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Analisi matematica. — *On the multiplicity of brake orbits and homoclinics in Riemannian manifolds.* Nota di Roberto Giambò, Fabio Giannoni e Paolo Piccione, presentata (*) dal Socio A. Ambrosetti.

Abstract. — Let (M,g) be a complete Riemannian manifold, $\Omega \subset M$ an open subset whose closure is diffeomorphic to an annulus. If $\partial\Omega$ is smooth and it satisfies a strong concavity assumption, then it is possible to prove that there are at least two geometrically distinct geodesics in $\overline{\Omega} = \Omega \bigcup \partial\Omega$ starting orthogonally to one connected component of $\partial\Omega$ and arriving orthogonally onto the other one. The results given in [5] allow to obtain a proof of the existence of two distinct homoclinic orbits for an autonomous Lagrangian system emanating from a nondegenerate maximum point of the potential energy, and a proof of the existence of two distinct *brake orbits* for a class of Hamiltonian systems. Under a further symmetry assumption, it is possible to show the existence of at least $\dim(M)$ pairs of geometrically distinct geodesics as above, brake orbits and homoclinics.

Key words: Brake orbits; Homoclinics; Variational methods.

Riassunto. — *Molteplicità di* brake orbits *e curve omocline su varietà Riemanniane.* Sia (M,g) una varietà Riemanniana completa, e $\Omega \subset M$ un aperto la cui chiusura è omeomorfa ad un anello. Se $\partial\Omega$ è liscio e soddisfa un'ipotesi di concavità forte, è possibile dimostrare che esistono almeno due geodetiche geometricamente distinte in $\overline{\Omega} = \Omega \bigcup \partial\Omega$, aventi gli estremi su componenti connesse distinte di $\partial\Omega$, e velocità iniziale e finale ortogonali a $\partial\Omega$. I risultati di [5] permettono di ottenere una dimostrazione, nel caso di un sistema Lagrangiano autonomo, dell'esistenza di due distinte curve omocline partenti da un punto di massimo non degenere dell'energia potenziale, e una dimostrazione dell'esistenza di due distinte *brake orbits* per una classe di sistemi Hamiltoniani. Sotto ulteriori ipotesi di simmetria, si ottiene l'esistenza di almeno $\dim(M)$ coppie di geodetiche geometricamente distinte, di *brake orbits* e di curve omocline.

In this *Note* we will describe a version of the Ljusternik-Schnirelman theory that can be used to prove the existence of multiple orthogonal geodesic chords in Riemannian manifolds with boundary. This fact, together with the results in [5], gives a multiplicity result for homoclinics and brake orbits of a class of Hamiltonian systems.

1. Geodesics in riemannian manifolds with boundary

Let (M,g) be a C^2-Riemannian manifold with $\dim(M) = m \geq 2$. The symbol ∇ will denote the covariant derivative of the Levi-Civita connection of g, as well as the gradient differential operator for smooth maps on M. The Hessian $\mathrm{H}^f(q)$ of a smooth map $f : M \to \mathbb{R}$ at a point $q \in M$ is the symmetric bilinear form $\mathrm{H}^f(q)(v,w) = g\big((\nabla_v \nabla f)(q), w\big)$ for all $v, w \in T_xM$; equivalently, $\mathrm{H}^f(q)(v,v) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big|_{s=0} f(\gamma(s))$, where $\gamma : \,]-\varepsilon, \varepsilon[\, \to M$ is the unique (affinely parameterized) geodesic in M with $\gamma(0) = q$ and

(*) Nella seduta dell'11 marzo 2005.

$\dot\gamma(0) = v$. We will denote by $\frac{\mathrm{D}}{\mathrm{d}t}$ the covariant derivative along a curve, in such a way that $\frac{\mathrm{D}}{\mathrm{d}t}\dot\gamma = 0$ is the equation of the geodesics. A basic reference on the background material for Riemannian geometry is [4].

Let $\Omega \subset M$ be an open subset; $\overline{\Omega} = \Omega \bigcup \partial\Omega$ will denote its closure. There are several notion of convexity and concavity in Riemannian geometry, extending the usual ones for subsets of the Euclidean space $\mathbb{R}^m$. In this paper we will use a somewhat strong concavity assumption for compact subsets of M, that we will call «strong concavity» below, and which is stable by C^2-small perturbations of the boundary. Let us first recall the following:

DEFINITION 1.1. $\overline{\Omega}$ is said to be *convex* if every geodesic $\gamma : [a, b] \to \overline{\Omega}$ whose endpoints $\gamma(a)$ and $\gamma(b)$ are in Ω has image entirely contained in Ω. Likewise, $\overline{\Omega}$ is said to be *concave* if its complement $M \setminus \overline{\Omega}$ is convex.

If $\partial\Omega$ is a smooth embedded submanifold of M, let $\mathbb{I}_{\mathfrak{n}}(x) : T_x(\partial\Omega) \times T_x(\partial\Omega) \to \mathbb{R}$ denote the *second fundamental form of $\partial\Omega$ in the normal direction* $\mathfrak{n} \in T_x(\partial\Omega)^{\perp}$. Recall that $\mathbb{I}_{\mathfrak{n}}(x)$ is a symmetric bilinear form on $T_x(\partial\Omega)$ defined by:

$$\mathbb{I}_{\mathfrak{n}}(x)(v, w) = g(\nabla_v W, \mathfrak{n}), \qquad v, w \in T_x(\partial\Omega),$$

where W is any local extension of w to a smooth vector field along $\partial\Omega$.

REMARK 1.2. Assume that it is given a smooth function $\phi : M \to \mathbb{R}$ with the property that $\Omega = \phi^{-1}\big(]-\infty, 0[\big)$ and $\partial\Omega = \phi^{-1}(0)$, with $\mathrm{d}\phi \neq 0$ on $\partial\Omega$ (1). The following equality between the Hessian H^{ϕ} and the second fundamental form (2) of $\partial\Omega$ holds:

$$\mathrm{H}^{\phi}(x)(v, v) = -\mathbb{I}_{\nabla\phi(x)}(x)(v, v), \quad x \in \partial\Omega,\ v \in T_x(\partial\Omega); \tag{1.1}$$

Namely, if $x \in \partial\Omega$, $v \in T_x(\partial\Omega)$ and V is a local extension around x of v to a vector field which is tangent to $\partial\Omega$, then $v\big(g(\nabla\phi, V)\big) = 0$ on $\partial\Omega$, and thus:

$$\mathrm{H}^{\phi}(x)(v, v) = v\big(g(\nabla\phi, V)\big) - g(\nabla\phi, \nabla_v V) = -\mathbb{I}_{\nabla\phi(x)}(x)(v, v).$$

Note that the second fundamental form is defined intrinsically, while there is general no natural choice for a function ϕ describing the boundary of Ω as above.

DEFINITION 1.3. We will say that $\overline{\Omega}$ is *strongly concave* if $\mathbb{I}_{\mathfrak{n}}(x)$ is negative definite for all $x \in \partial\Omega$ and all inward pointing normal direction $\mathfrak{n}$.

Observe that if $\overline{\Omega}$ is strongly concave, geodesics γ starting tangentially to $\partial\Omega$ move *inside* Ω, as we see looking at Taylor expansion of $s \mapsto \phi(\gamma(s))$.

(1) For example one can choose ϕ such that $|\phi(q)| = \mathrm{dist}(q, \partial\Omega)$ for all q in a (closed) neighborhood of $\partial\Omega$.
(2) Observe that, with our definition of ϕ, then $\nabla\phi$ is a normal vector to $\partial\Omega$ pointing *outwards* from Ω.

Remark 1.4. Strong concavity is evidently a C^2-*open condition*. Then, by 1.1, if $\overline{\Omega}$ is compact, we deduce the existence of $\delta_0 > 0$ such that $H^{\phi}(x)(v,v) < 0$ for all $x \in \phi^{-1}\big([-\delta_0,\delta_0]\big)$ and for all $v \in T_xM$, $v \neq 0$, such that $g\big(\nabla\phi(x),v\big) = 0$, (moreover $\nabla\phi \neq 0\, \forall x \in \phi^{-1}([-\delta_0,\delta_0])$).

The main objects of our study are geodesics in M having image in $\overline{\Omega}$ and with endpoints orthogonal to $\partial\Omega$, that will be called *orthogonal geodesic chords*:

Definition 1.5. A geodesic $\gamma : [a,b] \to M$ is called a *geodesic chord* in $\overline{\Omega}$ if $\gamma(]a,b[) \subset \Omega$ and $\gamma(a), \gamma(b) \in \partial\Omega$; by a *weak geodesic chord* we will mean a geodesic $\gamma : [a,b] \to M$ with image in $\overline{\Omega}$ and endpoints $\gamma(a), \gamma(b) \in \partial\Omega$. A (weak) geodesic chord is called *orthogonal* if $\dot\gamma(a^+) \in (T_{\gamma(a)}\partial\Omega)^\perp$ and $\dot\gamma(b^-) \in (T_{\gamma(b)}\partial\Omega)^\perp$, where $\dot\gamma(\,\cdot^{\pm})$ denote the lateral derivatives. An orthogonal geodesic chord in $\overline{\Omega}$ whose endpoints belong to distinct connected components of $\partial\Omega$ will be called a *crossing orthogonal geodesic chord* in $\overline{\Omega}$.

For shortness, we will write **OGC** for «orthogonal geodesic chord» and **WOGC** for «weak orthogonal geodesic chord». Although the general class of weak orthogonal geodesic chords are perfectly acceptable solutions of our initial geometrical problem, our suggested construction of a variational setup works well only in a situation where one can exclude *a priori* the existence in $\overline{\Omega}$ of orthogonal geodesic chords $\gamma : [a,b] \to \overline{\Omega}$ for which there exists $s_0 \in]a,b[$ such that $\gamma(s_0) \in \partial\Omega$.

One does not lose generality in assuming that there are no such WOGC's in $\overline{\Omega}$ by recalling the following result from [5, Proposition 2.6]:

Proposition 1.6. *Let $\Omega \subset M$ be an open set whose boundary $\partial\Omega$ is smooth and compact and with $\overline{\Omega}$ strongly concave. Assume that there are only a finite number of crossing orthogonal geodesic chords in $\overline{\Omega}$. Then, there exists an open subset $\Omega' \subset \Omega$ with the following properties*:

(1) *$\overline{\Omega'}$ is diffeomorphic to $\overline{\Omega}$ and it has smooth boundary*;

(2) *$\overline{\Omega'}$ is strongly concave*;

(3) *the number of crossing OGC's in $\overline{\Omega'}$ is less than or equal to the number of crossing OGC's in $\overline{\Omega}$*;

(4) *every crossing WOGC in $\overline{\Omega'}$ is a crossing OGC in $\overline{\Omega'}$.*

The central result described in this *Note* is a lower estimate on the number of distinct orthogonal geodesic chords under the strong concavity condition given in Definition 1.3. Concerning the convex case we recall that, in [2], Bos proved that if $\partial\Omega$ is smooth, $\overline{\Omega}$ convex and homeomorphic to the m-dimensional disk, then there are at least m distinct OGC's for $\overline{\Omega}$. Such a result is a generalization of a classical result by Ljusternik and Schnirelman (see [11]), where the same result was proven for convex subsets of $\mathbb{R}^m$ endowed with the Euclidean metric. Always in the convex case in [7] it was studied the dependence of the number of the OGC's by the topology of the domain $\overline{\Omega}$.

By an *m-dimensional annulus* we mean a topological space homeomorphic to topological product $\mathbb{S}^{m-1} \times [0,1]$, that can be thought as the subset of $\mathbb{R}^m$:

$$\mathbb{A} = \{p \in \mathbb{R}^m : 1 \le |p| \le 2\};$$

Our central result is the following:

THEOREM 1.7. *Let Ω be an open subset of M with smooth boundary $\partial\Omega$, such that $\overline{\Omega}$ is strongly concave and homeomorphic to an annulus. Suppose there are no crossing WOGC in $\overline{\Omega}$. Then there are at least two geometrically distinct* (3) *crossing orthogonal geodesic chords in $\overline{\Omega}$.*

2. The class of admissible deformations and the used functional

Multiplicity of OGC's in the case of compact manifolds having convex boundary is typically proven by applying a curve-shortening argument. From an abstract viewpoint, the curve-shortening process can be seen as the construction of a flow in the space of paths, along whose trajectories the length or energy functional is decreasing.

Shortening a curve having image in a closed convex subset $\overline{\Omega}$ of a Riemannian manifold produces another curve in $\overline{\Omega}$; in this sense, we think of the shortening flow as being «inward pushing» in the convex case. As opposite to the convex case, the shortening flow in the concave case will be «outwards pushing», and this fact requires the one should consider only those portions of a curve that remain inside $\overline{\Omega}$ when it is stretched outwards.

«Variational criticality» relatively to the energy functional can be defined in terms of «outwards pushing» infinitesimal deformations of the path space as follows.

For $x \in H^1([0,1],\mathbb{R}^m)$, let $\mathcal{V}^+(x)$ denote the following cone in $T_xH^1([0,1],\mathbb{R}^m)$:

$$(2.1)\quad \mathcal{V}^+(x) = \{V \in T_xH^1([0,1],\mathbb{R}^m) : g(V(s), \nabla\phi(x(s))) \ge 0 \ \text{ for } x(s) \in \phi^{-1}(0)\};$$

vector fields in $\mathcal{V}^+(x)$ are interpreted as infinitesimal variations of x by curves stretching outwards from the set $\overline{\Omega}$. Similarly, for $x \in H^1([0,1],\mathbb{R}^m)$ we define the cone:

$$(2.2)\quad \mathcal{V}^-(x) = \{V \in T_xH^1([0,1],\mathbb{R}^m) : g(V(s), \nabla\phi(x(s))) \le 0 \ \text{ for } x(s) \in \phi^{-1}(0)\}.$$

DEFINITION 2.1. Let $x \in H^1([0,1],\mathbb{R}^m)$ and $[a,b] \subset [0,1]$; we will say that $x|_{[a,b]}$ is a *variationally critical portion* of x if $x|_{[a,b]}$ is not constant and if

$$(2.3)\qquad \int_a^b g\left(\dot{x}, \frac{\mathrm{D}}{\mathrm{d}t}V\right)\mathrm{d}t \ge 0, \quad \forall\, V \in \mathcal{V}^+(x).$$

The integral in (2.3) gives precisely the first variation of the geodesic action functional $\frac{1}{2}\int_a^b g(\dot{x},\dot{x})\,\mathrm{d}s$ in (M,g) along $x|_{[a,b]}$. Hence, variationally critical portions are interpreted

(3) By *geometrically distinct* curves we mean curves having distinct images as subsets of $\overline{\Omega}$.

as those curves $x|_{[a,b]}$ whose geodesic energy is *not decreased* after infinitesimal variations by curves stretching outwards from the set $\overline{\Omega}$. The motivation for using outwards pushing infinitesimal variations is due to the concavity of $\overline{\Omega}$. In the convex case it is customary to use infinitesimal variations of x in $\mathcal{V}^-(x)$ (keeping the endpoints of x on $\partial\Omega$); the corresponding notion of criticality in the convex case gives orthogonal geodesic chords in $\overline{\Omega}$.

Unfortunately in the concave case the situation is much more complicated. Indeed the class of variationally critical portions contains *properly* the set of portions consisting of crossing OGC's; such curves can be defined as «geometrically critical» paths. In order to construct the shortening flow, an accurate analysis of all possible variationally critical paths is required, and the concavity condition guarantees that such paths are *well behaved* as pointed out by the following results.

Let δ_0 be as in Remark 1.4.

LEMMA 2.2. *Let $x \in H^1([0,1],\mathbb{R}^m)$ be fixed, and let $[a,b] \subset [0,1]$ be such that $x|_{[a,b]}$ is a (non-constant) variationally critical portion of x, with $x(a), x(b) \in \partial\Omega$ and $x([a,b]) \subset \overline{\Omega}$. Then*:

(1) *$x^{-1}(\partial\Omega) \cap [a,b]$ consists of a finite number of closed intervals and isolated points*;

(2) *x is constant on each connected component of $x^{-1}(\partial\Omega) \cap [a,b]$*;

(3) *$x|_{[a,b]}$ is piecewise C^2, and the discontinuities of $\dot{x}$ may occur only at points in $\partial\Omega$*;

(4) each C^2 *portion of $x|_{[a,b]}$ is a geodesic in $\overline{\Omega}$*;

(5) $\inf\{\phi(x(s)) \,:\, s \in [a,b]\} < -\delta_0$.

PROPOSITION 2.3. *Assume that there are not crossing WOGC's in the set $\overline{\Omega}$. Let $x \in H^1([0,1],\mathbb{R}^m)$ and $[a,b] \subset [0,1]$ such that $x|_{[a,b]}$ is a variationally critical portion of x, with $x(a), x(b)$ in different connected components of $\partial\Omega$ and $x([a,b]) \subset \overline{\Omega}$. Suppose that the restriction of x to $[a,b]$ is of class C^1. Then, $x|_{[a,b]}$ is a crossing orthogonal geodesic chord in $\overline{\Omega}$ with $x(]a,b[) \subset \Omega$.*

Variationally critical portions $x|_{[a,b]}$ of class C^1 will be called *regular variationally critical portions*; those critical portions that do not belong to this class will be called *irregular*. Irregular variationally critical portions of curves $x \in H^1([0,1],\mathbb{R}^m)$ are further divided into two subclasses, described below.

PROPOSITION 2.4. *Let $x \in H^1([0,1],\mathbb{R}^m)$ and let $[a,b] \subset [0,1]$ be such that $x|_{[a,b]}$ is an irregular variationally critical portion of x with $x(a), x(b)$ in different connected components of $\partial\Omega$ and $x([a,b]) \subset \overline{\Omega}$. Then, there exists a subinterval $[\alpha,\beta] \subset [a,b]$ such that $x|_{[a,\alpha]}$ and $x|_{[\beta,b]}$ are constant (in $\partial\Omega$), $\dot{x}(\alpha^+) \in T_{x(\alpha)}(\partial\Omega)^\perp$, $\dot{x}(\beta^-) \in T_{x(\beta)}(\partial\Omega)^\perp$, and one of the two mutually exclusive situations occurs*:

(1) *there exists a finite number of intervals $[t_1,t_2] \subset]\alpha,\beta[$ such that $x([t_1,t_2]) \subset \partial\Omega$ and that are maximal with respect to this property*; *moreover, x is constant on each such interval $[t_1,t_2]$, and $\dot{x}(t_1^-) \neq \dot{x}(t_2^+)$*;

(2) *$x|_{[\alpha,\beta]}$ is a crossing OGC in $\overline{\Omega}$.*

Thanks to the above results it is possible to move far away from critical portions which are not crossing OGC's, using «inward pushing» infinitesimal deformations (cf. (2.2)) for irregular variational critical portions. Indeed irregular variational critical portions are not of class C^1, while critical portions with respect to (2.2) are of class C^1. Then the only portions of curves which are critical with respect either (2.1) or (2.2) are crossing OGC's, and this is the key to avoid irregular variational critical portions by our flows.

The shortening flow is constructed locally around portions which are not crossing OGC, and then patched together obtaining homotopies included in the following class that we are going to describe.

Let $\psi : \mathbb{A} \to \overline{\Omega}$ a homeomorphism and take

$$\mathfrak{C}' = \{\psi \circ y \, : \, y(s) = (1+s)q, \; q \in \mathbb{R}^m, \; \|q\|_E = 1\},$$

where $\|\cdot\|_E$ denotes Euclidean norm in $\mathbb{R}^m$. Using piecewise geodesics the curves in $\mathfrak{C}'$ can be regularized to curves in H^1, obtaining a set $\mathfrak{C}$ homeomorphic to $\mathbb{S}^{m-1}$ and consisting of curves $\gamma \in H^1([0,1],\overline{\Omega})$, such that $\gamma(0) \in D_1$ and $\gamma(1) \in D_2$, where D_1 and D_2 are the two connected components of $\partial\Omega$.

Let δ_0 be as in Remark 1.4 and denote by $\mathcal{D}_i$ the connected components of $\phi^{-1}([0,\delta_0])$ that contains D_i. Set

$$\mathfrak{M} = \{x \in H^1([0,1], \phi^{-1}(]-\infty,\delta_0[)) \, : \, x(0) \in \mathcal{D}_1, \; x(1) \in \mathcal{D}_2\}.$$

Note that $\mathfrak{C} \subset \mathfrak{M}$. For all $x \in \mathfrak{M}$, let $\mathcal{I}_x$ denote the following collection of closed subintervals of $[0,1]$:

$$\mathcal{I}_x = \{[a,b] \subset [0,1] \, : \, x([a,b]) \subset \overline{\Omega}, \text{ and } [a,b] \text{ is maximal w.r.t. this property}\}.$$

If $x \in \mathfrak{M}$ and $[a,b] \in \mathcal{I}_x$, we say that $[a,b]$ is a *crossing interval* of Ω for x if $x(a) \in \mathcal{D}_1$ and $x(b) \in \mathcal{D}_2$, and we set

$$\mathcal{J}_x = \{[a,b] \in \mathcal{I}_x \, : \, [a,b] \text{ is a crossing interval of } \Omega \text{ for } x\}.$$

A crucial notion is the one of *h–genuine interval*:

Definition 2.5. Let $\mathcal{D} \subset \mathfrak{C}$, let $h : [0,1] \times \mathcal{D} \to \mathfrak{M}$ be a continuous map, $\gamma \in \mathcal{D}$, and $\tau \in [0,1]$. We say that an interval $[a_\tau, b_\tau] \in \mathcal{J}_{h(\tau,\gamma)}$ *is h-genuine* if for all $0 \le \tau' \le \tau$ there exists $[a_{\tau'}, b_{\tau'}] \in \mathcal{J}_{h(\tau',\gamma)}$ such that $[a_\tau, b_\tau] \subset [a_{\tau'}, b_{\tau'}]$ (see fig. 1).

Let $M_0 = \max\left\{ \int_0^1 g(\dot{x},\dot{x})\,ds \, : \, x \in \mathfrak{C} \right\}$ where g is the Riemannian structure of $\mathfrak{M}$. The class $\mathcal{H}$ of admissible homotopies consists of the maps $h : [0,1] \times \mathcal{D} \to \mathfrak{M}$ (with $\mathcal{D}$ closed subset of $\mathfrak{M}$) such that:

- $h(0,\cdot)$ is the inclusion $\mathcal{D} \hookrightarrow \mathfrak{M}$;
- h sends outside $\overline{\Omega}$ the pieces of curves which are outside $\overline{\Omega}$;
- any h–genuine interval $[a_\tau, b_\tau]$ related to $x = h(\tau,\gamma)$ is such that $\int_{a_\tau}^{b_\tau} \frac{1}{2} g(\dot{x},\dot{x})\,ds < M_0$;

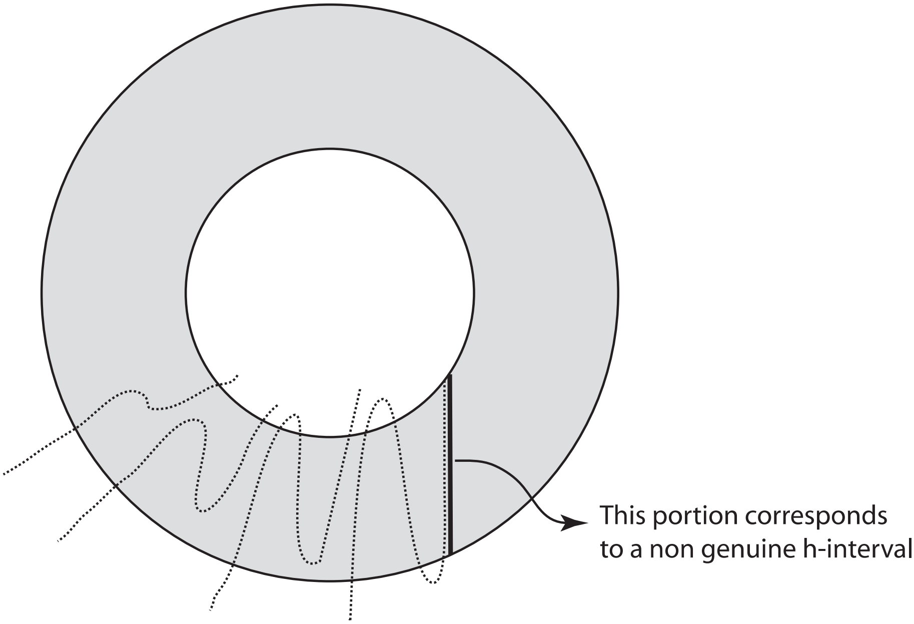

Fig. 1. – The dotted curves in the figure represent part of the «evolution» of the curve γ by the homotopy h. A portion of $h(\tau,\gamma)$ corresponds to a crossing interval of $h(\tau,\gamma)$ which is *not* h-genuine. It arises from a non crossing interval of γ.

- h moves far from variationally critical portions of curves (w.r.t. outward pushing infinitesimal deformations) that are not crossing OGC.

Along these homotopies, we define the functional

$$\mathcal{F}(h,\tau,\gamma) = \sup\left\{ \frac{(b_\tau - a_\tau)}{2} \int_{a_\tau}^{b_\tau} g(\dot{x},\dot{x})\,\mathrm{d}s : x = h(\tau,\gamma),\ [a_\tau,b_\tau] \in \mathcal{J}_x \text{ is } h\text{-genuine} \right\},$$

where $h \in \mathcal{H}$, $\tau \in [0,1]$, and $\gamma \in \mathcal{D}$.

Remark 2.6. Note that by the definition of $\mathcal{H}$, $\mathfrak{C}$ and $\mathfrak{M}$, for any h,τ,γ there exists at least one h-genuine interval $[a_\tau,b_\tau] \in \mathcal{J}_{h(\tau,\gamma)}$. Note also that the number of such intervals is finite. Moreover it should be pointed out that any $\frac{(b-a)}{2}\int_a^b g(\dot{y},\dot{y})\,\mathrm{d}t$ coincides with $\frac{1}{2}\int_0^1 g(\dot{y}_{a,b},\dot{y}_{a,b})\,\mathrm{d}t$, where $y_{a,b}$ is the affine reparameterization of y on the interval $[0,1]$.

If $\rho_0 = \inf\{dist(x_1,x_2) : x_1 \in D_1, x_2 \in D_2\}$ (where *dist* denotes the distance induced by the Riemann structure g), it is

$$\mathcal{F}(h,\tau,\gamma) \geq \frac{1}{2}\rho_0^2, \qquad \forall h \in \mathcal{H},\ \forall \gamma \in \mathcal{D},\ \forall \tau \in [0,1]. \tag{2.4}$$

Moreover, if $[a_\tau, b_\tau]$ is any interval where the supremum in the definition of $\mathcal{F}$ is attained, since it satisfied $\int_{a_\tau}^{b_\tau} \frac{1}{2} g(\dot{x}, \dot{x})\,\mathrm{d}s < M_0$, setting $x = h(\tau, \gamma)$, it is

$$\mathcal{F}(h, \tau, \gamma) = \frac{(b_\tau - a_\tau)}{2} \int_{a_\tau}^{b_\tau} g(\dot{x}, \dot{x})\,\mathrm{d}t < (b_\tau - a_\tau)\frac{M_0}{2} \le \frac{M_0}{2}. \tag{2.5}$$

3. The deformations results

We say that c is a *geometrically critical value* if there exists a crossing OGC γ such that $\frac{1}{2}\int_0^1 g(\dot{\gamma}, \dot{\gamma})\,\mathrm{d}s = c$. Otherwise c is called *geometrically regular value*. It is important to note that, thanks to transversality condition satisfied by OGC's at their endpoints, it is easy to prove that different geometrically critical values corresponds to geometrically distinct crossing OGC's. Therefore, the central issue becomes a problem of proving multiplicity of geometrically critical values. Towards this goal, the following deformation results can be proved.

Proposition 3.1 (First Deformation Lemma). *Let c be a geometrically regular value of $\mathcal{F}$. There exists $\varepsilon = \varepsilon(c) > 0$ such that for all compact subset $\mathcal{D} \subset \mathfrak{C}$ and for all $h \in \mathcal{H}$ with*

$$\mathcal{F}(h, 1, \mathcal{D}) \subset\,]-\infty, c + \varepsilon],$$

there exists a continuous map $\eta \in C^0\big([0,1] \times h(1, \mathcal{D}), \mathfrak{M}\big)$ such that $\eta \star h \in \mathcal{H}$ (here $\star$ is the homotopies concatenation operator) and

$$\mathcal{F}(\eta \star h, 1, \mathcal{D}) \subset\,]-\infty, c - \varepsilon].$$

Let now $r_* > 0$ be fixed and let us consider the set:

$$\mathcal{U}_{r_*} = \Big\{x \in \mathfrak{M} : \text{exists } [a,b] \in \mathcal{J}_x \{\text{and an OGC } \gamma : [a,b] \to \overline{\Omega} \text{ from } D_1 \text{ to } D_2 \text{ such that } \max_{s \in [a,b]} dist\big(x(s), \gamma([0,1])\big) < r_*\Big\}. \tag{3.1}$$

Note that $\mathcal{U}_{r_*}$ is open in $\mathfrak{M}$. Assume now that the number of OGC's from D_1 to D_2 is finite; then, r_* can be chosen small enough so that the following two facts hold true:

(3.2) for all $x \in \mathcal{U}_{r_*}$, {for all $[a,b] \in \mathcal{J}_x$ there exists at most one γ satisfying (3.1).

and

(3.3) the set $\Big\{A \in D_1 : \|A - \gamma(0)\| < 2r_* \text{ for some } \gamma \text{ OGC from } D_1 \text{ to } D_2\Big\}$ is contractible in D_1,

(here $\|\cdot\|$ denotes the norm induced by g).

PROPOSITION 3.2 (Second Deformation Lemma). *Let c be a geometrically critical value. Then, there exists* $\varepsilon_* = \varepsilon_*(c) > 0$ *such that, for all compact subset* $\mathcal{D} \subset \mathfrak{C}$ *and for all* $h \in \mathcal{H}$ *with*

$$\mathcal{F}(h, 1, \mathcal{D}) \subset]-\infty, c + \varepsilon_*],$$

there exists a continuous map $\eta \in C^0\big([0,1] \times h(1, \mathcal{D}), \mathfrak{M}\big)$ *such that* $\eta \star h \in \mathcal{H}$ *and*:

$$\mathcal{F}\big(\eta \star h, 1, \mathcal{D} \setminus h(1,\cdot)^{-1}(\mathcal{U}_{r_*})\big) \subset]-\infty, c - \varepsilon_*].$$

PROPOSITION 3.3. *Assume that there are only a finite number of OGC's from* D_1 *to* D_2*, and assume that* r_* *is a small positive number for which* (3.2) *and* (3.3) *are satisfied. Then, for all* $h \in \mathcal{H}$ *there exists an open set* $\mathcal{A}$ *of* $\mathfrak{C}$, $\mathcal{A} \supset h(1,\cdot)^{-1}\big(\mathcal{U}_{r_*}\big)$*, that is contractible in* $\mathfrak{C}$.

To prove Theorem 1.7, we set

$$\Gamma_i = \{\mathcal{D} \subset \mathfrak{C} : \mathcal{D} \text{ is closed and } \operatorname{cat}_{\mathfrak{C}}(\mathcal{D}) \geq i\}$$

where cat is the classical Ljusternik-Schnirelman category, and

$$c_i = \inf_{\mathcal{D} \in \Gamma_i,\, h \in \mathcal{H}} \left(\sup_{x \in \mathcal{D}} \mathcal{F}(h, 1, x) \right).$$

Note that $\Gamma_1, \Gamma_2 \neq \emptyset$ since $\operatorname{cat}_{\mathfrak{C}}(\mathfrak{C}) = \operatorname{cat}_{\mathbb{S}^{m-1}} \mathbb{S}^{m-1} = 2$.

By (2.4) $c_i \geq \dfrac{\rho^2}{2}$, while by (2.5), $c_i \leq \dfrac{M_0}{2}$ $(i = 1, 2)$. Finally, using classical arguments (cf. *e.g.* [12, 17]), thanks to Propositions 3.1, 3.2 and 3.3, we see that any c_i is a geometrically critical value and if the number of crossing OGC's is finite then $c_1 < c_2$, obtaining the existence of two geometrically distinct crossing OGC's.

4. BRAKE AND HOMOCLINIC ORBITS OF HAMILTONIAN SYSTEMS

The result of Theorem 1.7 can be applied to prove a multiplicity result for brake orbits and homoclinic orbits, as follows.

Let $p = (p_i)$, $q = (q^i)$ be coordinates on $\mathbb{R}^{2m}$, and let us consider a *natural* Hamiltonian function $H \in C^2\big(\mathbb{R}^{2m}, \mathbb{R}\big)$, *i.e.*, a function of the form

$$H(p, q) = \frac{1}{2} \sum_{i,j=1}^{m} a^{ij}(q) p_i p_j + V(q), \tag{4.1}$$

where $V \in C^2\big(\mathbb{R}^m, \mathbb{R}\big)$ and $A(q) = \big(a^{ij}(q)\big)$ is a positive definite quadratic form on $\mathbb{R}^m$:

$$\sum_{i,j=1}^{m} a^{ij}(q) p_i p_j \geq \upsilon(q) |q|^2$$

for some continuous function $\upsilon : \mathbb{R}^m \to \mathbb{R}^+$ and for all $(p, q) \in \mathbb{R}^{2m}$.

The corresponding Hamiltonian system is:

$$\begin{cases} \dot{p} = -\dfrac{\partial H}{\partial q} \\ \dot{q} = \dfrac{\partial H}{\partial p}, \end{cases} \tag{4.2}$$

where the dot denotes differentiation with respect to time.

For all $q \in \mathbb{R}^m$, denote by $\mathcal{L}(q) : \mathbb{R}^m \to \mathbb{R}^m$ the linear isomorphism whose matrix with respect to the canonical basis is $\big(a_{ij}(q)\big)$, the inverse of $\big(a^{ij}(q)\big)$; it is easily seen that, if (p, q) is a solution of class C^1 of (4.2), then q is actually a map of class C^2 and

$$p = \mathcal{L}(q)\dot{q}. \tag{4.3}$$

With a slight abuse of language, we will say that a C^2-map $q : I \to \mathbb{R}^m$ is a solution of (4.2) if (p, q) is a solution of (4.2) where p is given by (4.3). Since the system (4.2) is autonomous, *i.e.*, time independent, then the function H is constant along each solution, and it represents the total energy of the solution of the dynamical system. There exists a large amount of literature concerning the study of periodic solutions of autonomous Hamiltonian systems having energy H prescribed (see for instance [9] and the references therein).

We consider here a special kind of periodic solutions of (4.2), called *brake orbits*. A brake orbit for the system (4.2) is a non constant periodic solution $\mathbb{R} \ni t \mapsto \big(p(t), q(t)\big) \in \mathbb{R}^{2m}$ of class C^2 with the property that $p(0) = p(T) = 0$ for some $T > 0$. Since H is even in the variable p, a brake orbit (p, q) is $2T$-periodic, with p odd and q even about $t = 0$ and about $t = T$. Clearly, if E is the energy of a brake orbit (p, q), then $V\big(q(0)\big) = V\big(q(T)\big) = E$.

The link between brake orbits and orthogonal geodesic chords is obtained in [5, Theorem 5.9] working on the closure of the open set

$$\Omega_E = V^{-1}\big(]-\infty, E[\big) = \big\{x \in \mathbb{R}^m : V(x) < E\big\} \tag{4.4}$$

endowed with the *Jacobi metric*

$$g_E(x) = \big(E - V(x)\big)g_0(\dot{x}, \dot{x}) \tag{4.5}$$

where $g_0(\dot{x}, \dot{x}) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{m} a_{ij}(x)\, \mathrm{d}x^i\, \mathrm{d}x^j$.

More precisely for all $x \in \Omega_E$ consider

$$d_E(y) := \inf\left\{ \int_0^1 \big((E - V(x))g_0\big(\dot{x}, \dot{x}\big)\big)^{1/2} \mathrm{d}t : x \in H^1\big([0,1], \overline{\Omega}_E\big),\ x(0) = y,\ x(1) \in \partial\Omega \right\}$$

which is attained in a unique curve if y is sufficiently close to $\Omega^{-1}(E)$. In [5] the following result is proved:

THEOREM 4.1. *Let $d_E : \Omega \to [0, +\infty[$ be the map defined above, and assume that $\overline{\Omega}_E$ is compact. There exists a positive number δ_* such that, setting*:

$$\Omega_* = \big\{x \in \Omega_E\ :\ d_E(x) > \delta_*\big\},$$

the following statements hold:

(1) $\partial\Omega_*$ *is of class* C^2;

(2) $\overline{\Omega}_*$ *is homeomorphic to* $\overline{\Omega}_E$;

(3) $\overline{\Omega}_*$ *is strongly concave relatively to the Jacobi metric* g_E;

(4) *if* $x : [0,1] \to \overline{\Omega}_*$ *is an orthogonal geodesic chord in* $\overline{\Omega}_*$ *relatively to the Jacobi metric* g_E*, then there exists* $[\alpha,\beta] \supset [0,1]$ *and a unique extension* $\widehat{x} : [\alpha,\beta] \to \overline{\Omega}$ *of* x *with* $\widehat{x} \in H^1\big([\alpha,\beta],\overline{\Omega}\big)$ *satisfying*:

- $\widehat{x}(s) \in d_E^{-1}\big(]-\delta_*,0[\big)$ *for all* $s \in\,]\alpha,0[\bigcup]1,\beta[$;
- $V\big(\widehat{x}(\alpha)\big) = V\big(\widehat{x}(\beta)\big) = E$;
- $\widehat{x}$ *can be reparameterized to a brake orbit (Maupertuis Principle).*

Using Theorem 4.1 and Theorem 1.7, we get immediately the following:

THEOREM 4.2. *Let* $H \in C^2\big(\mathbb{R}^{2m},\mathbb{R}\big)$ *be a natural Hamiltonian function as in* (4.1), $E \in \mathbb{R}$ *and*

$$\Omega_E = V^{-1}\big(]-\infty,E[\big).$$

Assume that $\mathrm{d}V(x) \neq 0$ *for all* $x \in \partial\Omega_E$ *and that* $\overline{\Omega_E}$ *is homeomorphic to an* m*-dimensional annulus. Then, the Hamiltonian system* (4.2) *has at least two geometrically distinct brake orbits having energy* E *and endpoints in different connected components of* $V^{-1}(E)$.

Let us now go back to our Riemannian manifold (M,g) and assume that we are given a map $V \in C^2\big(M,\mathbb{R}\big)$; the corresponding second order Hamiltonian system is the equation:

$$\frac{D}{\mathrm{d}t}\dot{q} + \nabla V(q) = 0. \tag{4.6}$$

When $M = \mathbb{R}^m$ and g is the Riemannian metric

$$g = \frac{1}{2}\sum_{i,j=1}^{m} a_{ij}(x)\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{4.7}$$

where the coefficients a_{ij} are as above, then equation (4.6) is equivalent to (4.2), in the sense that x is a solution of (4.6) if and only if the pair $q = x$ and $p = \mathcal{L}(x)\dot{x}$ is a solution of (4.2).

Let $x_0 \in M$ be a critical point of V, *i.e.*, such that $\nabla V(x_0) = 0$. A *homoclinic orbit* for the system (4.6) emanating from x_0 is a solution $q \in C^2\big(\mathbb{R},M\big)$ of (4.6) such that:

$$\lim_{t\to-\infty} q(t) = \lim_{t\to+\infty} q(t) = x_0, \tag{4.8}$$

$$\lim_{t\to-\infty} \dot{q}(t) = \lim_{t\to+\infty} \dot{q}(t) = 0. \tag{4.9}$$

Multiplicity results for homoclinic orbits of (4.6) by variational methods were established mostly in the non autonomous periodic case (see for instance [3, 8, 15, 16]). The autonomous case is somewhat harder to treat, and, to the knowledge of the authors, the only results available in the literature are:

- [1], where it is considered a potential in $\mathbb{R}^m$ satisfying a pinching property, a superquadraticity condition and a suitable assumption on the second derivative;
- [13], where the results of [1] are improved taking off the superquadraticity condition and the assumption on the second derivative of the potential;

- [14], where the author considers the case of a potential in $\mathbb{R}^m$ periodic in all variables;
- [18], where it is considered a small perturbation of a radial potential.

Let (M, g) be a Riemannian manifold, $V \in C^2(M, \mathbb{R})$ and let $x_0 \in M$ be a nondegenerate maximum point of V, with $V(x_0) = E$. Assume that:

(*a*) $V^{-1}(]-\infty, E[) \bigcup \{x_0\}$ is homeomorphic to an open ball of $\mathbb{R}^m$;
(*b*) $\mathrm{d}V(x) \neq 0$ for all $x \in V^{-1}(E) \setminus \{x_0\}$.

Using again the Jacobi metric (considering also the distance from the nondegenerate maximum point with respect to the Jacobi metric) in [5, Theorem 5.19] it is shown that the study of the multiplicity of homoclinics under assumptions (*a*) and (*b*) can be reduced to the study of the multiplicity of crossing OGC's on a strongly concave domain homeomorphic to an annulus. Therefore by Theorem 1.7, we have the following Theorem, which gives a generalization of the results in [1] and in [18]:

THEOREM 4.3. *Let* (M, g) *be a Riemannian manifold,* $V \in C^2(M, \mathbb{R})$ *and let* $x_0 \in M$ *be a nondegenerate maximum point of* V*, with* $V(x_0) = E$*. Assume that (a) and (b) above are satisfied. Then, there are at least two geometrically distinct homoclinic orbits for the system* (4.6) *emanating from* x_0 *and reaching* $V^{-1}(E) \setminus \{x_0\}$.

Finally, we consider the case of orthogonal geodesic chords in an annulus under a further central symmetry assumption. We will say that a subset A of a Riemannian manifold (M, g) is *centrally symmetric around the point* $x_0 \in M$ if there exists an isometry $I : M \to M$ with $I^2 = \mathrm{Id}$ whose unique fixed point is x_0 and such that $I(A) = A$. Observe that if $\gamma : [a, b] \to M$ is a geodesic, then $I \circ \gamma$ is also a geodesic in (M, g), and if γ is orthogonal at the endpoints at some hypersurface $\Sigma \subset M$, then $I \circ \gamma$ is orthogonal at the endpoints to the hypersurface $I(\Sigma)$.

THEOREM 4.4. *Under the hypotheses of Theorem* 1.7, *assume further that* $\overline{\Omega}$ *is centrally symmetric around a point* $y_0 \in M \setminus \overline{\Omega}$*. Suppose there are not crossing WOGC in* $\overline{\Omega}$*. Then there are at least* m *crossing orthogonal geodesic chords* $\gamma_1, \ldots, \gamma_m$ *in* $\overline{\Omega}$ *such that each* γ_i *is geometrically distinct from* γ_j *and* $I \circ \gamma_j$*,* $\forall i \neq j$.

To prove the above result we use the category of the projective space $\mathbb{P}^{m-1}$. Accordingly, using again the results of [5] about Jacobi metric, we get similar multiplicity results for brake orbits and for homoclinic orbits under a central symmetry assumption.

THEOREM 4.5. *Under the assumptions of Theorem* 4.2*, assume further that the functions* a_{ij} *and* V *are centrally symmetric around some point* $y_0 \notin V^{-1}(]-\infty, E])$*. Then there are at least* m *brake orbits* $\gamma_1, \ldots, \gamma_m$ *of energy* E *for the Hamiltonian system* (4.2)*, having extreme points in different connected components of* $V^{-1}(E)$ *and such that each* γ_i *is geometrically distinct from* γ_j *and* $I \circ \gamma_j$*,* $\forall i \neq j$.

THEOREM 4.6. *Under the assumptions of Theorem* 4.3*, if* (M, g) *is centrally symmetric relatively to* x_0 *and the map* V *is also centrally symmetric around* x_0*, then there are at least* m

homoclinic orbits $\gamma_1, \dots \gamma_m$ *for the system* (4.6) *emanating from* x_0 *and such that each* γ_i *is geometrically distinct from* γ_j *and* $I \circ \gamma_j$, $\forall i \neq j$.

Note that in Theorem 4.5 the hypersuface $H^{-1}(E)$ is not the boundary of a convex set. So we can not compare this result with the results of [10] where $H^{-1}(E)$ is symmetric with respect to the origin and bounding a convex compact set.

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