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D-modules and Arrangement of Hyperplanes

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$\mathcal{D}$-modules and Arrangement of Hyperplanes

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by

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Introduction

Let $\mathcal{A}$ be an arrangement of hyperplanes in $V = \mathbb{C}^n$, all containing the origin. For each $H \in \mathcal{A}$, let α_H be a linear form whose kernel is H. Then $d_{\mathcal{A}} = \prod_{H \in \mathcal{A}} \alpha_A$ is a defining polynomial of $\mathcal{A}$ of degree $k = |\mathcal{A}|$. Let $Y_{\mathcal{A}} = V \setminus \bigcup_{H \in \mathcal{A}} H$ be the open connected submanifold of V determinated by $\mathcal{A}$. We may ask how various topological properties of $Y_{\mathcal{A}}$ may be determined from $\mathcal{A}$. This line of investigation began with work of Arnold [1], Brieskorn [6], and Deligne [10].

In this work we study the cohomology ring $H^*(Y_{\mathcal{A}}, \mathbb{C})$ with an approach of the $\mathcal{D}$-modules theory. We begin given a description of $H^*(Y_{\mathcal{A}}, \mathbb{C})$. Let $\omega_H = d\alpha_H / 2\pi i \alpha_H$ be a holomorphic 1-form on $Y_{\mathcal{A}}$ associated to $H \in \mathcal{A}$. Let $[\omega_H]$ denote the corresponding De Rham cohomology class. Let $\mathcal{R} = \mathcal{R}(\mathcal{A})$ be the graded $\mathbb{C}$-algebra of holomorphic differential forms on $Y_{\mathcal{A}}$ generated by the $[\omega_H]$ and the identity. Brieskorn [6] showed that $\mathcal{R} \simeq H^*(Y_{\mathcal{A}}, \mathbb{C})$ as graded vector space. Orlik and Solomon [20] gave a description of the ring structure of $\mathcal{R}$. Let $\mathcal{E} = \mathcal{E}(\mathcal{A})$ be the exterior algebra of a vector space with basis consisting of elements e_H in one to one correspondence with the hyperplanes $H \in \mathcal{A}$. We say that a subset S of $\mathcal{A}$ is independent if $\bigcap_{H \in S} H$ has codimension $|S|$, and is dependent otherwise. Thus S is independent when the hyperplanes of S are in general position. Define a $\mathbb{C}$-linear map $\partial : \mathcal{E} \to \mathcal{E}$ by $\partial 1 = 0$, $\partial e_H = 1$ and

$$\partial(e_{H_1} \dots e_{H_p}) = \sum_{j=1}^{p} (-1)^{j-1} e_{H_1} \dots \widehat{e_{H_j}} \dots e_{H_p} \ .$$

Let $\mathcal{I}$ be the ideal of $\mathcal{E}$ generated by all elements $\partial(e_{H_1} \dots e_{H_p})$ where $\{H_1, \dots, H_p\}$ is dependent. It is proved in [20] that the map $\overline{e_H} \to [\omega_H]$ defines an isomorphism between the graded algebras $\mathcal{E}/\mathcal{I}$ and $H^*(Y_{\mathcal{A}}, \mathbb{C})$. Denote the poset of intersections of elements of $\mathcal{A}$ by $L = L(\mathcal{A})$ ordered by reversed inclusion, and with a rank function defined by $r(X) = \text{codim} X$, $X \in L$. Orlik-Solomon [20] constructed $\mathcal{E}/\mathcal{I}$ using only $L(\mathcal{A})$.

Let $D_n = \mathbb{C}\langle x_1, \dots, x_n, \partial/\partial x_1, \dots, \partial/\partial x_n \rangle$ be the Weyl algebra of rank n over $\mathbb{C}$ and let $P = P(\mathcal{A}) = \mathbb{C}[x_1, \dots, x_n, d_{\mathcal{A}}^{-1}]$ be the algebra of rational functions on $Y_{\mathcal{A}}$. In the present work we construct a sequence of P as D_n-

modules, and obtain the direct sum decomposition of its D_n-modules (Chapter 4). Furthermore, using this decomposition, we compute, in the Chapter 5, the cohomology ring $H^*(Y_{\mathcal{A}})$. Finally, in Chapter 6, we get the Poincaré series of $P(\mathcal{A})$. All D_n-modules mentioned here are left D_n-modules. Let $r = r(\mathcal{A}) = r(\bigcap_{H\in\mathcal{A}} H)$ be the rank of the maximal element of $L(\mathcal{A})$, namely, the cardinality of a maximal linearly independent subset of $\mathcal{A}^* = \{\alpha_H \mid H \in \mathcal{A}\}$. Then each element of P can be written as a finite sum of quotients of the form $f/\prod_{j=1}^{h} \alpha_{i_j}^{m_j}$ where $0 \le h \le r$, $\{\alpha_{i_1}, \ldots, \alpha_{i_h}\}$ is a linearly independent subset of $\mathcal{A}^*$, $m_j \in \mathbb{N}$, $f \in \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_n]$ and $\prod_{j=1}^{0} \alpha_{i_j}^{m_j} := 1$. This allow us to get the following sequence of holonomic D_n-submodules of P : $0 = P_{-1} \subset \mathbb{C}[\mathbf{x}] = P_0 \subset P_1 \subset \ldots \subset P_r = P$, where

$$P_h = \left\{ \sum \frac{f^{m_1\ldots m_t}_{s_1\ldots s_t}}{\alpha_{s_1}^{m_1}\ldots\alpha_{s_t}^{m_t}} \;\middle|\; 0 \le t \le h,\ f^{m_1\ldots m_t}_{s_1\ldots s_t} \in \mathbb{C}[\mathbf{x}],\ m_i \in \mathbb{N} \right\}.$$

For each $X \in L_h = \{X \in L(\mathcal{A}) \mid r(X) = h\}$ consider its dual subspace X^* of $(\mathbb{C}^n)^*$ of dimension h. Let $\mathcal{B}_{X^*}$ be the set of all possible bases to X^* constituted with elements of $\mathcal{A}^*$. For each X^*, and for each basis $B = \{\alpha_{i_1}, \ldots, \alpha_{i_h}\} \in \mathcal{B}_{X^*}$ we define the following holonomic D_n-submodule of P_h/P_{h-1}

$$V^B_{X^*} = \left\{ \sum \left(\frac{f^{m_1\ldots m_h}_{i_1\ldots i_h}}{\alpha_{i_1}^{m_1}\ldots\alpha_{i_h}^{m_h}} \bmod P_{h-1} \right) \;\middle|\; f^{m_1\ldots m_h}_{i_1\ldots i_h} \in \mathbb{C}[\mathbf{x}],\ m_j \in \mathbb{Z}^+ \right\} .$$

We show in Proposition 4.2.11 that for each basis $B \in \mathcal{B}_{X^*}$ the D_n-module $V^B_{X^*}$ is isomorphic to each other, and after a linear change of coordinates in $(\mathbb{C}^n)^*$ such that $X^* = \langle y_1, \ldots, y_h\rangle$, $V^B_{X^*}$ is isomorphic, as D_n-module, to $M_{X^*} = \mathbb{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}]$ where $\partial_{y_j} = \partial/\partial y_j$. Now let $V^{\text{mod}}_{X^*}$ be the $\mathbb{C}$-subspace of P_h/P_{h-1} generated by all $\left[1/\prod_{\alpha\in B}\alpha\right]$, $B \in \mathcal{B}_{X^*}$, then the holonomic D_n-module P_h/P_{h-1} has the following decomposition

$$P_h/P_{h-1} = \bigoplus_{X\in L_h} \sum_{B\in\mathcal{B}_{X^*}} V^B_{X^*} = \bigoplus_{X\in L_h} M_{X^*} \otimes_{\mathbb{C}} V^{\text{mod}}_{X^*}.$$

It is possible to determine a basis to $V^{\text{mod}}_{X^*}$ applying the notion of *not broken circuit* (nbc) to $\mathcal{B}_{X^*}$. Let V_{X^*} be the $\mathbb{C}$-vector space generated by the set $\{1/\prod_{\alpha\in B}\alpha \mid B \in \mathcal{B}_{X^*}\}$, then $\{1/\prod_{\alpha\in B}\alpha \mid B \in \mathcal{B}_{X^*}$ and B is a nbc$\}$ is a basis to V_{X^*}, cf. Lemma 4.2.16, and

Theorem 4.2.23 For $1 \le h \le r$ we have $P_h = \bigoplus_{\substack{X\in L(\mathcal{A})\\ r(X)\le h}} M_{X^*} \otimes_{\mathbb{C}} V_{X^*}$. In particular $P = \bigoplus_{X\in L(\mathcal{A})} M_{X^*} \otimes_{\mathbb{C}} V_{X^*}$.

Theorem 4.2.24 The natural map $\bigoplus_{X\in L_h} M_{X^*} \otimes_{\mathbb{C}} V_{X^*} \xrightarrow{\psi} P_h/P_{h-1}$ is an isomorphism of D_n-modules.

This allow us to decompose the De Rham complex for $Y_{\mathcal{A}}$ as a direct sum of complexes with cohomology just in one degree and 1-dimensional: Define the following cochain complex $(\mathcal{L}^*_h, \delta_{\mathcal{L}^*_h})$:

$$\mathcal{L}_h^s = \mathcal{L}_h^s(\langle y_1,\ldots,y_h\rangle) = \left\{ \sum_{1\le i_1<\ldots<i_s\le n} f_{i_1\ldots i_s} \bullet \frac{1}{y_1\ldots y_h} dy_{i_1}\ldots dy_{i_s} \right\}$$

with $\delta_{\mathcal{L}_h^*} : \mathcal{L}_h^* \to \mathcal{L}_h^*$ the usual differential, and $f_{i_1\ldots i_s} \in \mathbb{C}[y_{h+1},\ldots,y_n,\partial_{y_1},\ldots,\partial_{y_h}]$. Thus, cf. Corollary 5.1.4, the groups of cohomology $H^*(\mathcal{L}_h^*)$ are $\mathbb{C}\cdot\frac{1}{y_1\ldots y_h}dy_1\ldots dy_h$ in dimension h and 0 elsewhere. Then for each $X \in L_h(\mathcal{A})$ we associate the following complex

$$\mathcal{L}_h(X) = \bigoplus_{\substack{\langle\alpha_{j_1},\ldots,\alpha_{j_h}\rangle = X^* \\ (j_1,\ldots,j_h)\ \mathbf{nbc}}} \mathcal{L}_h(\{\alpha_{j_1},\ldots,\alpha_{j_h}\})$$

where $\mathcal{L}_h(\{\alpha_{j_1},\ldots,\alpha_{j_h}\})$ is the same complex $\mathcal{L}_h^*$ but it is just defined for $\{\alpha_{j_1},\ldots,\alpha_{j_h}\}$. Finally, associated to the D_n-module $\mathcal{P}_h$, the complex $\mathcal{L}_h(\mathcal{P}_h) = \bigoplus_{X\in L_h}\mathcal{L}_h(X)$ allows us to calculate the h-th cohomology of $Y_{\mathcal{A}}$.

Theorem 5.2.8 *For $1 \le h \le r$ there exists an isomorphism between $H^h(Y_{\mathcal{A}})$ and $H^h(\mathcal{L}_h(\mathcal{P}_h))$:*

$$H_{DR}^h(Y_{\mathcal{A}}) \cong H^h(\mathcal{L}_h(\mathcal{P}_h)) = \bigoplus_{\substack{\langle\alpha_{j_1},\ldots,\alpha_{j_h}\rangle = X^* \\ (j_1,\ldots,j_h)\ \mathbf{nbc}}} \mathbb{C}\cdot\frac{1}{\alpha_{j_1}\ldots\alpha_{j_h}} d\alpha_{j_1}\wedge\ldots\wedge d\alpha_{j_h}$$

Let $P(\mathcal{A},t)$ be the Poincaré polynomial of the arrangement $\mathcal{A}$, cf. Definition 1.6.1, we see in Theorem 6.1.14 that the Poincaré series $Poin(P(\mathcal{A}),t)$ of the graded module $P(\mathcal{A})$ is equal to $(1-t)^{-n}Poin(\mathcal{A},t)$.

Chapter 1

Combinatorics of an arrangement

In this chapter we collect some definitions, notations and results about the combinatorics of an arrangement of hyperplanes that will be used in the rest of this work.

1.1 Definition of an arrangement.

We let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ the set of natural numbers and for $k \in \mathbb{N}$ let $[k] = \{1, 2, \ldots, k\}$ be the set of the k first non-negative integers (where $[0] \stackrel{\textbf{def}}{=} \phi$)

Definition 1.1.1 *A central arrangement of hyperplanes is a finite collection of codimension one subspaces of a complex vector space V. Let us denote it by $\mathcal{A}$ and call it simply an arrangement or n-arrangement if* $\dim V = n$.

The cardinality of $\mathcal{A}$ will be usually denoted by k, and very often we will fix an arbitrary linear order on $\mathcal{A}$, i.e., put $\mathcal{A} = \{H_1, \ldots, H_k\}$.

Sometimes, when it is convenient, we fix a linear basis $\{x_1, \ldots, x_n\}$ of V^* and identify V with $\mathbb{C}^n$ using the dual basis in V. Then in order to define a hyperplane H_i of $\mathcal{A}$ it suffices to fix a linear form $\alpha_i \in V^*$ such that $H_i = \ker(\alpha_i)$. This linear form is uniquely defined up to multiplication by a nonzero element of $\mathbb{C}$. We denote by $\mathcal{A}^* = \{\alpha_1, \ldots, \alpha_k\}$ the set of those linear forms and by $d_{\mathcal{A}} = \prod_{i=1}^{k} \alpha_i$ the homogeneous polynomial of degree k that also defines $\mathcal{A}$.

1.2 The intersection poset $L(\mathcal{A})$.

In order to define the Orlik-Solomon algebra of $\mathcal{A}$ we do not need to know the hyperplanes, it suffices to know the combinatorics of $\mathcal{A}$, i.e., its intersection poset $L(\mathcal{A})$. We will explain in details on this fact in Chapter 2.

Definition 1.2.1 *Let $\mathcal{A}$ be an arrangement and let $L = L(\mathcal{A})$ be the set of all nonempty subspaces of V that are intersections of some elements of $\mathcal{A}$. Define a* **partial order** *on L by*

$$X \leq Y \Longleftrightarrow Y \subseteq X$$

Note that

• V as the intersection of the empty set of hyperplanes of $\mathcal{A}$ is the unique minimal element of L.
• $T(\mathcal{A}) = \cap_{i=1}^{k} H_i$ is the unique maximal element of L because we consider only central arrangements.

Since each element of $\mathcal{A}^*$ is homogeneous, $T(\mathcal{A})$ contains $\mathbf{0}$.

Definition 1.2.2 *Define a rank function on L by $r(X) = codim X$. Thus $r(V) = 0$ and $r(H) = 1$ for every $H \in \mathcal{A}$. Call such an H an* **atom** *of L. Let $X, Y \in L$. Define their* **meet** *by*

$$X \wedge Y = \cap\{Z \in L \mid X \cup Y \subseteq Z\} \ .$$

If $X \cap Y \neq \emptyset$, we define their **join** *(the leats upper bound) by*

$$X \vee Y = X \cap Y \ .$$

The poset L has the following properties:

Lemma 1.2.3 *Let $\mathcal{A}$ be an arrangement and $L = L(\mathcal{A})$. Then*

1. *L is* **atomic**, *i.e., every element of $L \setminus \{V\}$ is a join of some atoms.*
2. *L is* **ranked**, *i.e., for every $X \in L$ all maximal linearly ordered subsets*

$$V = X_0 < X_1 < \ldots < X_r = X$$

   have the same cardinality, namely the codimension of X. Thus L is a geometric poset.

3. *All joins exist, so L is a lattice. For all $X, Y \in L$ the rank function satisfies*

$$r(X \wedge Y) + r(X \vee Y) \leq r(X) + r(Y) \ .$$

   Thus for a central arrangement, L is a geometric lattice.

Definition 1.2.4 *The* **rank** *of $\mathcal{A}$, $r(\mathcal{A})$, is the rank of the maximal element of $L(\mathcal{A})$: $T(\mathcal{A}) = \cap_{i=1}^{k} H_i$. We will call the n-arrangement $\mathcal{A}$* **essential** *if $r(\mathcal{A}) = n$.*

Clearly $r(\mathcal{A}) \leq n$ and $\mathcal{A}$ is essential if an only if it contains n linearly independent hyperplanes. For a central arrangement, this is equivalent to the condition $T(\mathcal{A}) = \{\mathbf{0}\}$.

Definition 1.2.5 *Let $L_p(\mathcal{A}) = \{X \in L \mid r(X) = p\}$. The* **Hasse diagram** *of L has vertices labeled by the elements of L and arranged on levels L_p for $p \geq 0$. Suppose $X \in L_p$ and $Y \in L_{p+1}$. An edge in the Hasse diagram connects X with Y if $X < Y$.*

1.3 Examples.

Example 1.3.1 Let $\mathcal{B}_0$ be the **Boolean** arrangement defined by

$$d_{\mathcal{B}_0} = x_1 x_2 \dots x_n$$

this is the arrangement of the coordinate hyperplanes in $\mathbb{C}^n$.

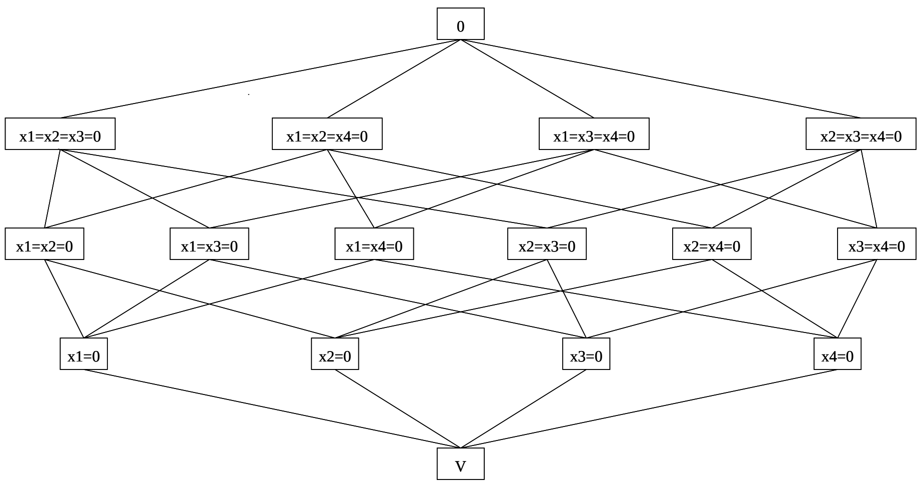

Figure 1.1: **The Hasse diagram of $\mathbf{d_{\mathcal{B}_0} = x_1 x_2 x_3 x_4}$.**

Example 1.3.2 Let $\mathcal{B}r$ be the **Braid** arrangement defined by

$$d_{\mathcal{B}r} = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

this is the arrangement such that $Y_{\mathcal{B}r} \stackrel{\mathbf{def}}{=} \mathbb{C}^n \setminus \bigcup_{1 \leq i < j \leq n} \ker(x_i - x_j)$ define the pure braid space contained in $\mathbb{C}^n$.

Example 1.3.3 Let $\mathcal{B}^+$ be the arrangement defined by

$$d_{\mathcal{B}^+} = \prod_{1 \leq i < j \leq n} (x_i + x_j)$$

The arrangements $\mathcal{B}_0$ and $\mathcal{B}^+$ are essential, $\mathcal{B}r$ is only central: $T(\mathcal{B}r)$ is the line $\{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_1 = x_2 = \dots = x_n\}$.

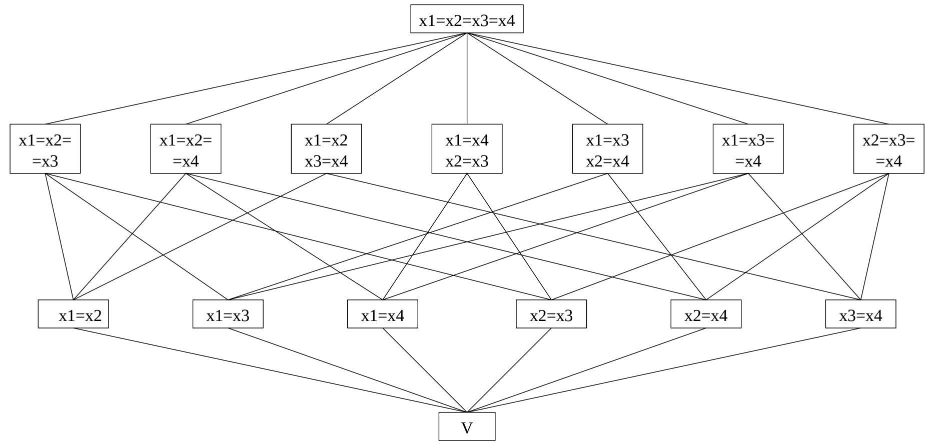

Figure 1.2: **The Hasse diagram of** $\mathbf{d}_{\mathcal{B}\mathbf{r}} = \prod_{\mathbf{1\le i<j\le 4}} (\mathbf{x_i} - \mathbf{x_j})$.

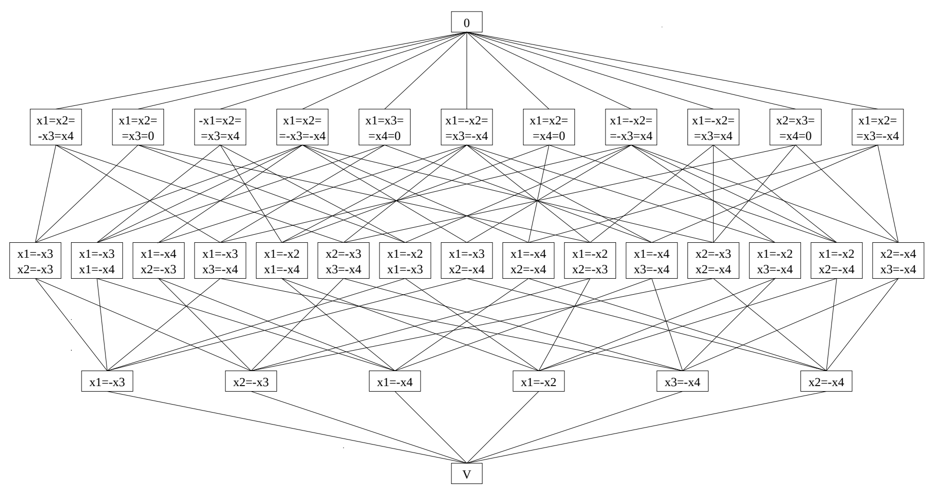

Figure 1.3: **The Hasse diagram of** $\mathbf{d}_{\mathcal{B}+} = \prod_{\mathbf{1\le i<j\le 4}} (\mathbf{x_1} + \mathbf{x_j})$.

1.4 Subarrangements.

Definition 1.4.1 *Let* $(\mathcal{A}, V)$ *be an arrangement. If* $\mathcal{M} \subseteq \mathcal{A}$ *is a subset, then* $(\mathcal{M}, V)$ *is called a* **subarrangement** *of* $\mathcal{A}$. *For* $X \in L(\mathcal{A})$ *define a subarrangement* $\mathcal{A}_X$ *of* $\mathcal{A}$ *by*

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}$$

Define an arrangement $(\mathcal{A}^X, X)$ *in* X *by*

$$\mathcal{A}^X = \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X \text{ and } X \cap H \neq \emptyset\}$$

Note that $A_V = \emptyset$ and if $X \neq V$, then $\mathcal{A}_X$ has center X in any arrangement. We call $\mathcal{A}^X$ the **restriction** of $\mathcal{A}$ to X. Note that $\mathcal{A}^V = \mathcal{A}$.

The method of **deletion and restriction** is a basic construction in [20], [21] to prove that the Orlik-Solomon algebra is isomorphic to the cohomology algebra of $Y_{\mathcal{A}} \stackrel{\text{def}}{=} V \setminus \cup_{H\in\mathcal{A}} H = V \setminus d_{\mathcal{A}}^{-1}(0)$, see Section 2.5. This method follows by induction in the cardinality of $\mathcal{A}$, and for this last one we give the following definition:

Definition 1.4.2 *Let* $\mathcal{A}$ *be a nonempty arrangement and let* $H \in \mathcal{A}$. *Let* $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ *and let* $\mathcal{A}'' = \mathcal{A}^H$. *We call* $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ *a* **triple of arrangements** *and* H *the* **distinguished** *hyperplane.*

1.5 The Möbius Function.

Definition 1.5.1 *Let* $\mathcal{A}$ *be an arrangement. Define the* **Möbius function** $\mu_{\mathcal{A}} = \mu : L \times L \to \mathbb{Z}$ *as follows:*

$$\begin{aligned} \mu(X,X) &= 1 && \text{if } X \in L, \\ \sum_{X \leq Z \leq Y} \mu(X,Z) &= 0 && \text{if } X, Y, Z \in L \text{ and } X < Y, \\ \mu(X,Y) &= 0 && \text{otherwise} \end{aligned}$$

Note that for fixed $X \in L$ the values of $\mu(X,Y)$ may be computed recursively. There are useful reformulations of $\mu(X,Y)$.

Lemma 1.5.2 (*see* [21]) *Let* $\mathcal{A}$ *be an arrangement. For* $X, Y \in L$ *with* $X \leq Y$, *let* $S(X,Y)$ *the set of central subarrangements* $\mathcal{M} \subseteq \mathcal{A}$ *such that* $\mathcal{A}_X \subseteq \mathcal{M}$ *and* $T(\mathcal{M}) = Y$. *Then*

$$\mu(X,Y) = \sum_{\mathcal{M} \in S(X,Y)} (-1)^{|\mathcal{M} \setminus \mathcal{A}_X|} \quad .$$

Definition 1.5.3 *Let* $\mathcal{A}$ *be an arrangement. Let* $ch(L)$ *be the set of all chains in* L*:*

$$ch(L) = \{(X_1, \ldots, X_p) \mid X_1 < \ldots < X_p\}$$

Let $ch[X,Y] = \{(X_1, \ldots, X_p) \in ch(L) \mid X_1 = X,\ X_p = Y\}$. *Denote the cardinality of* $c \in ch(L)$ *by* $|c|$.

Lemma 1.5.4 (*see* [21]) *For all* $X, Y \in L$

$$\mu(X,Y) = \sum_{c \in ch[X,Y]} (-1)^{|c|-1} \quad .$$

Theorem 1.5.5 (*see* [21]) *If* $X \leq Y$, *then* $\mu(X,Y) \neq 0$ *and* $sig\mu(X,Y) = (-1)^{r(X)-r(Y)}$.

1.5.1 The Function $\mu(X)$.

Definition 1.5.6 *For $X \in L$ define $\mu(X) = \mu(V, X)$.*

Clearly $\mu(V) = 1$, $\mu(H) = -1$, for all $H \in L$ and if $r(X) = 2$, then $\mu(X) = |\mathcal{A}_X| - 1$. In general is not possible to give a formula for $\mu(X)$.

Example 1.5.7 (*see [21]*) Consider the Boolean arrangement defined by $d_{\mathcal{B}_0} = x_1 x_2 \ldots x_n$. Then for $X \in L$: $\mu(X) = (-1)^{r(X)}$.

1.6 The Poincaré Polynomial.

Definition 1.6.1 *Let $\mathcal{A}$ be an arrangement with intersection poset L and Möbius function μ. Let t be an indeterminate. Define the Poincaré polynomial of $\mathcal{A}$ by*

$$Poin(\mathcal{A}, t) = \sum_{X \in L} \mu(X)(-t)^{r(X)} .$$

The Poincaré polynomial is one of the most important combinatorial invariants of an arrangement. It follows from Theorem 1.5.5 that $Poinc(\mathcal{A}, t)$ has nonnegative coefficients.

Example 1.6.2 Let $\mathcal{A}$ be the 3-arrangement defined by $d_{\mathcal{A}} = x_1 x_2 x_3 (x_1 + x_2 - x_3)$. Then

$$Poin(\mathcal{A}, t) = 1 + 4t + 6t^2 + 3t^3 = (1 + t)(1 + 3t + 3t^2) .$$

Example 1.6.3 The Poincaré polynomial of the Boolean arrangement $d_{\mathcal{B}0} = x_1 \ldots x_n$ is

$$Poin(\mathcal{B}_0, t) = \sum_{k=0}^{n} \binom{n}{k} t^k = (1 + t)^n .$$

Example 1.6.4 (*see [21]*) Let $\mathcal{B}r$ be the braid n-arrangement. Then

$$Poin(\mathcal{B}r, t) = (1 + t)(1 + 2t) \ldots (1 + (n - 1)t) .$$

Note that the factor $(1 + t)$ divides the Poincaré polynomial of every central arrangement (cf. [21, Proposition 2.54]), but more factors of the form $(1 + bt) \in \mathbb{Z}[t]$ do not exist in general.

We recall the following well known results (see [21]):

Lemma 1.6.5 *Let $\mathcal{A}$ be an arrangement. Then*

$$Poin(\mathcal{A}, t) = \sum_{\mathcal{M} \subseteq \mathcal{A}} (-1)^{|\mathcal{M}|} (-t)^{r(\mathcal{M})} ,$$

where the sum is over all central subarrangements $\mathcal{M}$ of $\mathcal{A}$.

Theorem 1.6.6 ($Deletion - Restriction$) *If $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ is a triple of arrangements, then*

$$Poin(\mathcal{A}, t) = Poin(\mathcal{A}', t) + tPoin(\mathcal{A}'', t) .$$

Chapter 2

The Orlik-Solomon algebra

In this chapter we associate to an arrangement $\mathcal{A}$ a graded anticommutative algebra $\mathbf{A}(\mathcal{A})$ over $\mathbb{C}$, which in the literature has become known as the **Orlik-Solomon algebra** . The algebra $A(\mathcal{A})$ was first defined in [20], where it was used to prove that for a complex arrangement $\mathcal{A}$, $A(\mathcal{A})$ is isomorphic as a graded algebra to the cohomology algebra of the complement $Y_{\mathcal{A}}$. This algebra is constructed using only $L(\mathcal{A})$.

2.1 Construction of the algebra $A(\mathcal{A})$.

Definition 2.1.1 *Let $\mathcal{A}$ be an arrangement over $\mathbb{C}$. Let $E_1 = \bigoplus_{H\in\mathcal{A}} \mathbb{C}e_H$ and let $E = E(\mathcal{A}) = \Lambda(E_1)$ be the exterior algebra of E_1.*

Note that E_1 has a $\mathbb{C}$-basis consisting of elements e_H, of degree 1, in one-to-one correspondence with the hyperplanes $H \in \mathcal{A}$. If we write $uv = u \wedge v$, then $e_H^2 = 0$, $e_H e_K = -e_K e_H$ for $H, K \in \mathcal{A}$. The algebra E is graded via

$$E = \bigoplus_{p=0}^{k} E_p,$$

where $E_0 = \mathbb{C}$, E_1 agrees with its earlier definition and $E_p = \Lambda^p E_1$ as $\mathbb{C}$-module is free and has the distinguished basis consisting of monomials $e_S = e_{i_1} \dots e_{i_p}$ where $S = \{i_1, \dots, i_p\}$ is running through all the subsets of $[k]$ of cardinality p, $i_1 < i_2 < \dots < i_p$, and e_{i_j} correspond to $H_{i_j} \in \mathcal{A}$: $e_{i_j} \stackrel{\mathbf{def}}{=} e_{H_{i_j}}$. Throughout we call this monomials **standard** and identify i_j with H_{i_j} .

Definition 2.1.2 *Define a $\mathbb{C}$-linear map $\partial = \partial_E : E \to E$ by $\partial 1 = 0$, $\partial e_H = 1$ and for every $S = \{i_1, \dots, i_p\} \subseteq [k]$ of cardinality $p \geq 2$*

$$\partial e_S = \partial(e_{i_1} \dots e_{i_p}) = \sum_{r=1}^{p} (-1)^{r-1} e_{i_1} \dots \widehat{e_{i_r}} \dots e_{i_p} = \sum_{r=1}^{p} (-1)^{r-1} e_{S_r} \ ,$$

where S_r is the complement in S to its r-th element.

Thus the graded algebra E is a differential graded algebra with respect to the differential ∂ of degree -1.

Recall two familiar properties of the exterior algebra.

Lemma 2.1.3 *The map $\partial : E \longrightarrow E$ satisfies*

1. $\partial^2 = 0$
2. *if $u \in E_p$ and $v \in E$, then $\partial(uv) = (\partial u)v + (-1)^p u(\partial v)$.*

We see from 1) that (E, ∂) is a chain complex. Part 2) says that ∂ is a derivation of the exterior algebra.

For every $S \subset [k]$, we denote $\cap S = \bigcap_{i \in S} H_i$. Since $\mathcal{A}$ is central, $\cap S \in L$ for all S. If $p = 0$, we agree that S is the empty set, $e_S = 1$ and $\cap S = V$. Since the rank function on L is codimension, it is clear that $r(\cap S) \leq |S|$.

Definition 2.1.4 *We call S* **independent** *if $r(\cap S) = |S|$ and* **dependent** *if $r(\cap S) < |S|$.*
Let $\mathcal{S}_p$ denote the set of all orderly p-tuples $(H_{i_1}, \ldots, H_{i_p})$ and let $\mathcal{S} = \cup_{p \geq 0} \mathcal{S}_p$.

Then the tuple S is independent if the corresponding linear forms $\alpha_{i_1}, \ldots, \alpha_{i_p}$ are linearly independent. Equivalently, the hyperplanes of S are in general position.

Definition 2.1.5 *Let $\mathcal{A}$ be an arrangement. The Orlik-Solomon (OS) ideal of $\mathcal{A}$ is the ideal $I = I(\mathcal{A})$ of E generated by ∂e_S for every dependent $S \in \mathcal{S}$.*

Clearly $I(\mathcal{A})$ is a graded ideal because it is generated by homogeneous elements. Let $I_p(\mathcal{A}) = I(\mathcal{A}) \cap E_p$. Since the elements of $\mathcal{S}_1$ are independent, $I_0 = 0$. The only dependent elements of $\mathcal{S}_2$ are of the form (H, H), so $e_S = e_H^2 = 0$ and we have $I_1 = 0$. Then

$$I(\mathcal{A}) = \bigoplus_{p \geq 2}^{k} I_p(\mathcal{A})$$

Definition 2.1.6 *Let $\mathcal{A}$ be an arrangement. The OS algebra of $\mathcal{A}$ is the graded algebra $A = A(\mathcal{A}) = E/I$. Let $\varphi : E \longrightarrow A$ be the natural homomorphism and let $A_p = \varphi(E_p)$. If $H \in \mathcal{A}$, let $a_H = \varphi(e_H)$ and if $S \in \mathcal{S}$, let $a_S = \varphi(e_S)$.*

Lemma 2.1.7 *If $S \in \mathcal{S}$ and $H \in S$, then $e_S = e_H \partial e_S$.*

Proof. If $H \in S$, then $e_H e_S = 0$. Thus $0 = \partial(e_H e_S) = e_S - e_H \partial e_S$. ■

In Definition 2.1.5, the set of generators can be made smaller.

Definition 2.1.8 *A p-tuple $S = (i_1, \dots, i_p) \subseteq [k]$ is a* **circuit** *if it is minimally dependent. Thus $(H_{i_1}, \dots, H_{i_p})$ is dependent, but for $1 \leq l \leq p$ the $(p-1)$-tuple $(H_{i_1}, \dots, \widehat{H_{i_l}}, \dots, H_{i_p})$ is independent.*

Lemma 2.1.9 *An OS ideal is generated by ∂e_T for every circuit $T \in \mathcal{S}$*

Proof. Let S be a dependent set and $T \subset S$ a circuit. Then $e_S = \pm e_T e_{S \setminus T}$. Using the Leibniz rule, Lemma 2.1.3(2), we have

$$\partial e_S = \pm \partial e_T e_{S\setminus T} \pm e_T \partial e_{S\setminus T}.$$

The result follows using Lemma 2.1.7 for the last term of the above sum. ■

Note that, by Lemma 2.1.7, I contains e_S for every dependent set S. This implies that A **is generated as a $\mathbb{C}$-module by the images of the e_S such that S is independent**.

Since $I_0 = 0$ we have $A_0 = \mathbb{C}$. Moreover the elements a_H are linearly independent over $\mathbb{C}$ because $I_1 = 0$. Hence $A_1 = \bigoplus_{H \in \mathcal{A}} \mathbb{C} a_H$. According to Definition 2.1.4, if $p > n$, then every element of $\mathcal{S}_p$ is dependent and it follows from the last observation that $A_p = 0$. Thus

$$A = \mathbb{C} \oplus \bigoplus_{H \in \mathcal{A}} \mathbb{C} a_H \oplus \bigoplus_{p=2}^{n} A_p$$

Example 2.1.10 (see [21]) Suppose $n = 2$ and $\mathcal{A} = \{H_1, \dots, H_k\}$. Write $a_i = a_{H_i}$. Then the OS algebra of $\mathcal{A}$ is

$$A(\mathcal{A}) = \mathbb{C} \oplus \bigoplus_{p=1}^{k} \mathbb{C} a_p \oplus \bigoplus_{p=1}^{k-1} \mathbb{C} a_p a_k.$$

We have computed A_0, A_1 and we know that $A_p = 0$ for $p > 2$. It remains to compute A_2. Since $\dim V = 2$, (H_i, H_j, H_l) is dependent for all (i, j, l). Thus I_2 contains the element

$$\partial(e_i e_j e_l) = e_j e_l - e_i e_l + e_i e_j = e_i e_j + e_j e_l + e_l e_i.$$

It follows that A_2 is spanned by $a_p a_q$ subject to the relations

$$a_i a_j + a_j a_l + a_l a_i = 0$$

for all (i, j, l). This shows that A_2 is spanned by $a_p a_k$ for $1 \leq p < k$. It remains to show that the sum is direct. Suppose $\sum_{p=1}^{k-1} c_p a_p a_k = 0$ with $c_p \in \mathbb{C}$. Then $\sum_{p=1}^{k-1} c_p e_p e_k \in I_2$. Recall that I_2 is spanned by the elements $\partial(e_i e_j e_l)$. Since $\partial\partial = 0$, we have $\partial I_2 = 0$ and hence

$$\partial(\sum_{p=1}^{k-1} c_p e_p e_k) = \sum_{p=1}^{k-1} c_p (e_k - e_p) = 0.$$

Since $e_1, \dots, e_k$ are linearly independent over $\mathbb{C}$, we get that $c_k = 0$ for all p.

Example 2.1.11 If $\mathcal{A}$ is the Boolean arrangement, then $S = (H_1, \dots, H_p)$ is independent if and only if $H_1, \dots, H_p$ are distint hyperplanes. Hence if S is dependent, then $e_S = 0$. Thus $I = 0$ and $A = E$.

2.2 A($\mathcal{A}$) is an acyclic complex.

Lemma 2.2.1 *$\partial I \subseteq I$.*

Proof. Since I is generated over $\mathbb{C}$ by elements of the form $e_T \partial e_S$, where $T, S \in \mathcal{S}$ and S is dependent, using the Leibniz rule we have

$$\partial(e_T \partial e_S) = \partial e_T \partial e_S \in I$$

whence I is invariant with respect to ∂. ∎

Now we can give the following

Definition 2.2.2 *Since $\partial_E I \subset I$, we may define $\partial_A : A \longrightarrow A$ by $\partial_A \varphi(u) = \varphi \partial_E(u)$ for $u \in E$*

Thus A receives a piece of structure from E, ∂_A defines the structure of non-commutative differential graded algebra on A. We have the following

Lemma 2.2.3 *The map $\partial_A : A \longrightarrow A$ satisfies*

1. *$\partial_A^2 = 0$,*
2. *if $a \in A_p$ and $b \in A$, then $\partial_A(ab) = (\partial_A a)b + (-1)^p a(\partial_A b)$,*
3. *if $\mathcal{A}$ is not empty, then the chain complex (A, ∂_A) is acyclic.*

Proof. Parts 1. and 2. follow from the corresponding facts for ∂_E .
Since ∂_A is homogeneous of degree -1, (A, ∂_A) is a chain complex.
It follows from 1. that $\mathrm{Im}\partial_A \subset \ker \partial_A$. To prove that the complex is acyclic we must show the reverse inclusion. Since $\mathcal{A}$ is not empty, we may choose $H \in \mathcal{A}$. Let $v = e_H$, $b = \varphi(v)$ and let $a \in A$. Choose $u \in E$ with $\varphi(u) = a$. Then $\partial_E(vu) = (\partial_E v)u - v(\partial_E u) = u - v(\partial_E u)$. Applying the $\mathbb{C}$-algebra homomorphism φ to the first and last terms gives $a = \partial_A(ba) + b\partial_A a$ for all $a \in A$. Thus $\mathrm{Im}\partial_A \supset \ker \partial_A$. ∎

2.3 The Structure of $A(\mathcal{A})$.

We decompose the algebra E into a direct sum indexed by elements of L whence we have a finest grading on E, the grading by the Boolean poset of all subsets of $[k]$.

Definition 2.3.1 *For $X \in L$ let $\mathcal{S}_X = \{S \in \mathcal{S} : \cap S = X\}$ and let*

$$E_X = \sum_{S \in \mathcal{S}_X} \mathbb{C} e_S .$$

Lemma 2.3.2 *Since $\mathcal{S} = \cup_{X \in L} \mathcal{S}_X$ is a disjoint union, $E = \oplus_{X \in L} E_X$ is a direct sum.*

Notice that this grading is in general incomparable with the standard grading by rk$(\cap S)$.

The algebra A has an analogous direct sum decomposition:

Definition 2.3.3 *If $X \in L$, let $A_X = \varphi(E_X)$.*

Theorem 2.3.4 *Let $\mathcal{A}$ be an arrangement and let $A = A(\mathcal{A})$. Then*

$$A = \bigoplus_{X \in L} A_X$$

and this grading is finer than the standard grading $A = \bigoplus_{p=0}^{n} A_p$.

Proof. Clearly any e_S is homogeneous. If T is a circuit in $[k]$ and $\cap T = X$ then $\cap T_i = X$ for every $i \in T$. Thus $\partial e_T \in E_X$. Let $I_X = I \cap E_X$. Using Lemma 2.1.9, this shows that $I = \oplus_{X \in L} I_X$. Thus $A = \oplus_{X \in L} A_X$.
The second statement follows from the fact that A is generated as $\mathbb{C}$-module by the images of e_S with S independent. For such an S we have $rk(\cap S) = |S|$. This shows that

$$A_p = \bigoplus_{X \in L_p} A_X.$$

■

2.3.1 Filtration of $A(\mathcal{A})$ by $L(\mathcal{A})$.

The above grading of A by L induces a filtration of A that can also be defined independently. Among subarrangements of an arrangement $\mathcal{A}$ there are the ones corresponding to elements of L. Recall that all the subarrangements $\mathcal{A}_X$ are central for every $X \in L$. They can be completely characterized also by the property of being closed. This means that with several hyperplanes they contain all hyperplanes dependent of them.

We want to show that the graded algebras $A(\mathcal{A}_X)$ form a filtration of $A(\mathcal{A})$. First, if $\widetilde{\mathcal{A}}$ is a subarrangement of $\mathcal{A}$, then we view $E(\widetilde{\mathcal{A}})$ as a subalgebra of $E(\mathcal{A})$ generated by all the e_H with $H \in \widetilde{\mathcal{A}}$, and $L(\widetilde{\mathcal{A}})$ as a sublattice of $L(\mathcal{A})$. Note that $\mathcal{S}(\widetilde{\mathcal{A}}) \subseteq \mathcal{S}(\mathcal{A})$ and an element $S \in \mathcal{S}(\widetilde{\mathcal{A}})$ is dependent viewed in $\mathcal{S}(\widetilde{\mathcal{A}})$ if and only if it is dependent in $\mathcal{S}(\mathcal{A})$. Notice that the map $\partial_{E(\widetilde{\mathcal{A}})}$ is the restriction of $\partial_{E(\mathcal{A})}$ to $E(\widetilde{\mathcal{A}})$, and $A(\widetilde{\mathcal{A}}) = E(\widetilde{\mathcal{A}})/I(\widetilde{\mathcal{A}})$. Clearly

$$I(\widetilde{\mathcal{A}}) \subseteq I(\mathcal{A}) \cap E(\widetilde{\mathcal{A}}) \qquad (2.1)$$

Definition 2.3.5 *Let $\widetilde{\mathcal{A}}$ be a subarrangement of $\mathcal{A}$. Since $I(\widetilde{\mathcal{A}}) \subseteq I(\mathcal{A}) \cap E(\widetilde{\mathcal{A}})$, the inclusion $E(\widetilde{\mathcal{A}}) \subseteq E(\mathcal{A})$ induces a $\mathbb{C}$-algebra homomorphism*
$\iota : A(\widetilde{\mathcal{A}}) \longrightarrow A(\mathcal{A})$ *such that for $H \in \widetilde{\mathcal{A}}$*

$$\iota(e_H + I(\widetilde{\mathcal{A}})) = e_H + I(\mathcal{A}).$$

Note that ι is a monomorphism precisely when in (2.1) holds the equality.

Lemma 2.3.6 *For every $X \in L(\mathcal{A})$ holds $I(\mathcal{A}_X) = I(\mathcal{A}) \cap E(\mathcal{A}_X)$.*

The next result follows from Lemma 2.3.6.

Proposition 2.3.7 *The map ι is a monomorphism for $\widetilde{\mathcal{A}} = \mathcal{A}_X$.*

Corollary 2.3.8 *The correspondence $X \longmapsto A(\mathcal{A}_X)$, $X \in L$, defines a monotone map of L to the poset of graded subalgebras of A ordered by inclusion, i.e., a filtration of A.*

Proposition 2.3.9 *The filtration $\{A(\mathcal{A}_X)\}_{X\in L}$ is induced by the grading $A = \oplus_{Y\in L} A_Y(\mathcal{A})$. More precisely, $A_Y(\mathcal{A}_X) = A_Y(\mathcal{A})$ for every $X, Y \in L$ such that $Y \leq X$, whence $A(\mathcal{A}_X) = \oplus_{Y\leq X} A_Y(\mathcal{A})$.*

Proof. Let $\iota : A(\mathcal{A}_X) \longrightarrow A(\mathcal{A})$ be the monomorphism of Proposition 2.3.7. The module $A_Y(\mathcal{A}) = \varphi(E_Y(\mathcal{A}))$ is spanned over $\mathbb{C}$ by all elements $e_S + I(\mathcal{A})$ with $S \in \mathcal{S}_Y(\mathcal{A})$. Similarly $A_Y(\mathcal{A}_X)$ is spanned over $\mathbb{C}$ by all elements $e_S + I(\mathcal{A}_X)$ with $S \in \mathcal{S}_Y(\mathcal{A}_X)$. Since $\mathcal{S}_Y(\mathcal{A}) = \mathcal{S}_Y(\mathcal{A}_X)$, we have $\iota(A_Y(\mathcal{A}_X)) = A_Y(\mathcal{A})$. Since ι is a monomorphism, this completes the proof. ∎

2.4 Gröbner basis for OS ideals.

Recall that we fixed an arbitrary linear order on an arrangement $\mathcal{A}$. This order induces the degree lexicografic order (**deg-lex**) $\prec$ on the set of all standard monomials e_S of E :
If $S = (i_1, \dots, i_p)$, $T = (j_1, \dots, j_q) \in \mathcal{S}$ where $i_1 < \dots < i_p$ and $j_1 < \dots < j_q$ then

$$e_S \prec e_T \iff \begin{cases} p < q \\ \text{or } p = q \text{ and } e_S \prec_{lex} e_T \ . \end{cases}$$

The basis of E consisting of standard monomials is multiplicative up to $\pm$, i.e., the product of two standard monomials is either 0 or a standard monomial perhaps with the negative sign; and the deg-lex order is multiplicative, i.e., invariant under multiplication by monomials in the same sense as above. Thus we can apply theory of Gröbner basis to the ideal $I = I(\mathcal{A})$, and hence show that the $\mathbb{C}$-algebra $A(\mathcal{A})$ is a free $\mathbb{C}$-module by constructing a standard $\mathbb{C}$-basis for $A(\mathcal{A})$.

Before our principal statement we recall and give some definitions.

Definition 2.4.1 *A standard p-tuple $S \in \mathcal{S}$ is a* **broken circuit** *if there exists $H \in \mathcal{A}$ such that $H \prec H_j$ for all $j \in S$ and (H, S) is a circuit.*

It is clear that every broken circuit is obtained by deleting the minimal element in a standard circuit, and every broken circuit is independent.

Definition 2.4.2 *A standard p-tuple S is called* **not broken circuit** (**nbc**) *if it does not contain any broken circuit. Define*

$$\mathcal{C}_p := \{S \in \mathcal{S}_p \mid S \text{ is standard and nbc}\}$$

Let $\mathcal{C} = \cup_{p\geq 0}\mathcal{C}_p$.

Note that if S is a nbc, then S is independent.

Definition 2.4.3 *Let $C = C(\mathcal{A})$ be the $\mathbb{C}$-module defined as follows. Let $C_0 = \mathbb{C}$, and for $p \geq 1$ let C_p be the free $\mathbb{C}$-module with basis $\{e_S \in E \mid S \in \mathcal{C}_p\}$. Let $C = C(\mathcal{A}) = \bigoplus_{p\geq 0} C_p$. Then $C(\mathcal{A})$ is a free graded $\mathbb{C}$-module.*

By definition, $C(\mathcal{A})$ is a submodule of $E(\mathcal{A})$ but in general $C(\mathcal{A})$ is not closed under multiplication in $E(\mathcal{A})$, so $C(\mathcal{A})$ is not a subalgebra.

Definition 2.4.4 *Let $C_X = C_X(\mathcal{A}) = C \cap E_X$. Then each C_X is a free $\mathbb{C}$-module for every $X \in L(\mathcal{A})$.*

Notice that since C is spanned by monomials it is naturally graduated by $[k]$ and $L(\mathcal{A})$. Since if $S \in \mathcal{C}_p$, S is independent, the latter grading is finer than the former, i.e., $C_p = \bigoplus_{X\in L_p} C_X$ for every $0 \leq p$, and hence $C = \bigoplus_{X\in L} C_X$.

Lemma 2.4.5 *Let H_1 be the minimal element of $\mathcal{A}$ and write $e_1 = e_{H_1}$. Then $e_1 C \subseteq C$, so C is closed under multiplication by e_1.*

Proof. Since a broken circuit is obtained from a standard circuit by deleting the minimal element, no broken circuit has the form (H_1, S). ∎

Lemma 2.4.6 *Suppose $\mathcal{A}$ is not empty. Let ∂_C denote the restriction of the map $\partial : E \longrightarrow E$ to C. Then $\partial_C(C) \subset C$ and (C, ∂_C) is an acyclic complex.*

Proof. Deleting an element of a nbc p-tuple result a nbc $(p-1)$-tuple. This shows that $\partial_C(C) \subset C$. It is clear that $Im\partial_C \subseteq \ker(\partial_C)$ because $\partial_C^2 = (\partial_E|_C)^2$. Now, suppose $c \in C$ and $\partial_C c = 0$. By the Lemma 2.4.5 $e_1 c \in C$ and $c = c - e_1(\partial_C c) = \partial_C(e_1 c) \in \partial_C C$. This shows that the complex is acyclic and if $X \in L_p$:

$$\partial_C(C_X) \subset \bigoplus_{\substack{Y<X \\ Y\in L_{p-1}}} C_Y \ .$$

∎

Lemma 2.4.7 *For every $X \in L(\mathcal{A}) \setminus \{\mathbb{C}^n\}$ the restriction of ∂ to C_X is injective.*

Proof. Let i be the minimal element of $[k]_X \stackrel{\mathbf{def}}{=} \{i \in [k] \mid \exists\, S \in \mathcal{S}_X \text{ and } i \in S\}$. It follows from definition of C_X that $i \in S$ for every $e_S \in C_X$ whence $e_i C_X = 0$. Thus $e_i \partial c = c - \partial(e_i c) = c$ for every $c \in C_X$. This shows that ∂ restrict to C_X is injective. ■

Theorem 2.4.8 *Let* $B = \{\partial e_S \mid \text{S is a circuit}\}$. *Then* B *is a Gröbner basis of* I.

Proof. Recall that the initial monomials of elements from B are their largest monomials in the deg-lex order. Thus $in_{\prec}(\partial e_S) = e_{S_1}$ where $S_1 = (i_2, \ldots, i_p)$ if $S = (i_1, i_2, \ldots, i_p)$, whence $In(B) \stackrel{\mathbf{def}}{=} \{in_{\prec}(\partial e_S) \mid \partial e_S \in B\}$ correspond to the broken circuits. Then the statement of our Theorem, by a known fact of the Gröbner basis theory, means that

$$In(I) = \langle In(B) \rangle \tag{2.2}$$

Now, the natural linear complement to $\langle In(B)\rangle$ $(In(I))$ namenaly the free $\mathbb{C}$-module spanned by all the monomials of E not divisible by any element of $In(B)$ (resp. no in $In(I)$) is C, the module of the Definition 2.4.3, (is denoted by $\overline{C}$). Clearly

$$E = I \bigoplus \overline{C}$$

(as $\mathbb{C}$-modules) whence the restriction of φ (see Definition 2.1.6) to $\overline{C}$ is a linear isomorphism $\overline{C} \longrightarrow A$. Since $\langle In(B)\rangle \subset In(I)$ it is always true that $\varphi(C) = A$ and (2.2) is equivalent to

$$\ker(\varphi_{|C}) = 0$$

i.e., nbc-monomials are independent in A.
Since φ is homogeneous with respect to the grading by $L(\mathcal{A})$ it is sufficient to prove that φ restrict to C_X is injective for every $X \in L(\mathcal{A})$. We use induction on $r(X)$:

- If $r(X) = 0$, i.e., $X = V$ then $C_X = \mathbb{C} = A_X$ and φ restricted to C_X is the identity map.
- Suppose $r(X) = r > 0$. Consider the commutative diagram

$$\begin{array}{ccc} C_X & \xrightarrow{\partial_{|C_X}} & C_{r-1} \\ \big\downarrow{\scriptstyle \varphi_{|C_X}} & & \big\downarrow{\scriptstyle \varphi_{|C_{r-1}}} \\ A_X & \xrightarrow{\partial_{A_X}} & A_{r-1} \end{array}$$

By the Lemma 2.4.7, ∂ is injective on C_X. Also the restriction of φ is injective on C_{r-1} by the inductive hypothesis. Thus φ is injective on C_X, which completes the proof. ■

Corollary 2.4.9 *The algebra $A(\mathcal{A})$ is a free graded $\mathbb{C}$-module. The $\mathbb{C}$-modules $A_X(\mathcal{A})$ for $X \in L$ and $A_p(\mathcal{A})$ for $p \geq 0$ are also free. Moreover the set*

$$\{e_S + I \mid S \text{ is standard and nbc}\}$$

is a basis for $A(\mathcal{A})$ as a graded $\mathbb{C}$-module.

Proof. The $\mathbb{C}$-modules $C_X(\mathcal{A})$ are free by definition. It follows from the Theorem 2.4.8 that $C_X(\mathcal{A}) \cong A_X(\mathcal{A})$. Thus $A_X(\mathcal{A})$ is a free $\mathbb{C}$-module. Since $A_p = \bigoplus_{X \in L_p} A_X$, it is also free. The remaining assertions follow from the facts that $C = \bigoplus_{X \in L} C_X$, $A = \bigoplus_{X \in L} A_X$ and the Theorem 2.4.8. ∎

We closed this section with a Theorem that connect the Orlik-Solomon algebras to a triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$.

Theorem 2.4.10 (see [21]) *Let $\mathcal{A}$ be an arrangement. Let $H_1 \in \mathcal{A}$ and let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be the corresponding triple. Let $i : A(\mathcal{A}') \longrightarrow A(\mathcal{A})$ be the natural homomorphism and let $j : A(\mathcal{A}) \longrightarrow A(\mathcal{A}'')$ be the $\mathbb{C}$-linear map defined by*

$$\begin{aligned} j(a_{H_{i_1}} \dots a_{H_{i_p}}) &= 0 \ , \\ j(a_{H_1} a_{H_{i_1}} \dots a_{H_{i_p}}) &= a_{H_1 \cap H_{i_1}} \dots a_{H_1 \cap H_{i_p}} \end{aligned}$$

for $(H_{i_1} \dots H_{i_p}) \in \mathcal{S}(\mathcal{A}')$, where $1 < i_1 < \dots < i_p \leq k$. Then the following sequence is exact:

$$0 \longrightarrow A(\mathcal{A}') \xrightarrow{i} A(\mathcal{A}) \xrightarrow{j} A(\mathcal{A}'') \longrightarrow 0 \ .$$

2.5 Differential Forms.

In this section we study the algebra $R(\mathcal{A})$ of differential forms generated by 1 and the differential forms $\omega_H = d\alpha_H/\alpha_H$ for $H \in \mathcal{A}$. This algebra was first computed by Arnold [1] for the braid arrangement. Brieskorn [6] defined it for all arrangements and showed that it is isomorphic to the cohomology algebra. Its isomorphism with $A(\mathcal{A})$ was established by Orlik-Solomon [20] for central arrangements. Here we show the isomorphism $A(\mathcal{A}) \cong R(\mathcal{A})$ by induction by means of the deletion and restriction method.

2.5.1 The de Rham Complex.

Let $(\mathcal{A}, V)$ be a central arrangement. Let S be the symmetric algebra of V^* and let F be the quotient field of S. Recall that we have chosen a basis $x_1, \dots, x_n$ for V^* so we get $S \cong \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \dots, x_n]$ and $F \cong \mathbb{C}(\mathbf{x}) = \mathbb{C}(x_1, \dots, x_n)$. We view $F \otimes_{\mathbb{C}} V^*$ as a vector space over F by defining $f(g \otimes \alpha) = fg \otimes \alpha$ where $f, g \in F$ and $\alpha \in V^*$. There exists a unique $\mathbb{C}$-linear map $d : F \longrightarrow F \otimes V^*$ such that $d(fg) = f(dg) + g(df)$ for $f, g \in F$ and $d\alpha \in \mathbb{C}$ for $\alpha \in V^*$. In terms of the above basis, the differential *df* is given by the usual formula

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \otimes x_i = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

Note that $F \otimes V^* = Fdx_1 \oplus \ldots \oplus Fdx_n$.

Definition 2.5.1 *Let $\Omega(V)$ be the exterior algebra of the F-vector space $F \otimes V^*$ graded by $\Omega(V) = \bigoplus_{p=0}^{n} \Omega^p(V)$ where*

$$\Omega^p(V) = \bigoplus_{1 \leq i_1 < \ldots < i_p \leq n} Fdx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

We write $\omega\eta = \omega \wedge \eta$ for $\omega, \eta \in \Omega(V)$, and identify Ω^0 with F. The elements of $\Omega^p(V)$ are called rational differential p-forms on V. We list some well-known properties of d.

Proposition 2.5.2 *The map $d : F \longrightarrow F \otimes V^*$ may be extended in an unique way to a $\mathbb{C}$-linear map $d : \Omega(V) \longrightarrow \Omega(V)$ with the following properties :*

1. $d^2 = 0$,
2. *if $\omega \in \Omega^p(V)$ and $\eta \in \Omega(V)$, then $d(\omega\eta) = (d\omega)\eta + (-1)^p\omega(d\eta)$,*
3. *if $\omega = \sum f_{i_1 \ldots i_p} dx_{i_1} \ldots dx_{i_p}$ where $1 \leq i_1 < \ldots < i_p \leq n$ and $f_{i_1 \ldots i_p} \in F$, then*

$$d\omega = \sum_{j=1}^{n} \sum (\partial f_{i_1 \ldots i_p} / \partial x_j) dx_j dx_{i_1} \ldots dx_{i_p} \ .$$

2.5.2 The Algebra R($\mathcal{A}$).

Definition 2.5.3 *Let $\mathcal{A}$ be an arrangement. For $H \in \mathcal{A}$, let $w_H = d\alpha_H / \alpha_H \in \Omega^1(V)$. Let $R = R(\mathcal{A})$ be the $\mathbb{C}$-subalgebra of $\Omega(V)$ generated by 1 and ω_H for $H \in \mathcal{A}$.*

Let $R_p = R \cap \Omega^p(V)$. Since R is generated by 1 and the 1-forms ω_H, it is naturally graded $R = \bigoplus_{p=0}^{n} R_p$.

Example 2.5.4 (see Example 2.1.10) Let $\mathcal{A} = \{H_1, \ldots, H_k\}$ be a central 2-arrangement. Write $\omega_i = w_{H_i}$. Then

$$R(A) = \mathbb{C} \oplus \bigoplus_{i=1}^{k} \mathbb{C}\omega_i \oplus \bigoplus_{i=1}^{k-1} \mathbb{C}\omega_i\omega_k \ .$$

We know that $R_0 = \mathbb{C}$ and that $R_p = 0$ for $p > 2$. By definition $\omega_1, \ldots, \omega_k$ span R_1 over $\mathbb{C}$. These 1-forms are linearly independent over $\mathbb{C}$ because the rational functions $1/\alpha_1, \ldots, 1/\alpha_k$ are linearly independent over $\mathbb{C}$. Since $\omega_i^2 = 0$ and $\omega_i\omega_j = -\omega_j\omega_i$, the space R_2 is spanned over $\mathbb{C}$ by the $\omega_i\omega_j$ with $i < j$. In order to discover the remaining relations among these generators, let x, y be a basis for V^* and write $\alpha_i = a_i x + b_i y$ with $a_i, b_i \in \mathbb{C}$. Then $\omega_i = (a_i/\alpha_i)dx + (b_i/\alpha_i)dy$ and we have

$$d\alpha_i d\alpha_j = (a_i b_j - b_i a_j) dx dy$$

Thus for any i, j, l we have

$$\alpha_l d\alpha_i d\alpha_j + \alpha_i d\alpha_j d\alpha_l + \alpha_j d\alpha_l d\alpha_i = det \begin{bmatrix} a_i & a_j & a_l \\ b_i & b_j & b_l \\ \alpha_i & \alpha_j & \alpha_l \end{bmatrix} dx dy = 0$$

because the third row is a linear combination of the first two. If we multiply this equation by $1/(\alpha_l \alpha_i \alpha_j)$ we get

$$\omega_i \omega_j + \omega_j \omega_l + \omega_l \omega_i = 0$$

In particular, we have $\omega_i \omega_j = \omega_i \omega_k - \omega_j \omega_k$ if $1 \leq i < j \leq k$, so R_2 is spanned by the elements $\omega_i \omega_k$ for $1 \leq i < k$. It remains to show that these elements are linearly independent over $\mathbb{C}$. Define an F-linear map $\partial : \Omega^2(V) \longrightarrow \Omega^1(V)$ by $\partial(f dx dy) = f x dy - f y dx$. Then $\partial(\omega_i \omega_j) = \omega_j - \omega_i$. If $\sum_{i=1}^{k-1} c_i \omega_i \omega_k = 0$ with $c_i \in \mathbb{C}$, then applying ∂ gives $\sum_{i=1}^{k-1} c_i(\omega_i - \omega_k) = 0$. Since $\omega_1, \ldots, \omega_k$ are linearly independent over $\mathbb{C}$, we get $c_1 = \ldots = c_{k-1} = 0$. This proves the assertion.

Lemma 2.5.5 (see [20], [21]) *There exists a surjective homomorphism* $\gamma : A(\mathcal{A}) \longrightarrow R(\mathcal{A})$ *of graded* $\mathbb{C}$*-algebras such that* $\gamma(a_H) = \omega_H$ *for all* $H \in \mathcal{A}$.

2.5.3 Deletion and Restriction.

Let $\mathcal{A}$ be a nonempty arrangement, let $H_1 \in \mathcal{A}$, and let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be the inductive triple with respect to H_1. Note that $R(\mathcal{A}')$ and $R(\mathcal{A})$ are both subalgebras of $\Omega(V)$ and that $R(\mathcal{A}') \subset R(\mathcal{A})$. We shall see that there is a short exact sequence of $\mathbb{C}$-modules

$$0 \longrightarrow R(\mathcal{A}') \xrightarrow{i} R(\mathcal{A}) \xrightarrow{j} R(\mathcal{A}'') \longrightarrow 0 \ .$$

We define the map j with the help of the Leray residue map on differential forms. Let $\alpha_1 = \alpha_{H_1}$ and let $S_{(\alpha_1)}$ be the localization of S at the prime ideal (α_1). By definition, $S_{(\alpha_1)}$ is the subring of F consisting of all f/g such that $f, g \in S$ and g is prime to α_1. Let $\rho : V^* \longrightarrow H_1^*$ be the restriction map and let $y_i = \rho(x_i)$. Let $\mathbb{C}(H_1)$ be the subfield of F associated to H_1^*, then we may extended ρ uniquely to a $\mathbb{C}$-algebra homomorphism $\rho : S_{(\alpha_1)} \longrightarrow \mathbb{C}(H_1)$. Both existence and uniqueness follow from the formula

$$\rho(f/g) = f(y_1, \ldots, y_n)/g(y_1, \ldots, y_n) \ .$$

Note that $g(y_1, \ldots, y_n) \neq 0$ because g is prime to α_1. Define a $\mathbb{C}$-subalgebra Ω_1 of $\Omega(V)$ by

$$\Omega_1 = \bigoplus_{p=0}^{n} \bigoplus_{i_1 < \ldots < i_p} S_{(\alpha_1)} dx_{i_1} \ldots dx_{i_p} \ .$$

This subalgebra does not depend on the basis for V^*.

Lemma 2.5.6 *The map $\rho : S_{(\alpha_1)} \longrightarrow \mathbb{C}(H_1)$ may be extended in a unique way to a $\mathbb{C}$-linear map $\rho : \Omega_1 \longrightarrow \Omega(H_1)$ such that for $\omega, \eta \in \Omega_1$, $f \in S_{(\alpha_1)}$, and $\beta \in V^*$ we have*

1. $\rho(\omega\eta) = \rho(\omega)\rho(\eta)$,
2. $\rho(f\omega) = \rho(f)\rho(\omega)$,
3. $\rho(d\beta) = d\rho(\beta)$,
4. *If* $\omega = \sum f_{i_1,\ldots,i_p} dx_{i_1} \ldots dx_{i_p}$, *then*

$$\rho(\omega) = \sum f_{i_1,\ldots,i_p}(y_1, \ldots, y_n) dy_{i_1} \ldots dy_{i_p} \ .$$

Lemma 2.5.7 *Suppose $\beta \in V^* \setminus \{0\}$. If $\omega \in \Omega_1$ and $(d\beta)\omega = 0$, then there exists $\psi \in \Omega_1$ with $\omega = (d\beta)\psi$.*

Proof. We can choose a basis $x_1, \ldots, x_n$ for V^* such that $\beta = x_1$. Assume that ω is a p-form : $\omega = \sum f_{i_1 \ldots i_p} dx_{i_1} \ldots dx_{i_p}$ where $f_{i_1 \ldots i_p} \in S_{\alpha_1}$ and the sum is over all $1 \le i_1 < \ldots < i_p \le n$. Then

$$0 = (dx_1)\omega = \sum f_{i_1 \ldots i_p} dx_1 dx_{i_1} \ldots dx_{i_p}$$

where the sum is over all $2 \le i_1 < \ldots < i_p \le n$. Thus $f_{i_1 \ldots i_p} = 0$ if $i_1 \ge 2$. ∎

Definition 2.5.8 *Say that $\phi \in \Omega(V)$ has at most a* **simple pole** *along H_1 if $\alpha_1\phi \in \Omega_1$.*

Lemma 2.5.9 *Suppose $\phi \in \Omega(V)$ has at most a simple pole along H_1 and that $d\phi = 0$. Then there exists $\psi, \theta \in \Omega_1$ such that*

$$\phi = (d\alpha_1/\alpha_1)\psi + \theta \ .$$

The form $\rho(\psi) \in \Omega(H_1)$ is uniquely determined by ϕ .

Proof. Since $d\phi = 0$, it follows from Proposition 2.5.2(2) that

$$d(\alpha_1\phi) = (d\alpha_1)\phi - \alpha_1 d\phi = (d\alpha_1)\phi$$

Since $\alpha_1\phi \in \Omega_1$ by hypothesis and $d\Omega_1 \subset \Omega_1$, it follows from Lemma 2.5.7 that there exists $\theta \in \Omega_1$ such that $d(\alpha_1\phi) = (d\alpha_1)\theta$. Thus $(d\alpha_1)\phi = (d\alpha_1)\theta$, which implies $(d\alpha_1)\alpha_1(\phi - \theta) = 0$. Since $\alpha_1(\phi - \theta) \in \Omega_1$, it follows from Lemma 2.5.7 that there exists $\psi \in \Omega_1$ such that $\alpha_1(\phi - \theta) = (d\alpha_1)\psi$. This proves the existence of θ and ψ .
To prove the uniqueness of $\rho(\psi)$, it suffices to show that if $\psi, \theta \in \Omega_1$ and $(d\alpha_1/\alpha_1)\psi + \theta = 0$, then $\rho(\psi) = 0$. First note that $(d\alpha_1)\theta = 0$. It follows from Lemma 2.5.7 that there exists $\theta' \in \Omega_1$ such that $\theta = (d\alpha_1)\theta'$. Now $(d\alpha_1)(\psi + \alpha_1\theta') = (d\alpha_1)\psi + \alpha_1\theta = 0$. Since $\psi + \alpha_1\theta' \in \Omega_1$, we may apply Lemma 2.5.7 again to conclude that there exists $\theta'' \in \Omega_1$ with $\psi + \alpha_1\theta' = (d\alpha_1)\theta''$. Since $\rho(\alpha_1) = 0$, it follows from Lemma 2.5.6 that $\rho(\alpha_1\theta') = 0$ and $\rho((d\alpha)\theta'') = 0$. Thus $\rho(\psi) = 0$. ∎

Definition 2.5.10 *The uniquely determined form $\rho(\psi)$ is called the* **residue** *of ϕ along H_1. We denote it* $\mathrm{res}(\phi)$.

If $H \in \mathcal{A}$, then $d\omega_H = 0$, so $d(\omega_{H_{i_1}} \dots \omega_{H_{i_p}}) = 0$ for all $H_{i_1}, \dots, H_{i_p} \in \mathcal{A}$. Thus $d\phi = 0$ for all $\phi \in R(\mathcal{A})$. It is clear from the definition that each $\phi \in R(\mathcal{A})$ has at most a simple pole along H_1. Thus $\mathrm{res}(\phi)$ is defined for all $\phi \in R(\mathcal{A})$.

Lemma 2.5.11 (see [21]) *Suppose $H_{i_1}, \dots, H_{i_p} \in \mathcal{A}'$, where $1 < i_1 < \dots < i_p \leq k$. Then*

1. $\mathrm{res}(\omega_{H_{i_1}} \dots \omega_{H_{i_p}}) = 0$,
2. $\mathrm{res}(\omega_{H_1}\omega_{H_{i_1}} \dots \omega_{H_{i_p}}) = \omega_{H_1 \cap H_{i_1}} \dots \omega_{H_1 \cap H_{i_p}}$,
3. $\mathrm{res} R(\mathcal{A}) \subseteq R(\mathcal{A}'')$.

Proof. Let $\phi = \omega_{H_{i_1}} \dots \omega_{H_{i_p}}$. We may choose $\psi = 0$ and $\theta = \phi$ in Lemma 2.5.9. This shows that $\mathrm{res}(\phi) = 0$ an proves 1. Now let $\phi = \omega_{H_1}\omega_{H_{i_1}} \dots \omega_{H_{i_p}}$. We may choose $\psi = \omega_{H_{i_1}} \dots \omega_{H_{i_p}}$ and $\theta = 0$ in Lemma 2.5.9. This shows that $\mathrm{res}\phi = \rho(\omega_{H_{i_1}} \dots \omega_{H_{i_p}})$. By Lemma 2.5.6(1), we have $\rho(\omega_{H_{i_1}} \dots \omega_{H_{i_p}}) = \rho(\omega_{H_{i_1}}) \dots \rho(\omega_{H_{i_p}})$. It remains to show that $\rho(\omega_{H_{i_j}}) = \omega_{H_1 \cap H_{i_j}}$. If $H \in \mathcal{A}'$, then it follows from Lemma 2.5.6 that $\rho(\omega_H) = \rho(d\alpha_H/\alpha_H) = d\rho(\alpha_H)/\rho(\alpha_H)$. Since $\rho(\alpha_H)$ is a polynomial function on H_1 which defines the hyperplane $H_1 \cap H \in \mathcal{A}''$, we have $\rho(\omega_H) = \omega_{H_1 \cap H}$. This proves 2. To prove 3., note that since $\omega_{H_1}^2 = 0$, it follows from the definition of $R(\mathcal{A})$ and $R(\mathcal{A}')$ that $R(\mathcal{A}) = R(\mathcal{A}') + \omega_{H_1} R(\mathcal{A}')$. Thus 3. follows from 1. and 2. ∎

2.5.4 The Isomorphism of R and A.

Theorem 2.5.12 *Let $\mathcal{A}$ be an arrangement. The map $\gamma : A(\mathcal{A}) \longrightarrow R(\mathcal{A})$, $a_H \longmapsto \alpha_H$, induces an isomorphism of graded $\mathbb{C}$-algebras.*

Theorem 2.5.13 *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of arrangement with respect to $H_1 \in \mathcal{A}$. Let $i : R(A') \longrightarrow R(A)$ be the inclusion map and define $j : R(A) \longrightarrow R(A'')$ by $j(\phi) = \mathrm{res}(\phi)$ for $\phi \in R(\mathcal{A})$. Then there is an exact sequence:*

$$0 \longrightarrow R(\mathcal{A}') \xrightarrow{i} R(\mathcal{A}) \xrightarrow{j} R(\mathcal{A}'') \longrightarrow 0.$$

Proof. We prove Theorems 2.5.12 and 2.5.13 simultaneously by induction on $|\mathcal{A}|$:
If $\mathcal{A} = \emptyset$, then $A(\mathcal{A}) = \mathbb{C} = R(\mathcal{A})$ and the first result holds. The second assumes that $\mathcal{A}$ is noempty.
If $\mathcal{A} \neq \emptyset$:
• If $|\mathcal{A}| = 1$, let $\mathcal{A} = \{H\}$, then $\mathcal{A}' = \mathcal{A}'' = \emptyset$ and $R(\mathcal{A}) = \mathbb{C} + \mathbb{C}\omega_H$, $R(\mathcal{A}') = R(\mathcal{A}'') = \mathbb{C}$, so both statements are clear.
• If $|\mathcal{A}| > 1$, then we see from Lemma 2.5.11(3) that $jR(\mathcal{A}) \subset R(\mathcal{A}'')$ and from Lemma 2.5.11(2) that j is surjective. It follows from Lemma 2.5.11(1)

that $ji = 0$, so $\mathrm{Im}(i) \subset \ker(j)$. To prove that $\ker(j) \subset \mathrm{Im}(i)$ we consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A(\mathcal{A}') & \xrightarrow{i_A} & A(\mathcal{A}) & \xrightarrow{j_A} & A(\mathcal{A}'') & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma''} & & \\
0 & \longrightarrow & R(\mathcal{A}') & \xrightarrow{i} & R(\mathcal{A}) & \xrightarrow{j} & R(\mathcal{A}'') & \longrightarrow & 0
\end{array}$$

The diagram is commutative. This is clear for the left square by the definition of i_A and i. For the right square it follows from Lemma 2.5.11. The top row is exact by Theorem 2.4.10. We may assume by the induction hypothesis in Theorem 2.5.12 that γ' and γ'' are isomorphisms. A diagram chase shows that $\ker(j) \subset \mathrm{Im}(i)$. This proves that second row of the diagram is exact. Thus Theorem 2.5.13 hold for $\mathcal{A}$. It follows from Five Lemma that γ is an isomorphism, so Theorem 2.5.12 is also established for $\mathcal{A}$. ■

Chapter 3

Basics of algebraic $\mathcal{D}$-modules

3.1 Systems of linear partial differential equations.

Let U be a complex domain in the n-dimensional complex affine space $\mathbb{C}^n$ and $\mathcal{D}(U)$ the ring of partial differential operators on U with holomorphic coefficients. Let $\mathcal{S}$ denote the system of linear partial differential equations

$$\mathcal{S}: \qquad P_1 \bullet u = \ldots = P_m \bullet u = 0$$

for $P_i \in \mathcal{D}(U)$.

Let $\mathcal{F}$ be a suitable function space on U stable by the action of $\mathcal{D}(U)$, e.g.,

- $\mathcal{O}(U)$ the space of holomorphic functions,
- $\mathbf{C}^\infty(U)$ the space of $\mathbf{C}^\infty$ functions, or
- $\mathcal{SD}(U)$ the space of Schwarz distributions.

If $\phi \in \mathcal{F}$ is a solution to the system $\mathcal{S}$, $P_i \bullet \phi = 0$ $(1 \leq i \leq m)$, then the map

$$\widetilde{\phi} \ : \ \mathcal{D}(U) \longrightarrow \mathcal{F} \quad , \quad Q \longmapsto Q \bullet \phi$$

is a left $\mathcal{D}(U)$-linear by definition and $Ker(\widetilde{\phi})$ contains the P_i's , $1 \leq i \leq m$. Then the $\mathcal{D}(U)$-homomorphism $\widetilde{\phi}$ factorizes to the $\mathcal{D}(U)$-homomorphism

$$\widetilde{\phi} : \mathcal{D}(U)/\mathcal{I} \longrightarrow \mathcal{F} \quad , \quad Q \ mod \ \mathcal{I} \longmapsto Q \bullet \phi$$

where $\mathcal{I} = \sum_{i=1}^{m} \mathcal{D}(U)P_i$ is the left ideal of the ring $\mathcal{D}(U)$ generated by the P_i's .

Thus if we denote by M the left $\mathcal{D}(U)$-module $\mathcal{D}(U)/\mathcal{I}$, let $Sol(\mathcal{S};\mathcal{F})$ denote the space of solutions to the system $\mathcal{S}$ in $\mathcal{F}$ and let $Hom_{\mathcal{D}(U)}(M,\mathcal{F})$ be the space of left $\mathcal{D}(U)$-module homomorphisms, we have the identification:

$$Sol(\mathcal{S},\mathcal{F}) \longleftrightarrow Hom_{\mathcal{D}(U)}(M;\mathcal{F}) \quad , \quad \phi \longleftrightarrow \widetilde{\phi}$$

There are several reasons why one can consider such algebraic objects, $\mathcal{D}$-modules. First of all, an interpretation of solution spaces as $Hom_{\mathcal{D}}(\ ,\ )$ prolongs naturally to use of homological algebra, which benefits us much enough. Secondly, as will be noted later, one of the basic invariants, the characteristic variety of a system can be correctly defined only when we consider the ideal generated by the P_i's, i.e., a fixed set of generators is not enough for the definition.

3.2 Algebraic differential operators

Since all $\mathcal{D}$-modules in this thesis are algebraic, we begin with basic notions on algebraic differential operators.

Simplest but important examples are linear differential operators with polynomial coefficients. The ring of differential operators with polynomial coefficients on the n-dimensional complex affine space $\mathbb{C}^n$, denoted by $\mathcal{D}(\mathbb{C}^n)$, is the Weyl algebra. The Weyl algebra $\mathcal{D}(\mathbb{C}^n)$ is a $\mathbb{C}$-algebra generated by

$$x_i, \qquad \partial_{x_i} = \frac{\partial}{\partial x_i}, \qquad 1 \le i \le n$$

with Heisenberg commutator relations

$$[\partial_{x_i}, x_j] = \delta_{ij} \cdot 1, \qquad [x_i, x_j] = [\partial_{x_i}, \partial_{x_j}] = 0 \ .$$

Even on general smooth algebraic varieties, the situation does not differ much from the above. Let X be a smooth affine algebraic variety over $\mathbb{C}$ and let $\mathbb{C}[X]$ be the algebra of regular functions on X: $f(x) \stackrel{\textbf{def}}{=} x(f)$ for $f \in \mathbb{C}[X]$, $x \in Hom_{\mathbb{C}-alg}(\mathbb{C}[X],\mathbb{C}) \longleftrightarrow X$. The family of subsets $X_f = \{x \in X \mid f(x) \neq 0\}$, $f \in \mathbb{C}[X]$, forms a basis of open sets in X, the Zariski topology of X. Note that $\mathbb{C}[X_f] = \mathbb{C}[X]_f = \mathbb{C}[X][f^{-1}]$ is the algebra of regular functions of an open affine subvariety X_f of X .

The correspondence

$$X_f \ \longmapsto \mathbb{C}[X_f]$$

gives rise to the structure sheaf $\mathcal{O}_X$ of X as a local ringed space:

$$\mathcal{O}_X(X_f) = \Gamma(X_f, \mathcal{O}_X) = \mathbb{C}[X_f] \ .$$

The stalk $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ at $x \in X$ is the localization of $\mathbb{C}[X]$ at the maximal ideal $\mathfrak{m}_x \in \mathrm{Specm}\mathbb{C}[X]$:

$$\mathcal{O}_{X,x} \stackrel{\mathbf{def}}{=} \mathcal{O}_{X,\mathfrak{m}_x} = \varinjlim_{x \in X_f} \mathbb{C}[X_f] \ .$$

In general, a smooth algebraic variety is defined to be a local ringed space $(X, \mathcal{O}_X)$ such that every $x \in X$ has an open neighborhood U such that $(U, \mathcal{O}_X|_U)$ is isomorphic to a smooth affine variety as local ringed spaces as above.

Linear differential operators are defined as follows in algebraic geometry.

Definition 3.2.1 *A $\mathbb{C}$-linear sheaf endomorphism $P \in End_{\mathbb{C}}(\mathcal{O}_X)$ is called a linear differential operator of order not greater than m if*

$$(ad\mathcal{O}_X)^{m+1} P = 0 \ .$$

More precisely, for every open $U \subset X$, P is a collection of $\mathbb{C}$-linear maps

$$P_U \in End_{\mathbb{C}}(\mathcal{O}_X(U))$$

compatible with all sheaf restriction data $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$, $V \subset U$, satisfying

$$[f_0, [f_1, [\ldots, [f_m, P_U] \ldots]]] = 0 \qquad \text{for every} \qquad f_0, f_1, \ldots, f_m \in \mathcal{O}_X(U) \ .$$

By definition, if X is affine, a linear differential operator P of order not greater than m is seen to be a $\mathbb{C}$-linear endomorphism $P \in End_{\mathbb{C}}(\mathbb{C}[X])$ such that $(ad\mathbb{C}[X])^{m+1} P = 0$.

Denote by $F_m^{\mathcal{D}}\mathcal{D}(X)$ the set of all linear differential operators on X of order not greater than m. Clearly

$$F_m^{\mathcal{D}}\mathcal{D}(X) \subset F_{m+1}^{\mathcal{D}}\mathcal{D}(X) \ , \qquad m \geq 0$$

and it is easily seen that $F_m^{\mathcal{D}}\mathcal{D}(X)\ F_l^{\mathcal{D}}\mathcal{D}(X) \subset F_{m+l}^{\mathcal{D}}\mathcal{D}(X)$. Thus the set of all linear differential operators on X forms a $\mathbb{C}$-algebra

$$\mathcal{D}(X) = \bigcup_{m=0}^{\infty} F_m^{\mathcal{D}}\mathcal{D}(X) \tag{3.1}$$

with filtration $F^{\mathcal{D}}$. Note also that $F_0^{\mathcal{D}}\mathcal{D}(X) = \mathcal{O}_X(X)$ by the correspondence $P \longmapsto P(1)$.

Definition 3.2.2 *The sheaf $\mathcal{D}_X$ of algebras of linear differential operators on X is defined by the functor*

$$\mathcal{D}_X : U \longmapsto \mathcal{D}(U) \qquad \textit{for every open} \qquad U \subset X$$

with obvious restriction maps.

Thus $\mathcal{D}_X(U) = \mathcal{D}(U) = \bigcup_{m=0}^{\infty} F_m^{\mathcal{D}}\mathcal{D}(U)$. The sheaf $\mathcal{D}_X$ also has the increasing filtration $F^{\mathcal{D}}$ by orders

$$(F_m^{\mathcal{D}}\mathcal{D}_X)(U) = F_m^{\mathcal{D}}\mathcal{D}(U)\ , \quad m \geq 0$$

The following lemma guarantees calculation in the algebraic case similar to the complex analytic case .

Lemma 3.2.3 (see [3, 7, 14]) *In a smooth n-dimensional algebraic variety X, every point $p \in X$ has an affine open neighborhood U with vector fields ∂_{x_i} and functions x_i, $1 \leq i \leq n$, on U such that*

$$[x_i, x_j] = [\partial_{x_i}, \partial_{x_j}] = 0\ , \qquad [\partial_{x_i}, x_j] = \delta_{ij} \cdot 1$$

$$F_m\mathcal{D}_X(U) = \bigoplus_{|\alpha| \leq m} \mathcal{O}_X(U)\partial^\alpha$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $|\alpha| = \sum_{i=1}^{n} \alpha_i$ *and* $\partial^\alpha = \prod_{i=1}^{n} \partial_{x_i}^{\alpha_i}$.

Remark. Let X_{an} be the underlying complex manifold of a smooth algebraic variety X and $i\ :\ X_{\mathrm{an}} \longrightarrow X$ the natural morphism of local ringed spaces: $i^{-1}\mathcal{O}_X \longrightarrow \mathcal{O}_{X_{\mathrm{an}}}$ is the identification of regular functions on X with holomorphic functions on X_{an}. Thus the sheaf $\mathcal{D}_{X_{\mathrm{an}}}$ of linear differential operators with holomorphic coefficients is regarded as $\mathcal{O}_{X_{\mathrm{an}}} \bigotimes_{i^{-1}\mathcal{O}_X} \mathcal{D}_X$. For a small open U in X_{an} (in the classical topology) the above choice of coordinates $\{x_i, \partial_{x_i}\ ; 1 \leq i \leq n\}$ is a standard one in $\mathcal{D}_{X_{\mathrm{an}}}$.

3.3 The Weyl algebra and its modules.

Recall $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, ..., x_n]$ denote the ring of polynomials in n commuting variables over $\mathbb{C}$ and $\mathcal{O}_{\mathbb{C}}(\mathbb{C}^n) = \mathbb{C}[\mathbf{x}]$. In this section we consider more close by the Weyl algebra because it is the $\mathcal{D}$-module that we shall use in the remainder of this thesis.

3.3.1 The Weyl algebra.

Definition 3.3.1 *The* **Weyl algebra** $D_n(\mathbb{C})$ *of rank n over $\mathbb{C}$ is the algebra of linear differential operators with coefficients in the polynomial algebra* $\mathbb{C}[\mathbf{x}]$:

$$D_n = D_n(\mathbb{C}) = \Gamma(\mathbb{C}^n,\ \mathcal{D}_{\mathbb{C}^n}) = \mathcal{O}_{\mathbb{C}}(\mathbb{C}^n)\langle\partial_{x_1}, \ldots, \partial_{x_n}\rangle\ .$$

Thus $D_n(\mathbb{C})$ is the algebra over $\mathbb{C}$ with generators $x_1, \ldots x_n, \partial_{x_1}, \ldots, \partial_{x_n}$ and relations

$$[x_i, x_j] = [\partial_{x_i}, \partial_{x_j}] = 0 \quad , \quad [\partial_{x_i}, x_j] = \delta_{ij} \cdot 1$$

where δ_{ij} is the Kronecker delta symbol and 1 is the identity operator. Hence this algebra is noncommutative.

As a $\mathbb{C}$-vector space the Weyl algebra admits a **canonical basis**. If $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, then $x^\alpha\partial^\beta$ denotes $x_1^{\alpha_1} \ldots x_n^{\alpha_n}\partial_{x_1}^{\beta_1} \ldots \partial_{x_n}^{\beta_n}$. Compare with Lemma 3.2.3 and (3.1) the following

Proposition 3.3.2 *The set* $\mathbf{B} = \{x^\alpha \partial^\beta : \alpha, \beta \in \mathbb{N}^n\}$ *is a basis of* D_n *as a vector space over* $\mathbb{C}$.

If an element of D_n is written as a linear combination of this basis then we say that it is in **canonical form**.

Definition 3.3.3 *Let* $D \in D_n$. *The* **degree** *of* D, *denoted* $deg(D)$, *is the largest length of the multi-indices* $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$ *for which* $x^\alpha \partial^\beta$ *appears with non-zero coefficient in the canonical form of* D. *We use the convention that the zero polynomial has degree* $-\infty$.

If $D, D' \in D_n$ are written in canonical form, then so is $D + D'$, and

$$deg(D + D') \leq \max\{deg(D), deg(D')\}. \tag{3.2}$$

The formula

$$deg(DD') = deg(D) + deg(D'). \tag{3.3}$$

also holds. Similarly we have

$$deg([D, D']) \leq deg(D) + deg(D') - 2. \tag{3.4}$$

As in the case of polynomial rings over a field, the formula (3.3), (3.4) above may be used to prove the following.

Proposition 3.3.4 *The algebra* D_n *is a simple domain.*

Because the kernel of an endomorphism of D_n is a two-sided ideal we have the following:

Corollary 3.3.5 *Every endomorphism of* D_n *is injective.*

Although D_n does not have any non-trivial two-sided ideals, it is not a division ring. In fact, the only elements of D_n that have an inverse are the constants. Thus every non-constant operator generates a non-trivial left ideal of D_n. However, every left ideal of D_n is generated by two elements. This a very important result, due to J.T. Stafford (see [23] or [4]).

3.3.2 D_n-modules.

Since $1/d_{\mathcal{A}}$ is defined as a regular function on the complement $Y_{\mathcal{A}}$ of the zero set of $d_{\mathcal{A}}$ in $\mathbb{C}^n$, we write $\mathbb{C}[\mathbf{x}]_{d_{\mathcal{A}}} = \mathbb{C}[x_1, ..., x_n, d_{\mathcal{A}}^{-1}]$ for the algebra generated by $\mathbb{C}[\mathbf{x}]$ and $1/d_{\mathcal{A}}$ and called it the localization of the ring $\mathbb{C}[\mathbf{x}]$ at $d_{\mathcal{A}}$. This is the algebra of rational functions (with denominator some nonnegative power of $d_{\mathcal{A}}$) defined in the open (quasi-)affine variety $Y_{\mathcal{A}}$.

The rings $\mathbb{C}[\mathbf{x}]$ and $\mathbb{C}[\mathbf{x}, d_{\mathcal{A}}^{-1}]$ have an obvious structure of left D_n-module. Indeed, x_i act by multiplication: $x_i \bullet F = x_i F$, and ∂_{x_i} act by differentiation with respect to $x_i : \partial_{x_i} \bullet F = \dfrac{\partial F}{\partial x_i}$, where $1 \leq i \leq n$ and $F \in \mathbb{C}[\mathbf{x}]$ or $\mathbb{C}[\mathbf{x}, d_{\mathcal{A}}^{-1}]$.

In fact $\mathbb{C}[\mathbf{x}, d_{\mathcal{A}}^{-1}]$ is a left D_n-submodule of $\mathbb{C}(\mathbf{x})$, the field of rational functions of $\mathbb{C}[\mathbf{x}]$.

Now we recall some theory about modules over a ring.

Lemma 3.3.6 *Let R be a ring and M an irreducible left R-module.*

1. *If $0 \neq u \in M$, then $M \cong R/\mathrm{Ann}_R(u)$.*
2. *If R is not a division ring, then M is a torsion module.*

Let us apply these results to the D_n-module $\mathbb{C}[\mathbf{x}]$.

Proposition 3.3.7 *The ring $\mathbb{C}[\mathbf{x}]$ is an irreducible, torsion D_n-module, and*

$$\mathbb{C}[\mathbf{x}] \cong D_n / \textstyle\sum_1^n D_n \partial_{x_i} \ .$$

Another module that is closely related to $\mathbb{C}[\mathbf{x}]$ is $D_n / \sum_1^n D_n \cdot x_i$. As a $\mathbb{C}$-vector space it is isomorphic to $\mathbb{C}[\partial] = \mathbb{C}[\partial_{x_1}, \ldots, \partial_{x_n}]$, the set of polynomials in $\partial_{x_1}, \ldots, \partial_{x_n}$. Using this isomorphism, we may identify the action of D_n directly on $\mathbb{C}[\partial]$: the $\partial' s$ act by multiplication, whereas x_i acts on ∂_{x_j} giving $-\delta_{i,j} \cdot 1$. Apart from the obvious similarities, the modules $\mathbb{C}[\mathbf{x}]$ and $\mathbb{C}[\partial]$ are related in a deeper way that will be explained in brief.

Let R be a ring and M a left R-module. Suppose that σ is an automorphism of R. We shall define a new left module M_σ, as follows. As an abelian group, $M_\sigma = M$. The difference lies in the action of R on M_σ. Let $a \in R$ and $u \in M$, define $a \bullet u = \sigma(a)u$. A routine calculation shows that M_σ is a left R-module. It is called the **twisted module** of M by σ. M_σ inherits many of the properties of M.

Proposition 3.3.8 *Let R be a ring, M a left R-module and σ an automorphism of R. Then*

1. *M_σ is irreducible if and only if M is irreducible.*
2. *M_σ is a torsion module if and only if M is a torsion module.*
3. *If N is a submodule of M then $(M/N)_\sigma \cong M_\sigma/N_\sigma$.*
4. *Let J be a left ideal of R. Set $\sigma(J) = \{\sigma(j) : j \in J\}$. Then $\sigma(J)$ is a left ideal of R and $(R/J)_\sigma \cong R/\sigma^{-1}(J)$.*

Let us apply this construction to D_n. An important example is the **Fourier transform**.

Definition 3.3.9 *Let $\mathcal{F}$ be the automorphism of D_n defined by*

$$\mathcal{F}(x_i) = \partial_{x_i} \qquad , \qquad \mathcal{F}(\partial_{x_i}) = -x_i.$$

Let M be a left module. The twisted module $M_{\mathcal{F}}$ is called the **Fourier transform** *of M. Clearly, $\mathcal{F}$ transform a differential operator with constant coefficients into a polynomial.*

Proposition 3.3.10 *The Fourier transform of* $\mathbb{C}[\mathbf{x}]$ *is* $\mathbb{C}[\partial]$.

Proof. It follows from Proposition 3.3.7 that $\mathbb{C}[\mathbf{x}] \cong D_n/J$, where $J = \sum_1^n D_n \cdot \partial_{x_i}$. Since $\mathcal{F}^{-1}(J) = \sum_1^n D_n \cdot x_i$ we may apply Proposition 3.3.8(4) to get the desired result. ■

It follows from Propositions 3.3.8(1),(2) and 3.3.10 that $\mathbb{C}[\partial]$ is irreducible and a torsion D_n-module. In fact, in this way, $\mathbb{C}[\mathbf{x}]_\sigma$ is irreducible for every automorphism σ of D_n.

Further applications of the Fourier transformations can be found in [7], [8], [15], [17].

Definition 3.3.11 *Let* R *be a* $\mathbb{C}$*-algebra. We say that* R *is* **graded** *if there exists* $\mathbb{C}$*-vector subspaces* R_i, $i \in \mathbb{N}$, *such that*

1. $R = \bigoplus_{i\in\mathbb{N}} R_i$,
2. $R_i \cdot R_j \subseteq R_{i+j}$.

The R_i *are called the* **homogeneous components** *of* R. *The elements of* R_i *are the* **homogeneous elements of degree** i. *If* $R_i = 0$ *when* $i < 0$ *then we say that the grading is* **positive**.

Note that we defined graded rings without assuming commutativity.

Now let $S = \bigoplus_{i\geq 0} S_i$ be another graded $\mathbb{C}$-algebra.

Definition 3.3.12 *A homomorphism of* $\mathbb{C}$*-algebras* $\Phi : R \longrightarrow S$ *is graded of degree 0 if* $\Phi(R_i) \subseteq S_i$.

Proposition 3.3.13 *Let* R *and* S *be graded algebras over* $\mathbb{C}$.

1. *The kernel of a graded homomorphism of* $\mathbb{C}$*-algebras* $\Phi : R \longrightarrow S$ *is a graded two-sided ideal of* R*; i.e.,* $\ker(\Phi) = \bigoplus_{i\geq 0}(\ker(\Phi) \cap R_i)$.
2. *If* I *is a graded two-sided ideal of* R *then* R/I *is a graded* $\mathbb{C}$*-algebra.*

A graded algebra admits a special kind of module.

Definition 3.3.14 *Let* $R = \bigoplus_{i\geq 0} R_i$ *be a graded* $\mathbb{C}$*-algebra. A left* R*-module* M *is a* **graded module** *if there exists* $\mathbb{C}$*-vector spaces* M_i, *for* $i \geq 0$, *such that*

1. $M = \bigoplus_{i\geq 0} M_i$,
2. $R_i \cdot M_j \subseteq M_{i+j}$.

The M_i *are the* **homogeneous components of degree** i *of* M.

Note that the definition of graded module depends on the graded structure chosen for the algebra R.

Definition 3.3.15 *Let R be a graded $\mathbb{C}$-algebra and M, M' be graded left R-modules. A submodule N of M is a* **graded submodule** *if $N = \bigoplus_{i\geq 0}(N \cap M_i)$. An R-module homomorphism $\theta : M \longrightarrow M'$ is graded of degree 0 if $\theta(M_i) \subseteq M'_i$.*

It follows that $\ker(\theta)$ is a graded submodule and that the quotient module M/N is a graded R-module.

Definition 3.3.16 *Let R be a $\mathbb{C}$-algebra. A family $F = \{F_i\}_{i\geq 0}$ of $\mathbb{C}$-vector spaces is a* **filtration** *of R if*

1. $F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots \subseteq R$,
2. $R = \bigcup_{i\geq 0} F_i$,
3. $F_i \cdot F_j \subseteq F_{i+j}$.

If an algebra has a filtration it is called a **filtered algebra**. *We use the convention that $F_j = \{0\}$ if $j < 0$.*

Note that every graded algebra is filtered. On the other hand there are filtered algebras which do not have a natural grading. This happens to the Weyl Algebra; which, however, admits many different filtrations: the usual filtration of $\mathcal{D}_{\mathbb{C}^n}$ by orders, given in Definition 3.2.2, defines one filtration on D_n. Here we will discuss the **Bernstein filtration** of D_n defined using the degree of operators in D_n. Denote by

$$B_k = \{a \in D_n \mid \text{degree of } a \leq k\} .$$

These are vector subspaces of D_n. Clearly $\mathcal{B} = \{B_k\}_{k\in\mathbb{N}}$ is a filtration of D_n. Notice that each B_k is a vector space of finite dimension. A basis for B_k is determined by the monomials $x^\alpha\partial^\beta$ with $|\alpha| + |\beta| \leq k$. In particular, $B_0 = \mathbb{C}$ and $\{1, x_1, \ldots, x_n, \partial_{x_1}, \ldots, \partial_{x_n}\}$ is a basis of B_1 .

Suppose now that $F = \{F_i\}_{i\in\mathbb{N}}$ is a filtration of R. We introduce the **symbol map of order** k, which is the canonical projection of vector spaces

$$\sigma_k : F_k \longrightarrow F_k/F_{k-1} .$$

Consider the $\mathbb{C}$-vector space

$$gr^F R = \bigoplus_{i\geq 1} (F_i/F_{i-1}) .$$

A homogeneous element of $gr^F R$ is of the form $\sigma_k(a)$ for some $a \in F_k$. Let $\sigma_m(b)$ be another homogeneous element, and define their product by

$$\sigma_k(a)\sigma_m(b) = \sigma_{k+m}(ab).$$

and extend it by linearity. $gr^F R$ with this multiplication is a graded $\mathbb{C}$-algebra, with homogeneous components F_i/F_{i-1}. This is called the **graded algebra of R associated with the filtration** F. Put $S_n = gr^{\mathcal{B}} D_n$, because $[P, Q] \in B_{i+j-1}$ for $P \in B_i$, $Q \in B_j$, it is easy to prove the following theorem:

Theorem 3.3.17 *The graded algebra S_n is isomorphic to the polynomial ring over $\mathbb{C}$ in $2n$ variables $\mathbb{C}[y_1, \ldots, y_{2n}]$, where, for $i = 1, \ldots, n$, $y_i = \sigma_1(x_i)$ and $y_{i+n} = \sigma_1(\partial_i)$.*

Like as graded modules, one may define filtered modules. For the sake of simplicity we shall give the definitions only for the Weyl algebra with the Bernstein filtration.

Definition 3.3.18 *Let M be a left D_n-module. A family $\Gamma = \{\Gamma_i\}_{i \geq 0}$ of $\mathbb{C}$-vector spaces of M is a* **filtration** *of M if it satisfies*

1. $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots \subseteq M$,
2. $\bigcup_{i \geq 0} \Gamma_i = M$,
3. $B_i \Gamma_j \subseteq \Gamma_{i+j}$.
4. *Γ_i is a $\mathbb{C}$-vector space of finite dimension.*

Note that 3., with $i = 0$, implies that each Γ_j is a $\mathbb{C}$-vector space. The convention that $\Gamma_i = \{0\}$ if $j < 0$ remains in force.

Of course $\mathcal{B}$ is a filtration of D_n as an D_n-module. A more interesting example is the D_n-module $\mathbb{C}[\mathbf{x}]$: the vector spaces Γ_i of all polynomials of degree $\leq i$ form a filtration of $\mathbb{C}[\mathbf{x}]$ for the Bernstein filtration $\mathcal{B}$.

Following the pattern above, we may define the graded module associated with a filtered module. Let M be a left D_n-module and let Γ be a filtration of M with respect to $\mathcal{B}$. Define the **symbol map of order** k of the filtration Γ to be the canonical projection

$$\mu_k : \Gamma_k \longrightarrow \Gamma_k / \Gamma_{k-1}.$$

Now put

$$gr^{\Gamma} M = \bigoplus_{i \geq 1} \left(\Gamma_i / \Gamma_{i-1} \right).$$

We will define an action of S_n on this vector space. If $a \in F_k$ and $u \in \Gamma_i$ let

$$\sigma_k(a) \cdot \mu_i(u) = \mu_{k+i}(au)$$

Extending this formula by linearity we get an action of S_n on $gr^{\Gamma} M$. The graded S_n-module $gr^{\Gamma} M$ is called the **graded module associated to the filtration** Γ.

Let us return to a previous example. Let Γ be the filtration of $\mathbb{C}[\mathbf{x}]$ with respect to $\mathcal{B}$ defined above. Then Γ_i / Γ_{i-1} is isomorphic to the vector space of all homogeneous polynomials of degree i. Hence $gr^{\Gamma} \mathbb{C}[\mathbf{x}]$ is isomorphic to $\mathbb{C}[\mathbf{x}]$ as a vector space. Recall, by Theorem 3.3.17, that $S_n \cong \mathbb{C}[y_1, \ldots, y_{2n}]$. We could determine the action of the $y's$ on a homogeneous polynomial f of degree r, which is to be thought of as an element of Γ_r / Γ_{r-1} :

• For $i = 1, \ldots, n$ we have that $y_i \cdot f = x_i f$.

• For $i = n+1, \ldots, 2n$ we have $y_i \cdot f = \mu_{r+1}(\partial_i(f))$. But $\partial_i(f)$ is homogeneous of degree $\leq r-1$, hence $y_i \cdot f = 0$. In particular $\mathrm{Ann}_{S_n}(gr^{\Gamma}\mathbb{C}[\mathbf{x}])$ is the ideal generated by $y_{n+1}, \ldots, y_{2n}$.

Let M be a left D_n-module with a filtration Γ with respect to $\mathcal{B}$. Suppose that N is a submodule of M. We may use Γ to construct filtrations induced for both N and M/N. To get a filtration for N put $\Gamma' = \{N \cap \Gamma_i\}_{i \geq 0}$. The inclusion $N \subseteq M$ allows us to get an injective homomorphism of S_n-modules:

$$\iota : gr^{\Gamma'} N \longrightarrow gr^{\Gamma} M \ .$$

We write $gr^{\Gamma'} N \subseteq gr^{\Gamma} M$, for short. Also, the surjective homomorphism $\pi_- : M \longrightarrow M/N$ induces a surjective homomorphism of S_n-modules:

$$\pi : gr^{\Gamma} M \longrightarrow gr^{\Gamma''} M/N \ .$$

where $\Gamma'' = \{\pi_-(\Gamma_i)\}_{i \geq 0}$.

Lemma 3.3.19 *Let M be an D_n-module with a filtration Γ compatible with $\mathcal{B}$. The sequence of S_n-modules*

$$0 \longrightarrow gr^{\Gamma'} N \xrightarrow{\iota} gr^{\Gamma} M \xrightarrow{\pi} gr^{\Gamma''} M/N \longrightarrow 0$$

is exact.

Example 3.3.20 Led d be an operator in D_n of degree r and put $M = D_n/D_n d$. Take $\mathcal{B}$ to be the filtration of D_n as a left D_n-module. The induced filtration in $D_n d$ is $B'_k = B_{k-r} d$. Thus

$$B'_k / B'_{k-1} = B_{k-r} d / B_{k-r-1} d \cong (B_{k-r}/B_{k-r-1}) \sigma_r(d).$$

Since B_k/B_{k-1} is the homogeneous component of degree k of S_n, then

$$gr^{\mathcal{B}'}(D_n d) \cong S_n \sigma_r(d).$$

By the Lemma 3.3.19, there is an exact sequence,

$$0 \to S_n \sigma_r(d) \to S_n \to gr^{\mathcal{B}'} M \to 0 \ .$$

Therefore, $gr^{\mathcal{B}'} M \cong S_n / S_n \sigma_r(d)$.

Theorem 3.3.21 *Let M be a left D_n-module with a filtration Γ with respect to the Bernstein filtration $\mathcal{B}$. If $gr^{\Gamma} M$ is a noetherian S_n-module, then M is noetherian.*

It is an easy consequence of Theorems 3.3.17 and 3.3.21 the following

Corollary 3.3.22 *D_n is a left noetherian ring. In particular every finitely generated left D_n-module is noetherian.*

Now, let M be a left D_n-module and Γ a filtration of M with respect to the Bernstein filtration $\mathcal{B}$. If $gr^{\Gamma}M$ is finitely generated, then it is noetherian because $S_n = gr^{\mathcal{B}}(D_n)$ is noetherian. Hence M is finitely generated, by Theorem 3.3.21. However, it is not always true that if M is finitely generated over D_n then $gr^{\Gamma}M$ is finitely generated over S_n .

Definition 3.3.23 *Let M be a finitely generated left D_n-module and Γ a filtration of M with respect to the Bernstein filtration $\mathcal{B}$. If $gr^{\Gamma}M$ is a finitely generated S_n-module we say that Γ is a* **good filtration of** *M.*

It is nonetheless true that every finitely generated D_n-module admits a good filtration. Indeed, if M is generated by $u_1, \ldots, u_s$ then the filtration Γ of M defined by $\Gamma_k = \sum_1^s B_k u_i$ is good. The graded module $gr^{\Gamma}M$ is generated over S_n by the symbols of $u_1, \ldots, u_s$.

There exists an easy criterion to check whether a filtration is good.

Proposition 3.3.24 *Let M be a left D_n-module. A filtration Γ of M with respect to $\mathcal{B}$ is good if and only if there exists k_0 such that $\Gamma_{i+k} = B_i\Gamma_k$, for all $k \geq k_0$.*

3.3.3 Invariants for D_n-modules.

Using the filtering and grading methods it is possible to define a dimension for D_n-modules. This is a very useful invariant and it comes naturally associated with another invariant: the multiplicity. To define this invariants we need a result in commutative algebra

Theorem 3.3.25 *Let $M = \bigoplus_{i\geq 0} M_i$ be a finitely generated graded module over the polynomial ring $\mathbb{C}[\mathbf{x}]$. There exists a numerical polynomial $\chi(t)$ and a positive integer N such that*

$$\chi(s) = \sum_0^s \dim_{\mathbb{C}}(M_i)$$

for every $s \geq N$.

The polynomial $\chi(t)$ is known as the **Hilbert polynomial** of M. Recall that a numerical polynomial is a polynomial $p(t)$ of $\mathbb{Q}[t]$ such that $p(n) \in \mathbb{Z}$ for all integers $n \gg 0$.

A central merit of the Bernstein filtration is that it admits use of the Hilbert polynomials. Let M be a finitely generated left D_n-module. Suppose that Γ is a good filtration of M with respect to the Bernstein filtration $\mathcal{B}$. Denote by $\chi(t, \Gamma, M)$ the Hilbert polynomial of the graded module $gr^{\Gamma}M$ over the polynomial ring S_n. By Theorem 3.3.25 we have, for $t \gg 0$,

$$\chi(t, \Gamma, M) = \sum_0^t \dim_{\mathbb{C}} (\Gamma_i/\Gamma_{i-1}) = \dim_{\mathbb{C}}(\Gamma_t) \ . \tag{3.5}$$

Definition 3.3.26 *The* **dimension d(M)** *of M is the degree of $\chi(t,\Gamma,M)$. Let $a_{d(M)}$ be the leading coefficient of $\chi(t,\Gamma,M)$. The* **multiplicity** *of M is $m(M) = d(M)!a_{d(M)}$. If $M \neq 0$, then both numbers are non-negative integers.*

The definitions of the dimension and multiplicity are independent of the choice of the good filtration Γ by which the Hilbert polynomial is calculated (see, e.g., [4, 9, 14]).

Example 3.3.27 Consider the following D_n-modules

- D_n : The Bernstein filtration $\mathcal{B}$ is a good filtration of D_n. The monomials $x^\alpha \partial^\beta$ with $|\alpha|+|\beta| \leq t$ form a basis of B_t as a $\mathbb{C}$-vector space. So we must count the non-negative solutions of the equation $\alpha_1+\ldots+\alpha_n+\beta_1+\ldots+\beta_n \leq t$. There are $\binom{t+2n}{2n}$ such solutions. Hence $\chi(t,\mathcal{B},D_n) = \binom{t+2n}{2n}$. As a polynomial in t it has degree $2n$ and leading coefficient $1/(2n)!$. Thus $d(D_n) = 2n$ and $m(D_n) = 1$.

- $\mathbb{C}[\mathbf{x}]$: We have defined a filtration Γ of $\mathbb{C}[\mathbf{x}]$ such that $\Gamma_t = \{f \in \mathbb{C}[\mathbf{x}] \mid deg(f) \leq t\}$. Since B_i contains the polynomials in $x_1,\ldots,x_n$ of degree i, we have that $B_i\Gamma_t = \Gamma_{i+t}$. Hence Γ is a good filtration and $\chi(t,\Gamma,\mathbb{C}[\mathbf{x}]) = \dim_{\mathbb{C}}\Gamma_t = \binom{n+t}{n}$ is a polynomial of degree n and leading coefficient $1/n!$. Hence $d(\mathbb{C}[\mathbf{x}]) = n$ and $m(\mathbb{C}[\mathbf{x}]) = 1$.

A large class of examples is obtained by twisting a module by an automorphism. But this will lead us to a different definition for the dimension. We begin with the Fourier transform of a module. In this case the automorphism $\mathcal{F}$ of D_n preserves the Bernstein filtration, thus $\mathcal{F}(B_i) = B_i$. This is convenient.

Proposition 3.3.28 *Let M be a finitely generated left D_n-module. Then M and $M_{\mathcal{F}}$ have the same dimension and multiplicity.*

Things are much more complicated if the automorphism does not preserve the filtration. To get around the problem we must give a different definition for the dimension. Start by choosing a finite number of elements which generate D_n as a $\mathbb{C}$-algebra and let V be the $\mathbb{C}$-vector space generated by these elements and by 1. Put $U_0 = \mathbb{C}$ and $U_k = V^k$. This is a filtration of D_n which satisfies $\dim_{\mathbb{C}} U_k < \infty$. Note that if $V = B_1$, then $U_k = B_k$ is the Bernstein filtration of D_n .

Now let M be a finitely generated left D_n-module, with a good filtration Γ with respect to the Bernstein filtration. Without loss of generality we may assume that $\Gamma_k = B_k\Gamma_0$, for $k \geq 0$. Put $\Omega_k = U_k\Gamma_0$ and

$$\delta(M,V) = \inf\{v \in \mathbb{N} \mid \dim_{\mathbb{C}}\Omega_t \leq t^v \text{ for } t \gg 0\} .$$

Proposition 3.3.29 (see [9]) *Let V be a vector space whose basis is a finite set of generators for D_n. Then $\delta(M,V) = d(M)$.*

As a simple corollary of this Proposition we have

Corollary 3.3.30 *Let M be a finitely generated left D_n-module and σ an automorphism of D_n, then $d(M_\sigma) = d(M)$.*

In ring theory $\delta(M, V)$ is called the **Gelfand-Kirillov dimension** of a module, see, for example, [18].

The following additivity is easily proved (compare with Lemma 3.3.19).

Theorem 3.3.31 *Let*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

be an exact sequence of finitely generated left D_n-modules. Take a good filtration Γ on M_2 and $\Gamma'_i = M_1 \cap \Gamma_i$, $\Gamma''_i = \mathrm{Im}\Gamma_i$ respectively filtrations (good) for M_1 and M_3. Then

$$0 \longrightarrow gr^{\Gamma'} M_1 \longrightarrow gr^{\Gamma} M_2 \longrightarrow gr^{\Gamma''} M_3 \longrightarrow 0$$

is exact and

1. $\chi(t, \Gamma, M_2) = \chi(t, \Gamma', M_1) + \chi(t, \Gamma'', M_3)$,
2. $d(M_2) = \max\{d(M_1), d(M_2)\}$,
3. *If* $d(M_1) = d(M_3)$, *then* $m(M_2) = m(M_1) + m(M_3)$.

This theorem is useful to know the dimension of some modules. For example, the dimension and multiplicity of a free module of finite rank r are $2n$ and r, respectively. This also follows from the following result:

Corollary 3.3.32 *Let $M_1, \ldots, M_k$ be finitely generated left D_n-modules, and $M = M_1 \oplus \ldots \oplus M_k$.*

1. $d(M) = \max\{d(M_1), \ldots, d(M_k)\}$.
2. *If* $d(M) = d(M_i)$ *for* $1 \leq i \leq k$, *then* $m(M) = \sum_1^k m(M_i)$.

We may also use the theorem to get an upper bound on the dimension of a finitely generated D_n-module.

Corollary 3.3.33 *Let M be a finitely generated D_n-module. Then $d(M) \leq 2n$.*

Proof. Suppose that M is generated by r elements. Then there exists a surjective homomorphism $\Phi : D_n^r \longrightarrow M$. It follows from the Theorem 3.3.31 that $d(D_n^r) = \max\{d(M), d(\ker(\Phi))\}$. Since $d(D_n^r) = 2n$ by Corollary 3.3.32, we conclude that $d(M) \leq 2n$. ∎

This upper bound may be sharpened if the module is a quotient of D_n by a left ideal.

Corollary 3.3.34 *Let I be a non-zero left ideal of D_n. Then $d(D_n/I) \leq 2n-1$.*

Next we establish a lower bound on the dimension of a finitely generated D_n-module.

Theorem 3.3.35 *If M is a non zero finitely generated left D_n-module, then $d(M) \geq n$.*

This inequality was first proved by I.N.Bernstein in [2], and is often called the **Bernstein inequality**. We have already seen that both bounds for the dimension are attained, for example $d(D_n) = 2n$ and $d(\mathbb{C}[X]) = n$. In fact there exists D_n-modules of dimension k for every k in the interval n to $2n$.

3.3.4 Holonomic D_n-modules.

Definition 3.3.36 *A finitely generated left D_n-module is* **holonomic** *if it is zero, or if it has dimension n.*

Notice that every holonomic left D_n-module is noetherian. We already know an example of a holonomic D_n-module, $\mathbb{C}[X]$, and that D_n itself is not a holonomic module. The knowledge of some holonomic D_n-modules enable us to get new examples with the help of the next proposition

Proposition 3.3.37 *Let n be a positive integer.*

1. *Submodules and quotients of holonomic D_n-modules are holonomic.*
2. *Finite sums of holonomic D_n-modules are holonomic.*

Proposition 3.3.38 *Holonomic D_n-modules are torsion modules.*

Proof. Let M be a holonomic left D_n-module. Pick $0 \neq u \in M$ and consider the map $\Phi : D_n \longrightarrow M$ defined by $\Phi(a) = au$. Since $\mathrm{Im}\Phi \subseteq M$, it follows that $d(\mathrm{Im}\Phi) = n$. Thus by Theorem 3.3.31

$$2n = d(D_n) = d(\ker(\Phi)).$$

In particular $\ker(\Phi) \neq 0$, and u is a torsion element of M. ■

Many interesting properties of holonomic modules follow from the fact that they are artinian.

Theorem 3.3.39 *Holonomic modules are artinian.*

Not all D_n-modules are artinian. For example, D_n is not artinian as a module over itself. It is easy to construct an infinite descending chain; take for instance

$$D_n x_n \supseteq D_n x_n^2 \supseteq D_n x_n^3 \supseteq \dots$$

A ring R that is artinian as a left R-module is called a **left artinian ring.** The argument above shows that D_n is not a left artinian ring.

Theorem 3.3.40 *Let R be a simple left noetherian ring and M a finitely generated left R-module. If M is artinian but R is not artinian (as a left R-module), then M is a cyclic module.*

Corollary 3.3.41 *Holonomic modules are cyclic.*

To finish this subsection we give a technical lemma which we must use for show (in Chapter 4) that $\mathbb{C}[X, d_A^{-1}]$ is holonomic.

Lemma 3.3.42 (see, for instance, [4], [9], [14]) *Let M be a left D_n-module with a filtration Γ with respect to the Bernstein filtration of D_n. Suppose that there exists constants c_1, c_2 such that for $j \gg 0$*

$$\dim_{\mathbb{C}}\Gamma_j \leq c_1 j^n/n! + c_2(j+1)^{n-1} \ .$$

Then M is a holonomic D_n-module whose multiplicity cannot exceed c_1. In particular M is finitely generated.

3.3.5 Characteristic varieties for D_n-modules.

Next we give a geometrical interpretation of the dimension of an D_n-module. Let M be a finitely generated left D_n-module with a good filtration Γ. Then $gr^\Gamma M$ is a finitely generated module over the polynomial ring S_n. Let $\mathrm{Ann}(M,\Gamma)$ stand for the annihilator of $gr^\Gamma M$ in S_n. Then $\mathrm{Ann}(M,\Gamma)$ is an ideal of S_n. It depends not only on M, but also on the choice of the good filtration Γ. The radical of $\mathrm{Ann}(M,\Gamma)$ however is independent of the filtration.

Lemma 3.3.43 *Let Ω be another good filtration of M. Then*

$$\mathrm{rad}(\mathrm{Ann}(M,\Gamma)) = \mathrm{rad}(\mathrm{Ann}(M,\Omega)).$$

Definition 3.3.44 *The ideal $\mathcal{I}(M) = \mathrm{rad}(\mathrm{Ann}(M,\Gamma))$ is called the* **characteristic ideal** *of M. The affine variety*

$$Ch(M) = \mathcal{Z}(\mathcal{I}(M)) \subseteq \mathbb{C}^{2n}$$

is called the **characteristic variety** *of M.*

It follows from Lemma 3.3.43 that $\mathcal{I}(M)$ is independent of the good filtration Γ used to calculate it. In other words, **it is an invariant of** M, and so is $Ch(M)$. Actually, $Ch(M)$ is the most important geometric invariant of a D_n-module. Since $\mathcal{I}(M)$ is a homogeneous ideal, the variety $Ch(M)$ is conic along the fibers $\mathbb{C}$. Note that $Ch(M)$ is a subvariety of $\mathbb{C}^{2n}$ because S_n is a polynomial ring in $2n$ variables.

Example 3.3.45 Let $d \in D_n$ be an element of degree r and put $M = D_n/D_n d$. If $\mathcal{B}'$ is the filtration of M induced by the Bernstein filtration of D_n, then, by the Example 3.3.20, $gr^{\mathcal{B}'} M = S_n/S_n\sigma_r(d)$. Therefore, $\mathrm{Ann}(M,\mathcal{B}') = S_n\sigma_r(d)$ and so $Ch(M) = \mathcal{Z}(\sigma_r(d))$ is a hypersurface.

The following Proposition is an easy consequence of Lemma 3.3.19.

Proposition 3.3.46 *Let M be a finitely generated left D_n-module and N a submodule of M. Then*

$$Ch(M) = Ch(N) \cup Ch(M/N) \ .$$

Let J be an ideal of $S_n = \mathbb{C}[y_1, \ldots, y_{2n}]$ and put $V = \mathcal{Z}(J)$. If p is a point of V then the **Zariski tangent** space of V at p is the linear subspace of $\mathbb{C}^{2n}$ defined by the equations

$$\sum_1^{2n} \frac{\partial F}{\partial y_i}(p) y_i = 0$$

for all $F \in J$. This subspace (of $\mathbb{C}^{2n}$) is denoted by $T_p(V)$. The **dimension** of V equals $\inf\{\dim_{\mathbb{C}} T_p(V) \mid p \in V\}$. Actually one does not need to look at every point of V. If p is a *non-singular point* of V, then $\dim(V) = \dim_{\mathbb{C}} T_p(V)$. This is equivalent to the definition in terms of heights of prime ideals, see [13].

The following theorem is an immediate consequence of the fact that if N is a graded module over S_n then the degree of its Hilbert polynomial is $\dim \mathcal{Z}(\mathrm{Ann}_{S_n} N)$, see [13].

Theorem 3.3.47 *Let M be a finitely generated left module over D_n. Then* $\dim Ch(M) = d(M)$.

It is now very easy to show that if $d \neq 0$ is an operator in D_n, then $D_n/D_n d$ has dimension $2n - 1$, compare with Example 3.3.45.

Chapter 4

The left D_n-module $P(\mathcal{A})$

This chapter is dedicate to get and study some algebraic properties of the left D_n-module $P = P(\mathcal{A}) \stackrel{\text{def}}{=} \mathbb{C}[\mathbf{x}, d_{\mathcal{A}}^{-1}] = \mathbb{C}[\mathbf{x}, \alpha_1^{-1}, \ldots, \alpha_k^{-1}]$. Some of these will enable us to calculate the De Rham Cohomology of the variety $Y_{\mathcal{A}}$ and get the fundamental Theorem 5.2.8.
Recall that if $\mathcal{A} = \{H_1, \ldots, H_k\}$ is our arrangement, then we denote by $\mathcal{A}^* = \{\alpha_1, \ldots, \alpha_k\}$ the set of linear forms that define $\mathcal{A}$. The homogeneous polynomial $d_{\mathcal{A}} = \prod_{i=1}^{k} \alpha_i$ also defines $\mathcal{A}$. Note that $\text{deg} d_{\mathcal{A}} = k = |\mathcal{A}|$.

4.1 Holonomicity of $P(\mathcal{A})$.

We began enunciating a well-known result about P, which will be useful in the following section

Proposition 4.1.1 *(see [9]) The left D_n-module $\mathbb{C}[\mathbf{x}, d_{\mathcal{A}}^{-1}]$ is holonomic with multiplicity $\leq (k+1)^n$.*

Proof. Denote $d_{\mathcal{A}}$ by d and set

$$\Gamma_t = \{f/d^t |\ deg(f) \leq (k+1)t\}$$

We check that Γ is a filtration for P. Let $f/d^t \in P$ and assume that $\deg(f) = s$. Then $f/d^t = f \cdot d^s/d^{t+s}$. But $\deg(fd^s) = s(k+1) \leq (k+1)(s+t)$. Hence $f/d^t \in \Gamma_{s+t}$. It follows that $P = \cup_{t \geq 0} \Gamma_t$.
Next suppose that $f/d^t \in \Gamma_t$. Equivalently, $\deg(f) \leq (k+1)t$. Multiplication by x_i increases the degree of f by 1, thus $x_i(f/d^t) = x_i f d/d^{t+1} \in \Gamma_{t+1}$. Differentiating f/d^t with respect to x_i we get

$$(d\partial_i(f) - tf\partial_i(d))/d^{t+1}$$

The numerator has degree $\leq (k+1)t + (k-1)$, so that $\partial_i(f/d^t) \in \Gamma_{t+1}$. This may be summed up as $B_1 \cdot \Gamma_t \subseteq \Gamma_{t+1}$. Since $B_i = B_1^i$ we also have that $B_i \cdot \Gamma_t \subseteq \Gamma_{i+t}$.

Finally, the dimension of Γ_t cannot exceed the dimension of the vector space of polynomials of degree $\leq (k+1)t$. Hence each Γ_t is finite dimensional. This concludes the proof that Γ is a filtration of P, and shows that

$$\dim_{\mathbb{C}}\Gamma_t \leq \binom{(k+1)t+n}{n}.$$

Since the two terms of highest degree in t of this binomial number are

$$(k+1)^n t^n/n! \text{ and } (k+1)^{n-1}(n+1)nt^{n-1}/2(n!)$$

it follows that

$$\dim_{\mathbb{C}}\Gamma_t \leq (k+1)^n t^n/n! + (k+1)^{n-1}(n+1)n(t+1)^{n-1}/n!$$

for every large values of t. By Lemma 3.3.42, P must be a holonomic module of multiplicity $\leq (k+1)^n$, as required. ■

More generally we have the following result:

Theorem 4.1.2 (see, for instance, [4], [9], [14]) *If M is a holonomic D_n-module and p a no constant polynomial in $\mathbb{C}[\mathbf{x}]$, then so is $M[p^{-1}] := M \otimes_{\mathbb{C}[\mathbf{x}]} \mathbb{C}[\mathbf{x}, p^{-1}]$, the localization of M at p.*

4.2 Structure of $P(\mathcal{A})$ as D_n-module

Recall that $r(\mathcal{A}) = r$, the rank of $\mathcal{A}$, denotes the cardinality of a maximal linearly independent subset of $\mathcal{A}^*$. The following lemma allows us to write in a very convenient way every element of D_n-module P. Next, with this in hand, we will be able to get a series of decomposition for P, cf. (4.1).

Lemma 4.2.1 *It is possible to write every element g of P as a finite sum of quotients of the form $\dfrac{f}{\prod_{j=1}^{h} \alpha_{i_j}^{m_j}}$, where $0 \leq h \leq r$, $\{\alpha_{i_1}, \ldots, \alpha_{i_h}\}$ is a linearly independent subset of $\mathcal{A}^*$, $m_1, \ldots, m_h \in \mathbb{N}$, $f \in \mathbb{C}[\mathbf{x}]$ and $\prod_{i=1}^{0} \alpha_{s_i}^{m_i} \stackrel{\mathbf{def}}{=} 1$.*

Proof. Note that if $h = 0$, then the element s belongs to $\mathbb{C}[\mathbf{x}]$. By induction on the number of the different linear factors on the denominator of each summand, we need only show the Lemma when g has the form

$$g = \frac{1}{\alpha_1^{m_1} \ldots \alpha_h^{m_h} \alpha_{h+1}^{m_{h+1}}}$$

where $\{\alpha_1, \ldots, \alpha_h\}$ is a linearly independent subset of $\mathcal{A}^*$, α_{h+1} is a $\mathbb{C}$-linear combination of them: $\alpha_{h+1} = \sum_{i=1}^{h} c_i \alpha_i$, $c_i \in \mathbb{C}$, and $m_1, \ldots, m_{h+1} \in \mathbb{Z}^+$. Then

$$g = \frac{1}{\alpha_1^{m_1} \dots \alpha_h^{m_h} \alpha_{h+1}^{m_{h+1}}} = \frac{\sum_{i=1}^{h} c_i \alpha_i}{\alpha_1^{m_1} \dots \alpha_h^{m_h} \alpha_{h+1}^{m_{h+1}+1}}$$

$$= \sum_{i=1}^{h} \frac{c_i}{\alpha_1^{m_1} \dots \alpha_i^{m_i-1} \dots \alpha_h^{m_h} \alpha_{h+1}^{m_{h+1}+1}} .$$

Now for each summand of the last sum above we repeat the initial process done with g until obtaining a zero exponent for some α_i in the denominator. In this way we get a new expression for g as a finite sum of quotients of the form $\frac{a_i}{\alpha_1^{r_1} \dots \widehat{\alpha_i^{r_i}} \dots \alpha_h^{r_h} \alpha_{h+1}^{r_{h+1}}}$, $a_i \in \mathbb{C}$, $i = 1, \dots, h$. If the sets $\{\alpha_1, \dots, \widehat{\alpha_i}, \dots, \alpha_h, \alpha_{h+1}\}$ are all linearly independent, for $i = 1, \dots, h$, then we have obtained the expression wished for g, otherwise, by induction we get the expression wished for the quotients that not yet verify the required condition, an thus the expression wished for g.

■

This Lemma inspires the following definition.

Definition 4.2.2 *For $h = 0, 1, \dots, r$, define the left D_n-submodules of P :*

$$P_h = \left\{ \sum_{finite} \frac{f_{s_1,\dots,s_t}^{m_1,\dots,m_t}}{\prod_{i=1}^{t} \alpha_{s_i}^{m_i}} \;\middle|\; 0 \le t \le h,\ f_{s_1,\dots,s_t}^{m_1,\dots,m_t} \in \mathbb{C}[\mathbf{x}],\ m_1, \dots, m_t \in \mathbb{N} \right\}$$

where $\{\alpha_{s_1}, \dots, \alpha_{s_t}\}$ varies between all the linearly independent subsets of $\mathcal{A}^*$ of cardinality t and $\prod_{i=1}^{0} \alpha_{s_i}^{m_i} \stackrel{\text{def}}{=} 1$.

Hence, by Lemma 4.2.1, we have the following finite ascending chain of left D_n-submodules of P:

$$0 =: P_{-1} \subseteq \mathbb{C}[\mathbf{x}] = P_0 \subseteq P_1 \subseteq P_2 \subseteq P_3 \subseteq \dots \subseteq P_r = P \tag{4.1}$$

By Proposition 3.3.37 those D_n-modules are holonomics since P is.

Now we can consider the left D_n-modules P_h/P_{h-1}, $h = 0, 1, \dots, r$. Again by Proposition **??** these are holonomic.

Our next aim is to get a decomposition of P_h/P_{h-1} as a direct sum of isotropy component left D_n-modules associated to each $X \in L_h(\mathcal{A})$. First we give some notations and definitions to use in which it follows.

For each $X \in L(\mathcal{A})$ consider the dual subspace X^* of $(\mathbb{C}^n)^*$ of dimension $r(X)$, namely, the necessary number of hyperplanes in general position to get X.

Definition 4.2.3 *For each X in $L_h(\mathcal{A})$, $1 \le h \le r$, let $\mathcal{B}_{X^*}$ be the set of all possible bases to X^* constituted with elements of $\mathcal{A}^*$.*

Definition 4.2.4 *For each X in $L_h(\mathcal{A})$, $1 \leq h \leq r$, and for each basis $B = \{\alpha_{i_1}, \ldots, \alpha_{i_h}\}$ in $\mathcal{B}_{X^*}$ define the holonomic D_n-submodule of P_h/P_{h-1}:*

$$V_B^{X^*} = \left\{ \sum_{finite} \left(\frac{f_{i_1,\ldots,i_h}^{m_1,\ldots,m_h}}{\alpha_{i_1}^{m_1} \cdots \alpha_{i_h}^{m_h}} \bmod P_{h-1} \right) \;\middle|\; f_{i_1,\ldots,i_h}^{m_1,\ldots,m_h} \in \mathbb{C}[\mathbf{x}],\ m_i \in \mathbb{Z}^+ \right\} .$$

Note that of $V_B^{X^*}$ definition follows that it is an irreducible left D_n-module. Then it is also cyclic (this is also a consequence of its holonomicity, cf. Corollary 3.3.41). A generator for $V_B^{X^*}$, as a D_n-module, is the class of $\dfrac{1}{\alpha_{i_1} \cdots \alpha_{i_h}}$, cf. Proposition 4.2.6.

Let $X \in L_h$ and let $B = \{\alpha_{i_1}, \ldots, \alpha_{i_h}\}$ be a basis for X^*. Then there exists a basis $\{y_1 := \alpha_{i_1}, \ldots, y_r := \alpha_{i_r}, y_{r+1}, \ldots, y_n\}$ of $(\mathbb{C}^n)^*$, where $\{y_1, \ldots, y_r\}$ is a maximal linearly independent subset of $\mathcal{A}^*$ such that $\{\alpha_{i_1}, \ldots, \alpha_{i_h}\} \subseteq \{y_1, \ldots, y_r\}$. After a linear change of the usual basis $\{x_1, \ldots, x_n\}$ to $(\mathbb{C}^n)^*$ by $\{y_1, \ldots, y_n\}$, we see clearly that the element $\left[\dfrac{1}{y_1 \cdots y_h}\right]$ in $V_{X^*}^{\{y_1,\ldots,y_h\}}$ is annihilated by the linear operators $y_1, \ldots, y_h, \partial_{y_{h+1}}, \ldots, \partial_{y_n}$, i.e. by the left D_n-ideal $I_B = D_n(y_1, \ldots, y_h, \partial_{y_{h+1}}, \ldots, \partial_{y_n})$. Actually it is very easy to see that

Lemma 4.2.5 *With the previous definitions we have*

$$\mathrm{Ann}_{D_n}\left(\frac{1}{y_1 \cdots y_h}\right) = I_B \ .$$

The ideal I_B plays an important role in which it follows. In the first place it allows us to get a way simple to write the module $V_{X^*}^{\{y_1,\ldots,y_h\}}$.

Proposition 4.2.6 *Let us denote with M_B the left D_n-module D_n/I_B. Then we have the isomorphism of left D_n-modules*

$$V_{X^*}^{\{y_1,\ldots,y_h\}} \cong M_B \cong D_n \bullet \left[\frac{1}{y_1 \cdots y_h}\right] . \tag{4.2}$$

Proof. The first isomorphism follows from Lemma 4.2.5 and Lemma 3.3.6. To the second one use the exact sequence $0 \to I_B \to D_n \to D_n \bullet \left[\dfrac{1}{y_1 \cdots y_h}\right]$. ■

Corollary 4.2.7 *Consider two different elements X_1, X_2 in L_h, $1 \leq h \leq r$. Then $V_{X_1^*}^{B_1} \cap V_{X_2^*}^{B_2} = \{[0]\}$ for each B_1 in $\mathcal{B}_{X_1^*}$ and for each B_2 in $\mathcal{B}_{X_2^*}$.*

Proposition 4.2.8 *There exists an isomorphism of D_n-modules between the left D_n-module M_B and the ring of polynomials $\mathbb{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}]$. This last one is an irreducible, holonomic D_n-module, and its characteristic variety is the conormal space defined by the system of equations $\xi_1 = \ldots = \xi_h = \xi_{n+h+1} = \ldots = \xi_{2n} = 0$ where for $i = 1, \ldots, h$ $\xi_i = \sigma_1(y_i)$, and for $i = 1, \ldots, n-h$, $\xi_{n+h+i} = \sigma_1(\partial_{y_i})$ (σ_1 is the symbol map of order 1, cf. Theorem 3.3.17) .*

Proof. Let $\mathcal{T}$ be the automorphism of D_n defined by

$$\begin{array}{rclcrclll} \mathcal{T}(y_i) & = & \partial_{y_i} & , & \mathcal{T}(\partial_{y_i}) & = & -y_i & \text{for} & 1 \leq i \leq h \\ \mathcal{T}(y_i) & = & y_i & , & \mathcal{T}(\partial_{y_i}) & = & \partial_{y_i} & \text{for} & h+1 \leq i \leq n \end{array}$$

Proposition 3.3.7 affirms that $\mathbb{C}[\mathbf{y}] = \mathbb{C}[y_1, \ldots, y_n] \cong D_n/J$ where $J = \sum_1^n D_n \cdot \partial_{y_i}$. It is easy to see that

$$\mathcal{T}^{-1}(J) = \textstyle\sum_1^h D_n \cdot y_i + \sum_{h+1}^n D_n \cdot \partial_i = I_B \ .$$

Then we can apply Proposition 3.3.8(4):

$$M_B = D_n/\mathcal{T}^{-1}(J) \cong \mathbb{C}[y_1, \ldots, y_n]_{\mathcal{T}} \cong \mathbb{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}]$$

Thus, by the Proposition 3.3.8(1) and Corollary 3.3.30, $\mathbb{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}]$ is an irreducible, holonomic left D_n-module and isomorphic to M_B.

Recall that the graded algebra $\mathrm{gr}^{\mathcal{B}} D_n$ is isomorphic to the polynomial ring in $2n$ variables $\mathbb{C}[\xi] = \mathbb{C}[\xi_1, \ldots, \xi_{2n}]$. Let Γ be a good filtration for $\mathbb{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}]$ with respect to the Bernstein filtration $\mathcal{B}$, for example the induced one by $\mathcal{B}$. By Lemma 3.3.19 the exact sequence of D_n-modules $0 \longrightarrow I_B \longrightarrow D_n \longrightarrow M_B \longrightarrow 0$ implies the following exact sequence of $\mathbb{C}[\xi]$-modules

$$0 \longrightarrow \mathrm{gr}^{\Gamma'} I_B \longrightarrow \mathrm{gr}^{\mathcal{B}} D_n \longrightarrow \mathrm{gr}^{\Gamma} M_B \longrightarrow 0$$

where Γ' is the filtration induced by $\mathcal{B}$ on I_B. Then $\mathrm{gr}^{\Gamma} M_B \cong \dfrac{\mathbb{C}[\xi]}{\mathrm{gr}^{\Gamma'} I_B}$ and

$$\begin{array}{rcl} \mathrm{Ann}(\mathbb{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}], \Gamma) & = & \mathrm{Ann}_{\mathbb{C}[\xi]}(\mathrm{gr}^{\Gamma} M_B) \\ & = & \mathrm{gr}^{\Gamma'} I_B \\ & = & \mathbb{C}[\xi](\xi_1, \ldots, \xi_h, \xi_{n+h+1}, \ldots, \xi_{2n}) \ . \end{array}$$

Since this last ideal is radical we have

$$\mathrm{Ch}(\mathbb{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}]) = \mathcal{Z}(\xi_1, \ldots, \xi_h, \xi_{n+h+1}, \ldots, \xi_{2n}) \ .$$

■

As a consequence of the isomorphism in (4.2) and Proposition 4.2.8 we have the following corollary:

Corollary 4.2.9 *There exists an isomorphism of irreducible left D_n-modules*

$$V_{X^*}^{\{y_1, \ldots, y_h\}} \cong \mathbb{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}]. \tag{4.3}$$

Next, in Proposition 4.2.12, we give a preliminary and important decomposition of P_h/P_{h-1}. For this is clearly necessary to consider the D_n-module V_{X*}^B associated to each $B \in \mathcal{B}_{X^*}$ when $X \in L_h(\mathcal{A})$. First we give the following lemma:

Lemma 4.2.10 *Let R be a ring on a field K of characteristic 0, let M be a left R-module and let $\{M_i\}_{i=1}^{s}$ be a family of irreducible left R-submodules of M each isomorphic to each other as left R-modules. Then the* **isotropy component** *$\sum M_i$ is expressible as a direct sum of some of them, namely*

$$\sum_{i=1}^{s} M_i = \bigoplus_{a=1}^{t} M_{j_a} \quad ,$$

where $\{M_{j_a}\}_{a=1}^{t} \subseteq \{M_i\}_{i=1}^{s}$ and $t \leq s$.

Proof. Since M_i is an irreducible R-module it is cyclic, i.e., there exists an element u_i in M_i such that $M_i = R \cdot u_i$. Let $\sum_{i=1}^{s} Ku_i$ be the K-vector space generates by $\mathcal{U} = \{u_i\}_{i=1}^{s}$. From $\mathcal{U}$ we can extract a K-basis $\widetilde{\mathcal{U}} = \{u_{j_a}\}_{a=1}^{t}$ to $\sum_{i=1}^{s} Ku_i$. The corresponding R-modules M_{j_a} to this basis are suitable to our affirmation. In fact, since $u_l \in \bigoplus_{a=1}^{t} Ku_{j_a} \subset \sum_{a=1}^{t} M_{j_a}$ for every $l \in [s]$ follows that $\sum_{i=1}^{s} M_i \subset \sum_{a=1}^{t} M_{j_a}$. So $\sum_{i=1}^{s} M_i = \sum_{a=1}^{t} M_{j_a}$. It remains to show that $\sum_{a=1}^{t} M_{j_a}$ is a direct sum. For this, since that $\bigcap_{a=1}^{t} M_{j_a} \subset M_{j_l}$ for every $l \in [t]$, then either each M_{j_a} is contained in the others or $\bigcap_{a=1}^{t} M_{j_a} = 0$. The first case implies that all M_{j_a} are equal. Thus the sum $\sum_{a=1}^{t} M_{j_a}$ is direct. ■

Proposition 4.2.11 *For each X in L_h, $1 \leq h \leq r$, and each basis B in $\mathcal{B}_{X^*}$:*

(1) *The vector spaces $V_{X^*}^{B}$ are isomorphic to each other as D_n-modules.*

(2) *The ideal $I_{X^*} := I_B$ is not dependent of B.*

Proof. Fixed a basis $B = \{\alpha_{i_1}, \ldots, \alpha_{i_h}\}$ of X^*, there exists a basis $\{y_1 := \alpha_{i_1}, \ldots, y_r := \alpha_{i_r}, y_{r+1}, \ldots, y_n\}$ of $(\mathbb{C}^n)^*$, where $\{y_1, \ldots, y_r\}$ is a maximal linearly independent subset of $\mathcal{A}^*$ such that $\{\alpha_{i_1}, \ldots, \alpha_{i_h}\} \subseteq \{y_1, \ldots, y_r\}$. After a linear change of the usual basis $\{x_1, \ldots, x_n\}$ of $(\mathbb{C}^n)^*$ by $\{y_1, \ldots, y_n\}$, every other basis $B' = \{\alpha_{j_1}, \ldots, \alpha_{j_h}\}$ of X^* satisfies $B' \subset \mathrm{Span}\{y_1, \ldots, y_h\}$ and $\{y'_1 := \alpha_{j_1}, \ldots, y'_h := \alpha_{j_h}, y'_{h+1} := y_{h+1}, \ldots, y'_n := y_n\}$ is a basis to $(\mathbb{C}^n)^*$. Then associate to each basis B' in $\mathcal{B}_{X^*}$ we have an $n \times n$ invertible matrix with entries in $\mathbb{C}$

$$C_{B'}^{B} = \begin{pmatrix} D & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{r-h} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix} ,$$

where $D \in GL_h(\mathbb{C})$ and $\mathbf{I}_s$ is the unit matrix of rank s, such that

$$(4.4) \qquad {}^t(y'_1, \ldots, y'_h, y'_{h+1}, \ldots, y'_n) = C_{B'}^{B}\ {}^t(y_1, \ldots, y_h, y_{h+1}, \ldots, y_n).$$

Thus the partial derivatives change linearly by means of

$$(4.5) \ {}^t\Big(\partial_{y'_1}, \ldots, \partial_{y'_h}, \partial_{y'_{h+1}}, \ldots, \partial_{y'_n}\Big) = \left({}^tC_{B'}^{B}\right)^{-1}\ {}^t\big(\partial_{y_1}, \ldots, \partial_{y_h}, \partial_{y_{h+1}}, \ldots, \partial_{y_n}\big)$$

Then we get $\mathbb{C}\left[\partial_{y'_1},\ldots,\partial_{y'_h},y'_{h+1},\ldots,y'_n\right] = \mathbb{C}[\partial_{y_1},\ldots,\partial_{y_h},y_{h+1},\ldots,y_n]$. It follows, by Corollary 4.2.9, that $V^B_{X^*}$ and $V^{B'}_{X^*}$ are isomorphic as D_n-modules. Moreover, it follows from (4.4) and (4.5) that I_{X^*} is not dependent of B. ■

Proposition 4.2.12 *For $1 \leq h \leq r$ the quotient of two consecutive D_n-modules of sequence (4.1) has the following decomposition*

$$P_h/P_{h-1} = \bigoplus_{X\in L_h}\sum_{B\in\mathcal{B}_{X^*}} V^B_{X^*} = \bigoplus_{X\in L_h}\left(\bigoplus_{B\in\widetilde{\mathcal{B}}_{X^*}} V^B_{X^*}\right) \tag{4.6}$$

where $\widetilde{\mathcal{B}}_{X^*}$ is a convenient subset of $\mathcal{B}_{X^*}$.

Proof. By Lemma 4.2.10 we have that the isotropy component $\sum\limits_{B\in\mathcal{B}_{X^*}} V^B_{X^*}$ associated to X^* is equal to $\bigoplus\limits_{B\in\widetilde{\mathcal{B}}_{X^*}} V^B_{X^*}$ for a convenient subset $\widetilde{\mathcal{B}}_{X^*}$ of $\mathcal{B}_{X^*}$. Thus the last equality in 4.6 is done.

Now let us consider two different elements X_1, X_2 of L_h. It follows by Corollary 4.2.7 that for the corresponding isotropy components we have $\sum\limits_{B\in\mathcal{B}_{X_1^*}} V^B_{X_1^*} \cap \sum\limits_{B\in\mathcal{B}_{X_2^*}} \cap V^B_{X_2^*} = \{[0]\}$. Then $\bigoplus\limits_{X\in L_h}\sum\limits_{B\in\mathcal{B}_{X^*}} V^B_{X^*} \subset P_h/P_{h-1}$. Moreover, by Lemma 4.2.1, every f in P_h/P_{h-1} can be written as a finite sum of elements of the form $\dfrac{f^{m_1,\ldots,m_h}_{i_1,\ldots,i_h}}{\alpha^{m_1}_{i_1}\ldots\alpha^{m_h}_{i_h}} \bmod P_{h-1} \in V^{\{\alpha_{i_1},\ldots,\alpha_{i_h}\}}_{X^*}$ for some $X \in L_h$ and $B = \{\alpha_{i_1},\ldots,\alpha_{i_h}\}$ a basis in $\mathcal{B}_{X^*}$. Then $P_h/P_{h-1} = \bigoplus\limits_{X\in L_h}\sum\limits_{B\in\mathcal{B}_{X^*}} V^B_{X^*}$. ■

Proposition 4.2.13 *Let X in L_h, $1 \leq h \leq r$, and let $V^{\text{mod}}_{X^*}$ be the $\mathbb{C}$-subspace of P_h/P_{h-1} annihilated by I_{X^*}. Then $V^{\text{mod}}_{X^*}$ is generated by*

$$\mathcal{U}^{\text{mod}}_{X^*} = \left\{ \frac{1}{\prod\limits_{\alpha\in B}\alpha} \bmod P_{h-1} \;\middle|\; B \in \mathcal{B}_{X^*} \right\}.$$

Proof. Consider two different elements X_1, X_2 in L_h. According to the definition of I_{X^*}, if $B_1 \in \mathcal{B}_{X_1^*}$ and $B_2 =\in \mathcal{B}_{X_2^*}$, then the corresponding annihilator ideal $I_{X_j^*}$ are such that

$$I_{X_j^*} \bullet \left(\frac{1}{\prod\limits_{\alpha\in B_i}\alpha} \bmod P_{h-1} \right)$$

is equal to $\{[0]\}$ if $i = j$ and diverse of $\{[0]\}$ if $i \neq j$.
This implies that $\mathrm{Ann}_{D_n}\left(\dfrac{1}{\alpha_{j_1}\ldots\alpha_{j_h}}\ \mathit{mod}\ P_{h-1}\right) = I_{X^*}$ if and only if $B =$

$\{\alpha_{j_1}, \ldots, \alpha_{j_h}\}$ is a basis to X^*. Then the $\mathbb{C}$-subspace $V_{X^*}^{\mathrm{mod}}$ of P_h/P_{h-1} annihilated by I_{X^*} is into an unique isotropy component in the decomposition of P_h/P_{h-1} as in (4.6) and is generated by $\mathcal{U}_{X^*}^{\mathrm{mod}}$. ■

By Proposition 4.2.11(2), for each $X \in L(\mathcal{A})\backslash\{\mathbb{C}^n\}$ we associated a canonical holonomic left D_n-module $M_{X^*} \overset{\mathbf{def}}{=} D_n/I_{X^*}$, and, by Proposition 4.2.8, $M_{X^*} \cong V_{X^*}^B$ for each B in $\mathcal{B}_{X^*}$. Then we have the following Proposition:

Proposition 4.2.14 *According to the notations previous*

$$P_h/P_{h-1} \cong \bigoplus_{X \in L_h(\mathcal{A})} M_{X^*} \otimes_{\mathbb{C}} V_{X^*}^{\mathrm{mod}}. \tag{4.7}$$

Proof. By Proposition 4.2.12 we need only show

$$\sum_{B \in \mathcal{B}_{X^*}} V_{X^*}^B \cong M_{X^*} \otimes V_{X^*}^{\mathrm{mod}}$$

This follows from the last remark above and the Proposition 4.2.13. ■

Now our next aim is to choose a basis for $V_{X^*}^{\mathrm{mod}}$. It is possible using the notion of not broken circuit (nbc), cf. Definition 2.4.2, to the set $\mathcal{B}_{X^*}$, consequently to $\mathcal{U}_{X^*}^{\mathrm{mod}}$.

Definition 4.2.15 *For every $X \in L_h(\mathcal{A})$, $1 \le h \le r$, define the left D_n-module*

$$R_{X^*} = M_{X^*} \otimes_{\mathbb{C}} V_{X^*} \ ,$$

where V_{X^*} is the $\mathbb{C}$-vector space generated by $\mathcal{U}_{X^*} = \left\{ \dfrac{1}{\prod_{\alpha \in B} \alpha} \;\middle|\; B \in \mathcal{B}_{X^*} \right\}.$
For $\mathbb{C}^n$ in $L_0(\mathcal{A})$, define $V_{(\mathbb{C}^n)^*} = \mathbb{C}$ and $R_{(\mathbb{C}^n)^*} = \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_n]$.

Lemma 4.2.16 *Let X in $L_h(\mathcal{A})$, $h \in [r]$, then the set*

$$\mathcal{B}_{X^*}^{\mathrm{nbc}} = \{\{\alpha_{j_1}, \ldots, \alpha_{j_h}\} \in \mathcal{B}_{X^*} \mid (j_1, \ldots, j_h) \text{ is a nbc}\}$$

is such that the corresponding set $\mathcal{U}_{X^*}^{\mathrm{nbc}}$ is a basis to V_{X^*}.

Proof. The set $\mathcal{U}_{X^*}^{\mathrm{nbc}}$ generate V_{X^*}: For each basis $\{\alpha_{i_1}, \ldots, \alpha_{i_h}\}$ of X^* there exist two possibilities: If $(i_1, \ldots, i_h)$ is a nbc, then $\dfrac{1}{\alpha_{i_1} \ldots \alpha_{i_h}} \in \mathcal{U}_{X^*}^{\mathrm{nbc}}$. Otherwise, there exists an m-subtuple $(j_1, \ldots, j_m)$ of $(i_1, \ldots, i_h)$, $1 < m < h$, such that $(j_1, \ldots, j_m)$ is a broken circuit. Thus there exists $1 \le l \le k$, $l < i_1$, such that $(l, j_1, \ldots, j_m)$ is a circuit. Equivalently we have the following relation $a_1\alpha_{j_1} + \ldots + a_m\alpha_{j_m} = \alpha_l$, for some $a_1, \ldots, a_m \in \mathbb{C}$. It implies that

$$\sum_{u=1}^{m} \frac{a_u}{\alpha_l \alpha_{j_1} \ldots \widehat{\alpha_{j_u}} \ldots \alpha_{j_m}} = \frac{1}{\alpha_{j_1} \ldots \alpha_{j_m}} \ . \tag{4.8}$$

Note that for each u in $[m]$ the set $\{\alpha_l, \alpha_{j_1}, \ldots, \widehat{\alpha_{j_u}}, \ldots, \alpha_{j_m}\}$ is linearly independent and $B_u = (\{\alpha_{i_1}, \ldots, \alpha_{i_h}\} \setminus \{\alpha_{j_u}\}) \cup \{\alpha_l\}$ is another basis to X^*. From (4.8) we get

$$\frac{a_1}{\alpha_l \alpha_{i_1} \ldots \widehat{\alpha_{j_1}} \ldots \alpha_{i_h}} + \ldots + \frac{a_m}{\alpha_l \alpha_{i_1} \ldots \widehat{\alpha_{j_m}} \ldots \alpha_{i_h}} = \frac{1}{\alpha_{i_1} \ldots \alpha_{i_h}} . \tag{4.9}$$

If the corresponding h-tuple from each basis B_u is a nbc we get from (4.9) that $\dfrac{1}{\alpha_{i_1} \ldots \alpha_{i_h}}$ is in $\langle \mathcal{U}_{X^*}^{\mathrm{nbc}} \rangle$. Otherwise, there exists at least one h-tuple $(l_1, \ldots, l_h)$ that is not a nbc. Then for each such $(l_1, \ldots, l_h)$ we can repeat the initial process, in the similar case, done with $(i_1, \ldots, i_h)$. This procedure shall finish after a finite number of steps because the cardinality of $\mathcal{U}_{X^*}$ is finite. Finally we get that $\dfrac{1}{\alpha_{i_1} \ldots \alpha_{i_h}} \in \langle \mathcal{U}_{X^*}^{\mathrm{nbc}} \rangle$.

The set $\mathcal{U}_{X^*}^{\mathrm{nbc}}$ is $\mathbb{C}$-linearly independent: Suppose that

$$\sum_{(i_1,\ldots,i_h) \in \mathcal{U}_{X^*}^{\mathrm{nbc}}} \frac{c_{i_1 \ldots i_h}}{\alpha_{i_1} \ldots \alpha_{i_h}} = 0$$

with $c_{i_1 \ldots i_h} \in \mathbb{C}$. Let l_X be the minimal among all the first entry of each h-tuple in $\mathcal{U}_{X^*}^{\mathrm{nbc}}$. Thus we can divide the last sum as

$$\frac{1}{\alpha_{l_X}} \cdot \sum_{(l_X, i_2, \ldots, i_h) \in \mathcal{U}_{X^*}^{\mathrm{nbc}}} \frac{c_{l_X i_2 \ldots i_h}}{\alpha_{i_2} \ldots \alpha_{i_h}} + \underbrace{\sum_{\substack{(i_1,\ldots,i_h) \in \mathcal{U}_{X^*}^{\mathrm{nbc}} \\ i_1 \neq l_X}} \frac{c_{i_1 \ldots i_h}}{\alpha_{i_1} \ldots \alpha_{i_h}}}_{T_X} = 0$$

or $\displaystyle\sum_{(l_X, i_2, \ldots, i_h) \in \mathcal{U}_{X^*}^{\mathrm{nbc}}} \frac{c_{l_X i_2 \ldots i_h}}{\alpha_{i_2} \ldots \alpha_{i_h}} + \alpha_{l_X} \cdot T_X = 0$. So $\displaystyle\sum_{(l_X, i_2, \ldots, i_h) \in \mathcal{U}_{X^*}^{\mathrm{nbc}}} \frac{c_{l_X i_2 \ldots i_h}}{\alpha_{i_2} \ldots \alpha_{i_h}} = 0$ within $\ker(\alpha_{l_X})$. Note that $\{\alpha_{i_2}, \ldots, \alpha_{i_h}\}$ is linearly independent modulo α_{l_X}, and $(i_2, \ldots, i_h) \in \mathcal{U}_{Y^*}^{\mathrm{nbc}}$ for some subspace $Y^* = \langle \alpha_{i_2}, \ldots, \alpha_{i_h} \rangle$ of X^* obtained after remove α_{l_X} from every basis $\{\alpha_{l_X}, \alpha_{i_2}, \ldots, \alpha_{i_h}\}$ in $\mathcal{B}_{X^*}^{\mathrm{nbc}}$. Thus we have

$$\sum_{(l_X, i_2, \ldots, i_h) \in \mathcal{U}_{X^*}^{\mathrm{nbc}}} \frac{c_{l_X i_2 \ldots i_h}}{\alpha_{i_2} \ldots \alpha_{i_h}} = 0 .$$

By induction on $\dim X^*$ we shall prove $c_{l_X i_2 \ldots i_h} = 0$ for all $(l_X, i_2, \ldots, i_h)$ in $\mathcal{U}_{X^*}^{\mathrm{nbc}}$ and $T_X = 0$. In fact, let

$$\mathcal{Z}_{X^*} = \{Y^* \subset X^* \mid Y^* = \langle \alpha_{i_2}, \ldots, \alpha_{i_h} \rangle \text{ if } (l_X, i_2, \ldots, i_h) \in \mathcal{U}_{X^*}^{\mathrm{nbc}}\}$$

and fix one Y^* in $\mathcal{Z}_{X^*}$. Then we might divide the last sum to get

$$\sum_{(l_Y, i_3, \ldots, i_h) \in \mathcal{U}_{Y^*}^{\mathrm{nbc}}} \frac{c_{l_X l_Y i_3 \ldots i_h}}{\alpha_{i_3} \ldots \alpha_{i_h}} + \alpha_{l_Y} \Bigg(\underbrace{\sum_{\substack{(i_2,\ldots,i_h) \in \mathcal{U}_{Y^*}^{\mathrm{nbc}} \\ i_2 \neq l_Y}} \frac{c_{l_X i_2 \ldots i_h}}{\alpha_{i_2} \ldots \alpha_{i_h}}}_{T_Y} + \sum_{\substack{\langle \alpha_{j_2}, \ldots, \alpha_{j_h} \rangle = Z^* \\ Z^* \in \mathcal{Z}_{X^*} \setminus \{Y^*\}}} \frac{c_{l_X j_2 \ldots j_h}}{\alpha_{j_2} \ldots \alpha_{j_h}} \Bigg) = 0$$

Then $\sum_{(l_Y,i_3,\ldots,i_h)\in\mathcal{U}^{\text{nbc}}_{Y^*}} \frac{c_{l_X l_Y i_3\ldots i_h}}{\alpha_{i_3}\ldots\alpha_{i_h}} = 0$ within $\ker(\alpha_{l_Y})$. By induction on $\dim X^*$, since $\dim Y^* < \dim X^*$, $c_{l_X l_Y i_3\ldots i_h} = 0$ for all $(l_Y, i_3, \ldots, i_h)$ in $\mathcal{U}^{\text{nbc}}_{Y^*}$ such that $(l_X, l_Y, \ldots, i_h)$ belongs to $\mathcal{U}^{\text{nbc}}_{X^*}$, and $T_Y = 0$. But this is true for every Y^* in $\mathcal{Z}_{X^*}$. Thus $c_{l_X i_2\ldots i_h} = 0$ for all $(l_X, i_2, \ldots, i_h)$ in $\mathcal{U}^{\text{nbc}}_{X^*}$. This implies that $T_X = 0$. Thus α_{l_X} appear in all basis of X^* in $\mathcal{B}^{\text{nbc}}_{X^*}$ and $\mathcal{U}^{\text{nbc}}_{X^*}$ is linearly independent. ■

The following Corollary follows from the proof of the last Lemma.

Corollary 4.2.17 *Let $X \in L_h$, $1 \le h \le r$, and let l_X be the minimal among all the first entry of each h-tuple $(i_1, \ldots, i_h)$ such that $\{\alpha_{i_1}, \ldots, \alpha_{i_h}\} \in \mathcal{B}_{X^*}$. Then $B \in \mathcal{B}^{\text{nbc}}_{X^*}$ if and only if $\alpha_{l_X} \in B$.*

Lemma 4.2.18 *Let X, Y be two elements in L_h, $1 \le h \le r$. Then $X \neq Y$ if and only if $V_{X^*} \cap V_{Y^*} = \{0\}$.*

Proof. Consider $X^* \neq Y^*$. From Lemma 4.2.16, a basis to V_{X^*} and V_{Y^*} is $\mathcal{U}^{\text{nbc}}_{X^*}$ and $\mathcal{U}^{\text{nbc}}_{Y^*}$ respectively. After a linear change of the basis $\{x_1, \ldots, x_n\}$ of $(\mathbb{C}^n)^*$ by $\{y_1, \ldots, y_n\}$ such that $X^* = \langle y_1, \ldots, y_h\rangle$, by Corollary 4.2.17 y_1 is in each basis B in $\mathcal{B}^{\text{nbc}}_{X^*}$ but it is not in any basis B' in $\mathcal{B}^{\text{nbc}}_{Y^*}$. Suppose that there exists a non-zero element v in $V_{X^*} \cap V_{Y^*}$. Then, by Proposition 4.2.14, $[v] = v$ mod P_{h-1} belongs to $M_{X^*} \otimes V^{\text{mod}}_{X^*} \cap M_{Y^*} \otimes V^{\text{mod}}_{Y^*}$ and $[v] \neq [0]$. Thus $[v]$ can be written as

$$[v] = \sum_{B\in\mathcal{B}^{\text{nbc}}_{X^*}} \left[\frac{a_B}{\prod\limits_{\alpha\in B}\alpha}\right] = \sum_{B'\in\mathcal{B}^{\text{nbc}}_{Y^*}} \left[\frac{b_{B'}}{\prod\limits_{\beta\in B'}\beta}\right] \quad , \qquad a_B,\ b_{B'} \in \mathbb{C}.$$

Clearly we get $y_1 \bullet \left[\frac{a_B}{\prod_{\alpha\in B}\alpha}\right] = [0]$ for each basis B in $\mathcal{B}^{\text{nbc}}_{X^*}$ but $y_1 \bullet \left(\sum_{B'\in\mathcal{B}^{\text{nbc}}_{Y^*}} \left[\frac{b_{B'}}{\prod_{\beta\in B'}\beta}\right]\right) \neq [0]$. It is a contradiction. ■

An immediate consequence of Lemma 4.2.18 is the following:

Corollary 4.2.19 *Let X, Y be two elements in L_h, $1 \le h \le r$. Then $X \neq Y$ if and only if $R_{X^*} \cap R_{Y^*} = \{0\}$.*

The next two lemmas enable us to write the D_n-module P as a direct sum of the R_{X^*}. We start with the following technical Lemma.

Lemma 4.2.20 *Fix two standard tuple $I = (i_1, \ldots, i_h)$ and $J = (j_1, \ldots, j_s)$ such that $h + s = n$ and $I \cup J = [n]$. Consider a polynomial f in $\mathbb{C}[y_{i_1}, \ldots, y_{i_h}, \partial_{y_{j_1}}, \ldots, \partial_{y_{j_s}}]$. Then*

(a) *If f is such that $f \bullet \dfrac{1}{y_{j_1}\cdots y_{j_s}} = 0$, then $f \equiv 0$.*

(*b*) *If* $f \cdot \partial_{y_{j_l}} \bullet \dfrac{1}{y_{j_1} \cdots y_{j_s}} = 0$, *for some* l *in* $[s]$, *then* $f \equiv 0$.

More generally, if the subset $\{\alpha_1, \ldots, \alpha_s\}$ *of* $\mathrm{Span}\{y_{j_1}, \ldots, y_{j_s}\}$ *is linearly independent, then* (a) *and* (b) *hold with* $\dfrac{1}{\alpha_1 \ldots \alpha_s}$ *instead of* $\dfrac{1}{y_{j_1} \cdots y_{j_s}}$.

Proof. We start to show (a) by induction on s: If $f \in \mathbb{C}[y_1, \ldots, y_n]$ $(s = 0)$, then it is clear that $f \equiv 0$. Now let $s > 0$. If there is not $u \in [s]$ such that $\deg_{\partial_{y_{j_u}}} f = m > 0$, then it is also clear that $f \equiv 0$, otherwise f can be written as

$$Q_m \partial^m_{y_{j_u}} + Q_{m-1} \partial^{m-1}_{y_{j_u}} + \ldots + Q_1 \partial_{y_{j_u}} + Q_0$$

where $Q_m, \ldots, Q_0 \in \mathbb{C}[y_{i_1}, \ldots, y_{i_h}, \partial_{y_{j_1}}, \ldots, \widehat{\partial_{y_{j_u}}}, \ldots, \partial_{y_{j_s}}]$ and $Q_m \neq 0$. Thus $f \bullet \dfrac{1}{y_{j_1} \cdots y_{j_s}} = 0$ is equivalent to

$$\left(\frac{(-1)^m m!}{y^{m+1}_{j_u}} Q_m + \frac{(-1)^{m-1}(m-1)!}{y^m_{j_u}} Q_{m-1} + \ldots + \frac{1}{y_{j_u}} Q_0 \right) \bullet \frac{1}{y_{j_1} \cdots \widehat{y_{j_u}} \cdots y_{j_s}} = 0$$

or

$$\left((-1)^m m! Q_m + (-1)^{m-1}(m-1)! y_{j_u} Q_{m-1} + \ldots + y^m_{j_u} Q_0\right) \bullet \frac{1}{y_{j_1} \cdots \widehat{y_{j_u}} \cdots y_{j_s}} = 0.$$

Denote by $\widetilde{f}$ the operator that acts on $\dfrac{1}{y_{j_1} \cdots \widehat{y_{j_u}} \cdots y_{j_s}}$ in the last equation. Note that $\widetilde{f}$ belongs to $\mathbb{C}[y_{i_1}, \ldots y_{i_h}, y_{j_u}, \partial_{y_{j_1}}, \ldots, \widehat{\partial_{y_{j_u}}}, \ldots, \partial_{y_{j_s}}]$. By induction on s we have $\widetilde{f} \equiv 0$. Then $Q_m = 0$ and $f \equiv 0$.

In order to show (b) note that $f \cdot \partial_{y_{j_l}} = \partial_{y_{j_l}} \cdot f$. Again, by induction on s, if $s = 0$ then $f = 0$. For $s > 0$, if there is not $u \in [s]$ such that $\deg_{\partial_{y_{j_u}}} f = m > 0$, then it is also clear that $f \cdot \partial_{y_{j_l}} \bullet \dfrac{1}{y_{j_1} \cdots y_{j_s}} = 0$ implies $f = 0$, otherwise $f \cdot \partial_{y_{j_l}}$ can be written as

$$(Q_m \partial_{y_{j_l}}) \partial^m_{y_{j_u}} + (Q_{m-1} \partial_{y_{j_l}}) \partial^{m-1}_{y_{j_u}} + \ldots + (Q_1 \partial_{y_{j_l}}) \partial_{y_{j_u}} + (Q_0 \partial_{y_{j_l}})$$

where $Q_m, \ldots, Q_0 \in \mathbb{C}[y_{i_1}, \ldots, y_{i_h}, \partial_{y_{j_1}}, \ldots, \widehat{\partial_{y_{j_u}}}, \ldots, \partial_{y_{j_s}}]$ and $Q_m \neq 0$. If $l \neq u$ then again $Q'_p = Q_p \partial_{y_{j_l}} \in \mathbb{C}[y_{i_1}, \ldots, y_{i_h}, \partial_{y_{j_1}}, \ldots, \widehat{\partial_{y_{j_u}}}, \ldots, \partial_{y_{j_s}}]$ for $p = 0, 1, \ldots, m$, and the result follows from (a). Otherwise $f \cdot \partial_{y_{j_l}} \bullet \dfrac{1}{y_{j_1} \cdots y_{j_s}} = 0$ is equivalent to

$$\left((-1)^{m+1}(m+1)! Q_m + (-1)^m m! y_{j_u} Q_{m-1} + \ldots - y^m_{j_u} Q_0\right) \bullet \frac{1}{y_{j_1} \cdots \widehat{y_{j_u}} \cdots y_{j_s}} = 0$$

and the result follows from (a) and induction on s.

The general case follows by induction on s and from relations (4.4) and (4.5). ■

Definition 4.2.21 *Let X in L_h, $1 \le h \le r$. Define the natural map of D_n-modules $\phi_X : R_{X^*} = M_{X^*} \otimes_{\mathbb{C}} V_{X^*} \longrightarrow P$ as $m \otimes v \mapsto m \bullet v$, $m \in M_{X^*}$, $v \in V_{X^*}$.*

Lemma 4.2.22 *Let X in L_h, $1 \le h \le r$. The map ϕ_X is injective.*

Proof. After a linear change of basis to $(\mathbb{C}^n)^*$ such that $X^* = \langle y_1, \ldots, y_h \rangle$, by Lemma 4.2.16, R_{X^*} can be written as $\mathbb{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}] \otimes_{\mathbb{C}} \langle \mathcal{U}_{X^*}^{\mathrm{nbc}} \rangle = \bigoplus_{B \in \mathcal{B}_{X^*}^{\mathrm{nbc}}} M_{X^*} \otimes_{\mathbb{C}} \left(\frac{1}{\prod_{\alpha \in B} \alpha} \right)$. Then ϕ_X injective is equivalent to show that $\phi_X^B : M_{X^*} \otimes_{\mathbb{C}} \left(\frac{1}{\prod_{\alpha \in B} \alpha} \right) \longrightarrow P$ is injective for each $B \in \mathcal{B}_{X^*}^{\mathrm{nbc}}$, i.e., if $Q \bullet \left(\frac{1}{\prod_{\alpha \in B} \alpha} \right) = 0$, where $Q \in \mathbb{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}]$ and $B \in \mathcal{B}_{X^*}^{\mathrm{nbc}}$, then $Q = 0$. It follows from Lemma 4.2.20. ■

An immediate consequence of Lemma 4.2.22, Corollary 4.2.19 and P_h definition is our first main result:

Theorem 4.2.23 *For $0 \le h \le r$, we have the following decomposition*

$$P_h = \bigoplus_{j=0}^{h} \bigoplus_{X \in L_j(\mathcal{A})} R_{X^*} \ .$$

In particular, since $P = P_r$, we have $P = \bigoplus_{X \in L(\mathcal{A})} R_{X^*}$.

Now we are ready to give out the following theorem:

Theorem 4.2.24 *For $0 \le h \le r$, the natural map induced by ϕ_X, $\psi : \bigoplus_{X \in L_h(\mathcal{A})} R_{X^*} \longrightarrow P_h / P_{h-1}$ is an isomorphism of D_n-modules.*

Proof. According to Proposition 4.2.14 follows that ψ is a D_n-morphism surjective.
In order to see that ψ is injective it is sufficient to demonstrate that the restricted map $\psi_X : R_{X^*} \longrightarrow P_h / P_{h-1}$ is injective for each $X \in L_h$. Recall that $\mathcal{A}_X = \{H \in \mathcal{A} \mid H \subseteq X\}$. Let $d_{\mathcal{A}_X} = \prod_{\alpha \in \mathcal{A}_X^*} \alpha$ be the homogeneous polynomial that defines the subarrangement $\mathcal{A}_X$. Define the D_n-submodule P^X of P by $\mathbb{C}[\mathbf{x}, d_{\mathcal{A}_X}^{-1}]$. By Lemma 4.2.1 P^X admits a finite ascending chain similar to one of (4.1) to P. Thus ψ_X injective is equivalent to show that $\overline{\psi}_X : R_{X^*} \longrightarrow P_h^X / P_{h-1}^X$ is injective, i.e., $V_{X^*} \cap P_{h-1}^X = \{0\}$. Let us suppose, by contradiction, that there exists a non-zero element v in $V_{X^*} \cap P_{h-1}^X$. Then after a linear change of the

basis $\{x_1, \ldots, x_n\}$ of $(\mathbb{C}^n)^*$ by $\{y_1, \ldots, y_n\}$ such that $X^* = \langle y_1, \ldots, y_h \rangle$, we can write v as

$$v = \sum_{B \in \mathcal{B}_{X^*}^{\rm nbc}} \frac{c_B}{\prod_{\alpha \in B} \alpha} = \sum \frac{a_{j_1 \ldots j_s}}{\alpha_{j_1}^{m_1} \ldots \alpha_{j_s}^{m_s}} ,$$

where the first sum belongs to V_{X^*} and $c_B \in \mathbb{C}$ for all basis B in $\mathcal{B}_{X^*}^{\rm nbc}$, the second to P_{h-1}^X, $0 \le s \le h-1$, $a_{j_1 \ldots j_s} \in \mathbb{C}[y_1, \ldots, y_n]$, $\{\alpha_{j_1}, \ldots, \alpha_{j_s}\}$ is a linear independent subset of $\mathrm{Span}\{y_1, \ldots, y_h\} \cap \mathcal{A}_X^*$ and $m_1, \ldots, m_s \in \mathbb{N}$. It is clear that $\sum_{B \in \mathcal{B}_{X^*}^{\rm nbc}} \left(c_B / \prod_{\alpha \in B} \alpha\right) \bmod P_{h-1}^X \neq [0]$, instead $\sum \left(a_{j_1 \ldots j_s} / \alpha_{j_1}^{m_1} \ldots \alpha_{j_s}^{m_s}\right) \bmod P_{h-1}^X = [0]$. It is a contradiction. ■

From Proposition 4.2.14, Lemma 4.2.16 and Theorem 4.2.24 we obtain

Corollary 4.2.25 *If $X \in L_h$, $1 \le h \le r$, then the set of coset*

$$\left\{ \frac{1}{\prod_{\alpha \in B} \alpha} \bmod P_{h-1} \mid B \in \mathcal{B}_{X^*}^{\rm nbc} \right\}$$

is a $\mathbb{C}$-basis to $V_{X^*}^{\rm mod}$.

Hence we have the completely decomposition of P_h/P_{h-1}, for every $0 \le h \le r$, after that one of the isomorphism given in (4.7).

Definition 4.2.26 *Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^n$ of rank r. Define the holonomic D_n-module $\mathcal{P} = \mathcal{P}(\mathcal{A}) = \bigoplus_{h=0}^r \mathcal{P}_h$, associated to the arrangement $\mathcal{A}$ and isomorphic to $P(\mathcal{A})$, as follows. Let $\mathcal{P}_0 = P_0 = \mathbb{C}[x_1, \ldots, x_n]$, and for h in $[r]$*

$$\mathcal{P}_h = P_h/P_{h-1} \cong \bigoplus_{X \in L_h} R_{X^*} = \bigoplus_{X \in L_h} M_{X^*} \otimes_{\mathbb{C}} \langle \mathcal{U}_{X^*}^{\rm nbc} \rangle \cong \bigoplus_{X \in L_h} M_{X^*}^{a(X^*)}$$

where $a(X^*) := \dim V_{X^*}$ is equal to $|\mathcal{U}_{X^*}^{\rm nbc}|$ $(= |\mathcal{B}_{X^*}^{\rm nbc}|)$, the multiplicity of M_{X^*}.

4.3 Examples.

In the following examples we compute the decomposition of the D_n-module $\mathcal{P}(\mathcal{A})$ for some arrangements $\mathcal{A}$. Consider the linear order on an arrangement $\mathcal{A}$

$$H_i = \ker(\alpha_i) \prec H_j = \ker(\alpha_j) \quad \Leftrightarrow \quad i < j \ .$$

Recall, by Proposition 4.2.11, that the vector spaces $V_{X^*}^B$, $B \in \mathcal{B}_{X^*}$, that correspond to a given space X^* are isomorphic to each other as D_n-modules and isomorphic to M_{X^*}, so we need compute only M_{X^*}.

Example 4.3.1 Consider a 2-arrangement $\mathcal{A}^* = \{\alpha_1, \ldots, \alpha_k\}$. A basis to $(\mathbb{C}^2)^*$ with elements of $\mathcal{A}^*$ is $\{\mathbf{y}_1 := \alpha_1, \mathbf{y}_2 := \alpha_2\}$. Each element of $\mathcal{A}^*$ can be written as a liner combination of y_1, y_2. It is easy to see that there is not any t-standard circuit for $t = 2$ or $t \ge 4$. The 3-standard circuit are (i, j, h) for all $1 \le i < j < h \le k$, then 2-broken circuit are (j, h), $2 \le j < h \le k$. The $\mathcal{C}_i$ are

$$\mathcal{C}_0 = \{1\} \ , \ \mathcal{C}_1 = \{(i) \mid i \in [k]\} \ , \ \mathcal{C}_2 = \{(1,j) \mid 2 \leq j \leq k\}$$

So

$$\mathcal{P}_0(\mathcal{A}) = \mathbb{C}[y_1, y_2] \ ,$$
$$\mathcal{P}_1(\mathcal{A}) = \mathbb{C}[y_2, \partial_{y_1}] \otimes_{\mathbb{C}} \langle 1/y_1 \rangle \oplus \bigoplus_{2 \leq i \leq k} \mathbb{C}[y_1, \partial_{y_2}] \otimes_{\mathbb{C}} \langle 1/\alpha_i \rangle \ ,$$
$$\mathcal{P}_2(\mathcal{A}) = \bigoplus_{2 \leq i \leq k} \mathbb{C}[\partial_{y_1}, \partial_{y_2}] \otimes_{\mathbb{C}} \langle 1/y_1, \alpha_i \rangle$$

Example 4.3.2 The homogeneous polynomial $d_{\mathcal{B}+} = \prod_{1 \leq i < j \leq 4}(x_i + x_j)$ defines a 4-arrangement. Put $\mathbf{y}_1 := x_1 + x_2, \ \mathbf{y}_2 := x_2 + x_3, \ \mathbf{y}_3 := x_3 + x_4, \ \mathbf{y}_4 := x_1 + x_3$, then $\{y_1, \ y_2, \ y_3, \ y_4\}$ is a basis to $(\mathbb{C}^4)^*$. The remaining linear forms in $(\mathcal{B}^+)^*$ have the following expression in this new basis: $\mathbf{y}_5 := x_1 + x_4 = y_1 - y_2 + y_3$, $\mathbf{y}_6 := x_2 + x_4 = y_1 - y_4 + y_3$. Clearly there is not any t-standard circuit for $t = 3$ or $t \geq 5$. The 4-standard circuit are $(1,2,3,5), \ (1,3,4,6), \ (2,4,5,6)$, then 3-broken circuit are $(2,3,5), \ (3,4,6), \ (4,5,6)$. Then the $\mathcal{C}_i$ are

$$\mathcal{C}_0 = \{1\} \ ,$$
$$\mathcal{C}_1 = \{(1),(2),(3),(4),(5),(6)\} \ ,$$
$$\mathcal{C}_2 = \{(1,2),(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6) \\ (4,5),(4,6),(5,6)\},$$
$$\mathcal{C}_3 = \{(1,2,3),(1,2,4),(1,2,5),(1,2,6),(1,3,4),(1,3,5),(1,3,6),(1,4,5), \\ (1,4,6),(1,5,6),(2,3,4),(2,3,6),(2,4,5),(2,4,6),(2,5,6),(3,4,5), \\ (3,5,6)\}$$
$$\mathcal{C}_4 = \{(1,2,3,4),(1,2,3,6),(1,2,4,5),(1,2,4,6),(1,2,5,6),(1,3,4,5), \\ (1,3,5,6)\}$$

So

$$\mathcal{P}_0(\mathcal{B}^+) = \mathbb{C}[y_1, y_2, y_3, y_4] \ ,$$
$$\mathcal{P}_1(\mathcal{B}^+) = \Big(\bigoplus_{1 \leq i \leq 4} \mathbb{C}\,[y_1, ..., \widehat{y_i}, ..., y_4, \partial_{y_i}] \otimes \langle 1/y_i \rangle \Big) \oplus$$
$$\oplus \mathbb{C}[y_1, y_3, y_4, \partial_{y_2}] \otimes \langle 1/y_1 - y_2 + y_3 \rangle \oplus$$
$$\oplus \mathbb{C}[y_1, y_2, y_3, \partial_{y_4}] \otimes \langle 1/y_1 - y_4 + y_3 \rangle$$

For $X_{12}^* = \langle y_1, y_2 \rangle : R_{X_{12}^*} = \mathbb{C}\,[y_3, y_4, \partial_{y_1}, \partial_{y_2}] \otimes \langle 1/y_1 y_2 \rangle$
For $X_{13}^* = \langle y_1, y_3 \rangle : R_{X_{13}^*} = \mathbb{C}\,[y_2, y_4, \partial_{y_1}, \partial_{y_3}] \otimes \langle 1/y_1 y_3 \rangle$
For $X_{14}^* = \langle y_1, y_4 \rangle : R_{X_{14}^*} = \mathbb{C}\,[y_2, y_3, \partial_{y_1}, \partial_{y_4}] \otimes \langle 1/y_1 y_4 \rangle$
For $X_{15}^* = \langle y_1, y_1 - y_2 + y_3 \rangle : R_{X_{15}^*} = \mathbb{C}\,[y_3, y_4, \partial_{y_1}, \partial_{y_2}] \otimes \langle 1/y_1(y_1 - y_2 + y_3) \rangle$
For $X_{16}^* = \langle y_1, y_1 - y_4 + y_3 \rangle : R_{X_{16}^*} = \mathbb{C}\,[y_2, y_4, \partial_{y_1}, \partial_{y_3}] \otimes \langle 1/y_1(y_1 - y_4 + y_3) \rangle$
For $X_{23}^* = \langle y_2, y_3 \rangle : R_{X_{23}^*} = \mathbb{C}\,[y_1, y_4, \partial_{y_2}, \partial_{y_3}] \otimes \langle 1/y_2 y_3 \rangle$
For $X_{24}^* = \langle y_2, y_4 \rangle : R_{X_{24}^*} = \mathbb{C}\,[y_1, y_3, \partial_{y_2}, \partial_{y_4}] \otimes \langle 1/y_2 y_4 \rangle$

For $X^*_{25} = \langle y_2, y_1 - y_2 + y_3\rangle$: $R_{X^*_{25}} = \mathbb{C}[y_3, y_4, \partial_{y_1}, \partial_{y_2}] \otimes \langle 1/y_2(y_1 - y_2 + y_3)\rangle$
For $X^*_{26} = \langle y_2, y_1 - y_4 + y_3\rangle$: $R_{X^*_{26}} = \mathbb{C}[y_1, y_4, \partial_{y_2}, \partial_{y_3}] \otimes \langle 1/y_2(y_1 - y_4 + y_3)\rangle$
For $X^*_{34} = \langle y_3, y_4\rangle$: $R_{X^*_{34}} = \mathbb{C}[y_1, y_2, \partial_{y_3}, \partial_{y_4}] \otimes \langle 1/y_3y_4\rangle$
For $X^*_{35} = \langle y_3, y_1 - y_2 + y_3\rangle$: $R_{X^*_{35}} = \mathbb{C}[y_1, y_4, \partial_{y_2}, \partial_{y_3}] \otimes \langle 1/y_3(y_1 - y_2 + y_3)\rangle$
For $X^*_{36} = \langle y_3, y_1 - y_4 + y_3\rangle$: $R_{X^*_{36}} = \mathbb{C}[y_1, y_2, \partial_{y_3}, \partial_{y_4}] \otimes \langle 1/y_3(y_1 - y_4 + y_3)\rangle$
For $X^*_{45} = \langle y_4, y_1 - y_2 + y_3\rangle$: $R_{X^*_{45}} = \mathbb{C}[y_1, y_2, \partial_{y_3}, \partial_{y_4}] \otimes \langle 1/y_4(y_1 - y_2 + y_3)\rangle$
For $X^*_{46} = \langle y_4, y_1 - y_4 + y_3\rangle$: $R_{X^*_{46}} = \mathbb{C}[y_1, y_2, \partial_{y_3}, \partial_{y_4}] \otimes \langle 1/y_4(y_1 - y_4 + y_3)\rangle$
For $X^*_{56} = \langle y_1 - y_2 + y_3, y_1 - y_4 + y_3\rangle$:

$$R_{X^*_{56}} = \mathbb{C}[y_1, y_3, \partial_{y_2}, \partial_{y_4}] \otimes \langle 1/(y_1 - y_2 + y_3)(y_1 - y_4 + y_3)\rangle$$

For $X^*_{123} = \langle y_1, y_2, y_3\rangle = \langle y_1, y_2, y_5\rangle = \langle y_1, y_3, y_5\rangle = \langle y_2, y_3, y_5\rangle$:

$$R_{X^*_{123}} = \mathbb{C}[y_4, \partial_{y_1}, \partial_{y_2}, \partial_{y_3}] \otimes \langle 1/y_1y_2y_3, 1/y_1y_2(y_1 - y_2 + y_3), 1/y_1y_3(y_1 - y_2 + y_3)\rangle$$

For $X^*_{124} = \langle y_1, y_2, y_4\rangle$: $R_{X^*_{124}} = \mathbb{C}[y_3, \partial_{y_1}, \partial_{y_2}, \partial_{y_4}] \otimes \langle 1/y_1y_2y_4\rangle$
For $X^*_{126} = \langle y_1, y_2, y_1 - y_4 + y_3\rangle$:

$$R_{X^*_{126}} = \mathbb{C}[y_4, \partial_{y_1}, \partial_{y_2}, \partial_{y_3}] \otimes \langle 1/y_1y_2(y_1 - y_4 + y_3)\rangle$$

For $X^*_{134} = \langle y_1, y_3, y_4\rangle = \langle y_1, y_3, y_6\rangle = \langle y_1, y_4, y_6\rangle = \langle y_3, y_4, y_6\rangle$:

$$R_{X^*_{134}} = \mathbb{C}[y_2, \partial_{y_1}, \partial_{y_3}, \partial_{y_4}] \otimes \langle 1/y_1y_3y_4, 1/y_1y_3(y_1 - y_4 + y_3), 1/y_1y_4(y_1 - y_4 + y_3)\rangle$$

For $X^*_{145} = \langle y_1, y_4, y_1 - y_2 + y_3\rangle$:

$$R_{X^*_{145}} = \mathbb{C}[y_2, \partial_{y_1}, \partial_{y_3}, \partial_{y_4}] \otimes \langle 1/y_1y_4(y_1 - y_2 + y_3)\rangle$$

For $X^*_{156} = \langle y_1, y_1 - y_2 + y_3, y_1 - y_4 + y_3\rangle$:

$$R_{X^*_{156}} = \mathbb{C}[y_3, \partial_{y_1}, \partial_{y_2}, \partial_{y_4}] \otimes \langle 1/y_1(y_1 - y_2 + y_3)(y_1 - y_4 + y_3)\rangle$$

For $X^*_{234} = \langle y_2, y_3, y_4\rangle$: $R_{X^*_{234}} = \mathbb{C}[y_1, \partial_{y_2}, \partial_{y_3}, \partial_{y_4}] \otimes \langle 1/y_2y_3y_4\rangle$
For $X^*_{236} = \langle y_2, y_3, y_1 - y_4 + y_3\rangle$:

$$R_{X^*_{236}} = \mathbb{C}[y_1, \partial_{y_2}, \partial_{y_3}, \partial_{y_4}] \otimes \langle 1/y_2y_3(y_1 - y_4 + y_3)\rangle$$

For $X^*_{245} = \langle y_2, y_4, y_5\rangle = \langle y_2, y_4, y_6\rangle = \langle y_2, y_5, y_6, \rangle = \langle y_4, y_5, y_6\rangle$:

$$R_{X^*_{245}} = \mathbb{C}[y_1, \partial_{y_2}, \partial_{y_3}, \partial_{y_4}] \otimes \langle 1/y_2y_4(y_1 - y_2 + y_3),\ 1/y_2y_4(y_1 - y_4 + y_3), 1/y_2(y_1 - y_2 + y_3)(y_1 - y_4 + y_3)\rangle$$

For $X^*_{345} = \langle y_3, y_4, y_1 - y_2 + y_3\rangle$:

$$R_{X^*_{345}} = \mathbb{C}[y_1, \partial_{y_2}, \partial_{y_3}, \partial_{y_4}] \otimes \langle 1/y_3y_4(y_1 - y_2 + y_3)\rangle$$

For $X^*_{356} = \langle y_3, y_1 - y_2 + y_3, y_1 - y_4 + y_3\rangle$:

$$R_{X^*_{356}} = \mathbb{C}[y_1, \partial_{y_2}, \partial_{y_3}, \partial_{y_4}] \otimes \langle 1/y_3(y_1 - y_2 + y_3)(y_1 - y_4 + y_3)\rangle$$

For $X^*_{1234} = \langle y_1, y_2, y_3, y_4\rangle = \langle y_1, y_2, y_3, y_6\rangle = \langle y_1, y_2, y_4, y_5\rangle = \langle y_1, y_2, y_4, y_6\rangle$
$= \langle y_1, y_2, y_5, y_6\rangle = \langle y_1, y_3, y_4, y_5\rangle = \langle y_1, y_3, y_5, y_6\rangle = \langle y_1, y_4, y_5, y_6\rangle$
$= \langle y_2, y_3, y_4, y_5\rangle = \langle y_2, y_3, y_4, y_6\rangle = \langle y_2, y_3, y_5, y_6\rangle = \langle y_3, y_4, y_5, y_6\rangle$,

we get $\mathcal{U}^{\text{nbc}}_{X^*_{1234}} = \{1/y_1y_2y_3y_4,\ 1/y_1y_2y_3(y_1 - y_4 + y_3),\ 1/y_1y_2y_4(y_1 - y_2 + y_3),$
$1/y_1y_2y_4(y_1 - y_4 + y_3),\ 1/y_1y_2(y_1 - y_2 + y_3)(y_1 - y_4 + y_3),$
$1/y_1y_3y_4(y_1 - y_2 + y_3),\ 1/y_1y_3(y_1 - y_2 + y_3)(y_1 - y_4 + y_3)\}$

and $M_{X^*_{1234}} = \mathbb{C}[\partial_{y_1}, \partial_{y_2}, \partial_{y_3}, \partial_{y_4}]$. So

$$\mathcal{P}_4(\mathcal{B}^+) = M_{X^*_{1234}} \otimes \langle V^{\text{nbc}}_{X^*_{1234}}\rangle\ .$$

Example 4.3.3 The 4-braid arrangement is defined by $d_{\mathcal{B}r} = \prod_{1\le i<j\le 4}(x_i - x_i)$. It is not possible to get a basis to $(\mathbb{C}^4)^*$ only with elements of $\mathcal{B}r^*$. If we put $\mathbf{y}_1 := x_1 - x_2$, $\mathbf{y}_2 := x_2 - x_3$, $\mathbf{y}_3 := x_3 - x_4$, $\mathbf{y}'_4 := x_1$ we have a basis $\{y_1, y_2, y_3, y'_4\}$ to $(\mathbb{C}^4)^*$. Note that $\{y_1, y_2, y_3\}$ is a maximal linearly independent subset of $\mathcal{B}r^*$, so $r(\mathcal{A}) = 3$ and the remaining linear forms in $\mathcal{B}r^*$ have the following expression $y_4 := x_1 - x_3 = y_1 + y_2$, $y_5 := x_2 - x_4 = y_2 + y_3$, $y_6 := x_1 - x_4 = y_1 + y_2 + y_3$. Thus the 3-standard circuit are $(1,2,4)$, $(1,5,6)$, $(2,3,5)$, $(3,4,6)$, then 2-broken circuit are $(2,4)$, $(3,5)$, $(4,6)$, $(5,6)$. The 4-standard circuit are $(1,2,3,6)$, $(1,3,4,5)$, $(2,4,5,6)$, then 3-broken circuit are $(2,3,6)$, $(3,4,5)$, $(4,5,6)$. There isn't any t-standard circuit for $t \ge 5$. Then the basis for every $\mathcal{C}_i$ are

$\mathcal{C}_0 = \{1\}$,
$\mathcal{C}_1 = \{(1),(2),(3),(4),(5),(6)\}$,
$\mathcal{C}_2 = \{(1,2),(1,3),(1,4),(1,5),(1,6),(2,3),(2,5),(2,6),(3,4),(3,6),(4,5)\}$
$\mathcal{C}_3 = \{(1,2,3),(1,2,5),(1,2,6),(1,3,4),(1,3,6),(1,4,5)\}$

$$\mathcal{P}_0(\mathcal{B}_r) = \mathbb{C}[y_1, y_2, y_3, y'_4]\ ,$$

$$\mathcal{P}_1(\mathcal{B}_r) = \left(\bigoplus_{1\le i\le 3} \mathbb{C}[y_1, ..., \widehat{y_i}, ...y_3, y'_4, \partial_{y_i}]\otimes\frac{1}{y_i}\right)\oplus\mathbb{C}[y_2, y_3, y'_4, \partial_{y_1}]\otimes\frac{1}{y_1+y_2}\oplus$$
$$\oplus\mathbb{C}[y_1, y_3, y'_4, \partial_{y_2}]\otimes\frac{1}{y_2+y_3}\oplus\mathbb{C}[y_2, y_3, y'_4, \partial_{y_1}]\otimes\frac{1}{y_1+y_2+y_3}\ ,$$

For $X^*_1 = \langle y_1, y_2\rangle = \langle y_1, y_4\rangle = \langle y_2, y_4\rangle$:

$$R_{X^*_1} = \mathbb{C}[y_3, y'_4, \partial_{y_1}, \partial_{y_2}] \otimes \langle 1/y_1y_2,\ 1/y_1(y_1 + y_2)\rangle$$

For $X^*_2 = \langle y_1, y_5\rangle = \langle y_1, y_6\rangle = \langle y_5, y_6\rangle$:

$$R_{X^*_2} = \mathbb{C}[y_3, y'_4, \partial_{y_1}, \partial_{y_2}] \otimes \langle 1/y_1(y_2 + y_3),\ 1/y_1(y_1 + y_2 + y_3)\rangle$$

For $X^*_3 = \langle y_3, y_4\rangle = \langle y_3, y_6\rangle = \langle y_4, y_6\rangle$:

$$R_{X^*_3} = \mathbb{C}[y_2, y'_4, \partial_{y_1}, \partial_{y_3}] \otimes \langle 1/y_3(y_1 + y_2),\ 1/y_3(y_1 + y_2 + y_3)\rangle$$

For $X_4^* = \langle y_2, y_3 \rangle = \langle y_2, y_5 \rangle = \langle y_3, y_5 \rangle$:

$$R_{X_4^*} = \mathbb{C}[y_1, y_4', \partial_{y_2}, \partial_{y_3}] \otimes \langle 1/y_2 y_3,\ 1/y_2(y_2 + y_3) \rangle$$

For $X_5^* = \langle y_1, y_3 \rangle : R_{X_5^*} = \mathbb{C}[y_2, y_4', \partial_{y_1}, \partial_{y_3}] \otimes \langle 1/y_1 y_3 \rangle$

For $X_6^* = \langle y_2, y_6 \rangle : R_{X_6^*} = \mathbb{C}[y_3, y_4', \partial_{y_1}, \partial_{y_2}] \otimes \langle 1/y_2(y_1 + y_2 + y_3) \rangle$

For $X_7^* = \langle y_4, y_5 \rangle : R_{X_7^*} = \mathbb{C}[y_3, y_4', \partial_{y_1}, \partial_{y_2}] \otimes \langle 1/(y_1 + y_2)(y_2 + y_3) \rangle$

For $X^* = \langle y_1, y_2, y_3 \rangle = \langle y_1, y_2, y_5 \rangle = \langle y_1, y_2, y_6 \rangle = \langle y_1, y_3, y_4 \rangle = \langle y_1, y_3, y_6 \rangle$
$= \langle y_1, y_4, y_5 \rangle = \langle y_2, y_3, y_6 \rangle = \langle y_3, y_4, y_5 \rangle = \langle y_4, y_5, y_6 \rangle$:

$$\mathcal{P}_3(\mathcal{B}_r) = \mathbb{C}[y_4', \partial_{y_1}, \partial_{y_2}, \partial_{y_3}] \otimes \langle 1/y_1 y_2 y_3,\ 1/y_1 y_2 y_5,\ 1/y_1 y_2 y_6,\ 1/y_1 y_3 y_4,\ 1/y_1 y_3 y_6,\ 1/y_1 y_4 y_5 \rangle$$

Chapter 5

Complexes and cohomology of $Y_{\mathcal{A}}$

5.1 Some Complexes.

We begin defining some useful cochain complexes $\mathcal{L}_h^*$, $\mathcal{G}_h^*$, $\mathcal{H}_h^*$, $0 \leq h \leq n$. The first complex $\mathcal{L}_h$, cf. (5.1), is associated to every basis B in $\mathcal{B}_{X^*}$, $X \in L_h$, and then we get a complex $\mathcal{L}(\mathcal{P}_h) = \bigoplus_{X \in L_h} \bigoplus_{B \in \mathcal{B}_{X^*}^{\mathrm{nbc}}} \mathcal{L}_h(B)$ associated to $\mathcal{P}_h$. The cohomology of $\mathcal{L}(\mathcal{P}_h)$ is the h-th De Rham cohomology of $Y_{\mathcal{A}}$, cf. Theorem 5.2.8.

Fixed h, $0 \leq h \leq n$, we define the following cochain complexes (5.1), (5.2) and (5.3):

$$\mathcal{L}_h^* = \mathcal{L}_h^*(\langle y_1, \ldots, y_h \rangle) : \ 0 \to \mathcal{L}_h^0 \overset{\delta_{\mathcal{L}}^0}{\to} \mathcal{L}_h^1 \overset{\delta_{\mathcal{L}}^1}{\to} \mathcal{L}_h^2 \to \quad \ldots \quad \to \mathcal{L}_h^{n-1} \overset{\delta_{\mathcal{L}}^{n-1}}{\to} \mathcal{L}_h^n \overset{\delta_{\mathcal{L}}^n}{\to} 0 \tag{5.1}$$

where

$$\mathcal{L}_h^0 = \mathbb{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}] \bullet \frac{1}{y_1 \cdots y_h} \ ,$$

$$\mathcal{L}_h^s = \left\{ \sum_{1 \leq i_1 < \ldots < i_s \leq n} f_{i_1 \ldots i_s} \bullet \frac{1}{y_1 \cdots y_h} dy_{i_1} \wedge \ldots \wedge dy_{i_s} \right\} \ , \ s = 1, \ldots, n,$$

$f_{i_1 \ldots i_s} \in \mathbb{C}[y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}]$. If we denote by $I = (i_1, \ldots, i_s)$ and $dy_I = dy_{i_1} \wedge \ldots \wedge dy_{i_s}$, then every element ω in $\mathcal{L}_h^s$ can be written as $\omega = \sum f_I \bullet \frac{1}{y_1 \ldots y_h} dy_I$. The differential $\delta_{\mathcal{L}} : \mathcal{L}_h \to \mathcal{L}_h$ is the usual differential defined as follows:

(i) If $\omega = f \bullet \dfrac{1}{y_1 \ldots y_h} \in \mathcal{L}_h^0$, then

$$\delta_{\mathcal{L}}^0 \omega = \sum_{i=1}^n \left(\partial_{y_i} \left(f \bullet \frac{1}{y_1 \ldots y_h} \right) \right) dy_i \ .$$

(ii) If $\omega = \sum f_I \bullet \frac{1}{y_1 \cdots y_h} dy_I \in \mathcal{L}_h^s$, where $s = card(I) = deg(\omega) > 0$, then

$$\delta_{\mathcal{L}}^s \omega = \sum \delta_{\mathcal{L}}^0 \left(f_I \bullet \frac{1}{y_1 \cdots y_h} \right) dy_I \; .$$

It is clear that $\delta_{\mathcal{L}} \circ \delta_{\mathcal{L}} = 0$. This is basically a consequence of the facts that $\delta_{\mathcal{L}}$ is an antiderivation, i.e., $\delta_{\mathcal{L}}(\tau \wedge \omega) = (\delta_{\mathcal{L}} \tau) \wedge \omega + (-1)^{deg \tau} \tau \wedge \delta_{\mathcal{L}} \omega$, and the mixed partials are equal.

$$(5.2) \qquad \mathcal{G}_h^* : \; 0 \longrightarrow \mathcal{G}_h^0 \xrightarrow{\delta_{\mathcal{G}}^0} \mathcal{G}_h^1 \xrightarrow{\delta_{\mathcal{G}}^1} \mathcal{G}_h^2 \longrightarrow \quad \ldots \quad \longrightarrow \mathcal{G}_h^{h-1} \xrightarrow{\delta_{\mathcal{G}}^{h-1}} \mathcal{G}_h^h \xrightarrow{\delta_{\mathcal{G}}^h} 0$$

where

$$\mathcal{G}_h^0 = \mathbb{C}[\partial_{y_1}, \ldots, \partial_{y_h}] \bullet \frac{1}{y_1 \cdots y_h} \; ,$$

$$\mathcal{G}_h^r = \left\{ \sum_{1 \leq i_1 < \ldots < i_r \leq h} f_{i_1 \ldots i_r} \bullet \frac{1}{y_1 \cdots y_h} dy_{i_1} \wedge \ldots \wedge dy_{i_r} \right\} \; , \; r = 1, \ldots, h,$$

$f_{i_1 \ldots i_r} \in \mathbb{C}[\partial_{y_1}, \ldots, \partial_{y_h}]$, and the differential $\delta_{\mathcal{G}} : \mathcal{G}_h \to \mathcal{G}_h$ is the usual differential defined in the same way of $\mathcal{L}_h$.

Finally, consider the de Rham subcomplex on $\mathbb{C}^{n-h}$:

$$(5.3) \quad \mathcal{H}_h^* : \; 0 \to \mathcal{H}_h^0 \xrightarrow{\delta_{\mathcal{H}}^0} \mathcal{H}_h^1 \xrightarrow{\delta_{\mathcal{H}}^1} \mathcal{H}_h^2 \to \quad \ldots \quad \to \mathcal{H}_h^{n-h-1} \xrightarrow{\delta_{\mathcal{H}}^{n-h-1}} \mathcal{H}_h^{n-h} \xrightarrow{\delta_{\mathcal{H}}^{n-h}} 0$$

where

$$\mathcal{H}_h^0 = \mathbb{C}[y_{h+1}, \ldots, y_n] \; ,$$

$$\mathcal{H}_h^t = \left\{ \sum_{h+1 \leq i_1 < \ldots < i_t \leq n} f_{i_1 \ldots i_t} dy_{i_1} \wedge \ldots \wedge dy_{i_t} \right\} \; , \; t = 1, \ldots, n-h,$$

$f_{i_1 \ldots i_t} \in \mathbb{C}[y_{h+1}, \ldots, y_n]$, and the differential $\delta_{\mathcal{H}} : \mathcal{H}_h \to \mathcal{H}_h$ is the usual differential defined in the same way of $\mathcal{L}_h$.

Lemma 5.1.1 *The complex $\mathcal{G}_h$ has cohomology*

$$H^*(\mathcal{G}_h) = \begin{cases} \mathbb{C} \cdot \frac{1}{y_1 \cdots y_h} dy_1 \wedge \ldots \wedge dy_h & \text{in dimension } h \; , \\ 0 & \text{elsewhere} \; . \end{cases}$$

Proof. For $r = 0$: Let $\omega = f \bullet \frac{1}{y_1 \cdots y_h} \in \mathcal{G}_h^0$. If $\delta_{\mathcal{G}}^0 \omega = \sum_{i=1}^h (f \cdot \partial_{y_i}) \bullet \frac{1}{y_1 \cdots y_h} dy_i = 0$, then we have that $\delta_{\mathcal{G}}^0 \omega \wedge (dy_1 \ldots \widehat{dy_i} \ldots dy_h) = (-1)^{i-1} (f \cdot \partial_{y_i}) \bullet \frac{1}{y_1 \cdots y_h} dy_1 \ldots dy_h = 0$ for all $i \in [h]$. It is possible if and only if $(f \cdot \partial_{y_i}) \bullet$

$\frac{1}{y_1 \ldots y_h} = 0$. By Lemma 4.2.20 (b) we have $f = 0$. Thus we have $\ker(\delta^0_{\mathcal{G}}) = \{0\}$ and $H^0(\mathcal{G}_h) = 0$.

For $0 < r < h$: Let $\omega = \sum_{1 \leq i_1 < \ldots < i_r \leq h} f_{i_1 \ldots i_r} \bullet \frac{1}{y_1 \ldots y_h} dy_{i_1} \ldots dy_{i_r}$ be an element in $\mathcal{G}^r_h$. If $\delta^r_{\mathcal{G}}\omega = \sum_{1 \leq l_1 < \ldots < l_r < l_{r+1} \leq h} \left(\sum_{j=1}^{r+1} (-1)^{j-1} f_{l_1 \ldots \widehat{l_j} \ldots l_{r+1}} \cdot \partial_{y_j} \right) \bullet \frac{1}{y_1 \ldots y_h} dy_{l_1} \ldots dy_{l_{r+1}} = 0$, where $\{l_1, \ldots, \widehat{l_j}, \ldots, l_{r+1}\}$ is equal to some $\{i_1, \ldots, i_r\}$, then, in analogue way for the case $r = 0$, we have $\left(\sum_{j=1}^{r+1} (-1)^{j-1} f_{l_1 \ldots \widehat{l_j} \ldots l_{r+1}} \cdot \partial_{y_j} \right) \bullet \frac{1}{y_1 \ldots y_h} = 0$ for all $1 \leq l_1 < \ldots < l_r < l_{r+1} \leq h$. By Lemma 4.2.20 it is possible if and only if $\sum_{j=1}^{r+1} (-1)^{j-1} f_{l_1 \ldots \widehat{l_j} \ldots l_{r+1}} \cdot \partial_{y_j} = 0$. The last equality above is true if and only if $f_{i_1 \ldots i_r} = 0$ for all $1 \leq i_1 < \ldots < i_r \leq h$. Thus we have again that $\ker(\delta^r_{\mathcal{G}}) = \{0\}$ and $H^r(\mathcal{G}_h) = 0$ for $0 < r < h$.

Finally, for $r = h$ let $\omega = f \bullet \frac{1}{y_1 \ldots y_h} dy_1 \ldots dy_h \in \mathcal{G}^h_h$, then $\delta^h_{\mathcal{G}}(\omega) = 0$ for all ω. Thus $\ker(\delta^h_{\mathcal{G}}) = \mathcal{G}^h_h$. Since

$$Im(\delta^{h-1}_{\mathcal{G}}) =$$
$$\left\{ \left(f_1 \cdot \partial_{y_1} - f_2 \cdot \partial_{y_2} + \ldots + (-1)^{h-1} f_h \cdot \partial_{y_h} \right) \bullet \frac{1}{y_1 \ldots y_h} dy_1 \ldots dy_h \right\},$$

we get that $H^h(\mathcal{G}_h) = \mathbb{C} \cdot \frac{1}{y_1 \ldots y_h} dy_1 \ldots dy_h$. ■

Lemma 5.1.2 *The complex $\mathcal{H}_h$ has cohomology*

$$H^*(\mathcal{H}_h) = \begin{cases} \mathbb{C} & \text{in dimension } 0, \\ 0 & \text{elsewhere} . \end{cases}$$

Proof. This is a consequence of the fact that $\mathcal{H}_h$ is the subcomplex of the de Rham complex $\mathbf{\Omega}_{DR}(\mathbb{C}^{n-h})$ on $\mathbb{C}^{n-h}$. ■

Proposition 5.1.3 *There exists the following relation between the complexes $\mathcal{L}_h$, $\mathcal{G}_h$ and $\mathcal{H}_h$:*

$$\mathcal{L}_h = \mathcal{G}_h \otimes_{\mathbb{C}} \mathcal{H}_h \ .$$

Proof. We will prove, cf. [11], that:

1. $\mathcal{L}^s_h = \bigoplus_{r+t=s} \mathcal{G}^r_h \otimes_{\mathbb{C}} \mathcal{H}^t_h$ $(= (\mathcal{G}_h \otimes_{\mathbb{C}} \mathcal{H}_h)^s)$, and
2. $\delta^s_{\mathcal{L}} = \delta^s_{\mathcal{G} \otimes \mathcal{H}} : (\mathcal{G}_h \otimes_{\mathbb{C}} \mathcal{H}_h)^s \to (\mathcal{G}_h \otimes_{\mathbb{C}} \mathcal{H}_h)^{s+1}$.

To prove 1. it is sufficient to see that every standard s-tuple $(i_1, \ldots, i_s)$ is decomposable in two standard tuples: an r-tuple $(i_1, \ldots, i_r)$, $1 \leq i_1 < \ldots < i_r \leq h$, and an $(s-r)$-tuple $(i_{r+1}, \ldots, i_s)$, $h+1 \leq i_{r+1} < \ldots < i_s \leq n$, for some $0 \leq r \leq h$. So every monomial of $f_{i_1 \ldots i_s}(y_{h+1}, \ldots, y_n, \partial_{y_1}, \ldots, \partial_{y_h}) \bullet \frac{1}{y_1 \ldots y_h} dy_{i_1} \ldots dy_{i_s} \in \mathcal{L}^s_h$ can be written as

$$c_{j_1\dots j_n} y_{h+1}^{j_{h+1}} \dots y_n^{j_n} \partial_{y_1}^{j_1} \dots \partial_{y_h}^{j_h} \bullet \frac{1}{y_1 \dots y_h} dy_{i_1} \dots dy_{i_r} dy_{i_{r+1}} \dots dy_{i_s}$$

where $(j_1, \dots, j_n) \in \mathbb{N}^n$ and $c_{j_1\dots j_n} \in \mathbb{C}$. It is possible to write out as:

$$\left(\partial_{y_1}^{j_1} \dots \partial_{y_h}^{j_h} \bullet \frac{1}{y_1 \dots y_h} dy_{i_1} \dots dy_{i_r}\right) \otimes_{\mathbb{C}} \left(c_{j_1\dots j_n} y_{h+1}^{j_{h+1}} \dots y_n^{j_n} dy_{i_{r+1}} \dots dy_{i_s}\right)$$

where the first factor belong to $\mathcal{G}_h^r$ and the second to $\mathcal{H}_h^{s-r}$. So $\mathcal{L}_h^s \subseteq \bigoplus_{r+t=s} \mathcal{G}_h^r \otimes_{\mathbb{C}} \mathcal{H}_h^t$. The second inclusion is obvious.

In order to show 2. we will show that if $s = r + t$ for some $0 \leq r \leq h$ then

$$\delta^s_{\mathcal{G}\otimes\mathcal{H}}|_{\mathcal{G}^r\otimes\mathcal{H}^t} = \delta^s_{\mathcal{L}}|_{\mathcal{G}^r\otimes\mathcal{H}^t} \ .$$

By definition of $\delta_{\mathcal{G}\otimes\mathcal{H}}$, $\delta_{\mathcal{L}}$, $\delta_{\mathcal{G}}$, $\delta_{\mathcal{H}}$, we have

$$\begin{array}{rcccc} \delta^s_{\mathcal{G}\otimes\mathcal{H}}|_{\mathcal{G}^r\otimes\mathcal{H}^t} & = & \delta^r_{\mathcal{G}^r} \otimes \mathrm{Id}_{\mathcal{H}^t} & + & (-1)^r \mathrm{Id}_{\mathcal{G}^r} \otimes \delta^t_{\mathcal{H}^t} \\ & = & \delta^r_{\mathcal{L}}|_{\mathcal{G}^r} \otimes \mathrm{Id}_{\mathcal{L}}|_{\mathcal{H}^t} & + & (-1)^r \mathrm{Id}_{\mathcal{L}}|_{\mathcal{G}^r} \otimes \delta^t_{\mathcal{L}}|_{\mathcal{H}^t} \\ & = & \delta^s_{\mathcal{L}}|_{\mathcal{G}^r_h\otimes\mathcal{H}^t_h} & & . \end{array}$$

■

Corollary 5.1.4 *The complex $\mathcal{L}_h = \mathcal{L}_h(\langle y_1, \dots, y_h\rangle)$ has cohomology*

$$H^*(\mathcal{L}_h(\langle y_1, \dots, y_h\rangle)) = \begin{cases} \mathbb{C} \cdot \dfrac{1}{y_1 \dots y_h} dy_1 \dots dy_h & \text{in dimension } h, \\ 0 & \text{elsewhere} \ . \end{cases}$$

Proof. Thanks to Proposition 5.1.3 and by the algebraic Künneth formula for the cohomology of a tensor product of a couple of complexes, we have that

$$H^s(\mathcal{L}_h) = \bigoplus_{r+t=s} H^r(\mathcal{G}_h) \otimes_{\mathbb{C}} H^t(\mathcal{H}_h) \ .$$

Hence the result follows from Lemmas 5.1.1 and 5.1.2 ■

This preliminary result enables us to calculate the cohomology of $Y_{\mathcal{A}}$.

5.2 Cohomology of $Y_{\mathcal{A}}$.

Definition 5.2.1 *Let $H = \ker(\alpha_H)$ and let $Y_H = \mathbb{C}^n \setminus H$. The map $\alpha_H : \mathbb{C}^n \longrightarrow \mathbb{C}$ restricts to $\alpha_H : Y_H \longrightarrow \mathbb{C}^*$. Choose the canonical generator of $H^*(\mathbb{C}^*)$ as $(1/2\pi i)(dz/z)$. Define a rational 1-form*

$$\eta_H = \frac{1}{2\pi i} \frac{d\alpha_H}{\alpha_H}$$

on $\mathbb{C}^n$. Let $\langle \eta_H \rangle$ be the cohomology class of η_H in $H^1(Y_H)$. Then

$$\langle \eta_H \rangle = \alpha_H^* \left(\frac{1}{2\pi i} \frac{dz}{z}\right) \in H^1(Y_H)$$

Denote the cohomology class of η_H in $H^1(Y_{\mathcal{A}})$ by $[\eta_H]$. Let $i : Y_{\mathcal{A}} \longrightarrow Y_H$ be the inclusion map. Then $[\eta_H] = i^*\langle \eta_H \rangle$

Recall the exact sequence of Theorem 2.5.13

Lemma 5.2.2 *There is a commutative diagram of exact sequences whose vertical maps $\eta : R_k(\mathcal{A}) \longrightarrow H^k(Y_{\mathcal{A}})$ are given by $\eta(\omega_H) = [\omega_H]$:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R_{k+1}(\mathcal{A}') & \xrightarrow{\ i\ } & R_{k+1}(\mathcal{A}) & \xrightarrow{\ j\ } & R_k(\mathcal{A}'') & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \eta'} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta''} & & \\ \dots & \longrightarrow & H^{k+1}(Y_{\mathcal{A}'}) & \xrightarrow{\ i^*\ } & H^{k+1}(Y_{\mathcal{A}}) & \xrightarrow{\ \vartheta\ } & H^k(Y_{\mathcal{A}''}) & \longrightarrow & \dots \end{array}$$

Theorem 5.2.3 (see [21]) *Let $\mathcal{A}$ be a nonempty complex arrangement.*

1. *The map $\eta : R_k(\mathcal{A}) \longrightarrow H^*(Y_{\mathcal{A}})$ is an isomorphism for $k \geq 0$.*
2. *$H^k(Y_{\mathcal{A}})$ are free abelian groups.*
3. *For $k \geq 0$ there exist split short exact sequences*

$$0 \longrightarrow H^{k+1}(Y_{\mathcal{A}'}) \xrightarrow{i^*} H^{k+1}(Y_{\mathcal{A}}) \xrightarrow{\vartheta} H^k(Y_{\mathcal{A}''}) \longrightarrow 0 \ .$$

Corollary 5.2.4 *The integral cohomology ring $H^*(\mathcal{A})$ is generated by 1 and the classes $[\eta_H]$ for $H \in \mathcal{A}$.*

Theorem 5.2.5 *The surjective map $\omega_H \longrightarrow [(1/2\pi i)\omega_H]$ induces an isomorphism of graded algebras $R(\mathcal{A}) \cong H^*(Y_{\mathcal{A}})$.*

This result shows there are no relations in cohomology other than those imposed by the algebraic relations. We showed in Theorem 2.5.12 that there is an isomorphism of algebras $A(\mathcal{A}) \cong R(\mathcal{A})$ which sends a_H to ω_H. We may apply this result when the coefficient ring is $\mathbb{Z}$ to obtain a structure theorem for $H^*(Y_{\mathcal{A}};\mathbb{Z})$ in terms of generators and the relation ideal.

Theorem 5.2.6 *Let $\mathcal{A}$ be a complex arrangement and A its OS algebra. The map $a_H \mapsto [(1/2\pi i)\omega_H]$ induces an isomorphism $A \longrightarrow H^*(Y_{\mathcal{A}})$ of graded $\mathbb{Z}$-algebras.*

Definition 5.2.7 *For each subspace X in L_h, define the following complex:*

$$\mathcal{L}_h(X^*) = \bigoplus_{\{\alpha_{j_1},\dots,\alpha_{j_h}\}\in\mathcal{B}^{\mathrm{nbc}}_{X^*}} \mathcal{L}_h(\{\alpha_{j_1},\dots,\alpha_{j_h}\})$$

where $\mathcal{L}_h(\{\alpha_{j_1},\dots,\alpha_{j_h}\})$ is the same complex $\mathcal{L}_h(X)$ defined in (5.1) for the set of generators $\{\alpha_{j_1},\dots,\alpha_{j_h}\}$ of X^*. Associated to the D_n-module $\mathcal{P}_h \cong \bigoplus_{X\in L_h} R_{X^*}$, define the complex

$$\mathcal{L}(\mathcal{P}_h) = \bigoplus_{X\in L_h} \mathcal{L}_h(X^*) \ .$$

Finally define the complex $\mathcal{L}(\mathcal{P}) = \mathcal{L}(\mathcal{P}(\mathcal{A})) = \bigoplus_{h=0}^{r} \mathcal{L}(\mathcal{P}_h)$ associated to $\mathcal{P}$.

Theorem 5.2.8 *Fixed $1 \leq h \leq r$, there exists an isomorphism between $H^h(Y_{\mathcal{A}})$ and $H^h(\mathcal{L}(\mathcal{P}_h))$:*

$$H^h(Y_{\mathcal{A}}) \cong H^h(\mathcal{L}(\mathcal{P}_h)) = \bigoplus_{X \in L_h} \bigoplus_{\{\alpha_{j_1},\ldots,\alpha_{j_h}\} \in \mathcal{B}^{\mathrm{nbc}}_{X^*}} \mathbb{C} \cdot \frac{1}{\alpha_{j_1} \ldots \alpha_{j_h}} d\alpha_{j_1} \wedge \ldots \wedge d\alpha_{j_h}.$$

Proof. Fix a subspace $X \in L_h(\mathcal{A})$. By Corollary 5.1.4 the associated complex $\mathcal{L}_h(X^*)$ has cohomology non-null only in dimension h. It is

$$H^h(\mathcal{L}_h(X^*)) = \bigoplus_{\{\alpha_{j_1},\ldots,\alpha_{j_h}\} \in \mathcal{B}^{\mathrm{nbc}}_{X^*}} \mathbb{C} \cdot \frac{1}{\alpha_{j_1} \ldots \alpha_{j_h}} d\alpha_{j_1} \wedge \ldots \wedge d\alpha_{j_h} \; .$$

Therefore the complex $\mathcal{L}(\mathcal{P}_h) = \bigoplus_{X \in L_h} \mathcal{L}_h(X^*)$ has cohomology non-null only in dimension h. Since the set $\{a_S \mid S = (j_1, \ldots, j_h)$ is standard and nbc$\}$ is a basis for the OS algebra $A_h(\mathcal{A})$ defined in Chapter 2, the map $a_S \mapsto \dfrac{d\alpha_{j_1} \wedge \ldots \wedge d\alpha_{j_h}}{\alpha_{j_1} \ldots \alpha_{j_h}}$ induces an isomorphism $A_h(\mathcal{A}) \to H^*(\mathcal{L}(\mathcal{P}_h))$. It follows, from Theorem 5.2.6, that $H^h(Y_{\mathcal{A}}) \cong H^h(\mathcal{L}(\mathcal{P}_h))$. ■

Corollary 5.2.9 *Let $b_h(Y_{\mathcal{A}}) = rank H^h(Y_{\mathcal{A}})$ be the Betti numbers of $Y_{\mathcal{A}}$. Then*

$$b_h = \sum_{X \in L_h} a(X^*) \; .$$

Proof. It is a consequence of Theorem 5.2.8 that

$$\mathrm{rank} H^h(Y_{\mathcal{A}}) = \mathrm{dim} H^k(Y_{\mathcal{A}}) = \sum_{X \in L_h} |\mathcal{U}^{\mathrm{nbc}}_{X^*}| = \sum_{X \in L_h} a(X^*) \; ,$$

where the last equality is by Definition 4.2.26. ■

Chapter 6

The Poincaré series of $P(\mathcal{A})$

In this last chapter we compute the Poincaré series of the D_n-module $P(\mathcal{A})$.

Definition 6.1.10 *If* $M = \bigoplus_{i\geq 0} M_i$ *is a graded vector space with* $\dim M_i < +\infty$*, for all* $i \geq 0$*, we let*

$$\text{Poin}(M,t) = \sum_{i=0}^{\infty} (\dim M_i) t^i$$

be its Poincaré series.

By Definition 4.2.15, for each $X \in L(\mathcal{A}) \setminus \{\mathbb{C}^n\}$, we have associated a $\mathbb{C}$-vector space V_{X^*} generated by $\mathcal{U}_{X^*} = \left\{ \frac{1}{\prod_{\alpha \in B} \alpha} \mid B \in \mathcal{B}_{X^*} \right\}$. Follows, by Lemma 4.2.18, that if $X, Y \in L \setminus \{\mathbb{C}^n\}$, $X \neq Y$, then $V_{X^*} \cap V_{Y^*} = \{0\}$. Thus we can give out the following definition.

Definition 6.1.11 *Let* $\mathcal{A}$ *be an arrangement of hyperplanes. Define the finite dimensional graded* $\mathbb{C}$*-vector space*

$$V(\mathcal{A}) = \bigoplus_{h=0}^{r} \bigoplus_{X \in L_h} V_{X^*}$$

For $X \in L(\mathcal{A})$ *let* $V(\mathcal{A})_{X^*} = V_{X^*}$.

Recall, by Lemma 4.2.16, that $\mathcal{U}_{X^*}^{\text{nbc}} = \left\{ \frac{1}{\prod_{\alpha \in B} \alpha} \mid B \in \mathcal{B}_{X^*}^{\text{nbc}} \right\}$ is a basis to V_{X^*}, when $X \in L(\mathcal{A}) \setminus \{\mathbb{C}^n\}$. Then we have the following Lemma:

Lemma 6.1.12 *The set*

$$\{1\} \cup \bigcup_{h=1}^{r} \bigcup_{X \in L_h} \mathcal{U}_{X^*}^{\text{nbc}}$$

is a basis to $V(\mathcal{A})$.

We must express the dimension of V_{X^*} $(= |\mathcal{U}^{\mathrm{nbc}}_{X^*}|)$ by using the function $\mu(X)$ defined in Chapter 1.

Theorem 6.1.13 *For $X \in L$ we have $\dim V_{X^*} = (-1)^{r(X)}\mu(X)$, and the Poincaré series Poin$(V(\mathcal{A}),t)$ of the finite dimensional graded $\mathbb{C}$-vector space $V(\mathcal{A})$ is equal to Poin$(\mathcal{A},t)$.*

Proof. It is clear that there exists an isomorphism of graded vector spaces between $A(\mathcal{A})$ and $V(\mathcal{A})$ and $A(\mathcal{A})_{X^*} \cong V_{X^*}$ for every $X \in L$. Moreover, since $\dim A(\mathcal{A})_{X^*} = |\mu_{X^*}|$, *see* [21]. Then the theorem follows. ■

By Theorem 4.2.23, we have that $P(\mathcal{A})$ is a graded D_n-module, infinite dimensional. Then its Poincareé series is a formal power series. The following theorem give us a combinatorial formula for it.

Theorem 6.1.14 *The Poincaré series Poin$(P(\mathcal{A}),t)$ of the graded D_n-module $P(\mathcal{A})$ is equal to $(1-t)^{-n}$Poin$(\mathcal{A},t)$.*

Proof. According to Theorem 4.2.23 we have

$$\mathrm{Poin}(P_{\mathcal{A}},t) = \sum_{X\in L} Poin(R_{X^*},t) = \sum_{X\in L} \mathrm{Poin}(M_{X^*},t)\mathrm{Poin}(V_{X^*},t)$$

Since the $\mathbb{C}$-algebra M_{X^*} is isomorphic to the polynomial algebra with n variables, we have $\mathrm{Poin}(M_{X^*},t) = (1-t)^{-n}$. Moreover, by the Theorem 6.1.13, we have $\mathrm{Poin}(V_{X^*},t) = \dim V_{X^*} t^{r(X)} = (-1)^{r(X)}\mu(X)t^{r(X)}$. Thus

$$\begin{aligned}\mathrm{Poin}(P_{\mathcal{A}},t) &= \sum_{X\in L}(1-t)^{-n}(-1)^{r(X)}\mu(X)t^{r(X)}\\ &= (1-t)^{-n}\mathrm{Poin}(\mathcal{A},t)\end{aligned}$$

■

It follows from Theorem 4.2.24 the Corollary

Corollary 6.1.15 *The Poincaré series $P(\mathcal{P}(\mathcal{A}),t)$ of $\mathcal{P}(\mathcal{A}) = \bigoplus_{h=0}^{r} P_h/P_{h-1}$ is equal to* $\mathrm{Poin}(P_{\mathcal{A}},t)$.

An interesting type of arrangements are the free arrangements.

Definition 6.1.16 *Let $Der_{\mathbb{C}}(\mathbb{C}[\mathbf{x}])$ be the $\mathbb{C}[\mathbf{x}]$-module of derivations:*

$$Der_{\mathbb{C}}(\mathbb{C}[\mathbf{x}]) = \{\theta \mid \theta : \mathbb{C}[\mathbf{x}] \to \mathbb{C}[\mathbf{x}] \text{ is a } \mathbb{C}-\text{linear derivation}\}.$$

It is immediately to see that $Der_{\mathbb{C}}(\mathbb{C}[\mathbf{x}])$ is a free $\mathbb{C}[\mathbf{x}]$-module of rank n, naturally isomorphic to $\mathbb{C}[\mathbf{x}]\otimes_{\mathbb{C}}\mathbb{C}^n$. The usual derivations $\partial_{x_1},\ldots,\partial_{x_n}$ is a basis for $Der_{\mathbb{C}}(\mathbb{C}[\mathbf{x}])$.

Definition 6.1.17 *A nonzero element $\theta \in Der_{\mathbb{C}}(\mathbb{C}[\mathbf{x}])$ is homogeneous of degree d if $\theta(f) \in \mathbb{C}[\mathbf{x}]_d$ for all $f \in (\mathbb{C}^n)^*$.*

Definition 6.1.18 *Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^n$. Define the module of $\mathcal{A}$-derivations by*

$$Der_{\mathbb{C}}(\mathcal{A}) = \{\theta \in Der_{\mathbb{C}}(\mathbb{C}[\mathbf{x}]) \mid \theta(\alpha) \in \alpha\mathbb{C}[\mathbf{x}] \text{ for any } \alpha \in \mathcal{A}^*\}.$$

The arrangement $\mathcal{A}$ is called a free arrangement if $Der_{\mathbb{C}}(\mathcal{A})$ is a free $\mathbb{C}[\mathbf{x}]$-module.

Proposition 6.1.19 (*see [21]*) *If $\mathcal{A}$ is a free n-arrangement, then $D_{\mathbb{C}}(\mathcal{A})$ has a basis consisting of n homogeneous elements.*

Definition 6.1.20 *Let $\mathcal{A}$ be a free arrangement and let $\{\theta_1, \dots, \theta_n\}$ be a homogeneous basis for $D_{\mathbb{C}}(\mathcal{A})$. The n nonnegative integers $\{\deg\theta_1, \dots, \deg\theta_n\}$ are called the exponents of $\mathcal{A}$.*

Notice that the exponents depend only on $\mathcal{A}$.

Proposition 6.1.21 (*see [21]*) *If $\mathcal{A}$ is a free arrangement when exponents $d_1, \dots, d_n$, then*

$$Poin(\mathcal{A}, t) = \prod_{i=1}^{n} (1 + d_i t).$$

Corollary 6.1.22 *Let $\mathcal{A}$ be a free arrangement with exponents $d_1, \dots, d_n$, then*

$$Poin(P(\mathcal{A}), t) = (1-t)^{-n} \prod_{i=1}^{n} (1 + d_i t).$$

Proof. It follows from Theorem 6.1.14 ■

Note that this is the case when $\mathcal{A}$ is the set of reflecting hyperplanes of any (real or complex) reflection group with exponents $d_1, \dots, d_n$ because $\mathcal{A}$ is a free arrangement.

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