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Polynomials and the art of counting: some instances of the Cyclic Sieving Phenomenon

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Sommario: *Uno dei molti aspetti affascinanti della combinatoria enumerativa è quello di trovare contatti fra varie aree della matematica, e di rivelare relazioni insospettate. Il Cyclic Sieving Phenomenon (CSP), introdotto da Reiner, Stanton e White nel 2004, è un recente capitolo di ricerca in questo campo. Lo scopo di questo articolo è quello di offrire un'introduzione breve ed elementare al CSP attraverso alcuni esempi. In sintesi, il CSP consiste in questo: si parte da un insieme su cui c'è una azione di un gruppo ciclico con* n *elementi, e si associa in modo naturale a questo insieme un polinomio. Il punto fondamentale è che questo polinomio ha una proprietà "magica": se si valuta nelle radici ennesime dell'unità, si ottengono dei numeri naturali che contano i punti fissi dell'azione del gruppo ciclico. Nei nostri esempi compariranno molti oggetti combinatori interessanti, legati ai numeri di Catalan, di Kirkman-Cayley e di Narayana, come le triangolazioni e le dissezioni di poligoni regolari, le partizioni non incrociate, le parentesizzazioni di liste e i grafi ad albero con radice.*

Abstract: *One of the many fascinating aspects of Enumerative Combinatorics is that it often finds contacts between different areas of mathematics, and reveals unsuspected relations. The Cyclic Sieving Phenomenon (CSP), introduced by Reiner, Stanton and White in 2004, is a recent chapter in this field. The purpose of this paper is to give a short and elementary introduction to the CSP by some examples. The gist of the story is that one starts from a set equipped with a cyclic group action, and finds a natural way to associate a polynomial to this set, with the following 'magic' property: if one evaluates this polynomial at some suitable roots of 1, one gets nonnegative integers that enumerate the fixed points of the group action. In our examples many interesting combinatorial objects will come into play, like triangulations and dissections of regular polygons, noncrossing partitions, parenthesizations of lists and rooted ordered plane trees.*

1. – Premise: a polynomial that knows a lot about the rotations of triangulated hexagons...

We start by dealing with a very specific example. There are 14 different ways to *triangulate* an hexagon, i.e., to draw three nonintersecting diagonals:

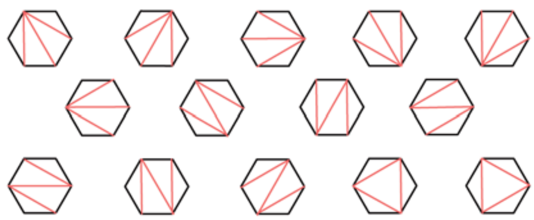

Fig. 1. – All the triangulations of a regular hexagon.

This is just an instance of a famous formula on the triangulations of a regular $(n+2)$-agon: the number

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of its different triangulations is equal to the n-th Catalan number

$$(1)\quad \mathrm{Cat}(n) := \frac{1}{n+1}\binom{2n}{n} = \frac{2n(2n-1)\cdots(n+2)}{n(n-1)\cdots 1}$$

We will give a closer look to the Catalan numbers in Section 5.

Let us now consider the group of rotations of the hexagon, that is generated by the rotation R through the angle $\frac{2\pi}{6}$ (say clockwise) around the barycenter.

This group contains 6 elements, including the identity, and when it acts on the set of triangulated hexagons represented in Figure 1 it splits the set into parts, called *orbits*. There is one orbit made by 6 elements, two orbits made by 3 elements and one orbit made by two elements (see Figure 2).

We will now show a natural way to associate a polynomial to the set of triangulations of a regular polygon. This will provide an algebraic counterpart of the geometric picture of rotations and their orbits described above. As a first step we recall what is the q-analog of a positive integer.

DEFINITION 1.1. – Given a positive integer n, we say that its *q-analog* is the following polynomial:

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}$$

We notice that, according to the definition above, $[1]_q = 1$. A relevant property of the polynomial $[n]_q$ is that if we evaluate it at $q = 1$ we obtain n. In other words, if one considers the variable q as a complex number, $[n]_q$ can be interpreted as a *continous deformation* of n, that gives back the value n when q is equal to 1.

Let us focus again on the equation (1). Let us substitute the integers $2n, 2n-1, ..., n+2$ that appear in the numerator on the right by their q-analogs, and let us do the same with the integers $n, n-1, ..., 2, 1$ that appear in the denominator. We obtain a quotient of polynomials that can be called, in a natural way, a q-analog of the Catalan number $\mathrm{Cat}(n)$:

$$\mathrm{Cat}_q(n) = \frac{[2n]_q[2n-1]_q \cdots [n+2]_q}{[n]_q[n-1]_q \cdots [1]_q}$$

One immediately observes that if we evaluate this rational function at $q = 1$ we get the number $\mathrm{Cat}(n)$. More surprisingly, one can prove that $\mathrm{Cat}_q(n)$ is in fact a polynomial with nonnegative integer coefficients.

Let us check this in our case when $n = 4$, i.e. in the case associated to triangulated hexagons:

$$\mathrm{Cat}_q(4) = \frac{[8]_q[7]_q[6]_q}{[4]_q[3]_q[2]_q[1]_q} = \frac{(q^8-1)(q^7-1)(q^6-1)}{(q^4-1)(q^3-1)(q^2-1)} =$$

$$= q^{12} + q^{10} + q^9 + 2q^8 + q^7 + 2q^6 + q^5 + 2q^4 + q^3 + q^2 + 1$$

Now we are getting to the crucial point of our example. Let us consider the complex number $\zeta_6 = \frac{1}{2} + \frac{\sqrt{3}}{2}\imath$, that is a primitive 6-th root of 1. One can observe that:

- If we evaluate $\mathrm{Cat}_q(4)$ at $q = 1 = \zeta_6^0$ we obtain 14.
- If we evaluate $\mathrm{Cat}_q(4)$ at $q = \zeta_6$ we obtain 0.
- If we evaluate $\mathrm{Cat}_q(4)$ at $q = \zeta_6^2$ we obtain 2.
- If we evaluate $\mathrm{Cat}_q(4)$ at $q = \zeta_6^3$ we obtain 6.
- If we evaluate $\mathrm{Cat}_q(4)$ at $q = \zeta_6^4$ we obtain 2.
- If we evaluate $\mathrm{Cat}_q(4)$ at $q = \zeta_6^5$ we obtain 0.

Why have we obtained nonnegative integers? Are we *counting* something?

The (beautiful) answer is that the polynomial $\mathrm{Cat}_q(4)$ is strictly related to the action of the group

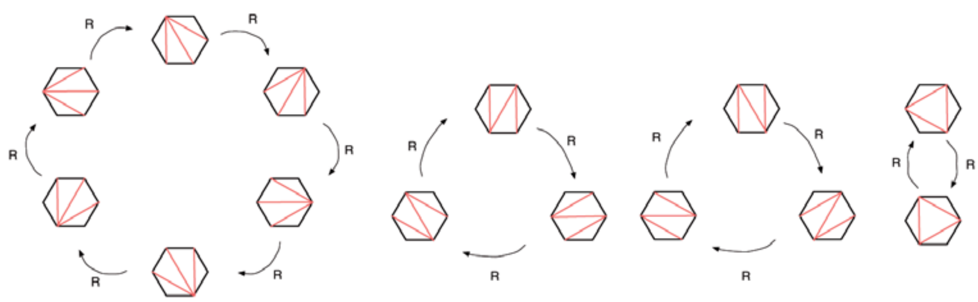


Fig. 2. – The orbits of the set of triangulated hexagons under the action of rotations.

of rotations genarated by R on the set of triangulated hexagons. In fact, looking at Figure 2 one observes that in the set of triangulated hexagons:

- there are 14 elements fixed by the rotation R^0 (i.e., by the identity);
- there are 0 elements that are fixed by the rotation $R^1 = R$;
- there are 2 elements that are fixed by the rotation $R^2 = R \circ R$;
- there are 6 elements that are fixed by the rotation $R^3 = R \circ R \circ R$;
- there are 2 elements that are fixed by the rotation R^4;
- there are 0 elements that are fixed by the rotation R^5.

By comparing the two lists above one concludes that for every $i = 0, \ldots, 5$ the number of triangulated hexagons fixed by the rotation R^i is equal to the evaluation of the polynomial $\mathrm{Cat}_4(q)$ in $q = \zeta_6^i$.

We are in the presence of an instance of a remarkable combinatorial phenomenon, the CSP (*Cyclic Sieving Phenomenon*). It was pointed out by Reiner, Stanton and White in 2004 in their paper [10] (generalizing the Stembridge's $q = -1$ phenomenon, see [19], [20]), and since then many instances have been discovered, like precious jewels, involving many interesting objects in combinatorics, algebra and geometry.

This paper is devoted to a first introduction to the CSP: in particular we will focus on a family of examples that include the triangulations of regular polygons.

We will start by recalling, in the next section, some basic definitions and some properties of q-analogs. But before doing this, we would like to intrigue the reader with another remark on the polynomial $\mathrm{Cat}_4(q)$. Let us consider the complex number $\zeta_8 = \cos\frac{2\pi}{8} + \imath \sin\frac{2\pi}{8}$, that is a primitive 8-th root of 1. If we evaluate $\mathrm{Cat}_4(q)$ at $q = \zeta_8$, $q = \zeta_8^3$, $q = \zeta_8^5$, $q = \zeta_8^7$ we obtain 0, while evaluating $\mathrm{Cat}_4(q)$ at $q = \zeta_8^2$ we obtain 2, at $q = \zeta_8^4$ we obtain 4 and at $q = \zeta_8^6$ we obtain 2. Again, the results of the evaluations are nonnegative integers. Does this mean that $\mathrm{Cat}_4(q)$ can also tell us another story, involving the group of rotations of a regular octagon?

2. – A formal definition of cyclic sieving

As illustrated by the example in the preceding section, the basic ingredients of the cyclic sieving phenomenon are a set equipped with an action of a cyclic group and a polynomial which naturally returns the cardinalities of the sets of fixed points when evaluated at appropriate roots of unit.

Let X be a finite set, let $C = \langle c \rangle$ a finite cyclic group acting on X, and let $\zeta \in \mathbb{C}$ be a root of unit having the same multiplicative order as c. Let $X(q) \in \mathbb{Z}[q]$ be a polynomial with nonnegative integer coefficients and let us denote by X^{c^d}, for every integer d, the set of elements of X fixed by c^d.

Definition 2.1. – The triple $(X, C, X(q))$ is said to exhibit the *cyclic sieving phenomenon* (or *CSP*) if, for every integer d,

$$\#X^{c^d} = X(\zeta^d).$$

Of course it's always possible, given a cyclic group C acting on a set X, to find a polynomial $X(q)$ that satisfies the cyclic sieving condition. Furthermore, via representation theoretic tools, one can show that such a polynomial can be taken with nonnegative integer coefficients. The point of this definition is that the polynomial $X(q)$ should be somehow naturally associated to the set X, and there are several ways to do that.

3. – Basic introduction to *q*-analogs

The most widespread way to associate a polynomial to a set of combinatorial objects involves q-analogs.

In a first, still vague, attempt of definition, a q-analog of an expression is any generalization involving a new parameter q that returns the original expression in the limit as $q \to 1$. Notice that, if $(X, C, X(q))$ exhibits the cyclic sieving, then $X(1) = \#X$, so $X(1)$ is an enumerator for X. It follows that $X(q)$ is a q-analog of an enumerator, usually called q-enumerator, for the set X.

Often q-analogs are based on the fact that

$$\lim_{q \to 1} \frac{q^\alpha - 1}{q - 1} = \alpha$$

as we have seen in the example of Section 1, where we introduced the definition of the q-analog $[n]_q$ of a positive integer n: $[n]_q = \frac{q^n-1}{q-1} = 1 + q + \cdots + q^{n-1}$. This can be extended to natural numbers by putting $[0]_q = 0$.

Despite this choice for $[n]_q$ seems to be arbitrary, it arises spontaneously in several contexts. It also allows for a natural definition of q-factorial and q-binomial. Namely, given $n, k \in \mathbb{N}$, with $k \leq n$, we define their q-factorial and q-binomial as

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

where the symbol $[0]_q!$ means 1 (this reminds the convention $0! = 1$).

As we are going to show, it is not hard to prove that the q-binomial is a polynomial with nonnegative integer coefficients. This might not be true if we choose some different polynomial with nonnegative integer coefficients as q-analog of a natural number.

Let us start by noticing that many identities among binomial coefficients have their q-analog. For example, the well known binomial identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

generalizes to the following:

PROPOSITION 3.1. – *For every $n, k \in \mathbb{N}$ with $0 < k < n$ we have the equalities*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$$

PROOF. – We prove the first equality (the second equality can be easily obtained from the first one using the relation $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$).

We observe that by definition

$$\begin{bmatrix} n \\ k \end{bmatrix}_q - \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q = \left(\frac{q^n - 1}{q^k - 1} - 1 \right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

Since

$$\left(\frac{q^n - 1}{q^k - 1} - 1 \right) = q^k \left(\frac{q^{n-k} - 1}{q^k - 1} \right)$$

we can write

$$\begin{bmatrix} n \\ k \end{bmatrix}_q - \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q = q^k \left(\frac{q^{n-k} - 1}{q^k - 1} \right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \qquad \square$$

The fact that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial with nonnegative integers now easily follows by induction, using the proposition above.

To keep getting some practice with q-binomials (for a deeper introduction see for instance Section 1.7 of [17]), let us show the q-analog of the well-known binomial theorem, that can be stated as: for every $n \in \mathbb{N}$ we have

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

The q-analog of this identity is the following, which of course specializes to the standard one for $q = 1$:

THEOREM 3.2 (The q-binomial theorem). – *For all $n \geq 1$ we have*

$$\prod_{j=1}^{n} (1 + xq^j) = \sum_{k=0}^{n} q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k.$$

PROOF. – There are many nice ways to obtain and prove this formula (see for instance Section 1.8 of [17]). Here we simply show how, once we have the formula, we can prove it by a straightforward induction on n. For $n = 0$, the thesis is obvious. Suppose that the thesis holds for n. We have

$$\begin{aligned}
\prod_{j=1}^{n+1} (1 + xq^j) &= (1 + xq^{n+1}) \cdot \prod_{j=1}^{n} (1 + xq^j) \\
&= (1 + xq^{n+1}) \cdot \sum_{k=0}^{n} q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \\
&= \sum_{k=0}^{n+1} \left(q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{\binom{k}{2} + n + 1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \right) x^k \\
&= \sum_{k=0}^{n+1} q^{\binom{k+1}{2}} \left(\begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \right) x^k
\end{aligned}$$

Now by applying the second equality of Proposition 3.1 we obtain

$$\sum_{k=0}^{n+1} q^{\binom{k+1}{2}} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q x^k$$

which is the formula for $n+1$, so the thesis follows. □

There are several tools that allow us to evaluate q-analogs of certain expressions at roots of unit. The most basic one is the following.

THEOREM 3.3. – *Let $m \equiv n \bmod d$, and let $\zeta = \zeta_d$. Then*

$$\frac{[m]_\zeta}{[n]_\zeta} = \begin{cases} \dfrac{m}{n} & \text{if } m \equiv n \equiv 0 \mod d \\ 1 & \text{otherwise.} \end{cases}$$

where the notation $[m]_\zeta$ indicates the evaluation of the polynomial $[m]_q$ in $q = \zeta$.

PROOF. – Let $m \equiv n \equiv r \bmod d$, with $0 \le r < d$. Since $1 + \zeta + \zeta^2 + \cdots + \zeta^{d-1} = 0$, we may delete any d consecutive terms in $1 + \zeta + \cdots + \zeta^{m-1}$ (or ζ^{n-1}), so

$$[m]_\zeta = [n]_\zeta = 1 + \zeta + \zeta^2 + \cdots + \zeta^{r-1}.$$

It follows that if $r \neq 0$ then $[m]_\zeta = [n]_\zeta = r$, as desired.

Suppose $r = 0$. Then we can write $n = hd$ and $m = kd$. Hence we have

$$\frac{[m]_q}{[n]_q} = \frac{(1 + q + \cdots + q^{d-1})(1 + q^d + \cdots + q^{d(k-1)})}{(1 + q + \cdots + q^{d-1})(1 + q^d + \cdots + q^{d(h-1)})}$$

so canceling the $1 + q + \cdots + q^{d-1}$ factor and recalling that $\zeta^d = 1$, we have

$$\frac{[m]_\zeta}{[n]_\zeta} = \frac{k}{h} = \frac{m}{n}$$

as desired. □

For instance, if we evaluate $\frac{[8]_q}{[4]_q}, \frac{[7]_q}{[3]_q}, \frac{[6]_q}{[2]_q}$ at $q = \zeta_6^3 = \zeta_2$ we obtain $\frac{8}{4}, 1, \frac{6}{2}$ respectively. Therefore the evaluation of $\mathrm{Cat}_q(4) = \frac{[8]_q[7]_q[6]_q}{[4]_q[3]_q[2]_q}$ at $q = \zeta_6^3 = \zeta_2$ is 6, in accordance with the computation in Section 1.

COROLLARY 3.4. – *If $\zeta = \zeta_d$ and $d \mid n$, then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_\zeta = \begin{cases} \dbinom{n/d}{k/d} & \text{if } d \mid k \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. – In the equality above, consider the numerator and denominator of the left hand side $\begin{bmatrix} n \\ k \end{bmatrix}_\zeta$ after canceling factorials. Since $d \mid n$, the product $[n]_\zeta[n-1]_\zeta \cdots [n-k+1]_\zeta$ starts with a zero factor and has another zero exactly every d factors; the product $[1]_\zeta[2]_\zeta \cdots [k]_\zeta$ starts with $d-1$ nonzero factors, and then has a zero every d factors.

Since the number of factors is the same, it follows that the numerator has always at least as zero factors as the denominator, with equality if and only if $d \mid k$.

It follows that if d does not divide k, the equality holds. If $d \mid k$ then $d \mid (n-k)$ and we have, applying Theorem 3.3,

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_\zeta &= \frac{[n-k+1]_\zeta}{[1]_\zeta} \cdot \frac{[n-k+2]_\zeta}{[2]_\zeta} \cdot \cdots \cdot \frac{[n]_\zeta}{[k]_\zeta} \\ &= 1 \cdots 1 \cdot \frac{n-k+d}{d} \cdot 1 \cdots 1 \cdot \frac{n-k+2d}{2d} \cdot 1 \cdot \cdots \cdot 1 \cdot \frac{n}{k} \\ &= \frac{n/d - k/d + 1}{1} \cdot \frac{n/d - k/d + 2}{2} \cdots \frac{n/d}{k/d} \\ &= \binom{n/d}{k/d} \end{aligned}$$

as desired. □

Provided these tools, we are ready to approach some easy cyclic sieving problems.

4. – A first example of cyclic sieving phenomenon

The most straightforward approach to prove that a given triple $(X, C, X(q))$ exhibits the CSP is by brute force evaluation. If the action of C on X is simple enough, one may directly compute the cardinalities of the fixed point set X^{c^d}. Moreover, if it is also possible to evaluate the q-enumerator at the roots of unit, one can show that

$$\#X^{c^d} = X(\zeta^d).$$

The other major approach used to prove that a triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon is algebraic, and it involves representation theory. Such kind of proofs tend to be more elegant, and they also give an insight into why a given triple should exhibit the CSP. Conversely, an instance of the CSP may suggest various new results in representation theory.

We will now briefly explain how representation theory can be used to prove an instance of cyclic sieving. The reader who is not familiar with group actions on vector spaces may skip a few lines.

In order to prove that a given triple $(X, C, X(q))$ exhibits the CSP via representation theory, we have to find a vector space V with a distinguished basis $\mathcal{B} = \{e_x \mid x \in X\}$ and a group G which acts linearly on V. We also need an element $g \in G$ whose action on V can be easily described in terms of the basis $\mathcal{B}$ and the group $C = \langle c \rangle$.

For instance, let us suppose that for every $x \in X$ we have $g \cdot e_x = e_{cx}$. Let us denote by $\chi\colon G \to \mathbb{C}$ the character of the G-representation V, defined by $\chi(g) = \mathrm{Tr}(\rho_g)$, where ρ_g is the matrix associated to g.

Then for every $d \in Z$ we have $\#X^{c^d} = \chi(g^d)$. One then only needs to show that $X(\zeta^d) = \chi(g^d)$, and this can be done using several tools from algebra.

We will now prove an instance of the CSP for the set $X = \mathcal{P}_k([n])$ of subsets of $[n] := \{1, \cdots, n\}$ of cardinality k.

The group S_n acts naturally on $\{1, \cdots, n\}$. This induces an action on X in this way: if $S = \{i_1, \cdots, i_k\}$ then $\sigma S = \{\sigma(i_1), \cdots, \sigma(i_k)\}$. Then we consider the cyclic group generated by an n-cycle: take $c = (1, \cdots, n)$ and $C = \langle c \rangle$. We only need a q-enumerator for X. An obvious choice is $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$.

Now we are ready to state our first real cyclic sieving theorem, which is a special case of a more general result proved by Reiner, Stanton and White in their paper [10].

THEOREM 4.1. – *The triple*

$$\left(\mathcal{P}_k([n]), \langle (1, \cdots, n) \rangle, \begin{bmatrix} n \\ k \end{bmatrix}_q \right)$$

exhibits the cyclic sieving phenomenon.

We propose two different proofs of this theorem, a brute force one and an algebraic one.

Brute force proof.

Let's start by counting fixed points. First of all we need to introduce some notation. We recall that every element $\sigma \in S_n$ can be written as a product of disjoint cycles: we will say that the set of numbers that appear in a cycle is the *support* of the cycle. For instance if in S_6 we consider the permutation

$$\sigma = (1, 2, 4)(3, 5, 6)$$

the supports of the cycles of σ are the sets $\{1, 2, 4\}$ and $\{3, 5, 6\}$. Now we need a lemma.

LEMMA 4.2. – *Let $\sigma \in S_n$, $S \subseteq [n]$. Then $\sigma S = S$ if and only if S is a disjoint union of supports of cycles of σ.*

PROOF. – If S is a disjoint union of supports of cycles of σ, then clearly $\sigma S = S$. Conversely, if S is not a disjoint union of supports of cycles of σ, then there must be some cycle τ of σ, and some $i, j \in [n]$ such that $i \in S$, $j \notin S$ and $\tau(i) = j$, so $\sigma S \neq S$. □

COROLLARY 4.3. – *If $g \in C$ is an element of order d, then*

$$\#X^g = \begin{cases} \dbinom{n/d}{k/d} & \text{if } d \mid k \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. – Since g is a power of $(1, \cdots, n)$ of order d, then the disjoint cycle decomposition of g consists of n/d cycles of length d. If d does not divide k, then no subset of k elements can be a disjoint union of supports of cycles of g, so there are no fixed points. If $d \mid k$, then the fixed points are the sets obtained by choosing k/d of the n/d supports of cycles of g, which can be done in the number of ways stated above. □

We now have to evaluate $X(\zeta_d)$, where ζ_d is a root of unit of order d. This is easily done using the results shown in Section 3. Then the thesis immediately follows comparing Corollary 4.3 and Corollary 3.4. □

Algebraic proof.

This proof is less elementary and the reader who is not familiar with multilinear algebra can skip to the next section.

At first we need a vector space with a basis indexed by $X = \mathcal{P}_k([n])$. A quite natural choice is $V = \Lambda^k \mathbb{C}[n]$, because not only it has the right dimension as $\mathbb{C}$-vector space, but it also has a basis $\mathcal{B} = \{e_S \mid S \in X\}$ indexed by X, where

$$e_S = \bigwedge_{i \in S} e_i.$$

Notice that if M is a G-module, then $\Lambda^k M$ is also a G-module, with the action defined by

$$g(v_1 \wedge \cdots \wedge v_k) = g(v_1) \wedge \cdots \wedge g(v_k)$$

extended by linearity.

So our vector space V is a S_n module.

Let $\chi = \chi_V$. At first we want to evaluate $\chi(c^d)$, where $c = (1, \ldots, n)$. Since c has order n, we know that its eigenvalues on $\mathbb{C}[n]$ are $\zeta, \zeta^2, \ldots, \zeta^{n-1}, 1$. Now, let $v_1, \ldots, v_n$ be a basis of $\mathbb{C}[n]$ of eigenvectors for c. Then $\{v_S \mid S \in X\}$, where

$$v_S = \bigwedge_{i \in S} v_i,$$

is a basis of V of eigenvectors for c. The corresponding eigenvalues are $\zeta^{\Sigma S}$, where $\Sigma S = \sum_{i \in S} i$, therefore we have

$$\chi(c^d) = \sum_{S \in X} \zeta^{d(\Sigma S)}.$$

Now we notice that the coefficient of x^k in the left hand side of Theorem 3.2 is equal to $\sum_{S \in X} q^{\Sigma S}$. It follows that

$$\chi(c^d) = \zeta^{d\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{\zeta^d}$$

so we need to prove that, if $c^d S = S$, then $c^d \cdot e_S = \zeta^{d\binom{k+1}{2}} e_S$. If this statement holds, in fact, then $\zeta^{-d\binom{k+1}{2}} \chi(c^d) = \# X^{c^d}$, because the trace is invariant by base change. Since it agrees with the q-binomial evaluation, the thesis will follow.

Let $c^d S = S$. Then c^d is a product of d cycles of length $r = n/d$, and S is a disjoint union of the supports of b of these cycles, with $k = br$. We have $c^d \cdot e_S = (-1)^{b(r-1)} e_S$, so we have to prove that $(-1)^{b(r-1)} = \zeta^{d\binom{k+1}{2}}$. It holds

$$(x^r - 1)^b = \prod_{i=1}^{k} (x - \zeta^{di})$$

so, evaluating at $x = 0$, we have $(-1)^b = (-1)^k \zeta^{d\binom{k+1}{2}}$, and the thesis follows immediately. □

5. – Catalan numbers, Narayana numbers and Kirkman-Cayley numbers

Before discussing some other instances of cyclic sieving, we need to give a closer look to the Catalan numbers, that were introduced in the Premise and appeared also in the previous section:

$$\mathrm{Cat}(n) = \frac{1}{n+1} \binom{2n}{n}.$$

The Catalan numbers count an impressive amount of apparently unrelated things. More than 200 different combinatorial interpretations can be found in [18]: we now show one of these interpretations.

Let us say that there is a special all-you-can-eat offer at the restaurant, for 10 euros only, and that $2n$ people go to the restaurant. Half of them have a 10 euros bill, while the other half have a 20 euros bill. Unfortunately the cashier has no available change, but if enough people with the 10 euros bill pay first, then everything will be ok, right? In how many ways can it be done? The answer is essentially based on these omnipresent numbers, as we'll see in a moment. But let us start with a different problem.

DEFINITION 5.1. – *A* Dyck word *of length* $2n$ *is a sequence consisting of* n *X's and* n *Y's, such that no initial segment of the sequence has more X's than Y's.*

For instance, *YYXYXX* is a Dyck word of length 6, while *YYXXXY* is not. How many Dyck words of length $2n$ are there? This is related to the restaurant problem: it is sufficient to interpret the people with the 10 euros bill as Y's, and the people with the 20 euros bill as X's. Then the answer to the restaurant problem is the number of Dyck words of length $2n$ multiplied by $(n!)^2$ (because the people are all different, but the letters aren't). Now it is time to count Dyck words.

THEOREM 5.2. – *The number of Dyck words of length* $2n$ *is* $\mathrm{Cat}(n)$.

PROOF. – Let us consider a sequence w of n X's and n Y's that is not a Dyck word. Therefore there is a first 'bad' X that violates the Dyck condition: let us denote by w' the subsequence of w that starts after this bad X. Now we modify w by interchanging all the X's and Y's in the subsequence w': we get a sequence of $(n+1)$ X's and $(n-1)$ Y's. Conversely, given a sequence γ of $(n+1)$ X's and $(n-1)$ Y's we can obtain, by reversing the procedure above (notice that in γ there must be a first bad X), a sequence w of n X's and n Y's that is not a Dyck word.

Now we observe that the total number of sequences of n X's and n Y's is $\binom{2n}{n}$, while the argument above shows that the sequences that are not Dyck words are as many as the sequences of $(n+1)$ X's and $(n-1)$ Y's, that is to say, $\binom{2n}{n-1}$.

Therefore the number of Dyck words is

$$\binom{2n}{n} - \binom{2n}{n-1}$$

that is equal to $\mathrm{Cat}(n)$, as it is shown by an easy computation. □

Definition 5.3. – A *Dyck path* is a lattice path from $(0,0)$ to (n,n) that lies (weakly) above the line $x = y$.

Is it obvious that Dyck paths and Dyck words are in a bijective correspondence, described as follows. Let us consider a Dyck word w: if we start from $(0,0)$ and increase the x coordinate by one each time we read an X in w and the y coordinate by one each time we read a Y, we end up with a Dyck path.

Now, we could go a little further. It's obvious that a Dyck path is composed of an even number of segments. Given the integer k, with $1 \le k \le n$, how many Dyck paths with exactly $2k$ segments are there? The answer is given by the Narayana number

$$\mathrm{Nar}(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

For a proof of this fact, together with more informations about Dyck paths, one can see Excercise 6.36 of [16].

A generalization of the Catalan numbers is provided by the Kirkman-Cayley numbers. To introduce them, we go back to the problem of dissecting a polygon by its diagonals, that was discussed in Section 1 in the particular case of the triangulations of a hexagon.

Definition 5.4. – A *dissection* of a regular n-gon with k diagonals is a way to draw k diagonals of the n-gon such that any couple of diagonals do not intersect in their interior (they may intersect in a vertex of the n-gon, though).

Theorem 5.5. – *The number of dissections of a regular n-gon with k diagonals is the Kirkman-Cayley number*

$$\mathrm{D}(n,k) = \frac{1}{k+1}\binom{n-3}{k}\binom{n+k-1}{k}.$$

One can find a short proof of this in [16] (Exercise 6.33); other direct proofs by bijections may be found for instance in [4], [15], [7].

Notice that if $k = n-3$, that is to say, if the number of diagonals is maximal, the dissections are in fact triangulations. One can check by a simple computation that $\mathrm{D}(n, n-3)$, the number of triangulations of a regular n-gon, is equal to $\mathrm{Cat}(n-2)$, as we announced in Section 1. Even if we do not prove Theorem 5.5 in this paper, we will suggest later a way to prove that the Catalan numbers count the triangulations of polygons, as a consequence of a 'chain' of bijections (see Remark 8.4). This is one of the many simple proofs that can be found in the literature.

6. – An instance of the CSP regarding polygon dissections

As we said, the cyclic sieving phenomenon has been observed many times on sets of combinatorial objects equipped with a natural action of a cyclic group. A very interesting instance involves polygon dissections. It turns out that

$$\mathrm{D}_q(n,k) = \frac{1}{[k+1]_q}\begin{bmatrix} n-3 \\ k \end{bmatrix}_q \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q$$

is a polynomial with nonnegative integer coefficients, which is called the q-Kirkman-Cayley number.

This polynomial is the main character in the following:

Theorem 6.1 (Reiner, Stanton, White [10, Theorem 7.1]). – *Let X be the set of dissections of a* *regular n-gon using k noncrossing diagonals. Let $X(q) := \mathrm{D}_q(n,k)$ and let be $C = C_n$ be the cyclic group of order n acting on X by cyclic rotation of the polygon. Then the triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon.*

As we know, when $n = 6$ and $k = 3$ the theorem above deals with the set of triangulations of a regular hexagon: therefore this set provides an instance of the CSP with respect to the action of the groups C_6 of the rotations of the hexagon and the polynomial $\mathrm{D}_q(6,3) = \mathrm{Cat}_q(4)$. This was exactly our starting example in Section 1.

We give here an account of the proof provided by V. Reiner, D. Stanton and D. White, that is by brute force enumeration. No representation theoretical proof is known yet. The readers who are more interested in a overview can skip to the next section.

PROOF. – We will use Lemma 3.3 several times. Let ζ be a n^{th} root of unit, and let $d \mid n$. At first we evaluate $\mathrm{D}_{\zeta^{\frac{n}{d}}}(n,k)$.

By straightforward computation, we have

$$\mathrm{D}_{\zeta^{\frac{n}{d}}}(n,k) = \begin{cases} \mathrm{D}(n,k) & \text{if } d=1, \\ \dfrac{\lfloor\frac{n+k-1}{2}\rfloor!}{\frac{n-2}{2}\lfloor\frac{n-k-3}{2}\rfloor!\lfloor\frac{k+1}{2}\rfloor!\lfloor\frac{k}{2}\rfloor!} & \text{if } d=2, \\ \dfrac{(\frac{n+k}{d}-1)!}{(\frac{n-k}{d}-1)!(\frac{k}{d})!^2} & \text{if } d=3 \text{ and } d|k, \\ 0 & \text{otherwise.} \end{cases}$$

Now we need to count polygon dissections which are $\frac{n}{d}$-fold symmetric. The evaluation is split in three cases.

Case 1. $d = 2$ and k odd. In this case, a k-dissection must contain a unique diameter, which can be chosen in $n/2$ ways. The rest of the dissection is completely determined by the dissection of the $(n/2+1)$-gon with $(k-1)/2$ diagonals on either side of the diameter. Hence the number of such dissections is

$$\frac{n}{2}\cdot \mathrm{D}\left(\frac{n}{2}+1, \frac{k-1}{2}\right) = \frac{\lfloor\frac{n+k-1}{2}\rfloor!}{\frac{n-2}{2}\lfloor\frac{n-k-3}{2}\rfloor!\lfloor\frac{k+1}{2}\rfloor!\lfloor\frac{k}{2}\rfloor!}$$

as desired.

Case 2. $d = 2$ and k even. Centrally symmetric dissections with p antipodal pairs of diagonals (where the diameter count as a pair) were counted by R. Simion in [14, Proposition 1.1], and their number is

$$\binom{n/2-1}{p}\binom{n/2+p-1}{p}.$$

For $p = k/2$, subtracting those with $k-1$ diagonals (so, with a diameter), we have that the number of centrally symmetric dissections with k diagonals is

$$\frac{n}{2}\cdot \mathrm{D}\left(\frac{n}{2}+1, \frac{k-1}{2}\right) = \frac{\frac{n+k-2}{2}!}{\frac{n-2}{2}\frac{n-k-4}{2}!\left(\frac{k}{2}!\right)^2}$$

which agrees with the q-enumerator evaluation.

Case 3. $d \geq 3$. In this case, it's obvious that any diagonal lies in a free orbit under the action of $c^{n/d}$, so k must be divisible by d.

When $d \mid k$, as we did for $d = 2$, we decompose d-fold rotationally symmetric k-dissections into two sets: those for which the central polygon is a d-gon, and those for which is not. As in Case 1, the former set is counted by

$$\frac{n}{d}\mathrm{D}\left(\frac{n}{d}+1, \frac{k}{d}-1\right).$$

For the latter set, in [8, Proposition 4.2] an explicit bijection with the set of centrally symmetric subdivisions of a $2n/d$-gon with $2k/d$ diagonals is given, and these are counted in Case 2. Totalling the cardinalities gives

$$\frac{(\frac{n+k}{d}-1)!}{(\frac{n-k}{d}-1)!(\frac{k}{d})!^2},$$

as desired. □

The instance of cyclic sieving provided by Theorem 6.1 is especially interesting because it seems to relate with a different, more complicated one, as suggested also by the final remark in Section 1. We will discuss it in Section 8, but let us proceed step by step.

7. – Two instances of the CSP regarding noncrossing partitions

This section is devoted to set partitions, in particular to noncrossing partitions. We start by giving the definitions, then we will show that noncrossing partitions are enumerated by the Catalan and Narayana numbers. In this picture, two beautiful instances of the CSP will appear.

DEFINITION 7.1. – A *partition* of a set X is a set $\pi = \{X_\alpha \mid \alpha \in A\}$ such that $X_\alpha \neq \emptyset$, $X_\alpha \subseteq X$ for all $\alpha \in A$, $X_\alpha \cap X_\beta = \emptyset$ for all $\alpha \neq \beta$ and

$$\bigcup_{\alpha \in A} X_\alpha = X.$$

Elements of a partition π are said to be *blocks*. Elements in the same block are called *blockmates*. If a block has only one element, it's said to be a *singleton*.

Let us denote by $\Pi(n)$ the set of the partitions of $\{1, 2, ..., n\}$. There is a convenient way to represent a partition $\pi \in \Pi(n)$ on a disk, as follows. Draw n points on a circumference, label them clockwise, and then highlight the polygons whose vertices are labeled with elements in the same block of π.

EXAMPLE 7.2 – *Take, as an example, the partition* $\{\{1, 2, 4\}, \{3, 5\}, \{6\}\}$ *and the partition* $\{\{1, 6\}, \{2, 4, 5\}, \{3\}\}$ *in* $\Pi(6)$. *These are represented below.*

Given $\pi \in \Pi(n)$, let us denote as $B_i(\pi)$ the block of π such that $i \in B_i(\pi)$.

DEFINITION 7.3. – A partition $\pi \in \Pi(n)$ is called *noncrossing* if, whenever we have $1 \leq a < b < c < d \leq n$, then

$$B_a(\pi) = B_c(\pi),\ B_b(\pi) = B_d(\pi) \Longrightarrow B_a(\pi) = B_b(\pi) = B_c(\pi) = B_d(\pi).$$

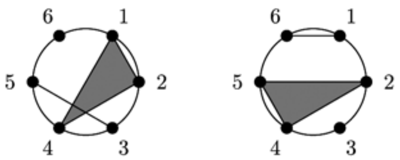


Fig. 3. – A crossing partition (left) and a noncrossing one (right).

It's pretty easy to check that a partition is noncrossing if and only if its blocks do not intersect when represented on a disk (so the naming makes some sense). In example 7.2, the partition on the left is crossing, while the one on the right is noncrossing.

Let $n, k, s \geq 0$. We define three collections of set partitions as follows.

$$\begin{aligned} \mathrm{NC}(n) &:= \{\pi \in \Pi(n) \mid \pi \text{ is noncrossing}\}, \\ \mathrm{NC}(n,k) &:= \{\pi \in \mathrm{NC}(n) \mid \pi \text{ has } k \text{ blocks}\}, \\ \mathrm{NC}(n,k,s) &:= \{\pi \in \mathrm{NC}(n,k) \mid \pi \text{ has } s \text{ singletons}\}. \end{aligned}$$

We have the disjoint union decompositions

$$\mathrm{NC}(n) = \bigsqcup_{k \leq n} \mathrm{NC}(n,k)$$

and

$$\mathrm{NC}(n,k) = \bigsqcup_{s \leq k} \mathrm{NC}(n,k,s).$$

The set $\mathrm{NC}(n)$ of the noncrossing partitions on n elements has cardinality given by the Catalan number $\mathrm{Cat}(n)$. In fact there is a bijection between Dyck paths and noncrossing partitions, that can be described as follows. Let us label the "Y steps" of a Dyck path from 1 to n as we read them, and the "X steps" with the greatest label among those already assigned to some Y step, but not yet assigned to any X step, as illustrated by Figure 4; then if we consider the partition whose blocks are composed by the labels of the sequences of consecutive X steps, we get a noncrossing partition.

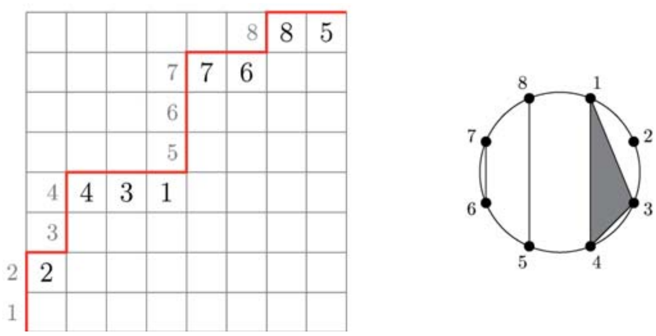


Fig. 4. – A Dyck path and its corresponding noncrossing partition.

We also have that the set $\mathrm{NC}(n,k)$ of the noncrossing partitions on n elements with k blocks has cardinality given by the Narayana number $\mathrm{Nar}(n,k)$.

Through the representation of partitions on a disk, one may define a natural action of the cyclic group C_n on the set of partitions, by rotation. There are several known instances of cyclic sieving related to this action.

The first theorem states an instance of the CSP for the whole set of noncrossing partitions.

THEOREM 7.4 (Reiner, Stanton, White [10]). – *Let $X = \mathrm{NC}(n)$, and let $C = C_n$ acting on X by rotation. The triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon, where*

$$X(q) = \mathrm{Cat}_q(n) := \frac{1}{[n+1]_q} \begin{bmatrix} 2\,n \\ n \end{bmatrix}_q$$

is a polynomial with nonnegative integer coefficients called the q-Catalan number.

As one could guess observing that the action by rotation preserves the number of blocks, another instance of the CSP appears.

THEOREM 7.5 (Reiner, Stanton, White [10, Theorem 7.2]). – *Let $X = \mathrm{NC}(n, k)$, and let $C = C_n$ acting on X by rotation. The triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon, where*

$$X(q) = \mathrm{Nar}_q(n, k) := q^{(n-k)(n-k-1)} \frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$$

is a polynomial with nonnegative integer coefficients called the q-Narayana number.

Actually, there is a third, even more important, instance of cyclic sieving phenomenon related to noncrossing partitions, involving the set $\mathrm{NC}(n, k, 0)$ of noncrossing partitions on n elements, with k blocks and no singletons. We will discuss it in the next section.

8. – A CSP instance that springs from dissections, parenthesizations, rooted ordered plane trees and noncrossing partitions

This section is a ride between some beautiful combinatorial objects that are related by a CSP instance. We will start from polygon dissections and by showing suitable bijections we will come to consider first parenthesizations of lists and then rooted ordered plane trees; in the end we will come across noncrossing partitions with 0 singletons.

We start by remarking that dissections of polygons are in bijective correspondence with *parenthesizations* of the list of n numbers $12\ldots n$. Let us explain this more in detail.

DEFINITION 8.1. – A *valid parenthesization* of the list of n numbers $12\ldots n$ with k pairs of parentheses is a way to put k pairs of parentheses among the numbers such that each pair include at least two elements, different pairs contain different numbers, and the maximal pair $(1\ldots n)$ is included.

For example,

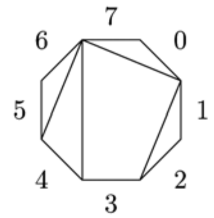
$$(((12)3)(45(67)))$$

is a valid parenthesization with 5 pair of parentheses of the list of seven numbers 1234567.

LEMMA 8.2. – *There is a bijective map between dissections of a regular $n+1$-gon with $k-1$ diagonals and valid parenthesizations of $1\ldots n$ with k pairs of parentheses.*

PROOF. – We define an explicit bijection as follows. At first, we label the edges of the $n+1$-gon as $0, 1, \ldots, n$, and then, for each diagonal, we enclose in a pair of matching parentheses all the numbers subtended by that diagonal on the side that does not contain 0. Since it admits an obvious inverse, this construction is bijective, so the thesis follows. □

EXAMPLE 8.3 – *Take the dissection*



Starting from 0 and going clockwise, the diagonals are associated to the pairs of parentheses (12), (456), (56), and (123456), so the dissection corresponds to the parenthesization (((12)3(4(56))7).

REMARK 8.4. – *Let us briefly explain why this bijection can be used to show that the number of triangulations of a* $n+1$*-gon is* $\mathrm{Cat}(n-1)$. *We observe that in this case the number of diagonals is* $n-2$, *therefore all we need is a bijection between the set of valid parenthesizations of* $1\dots n$ *with* $n-1$ *pairs of parentheses and the set of Dyck words of length* $2n-2$, *whose cardinality is* $\mathrm{Cat}(n-1)$ *by Theorem* 5.2.

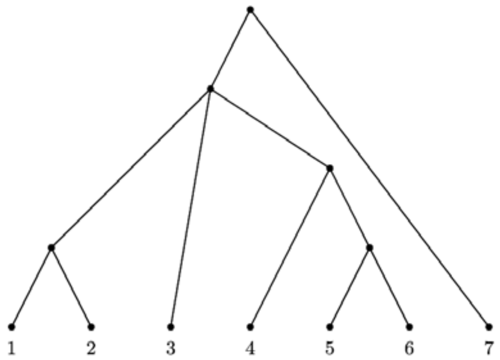
Now, given a valid parenthesization of $1\dots n$ *with* $n-1$ *pairs of parentheses, first we remove the number n, then we remove all the right parentheses, and finally we substitute every left parenthesis with a Y and every number with a X. For instance, if we start from the parenthesization* ((((12)3)(4(56))7), *we get* ((((123(4(56 *and finally the word YYYYXXXYXYXX. The map associated to this procedure turns out to be the desired bijection.*

A valid parenthesization of the list $1\dots n$ with k pair of parentheses may be represented by a rooted ordered plane tree with n leaves and $n+k$ vertices, where vertices represent pair of parentheses. In particular, each nonleaf vertex represents the pair of parentheses enclosing all the numbers that label the leaves below that vertex, as in the following example.

EXAMPLE 8.5 – *The rooted ordered plane tree associated to the parenthesization*

(((12)3(4(56))7)

is the following:

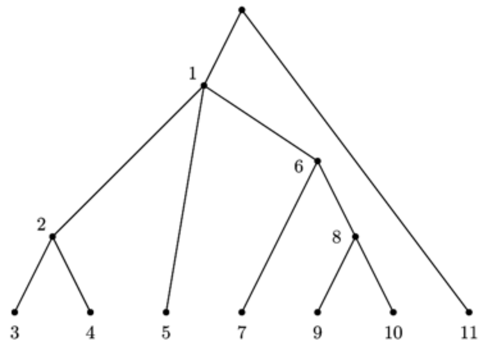


We notice that this establishes a bijection between the set of dissections of a $n+1$-gon with $k-1$ diagonals and the set $\mathcal{T}(n,k)$ of rooted ordered plane trees with n leaves and $n+k$ vertices that have the following further property: there are at least two edges that go down from every nonleaf vertex.

Let us now take a tree in $\mathcal{T}(n,k)$ and relabel it conveniently, so that it may be associated to a noncrossing partition. We do this as follows: we start from the root, and then we explore the graph turning right every time, labeling the vertices as we visit them (except the root vertex), as in Example 8.6.

Finally, we associate such a graph with a set partition of $[n+k-1]$ with k blocks and no singletons as follows: we associate each nonleaf vertex v (including the root) with the block made by all the numbers that label the vertices immediately below v. It's easy to check that such a partition belongs to $\mathrm{NC}(n+k-1,k,0)$ and that the map is bijective, since one can easily reverse the construction (this bijection was described by N. Dershowitz and S. Zaks in their paper [3]).

EXAMPLE 8.6 – *For the previous graph, the result is the following.*

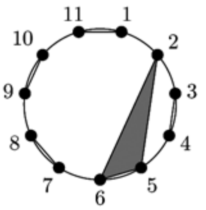


This graph is associated to the noncrossing partition of $\{1,\dots,11\}$ *with* 5 *blocks and no singletons whose blocks are made by numbers immediately below the same vertex. For example, the vertex labeled by* 1 *has* 2, 5, *and* 6 *immediately below it; the vertex labeled by* 2 *has* 3 *and* 4, *and so on.*

The result is the partition

$$\{\{1,11\},\{2,5,6\},\{3,4\},\{7,8\},\{9,10\}\}$$

represented below.



THEOREM 8.7. – *All the maps described above define a bijection between the set of dissections of a $n+1$-gon with $k-1$ diagonals and* $\mathrm{NC}(n+k-1,k,0)$.

PROOF. – One can observe that each step admits an inverse procedure. □

As a corollary of Theorem 8.7, we deduce that $\mathrm{NC}(n,k,0)$ has cardinality $\mathrm{D}(n-k+2,k-1)$. It is here that the CSP appears by means of the following deep result:

THEOREM 8.8 (Pechenik, [6, Theorem 1.4]). – *Let* $X=\mathrm{NC}(n,k,0)$, *and let* $C=C_n$ *acting on* X *by rotation. The triple* $(X,C,X(q))$ *exhibits the cyclic sieving phenomenon, where* $X(q)=\mathrm{D}_q(n-k+2,k-1)$.

We can finally go back to the example discussed in Section 1.

We notice that when $n=5$ and $k=4$, Theorem 8.7 provides us with a bijection between the set of triangulations of the hexagon and the set $\mathrm{NC}(8,4,0)$. By this bijection we obtain an unexpected action of the cyclic group C_8 on the set of triangulations of the hexagon. By Theorem 8.8 we know that this action of C_8 satisfies the CSP with the polynomial $\mathrm{D}_q(6,3)$, that is equal to $\mathrm{Cat}_q(4)$.

This explains why $\mathrm{Cat}_q(4)$ evaluated at the 8-th roots of 1 provides nonnegative numbers, which is the 'strange' phenomenon observed in the end of Section 1. The nonnegative integers that we computed in Section 1 are the cardinalities of the sets of fixed points with respect to the 'hidden' C_8 action on triangulated hexagons. More precisely, we deduce that this action partitions the set of 14 triangulations into three orbits, with respectively 8, 4, and 2 elements, while, as we know from Figure 2, the natural action of the group C_6 of the rotations of the hexagon produces four orbits with respectively 6, 3, 3, 2 elements. More in general, as Theorems 6.1 and 8.8 show, the q-Kirkman-Cayley numbers come into play in two different CSP instances, that involve two different cyclic groups.

The instance described in Theorem 8.8 is especially important not only because the q-enumerator is the same that appears in the instance involving dissections, but also because it can be used to deduce Theorems 7.4 and 7.5. In [12], B. Rhoades provides $\mathbb{Q}[\mathrm{NC}(n)]$ with a S_n-module structure which commutes with the permutation one, thereby giving a good proof of this instance of cyclic sieving.

9. – Suggestions for further readings

We point out to the interested reader the short presentation "What is... cyclic sieving" by Reiner, Stanton and White, in the Notices of AMS 2014 (see [11]). To the readers who want to learn about more instances of the CSP we suggest Sagan's survey [13] where one can also find an overview on several connections with recent research. A survey that gives more advanced information and more details on the instances of the CSP described in this paper is the master degree thesis [5]. Another instance of the CSP regarding the q-Catalan numbers can be found in the short paper [21].

Finally, we would like to point out to expert mathematicians that, among the recent advances in this field, there are some deep and beautiful instances of the CSP involving the generalizations of noncrossing partitions and Catalan numbers to the setting of complex reflection groups: to have an idea of these developments, one can read the papers [1], [2], [9] (and one can find further references in the bibliography there).

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