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MARTIN MOSKOWITZ

AN EXTENSION OF MAHLER'S THEOREM TO SIMPLY CONNECTED NILPOTENT GROUPS

To Guido Zappa on the occasion of his 90^{th} birthday

ABSTRACT. — This *Note* gives an extension of Mahler's theorem on lattices in $\mathbb{R}^n$ to simply connected nilpotent groups with a $\mathbb{Q}$-structure. From this one gets an application to groups of Heisenberg type and a generalization of Hermite's inequality.

KEY WORDS: Log lattice; Subgroup of finite index; Fundamental domain; Measure preserving automorphism; Equivariant map.

In 1946 Mahler [8] proved the following result which bears a striking resemblance to the classical theorem of Ascoli. It concerns lattices Γ in $\mathbb{R}^n$. Here by a lattice is meant a discrete subgroup with compact quotient; in other words a subgroup of $\mathbb{R}^n$ with n linearly independent generators. Given a lattice and linearly independent generators $v_1, \dots v_n$ the parallelepiped formed by the generators is called a fundamental domain. If $\mathcal{L}(\mathbb{R}^n) = \mathcal{L}$ denotes the space of all lattices (with a natural topology), Mahler's theorem is the following: A subset $\mathcal{S}$ of $\mathcal{L}$ has compact closure if and only if

1. The volumes of all the fundamental domains as Γ varies over $\mathcal{S}$ is bounded.
2. There exists some neighborhood U of 0 in $\mathbb{R}^n$ so that $\Gamma \cap U = (0)$ for all $\Gamma \in \mathcal{S}$.

The first condition is analogous to uniform boundedness while the second (often described as $\mathcal{S}$ being uniformly discrete) is analogous to equicontinuity in Ascoli's theorem.

We denote by $\mathcal{L}_0$, the subspace of $\mathcal{L}$ consisting of lattices whose fundamental domain has volume 1. Once we explain what the topology on $\mathcal{L}$ is we will see that $\mathcal{L}_0$ is closed in $\mathcal{L}$. A direct corollary of Mahler's theorem is then:

A subset $\mathcal{S}$ of $\mathcal{L}_0$ has compact closure if and only if there exists some neighborhood of 0 *in* $\mathbb{R}^n$ *so that* $\Gamma \cap U = (0)$ *for all* $\Gamma \in \mathcal{S}$.

$\mathcal{L}$ can be topologized by observing that GL $(n, \mathbb{R})$ acts transitively on it. Therefore we can choose any lattice as a base point for this orbit. Choosing the standard lattice, $\mathbb{Z}^n$, we see that the isotropy group is GL $(n, \mathbb{Z})$. Thus $\mathcal{L}$ can be identified in a natural way with the homogeneous space GL $(n, \mathbb{R})/$ GL $(n, \mathbb{Z})$. The (natural) topology is then that of this coset space. It does not depend on a choice of generators in the lattice. A short proof of Mahler's theorem using Siegel domains in GL $(n, \mathbb{R})$ is given in Borel [3, p. 16].

It is our goal to establish a non-abelian generalization of Mahler's theorem. To this end one can consider other simply connected solvable groups G rather than just $\mathbb{R}^n$. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. One can topologize Aut (G), the group of all bicontinuous automorphisms (actually smooth automorphisms) of G as in ([7, pp. 40 and 97-99)]. The topology on Aut (G) makes it into a locally compact group and actually a Lie group by Cartan's theorem since it has a faithful continuous representation into the linear group Aut $(\mathfrak{g})$. Moreover, the action $\mathrm{Aut}(G) \times G \to G$ is jointly continuous (actually smooth). Let $\mathrm{M}(G) = \{\alpha \in \mathrm{Aut}(G) : \Delta(\alpha) = 1\}$ stand for the group of measure preserving automorphisms of G, where $\Delta(\alpha)$ is the common ratio $\frac{\mu(\alpha(F))}{\mu(F)}$, for any measurable set $F \subset G$ of positive, finite left Haar measure, μ. Δ is a smooth homomorphism $\mathrm{Aut}(G) \to \mathbb{R}_+^\times$. Hence $\mathrm{M}(G)$ is a closed normal subgroup of Aut (G).

In general G might not have any lattices at all so now we shall have to assume G contains a lattice Γ. Here Γ is again a discrete subgroup and G/Γ is compact. Just as before Aut (G) acts on the set $\mathcal{L}(G)$ of all lattices. However, now it may not act transitively. For example, (see [1]) in the orbit space of the Heisenberg group N_n of dimension $2n+1$, any lattice Γ is isomorphic under an automorphism of N_n to a lattice of the form $\Gamma_{k,\vec{r}}$ for some $k \in \mathbb{Z}^+$ and $\vec{r} = (r_1, , r_n) \in (\mathbb{Z}^+)^n$ satisfying $r_1 = 1$, and $r_i | r_{i+1}$ for all i. $\Gamma_{k,\vec{r}}$ is defined by the conditions $x_i \in r_i\mathbb{Z}$, $y_i \in \mathbb{Z}$, and $z \in \frac{1}{k}\mathbb{Z}$. For this reason we shall have to assume that $\mathcal{S}$ is contained in a single orbit and, if successful in that case, any conclusion would evidently also be valid if $\mathcal{S}$ were contained in a finite union of orbits.

For any connected Lie group $\mathcal{L}$ can be topologized by the Chabauty topology (see [5]). If G is a simply connected and solvable linear group with real eigenvalues and Γ is a lattice, then the orbits, Aut $(G)(\Gamma)$, are closed in this topology and are homeomorphic with $\mathrm{Aut}(G)/\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$ (see [12, Theorem (3.1)]). Since we are going to be dealing with subsets of a single orbit we can consider the topology to be that of $\mathrm{Aut}(G)/\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$. Moreover, this stability subgroup is discrete whenever G has no non-trivial automorphisms of bounded displacement (see [13, Proposition (1.1)]). In particular, this holds for any simply connected solvable group of type E (see [14, Corollary (1.3)]). An important special case of all this is when G is simply connected and nilpotent, since such a group has a faithful unipotent representation (see [7, Theorem (3.1), p. 219]).

We now suppose that G is a simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ has a $\mathbb{Q}$-structure. Then by Malcev's theorem [9], G has a lattice Γ. Hence G is unimodular so we can just speak of Haar measure. Since the center $Z(G)$ of G is non-trivial, using induction on $\dim G$ and the formula $\int_G dg = \int_{G/Z(G)} d\dot{g} \int_{Z(G)} dz$ shows that Haar measure is Lebesgue measure in appropriate global coordinates. A well known formula for the derivative of the exponential map of a Lie group is:

$$d(\exp)_X = \sum_{n=0}^{\infty} \frac{(-1)^n \operatorname{ad}_X^n}{(n+1)!}.$$

Since G is nilpotent, each ad_X is simultaneously nil-triangular and this analytic function is actually a polynomial, hence we see $d(\exp)_X$ is unipotent. Hence $\det(d(\exp)_X) \equiv 1$. Because G is simply connected and nilpotent, $\exp : \mathfrak{g} \to G$ is a global diffeomorphism and since $\det(d(\exp)_x) \equiv 1$, the change of variable formula for multiple integrals tells us that Haar measure μ on G corresponds to Lebesgue measure υ on $\mathfrak{g}$. If $a \in \mathrm{Aut}(G)$ and $a^\bullet \in \mathrm{Aut}(\mathfrak{g})$ is its derivative, then for a subset F of finite positive measure in G we have $\mu(a(F)) = \Delta(a)\mu(F)$ and $\upsilon(a^\bullet(\log(F))) = |\det(a^\bullet)|\upsilon(\log(F))$. Hence after identifications, $\Delta(a) = |\det(a^\bullet)|$.

Our extension of Mahler's theorem is the following.

THEOREM 1. *Let G be a simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ has a $\mathbb{Q}$-structure. A subset $\mathcal{S} \subseteq \mathcal{L}$ which is contained in a finite number of* Aut (G)*-orbits has compact closure if and only if*

1. For all Γ in $\mathcal{S}$ the measures of the fundamental domains are bounded.
2. There exists a fixed neighborhood U of 1 in G such that for all Γ in $\mathcal{S}$, $U \cap \Gamma = (1)$.

Before proving the theorem we state the following result due to Barbano in [2]. There the interest was measure preserving automorphisms so that was the way it was formulated. However, as the reader can check, the argument works perfectly well for arbitrary automorphisms. A lattice Γ in a simply connected nilpotent group is called a *log-lattice* if $\log(\Gamma)$ is a lattice in $\mathfrak{g}$.

LEMMA 2. *Let G be a simply connected nilpotent group with $\mathbb{Q}$-structure and Γ be a non-log lattice in G. Then there exists a log-lattice $\Gamma_* \subseteq \Gamma$ in G with* $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$ *a subgroup of finite index in* $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma_*)$.

This lemma is Theorem (5.1) in [2]. We now turn to the proof of our theorem.

PROOF. In either direction we may assume $\mathcal{S}$ is contained in a single orbit, $\{a(\Gamma) : a \in \mathrm{Aut}(G)\}$. We first assume the two conditions and show $\mathcal{S}$ is relatively compact. Since G is simply connected $\mathrm{Aut}(G)^\bullet = \mathrm{Aut}(\mathfrak{g})$, so we can identify these. Moreover, since G is nilpotent exp is a global diffeomorphism. Hence taking derivatives gives an equivariant equivalence between the action of Aut (G) on G and the associated linear action on $\mathfrak{g}$,

$$\mathrm{Aut}(G) \times G \to G$$

and

$$\mathrm{Aut}(\mathfrak{g}) \times \mathfrak{g} \to \mathfrak{g}.$$

Also Aut $(\mathfrak{g})$ is a closed subgroup of GL $(\mathfrak{g})$. Let π denote the orbit map and consider $\mathcal{S}' \subseteq \mathrm{Aut}(\mathfrak{g}) \subseteq \mathrm{GL}(\mathfrak{g})$, where $\mathcal{S}' = \pi^{-1}(\mathcal{S})$. The first condition tells us Δ is bounded on $\mathcal{S}'$ and hence so is $|\det(a^\bullet)|$.

By [11] Γ has a subgroup Γ^* of finite index which is a log lattice, *i.e.*, $\log(\Gamma^*)$ is a lattice in $\mathfrak{g}$. Hence each $a(\Gamma)$ has a subgroup $a^\bullet(\Gamma^*)$ of finite index which is also a log lattice. Therefore there is a neighborhood U^* of 0 in $\mathfrak{g}$ which meets no other points of $\log(a^\bullet(\Gamma^*))$. By Mahler's theorem, in this linear action $\mathcal{S}'$ must act boundedly. Hence this must also be so in the equivariantly equivalent action of Aut (G) on G. Thus $\mathcal{S}$ has compact closure.

We now show relative compactness of $\mathcal{S}$ implies both conditions. For the first condition, since $\Delta(a)\mu(F) = \mu(a \cdot F)$, for $F \subseteq G$. If we take F to be a fundamental domain for Γ we see the first condition is equivalent to $\Delta(a)$ being bounded on $\mathcal{S}$. But Δ is continuous and $\mathcal{S}^-$ is compact so this is true.

For the second condition we apply Lemma 2 and choose a log-lattice Γ_* in G such that $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$ a subgroup of finite index in $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma_*)$. Therefore the projection

$$p : \mathrm{Aut}(G)/\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma) \to \mathrm{Aut}(G)/\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma_*)$$

is a covering map. The following lemma shows it is proper and applies whenever H is a (not necessarily connected) Lie group, D and D_* discrete subgroups with the index $[D_* : D]$ finite, $X = H/D$, $X_* = H/D_*$ and $p : X \to X_*$ is the natural map.

Lemma 3. *Let $p : X \to X_*$ be a covering space of the (not necessarily connected) manifolds X and X_* with the property that each fiber is finite. Then p is a proper map. In particular, S is relatively compact in X if and only if $p(S)$ is relatively compact in X_*.*

Proof. Let C be a compact set in X_* and $\{V_a\}$ be a covering of $p^{-1}(C)$ by open sets in X for which the open sets $p(V_a)$ are evenly covered. Thus $p^{-1}(p(V_a))$ is a finite union of open sets homeomorphic with $p(V_a)$. Since by compactness C is a finite union of the $p(V_a)$ it follows that $p^{-1}(C)$ is itself a finite union of the $\{V_a\}$.

Continuing the proof of the theorem we see that by Lemma 2 $\mathcal{S}$ is relatively compact in $\mathrm{Aut}(G)/\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$ if and only if $p(\mathcal{S}) = \mathcal{S}_*$ is relatively compact in $\mathrm{Aut}(G)/\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma_*)$. In the proof of the converse statement given just above we can take for Γ^* the lattice Γ_* which is also log-lattice subgroup of Γ. There we were reduced to the case of Mahler's theorem itself. Hence, by that argument there is a neighborhood U_* of 0 in $\mathfrak{g}$ which meets no other points of $\log(a^\bullet(\Gamma_*))$.

Lemma 4. *Let Γ be a finitely generated group and Γ_* be a subgroup of finite index. Then there exists a fixed integer k so that $\gamma^k \in \Gamma_*$ for every $\gamma \in \Gamma$.*

Proof. Since, as is well known, Γ is a finitely generated group and Γ_* has finite index in it there is a normal subgroup Γ_{**} of Γ contained in Γ_* which also has finite index, say k in Γ. In the finite group Γ/Γ_{**} Lagrange's theorem tells us that $\gamma^k \in \Gamma_{**}$ for every $\gamma \in \Gamma$.

Taking logs we see that the k-multiples of every element of $\log(\Gamma)$ lies in $\log(\Gamma_*)$. It follows that if we take a smaller neighborhood W_* of 0 in $\mathfrak{g}$ with $kW_* \subseteq U_*$, then $\exp(W_*)$ is a neighborhood of 1 in G which meets $a(\Gamma)$ only at the identity as a varies.

The following inequality is a variant of one in Margulis [10, p. 169], whose proof we leave to the reader.

Lemma 5. *Let T be a linear transformation on a finite dimensional real or complex vector space V of dimension n and $\|\cdot\|$ be the Hilbert-Schmidt norm on* End(V). *Then* $|\det T| \leq \|T\|^n$.

This inequality, together with our method of passing to log-lattices in the Lie algebra, gives the following sufficient condition.

COROLLARY 6. *Let G be a simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ has a $\mathbb{Q}$-structure. A subset $\mathcal{S} \subseteq \mathcal{L}$ contained in a finite number of* Aut (G)*-orbits has compact closure if*

1. *For all $\alpha(\Gamma)$ in $\mathcal{S}$ the set $\|\alpha^\bullet\|$ is bounded.*
2. *There exists a fixed neighborhood U of 1 in G such that for all Γ in $\mathcal{S}$, $U \cap \Gamma = (1)$.*

An immediate corollary of the theorem is

COROLLARY 7. *Let G be a simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ has a $\mathbb{Q}$-structure. A subset $\mathcal{S} \subseteq \mathcal{L}$ which is contained in a finite number of* $\mathrm{M}(G)$ *orbits has compact closure if and only if there exists a fixed neighborhood U of 1 in G such that for all lattices Γ in $\mathcal{S}$ we have $U \cap \Gamma = (1)$.*

We now apply this corollary to the Heisenberg groups N_n, their quaternionic analogues H_n, and the groups C_n built on the Cayley numbers. In all these cases, and certain others, the stabilizer of a lattice in G is again a *lattice* in $\mathrm{M}(G)$ and sometimes is even a *uniform lattice*. Also, the Lie algebras of groups of Heisenberg type all have $\mathbb{Q}$-structure by [6]. But here we have an interesting dichotomy between $\mathbb{R}^n$ and N_n on the one hand, and H_n and C_n on the other. In the former the stabilizer is a non-uniform lattice in $\mathrm{M}(G)$ while in the latter it is uniform (see [13] and [2]). Here since $\mathrm{M}(G)$ is the $\mathbb{R}$-points of an algebraic $\mathbb{Q}$-group and $\mathrm{Stab}_{\mathrm{M}(G)_0}(\Gamma)$ is an arithmetic subgroup (see [13] and [2]), $\mathrm{M}(G)_0$ has finite index in $\mathrm{M}(G)$ by [15]. Therefore we need not make any fundamental distinction between these.

Our next corollary follows from this dichotomy.

COROLLARY 8. *Let $G = H_n$ or C_n and Γ be a lattice in G. Then there exists a neighborhood U of* 1 *in G such that $\alpha(\Gamma) \cap U = (1)$ for all $\alpha \in \mathrm{M}(G)$. On the other hand, if $G = \mathbb{R}^n$, for $n \geq 2$ or N_n, then there can be no such neighborhood.*

One can check the non-compactnesss of the latter two homogeneous spaces by applying [4] and observing that $\mathrm{SL}(n, \mathbb{R})$ has non-trivial unipotent elements while in N_n there are unipotent elements in $\mathrm{M}(G)_0$ not in the unipotent radical. The compactness of the former two follows from [2] together with [13].

In this connection we remark that the fact that $\mathrm{M}(G)/\mathrm{Stab}_{\mathrm{M}(G)}(\Gamma)$ has finite volume is rather special. In general it only applies to irreducible groups of Heisenberg type (see [2] and [13]). These simply connected groups are, for example, all 2 step nilpotent. If G is the full real unitriangular group of order $n \geq 4$, that is, anything other than the Heisenberg group, this fails (see [13, p. 13]).

Finally, by passing to a log-lattice $\Gamma_* \subseteq \Gamma$ and using equivariance, Hermite's inequality (see [3]) can be generalized. We denote by $\|\cdot\|$ the Euclidean norm on $\mathfrak{g}$ transfered to G.

Corollary 9. *Let G be a simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ has a $\mathbb{Q}$-structure and Γ be a lattice in G. Then there is a positive constant $c(G)$ such that $min_{\gamma\in\Gamma,\gamma\neq 1}\|\alpha(\gamma)\| \leq c(G)\Delta(\alpha)^{\frac{1}{\dim G}}$, for every $\alpha \in \mathrm{Aut}\,(G)$.*

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