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Identification of a localized source in an interstellar cloud: an inverse problem

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Fisica matematica. — *Identification of a localized source in an interstellar cloud: an inverse problem*. Nota (*) di Meri Lisi e Silvia Totaro, presentata dal Socio M. Primicerio.

Abstract. — We study an inverse problem for photon transport in an interstellar cloud. In particular, we evaluate the position x_0 of a localized source $q(x) = q_0\delta(x - x_0)$, inside a nebula (for example, a star). We assume that the photon transport phenomenon is one-dimensional. Since a nebula moves slowly in time, the number of photons U inside the cloud changes slowly in time. For this reason, we consider the so-called quasi-static approximation u to the exact solution U. By using semigroup theory, we prove existence and uniqueness of u. Because of some monotonic properties of the operator which describes u as a function of q, the «position» of the source can be evaluated.

Key words: Inverse problems; Quasi-static approximation; Semigroup theory.

Riassunto. — *Identificazione di una sorgente in una nube interstellare: un problema inverso*. Viene studiato un problema inverso per il trasporto di fotoni in una nube interstellare. In particolare, è valutata la posizione x_0 di una sorgente $q(x) = q_0\delta(x - x_0)$, localizzata all'interno della nube (per esempio, una stella). Nel fare ciò, si suppone che il fenomeno del trasporto di fotoni sia monodimensionale. Inoltre, poiché la nube si muove lentamente, anche il numero di fotoni U al suo interno cambia lentamente nel tempo. Per questo motivo, è possibile considerare la cosiddetta approssimazione quasi-statica u della soluzione esatta U e, per mezzo della teoria dei semigruppi, dimostrare l'esistenza e l'unicità di u. Grazie ad alcune proprietà di monotonia dell'operatore che descrive u come una funzione di q, è infine possibile valutare la «posizione» della sorgente.

1. Introduction

Inverse problems are very relevant in order to compute many physical quantities which cannot be obtained by direct measurements (see, for instance, [4, 7, 10] and the references quoted therein).

In fact, the general nature of an inverse problem is to deduce a cause from an effect. Consider a physical system, depending on a collection of parameters, in which one can speak of inputs to the system and outputs from the system. If all of the parameters were known perfectly, then, for a given input, we could predict the output. It may happen, however, that some of the parameters characterizing the system are not known, being inaccessible to direct measurement. If it is important to know what these parameters are, in order to understand the system as completely as possible, we might try to infer them by observing the outputs from the system corresponding to special inputs. Thus, we seek the cause (the system parameters) given the effect (the output of the system for a given input).

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Problems of this type arise in several applications areas, such as geophysics, optics, quantum mechanics, astronomy, medical imaging and materials testing. In particular, we are going to study an inverse problem in astrophysics (see, for instance, [1, 5] and the references quoted therein), in order to compute some physical quantities which cannot be obtained by direct measurements.

Interstellar clouds are mainly composed by molecular gases (90% hydrogen), by more complex molecules and by grains of «dust» of silicon and carbonates. The dimension of an interstellar cloud (nebula) is of the order of ten light years, *i.e.*, between 10^{-1} and 10 parsec (one parsec is about $3 \cdot 10^{13}$ kilometres). We note that the diameter of the solar system is of the order of 10^{-4} parsec, [3].

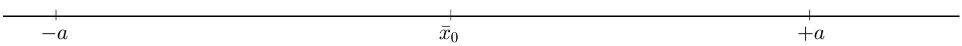
The numerical density of the particles inside an interstellar cloud ranges from 10^3 to 10^6 particles per cubic centimetre, (earth atmosphere density, at sea level, is approximately 10^{19} particles per cubic centimetre, whereas in the intergalactic vacuum one can find 10^0 particles/cm^3).

There are many kind of interstellar clouds: dark nebulae, nebulae which reflect light and nebulae which emit light.

Let us consider this last type of nebulae. They are «able» to emit light, because one or more photon sources (for example, some stars) are present inside the clouds. An example of this kind of nebulae is given by the Orion Nebula.

By means of some astronomical instruments, it is possible to make a «direct measurement» of the light emitted from a cloud of this kind, *i.e.*, the intensity of the unknown sources («far field measures»), but it is not possible to localize the position of the sources. In this contest, inverse problems are very relevant.

In this paper, we consider a rod model of a homogeneous cloud, containing a localized source. In particular, we shall study a one-dimensional particle transport problem in a homogeneous rod of thickness 2a, with a localized photon source $q(x, t)$ inside and surrounded by vacuum, *i.e.*, we assume that the photon number density U depends on the space variable x and on time t, [8].



We note that the rod model is a very naive picture of an interstellar cloud. This means that our results just give some general hints on what happens in the real world.

However, it is a first approach to this kind of problems. The study of a slab model is in progress, while the three-dimensional case model is one of our future aims.

According to our model, photons are monocromatic and can be captured or scattered by the particles of the nebula.

The paper is organized as follows. In Section 2, we shall study the mathematical problem in the Banach spaces setting, by considering the so called quasi-static equation. In Section 3, the inverse problem is investigated and we shall prove the monotonicity of a suitable operator.

2. The mathematical problem

There are many aspects concerning the physics of interstellar clouds which lead to mathematical models of kinetic type. The most classical among these is certainly radiative transport, [8], both because the radiation field (especially ultraviolet) within the clouds induces reactions which affect their chemical evolution, and because earth-based observations of the spectral lines yield informations on the chemical composition of the cloud itself.

Since the boundaries of an interstellar cloud move slowly in time [9], an appreciable movement takes many years to occur and the critical conditions, which bring a cloud to collapse, are reached after about a million of years.

This means that the boundaries of the cloud can be considered slowly varying functions of time. Thus, it is reasonable to assume that the number of photons inside the cloud changes slowly in time. Of course, photon sources are assumed to be slowly varying in time.

Thus, we can consider the so called quasi-static approximation u to the exact solution U of the transport equation, [5, 6]:

$$\begin{cases} 0 = -\dfrac{\partial u_+(x,t)}{\partial x} - \sigma u_+(x,t) + \sigma_s u_-(x,t) + q_+(x,t), \\[2ex] 0 = \dfrac{\partial u_-(x,t)}{\partial x} - \sigma u_-(x,t) + \sigma_s u_+(x,t) + q_-(x,t), \end{cases} \tag{1}$$

where $u(x,t) = \begin{pmatrix} u_+(x,t) \\ u_-(x,t) \end{pmatrix} \in X$ and $q(x,t) = \begin{pmatrix} q_+(x,t) \\ q_-(x,t) \end{pmatrix}$ is the source.

In particular, $u_+(x,t)$ represents the density of those particles which move from left to right in the rod, whereas $u_-(x,t)$ represents the density of those particles which move from right to left in the rod. Moreover σ and σ_s are, respectively, the total and the scattering cross sections of the particles in the nebula ($\sigma > \sigma_s > 0$).

Note that u still depends on t because of the presence of $q(x,t)$.

Associated with the previous system, we consider the non-reentry boundary conditions:

$$\begin{cases} u_+(-a,t) = 0, \\ u_-(+a,t) = 0, \end{cases} \quad t > 0.$$

In order to study the problem, let us consider the Banach space $X = L^1(-a,a) \times L^1(-a,a)$, with the norm

$$\|f\|_X = \int_{-a}^{a} |f_+(x)|dx + \int_{-a}^{a} |f_-(x)|dx, \quad \forall f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in X.$$

Following the classical ideas of [2] and multiplying the first equation of system (1) by $\exp(\sigma x)$ and the second equation by $\exp(-\sigma x)$, we obtain:

$$\begin{cases} 0 = -\dfrac{\partial}{\partial x}[u_+(x,t)\exp(\sigma x)] + \sigma_s \exp(\sigma x) u_-(x,t) + \exp(\sigma x) q_+(x,t), \\ 0 = \dfrac{\partial}{\partial x}[u_-(x,t)\exp(-\sigma x)] + \sigma_s \exp(-\sigma x) u_+(x,t) + \exp(-\sigma x) q_-(x,t). \end{cases}$$

Integrating with respect to x, the first equation between $-a$ and x, and the second equation between x and a, we obtain:

$$\begin{cases} u_+(x,t) = \int\limits_{-a}^{x} \exp[-\sigma(x-x')][\sigma_s u_-(x',t) + q_+(x',t)]dx', \\ u_-(x,t) = \int\limits_{x}^{a} \exp[-\sigma(x'-x)][\sigma_s u_+(x',t) + q_-(x',t)]dx', \end{cases} \tag{2}$$

because of the boundary conditions.

If we define

$$Q(x) = \begin{pmatrix} Q_+(x) \\ Q_-(x) \end{pmatrix} = \begin{pmatrix} \int\limits_{-a}^{x} \exp[-\sigma(x-x')]q_+(x')dx' \\ \int\limits_{x}^{a} \exp[-\sigma(x'-x)]q_-(x')dx' \end{pmatrix}, \tag{3}$$

system (2) reads as follows:

$$u(t) = (Ku)(t) + Q(t), \tag{4}$$

where $K : X \to X$ is the following operator:

$$(Kf)(x) = \sigma_s \begin{pmatrix} \int\limits_{-a}^{x} \exp[-\sigma(x-x')]f_-(x')dx' \\ \int\limits_{x}^{a} \exp[-\sigma(x'-x)]f_+(x')dx' \end{pmatrix}, \quad \forall f \in X.$$

It is easy to prove that

$$\|K\| \leq \frac{\sigma_s}{\sigma} < 1.$$

Hence

$$u(t) = (I-K)^{-1}Q(t), \tag{5}$$

where t may be considered as a «parameter».

Equation (5) represents the quasi-static transport equation with a generic source term. In our particular case, we want to consider a localized source, for example a source situated in the point x_0.

Thus, it can be modelled as follows:

$$q_+(x) = q_0\delta(x-x_0) = q_-(x), \tag{6}$$

where $q_0 = q_0(t)$ is a positive quantity and $\delta(x-x_0)$ is the Dirac delta functional at $x = x_0$.

Hence, because of the properties of the Dirac delta functional at $x = x_0$, we have from definition of Q (see (3)):

$$Q_+(x) = \begin{cases} q_0 \exp[-\sigma(x - x_0)], & x > x_0, \\ q_0, & x = x_0, \\ 0, & x < x_0, \end{cases} \tag{7}$$

and

$$Q_-(x) = \begin{cases} 0, & x > x_0, \\ q_0, & x = x_0, \\ q_0 \exp[-\sigma(x_0 - x)], & x < x_0. \end{cases} \tag{8}$$

REMARK 2.1. We can notice that, in the case of a localized source (6), the integration made in (2) with respect to x is essential, because of the presence of the Dirac delta functional δ. In fact, whereas δ does not belong to X, because it is a functional, its integral belongs to X. By making this procedure, we are able to study the quasi-static equation in the Banach spaces settings. Otherwise, we should embed the space X, for example, into a space of distributions and extend the problem to this space (see [6], for details). □

3. THE INVERSE PROBLEM

Now, since we want to identify the position of the localized source q (see (6)), that is we want to find the point x_0 belonging to $(-a, +a)$, we have to prove the monotonicity of the operator

$$T = (I - K)^{-1} Q \tag{9}$$

with respect to x_0.

We may assume that the radiation produced by the cloud is measured at $x = a$.

Since, from (5), we obtain:

$$u = (I - K)^{-1} Q = Q + KQ + K^2 Q + \ldots, \tag{10}$$

we study the operators Q, KQ, $K^2 Q$ and so on.

First of all, from relations (7) and (8), we have that:

$$Q(a) = \begin{pmatrix} Q_+(a) \\ Q_-(a) \end{pmatrix} = \begin{pmatrix} q_0 \exp[-\sigma(a - x_0)] \\ 0 \end{pmatrix}$$

and

$$\frac{dQ_+(a)}{dx_0} = \sigma q_0 \exp[-\sigma(a - x_0)] > 0.$$

Now, we have to compute KQ, *i.e.*,

$$(KQ)(x) = \begin{pmatrix} (KQ)(x)_+ \\ (KQ)(x)_- \end{pmatrix} = \sigma_s \begin{pmatrix} \int_{-a}^{x} \exp[-\sigma(x - x')] Q_-(x') dx' \\ \int_{x}^{a} \exp[-\sigma(x' - x)] Q_+(x') dx' \end{pmatrix}.$$

After many computations, for $x > x_0$, we obtain:

$$(KQ)(x) = \frac{q_0}{2}\frac{\sigma_s}{\sigma}\begin{pmatrix} \{\exp[-\sigma(x - x_0)] - \exp[-\sigma(2a + x + x_0)]\} \\ \\ \{\exp[-\sigma(x - x_0)] - \exp[-\sigma(2a - x - x_0)]\} \end{pmatrix},$$

whereas for $x < x_0$, we have:

$$(KQ)(x) = \frac{q_0}{2}\frac{\sigma_s}{\sigma}\begin{pmatrix} \{\exp[-\sigma(x_0 - x)] - \exp[-\sigma(2a + x + x_0)]\} \\ \\ \{\exp[-\sigma(x_0 - x)] - \exp[-\sigma(2a - x - x_0)]\} \end{pmatrix}.$$

In particular, for $x = a$, we obtain from the above considerations:

$$(KQ)(a) = \begin{pmatrix} \frac{q_0}{2}\frac{\sigma_s}{\sigma}\{\exp[-\sigma(a - x_0)] - \exp[-\sigma(3a + x_0)]\} \\ \\ 0 \end{pmatrix}.$$

Hence

$$\frac{d(KQ)(a)_+}{dx_0} > 0.$$

We can do the same thing for K^2Q and we obtain:

$$\left(K^2Q\right)(a)_- = 0,$$

$$\frac{d(K^2Q)(a)_+}{dx_0} > 0.$$

Thus, we have:

$$\frac{d}{dx_0}\left[(KQ)(a)_+ + \left(K^2Q\right)(a)_+\right] > 0.$$

Going on with this procedure, because of the form of $(K^nQ)(a)$, for any n, we are able to prove the monotonicity of T with respect to x_0.

Hence, the operator T^{-1} is defined and we have:

$$x_0 = T^{-1}(u).$$

Remark 3.1. We have to notice that computations made to obtain Q, KQ, K^2Q, and so on, become more and more complicated. Another way to prove the monotonicity of the operator T is to derivate directly equations of system (2) with respect to x_0.

By means of this tecnique, we are also able to obtain an estimate of the ratio $\frac{\sigma_s}{\sigma}$. We have that $\left[\frac{\sigma_s}{\sigma}\right]^2 < \frac{1}{2}$.

It is also interesting to note that the depencence of the quantity K^nQ on x_0 vanishes as n increases. This means that its influence on the sum (10), which defines the operator T, decreases as n increases.

From a physical point of view, we can interpret this fact, noting that, in some sense, the presence of the localized source on the photon transport affects mostly the so called «first-flight» photons, *i.e.*, those ones wich are directly emitted from the source itself or which have not undergone any scattering events. □

By a direct measurement of the light $\bar{u}$, it is possible to identify the position $\bar{x}_0$ belonging to the real interval $(-a, a)$.

A simple procedure based, for example, on a sort of bisection method may permit to identify the position x_0 of the localized source q.

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