solvable for any ν. This is no longer the case if $q \geqslant q_c$. The study of the boundary trace problem is extended in [8] to

$$-\Delta u + e^u = 0, \tag{1.7}$$

and in [6] to

$$-\Delta u + u \ln_+^\alpha(u) = 0, \tag{1.8}$$

for $\alpha > 0$.

- The second method is introduced in [18] to handle equations with a non-uniform absorption term. In such equations the duality-convexity argument is no longer valid because of the boundary degeneracy of the non-linear term, and it has to be replaced by a localization principle called the strong barrier property. The typical case is

$$-\Delta u + \varrho(x)^\alpha u^q = 0, \tag{1.9}$$

with $\alpha > -2$ and $q > 1$. For such an equation, many of the results obtained for (1.5), (1.6) are extendable, but their proofs are much more intricate.

- The last method is intended to treat not only equations, but inequalities such as

$$-\Delta u + g(x, u) \geqslant 0, \tag{1.10}$$

where $g(x, r) \geqslant 0$ for $(x, r) \in \Omega \times \mathbb{R}_+$. It is no longer based upon localization, but on a balayage principle in which the main role is handled by the solutions v_μ (whenever they exist) of

$$(1.11) \qquad \begin{cases} -\Delta v_\mu + g(x, v_\mu) = 0 & \text{in } \Omega, \\ v_\mu = \mu & \text{on } \partial\Omega, \end{cases}$$

where μ is a positive Radon measure on $\partial\Omega$. Let $\mathfrak{M}_+^g(\partial\Omega)$ be the set of measures such that problem (1.11) is solvable (always uniquely if $g(., r)$ is nondecreasing with respect to r). Then $\min\{u, u_\mu\}$ is a supersolution of (1.10) which admits a boundary trace in $\mathfrak{M}_+^g(\partial\Omega)$. If we denote by $\gamma_u(\mu)$ this boundary trace, it is proven that the formula

$$(1.12) \qquad \nu = \sup_{\mu \in \mathfrak{M}_+^g(\partial\Omega)} Tr_{\partial\Omega}(\gamma_u(\mu)),$$

defines a Borel measure, not necessarily regular, that we call the extended boundary trace and denote by $Tr^e_{\partial\Omega}(u)$. This method, developed in [19] (and in [20] for the associated parabolic inequalities), is well adapted to treat highly degenerate inequalities such as

$$(1.13) \qquad -\Delta u + \exp(-1/\varrho(x))\, f(u) \geqslant 0,$$

under a very weak assumption on f.

Our article is organized as follows: The power case; The strong barrier method; The balayage method.

2. The power case

Let $q > 1$ and $\Omega \subset \mathbb{R}^N$ be any domain. By a solution of

$$(2.1) \qquad -\Delta u + |u|^{q-1} u = 0 \quad \text{in } \Omega.$$

We mean a $C^2(\Omega)$ function. Keller [9] and Osserman [21] proved independently that the set of solutions of (2.1) is locally uniformly bounded, and more precisely that

$$(2.2) \qquad |u(x)| \leqslant C(N, q)\varrho(x)^{-2/(q-1)}, \quad \forall x \in \Omega.$$

The result is more general since it holds for subsolutions with a much larger class of nonlinearities. As a consequence there exists always a maximal solution u_M to (2.1). If Ω is smooth enough, the maximal solution is unique and satisfies [1, 23]

$$(2.3) \qquad \lim_{\varrho(x)\to 0} \varrho(x)^{2/(q-1)} u(x) = \left(\frac{2(q+1)}{(q-1)^2}\right)^{1/(q-1)}.$$

In the case $q = 2$, and using probabilistic methods Le Gall [10] obtained the first boundary trace result for positive solutions of (2.1) in the unit ball of $\mathbb{R}^2$. Three years after Marcus and Véron [13] extended Le Gall's result, using only analytic tools, to any exponent $q > 1$, in any space-dimension. The result is the following.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a smooth domain and $q > 1$. Let u be a positive solution of* (2.1). *Then for any $\omega \in \partial\Omega$ the following alternative occurs,* (*i*) *either for every relatively open subset* $\mathcal{O} \subset \Omega$ *containing* ω

(2.4)
$$\lim_{t \to 0} \int_{\mathcal{O}_t} u(\sigma, t)\, dS_t = \infty\,,$$

(*ii*) *or there exist a relatively open subset* $\mathcal{O} \subset \Omega$ *containing* ω *and a positive linear functional* ℓ *on* $C_c^\infty(\mathcal{O})$ *such that for every* $\zeta \in C_c^\infty(\mathcal{O})$,

(2.5)
$$\lim_{t \to 0} \int_{\mathcal{O}_t} u(\sigma, t)\, \zeta_t(\sigma, t)\, dS_t = \ell(\zeta).$$

We write $\partial\Omega = \mathcal{S}(u) \cup \mathcal{R}(u)$ where $\mathcal{S}(u)$ is the closed subset of boundary points where (*i*) occurs, and $\mathcal{R}(u) = \partial\Omega \setminus \mathcal{S}(u)$. By using a partition of unity, there exists a unique positive Radon measure μ on $\mathcal{R}(u)$ such that

(2.6)
$$\lim_{t \downarrow 0} \int_{\mathcal{R}(u)} u(\sigma, t)\, \zeta_t(\sigma, t)\, dS_t = \int_{\mathcal{R}(u)} \zeta(\sigma)\, d\mu\,,$$

for every $\zeta \in C_c(\mathcal{R}(u))$. Thus we define the boundary trace by the following couple

(2.7)
$$Tr_{\partial\Omega}(u) = (\mathcal{S}(u), \mu).$$

The set $\mathcal{S}(u)$ is called the *singular part* of the boundary trace of u, while $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$ is the *regular part*. To the couple $(\mathcal{S}(u), \mu)$ is associated in a unique way an outer regular positive Borel measure ν (an element of $\mathfrak{B}_+^{reg}(\partial\Omega)$), with singular part $\mathcal{S}(u)$ and regular part μ.

Proof of Theorem 2. The following dichotomy holds for every boundary point ω.

(*i*) Either there exists an open ball $B_{r_0}(\omega)$ such that

(2.8)
$$\int_{B_{r_0}(\omega) \cap \Omega} u^q \varrho\, dx = \infty\,,$$

(*ii*) or for any $r > 0$,

(2.9)
$$\int_{B_r(\omega) \cap \Omega} u^q \varrho\, dx < \infty\,.$$

Then $\mathcal{S}(u)$ is precisely the set of points where (*i*) occurs, and $\mathcal{R}(u)$ the complement, where (*ii*) occurs. The original way to prove that (2.8) implies (2.5) is based upon the fact that for any smooth open subset $G \subset \Omega$ and any $\phi \in C_c^2(\overline{G})$,

(2.10)
$$\int_G \left(-u\Delta\phi + u^q \phi \right) dx = -\int_{\partial G} \frac{\partial\phi}{\partial \boldsymbol{n}_G}\, u\, dS.$$

There exist test functions $\phi > 0$ in G such that,

(2.11)
$$\int_G \phi^{-q/q'} |\Delta\phi|^{q'}\, dx < \infty\,,$$

(with $q' = q/(q-1)$. By Hölder's inequality

$$\left| \int_G u\Delta\phi dx \right| \leqslant \left(\int_G \phi^{-q/q'} |\Delta\phi|^{q'} dx \right)^{1/q'} \left(\int_G u^q \phi dx \right)^{1/q}.$$

Now (2.10) implies the next two inequalities

$$\int_G u^q \phi dx + \left(\int_G \phi^{-q/q'} |\Delta\phi|^{q'} dx \right)^{1/q'} \left(\int_G u^q \phi dx \right)^{1/q} \geqslant - \int_{\partial G} \frac{\partial \phi}{\partial \boldsymbol{n}_G} udS, \tag{2.12}$$

and

$$\int_G u^q \phi dx - \left(\int_G \phi^{-q/q'} |\Delta\phi|^{q'} dx \right)^{1/q'} \left(\int_G u^q \phi dx \right)^{1/q} \leqslant - \int_{\partial G} \frac{\partial \phi}{\partial \boldsymbol{n}_G} udS. \tag{2.13}$$

Next we choose $G = G_{\geqslant} = \{x = (\sigma, t) : \sigma \in \mathcal{O}, \tau < t \leqslant \tau_0\}$ for some $0 < \tau < \tau_0$ and $\mathcal{O} \subset \partial\Omega$ relatively open and smooth, and $\phi(\sigma, t) = \varphi_1^\alpha(\sigma)(t - \tau)$ where φ_1 is the first positive eigenfunction of the Laplace Beltrami operator in $W_0^{1,2}(\mathcal{O})$ and α is a positive real number larger than $(q+1)/(q-1)$. Then $-\partial\phi/\partial\boldsymbol{n}_G \approx \phi_1^\alpha$ on $\{x = (\sigma, \tau) : \sigma \in \mathcal{O}\}$ and (2.11) holds uniformly with respect to τ. By using inequalities (2.12) and (2.13), it is clear that the behaviour (blow-up, or boundedness) of the boundary term as $\tau \to 0$ is governed by the integrability or nonintegrability of $u^q \varrho$ in G_0 and the remaining of the proof is straightforward. □

The reverse problem is to find a function u, solution of (2.1) in Ω with a given boundary trace in $\mathfrak{B}_+^{reg}(\partial\Omega)$. The following result is due to Marcus and Véron [14, 15].

THEOREM 2.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^2 and $1 < q < (N+1)/(N-1)$. Then for any $\nu \in \mathfrak{B}_+^{reg}(\partial\Omega)$, there exists a unique $u \in C^2(\Omega)$ solution of the problem*

$$\begin{cases} -\Delta u + |u|^{q-1} u = 0 & \text{in } \Omega, \\ Tr_{\partial\Omega}(u) = \nu. \end{cases} \tag{2.14}$$

REMARK 2.1. (*i*) The case $q = N = 2$ was first treated by Le Gall [10], by probabilistic methods.

(*ii*) When ν is a Radon measure, a former result of Gmira and Véron [7] gives existence and uniqueness, always in the range $0 < q < (N+1)/(N-1)$. This is different from the L^1 case which was solved by Brezis (1975) under the mere assumption $q > 0$.

(*iii*) In the range $1 < q < (N+1)/(N-1)$ the main point is that the local average blow-up (2.4) which occurs for any $\omega \in \mathcal{S}(u)$ implies a pointwise blow-up, namely, for every compact cone $c_\omega \subset \Omega \setminus \{\omega\}$ with vertex ω, there exist a constant C depending on the opening of the cone c_ω, on q and N, but not on u and ω, such that

$$u(x) \geqslant C|x - \omega|^{-2/(q-1)}, \qquad \forall x \in \Omega. \tag{2.15}$$

(*iv*) When $q \geqslant (N+1)/(N-1)$ existence does not hold for any Borel measure, even any Radon measure, as observed by Gmira and Véron [7]. Furthermore (2.15) does not hold, and as a consequence uniqueness does not hold too. Necessary and sufficient conditions for existence of a maximal solution have been found separately by Marcus and Véron [14, 15, 17], and Dynkin and Kuznetsov [3, 4].

REMARK 2.2. By using a similar convexity method Grillot and Véron studied the boundary trace of the solutions of the 2-dimensional conformal Gaussian equation

$$-\Delta u + K(x)e^{2u} = 0 \quad \text{in } \Omega. \tag{2.16}$$

Assuming that $K > 0$ is bounded from below, they showed the existence of a boundary trace in $\mathfrak{M}^{reg}(\partial\Omega)$ and gave sufficient conditions for solving the corresponding Dirichlet problem

$$\begin{cases} -\Delta u + K(x)e^{2u} = 0 & \text{in } \Omega, \\ Tr_{\partial\Omega}(u) = \nu \in \mathfrak{M}^{reg}(\partial\Omega). \end{cases} \tag{2.17}$$

In [6] Fabbri and Licois obtained somewhat similar results for the weakly super-linear equation

$$-\Delta u + u \ln_+ u^\alpha = 0 \quad \text{in } \Omega \tag{2.18}$$

for $\alpha > 0$, and the associated generalized Dirichlet problem.

3. THE STRONG BARRIER METHOD

In this section we assume that Ω is a smooth bounded domain and $g : (x, r) \mapsto g(x, r)$ is a continuous function defined on $\Omega \times \mathbb{R}$ such that $g(x, r) \geqslant 0$ for $r \geqslant 0$. The method for proving the existence of a boundary trace of positive solutions of the semilinear equation

$$-\Delta u + g(x, u) = 0 \quad \text{in } \Omega. \tag{3.1}$$

Relies on two notions: the *coercivity property* and *the strong barrier property* which allows us to define a boundary trace in the class of outer regular positive Borel measures.

From the linear theory, it is known that, if $u \in C^2(\Omega)$ is a positive solution of (3.1) in Ω such that $g(., u)\varrho \in L^1(\Omega)$, then u admits a boundary trace on $\partial\Omega$ in the class of Radon measures. Moreover a representation formula similar to (1.4) holds. It is not difficult to prove the following local version of this result.

PROPOSITION 3.1. *Let $u \in C^2(\Omega)$ be a positive solution of* (3.1). *Suppose that for some point $\omega \in \partial\Omega$ there exists an open neighborhood U such that*

$$\int_{U \cap \Omega} g(x, u)\varrho(x)\, dx < \infty. \tag{3.2}$$

Then $u \in L^1(K \cap \Omega)$ *for any compact subset* $K \subset U$, *and there exists a Radon measure* μ *on* $U \cap \partial\Omega$ *such that*

$$\lim_{t \downarrow 0} \int_{U \cap \Sigma_t} u(\sigma, t)\, \zeta_t(\sigma, t)\, dS_t = \int_{U \cap \partial\Omega} \zeta(\sigma)\, d\mu, \tag{3.3}$$

for every $\zeta \in C_c(U \cap \partial\Omega)$.

As a consequence we are led to the following definition.

DEFINITION 3.1. Let u be a nonnegative solution of (3.1). A point $\omega \in \partial\Omega$ is called a *regular point* of u if there exists an open neighborhood U of ω such that (3.2) holds. The set of regular points is denoted by $\mathcal{R}(u)$. It is a relatively open subset of $\partial\Omega$. Its complement, $\mathcal{S}(u) = \partial\Omega \setminus \mathcal{R}(u)$ is the *singular set* of u.

By a partition of unity, it exists a positive Radon measure μ on $\mathcal{R}(u)$ such that

$$\lim_{t \downarrow 0} \int_{\mathcal{R}(u)_t} u(\sigma, t)\, \zeta_t(\sigma, t)\, dS_t = \int_{\mathcal{R}(u)} \zeta(\sigma)\, d\mu, \tag{3.4}$$

for every $\zeta \in C_c(\mathcal{R}(u))$. In order to be able to consider solutions of (3.1) with a strong blow-up localized on a part the boundary, we introduce the following notions.

DEFINITION 3.2. A function g is a *coercive nonlinearity* in Ω if, for every compact subset $K \subset \Omega$, the set of positive solutions of (3.1) is uniformly bounded on K.

A model example of coercive nonlinearity is the following:

$$g(x, r) \geq h(x) g(r), \qquad \forall (x, r) \in \Omega \times \mathbb{R}_+ \tag{3.5}$$

where $h \in C(\Omega)$ is continuous and positive, and $f \in C(\mathbb{R}_+)$ is nondecreasing, and satisfies the Keller-Osserman assumption

$$\int_\theta^\infty \left(\int_0^t f(s)\, ds \right)^{-1/2} dt < \infty, \qquad \forall \theta > 0. \tag{3.6}$$

The verification of this property is based upon the maximum principle and the construction of local super solutions by the Keller-Osserman method.

DEFINITION 3.3. A function g possesses the *strong barrier property* at $\omega \in \partial\Omega$ if there exists $r_0 > 0$ such that for any $0 < r \leq r_0$ there is a positive super solution $v = v_{\omega, r}$ of (3.1) in $B_r(\omega) \cap \Omega$ such that $v \in C(B_r(\omega) \cap \overline{\Omega})$ and

$$\lim_{\substack{y \to x \\ y \in \Omega}} v(y) = \infty, \qquad \forall x \in \Omega \times \partial B_r(\omega). \tag{3.7}$$

If $g(x, r) = f(r)$ where f satisfies the Keller-Osserman assumption, then it possesses the strong barrier property at any boundary point. If

$$g(x, r) = \varrho(x)^\alpha r^q, \qquad \forall (x, r) \in \Omega \times \mathbb{R}_+$$

for some $\alpha > -2$ and $q > 1$, it possesses also the strong barrier property, but the proof, due to Du and Guo [2], is difficult in the case $\alpha > 0$ (the nonlinearity is degenerate at the boundary). Finally, if

$$g(x,r) = \exp(-1/\varrho(x))\, r^q, \qquad \forall (x,r) \in \Omega \times \mathbb{R}_+,$$

for $q > 1$, then Marcus and Véron proved in [20] that the strong barrier property does not hold.

The strong barrier property is used to derive that on the singular set $\mathcal{S}(u)$ of any positive solution of (3.1) the integral blow-up occurs in the sense of (2.4), without using the duality-convexity argument associated to Hölder's inequality, and more precisely.

PROPOSITION 3.2. *Let* $u \in C^2(\Omega)$ *be a positive solution of* (3.1) *and suppose that* $\omega \in \mathcal{S}(u)$. *Suppose that at least one of the following sets of conditions holds*:
I. *There exists an open neighborhood* U' *of* ω *such that* $u \in L^1(U' \cap \Omega)$.
II. (*a*) $g(x,\cdot)$ *is non-decreasing in* $\mathbb{R}_+$, *for every* $x \in \Omega$;
(*b*) $\exists U_\omega$, *an open neighborhood of* ω, *such that* g *is coercive in* $U_\omega \cap \Omega$;
(*c*) g *possesses the strong barrier property at* ω.
Then, for every open neighborhood U *of* ω,

$$\lim_{t \to 0} \int_{U \cap \Sigma_t} u(x)\, dS_t = \infty. \tag{3.8}$$

This proposition, jointly with Proposition 3.1, yields to the following trace result.

THEOREM 3.1. *Let* g *be a coercive nonlinearity which has the strong barrier property at any boundary point. Assume also that* $r \mapsto g(x,r)$ *is nondecreasing on* $\mathbb{R}_+$ *for every* $x \in \Omega$. *Then any nonnegative solution* u *of* (3.1) *possesses a boundary trace* ν *in* $\mathfrak{B}_+^{reg}(\partial\Omega)$ *with*

$$\nu = Tr_{\partial\Omega}(u) \approx (\mathcal{S}(u), \mu), \quad where\ \mu \in \mathfrak{M}_+(\mathcal{R}(u)). \tag{3.9}$$

This result applies in the particular case where $g(x,r) = \varrho(x)^\alpha r^q$. Actually, in this case, the following extension of Theorem 2.1 holds, with a more difficult proof.

THEOREM 3.2. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded open domain of class* C^2, $\alpha > -2$ *and* $1 < q < (N+1+\alpha)/(N-1)$. *Then for any* $\nu \in \mathfrak{B}_+^{reg}(\partial\Omega)$, *there exists a unique* $u \in C^2(\Omega)$ *solution of the problem*

$$\begin{cases} -\Delta u + \varrho^\alpha |u|^{q-1} u = 0 & in\ \Omega, \\ Tr_{\partial\Omega}(u) = \nu. \end{cases} \tag{3.10}$$

Again $q = (N+1+\alpha)/(N-1)$ is a critical value, above which conditions have to be put on the Borel measure ν in order the problem (3.10) be solvable.

4. The balayage method

The method developed below, due to Marcus and Véron [19, 20], deals with the definition of an extended boundary trace for positive solutions of

$$-\Delta u + g(x, u) \geqslant 0. \tag{4.1}$$

The nonlinearity g is always supposed to be be continuous in $\Omega \times \mathbb{R}$ and to satisfy

$$g(x, 0) = 0, \; \forall x \in \Omega; \; r \mapsto g(x, r) \text{ is nondecreasing}. \tag{4.2}$$

The solvability of the nonlinear Dirichlet problem with Radon measures as boundary data plays a key role in this approach.

Definition 4.1. Let $v \in \mathfrak{M}(\partial\Omega)$. A function $u = u_\mu$ defined in Ω is a solution of the problem

$$\begin{cases} -\Delta u + g(x, u) = 0 & \text{in } \Omega, \\ u = \mu & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

if $u \in L^1(\Omega)$, $g(., u)\varrho \in L^1(\Omega)$ and the equality

$$\int_\Omega (-u\Delta\zeta + g(x, u)\zeta)\,dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial \boldsymbol{n}} \frac{\partial\zeta}{\partial \boldsymbol{n}}\,d\mu, \tag{4.4}$$

holds for every $\zeta \in C_c^2(\overline{\Omega})$.

Existence may not hold for any measure, but the monotonicity implies uniqueness. Since $g(x, r) \geqslant 0$ for $(x, r) \in \Omega \times \mathbb{R}_+$, u_μ is nonnegative whenever $\mu \geqslant 0$. Moreover u_μ satisfies (1.2) for any $\zeta \in C(\partial\Omega)$, thus admits μ as boundary trace.

Definition 4.2. Given a function g as above we denote by class $\mathfrak{M}_+^g(\partial\Omega)$ the set of positive Radon measures such that problem (4.3) is solvable. The function g is positively subcritical if (4.3) is solvable for any measure $\mu \in \mathfrak{M}_+(\partial\Omega)$.

We give below some examples of positive measures belonging to the class $\mathfrak{M}_+^g(\partial\Omega)$ and some functions positively subcritical.

- The following implication holds

$$g(., \mathbb{P}_\mu) \in L^1(\Omega; \varrho dx) \Rightarrow \mu \in \mathfrak{M}_+^g(\partial\Omega),$$

since $\mathbb{P}_\mu$ is a positive super-solution such that $g(., \mathbb{P}_\mu) \in L^1(\Omega; \varrho dx)$ of (4.1), and 0 a solution.

- Let $g(x, .)$ satisfy the Δ_2 condition, uniformly with respect to x, that is there exists $K > 0$ such that

$$g(x, r+s) \leqslant K(g(x, r) + g(x, s)) \qquad \forall x \in \Omega, \quad \forall r, s \geqslant 0.$$

Let $\mu = \mu_s + \mu_r \in \mathfrak{M}_+^g(\partial\Omega)$, with singular part μ_s and regular one μ_r with respect to the $(N-1)$-dimensional Hausdorff measure, be such that

$$g(., \mathbb{P}_{\mu_s}) \in L^1(\Omega; \varrho dx) \Rightarrow \mu \in \mathfrak{M}_+^g(\partial\Omega),$$

then $\mu \in \mathfrak{M}_+^g(\partial\Omega)$. This follows from the fact that $W = \mathbb{P}_{\mu_s} + u_{\mu_r}$ is a super-solution and $g(., W) \in L^1(\Omega; \varrho dx)$.

• If there exist two continuous and nondecreasing functions h and f defined on $\mathbb{R}_+$ such that

$$\left\{\begin{array}{l} 0 \leqslant g(x, r) \leqslant h(\varrho(x))\, f(r), \quad \forall (x, r) \in \Omega \times \mathbb{R}_+, \\ \int_0^1 h(s)\, f(\sigma s^{1-N}) s^N\, ds < \infty, \quad \forall \sigma \geqslant 0, \\ \text{either } h(s) = s^{\alpha}, \text{ for some } \alpha \geqslant 0, \text{ or } f \text{ is convex}. \end{array}\right. \tag{4.5}$$

It is proven in [19] that the following existence and stability theorem holds.

THEOREM 4.1. *For any $\mu \in \mathfrak{M}_+(\partial\Omega)$, Problem* (4.3) *admits a unique solution $u = u_\mu$. Moreover the problem is stable, in the sense that if $\{\mu_n\} \subset \mathfrak{M}_+(\partial\Omega)$ converges to μ in the weak sense of measures on $\partial\Omega$, the sequence of corresponding solutions u_{μ_n} converge to u_μ, locally uniformly in Ω.*

The balayage method is based upon the following result the proof of which follows by Kato's inequality, by using the monotonicity of $g(x,.)$ and Doob's theorem on super-harmonic functions.

PROPOSITION 4.1. *Let g satisfy* (4.2) *and $u \in C(\Omega)$ satisfy* (4.1), *then for any $\mu \in \mathfrak{M}_+^g(\partial\Omega)$,*

$$w_\mu = \min\{u, u_\mu\}$$

is a nonnegative super solution of (4.1) *which admits a boundary trace $\gamma_u(\mu) \in \mathfrak{M}_+^g(\partial\Omega)$. Moreover the correspondence $\mu \mapsto \gamma_u(\mu)$ is nondecreasing and*

$$0 \leqslant \gamma_u(\mu) \leqslant \mu.$$

As a consequence there holds,

THEOREM 4.2. *Let g and u be as in Proposition* 4.1. *Then the formula*

$$\nu = \sup_{\mu \in \mathfrak{M}_+^g(\partial\Omega)} \gamma_u(\mu), \tag{4.6}$$

defines a generalized positive Borel measure on $\partial\Omega$.

This measure ν may not be a regular one since the localization property may not hold. It is by definition the *extended boundary trace* of u and denoted by

$$\nu = Tr^e_{\partial\Omega}(u). \tag{4.7}$$

If we denote by u^* the largest solution of (3.1) dominated by u, there holds

$$Tr^e_{\partial\Omega}(u) = Tr^e_{\partial\Omega}(u^*). \tag{4.8}$$

REMARK 4.1. But for some particular cases that we shall see later on, it is unknown if the extended boundary trace is a boundary trace in the previous sense.

If we assume that g is positively subcritical, all the positive Radon measures μ can be used to define the extended boundary trace, in particular the Dirac masses. Since $g(x,.)$ is nondecreasing, the same property holds for $\mu \mapsto u_\mu$, the solution of (4.3). If

$\omega \in \partial\Omega$ we put

(4.9) $$u_{\infty,\omega} = \lim_{k\to\infty} u_{k\delta_\omega}.$$

Then $u_{\infty,\omega}$ is a solution of (3.1) on $\{x \in \Omega : u_{\infty,\omega}(x) < \infty\}$. This set may be whole Ω if g satisfies the Keller-Osserman condition. Thus

$$w_{k\delta_\omega} = \min\{u, u_{k\delta_\omega}\} \leqslant u_{k\delta_\omega} \Rightarrow supp.(\gamma_u(k\delta_\omega)) = \{\omega\}.$$

Then

$$\gamma_u(k\delta_\omega) = \tilde{\gamma}_u(k, \omega)\delta_\omega,$$

where $k \mapsto \tilde{\gamma}_u(k, \omega)$ is nondecreasing. We set

(4.10) $$\tilde{\gamma}_u(\omega) = \lim_{k\to\infty} \tilde{\gamma}_u(k, \omega).$$

Since $w_{k\delta_\omega}$ is a super solution with boundary trace $\tilde{\gamma}_u(k, \omega)\delta_\omega$ it dominates $u_{\tilde{\gamma}_u(k,\omega)\delta_\omega}$. Therefore

$$u_{\tilde{\gamma}_u(k,\omega)\delta_\omega} \leqslant u, \qquad \forall k \geqslant 0 \Rightarrow u_{\tilde{\gamma}_u(\omega)\delta_\omega} \leqslant u,$$

for every $\omega \in \partial\Omega$.

PROPOSITION 4.2. *Let g be positively subcritical and u satisfy* (4.1) *with extended boundary trace ν. Then*

(4.11) $$u \geqslant u_{\infty,\omega} \Rightarrow \nu(\omega) = \infty.$$

If we suppose moreover that g satisfies (4.5), *then*

(4.12) $$\nu(\omega) = \infty \Rightarrow u \geqslant u_{\infty,\omega}.$$

We define the *atoms* of u as the boundary points ω such that $\tilde{\gamma}_u(\omega) > 0$, the *singular set* of u as the closed subset $\mathcal{S}(u)$ of $\omega \in \partial\Omega$ such that

(4.13) $$\sum_{\theta\in\mathcal{O}} \tilde{\gamma}_u(\theta) = \infty, \qquad \forall \mathcal{O} \in \mathcal{N}_\omega,$$

where $\mathcal{N}_\omega$ is the set of relatively open neighborhoods of ω included in $\partial\Omega$, and the *regular set* of u as the relatively open subset $\mathcal{R}(u)$ of $\omega \in \partial\Omega$ such that there exists $\mathcal{O} \in \mathcal{N}_\omega$ such that

(4.14) $$\sum_{\theta\in\mathcal{O}} \tilde{\gamma}_u(\theta) < \infty.$$

The real numbers $\tilde{\gamma}_u(\omega)$ play also an important role in the description of the pointwise boundary behaviour of any positive solution of (4.1). We recall the following notion.

DEFINITION 4.3. A set of μ-measurable functions $x \mapsto \psi_r(x)$ ($r > 0$), defined over a measured space (E, Σ, μ), with finite μ-mass, converges in measure to ψ when $r \to 0$, if for any $\varepsilon > 0$ there holds

$$\lim_{r\to 0} \mu\{x \in E : |\psi_r(x) - \psi(x)| > \varepsilon\} = 0.$$

The functions ψ_r converges in measure to ∞, if for any $k > 0$,

$$\lim_{r\to 0} \mu\{x \in E : \psi_r(x) \leqslant k\} = 0.$$

The convergence is equivalent to the following statement: from any sequence $\{r_n\}$ converging to 0 one can extract a subsequence $\{r_{n_k}\}$ such that $\psi_{r_{n_k}}$ converges to ψ (or ∞), μ-a.e. in E.

DEFINITION 4.4. We say that the coordinates are proper at $\omega = (\omega_1, \dots, \omega_N) \in \partial\Omega$ relatively to Ω if the plane $x_1 - \omega_1 = 0$ is tangent to $\partial\Omega$ at ω, and that the inward pointing vector to $\partial\Omega$ is the direction $x_1 - \omega_1 > 0$.

THEOREM 4.3. *Assume g satisfies* (4.2) *is positively subcritical, u is a nonnegative solution of* (4.1) *and* $\omega \in \partial\Omega$. *If the coordinates are proper at* ω *relatively to* Ω, *the following alternative holds. Either*

(*i*) $\tilde{\gamma}_u(\omega)$ *is finite and the following convergence holds*

$$\lim_{\substack{x \to \omega \\ (x_1 - \omega_1)/|x-\omega| \to \eta_1}} |x-\omega|^{N-1} u(x) - C(N)\,\tilde{\gamma}_u(\omega)\,\eta_1 = 0, \tag{4.15}$$

in measure on S_+^{N-1},
or

(*ii*) $\tilde{\gamma}_u(\omega)$ *is infinite and*

$$\lim_{x \to \omega} |x-\omega|^{N-1} u(x) = \infty, \tag{4.16}$$

in measure on S_+^{N-1}.

Finally, we recover the classical definition of the boundary trace if we assume that g is not degenerate near the boundary in the sense that there exists a continuous nondecreasing function f defined on $\mathbb{R}_+$ such that

$$\begin{cases} 0 \leqslant g(x, r) \leqslant f(r), & \forall (x, r) \in \Omega \times \mathbb{R}_+, \\ \int_0^1 f(s^{1-N})\, s^N\, ds < \infty. \end{cases} \tag{4.17}$$

THEOREM 4.4. *Assume g satisfies* (4.2) *and* (4.17) *and u is a nonnegative solution of* (4.1) *with extended boundary trace* ν. *Then for any* $\omega \in \partial\Omega$ *the following dichotomy occurs. Either,*

(*i*) $\nu(\mathcal{O}) = \infty$ *for any* $\mathcal{O} \in \mathcal{N}_\omega$. *In this case* $\omega \in \mathcal{S}(u)$ *and* $u \geqslant u_{\infty,\omega}$. *Consequently*

$$\lim_{t \to 0} \int_{\mathcal{O}_t} u(y)\, dS_t = \infty, \quad \forall \mathcal{O} \in \mathcal{N}_a. \tag{4.18}$$

Or

(*ii*) *there exists* $\mathcal{O} \in \mathcal{N}_\omega$ *such that* $\nu(\mathcal{O}) < \infty$. *In this case* $\omega \in \mathcal{R}(u)$ *and*

$$\sup_{0 < t \leqslant \beta_0} \int_{\mathcal{O}'_t} u(y)\, dS_t < \infty, \tag{4.19}$$

for relatively every open subset $\mathcal{O}' \subset \overline{\mathcal{O}'} \subset \mathcal{O}$. *Furthermore*

$$\lim_{t \to 0} \int_{\Sigma_t} u(y)\,\phi(\sigma(y))(y)\,dS_t = \int_{\mathcal{R}(u)} \phi(y)\,d\nu(y), \tag{4.20}$$

for every $\phi \in C_c(\mathcal{R}(u))$.

A consequence of this result due to Marcus and Véron is that the extended boundary trace is an outer regular Borel measure which coincides with the usual boundary trace.

REMARK 4.2. (*i*) If $g(x, r) = r^q$, inequality (4.17) is verified if and only if $1 < q < (N+1)/(N-1)$.

(*ii*) If $g(x, r) = \varrho(x)^\alpha r^q$, condition (4.5) holds if and only if $\alpha > -2$ and $1 < q < (N+1+\alpha)/(N-1)$. In that case the it follows from Section 2 that the extended boundary trace coincides with the boundary trace.

(*iii*) If $g(x, r) = \exp(-1/\varrho(x))\,r^q$ with $q > 1$, then the condition (4.5) holds, but not the barrier property. More precisely (see [19]), for any $\omega \in \partial\Omega$, the function $u_{\infty,\omega} = \lim_{k \to \infty} u_{k\delta_\omega}$ satisfies

$$\begin{cases} -\Delta u + \exp(-1/\varrho(x))\,u^q = 0 \quad \text{in } \Omega, \\ \lim_{\varrho(x) \to 0} u(x) = \infty. \end{cases} \tag{4.21}$$

Thus $u_{\infty,\omega}$ is a large solution. Moreover, by using the techniques introduced in [12]. It can be proved that this problem admits a unique solution $u = u_M$. Thus either the extended boundary trace is a bounded Borel measure, or $u = u_M$.

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