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Dottorato in Matematica, Roma «La Sapienza» (2005).

<http://www.bdim.eu/item?id=tesi_2005_FalcoPierluigi_1>

UNIVERSITÀ DEGLI STUDI DI ROMA "LA SAPIENZA"

FACOLTÀ DI SCIENZE MATEMATICHE FISICHE E NATURALI

TESI DI DOTTORATO IN MATEMATICA

Rigorous construction of the Thirring model: Ward-Takahashi Identities, Schwinger-Dyson Equations and New Anomalies

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Anno Accademico 2004-2005

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Introduction

Historical outlook. The Thirring model was proposed in [T58]. It describes Dirac fermions in $d = 1+1$ spacetime dimensions with local current-current interaction. With summation over repeated indices, the classical Action for mass μ and coupling λ reads:

$$\int \mathrm{d}^2x \; \overline{\psi}_x \left(i \;\partial\!\!\!/ + \mu\right) \psi_x - \frac{\lambda}{2} \int \mathrm{d}^2x \; \rho_\nu(x)\rho^\nu(x) \;, \tag{0.0.1}$$

where ψ and $\overline{\psi} \stackrel{def}{=} \psi^\dagger \gamma^0$ are 2-spinors; $x \stackrel{def}{=} (x_0, x_1)$; $\rho^\mu(x) \stackrel{def}{=} \overline{\psi}_x \gamma^\mu \psi_x$ is the current; and the γ's matrices are a realization of the Clifford algebra

$$\gamma^0 \stackrel{def}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;, \qquad \gamma^1 \stackrel{def}{=} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \;, \qquad \gamma^5 \stackrel{def}{=} i\gamma^0\gamma^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \;, \tag{0.0.2}$$

which, for $\eta^{\mu,\nu} \stackrel{def}{=} \delta_{\mu,\nu}(1 - 2\delta_{\mu,1})$, do satisfy the properties

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu,\nu} \;, \qquad (\gamma^\mu)^\dagger = -\gamma^0_M \gamma^\mu_M \gamma^0_M \;,$$
$$\{\gamma^5, \gamma^\mu\} = 0 \;, \qquad \left(\gamma^5\right)^2 = -1 \;, \qquad \left(\gamma^5\right)^\dagger = -\gamma^0\gamma^5\gamma^0 = -\gamma^5 \;.$$

This model is enough simple to be analysed in full details; and yet it contains many of the typical features of the quantization of relativistic *quantum field theories* (QFT), such as the *anomalous scaling* – as conjectured in QED, [JZ]; and the *anomalous phase and chiral symmetries* – like the anomalous chiral symmetry of QED or Standard Model.

As peculiarity of the $1+1$ spacetime dimension, since there are only two independent component of the current, the invariance of the classical massless Lagrangian under phase transformation $\psi_x \to e^{i\alpha}\psi_x$ and under chiral transformation $\psi_x \to e^{i\gamma^5\alpha_x}\psi_x$ led to the hope to find an exact solution also for the quantum massless model.

First, Thirring, [T58], derived many matrix elements of the interacting field; then, Glaser, [G58], gave an explicit formula for such a field operator, arising the criticism of Pradhan and Scarf. The breakthrough had place with Johnson, [J61], who first found the expression for the two point Schwinger functions which, until nowadays, has been accepted as the *exact solution*. In the end, Klaiber, [K64], with a slightly different technique, wrote out the general formula for all the Schwinger functions. All this story is commented upon in [W64]; here it is worthwhile to stress that *all above papers* were plagued by the typical infinities of relativistic QFT: the virtue of Johnson's development merely was a greater solidity of the final result.

A remarkable feature in [J61] is the presence of *anomalies* in the Ward-Takahashi identities (WTi): they occur – some years *before* the discovery of Adler, [A70] – as a modification of the field-current commutation relations, simply guessed in order to avoid triviality of the identities.

Remarkable as well is the procedure of joining of the Schwinger-Dyson equation (SDe) together to the phase and chiral WTi, in order to obtain a Closed Equation (CE) for the two point Schwinger function which can be solved straightforwardly.

In order to clear the result of all the surreptitious calculations with infinities, Wightman, [W64], stressed that the set of Schwinger function of Johnson and Klaiber, no matter how they were derived, only represent *good candidates*: if they verified the requirements of an axiomatic program, they would define a QFT to be called "Thirring model" essentially by definition. But no kind of *positive definiteness* of inner product of physical Hilbert space has ever been possible to prove; up to recent years, when in [M93] the *reflection positivity* was obtained as consequence of the Hamiltonian formulation of a many particle model, the Luttinger model, exactly soluble as showed in [ML65] and in a sense close to the massless Thirring model.

The massive theory is much less analysed, [GL72]. In such a case no "exact solution" was ever found; as well as no physical positive metric.

Now, a different point of view can be considered, the Renormalization Group (RG) approach *à la Wilson*. Such a technique has been revealed very profitable for certain QFT, like the Yukawa_2 model, [S75] and [MS76], or the ultraviolet part of Gross-Neveu model, [GK85] and [FMRS85]; the subtle point being that all such models are superrinormalizable, or were studied in asymptotically free regimes.

The Thirring model, instead, is renormalizable, but not superrinormalizable; and no regime is asymptotically free, since the effective coupling remains essentially constant over every regime. This property, called *vanishing of Beta function*, was already used in [BoM97] to point out the critical behavior of the infrared regime of Yukawa_2 model; and it is a consequence of the phase and chiral WTi – in agreement with the general belief that, without the aid of symmetries, RG can be effective only in constructing *trivial* theories.

As matter of fact, there is a basic conflict between the regularization of the theory and the phase and chiral symmetries. The situation is very similar to the scaling transformation: the classical theory is scale invariant; the theory regularized with a cutoff is no longer; removing the cutoff, scale invariance is recovered, but with a different exponent, called anomalous. In the same way, removing the cutoff, the WTi are recovered, but a change in the factor in front of the currents makes such identities anomalous.

In recent times, Benfatto and Mastropietro, [BM01],[BM02],[BM04],[BM05], have developed a technique to complete construction of Luttinger liquids without any reference to the exact solution of the Luttinger model. As byproduct of their developments, the anomaly of the WTi arose.

The aim of this thesis is to use such a technique to construct, by a *self-consistent* RG *approach*, uniform in the mass, the Thirring model at imaginary time. And then to make a continuation to Minkowskian spacetime by verifying the Osterwalder and Schrader axioms, (OSa). The occurrence of the phenomenon of *fermion doubling* – peculiar of the discretization on a lattice – has been solved introducing a momentum dependent mass term, as suggested in [W76], but also a mass counterterm which avoids the generation of mass in the massless theory.

As main applications, the anomalous WTi stated by Johnson are derived and, as consequence, the current operator is proved not to need any renormalization. Anyway, the explicit value of the anomaly obtained by Johnson are wrong by lowest order calculation, and this is *in violation of the Adler-Bardeen's theorem*, [A69]. Also the rigorous implementation of the Johnson's closure of the SDe is proved: it will be showed, anyway, the arising of a *new anomaly*, missed in the formal developments, which have driven Johnson to a *wrong anomalous exponent*.

Chapter 1:

Definitions and Main Results

1.1 Euclidean Thirring Model

Many properties of a quantum field theory can be obtained from the *Schwinger functions*, the "cumulants", or the "truncated expectations" of a statistical measure which correspond to the *imaginary-time version* of the model. Such a measure can be conveniently formulated in terms of a "path integral" on a lattice spacetime. Since the fields dealt with are *fermions* – namely only the case of anticommuting fields is considered – they are represented in the path integral formulation by Grassmannian variables.

1.1.1 Weyl formalism. While in Dirac notation of (0.0.1) the independent fields are the 2-spinor $\overline{\psi}$ and ψ, in Weyl notation they are $\widehat{\psi}_k \stackrel{def}{=} (\widehat{\psi}^-_{k,+}, \widehat{\psi}^-_{k,-})^T, \widehat{\psi}^\dagger_k \stackrel{def}{=} (\widehat{\psi}^+_{k,+}, \widehat{\psi}^+_{k,-})$. The Euclidean Clifford Algebra is defined to be:

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu,\nu} , \qquad (\gamma^\mu)^\dagger = \gamma^\mu ,$$
$$\{\gamma^5, \gamma^\mu\} = 0 , \qquad (\gamma^5)^2 = 1 , \qquad (\gamma^5)^\dagger = \gamma^5.$$

Such requirements are fulfilled by the same γ's matrices in (0.0.2), by multiplying γ^1 and γ^5 by the imaginary unity; namely, from now on the definitions in (0.0.2) are turned into:

$$\gamma^0 \stackrel{def}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \gamma^1 \stackrel{def}{=} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \gamma^5 \stackrel{def}{=} - i\gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Accordingly, the Euclidean Action, for mass μ and coupling λ, is defined to be:

$$\sum_{\omega,\sigma=\pm}\int\frac{\mathrm{d}^2k}{(2\pi)}\ \widehat{\psi}^+_{k,\omega}T_{\omega,\sigma}(k)\widehat{\psi}^-_{k,\sigma}\\ -\frac{\lambda}{2}\sum_{\omega=\pm}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\ \widehat{\psi}^+_{p,\omega}\widehat{\psi}^-_{q,\omega}\widehat{\psi}^+_{k,-\omega}\widehat{\psi}^-_{p+k-q,-\omega}\ , \tag{1.1.1}$$

where the coefficients of the quadratic part are

$$T_{\omega,\sigma}(k)\stackrel{def}{=}\begin{pmatrix}D_+(k) & -\mu\\ -\mu & D_-(k)\end{pmatrix}_{\omega,\sigma}\ ,\qquad \text{with}\quad D_\omega(k)\stackrel{def}{=}-ik_0+\omega k_1\ .$$

1.1.2 Spacetime Lattice. Let a and L be respectively the spacing and the side length of the lattice to be constructed, such that L/a is an integer. Then, in correspondence of such parameters, let the quotient set Q be defined as

$$Q\stackrel{def}{=}\left\{(n_0,n_1)\in\mathbb{Z}^2\middle|\ n\sim n' \text{ if } n-n'\in\frac{L}{a}\mathbb{Z}^2\right\}\ ;$$

the spacetime lattice, Λ, and its reciprocal one, D, are defined as

$$\Lambda\stackrel{def}{=}\{an_0,an_1\,|n\in Q\}\ ,\qquad D\stackrel{def}{=}\left\{\frac{2\pi}{L}\left(m_0+\frac{1}{2}\right),\frac{2\pi}{L}\left(m_1+\frac{1}{2}\right)\middle|\, m\in Q\right\}\ .$$

To shorten the notation, the Riemann sums on the lattices are denoted with integrals

$$\int_\Lambda \mathrm{d}^2x\ f(x)\stackrel{def}{=}a^2\sum_{x\in\Lambda}f(x)\ ,\qquad \int_D\mathrm{d}^2k\ \widehat{f}(k)\stackrel{def}{=}\left(\frac{2\pi}{L}\right)^2\sum_{k\in D}\widehat{f}(k)\ . \tag{1.1.2}$$

1.1.3 Grassmann Algebra. In correspondence of the fields in (1.1.1), there are four sets of Grassmann variables that, *with abuse of notation*, are called $\{\widehat{\psi}^\sigma_{k,\omega}\}^{k\in D}_{\sigma,\omega=\pm}$ as well. The integration in such a *finite algebra* is defined so that the integral of a constant is zero, while

$$\int\mathrm{d}\widehat{\psi}^{\sigma'}_{k',\omega'}\ \widehat{\psi}^\sigma_{k,\omega}=\delta_{\sigma',\sigma}\delta_{k',k}\delta_{\omega',\omega}\ ;$$

then the operation is extended by linearity to any polynomial of fields, considering $\left\{\mathrm{d}\widehat{\psi}^\sigma_{k,\omega}\right\}^{k\in D}_{\omega,\sigma}$ anticommuting with themselves and with all the fields. As consequence, the integration of the monomial $\mathcal{Q}(\psi)$, $\int\prod_{k\in D}\prod_{\omega=\pm}\mathrm{d}\widehat{\psi}^+_{k,\omega}\mathrm{d}\widehat{\psi}^-_{k,\omega}\mathcal{Q}(\psi)$, assigns 1 to $\mathcal{Q}(\psi)=\prod_{k\in D}\prod_{\omega=\pm}\widehat{\psi}^-_{k,\omega}\widehat{\psi}^+_{k,\omega}$, and 0 to all the other $\mathcal{Q}'(\psi)$ which cannot be obtained as permutation of fields in $\mathcal{Q}(\psi)$.

The derivative in the Grassmann algebra is defined to be equivalent to the integration:

$$\frac{\partial\mathcal{Q}(\psi)}{\partial\widehat{\psi}^+_{k,\omega}}\stackrel{def}{=}\int\mathrm{d}\widehat{\psi}^+_{k,\omega}\ \mathcal{Q}(\psi)\ ,\qquad \frac{\partial\mathcal{Q}(\psi)}{\partial\widehat{\psi}^-_{k,\omega}}\stackrel{def}{=}-\int\mathcal{Q}(\psi)\ \mathrm{d}\widehat{\psi}^-_{k,\omega}$$

– hence the derivative in $\widehat{\psi}^-_{k,\omega}$ *acts from the right.*

1.1.4 Schwinger functions. In order to give a meaning to the path integral formulation of the Schwinger function, it is necessary to introduce a "cutoff function", $\chi_N(k)$, made as follows. Let a momentum unity, κ, be fixed. Chosen any $\gamma > 1$, let N be any integer such that $\kappa\gamma^{N+1} \leq 3\pi/4a$. Then, let $\widehat{\chi}_N(t)$ be a $C_0^\infty(\mathbb{R})$ function with compact support $\{t \in \mathbb{R} : |t| \leq \kappa\gamma^{N+1}\}$ and $\widehat{\chi}_N(t) \equiv 1$ in $\{t \in \mathbb{R} : |t| \leq \kappa\gamma^N\}$. Besides, because of technical reason, it is convenient to take $\widehat{\chi}_N$ in the Gevrey class α: for a positive constant C,

$$\sup_{t\in\mathbb{R}} \left| \frac{\mathrm{d}^n \widehat{\chi}_N}{\mathrm{d}t^n}(t) \right| \leq C^n (n!)^\alpha \ ;$$

in particular, $\alpha = 2$ will be good enough. The possibility of constructing such a compact support function is discussed in A1.2. Finally, $\chi_N(k) \stackrel{def}{=} \widehat{\chi}_N(k_0)\widehat{\chi}_N(k_1)$. Calling $D_N \subset D$ the support of $\chi_N(k)$, the *Generating Functional* of the Schwinger functions of the Thirring model is defined to be $\mathcal{W}(\jmath,\varphi)$: in correspondence of certain parameters λ_N, μ_N, Z_N and $\zeta_N^{(2)}$, it is such that

$$e^{\mathcal{W}(\jmath,\varphi)} \stackrel{def}{=} \int \mathrm{d}P^{(\leq N)}(\psi) \exp\left\{ -\lambda_N \mathcal{V}\left(\sqrt{Z_N}\psi\right) + \zeta_N^{(2)} \mathcal{J}(\jmath, \sqrt{Z_N}\psi) + \mathcal{F}(\varphi,\psi) \right\} \ . \qquad (1.1.3)$$

The explanation of the above formula is the following. The integration is done w.r.t. the normalized Gaussian measure given by

$$\mathrm{d}P^{(\leq N)}(\psi) \stackrel{def}{=} \exp\left\{ L^2 O_N - Z_N \sum_{\alpha,\beta=\pm} \int_{D_N} \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{T_{\omega,\sigma}(k)}{\chi_N(k)} \widehat{\psi}^+_{k,\omega} \widehat{\psi}^-_{k,\sigma} \right\} \prod_{k\in D_N} \prod_{\omega=\pm} \mathrm{d}\widehat{\psi}^+_{k,\omega} \mathrm{d}\widehat{\psi}^-_{k,\omega} \ , \qquad (1.1.4)$$

where the *covariance* $\widehat{g}_{\omega,\sigma}(k)$ is such that:

$$\widehat{g}^{-1}(k) \stackrel{def}{=} \frac{T(k)}{\chi_N(k)} \ , \qquad \text{with} \quad T_{\omega,\sigma}(k) \stackrel{def}{=} \begin{pmatrix} D_+(k) & -\mu_N \\ -\mu_N & D_-(k) \end{pmatrix}_{\omega,\sigma} \ ;$$

hence $\widehat{g}(k)$ is periodic by the compact support of χ_N and well defined for any $k \in D$, also for $\mu_N = 0$, since the point $(0,0)$ does not belong to D. As well as $\widehat{g}^{-1}(k)$ is well defined in D_N, since the points in which the cutoff is zero do not belong to D_N. The factor e^{O_N} is the normalization of the Gaussian measure:

$$O_N \stackrel{def}{=} \int_{D_N} \frac{\mathrm{d}^2 k}{(2\pi)^2} \ \ln\left(\frac{L^4 |k|^2}{\chi_N^2(k)} \right) \ .$$

The self-interaction is given by the potential

$$\mathcal{V}(\psi) \stackrel{def}{=} \frac{1}{2} \sum_\omega \int_D \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{\mathrm{d}^2 k}{(2\pi)^2} \ \widehat{\psi}^+_{p,\omega} \widehat{\psi}^-_{q,\omega} \widehat{\psi}^+_{k,-\omega} \widehat{\psi}^-_{p+k-q,-\omega} \ ;$$

while the interaction with the external sources are

$$\mathcal{J}_\sigma(\jmath,\psi)\stackrel{def}{=}\sum_\omega\int_D\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\ \widehat{\jmath}_{p-k,\omega}\widehat{\psi}^+_{k,\sigma\omega}\widehat{\psi}^-_{p,\sigma\omega}\ ,$$

$$\mathcal{F}(\varphi,\psi)\stackrel{def}{=}\sum_\omega\int_D\frac{\mathrm{d}^2k}{(2\pi)^2}\ \left[\widehat{\varphi}^+_{k,\omega}\widehat{\psi}^-_{k,\omega}+\widehat{\psi}^+_{k,\omega}\widehat{\varphi}^-_{k,\omega}\right]\ ;$$

and $\{\widehat{\jmath}_{k,\omega}\}_{k,\omega}$ are a commuting variable, while $\{\widehat{\varphi}^\sigma_{k,\omega}\}_{k,\omega,\sigma}$ are anticommuting.

Finally, w.r.t. the classical Action (0.0.1), λ has been replaced with $\lambda_N Z_N^2$, the "bare coupling"; μ with μ_N, the "bare mass"; the free action was multiplied times Z_N, the "field strength"; and the interaction with the external source $\jmath$ brings a coupling $Z_N^{(2)}\stackrel{def}{=}\zeta_N^{(2)}Z_N$, the "density strength": such parameters are essential in order to have a finite interactive quantum theory, see Theorem 1.1. Besides, in has to be remarked that the introduction of the cutoff has required a reference momentum, κ, absent in the classical action of the massless theory, which will allow the arising of the anomalous dimension without violating the scaling symmetry.

The Fourier transform of the fields defines a Grassmann algebra also in the lattice Λ. The conventions are:

$$\psi^\sigma_{x,\omega}\stackrel{def}{=}\int_D\frac{\mathrm{d}^2k}{(2\pi)^2}\ e^{i\sigma kx}\widehat{\psi}^\sigma_{k,\omega}\ ;\qquad \varphi^\sigma_{x,\omega}\stackrel{def}{=}\int_D\frac{\mathrm{d}^2k}{(2\pi)^2}\ e^{i\sigma kx}\widehat{\varphi}^\sigma_{k,\omega}\ ;$$

$$\jmath_{x,\omega}\stackrel{def}{=}\int_D\frac{\mathrm{d}^2k}{(2\pi)^2}\ e^{ikx}\widehat{\jmath}_{k,\omega}\ .$$

The definition of derivative extends also to the fields $\left\{\psi^\sigma_{x,\omega}\right\}^{x\in\Lambda}_{\omega,\sigma=\pm}$, $\left\{\widehat{\varphi}^\sigma_{k,\omega}\right\}^{k\in D}_{\omega,\sigma=\pm}$ and $\left\{\varphi^\sigma_{x,\omega}\right\}^{x\in\Lambda}_{\omega,\sigma=\pm}$; while the derivative w.r.t. the fields $\left\{\widehat{\jmath}_{k,\omega}\right\}^{k\in D}_{\omega=\pm}$ and $\left\{\jmath_{x,\omega}\right\}^{x\in\Lambda}_{\omega=\pm}$ is the conventional one.

Well then, setting $\underline{x}\stackrel{def}{=}x^1,\ldots,x^n$, and $\underline{z}\stackrel{def}{=}z^1,\ldots,z^m$, collections of points in Λ, for any given choice of the labels $\underline{\sigma}\stackrel{def}{=}(\sigma_1\ldots,\sigma_m)$, $\underline{\omega}\stackrel{def}{=}(\omega_1\ldots,\omega_n)$ and $\underline{\varepsilon}\stackrel{def}{=}(\varepsilon_1\ldots,\varepsilon_n)$, the Schwinger functions are defined as

$$S^{(m;n)(\underline{\varepsilon})}_{\underline{\sigma};\underline{\omega}}(\underline{z};\underline{x})\stackrel{def}{=}\frac{\partial^{n+m}\mathcal{W}}{\partial\jmath_{z^1,\sigma_1}\cdots\partial\jmath_{z^m,\sigma_m}\partial\varphi^{\varepsilon_1}_{x^1,\omega_1}\cdots\partial\varphi^{\varepsilon_n}_{x^n,\omega_n}}(0,0)\ . \tag{1.1.5}$$

In order to shortening the notations of the Schwinger functions which will be most used in the following, let

$$S^{(2)}_\omega(x-y)\stackrel{def}{=}S^{(0;2)(-,+)}_{\omega,\omega}(x,y)\ ,\qquad S^{(1;2)}_{\sigma;\omega}(z;x-y)\stackrel{def}{=}S^{(1;2)(-,+)}_{\sigma;\omega,\omega}(z;x,y)\ .$$

1.1.5 Remarks. The role of the lattice discretization is only to have a finite Grassmann algebra: the *continuous limit*, $\kappa L,(\kappa a)^{-1}\to\infty$ is taken as soon as the Schwinger function are derived; it is trivial, since, on the other hand, the use of the functional integral suggest, but it is not strictly necessary to, the developments.

On the contrary, the function χ_N is an essential cutoff on the large momenta: the parameters λ_N, μ_N, Z_N and $\zeta_N^{(2)}$ will be chosen in a way to compensate the divergences of the *limit of removed cutoff*, $N \to +\infty$, of the Schwinger functions.

Theorem 1.1. *There exists $\varepsilon > 0$ and two positive constant, c and C, such that, for any $\lambda : |\lambda| \le \varepsilon$ and $\mu : 0 \le \mu \le \kappa\gamma^{-1}$, and for suitable λ_N, μ_N, Z_N and $Z_N^{(2)}$, analytic function of λ, the following properties of the Schwinger functions hold.*

1. *There exist three critical indices, η_λ, $\eta_\lambda^{(2)}$, and $\overline{\eta}_\lambda$, independent from the cutoff scale N and from the mass μ, analytic functions of λ and such that*

$$\eta_\lambda = \eta_2 \lambda^2 + \mathrm{O}(\lambda^3)\,, \qquad \eta_\lambda^{(2)} = \eta_2^{(2)} \lambda^2 + \mathrm{O}(\lambda^3)\,,$$

$$\overline{\eta}_\lambda = -\overline{\eta}_1 \lambda + \mathrm{O}(\lambda^2)\,,$$

   with η_2, $\eta_2^{(2)}$ and $\overline{\eta}_1$ strictly positive; and, for any N,

$$Z_N = \gamma^{-N\eta_\lambda}\left(1 + \mathrm{O}(\lambda^2)\right)\,, \qquad Z_N^{(2)} = \gamma^{-N\eta_\lambda^{(2)}}\left(1 + \mathrm{O}(\lambda^2)\right)\,,$$

$$\mu_N = \mu\gamma^{-N\overline{\eta}_\lambda}\left(1 + \mathrm{O}(\lambda)\right)\,,$$

   where $\mathrm{O}(\lambda)$ are finite in N.
2. *In the limit of removed cutoff, the Schwinger function are well defined distribution, fulfilling the* OSA.
3. *In the limit of removed cutoff, the two point Schwinger function verifies the bound*

$$\left| S_\omega^{(2)}(x-y) \right| \le \frac{\kappa C}{\left(\kappa|x-y|\right)^{1+\eta_\lambda}} e^{-c\sqrt{\left(\frac{\mu}{\kappa}\right)^{1+\overline{\tau}}\kappa|x-y|}}\,,$$

   for $\overline{\tau} \stackrel{def}{=} -\overline{\eta}_\lambda/(1+\overline{\eta}_\lambda)$. The same bound holds also for $S_{\omega,-\omega}^{(0;2)(-,+)}(x,y)$.
4. *In the limit of removed cutoff and of vanishing mass, i.e. $\mu = 0$,*

$$S_\omega^{(2)}(x-y) = (1 + \lambda B_\lambda) \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\, e^{-ik(x-y)} \frac{1}{D_\omega(k)} \left(\frac{\kappa}{|k|}\right)^{\eta_\lambda}\,, \tag{1.1.6}$$

   with B_λ analytic and $O(1)$ in λ. While $S_{\omega,-\omega}^{(0;2)(-,+)}(x,y) \equiv 0$.

The proof of the first three statements is obtained by the analysis in Chapter 3, the study of the flows of the effective couplings in 3.4, the convergence of the Schwinger functions, A.3.3 and A3.6, and by the equivalence of the Euclidean and Hamiltonian regularization, 3.5. The fourth statement is consequence of symmetries: see 4.3.

The OSA are reported in Appendix A2. When they hold, the Osterwalder-Schrader reconstruction theorem guarantees the possibility of analytically continuing the set of Schwinger functions to a set of functions obeying the Wightman axioms: this means the construction of a consistent relativistic and quantum field theory.

By item 2., the parameters Z_N and $Z_N^{(2)}$ are vanishing in the limit of removed cutoff; whereas μ_N is vanishing or diverging according to the sign of λ.

1.1.6 Ward-Takahashi identities: first anomaly. In the massless case, the phase and chiral symmetry makes current expectations and field expectations strictly related. By neglecting *formally* the presence of the cutoffs, and performing a combination of the phase and chiral transformation of the fields, it holds the following identity for the Fourier transform of such Schwinger functions:

$$\frac{D_\sigma(p)}{\zeta_N^{(2)}}\widehat{S}^{(1;2)}_{\sigma;\omega}(p;k) = \delta_{\sigma,\omega}\left[\widehat{S}^2_\omega(k) - \widehat{S}^2_\omega(k+p)\right] \ . \tag{1.1.7}$$

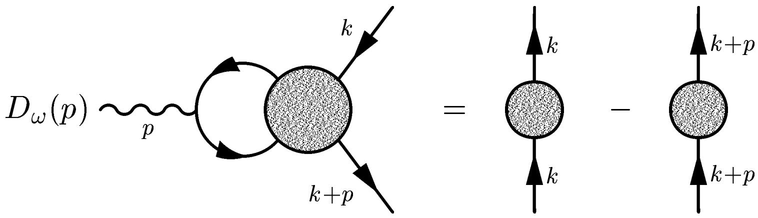

Fig 1: Graphical representation of (1.1.7)

This relation is actually *wrong*. Indeed, the presence of the cutoff – *essential ingredient* of meaningful QFT's – breaks the symmetries and generates a correction term $\widehat{H}^{(1;2)}_{\sigma;\omega}$:

$$\frac{D_\sigma(p)}{\zeta_N^{(2)}}\widehat{S}^{(1;2)}_{\sigma;\omega}(p;k) = \delta_{\sigma,\omega}\left[\widehat{S}^{(2)}_\omega(k) - \widehat{S}^{(2)}_\omega(k+p)\right] + \widehat{H}^{(1;2)}_{\sigma;\omega}(p;k) \ . \tag{1.1.8}$$

What is at first sight surprising is that in the limit of removed cutoff *the corrections are not vanishing*; and yet *anomalous* WTI, *strictly different from (1.1.7)*, are valid.

Theorem 1.2. *There exists $\varepsilon > 0$ and two positive constants, c and C, such that, for any $\lambda : |\lambda| \le \varepsilon$ and $\mu : 0 \le \mu \le \kappa\gamma^{-1}$, the following properties hold.*

1. *For $\mu = 0$, there exists two "bare parameters", λ_b and $\zeta_b^{(2)}$, analytic in λ, such that the coupling λ_N and the field strength $\zeta_N^{(2)}$, as chosen in Theorem 1.1, are independent form the scale of the cutoff, N; and are $\lambda_N = \lambda_b$, $\zeta_N^{(2)} = \zeta_b^{(2)}$.*
2. *For $\mu = 0$, there exist two coefficients, a and $\overline{a}$, analytically dependent on λ, such that*

$$\frac{1}{\zeta_b^{(2)}}\widehat{S}^{(1;2)}_{\sigma;\omega}(p,k) = \frac{a+\overline{a}\sigma\omega}{2}\ \frac{\widehat{S}^{(2)}_\omega(k) - \widehat{S}^{(2)}_\omega(k+p)}{D_\sigma(p)} \ , \tag{1.1.9}$$

with $(a+\overline{a}\sigma\omega)/2 \neq \delta_{\omega,\sigma}$ whenever $\lambda \neq 0$.

3. *The current-current correlation satisfies the bound*

$$\left|S^{(2;0)}_{\sigma,\omega}(x,y)\right| \leq \frac{C}{(\kappa|x-y|)^2} e^{-c\sqrt{\kappa\left(\frac{\mu}{\kappa}\right)^{1+\overline{\tau}\lambda}|x-y|}} , \tag{1.1.10}$$

for any allowed value of the mass μ.

The coupling λ_N and the density strength $\zeta^{(2)}_N$ do not depend on the cutoff scale since, the mass being zero, the theory is scaling invariant. The second statement is a sub-case of Theorem 4.2; while the third is proved in A3.7.

By item 3, the short distance behavior is the same as in the free theory: no critical index occurs and changes the exponent 2 of $1/(\kappa|x-y|)$.

It is interesting to see how the anomalous WTi arises. It is possible to find two finite counterterms, $\nu^{(+)}$ and $\nu^{(-)}$, analytically dependent on λ and independent on N, such that the correction can be decomposed as

$$\begin{aligned}\widehat{H}^{(1;2)}_{\sigma,\omega}(p;k) =& \nu^{(+)} D_\sigma(p) \widehat{S}^{(1;2)}_{\sigma;\omega}(p;k) + \nu^{(-)} D_{-\sigma}(p) \widehat{S}^{(1;2)}_{-\sigma;\omega}(p;k) \\ &+ \Delta \widehat{H}^{(1;2)}_{\sigma;\omega}(p;k) \; ;\end{aligned} \tag{1.1.11}$$

and, for p and k *fixed independently from N*, the rest $\Delta\widehat{H}^{(1;2)}_{\sigma;\omega}(p;k)$ is now really vanishing. To adhere to the Johnson's notation, let

$$a \overset{def}{=} \frac{1}{1-\left(\nu^{(-)}+\nu^{(+)}\right)} , \qquad \overline{a} \overset{def}{=} \frac{1}{1-\left(\nu^{(-)}-\nu^{(+)}\right)} \; ;$$

then, replacing (1.1.11) in (1.1.8), and taking the limit of removed cutoff, gives (1.1.9). Johnson's WTi is precisely given by (1.1.9); and his explicit values for a and $\overline{a}$ are in agreement with the Adler-Bardeen theorem on absence of radiative correction to the anomaly. Anyway, these values are *wrong*: while Johnson states $\nu^{(+)} = 0$, by lowest order computation, for λ small enough, $\nu^{(+)} < 0$ (see A9).

Despite the anomaly, and despite the phase and chiral symmetry hold only in the massless case, it is possible to prove the finiteness of the limit value of $\zeta^{(2)}_N$, *even in the massive model*; and accordingly the finiteness of the current-current Schwinger function, with no arising of an anomalous exponent.

1.1.7 Closed equation: new anomaly. The fields equation can be turned into an equation for the Schwinger function, the *Dyson-Schwinger equation*. In the massless case, the one for the two point Schwinger function reads

$$\frac{\widehat{S}^{(2)}_\omega(k)}{g_\omega(k)} = \frac{1}{Z_N} - \frac{\lambda_b}{\zeta^{(2)}_b} \int_D \frac{\mathrm{d}^2 p}{(2\pi)^2} \; \widehat{S}^{(1;2)}_{-\omega;\omega}(p;k-p) \; . \tag{1.1.12}$$

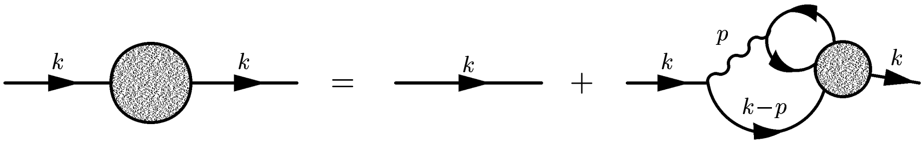

Fig 2: Graphical representation of (1.1.12)

Inserting the WTI (1.1.8) and the identity (1.1.11) in (1.1.12), since $\int_D d^2p\; D^{-1}_{-\omega}(p) = 0$ by oddness,

$$\frac{\lambda_b}{\zeta_b^{(2)}}\int_D \frac{d^2p}{(2\pi)^2}\;\widehat{S}^{(1;2)}_{-\omega;\omega}(p;k-p) = \frac{a-\overline{a}}{2}\lambda_b\int_D\frac{d^2p}{(2\pi)^2}\;\frac{\widehat{S}^{(2)}_\omega(k-p)}{D_{-\omega}(p)} + \sum_\mu \frac{a-\mu\omega\overline{a}}{2}\lambda_b\int_D\frac{d^2p}{(2\pi)^2}\;\Delta\widehat{H}^{(1;2)}_{\mu;\omega}(p;k-p)\;. \tag{1.1.13}$$

In the limit of removed cutoff, *if* the integral of $\Delta\widehat{H}^{(1;2)}_{\mu;\omega}$ had been vanishing, (1.1.13) would have been turned into

$$\frac{\lambda_b}{\zeta_b^{(2)}}\int_D \frac{d^2p}{(2\pi)^2}\;\widehat{S}^{(1;2)}_{-\omega;\omega}(p;k-p) = \frac{a-\overline{a}}{2}\lambda_b\int_D\frac{d^2p}{(2\pi)^2}\;\frac{\widehat{S}^{(2)}_\omega(k-p)}{D_{-\omega}(p)}\;. \tag{1.1.14}$$

Replacing it into (1.1.12), it would have held the equation

$$\frac{\widehat{S}^{(2)}_\omega(k)}{g_\omega(k)} = \frac{1}{Z_N} - \frac{a-\overline{a}}{2}\lambda_b\int_D\frac{d^2p}{(2\pi)^2}\;\frac{\widehat{S}^{(2)}_\omega(k-p)}{D_{-\omega}(p)}\;, \tag{1.1.15}$$

where $1/Z_N$ is divergent and should compensate the divergence of the integral. The above equation, *in a sense stated by Johnson* – actually his operations were even more formal; but his final finite solution is exactly the solution of (1.1.15) – is *wrong*. Indeed, $\Delta\widehat{H}^{(1;2)}_{\mu;\omega}$ was said to be vanishing only for fixed arguments, while here it is integrated over all the scales allowed by the cutoff. This seems to waste the possibility of the closure of the SDE; and yet, again, an *anomalous* CE still holds.

Theorem 1.3. *Under the same assumptions of Theorem 1.1:*

1. *The following equation holds, asymptotically in the limit of removed cutoff*

$$\frac{\widehat{S}^{(2)}_\omega(k)}{g_\omega(k)} = \frac{B_N}{Z_N} - A\lambda_b\frac{a-\overline{a}}{2}\int_D\frac{d^2p}{(2\pi)^2}\;\frac{\widehat{S}^{(2)}_\omega(k-p)}{D_{-\omega}(p)}\;, \tag{1.1.16}$$

where A, the "new anomaly", is analytic and $O(1)$ in λ; while B is $1+O(\lambda)$ and analytic in λ as well.

2. *It holds the following relations between the anomalous exponent and the coefficients in the first and second anomaly:*

$$\eta_\lambda = A\frac{\lambda_b}{2\pi}\frac{a-\overline{a}}{2}\;. \tag{1.1.17}$$

This result is a sub-case of Theorem 4.5, with the explicit expression of η_λ is discussed in 4.3.

The name "new anomaly" is justified since such is an effect of using a symmetry, exact only at removed cutoff, inside an integral which in the same limit is divergent; it has been overlooked not only in rigorous works, but even in the physical literature. In particular, $A \neq 1$ would imply a *striking and net difference w.r.t. the Johnson critical index.*

Such a difference could have been checked directly by lowest order computation of η_λ itself; but, since the fourth is the first non-trivial order, the actual computation is almost prohibitive. Therefore (1.1.17) is a shortcut, since it gives η_λ in terms of the easier calculations of $a - \overline{a}$ and A.

Now, by symmetry reasons, the first order of A is equal to 1, while $a - \overline{a} = \mathrm{O}(\lambda)$: this is in agreement with the the fact that η_λ is an even function of λ – as can be easily proved by transformation $\widehat{\psi}^\sigma_{k,\omega} \to \widehat{\psi}^\sigma_{\sigma k,\sigma\omega}$ in the functional-integral measure. But there is no general reason for which this result should survive also at the second order, at least for a *generic* choice of the cutoff function: in A9 there is a Montecarlo simulation which *does not prove,* but enforces the clue that $A \neq 1$.

It is appropriate to disclose here the developments leading to (1.1.16), leaving to the next chapters the proofs and the generalizations to the multi-point Schwinger functions. For a suitable choice of four counterterms, $\{\alpha^{(\mu)}\}_{\mu=\pm}$ and $\{\sigma^{(\mu)}\}_{\mu=\pm}$, analytically dependent on λ,

$$\begin{aligned}\lambda_b \sum_\mu \frac{a-\overline{a}\mu\omega}{2} \int_D \frac{\mathrm{d}^2 p}{(2\pi)^2}\, \Delta\widehat{H}^{(1;2)}_{\mu;\omega}(p;k-p) = \left(\sum_\mu \frac{a-\overline{a}\mu\omega}{2}\sigma^{(\mu\omega)}\lambda_b\right) \frac{\widehat{S}^{(2)}_\omega(k)}{\widehat{g}_\omega(k)} \\ + \left(\sum_\mu \frac{a-\overline{a}\mu\omega}{2}\alpha^{(\mu\omega)}\lambda_b\right) \frac{\lambda_b}{\zeta_b^{(2)}} \int_D \frac{\mathrm{d}^2 p}{(2\pi)^2}\, \widehat{S}^{(1;2)}_{-\omega;\omega}(p;k-p) + \Delta\widehat{K}_\omega(k)\end{aligned},$$

where, for k *fixed independently from* N, the rest $\Delta\widehat{K}_\omega(k)$ is vanishing. Putting together the above identity with (1.1.13) and (1.1.12), (1.1.16) holds for

$$\begin{aligned} A &\stackrel{def}{=} \frac{1}{1-(\lambda_b/2)\sum_\mu (a-\overline{a}\mu)\left(\alpha^{(\mu)}-\sigma^{(\mu)}\right)}\ , \\ B &\stackrel{def}{=} \frac{1-(\lambda_b/2)\sum_\mu (a-\overline{a}\mu)\alpha^{(\mu)}}{1-(\lambda_b/2)\sum_\mu (a-\overline{a}\mu)\left(\alpha^{(\mu)}-\sigma^{(\mu)}\right)}\ . \end{aligned} \tag{1.1.18}$$

Chapter 2:

Hamiltonian Regularization

Two different *regularizations* of the Thirring model can be considered: the Euclidean one, depicted in the previous Chapter, and the Hamiltonian one, introduced in the present Chapter. As well as, two are the main requirements of the OSA: the *Euclidean invariance* and the the *reflection positivity*.

Well then, the former property is evident only in the former regularization – and even false in the latter, if the limit of removed cutoff is not taken; while the latter property is built-in in the latter, and not so clear in the former.

But it is possible to prove that, for two (in general) *different* choices of the parameters of the Lagrangian, the two regularization, in the limit of removed cutoff, are equivalent, in the sense that the Schwinger function derived in the one or in the other scheme are *exactly the same*. And therefore they fulfill both the crucial properties.

This theorem is a first example of the effectiveness of the RG approach, which is introduced in the next Chapter.

2.1 Hamiltonian Thirring Model

This time only the space is discretized. Then, a finite dimensional Fock space, together to a many-body Hamiltonian, is built, guaranteeing *a priori* the validity of the *reflection positivity* (see A2.2) also after taking the continuum space limit.

Other constructions, different from the Hamiltonian formalism and verifying such positivity property, would have been possible: *e.g.* a certain lattice discretization of both space and time (different from the one in Chapter 1) would have turned the quantum field model into a statistical mechanical lattice model, nearest neighbours interactive, which is reflection positive by standard proof, [OS77]. Anyway, despite of the popularity of the latter route, here the former is preferred, since the consequent integration of the *hard fermions* (see later) was called upon, but never explicitly proved in [BM01] and in the following papers – where the setting can only be Hamiltonian, since they deal with many-body quantum models. As consequence, space and time are not managed on the same ground, and the phenomenon of *light velocity modification* occurs (as first noticed in [M93]): it is necessary to introduce a *counterterm* to fix the light velocity to 1.

In any case, lattice discretization of fermionic QFT – no matter if it affects only the space or both space and time – encounters the well known problem of the *doubling of fermions*. In order to make the effect of the double fermions to vanish, a possibility is to use a momentum dependent mass term, [W76]; but it destroys the symmetries of the propagators and generates a mass term even in the massless theory: a counterterm also for the mass is necessary, so that the mass on physical scale can be fixed to chosen value $\mu \geq 0$.

2.1.1 Hamiltonian. A finite dimensional Fock space is constructed in terms of the periodic spatial lattice, Λ_1, as follows. Let κ be fixed. Choosing $\gamma > 1$ and integer, let a and L be respectively the lattice spacing and the lattice side length, s.t. $4\kappa a \overset{def}{=} \gamma^{-N}$ and $4\kappa L \overset{def}{=} \gamma^{-h}$, for $N, -h$ large positive integers; then, the periodicity of the lattice is given by the quotient set

$$Q_1 \overset{def}{=} \left\{ n \in \mathbb{Z} \mid n \sim n' \text{ if } n - n' \in \frac{L}{a}\mathbb{Z} \right\},$$

so that the lattice Λ_1 and its reciprocal D_1 are

$$\Lambda_1 \overset{def}{=} \{ a n_1 \mid n_1 \in Q_1 \} \ , \qquad D_1 \overset{def}{=} \left\{ \frac{2\pi}{L}\left(m_1 + \frac{1}{2} \right) \mid m_1 \in Q_1 \right\} .$$

Now, let two couples of fermionic creation and destruction operators $\{a^{\sigma}_{k_1,\omega}\}^{\sigma,\omega=\pm}_{k_1 \in \Lambda_1}$ be defined with empty state $|0\rangle$; setting $c(k_1) \overset{def}{=} [1 - \cos(k_1 a)]/2a$, $e(k_1) \overset{def}{=} \sin(k_1 a)/a$ – the Fourier transform of the discrete derivative in x_1 – and, for any choice of the mass $\mu \geq 0$, letting $\mu(k_1) \overset{def}{=} \mu + c(k_1)$ be the "momentum dependent mass term", the free Hamiltonian is

$$H_0 \overset{def}{=} \frac{1}{L} \sum_{\omega} \sum_{k_1 \in \Delta_1} \omega e(k_1) a^+_{k_1,\omega} a^-_{k_1,\omega} + \frac{1}{L} \sum_{\omega} \sum_{k_1 \in \Delta_1} \mu(k_1) a^+_{k_1,\omega} a^-_{k_1,-\omega} .$$

In the limit $a \to 0$, the energy dispersion $e(k_1)$ is asymptotic to *two* linear dispersion: one containing $k_1 = 0$, which is the Euclidean Thirring dispersion; another one containing $k_1 = \pi/a$, and representing the double fermions: the role of $\mu(k_1)$ is to assign to the doubles a mass which is diverging with N.

The Hamiltonian is made interactive by the term

$$\frac{\lambda}{2}\frac{1}{L^3}\sum_{\omega}\sum_{k_1,p_1,q_1\in D_1} a^+_{k_1,\omega}a^-_{p_1,\omega}a^+_{q_1,-\omega}a^-_{k_1+q_1-p_1,-\omega}\ . \tag{2.1.1}$$

As in the Euclidean regularization, the parameter of the Lagrangian have to be tuned so to have a finite theory. Then, λ and μ are replaced with λ_N and μ_N; and H_0 is multiplied times the *field strength* Z_N.

Furthermore, to fix the mass to the chosen value and to have Schwinger functions with light velocity equal to 1 (as in the Euclidean regularization), it is necessary to introduce two further counterterms d_N and $\gamma^N n_N$, such that, setting $\nu_N \stackrel{def}{=} n_N/Z_N$ and $\delta_N \stackrel{def}{=} d_N/Z_N$, the interactive Hamiltonian finally reads

$$\begin{aligned} H \stackrel{def}{=} & \frac{1}{L}\sum_{\omega}\sum_{k_1\in D_1} \omega e(k_1) Z_N(1+\delta_N) a^+_{k_1,\omega}a^-_{k_1,\omega} \\ & + \frac{1}{L}\sum_{\omega}\sum_{k_1\in D_1}\Big(\mu(k_1)+\gamma^N\nu_N\Big) Z_N a^+_{k_1,\omega}a^-_{k_1,-\omega} \\ & + \frac{\lambda_N Z_N^2}{2}\frac{1}{L^3}\sum_{\omega}\sum_{k_1,p_1,q_1\in D_1} a^+_{k_1,\omega}a^-_{p_1,\omega}a^+_{q_1,-\omega}a^-_{k_1+q_1-p_1,-\omega}\ . \end{aligned} \tag{2.1.2}$$

2.1.2 Correlations. Let the fields and the density be defined

$$\psi^\sigma_{x,\omega} \stackrel{def}{=} e^{-x_0 H}\left(\frac{1}{L}\sum_{k_1\in D_1} e^{i\sigma k_1 x_1} a^\sigma_{k_1,\omega}\right) e^{x_0 H}\ , \qquad \rho^R_{x,\omega} \stackrel{def}{=} Z_N^{(2,+)}\psi^+_{x,\omega}\psi^-_{x,\omega} + Z_N^{(2,-)}\psi^+_{x,-\omega}\psi^-_{x,-\omega}\ ,$$

where $Z_N^{(2,+)}$ and $Z_N^{(2,-)}$ are the density strengths: they are two, rather than one as in the Euclidean regularization, since in this setting space and time are on different ground and the symmetry which make $Z_N^{(2,+)} = Z_N^{(2,-)}$ is missing.

For any $\underline{z} \stackrel{def}{=} z^{(1)},\dots,z^{(m)}$ and $\underline{x} \stackrel{def}{=} x^{(1)},\dots,x^{(n)}$, fixed set on spacetime points such that $0 < z_0^{(1)} < z_0^{(2)} < \cdots < x_0^{(1)} < \cdots < x_0^{(n)}$, the *correlations* are defined to be,

$$G^{(m;n)(\underline{\varepsilon})}_{\underline{\sigma};\underline{\omega}}(\underline{z};\underline{x}) \stackrel{def}{=} \frac{\mathrm{Tr}\left[e^{-LH}\rho^R_{z^{(1)},\sigma_1}\cdots\rho^R_{z^{(m)},\sigma_m}\psi^{\varepsilon_1}_{x^{(1)},\omega_1}\cdots\psi^{\varepsilon_n}_{x^{(n)},\omega_n}\right]}{\mathrm{Tr}\left[e^{-LH}\right]}\ , \tag{2.1.3}$$

where Tr is the trace over a complete set of states of the quantum lattice model.

2.1.3 Propagator. Also in this case the Schwinger function can be obtained in terms of a path integral formula, and a Grassmannian integration. The free Hamiltonian can be diagonalized in terms of a set of new creation and destruction operators, $\{b^\sigma_{k_1,\omega}\}^{\sigma,\omega=\pm}_{k_1\in\Lambda_1}$, and energy dispersion $E(k_1) \stackrel{def}{=} \sqrt{e^2(k_1)+\mu^2(k_1)}$:

$$H_0 = \frac{1}{L}\sum_{\omega}\sum_{k_1\in\Lambda_1}\omega E(k_1) b^+_{k_1,\omega}b^-_{k_1,\omega}\ ,$$

where $b^{\sigma}_{k_1,\mu} \stackrel{def}{=} \sum_\nu a^{\sigma}_{k_1,\nu}\left(C^{-1}(k_1)\right)_{\nu,\mu}$ for

$$C(k_1) \stackrel{def}{=} \begin{pmatrix} \mu(k_1) & E(k_1)-e(k_1) \\ e(k_1)-E(k_1) & \mu(k_1) \end{pmatrix} \frac{1}{\sqrt{\mu^2(k_1)+\left[E(k_1)-e(k_1)\right]^2}} \ .$$

Calling T the time ordering, it is useful to define the *propagator* as

$$\begin{aligned} g_{\alpha,\beta}(x) &\stackrel{def}{=} \frac{\mathrm{Tr}\left[e^{-LH_0}T\left(a^+_{k_1,\alpha}a^-_{k_1,\beta}\right)\right]}{\mathrm{Tr}\left[e^{-LH_0}\right]} = \sum_\omega \frac{\mathrm{Tr}\left[e^{-LH_0}T\left(b^+_{k_1,\omega}b^-_{k_1,\omega}\right)\right]}{\mathrm{Tr}\left[e^{-LH_0}\right]} C(k_1)_{\omega,\alpha}C(k_1)_{\omega,\beta} \\ &= \frac{1}{L}\sum_{k_1\in D_1} e^{-ix_1k_1-x_0\omega E(k_1)} \\ &\qquad \cdot \sum_\omega \left\{ \frac{\chi(x_0>0)}{1+e^{-\omega E(k_1)L}} - \frac{\chi(x_0\le 0)e^{-\omega E(k_1)L}}{1+e^{-\omega E(k_1)L}} \right\} C(k_1)_{\omega,\alpha}C(k_1)_{\omega,\beta} \ . \end{aligned}$$

By *partial-fraction expansion* of the meromorphic functions in the curl brackets (see A1.1), the propagator is turned into:

$$g_{\alpha,\beta}(x) = \lim_{M\to\infty}\frac{1}{L\beta}\sum_{k\in D} e^{-ik\cdot x}\frac{\widehat{\chi}_M(k_0)}{\mu_N^2(k_1)+k_0^2+e^2(k_1)} \begin{pmatrix} ik_0+e(k_1) & \mu_N(k_1) \\ \mu_N(k_1) & ik_0-e(k_1) \end{pmatrix}_{\alpha,\beta} \ , \qquad (2.1.4)$$

with $D \stackrel{def}{=} D_0\times D_1$ and $D_0 \stackrel{def}{=} \left\{\frac{2\pi}{\beta}(m+\frac{1}{2})\right\}_{m\in\mathbb{Z}}$ (namely D is the product of a periodic lattice in the space direction times an unbounded one in the time direction); $\widehat{\chi}_M(k_0)$ a non-negative, smooth cutoff, introduced to give a meaning to the above expression – which is a *generalized summation* of a series which does not converge in absolute sense. Specifically, with reference to the function $\widehat{\chi}_N(t)$ defined in 1.1.4, the cutoff is defined to be $\widehat{\chi}_M \stackrel{def}{=} \widehat{\chi}_N\left(\gamma^{-M+N}t\right)$.

2.1.4 Schwinger functions. As well know consequence of the Trotter formula for the expansion of the evolution operator, e^{x_0H}, and the Wick theorem (see for instance [FW]), the correlations in (2.1.3) can be generated from the functional $\mathcal{Z}(\jmath,\varphi) \stackrel{def}{=} e^{\mathcal{W}(\jmath,\varphi)}$, where, in its turn, $\mathcal{W}(\jmath,\varphi)$ is defined to be the *generating functional of the Schwinger function in the Hamiltonian regularization*:

$$\begin{aligned} e^{\mathcal{W}(\jmath,\varphi)} \stackrel{def}{=} \int \mathrm{d}P^{(\le M)}(\psi)\exp\Big\{ &-\lambda_N\mathcal{V}\left(\sqrt{Z_N}\psi\right) + \gamma^N\nu_N\mathcal{N}\left(\sqrt{Z_N}\psi\right) + \delta_N\mathcal{D}\left(\sqrt{Z_N}\psi\right) \\ &+ \sum_\sigma \zeta_N^{(2,\sigma)}\mathcal{J}_\sigma(\jmath,\sqrt{Z_N}\psi) + \mathcal{F}(\varphi,\psi)\Big\} \ . \end{aligned} \qquad (2.1.5)$$

The settings are the following. The Gaussian free measure is given by

$$\begin{aligned} \mathrm{d}P^{(\le M)}(\psi) \stackrel{def}{=} \exp\left\{ L^2 O_N - Z_N \sum_{\alpha,\beta=\pm}\int_{D_M}\frac{\mathrm{d}^2k}{(2\pi)^2}\,\frac{T_{\alpha,\beta}(k)}{\widehat{\chi}_M(k_0)}\widehat{\psi}^+_{k,\omega}\widehat{\psi}^-_{k,\omega} \right\} \\ \prod_{k\in D_M}\prod_{\omega=\pm}\mathrm{d}\widehat{\psi}^+_{k,\omega}\mathrm{d}\widehat{\psi}^-_{k,\omega} \ , \end{aligned} \qquad (2.1.6)$$

where O_M is the normalization, $\zeta_N^{(2,\sigma)} \stackrel{def}{=} Z_N^{(2,\sigma)}/Z_N$ and the covariance $\widehat{g}_{\mu,\nu}(k)$ is:

$$\widehat{g}^{-1}(k) \stackrel{def}{=} \frac{T(k)}{\widehat{\chi}_M(k_0)} \;, \qquad T(k) \stackrel{def}{=} \begin{pmatrix} -ik_0 + e(k_1) & \mu_N(k_1) \\ \mu_N(k_1) & -ik_0 - e(k_1) \end{pmatrix} \;,$$

with

$$e(k_1) \stackrel{def}{=} \frac{\sin(k_1 a)}{a} \;, \qquad \mu_N(k_1) \stackrel{def}{=} \frac{1-\cos(k_1 a)}{a} + \mu_N \;; \tag{2.1.7}$$

the lattice $D_M \stackrel{def}{=} \{k \in D : \widehat{\chi}_M(k_0) \neq 0\}$; the self-interaction is given by the potentials

$$\mathcal{V}(\psi) \stackrel{def}{=} \frac{1}{2} \sum_\omega \int_D \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{d^2k}{(2\pi)^2} \; \widehat{\psi}^+_{p,\omega} \widehat{\psi}^-_{q,\omega} \widehat{\psi}^+_{k,-\omega} \widehat{\psi}^-_{p+k-q,-\omega} \;,$$

and

$$\mathcal{D}(\psi) \stackrel{def}{=} \sum_\omega \int_D \frac{d^2k}{(2\pi)^2} \; \omega\varepsilon(p_1) \widehat{\psi}^+_{p,\omega} \widehat{\psi}^-_{p,\omega} \;, \qquad \mathcal{N}(\psi) \stackrel{def}{=} \sum_\omega \int_D \frac{d^2k}{(2\pi)^2} \; \widehat{\psi}^+_{p,\omega} \widehat{\psi}^-_{p,-\omega} \;.$$

In order to generate the Schwinger functions, there are also interactions with external sources:

$$\mathcal{J}_\sigma(\jmath, \psi) \stackrel{def}{=} \sum_\omega \int_D \frac{d^2k}{(2\pi)^2} \frac{d^2p}{(2\pi)^2} \; \widehat{\jmath}_{p-k,\omega} \widehat{\psi}^+_{k,\sigma\omega} \widehat{\psi}^-_{p,\sigma\omega} \;.$$

Theorem 2.1. *There exists $\varepsilon > 0$, a suitable choice of the parameters of the Hamiltonian model, λ_N, μ_N, Z_N, $Z_N^{(2,+)}$, $Z_N^{(2,-)}$, ν_N, δ_N, and a suitable choice of the parameters of the Euclidean model, λ_N, μ_N Z_N, $Z_N^{(2)}$ – the analogous parameters of the two model being, in general, different – such that, in the limit of removed cutoff, each Schwinger function in the former regularization coincides with the analogous Schwinger function in the latter one.*

The proof is deferred to the next Chapter: see 3.5.

Chapter 3:

Renormalization Group Analysis

After slicing the momenta in scales, the parameters of the generating functional are turned into *effective parameters* for each given momentum scale; in this way obtaining a sequence, the *flow of the running coupling constants*, which is controlled by the *vanishing of the Beta function*.

3.1 Renormalization Group Analysis for Hard Fermions

3.1.1 Momenta slicing. From now on, to be definite, the *scaling parameter* γ is fixed to be equal to 3 – but any other value would be fine, suitable changing the following definition of the cutoff. Then, $\kappa\gamma^{N+1} = 3\pi/4a$, and the *cutoff function* $\widehat{\chi}_0(k_0)$ is defined, for $t \in \mathbb{R}$,

$$\widehat{\chi}_0(t) \stackrel{def}{=} \begin{cases} 1 & \text{for } |t| \le \kappa \\ 0 & \text{for } 3\kappa \le \kappa|t| \le 4\kappa \\ \in (0,1) & \text{otherwise ;} \end{cases}$$

the actual shape in the third domain is here inessential: it will be chosen in 3.3.1. Accordingly, for $h = N, \ldots, M$, it is set $\widehat{\chi}_h(t) \stackrel{def}{=} \widehat{\chi}_0\left(\gamma^{-h} t\right)$. With $\widehat{\chi}_0$ it is possible to make a partition of the momenta scales: for any $h = N, \ldots, M$,

$$\widehat{\chi}_M(t) = \widehat{\chi}_h(t) + \sum_{k=h+1}^{M} \widehat{f}_k(t) \ , \qquad \text{with } \widehat{f}_k(t) \stackrel{def}{=} \widehat{\chi}_k(t) - \widehat{\chi}_{k-1}(t). \tag{3.1.1}$$

It is worthwhile to remark $\widehat{f}_k$ has compact support $\{t : \kappa\gamma^{k-1} \le |t| \le \kappa\gamma^{k+1}\}$.

3.1.2 Multiscale integration. The decomposition (3.1.1) has the purpose to obtain the following scale integration of $\mathcal{W}(\varphi, \jmath)$: for any integer $h : N, \dots, M$,

$$e^{\mathcal{W}(\jmath,\varphi)} = e^{E_h} \int \mathrm{d}\widehat{P}^{(\le h)}(\psi)\; e^{\mathcal{W}^{(h)}\left(\varphi,\jmath,\sqrt{Z_N}\psi\right)} \;, \tag{3.1.2}$$

where the *vacuum energy* on scale h, E_h, do not depend on the fields; the measure $\mathrm{d}\widehat{P}^{(\le h)}$ is the same as (1.1.4), with $\widehat{\chi}_M(k_0)$ replaced by $\widehat{\chi}_h(k_0)$; the *effective potential* on scale h, $\mathcal{W}^{(h)}$, is a functional of the fields:

$$\begin{aligned}\mathcal{W}^{(h)}\left(\varphi,\jmath,\sqrt{Z_N}\psi\right) \overset{def}{=} &-\lambda_N \mathcal{V}\left(\sqrt{Z_N}\psi\right) + \gamma^N \nu_N \mathcal{N}\left(\sqrt{Z_N}\psi\right) + \delta_N \mathcal{D}\left(\sqrt{Z_N}\psi\right) \\ &+ \sum_{\sigma=\pm} \zeta_N^{(2,\sigma)} \mathcal{J}_\sigma\left(\jmath,\sqrt{Z_N}\psi\right) + \mathcal{F}(\varphi,\jmath) + \mathcal{W}^{(h)}_{\mathrm{irr}}\left(\varphi,\jmath,\sqrt{Z_N}\psi\right) \;;\end{aligned} \tag{3.1.3}$$

namely it has the same expression of the argument of the exponential in the r.h.s. member of (1.1.3), apart from the *irrelevant contribution* $\mathcal{W}^{(h)}_{\mathrm{irr}}$.

Scale integration (3.1.2) can be verified by induction. Indeed, it is true for $h = M$, with $E_M = 0$ and $\mathcal{W}^{(M)}_{\mathrm{irr}} \equiv 0$; while the procedure to obtain E_{h-1}, $\mathcal{W}^{(h-1)}$ and $\mathcal{W}^{(h-1)}_{\mathrm{irr}}$ is the following.

The field ψ is decomposed into the sum of fields $\psi \to \psi + (Z_N)^{-1/2}\xi$, both with Gaussian distribution. The propagator on scale h of ξ, the *hard fermion field on scale* h, is given by

$$g^{(h)}_{\alpha,\beta}(x) \overset{def}{=} \int_D \frac{\mathrm{d}^2 k}{(2\pi)^2}\, e^{-ik\cdot x} \frac{\widehat{f}_h(k_0)}{\mu_N^2(k_1) + k_0^2 + e^2(k_1)} \begin{pmatrix} ik_0 + e(k_1) & -\mu_N(k_1) \\ -\mu_N(k_1) & ik_0 - e(k_1)\end{pmatrix}_{\alpha,\beta} \;; \tag{3.1.4}$$

hence, by decomposition (3.1.1), ψ is left with propagator

$$g^{(\le h-1)}_{\alpha,\beta}(x) \overset{def}{=} \int_D \frac{\mathrm{d}^2 k}{(2\pi)^2}\, e^{-ik\cdot x} \frac{\widehat{\chi}_{h-1}(k_0)}{\mu_N^2(k_1) + k_0^2 + e^2(k_1)} \begin{pmatrix} ik_0 + e(k_1) & -\mu_N(k_1) \\ -\mu_N(k_1) & ik_0 - e(k_1)\end{pmatrix}_{\alpha,\beta} . \tag{3.1.5}$$

Then, calling $\mathrm{d}\widehat{P}^{(\le h-1)}(\psi)$ and $\mathrm{d}\widehat{P}^{(h)}(\xi)$ the measure (1.1.4), with propagators (3.1.4) and (3.1.5) respectively, the hard fermion is integrated out:

$$\begin{aligned}\int \mathrm{d}\widehat{P}^{(\le h)}(\psi)\; e^{\mathcal{W}^{(h)}\left(\varphi,\jmath,\sqrt{Z_N}\psi\right)} &= \int \mathrm{d}\widehat{P}^{(\le h-1)}(\psi) \int \mathrm{d}\widehat{P}^{(h)}(\xi)\; e^{\mathcal{W}^{(h)}\left(\varphi,\jmath,\sqrt{Z_N}\psi+\xi\right)} \\ &\overset{def}{=} e^{\Delta E_{h-1}} \int \mathrm{d}\widehat{P}^{(\le h-1)}(\psi)\; e^{\mathcal{W}^{(h-1)}\left(\varphi,\jmath,\sqrt{Z_N}\psi\right)} \,,\end{aligned} \tag{3.1.6}$$

where ΔE_{h-1} is the part of the integration constant the fields. Therefore, the vacuum energy on scale $h-1$ is defined to be:

$$E_{h-1} \overset{def}{=} E_h + \Delta E_{h-1} \;;$$

while,

$$
\begin{aligned}
&\mathcal{W}_{\rm irr}^{(h-1)}\left(\varphi, \jmath, \sqrt{Z_N}\psi\right) \\
&\stackrel{def}{=} \ln \int {\rm d}\widehat{P}^{(h)}(\xi)\; e^{\mathcal{W}^{(h)}\left(\varphi,\jmath,\sqrt{Z_N}\psi+\xi\right)} - \Delta E_{h-1} \\
&= \sum_{n^\psi, n^\varphi, n^\jmath \geq 1}^{n^\psi+n^\varphi+n^\jmath\neq 0} \sum_{\underline{\omega},\underline{\sigma}} \int_\Lambda {\rm d}^2\underline{x}\; {\rm d}^2\underline{y}\; {\rm d}^2\underline{z} \\
&\left(\prod_{i=1}^{n^\psi} \sqrt{Z_N}\psi^{\sigma_i}_{x^{(i)},\omega_i}\right)\left(\prod_{i=1}^{n^\varphi} \frac{\varphi^{\sigma'_i}_{y^{(i)},\omega'_i}}{\sqrt{Z_N}}\right)\left(\prod_{j=1}^{n^\jmath} \jmath_{z^{(i)},\omega''_i}\right) W^{(h-1)}_{n^\psi;n^\varphi;n^\jmath,\underline{\omega},\underline{\sigma}}(\underline{x},\underline{y},\underline{z})\;,
\end{aligned}
\tag{3.1.7}
$$

where $\underline{x}$, $\underline{y}$ and $\underline{z}$ are short notations for $x^{(1)},\ldots,x^{(n^\psi)}$, $y^{(1)},\ldots,y^{(n^\varphi)}$ and $z^{(1)},\ldots,z^{(n^\jmath)}$ respectively. By the well known formula for the truncated expectation w.r.t. a Gaussian measure, the function $W^{(h-1)}_{n^\psi;n^\varphi;n^\jmath,\underline{\omega},\underline{\sigma}}(\underline{x},\underline{y},\underline{z})$ is a power series in the couplings $\lambda_N, \nu_N, \delta_N$, and coefficient given by all the Feynman graphs with $n^\psi + n^\varphi + n^\jmath$ external legs of kind $n^\psi, n^\varphi, n^\jmath$ attached respectively to the points $\underline{x}, \underline{y}, \underline{z}$, with eventually a constraint that some among the point in $\underline{x}$ may coincide: this is explained in more details in Appendix A3. The remarkable fact is that the number of the Feynman graphs at n-the order expansion is about $n!$; and yet, by *cluster expansion* and anticommutativity of the fermion fields, it is possible to prove a C^n-bound, making the power series defining $W^{(h-1)}_{n^\psi;n^\varphi;n^\jmath,\underline{\omega},\underline{\sigma}}(\underline{x},\underline{y},\underline{z})$ *absolutely convergent* for $\lambda_N, \nu_N, \delta_N$ small enough (see A3.2).

Finally, $\mathcal{W}^{(h-1)}_{\rm irr}$ is defined by (3.1.3): in power series expansion, it corresponds to the terms in (3.1.7) which are at least $\mathrm{O}(\lambda_N)$, except the terms for $n^\psi = 4$, $n^\varphi = n^\jmath = 0$ and linear in λ_N.

3.1.3 Dimensional bounds. In order to have a bound for $W^{(h)}_{n^\psi;n^\varphi;n^\jmath,\underline{\omega},\underline{\sigma}}$, it is possible to prove the following decay property of the diagonal and antidiagonal propagators: there exist two positive constants c and C such that

$$
\begin{aligned}
\left|g^{(h)}_{\omega,\omega}(x)\right| &\leq C\gamma^N e^{-c\sqrt{\gamma^N\kappa|x|}} e^{-c\sqrt{\gamma^h\kappa|x_0|}}\;, \\
\left|g^{(h)}_{\omega,-\omega}(x)\right| &\leq \gamma^{-(h-N)} C\gamma^N e^{-c\sqrt{\gamma^N\kappa|x|}} e^{-c\sqrt{\gamma^h\kappa|x_0|}}\;.
\end{aligned}
\tag{3.1.8}
$$

Since $h > N$, the more factor $\gamma^{-(h-N)}$ in the bound of the antidiagonal propagator represents a “gain factors” w.r.t. the bound of the diagonal one.

In the end of the integration of all hard fermions scales, (3.1.2) reads

$$
e^{\mathcal{W}(\jmath,\varphi)} = e^{E_N} \int {\rm d}\widehat{P}^{(\leq N)}(\widehat{\psi})\; e^{\mathcal{W}^{(N)}\left(\varphi,\jmath,\sqrt{Z_N}\widehat{\psi}\right)}\;, \tag{3.1.9}
$$

which is the starting point of the analysis of the double and light fermions in the next sections. Let $d(\underline{x})$ be the *tree distance* of the points $\underline{x}$, namely the length of the shortest tree path connecting every point in $\underline{x}$.

Lemma 3.1. *There exist $\varepsilon > 0$ and the positive constants c and C s.t., for any choice of the couplings $|\lambda_N|, |\delta_N|, |\nu_N| < 2\varepsilon$, the following bounds hold.*

1. *If $n^\varphi + n^\jmath \neq 0$,*

$$\int_\Lambda \mathrm{d}^2\underline{x}\ \left|W^{(N)}_{n^\psi;n^\varphi;n^\jmath,\underline{\omega},\underline{\sigma}}(\underline{x},\underline{y},\underline{z})\right| \leq C\frac{\gamma^{N\left(2-(1/2)n^\psi-(3/2)n^\varphi-n^\jmath\right)}}{e^{\frac{c}{2(n^\varphi+n^\jmath)}\sqrt{\gamma^N\kappa d(\underline{y},\underline{z})}}} .$$

2. *If $n^\varphi + n^\jmath = 0$,*

$$\int_\Lambda \mathrm{d}^2_*\underline{x}\ \left|W^{(N)}_{n^\psi;0;0,\underline{\omega},\underline{\sigma}}(\underline{x})\right| \leq C\gamma^{N\left(2-(1/2)n^\psi\right)} ,$$

where $\mathrm{d}^2_*\underline{x}$ means that the integration is performed w.r.t. all but any one variable among $x^{(1)},\ldots,x^{(n)}$.

The proof is the same of Lemma A.3.1.

3.1.4 Remark: superrinormalizability. The key feature, here, is the scaling $(Z_N)^{-1/2}$ of hard fermion in the decomposition $\widehat{\psi} \to \psi + (Z_N)^{-1/2}\,\xi$: this factor is *the same for all the scales* $h > N$, so that there is no generation of anomalous dimension in the hard fermion regime.

3.2 Renormalization Group Analysis for Double Fermions

3.2.1 Momenta slicing. At this point it is convenient to choose the image in $(0,1)$ of the cutoff function so that the constant function $I \equiv 1$ on the periodic lattice D_1 is equal to the sum of two $\widehat{\chi}_N$ functions, the former centred in $k_1 = 0$, and the latter centred in $k_1 = \pi/a$:

$$\widehat{\chi}_N(t) + \widehat{\chi}_N\left(t - \frac{\pi}{a}\right) \equiv 1 \tag{3.2.1}$$

(and such that $\widehat{\chi}_0$ is a Gevrey function: see A1.2). After the integration of the hard fermions, it was left the measure $\mathrm{d}\widehat{P}^{(\leq N)}(\psi)$, with propagator given by (3.1.5) for $h = N$: it is possible now to decompose the fields ψ into the sum $\psi \to \psi + (Z_N)^{-1/2}\,\xi$, where the *double fermion field*, ξ, has propagator

$$g^{(\mathrm{D})}_{\alpha,\beta}(x) \stackrel{def}{=} \int_D \frac{\mathrm{d}^2k}{(2\pi)^2}\ e^{-ik\cdot x}\frac{\widehat{\chi}_N(k_0)\widehat{\chi}_N\left(k_1-(\pi/a)\right)}{\mu_N^2(k_1)+k_0^2+e^2(k_1)}\begin{pmatrix} ik_0+e(k_1) & -\mu_N(k_1)\\ -\mu_N(k_1) & ik_0-e(k_1)\end{pmatrix}_{\alpha,\beta} ; \tag{3.2.2}$$

therefore, because of (3.2.1) and setting $\chi_N(k) \stackrel{def}{=} \widehat{\chi}_N(k_0)\widehat{\chi}_N(k_1)$, ψ is left with propagator

$$g^{(\leq N,D)}_{\alpha,\beta}(x) \stackrel{def}{=} \int_D \frac{\mathrm{d}^2k}{(2\pi)^2}\ e^{-ik\cdot x}\frac{\chi_N(k)}{\mu_N^2(k_1)+k_0^2+e^2(k_1)}\begin{pmatrix} ik_0+e(k_1) & -\mu_N(k_1)\\ -\mu_N(k_1) & ik_0-e(k_1)\end{pmatrix}_{\alpha,\beta} . \tag{3.2.3}$$

3.2.2 Dimensional bounds. Because of the definition of $\mu_N(k_1)$, the propagator $g^{\mathrm{D}}_{\mu,\nu}(x)$ is massive, and hence, without decomposition of $\widehat{\chi}_N(k_0)\widehat{\chi}_N\left(k_1-(\pi/a)\right)$ into scales, it enjoys the bound, for c and C two positive constants,

$$\left|g^{(\mathrm{D})}_{\alpha,\beta}(x)\right| \leq C\gamma^N e^{-c\sqrt{\gamma^N\kappa|x|}} . \tag{3.2.4}$$

Indeed in the support of $\widehat{\chi}_N(k_0)\widehat{\chi}_N\left(k_1-(\pi/a)\right)$, it holds $\pi/4a \leq |k_1| \leq \pi/4$, while $|k_0|$ can be very small: since the mass μ_N is supposed non-negative, the denominator is not lower than $\mu^2_N(k_1) \geq c^2(k_1) \geq \left(\kappa\gamma^N(2-\sqrt{2})/2\pi\right)^2$. And the bound follows by dimensionality argument. In this way the effects of the second pole are confined on the scale of the cutoff, N: since it will be proved that the Schwinger functions do not depend on contribution on such scales, the addition of $c(k_1)$ to the mass has had the effect to suppress the effects of the double fermions.

Integrating out the double field now requires a localization, which will be explained in the next section.

3.3 Renormalization Group Analysis for Soft Fermions

3.3.1 Momenta slicing. The last, more involved regime to be studied is the set of momentum scales below N. Let $\chi_N(k)$ be decomposed over the scales

$$\chi_N(t_0,t_1) = \chi_h(t_0,t_1) + \sum_{k=h+1}^{N} f_k(t_0,t_1) , \tag{3.3.1}$$

where the function $f_k(t_0,t_1)$ is defined to be $\chi_k(t_0,t_1) - \chi_{k-1}(t_0,t_1)$ and has squared support $\left\{(t_0,t_1) : \kappa\gamma^{k-1} \leq \max\{|t_0|,|t_1|\} \leq \kappa\gamma^{k+1}\right\}$.

3.3.2 Multiscale integration. As for the hard fermions, the functional integration of the soft fermions is performed scale by scale. By induction, for any integer $h : h \leq N$, it holds:

$$e^{\mathcal{W}(\jmath,\varphi)} = e^{E_h} \int \mathrm{d}\widetilde{P}^{(\leq h)}(\psi)\; e^{\mathcal{W}^{(h)}\left(\varphi,\jmath,\sqrt{Z_h}\psi\right)} , \tag{3.3.2}$$

where the effective potential on scale h is

$$\begin{aligned}\mathcal{W}^{(h)}\left(\varphi,\jmath,\sqrt{Z_h}\psi\right) \stackrel{def}{=} &-\lambda_h\mathcal{V}\left(\sqrt{Z_h}\psi\right) + \gamma^h\nu_h\mathcal{N}\left(\sqrt{Z_h}\psi\right) + \delta_h\mathcal{D}\left(\sqrt{Z_h}\psi\right)\\ &+\sum_{\sigma=\pm}\zeta^{(2,\sigma)}_h\mathcal{J}_\sigma\left(\jmath,\sqrt{Z_h}\psi\right) + \mathcal{F}(\varphi,\jmath) + \mathcal{W}^{(h)}_{\mathrm{irr}}\left(\varphi,\jmath,\sqrt{Z_h}\psi\right) ;\end{aligned} \tag{3.3.3}$$

the measure $\mathrm{d}\widetilde{P}^{(\leq h)}$, the couplings λ_h,ν_h, δ_h, $\zeta^{(2,\sigma)}_h$ and the irrelevant potential $\mathcal{W}^{(h)}_{\mathrm{irr}}$ are inductively specified by the procedure to construct $\mathcal{W}^{(h-1)}$.

The field ψ is decomposed into the sum of two fields, $\psi \to \psi + (Z_h)^{-1/2}\,\xi$, both with Gaussian distribution. The propagator of the *soft fermion* field, ξ is, *for* $h \neq N$:

$$g^{(h)}_{\alpha,\beta}(x) = \int_D \frac{\mathrm{d}^2 k}{(2\pi)^2}\; e^{-ik\cdot x} \frac{\widetilde{f}^{(h)}(k)}{\widetilde{\mu}_h^2(k) + k_0^2 + e^2(k_1)} \begin{pmatrix} ik_0 + e(k_1) & -\widetilde{\mu}_h(k) \\ -\widetilde{\mu}_h(k) & ik_0 - e(k_1) \end{pmatrix}_{\alpha,\beta} , \qquad (3.3.4)$$

with

$$\widetilde{f}^{(h)}(k) \stackrel{def}{=} f_h(k)\widetilde{C}^{(1)}_h(k) , \qquad \widetilde{c}_h(k) \stackrel{def}{=} \frac{Z_N}{Z_h} c(k_1)\widetilde{C}^{(1)}_h(k) ,$$

$$\widetilde{\mu}_h(k) \stackrel{def}{=} \mu_h \widetilde{C}^{(2)}_h(k) + \widetilde{c}_h(k_1) ,$$

and the quantities Z_h, μ_h, $\widetilde{C}^{(1)}_h(k)$ and $\widetilde{C}^{(1)}_h(k)$ will be constructed in the following *localization*. For $h = N$, to the above expression for the propagator it has to be added the propagator deriving from the the double fermions, $g^{(\mathrm{D})}_{\alpha,\beta}(k)$.

Since in presence of $\chi_{h-1}(k)$, by simply support compatibility, $\widetilde{C}^{(1)}_h(k) \equiv \widetilde{C}^{(2)}_h(k) \equiv 1$, by (3.3.1), ψ is left with propagator:

$$g^{(\le h-1)}_{\alpha,\beta}(x) \stackrel{def}{=} \int_D \frac{\mathrm{d}^2 k}{(2\pi)^2}\; e^{-ik\cdot x} \frac{\chi_{h-1}(k)}{\mu_{h-1}^2(k_1) + k_0^2 + e^2(k_1)} \begin{pmatrix} ik_0 + e(k_1) & -\mu_{h-1}(k_1) \\ -\mu_{h-1}(k_1) & ik_0 - e(k_1) \end{pmatrix}_{\alpha,\beta} , \qquad (3.3.5)$$

with

$$\mu_h(k_1) \stackrel{def}{=} \mu_h + \frac{Z_N}{Z_h} c(k_1) ,$$

without any residue of $\widetilde{C}^{(1)}_h(k)$ or $\widetilde{C}^{(2)}_h(k)$.

The soft fermions can be integrated out, scale by scale; this time this operation does not give directly $\mathcal{W}^{(h-1)}$, but rather $\widetilde{\mathcal{W}}^{(h-1)}$. Calling $\mathrm{d}P^{(\le h-1)}(\psi)$ and $\mathrm{d}P^{(h)}(\xi)$ the measure (1.1.4), with Z_N replaced by Z_h and propagators respectively given by (3.3.5) and (3.3.4)

$$\begin{aligned} \int \mathrm{d}\widetilde{P}^{(\le h)}(\psi)\; e^{\mathcal{W}^{(h)}\left(\varphi,\jmath,\sqrt{Z_h}\psi\right)} &= \int \mathrm{d}P^{(\le h-1)}(\psi) \int \mathrm{d}P^{(h)}(\xi)\; e^{\mathcal{W}^{(h)}\left(\varphi,\jmath,\sqrt{Z_h}\psi+\xi\right)} \\ &\stackrel{def}{=} \int \mathrm{d}P^{(\le h-1)}(\psi)\; e^{\widetilde{\mathcal{W}}^{(h-1)}\left(\varphi,\jmath,\sqrt{Z_h}\psi\right)+\Delta E_{h-1}} , \end{aligned} \qquad (3.3.6)$$

where ΔE_{h-1} is the part of the integration constant in the fields. Again, by the well known formulas of the truncated expectations:

$$\begin{aligned} &\widetilde{\mathcal{W}}^{(h-1)}\left(\varphi,\jmath,\sqrt{Z_h}\psi\right) \\ &= \sum_{n^\psi,n^\varphi,n^\jmath\ge 1}^{n+n^\varphi+n^\jmath\neq 0} \sum_{\underline{\omega},\underline{\sigma}} \int_\Lambda \mathrm{d}^2\underline{x}\,\mathrm{d}^2\underline{y}\,\mathrm{d}^2\underline{z} \\ &\left(\prod_{i=1}^{n} \sqrt{Z_h}\psi^{\sigma_i}_{x^{(i)},\omega_i}\right) \left(\prod_{i=1}^{n^\varphi} \frac{\varphi^{\sigma'_i}_{y^{(i)},\omega'_i}}{\sqrt{Z_h}}\right) \left(\prod_{j=1}^{n^\jmath} \jmath_{z^{(i)},\omega''_i}\right) \widetilde{W}^{(h-1)}_{n^\psi;n^\varphi;n^\jmath,\underline{\omega},\underline{\sigma}}(\underline{x},\underline{y},\underline{z}) . \end{aligned} \qquad (3.3.7)$$

For the light fermions a further step is necessary to extract parts of $\widetilde{\mathcal{W}}^{(h-1)}$ that can be absorbed either into the free measure $\mathrm{d}P^{(\le h-1)}$, or in the couplings; this is the *Localization*. In the end of this operation they are left a potential $\mathcal{W}^{(h-1)}$ and a measure $\mathrm{d}\widetilde{P}^{(\le h-1)}$, which fulfil (3.3.3).

3.3.3 Dimensional bounds. It is convenient to decompose the propagator $g^{(h)}_{\omega,\sigma}$ into the one of the Euclidean Model, $g^{(\mathrm{E},h)}_{\omega,\sigma}$, plus the rest, $g^{(\mathrm{R},h)}_{\omega,\sigma}$, plus the eventual contribution of the double fermion, $g^{(\mathrm{D})}_{\omega,\sigma}$; in their turn, let $g^{(\mathrm{E1},h)}_{\omega,\sigma}$, $g^{(\mathrm{R1},h)}_{\omega,\sigma}$ and $g^{(\mathrm{D1})}_{\omega,\sigma}$ be respectively the part of $g^{(\mathrm{E},h)}_{\omega,\sigma}$, $g^{(\mathrm{R},h)}_{\omega,\sigma}$ and $g^{(\mathrm{D})}_{\omega,\sigma}$ which is constant or linear in the mass. Finally:

$$g^{(h)}_{\omega,\sigma}(x)\stackrel{def}{=}g^{(\mathrm{E1},h)}_{\omega,\sigma}(x)+g^{(\mathrm{R1},h)}_{\omega,\sigma}(x)+\delta_{h,N}g^{(\mathrm{D1})}_{\omega,\sigma}(x)+r^{(1,h)}_{\omega,\sigma}(x)+r^{(2,h)}_{\omega,\sigma}(x)\ , \tag{3.3.8}$$

with the following definitions

$$\begin{aligned}
&g^{(\mathrm{E1},h)}_{\omega,\omega}(x)\stackrel{def}{=}\int_D\frac{\mathrm{d}^2k}{(2\pi)^2}\,e^{-ik\cdot x}\frac{\widetilde f^{(h)}(k)}{D_\omega(k)}\ ,\qquad g^{(\mathrm{E1},h)}_{\omega,-\omega}(x)\stackrel{def}{=}\int_D\frac{\mathrm{d}^2k}{(2\pi)^2}\,e^{-ik\cdot x}\frac{-\widetilde\mu_h(k)}{k_0^2+k_1^2}\widetilde f^{(h)}(k)\ ,\\
&g^{(\mathrm{R1},h)}_{\omega}(x)\stackrel{def}{=}\int_D\frac{\mathrm{d}^2k}{(2\pi)^2}\,e^{-ik\cdot x}\left[\frac{ik_0+\omega e(k_1)}{\widetilde c_h^2(k)+k_0^2+e^2(k_1)}-\frac{-D_{-\omega}(k)}{k_0^2+k_1^2}\right]\widetilde f^{(h)}(k)\ ,\\
&g^{(\mathrm{R1},h)}_{\omega,-\omega}(x)\stackrel{def}{=}\int_D\frac{\mathrm{d}^2k}{(2\pi)^2}\,e^{-ik\cdot x}\left[\frac{-\widetilde\mu_h(k)}{\widetilde c_h^2(k)+k_0^2+e^2(k_1)}-\frac{-\widetilde\mu_h(k)}{k_0^2+k_1^2}\right]\widetilde f^{(h)}(k)\ ,\\
&r^{(1,h)}_{\omega,\omega}(x)\stackrel{def}{=}\int_D\frac{\mathrm{d}^2k}{(2\pi)^2}\,e^{-ik\cdot x}\left[\frac{-D_{-\omega}(k)}{\widetilde\mu_h^2(k)+k_0^2+k_1^2}-\frac{-D_{-\omega}(k)}{k_0^2+k_1^2}\right]\widetilde f^{(h)}(k)\ ,\\
&r^{(1,h)}_{\omega,-\omega}(x)\stackrel{def}{=}\int_D\frac{\mathrm{d}^2k}{(2\pi)^2}\,e^{-ik\cdot x}\left[\frac{-\widetilde\mu_h(k)}{\widetilde\mu_h^2+k_0^2+k_1^2}-\frac{-\widetilde\mu_h(k)}{k_0^2+k_1^2}\right]\widetilde f^{(h)}(k)\ ;
\end{aligned}$$

then $g^{(\mathrm{D1})}_{\omega,\sigma}$ is given by the sum of $g^{(\mathrm{E1},N)}_{\omega,\sigma}$ and $g^{(\mathrm{R1},N)}_{\omega,\sigma}$, with the cutoff $\widehat f_N(k)$ replaced by $\widehat\chi_N(k_0)\widehat\chi_N(k_1-(\pi/a))$; and $r^{(2,h)}_{\omega,\sigma}(x)$ is defined in consequence of (3.3.8).

For ε small enough, (so that, by the inductive hypothesis (3.3.14) $1-c_0\varepsilon\ge 3/4$), there exists two positive constants, c and C s.t.:

$$\begin{aligned}
&\left|g^{(\mathrm{E1},h)}_{\omega,\omega}(x)\right|\le\frac{C\gamma^h}{e^{c\sqrt{\gamma^h\kappa|x|}}}\ ,\qquad &&\left|g^{(\mathrm{R1},h)}_{\omega,\omega}(x)\right|\le\gamma^{-(3/4)(N-h)}\frac{C\gamma^h}{e^{c\sqrt{\gamma^h\kappa|x|}}}\ ,\\
&\left|g^{(\mathrm{E1},h)}_{\omega,-\omega}(x)\right|\le\left|\frac{\mu_h}{\gamma^h\kappa}\right|\frac{C\gamma^h}{e^{c\sqrt{\gamma^h\kappa|x|}}}\ ,\qquad &&\left|g^{(\mathrm{R1},h)}_{\omega,-\omega}(x)\right|\le\left|\frac{\mu_h}{\gamma^h\kappa}\right|\gamma^{-(3/4)(N-h)}\frac{C\gamma^h}{e^{c\sqrt{\gamma^h\kappa|x|}}}\ ,\\
&\left|g^{(\mathrm{D1})}_{\omega,\omega}(x)\right|\le\frac{C\gamma^N}{e^{c\sqrt{\gamma^N\kappa|x|}}}\ ,\qquad &&\left|g^{(\mathrm{D1})}_{\omega,-\omega}(x)\right|\le\left|\frac{\mu_N}{\gamma^N\kappa}\right|\frac{C\gamma^N}{e^{c\sqrt{\gamma^N\kappa|x|}}}\ ,\\
&\left|r^{(1,h)}_{\omega,\sigma}(x)\right|\le\left|\frac{\mu_h}{\gamma^h\kappa}\right|^2\frac{C\gamma^h}{e^{c\sqrt{\gamma^h\kappa|x|}}}\ ,\qquad &&\left|r^{(2,h)}_{\omega,\sigma}(x)\right|\le\left|\frac{\mu_h}{\gamma^h\kappa}\right|^3\gamma^{-(3/4)(N-h)}\frac{C\gamma^h}{e^{c\sqrt{\gamma^h\kappa|x|}}}\ .
\end{aligned}\tag{3.3.9}$$

It is remarkable the propagators $g^{(\mathrm{R1},h)}_{\omega}$ and $r^{(2,h)}_{\omega,\sigma}$ have a gain factor $\gamma^{-(3/4)(N-h)}$ more than the standard bounds. Clearly, the above bounds are useful whenever $\mu_h\le\kappa\gamma^h$: when this condition is not satisfied, then the mass in the propagator is so large that it is possible to integrate the remaining scales all at once, as it was done for the double fermion propagator (see later the definition of the scale h^*).

3.3.4 Localization. The contribution to $\widetilde{\mathcal W}^{(h-1)}$ of certain kinds of Feynman graphs is extracted from the rest by *localization*: it extracts the 0-th or the 1-th order Taylor expansion in the momenta and the 0-th or the 1-th order expansion in the mass parameters $\{\mu_k\}_k$. Since the space of the momenta, D, does not contain $(0,0)$, and is not continuous, the Taylor expansion should be done taking *discrete derivatives* in the four nearest neighbour lattice site surrounding 0. This subtlety cannot be very important, since the continuous limit (for the lattice D only), $L \to \infty$, was not taken since the beginning, not to be involved with an infinite Grassmannian algebra. (The analogous argument is not valid also for the lattice Λ, since it is essential to make the limit $N \to +\infty$ *after* the renormalization has taken place.) Therefore, for shake of simplicity, the following developments, are *as if the lattice D were continuous rather than discrete*, leaving the correct technicality to [BM01].

Well then, it is convenient to introduce the directional derivatives

$$\partial^k_\omega \stackrel{def}{=} \frac{1}{2}\left[i\frac{\partial}{\partial_{k_0}} + \omega\frac{\partial}{\partial_{k_1}}\right] ,$$

which are orthogonal is the sense that the two relations are true: $\Big(\partial_\omega D_\sigma\Big)(k) = \delta_{\omega,\sigma}$ and $\sum_{\omega=\pm} D_\omega(k)\partial_\omega \equiv k_0\partial_{k_0} + k_1\partial_{k_1}$.

1. Let $\widehat{W}^{(h-1)}_{2,\alpha,\beta}(k)$ be considered. If $\alpha=\beta$, $\widehat{W}^{(h-1)}_{2,\alpha,\alpha}(0)$ =0 by (A4.3); if $\beta=-\alpha$, independently on α by (A4.4), it is possible to define

$$\widehat{W}^{(h-1)}_{2,\alpha,-\alpha}(0) = s_{h-1} + \gamma^{h-1}\Delta n_{h-1} + \Delta s^{(\mu)}_{h-1} ,$$

where, $\Delta s^{(\mu)}_{h-1}$ is the sum of the graphs in the expansion of $\widehat{W}^{(h-1)}_{2,\alpha,-\alpha}(0)$ which are at least quadratic in the masses $\{\mu_k\}_k$; while s_{h-1} is the sum of all the graphs linear in the masses, and therefore made with only antidiagonal propagator $g^{(\mathrm{E1},k)}_{\omega,-\omega}$, $g^{(\mathrm{R1},k)}_{\omega,-\omega}$ or $g^{(\mathrm{D1})}_{\omega,-\omega}$; finally, the sum of the graphs which are independent on the masses is in $\gamma^{h-1}\Delta n_{h-1}$. Then, let $\Big(\partial_\sigma \widehat{W}^{(h-1)}_{2,\alpha,\beta}\Big)(k)$ be considered. By (A4.3), for $\beta=-\alpha$, $\Big(\partial_\sigma \widehat{W}^{(h-1)}_{2,\alpha,-\alpha}\Big)(0) = 0$; while, for $\alpha=\beta$, it is possible to define, independently on α by (A4.4),

$$\Big(\partial_\sigma \widehat{W}^{(h-1)}_{2,\alpha,\alpha}\Big)(0) \begin{cases} \stackrel{def}{=} d^{(+)}_{h-1} + \Delta d^{(1,+)}_{h-1} & \text{for } \sigma=\alpha \\ \stackrel{def}{=} d^{(-)}_{h-1} + \Delta d^{(1,-)}_{h-1} & \text{for } \sigma=-\alpha ,\end{cases}$$

where $\Delta d^{(1,\sigma)}_{h-1}$ is the sum of the graphs which are at least linear in the masses; while $d^{(\sigma)}_{h-1}$ is the sum of the masses independent graphs. Defining $z_{h-1} \stackrel{def}{=} d^{(+)}_{h-1} + d^{(-)}_{h-1}$, and $\Delta d_{h-1} \stackrel{def}{=} -2d^{(-)}_{h-1}$ and, accordingly,

$$\Delta t_{h-1}(k) \stackrel{def}{=} \begin{pmatrix} z_{h-1}\big(-ik_0 + e(k_1)\big) & s_{h-1} \\ s_{h-1} & z_{h-1}\big(-ik_0 - e(k_1)\big) \end{pmatrix} ,$$

the localization is:

$$\mathcal{L}\left[\sum_{\alpha,\beta}\int_{D_{h-1}} \frac{\mathrm{d}^2k}{(2\pi)^2}\ \widehat{\psi}^+_{k,\alpha}\widehat{\psi}^-_{k,\beta}\widehat{W}^{(h-1)}_{2,\alpha,\beta}(k)\right] = \gamma^{h-1}\Delta n_{h-1}\sum_\omega \int_{D_{h-1}} \frac{\mathrm{d}^2k}{(2\pi)^2}\ \widehat{\psi}^+_{k,\omega}\widehat{\psi}^-_{k,-\omega}$$
$$+ \Delta d_{h-1}\sum_\omega \int_{D_{h-1}} \frac{\mathrm{d}^2k}{(2\pi)^2}\ \widehat{\psi}^+_{k,\omega}\widehat{\psi}^-_{k,\omega}\omega e(k) + \sum_{\alpha,\beta}\int_{D_{h-1}} \frac{\mathrm{d}^2k}{(2\pi)^2}\ \widehat{\psi}^+_{k,\alpha}\widehat{\psi}^-_{k,\beta}\big(\Delta t_{h-1}\big)_{\alpha,\beta}(k) .$$

Setting $\mathcal{R} \stackrel{def}{=} 1 - \mathcal{L}$:

$$\mathcal{R}\left[\sum_{\alpha,\beta}\int_{D_{h-1}}\frac{\mathrm{d}^2k}{(2\pi)^2}\;\widehat{\psi}^+_{k,\alpha}\widehat{\psi}^-_{k,\beta}\widehat{W}^{(h-1)}_{2,\alpha,\beta}(k)\right]$$
$$= \Delta s^{(\mu)}_{h-1}\sum_{\omega}\int_{D_{h-1}}\frac{\mathrm{d}^2k}{(2\pi)^2}\;\widehat{\psi}^+_{k,\omega}\widehat{\psi}^-_{k,-\omega} + \sum_{\sigma,\omega}\Delta d^{(\mu,\sigma)}_{h-1}\int_{D_{h-1}}\frac{\mathrm{d}^2k}{(2\pi)^2}\;\widehat{\psi}^+_{k,\omega}\widehat{\psi}^-_{k,\omega}D_{\sigma\omega}(k)$$
$$+ z_{h-1}\sum_{\alpha,\sigma}\int_{D_{h-1}}\frac{\mathrm{d}^2k}{(2\pi)^2}\;\widehat{\psi}^+_{k,\alpha}\widehat{\psi}^-_{k,\alpha}\Big[D_\sigma(k) - \big(-ik_0 + \sigma e(k_1)\big)\Big]$$
$$+ \sum_{\alpha,\beta,\omega,\sigma}\int_{D_{h-1}}\frac{\mathrm{d}^2k}{(2\pi)^2}\;\widehat{\psi}^+_{k,\alpha}\widehat{\psi}^-_{k,\beta}D_\omega(k)D_\sigma(k)\int_0^1 \mathrm{d}\tau\;(1-\tau)\left(\partial_\omega\partial_\sigma\widehat{W}^{(h-1)}_{2,\alpha,\beta}\right)(\tau k)\;.$$

The local part Δt_{h-1} is absorbed in the free measure. Calling:

$$\widetilde{C}^{(1)}_{h-1}(k) \stackrel{def}{=} \frac{1 + z_{h-1} + \Delta z_{h-1}}{1 + \chi_{h-1}(k)z_{h-1} + \chi_{h-1}(k)\Delta z_{h-1}}\;,$$

$$\widetilde{C}^{(2)}_{h-1}(k) \stackrel{def}{=} \frac{1 + z_{h-1} + \Delta z_{h-1}}{1 + \chi_{h-1}(k)z_{h-1} + \chi_{h-1}(k)\Delta z_{h-1}}\,\frac{1 + \chi_h(k)\,(s_{h-1}/\mu_{h-1})}{1 + (s_{h-1}/\mu_{h-1})}\;,$$

and, since s_{h-1} is linear in the masses, $m_{h-1} \stackrel{def}{=} s_{h-1}/\mu_{h-1}$, the *effective field strength* and the *effective mass* on scale $h-1$ are:

$$Z_{h-1} \stackrel{def}{=} Z_h(1 + z_{h-1})\;, \qquad \mu_{h-1} \stackrel{def}{=} \mu_h\frac{Z_h}{Z_{h-1}}(1 + m_{h-1})\;. \tag{3.3.10}$$

Then, in the same way, the local parts Δn_{h-1} and Δd_{h-1} are absorbed in the *effective counterterms* on scale $h-1$, ν_{h-1} and δ_{h-1}:

$$\delta_{h-1} \stackrel{def}{=} \left(\frac{Z_h}{Z_{h-1}}\right)(\delta_h + \Delta d_{h-1})\;, \qquad \nu_{h-1} \stackrel{def}{=} \left(\frac{Z_h}{Z_{h-1}}\right)\gamma(\nu_h + \Delta n_{h-1})\;. \tag{3.3.11}$$

A remarkable feature is that Z_{h-1}, ν_{h-1} and δ_{h-1} are *independent from the mass flow,* $\{\mu_k\}_k$. Finally, in changing free measure on scale $h-1$ from $\mathrm{d}P^{(\leq h-1)}$ to $\mathrm{d}\widetilde{P}^{(\leq h-1)}$, it has to be taken into account the change of the normalization:

$$\Delta\widetilde{E}_{h-1} \stackrel{def}{=} -\ln\left\{\left(\frac{Z_{h-1}}{Z_h}\right)^2\int_{D_{h-1}}\frac{\mathrm{d}^2k}{(2\pi)^2}\left[\frac{k_0^2 + e^2(k_1) + \widetilde{\mu}^2_{h-1}(k_1)}{k_0^2 + e^2(k_1) + \widetilde{\mu}^2_h(k)}\right]\left(\frac{1}{\widetilde{C}^{(1)}_{h-1}(k)}\right)^2\right\}\;.$$

so that the *effective vacuum energy* on scale $h-1$ is

$$E_{h-1} \stackrel{def}{=} E_h + \Delta E_{h-1} + \Delta\widetilde{E}_{h-1}\;.$$

2. Let $\widehat{W}^{(h-1)}_{4,\omega,-\omega}(k,p,q)$ be considered; and let

$$\widehat{W}^{(h-1)}_{4,\omega,-\omega}(0,0,0) \stackrel{def}{=} \Delta l_{h-1} + \Delta l^{(1)}_{h-1}\;,$$

where $\Delta l^{(1)}_{h-1}$ is the sum of all the graphs in the expansion of $\widehat{W}^{(h-1)}_{4,\omega,-\omega}(0,0,0)$ which are at least linear in the masses. Then

$$\begin{aligned}
&\mathcal{L}\left[\sum_\omega \int_{D_{h-1}} \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\ \psi^+_{k,\omega}\psi^-_{k+p-q,\omega}\psi^+_{p,-\omega}\psi^-_{q,-\omega}\widehat{W}^{(h-1)}_{4,\omega,-\omega}(k,p,q)\right]\\
&= \Delta l_{h-1}\sum_\omega \int_{D_{h-1}} \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\ \psi^+_{k,\omega}\psi^-_{k+p-q,\omega}\psi^+_{p,-\omega}\psi^-_{q,-\omega}\ ,\\
&\mathcal{R}\left[\sum_\omega \int_{D_{h-1}} \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\ \psi^+_{k,\omega}\psi^-_{k+p-q,\omega}\psi^+_{p,-\omega}\psi^-_{q,-\omega}\widehat{W}^{(h-1)}_{4,\omega,-\omega}(k,p,q)\right]\\
&= \Delta l^{(1)}_{h-1}\sum_\omega \int_{D_{h-1}} \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\ \psi^+_{k,\omega}\psi^-_{k+p-q,\omega}\psi^+_{p,-\omega}\psi^-_{q,-\omega}\\
&+\sum_{\omega,\sigma}\sum_{p'=k,p,q}\int_{D_{h-1}} \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\ \psi^+_{k,\omega}\psi^-_{k+p-q,\omega}\psi^+_{p,-\omega}\psi^-_{q,-\omega}D_\sigma(p')\\
&\qquad\qquad\cdot\int_0^1 \mathrm{d}\tau\left(\partial^{p'}_\sigma \widehat{W}^{(h-1)}_{4,\omega,-\omega}\right)(\tau k,\tau p,\tau q)\ .
\end{aligned}$$

The local part Δl_{h-1} is absorbed in the effective coupling on scale $h-1$:

$$\lambda_{h-1}\stackrel{def}{=}\left(\frac{Z_h}{Z_{h-1}}\right)^2(\lambda_h+\Delta l_{h-1})\ , \tag{3.3.12}$$

and also λ_{h-1} is independent from the flow $\{\mu_k\}_k$.

3. Let $\widehat{W}^{(h-1)}_{1;2,\mu;\nu}(0,0)$ be considered; since by (A4.5), it does not depend on σ, it is possible to define

$$\widehat{W}^{(h-1)}_{1;2,\sigma;\omega}(0;0)\stackrel{def}{=}\begin{cases} z^{(2)}_{h-1}+\Delta z^{(2,+)}_{h-1}+\Delta d^{(2,+)}_{h-1} & \text{for } \sigma=\omega\\ \Delta z^{(2,-)}_{h-1}+\Delta d^{(2,-)}_{h-1} & \text{for } \sigma=-\omega\ ;\end{cases}$$

where $\Delta d^{(2,\sigma)}_{h-1}$ is the sum of the graphs at least linear in the masses; then $z^{(2)}_{h-1}$ and $\Delta z^{(2,+)}_{h-1}$ are mass independent: the former is the sum of all the graphs made only with (diagonal) propagators $\{g^{(\mathrm{E1},k)}_{\omega,\omega}\}_k$, and interaction $\mathcal{V}$ (namely all the mass-independent graphs obtained in the case of the Euclidean model for such a kernel); while $\Delta z^{(2,\sigma)}_{h-1}$ is the sum of the graphs made with least one propagator $\{g^{(\mathrm{R},k)}_{\omega,\sigma}\}_k$ or an interaction $\mathcal{N}$ or

$\mathcal{D}$. Then

$$\mathcal{L}\left[\sum_{\sigma,\omega}\int_{D_{h-1}}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\jmath_{p-k,\sigma}\psi^+_{k,\omega}\psi^-_{p,\omega}\widehat{W}^{(h-1)}_{1;2,\sigma;\omega}(k,p)\right]$$
$$=\left(z^{(2)}+\Delta z^{(2,+)}_{h-1}\right)\sum_{\sigma}\int_{D_{h-1}}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\jmath_{p-k,\sigma}\psi^+_{k,\sigma}\psi^-_{p,\sigma}$$
$$+\Delta z^{(2,-)}_{h-1}\sum_{\sigma}\int_{D_{h-1}}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\jmath_{p-k,\mu}\psi^+_{k,-\sigma}\psi^-_{p,-\sigma}\ ,$$
$$\mathcal{R}\left[\sum_{\sigma,\omega}\int_{D_{h-1}}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\jmath_{p-k,\sigma}\psi^+_{k,\omega}\psi^-_{p,\omega}\widehat{W}^{(h-1)}_{1;2,\sigma;\omega}(k,p)\right]$$
$$=\sum_{\sigma,\omega}\Delta z^{(2,\omega)}_{h-1}\sum_{\omega}\int_{D_{h-1}}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\jmath_{p-k,\omega}\psi^+_{k,\omega\sigma}\psi^-_{p,\omega\sigma}$$
$$+\sum_{\mu,\nu,\sigma}\sum_{q=k,p}\int_{D_{h-1}}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\jmath_{p-k,\mu}\psi^+_{k,\nu}\psi^-_{p,\nu}D_\sigma(q)\int_0^1\mathrm{d}\tau\left(\partial^q_\sigma\widehat{W}^{(h-1)}_{1;2,\mu;\nu}\right)(\tau k,\tau p)\ .$$

The local parts are absorbed into the *effective density strength* on scale $h-1$, $\zeta^{(2,\sigma)}_{h-1}$:

$$\begin{pmatrix}\zeta^{(2,+)}_{h-1}\\ \zeta^{(2,-)}_{h-1}\end{pmatrix}\stackrel{def}{=}\left(\frac{Z_h}{Z_{h-1}}\right)\begin{pmatrix}1+z^{(2)}_{h-1}+\Delta z^{(2,+)}_{h-1} & \Delta z^{(2,-)}_{h-1}\\ \Delta z^{(2,-)}_{h-1} & 1+z^{(2)}_{h-1}+\Delta z^{(2,+)}_{h-1}\end{pmatrix}\begin{pmatrix}\zeta^{(2,+)}_{h}\\ \zeta^{(2,-)}_{h}\end{pmatrix}\ . \tag{3.3.13}$$

Multiscale integration goes on over all the scales k s.t. $\mu_k\le\kappa\gamma^k$, the first scale for which this is not true being $k=h^*$. It is simply to verify that , for $h=h^*+1$ the propagator (3.3.5) has the same dimensional bound of (3.3.4)

$$\left|g^{(\le h^*)}_{\omega,\sigma}(x)\right|\le C\gamma^{h^*}e^{-c\sqrt{\gamma^{h^*-1}\kappa|x|}}\ .$$

Finally, it holds the following theorem.

Theorem 3.1. *Let it be supposed there exists $\varepsilon>0$ and the constants $c_0>0$ such that at any RG step $h: h^*\le h\le N$, the effective parameters satisfy:*

$$\gamma^{-c_0\varepsilon^2}\le\frac{Z_h}{Z_{h+1}}\le\gamma^{c_0\varepsilon^2}\ ,\qquad\gamma^{-2c_0\varepsilon}\le\frac{\mu_h}{\mu_{h+1}}\le\gamma^{2c_0\varepsilon}\ ,\qquad\gamma^{-2c_0\varepsilon}\le\frac{\zeta^{(2,\sigma)}_h}{\zeta^{(2,\sigma)}_{h+1}}\le\gamma^{2c_0\varepsilon}\ , \tag{3.3.14}$$

$$|\nu_h|,|\delta_h|,|\lambda_h|\le 2\varepsilon\ . \tag{3.3.15}$$

Then, for suitable positive constants C,c:

1. *If $n^\varphi+n^\jmath\ne 0$,*

$$\int_\Lambda\mathrm{d}^2\underline{x}\ \left|W^{(h)}_{n^\psi;n^\varphi;n^\jmath,\underline{\omega},\underline{\sigma}}(\underline{x},\underline{y},\underline{z})\right|\le C\frac{\gamma^{h\left(2-(1/2)n^\psi-(3/2)n^\varphi-n^\jmath\right)}}{e^{\frac{c}{2(n^\varphi+n^\jmath)}\sqrt{\gamma^h\kappa d(\underline{y},\underline{z})}}}\ ;$$

2. *if* $n^{\varphi} + n^{\jmath} = 0$,

$$\int_{\Lambda} \mathrm{d}^2_* \underline{x} \; \left| W^{(h)}_{n^{\psi};0;0,\underline{\omega},\underline{\sigma}}(\underline{x}) \right| \leq C\gamma^{h\left(2-(1/2)n^{\psi}\right)} \; ;$$

The proof is follows by simple dimensional analysis, and is consequence of the Appendices A3 and A5. Since, by the first item, μ_h/γ^h is strictly decreasing in h, for any choice of the mass $0 \leq \mu \leq \gamma^{-1}\kappa$, the scale h^* is negative; and:

$$\frac{\log_\gamma(\mu/\kappa)}{1-2c_0\varepsilon} - 1 \leq h^* \leq \frac{\log_\gamma(\mu/\kappa)}{1+2c_0\varepsilon} \; ;$$

hence, in the massless case, $h^* = -\infty$.

3.4 Flows of the Running Coupling Constants

A remarkable feature of the Localization is that among the flows of the effective parameters, only the one for the mass is constructed with massive propagator; the others are constructed with propagators $\{g^{(\mathrm{E1},k)}_{\omega,\omega}\}_k$, $\{g^{(\mathrm{R1},k)}_{\omega,\omega}\}_k$ or $\{g^{(\mathrm{D1},k)}_{\omega,\omega}\}_k$, and therefore are independent on the mass flow. Since the scale h^* was introduced only to avoid bad bound on the massive propagators, all the flow, except $\{\mu_k\}_k$, can be extended from the range of scales $h^* \leq k \leq N$, to the range $k \leq N$.

Other features of the flows of the effective parameters are depicted in the following Theorem.

Theorem 3.2. *Fixed any* $\vartheta : 0 < \vartheta < 1/16$, *there exists* $\varepsilon > 0$ *and two positive constants* c *and* c_2, *such that in correspondence of any parameters* μ *and* λ *satisfying* $0 \leq \mu \leq \kappa\gamma^{-1}$ *and* $|\lambda| \leq \varepsilon$, *there exist the parameters* λ_N, μ_N, Z_N, $Z^{(2,+)}_N$, $Z^{(2,-)}_N$ *and* δ_N, ν_N, *such that the following properties hold.*

1. *The flow of* λ_N *is such that*

$$\lim_{h\to-\infty} \lambda_h = \lambda \; ; \qquad |\lambda_{h-1} - \lambda_h| \leq c\varepsilon^2 \gamma^{-(\vartheta/2)(N-h)} \; . \tag{3.4.1}$$

2. *The flows of* Z_N *and* μ_N *are such that* $\mu_0 = \mu$ *and* $Z_0 = 1$; *furthermore there exist* η_λ *and* $\overline{\eta}_\lambda$, *independent from the regularization used (Euclidean or Hamiltonian) from the cutoff* N, *and from the mass* μ, *such that*

$$Z_h = \gamma^{-h\eta_\lambda + \Delta G_h} \; , \qquad \mu_h = \mu\gamma^{-h\overline{\eta}_\lambda + \Delta\overline{G}_h} \; , \tag{3.4.2}$$

with the rests, ΔG_h *and* $\Delta\overline{G}_h$, *summable in* h: $|\Delta G_h|, |\Delta\overline{G}_h| \leq c_2\varepsilon^2\gamma^{-(\vartheta/2)\left(N-h\right)}$.

3. *The flows of* $Z^{(2,+)}_N$ *and* $Z^{(2,-)}_N$ *are such that* $Z^{(2,+)}_0 = Z^{(2,-)}_0 = 1$; *furthermore there exist* $\eta^{(2)}_\lambda$ *independent from the regularization, as well as from the mass* μ *and the cutoff* N, *such that*

$$Z^{(2,+)}_h = \gamma^{-h\eta^{(2)}_\lambda + \Delta G^{(2,+)}_h} \; , \qquad Z^{(2,-)}_h = \gamma^{-h\eta^{(2)}_\lambda + \Delta G^{(2,-)}_h} \; , \tag{3.4.3}$$

with the rests $\{\Delta G_h^{(2,\sigma)}\}_{\sigma=\pm}$ *summable in h:* $\left|\Delta G_h^{(2,\sigma)}\right| \le c_2\varepsilon^2\gamma^{-(\vartheta/2)(N-h)}$.

4. *The flows of* δ_N *and* ν_N *are such that* $|\delta_h|, |\nu_h| \le 2\varepsilon\gamma^{-\vartheta(N-h)}$.

The proof is given in Appendix A5. It is based on the vanishing of the Beta function of massless Thirring model.

3.5 Equivalence of the Euclidean and Hamiltonian Regularization

Proof of Theorem 2.1. It is a corollary of the Theorem 3.2. It can be obtained in the same way as the proof of Lemma A.3.4. Anyway, using the*short memory property* (see A3.5), and the compact support of the propagators, a slightly easier proof is available for the Fourier transform of the Schwinger functions with at least one field insertion. Indeed, the $(m; n+1)$-Schwinger functions calculated at fixed momenta $p_1, \dots, p_m$, $q_1, \dots, q_n$, no matter if they are obtained from the Hamiltonian or the Euclidean regularization, asymptotically in the limit of removed cutoff are equal to the sum of the following Feynman graphs: all the graphs found in the expansion of the Schwinger functions, excluding those ones having an interaction on scale $m \ge N$, or an interaction $\mathcal{D}$ or $\mathcal{N}$, or a propagator $\{g^{(\mathrm{R},k)}\}_k$, and replacing the parameters λ_k, Z_k, $Z_k^{(2,\sigma)}$ and μ_k, respectively with λ, $\gamma^{k\eta_\lambda}$, $\gamma^{k\eta_\lambda^{(2)}}$ and $\mu\gamma^{k\overline{\eta}_\lambda}$. Indeed, these graphs do not depend on the regularization; then, the difference between the sum of such graphs and the corresponding Schwinger function is bounded by the modulus of the sum of the graphs with one external fermionic propagator on the scale of the momentum q_1, called h_1 – fixed q_1, by compact support function, h_1 can be chosen between two adjoining momenta scales – an effective parameter or propagator on scale m, and falling in one of the following cases.

i. It is $m \ge N$. Then, by the short memory property, the sum of such graphs is bounded, up to a constant, by $\gamma^{-\vartheta(N-h_1)}$.

ii. It is $m < N$ and the parameter is δ_m or ν_m. By the property of the flows of δ_N and ν_N, and by the short memory property, the sum of such graphs is bounded, up to a constant, by $\gamma^{-\vartheta|m-h_1|}\gamma^{-(\vartheta/2)(N-m)} \le \gamma^{-(\vartheta/2)(N-h_1)}\gamma^{-(\vartheta/2)|m-h_1|}$.

iii. There is a propagator $g_\omega^{(R,m)}$ on scale $m < N$. By the bound of such a propagator and the short memory property, the sum of such graphs is bounded by $\gamma^{-\vartheta|m-h_1|}\gamma^{-(3/4)(N-m)} \le \gamma^{-(\vartheta/2)(N-h_1)}\gamma^{-(\vartheta/2)|m-h_1|}$, for $\vartheta < 3/4$.

iv. It is $m < N$ and effective parameter $\lambda_m - \lambda$, or $Z_m - \gamma^{m\eta_\lambda}$, or $\mu_m - \gamma^{m\overline{\eta}_\lambda}$, or $Z_m^{(2,\sigma)} - \gamma^{m\eta_\lambda^{(2)}}$. By the property of the flows, and by the short memory property, the sum of such graphs is bounded, up to a constant, by $\gamma^{-\vartheta|m-h_1|}\gamma^{-(\vartheta/2)(N-m)} \le \gamma^{-(\vartheta/2)(N-h_1)}\ \gamma^{-(\vartheta/2)|m-h_1|}$.

Furthermore the scale h^*, in the limit of removed cutoff, only depends on λ, μ. Therefore, it is possible to perform the sum over m and to get for the difference of the Schwinger function derived in the two different settings a bound $\gamma^{-(\vartheta/2)(N-h_1)}$, for $0 < \vartheta < 1/16$, up to a constant. Anyway, in order to have, for different regularizations, identical values of λ and μ (and consequently also of η_λ, $\overline{\eta}_\lambda$ and $\eta_\lambda^{(2)}$), the initial parameters will be generally different. ■

Chapter 4:

Phase and Chiral Symmetries

4.1 Ward-Takahashi Identities

The classical Lagrangian is invariant under the *global* transformations of the fields:

$$\psi^{\sigma}_{x,\omega} \to e^{i\sigma\alpha_\omega}\psi^{\sigma}_{x,\omega} \; ; \tag{4.1.1}$$

as the phase, $\{\alpha_\omega\}_{\omega=\pm}$ does depend on the component of the fermion fields, ω, this transformation is a combination of the phase and chiral transformations in the Dirac notation.

This symmetry can be implemented in the generating functional of the Euclidean Thirring model; and in particular, in order to obtaining the identity $\eta_\lambda = \eta^{(2)}_\lambda$ and the vanishing of the Beta function it will be useful to consider the generating functional with infrared cutoff on scale h. It has to be performed a real exponential transformation and to allow a dependence of the parameter $\{\alpha_\omega\}_{\omega=\pm}$ on the space points: a new real field, $\{\alpha_{x,\omega}\}^{x\in\Lambda}_{\omega=\pm}$ arises – this prescription looks like, but has not to be confused with, the implementation of a *gauge symmetry*.

4.1.1 WTI **for the Schwinger functions.** An essential condition to get the consequences of the WTI in the functional integration framework is to transform the field in *every site* of Λ. This seems to be forbidden by the choice of a compact support cutoff function, ad the consequent restriction to the momenta in D_N. Therefore, let $\chi^{\delta}_N(k)$ be the cutoff function obtained adding to $\chi_N(k)$ an exponential decaying tail $\delta\Delta\chi_N(k)$, alway strictly positive.

Hence, let the following transformation of the integration variables in Fourier space be considered

$$\widehat{\psi}^{\sigma}_{k,\omega} \longrightarrow \widehat{\psi}^{\sigma}_{k,\omega} - \sigma \int_D \frac{\mathrm{d}^2 p}{(2\pi)^2}\ \widehat{\alpha}_{p,\omega}\widehat{\psi}^{\sigma}_{k-\sigma p,\omega}\ . \tag{4.1.2}$$

Calling $\chi^{\delta}_{h,N}(k) \stackrel{def}{=} \chi^{\delta}_{N}(k) - \chi^{\delta}_{h}(k)$, the (4.1.2) implies the following transformation of the kernel of the free measure

$$\int_D \frac{\mathrm{d}^2 k}{(2\pi)^2}\ \widehat{\psi}^{+}_{k,\omega}\frac{D_\omega(k)}{\chi^{\delta}_{h,N}(k)}\widehat{\psi}^{-}_{k,\omega} \longrightarrow \int_D \frac{\mathrm{d}^2 k}{(2\pi)^2}\ \widehat{\psi}^{+}_{k,\omega}\frac{D_\omega(k)}{\chi^{\delta}_{h,N}(k)}\widehat{\psi}^{-}_{k,\omega}$$
$$+ \int_D \frac{\mathrm{d}^2 p}{(2\pi)^2}\frac{\mathrm{d}^2 k}{(2\pi)^2}\ \widehat{\alpha}_{p,\omega}\widehat{\psi}^{+}_{k,\omega}\widehat{\psi}^{-}_{k+p,\omega}\left[\frac{D_\omega(k)}{\chi^{\delta}_{h,N}(k)} - \frac{D_\omega(k+p)}{\chi^{\delta}_{h,N}(k+p)}\right]\ ,$$

and

$$\frac{D_\omega(k)}{\chi^{\delta}_{h,N}(k)} - \frac{D_\omega(k+p)}{\chi^{\delta}_{h,N}(k+p)} \stackrel{def}{=} -D_\omega(p) - C^{\delta}_{\omega}(k,k+p)$$
$$= -D_\omega(p) - \left[D_\omega(k)\left(1-\left(\chi^{\delta}_{h,N}\right)^{-1}(k)\right) - D_\omega(k+p)\left(1-\left(\chi^{\delta}_{h,N}\right)^{-1}(k+p)\right)\right]\ .$$

It is suitable to introduce the interactions with the external source $\widehat{\alpha}_\omega$:

$$\mathcal{A}_0(\alpha,\psi) \stackrel{def}{=} \sum_{\omega=\pm}\int_D \frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{\mathrm{d}^2 p}{(2\pi)^2}\ C^{\delta}_{\omega}(q,p)\widehat{\alpha}_{p-q,\omega}\widehat{\psi}^{+}_{q,\omega}\widehat{\psi}^{-}_{p,\omega}\ ,$$
$$\mathcal{A}_\sigma(\alpha,\psi) \stackrel{def}{=} \sum_{\omega=\pm}\int_D \frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{\mathrm{d}^2 p}{(2\pi)^2}\ D_{\sigma\omega}(p-q)\widehat{\alpha}_{p-q,\omega}\widehat{\psi}^{+}_{q,\sigma\omega}\widehat{\psi}^{-}_{p,\sigma\omega}\ ,\quad \text{for } \sigma=\pm\ ,$$

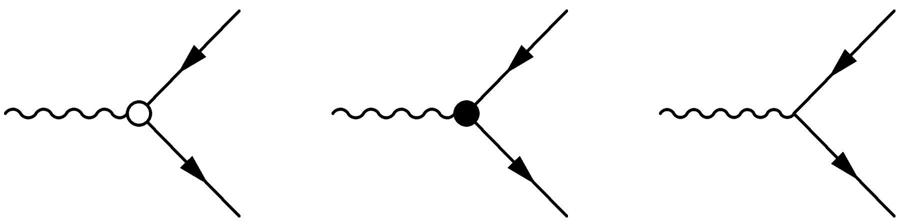

Fig 3: Graphical representation of $\mathcal{A}_0$, $\mathcal{A}_-$ and $\mathcal{A}_+$

so that, the transformation of $\mathcal{W}^{(h)}$ reads

$$e^{\mathcal{W}^{(h)}(\jmath,\varphi)} = \lim_{\delta\to 0}\int \mathrm{d}P^{[h,N]}(\psi)\exp\left\{-l_N\mathcal{V}(\psi) + Z^{(2)}_N\mathcal{J}(\jmath,\psi) + \mathcal{F}(\varphi,\psi)\right\}$$
$$\cdot\exp\left\{Z_N\mathcal{A}_+(\alpha,\psi) + Z_N\mathcal{A}_0(\alpha,\psi)\right\} \tag{4.1.3}$$
$$\cdot\exp\left\{\sum_{\omega=\pm}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\frac{\mathrm{d}^2 k}{(2\pi)^2}\ \widehat{\alpha}_{p,\omega}\left[\widehat{\varphi}^{+}_{k,\omega}\widehat{\psi}^{-}_{k+p,\omega} - \widehat{\psi}^{+}_{k,\omega}\widehat{\varphi}^{-}_{k+p,\omega}\right]\right\}\ .$$

Being that $\mathcal{W}^{(h)}$ is independent of α, summing and subtracting in the argument of the exponential $Z_N \sum_{\mu=\pm} \nu_N^{(\mu)} \mathcal{A}_\mu(\alpha,\psi)$, and then taking a derivative in $\widehat{\alpha}_{p,\mu}$ for $\widehat{\alpha}=0$, it yields:

$$\begin{aligned}\left(\frac{1-\nu_N^{(+)}}{\zeta_N^{(2)}}\right) D_\mu(p)\frac{\partial \mathcal{W}^{(h)}}{\partial \widehat{\jmath}_{p,\mu}}(\jmath,\varphi) - \frac{\nu_N^{(-)}}{\zeta_N^{(2)}} D_{-\mu}(p)\frac{\partial \mathcal{W}^{(h)}}{\partial \widehat{\jmath}_{p,-\mu}}(\jmath,\varphi) \\ = \int_D \frac{\mathrm{d}^2k}{(2\pi)^2}\left[\frac{\partial \mathcal{W}^{(h)}}{\partial \widehat{\varphi}^-_{k,\mu}}\widehat{\varphi}^-_{k+p,\mu} - \widehat{\varphi}^+_{k,\mu}\frac{\partial \mathcal{W}^{(h)}}{\partial \widehat{\varphi}^+_{k+p,\mu}}\right] - \frac{\partial \mathcal{W}_{\mathcal{A}}^{(h)}}{\partial \widehat{\alpha}_{p,\mu}}(0,\jmath,\varphi)\ ,\end{aligned} \tag{4.1.4}$$

where the last term is given is the derivative of the functional

$$\begin{aligned} e^{\mathcal{W}_{\mathcal{A}}^{(h)}(\alpha,\jmath,\varphi)} \stackrel{def}{=} \int \mathrm{d}P^{[h,N]}(\psi) \exp\left\{-l_N\mathcal{V}(\psi) + Z_N^{(2)}\mathcal{J}(\jmath,\psi) + \mathcal{F}(\varphi,\psi)\right\} \\ \exp\left\{Z_N\left[\mathcal{A}_0 + \sum_{\mu=\pm}\nu_N^{(\mu)}\mathcal{A}_\mu\right](\alpha,\psi)\right\}\ .\end{aligned} \tag{4.1.5}$$

Its derivatives are *remainders* which will be proved to vanish in the limit of removed cutoff. Anyway, this holds for $\{\nu_N^{(\sigma)}\}_{\sigma=\pm}$ having *non-vanishing limit*: w.r.t. the formal WTI, they represent an *anomaly*. Adhering to the Johnson's notations, let the following definitions be considered:

$$a_N \stackrel{def}{=} \frac{1}{1-\left(\nu_N^{(-)}+\nu_N^{(+)}\right)}\ , \qquad \overline{a}_N \stackrel{def}{=} \frac{1}{1+\left(\nu_N^{(-)}-\nu_N^{(+)}\right)}\ ;$$

now, the WTI due to the *phase symmetry* (to be compared with formula (16) of [J61]) is obtained summing (4.1.4) over μ:

$$\begin{aligned}\sum_\mu D_\mu(p)\frac{1}{\zeta_N^{(2)}}\frac{\partial \mathcal{W}^{(h)}}{\partial \widehat{\jmath}_{p,\mu}}(\jmath,\varphi) = a_N\sum_\mu \int_D \frac{\mathrm{d}^2k}{(2\pi)^2}\left[\frac{\partial \mathcal{W}^{(h)}}{\partial \widehat{\varphi}_{k,\mu}}\widehat{\varphi}^-_{k+p,\mu} - \widehat{\varphi}^+_{k,\mu}\frac{\partial \mathcal{W}^{(h)}}{\partial \widehat{\varphi}_{k+p,\mu}}\right] \\ - a_N\sum_\mu \frac{\partial \mathcal{W}_{\mathcal{A}}^{(h)}}{\partial \widehat{\alpha}_{p,\mu}}(0,\jmath,\varphi)\ ;\end{aligned}$$

whereas the one due to the *chiral symmetry* (to be compared with formula (17) of [J61]) is obtained multiplying both members of (4.1.4) times μ and summing over μ:

$$\begin{aligned}\sum_\mu \mu D_\mu(p)\frac{1}{\zeta_N^{(2)}}\frac{\partial \mathcal{W}^{(h)}}{\partial \widehat{\jmath}_{p,\mu}}(\jmath,\varphi) = \overline{a}_N\sum_\mu \mu\int_D \frac{\mathrm{d}^2k}{(2\pi)^2}\left[\frac{\partial \mathcal{W}^{(h)}}{\partial \widehat{\varphi}^-_{k,\mu}}\widehat{\varphi}^-_{k+p,\mu} - \widehat{\varphi}^+_{k,\mu}\frac{\partial \mathcal{W}^{(h)}}{\partial \widehat{\varphi}^+_{k+p,\mu}}\right] \\ - \overline{a}_N\sum_\mu \mu\frac{\partial \mathcal{W}_{\mathcal{A}}^{(h)}}{\partial \widehat{\alpha}_{p,\mu}}(0,\jmath,\varphi)\ .\end{aligned}$$

Finally, being that $(1+\sigma\mu)/2 = \delta_{\sigma,\mu}$, summing the two above equations, the final expression for the WTI reads:

$$\begin{aligned} D_\sigma(p)\frac{1}{\zeta_N^{(2)}}\frac{\partial \mathcal{W}^{(h)}}{\partial \widehat{\jmath}_{p,\sigma}}(\jmath,\varphi) = \sum_\mu \frac{a_N+\overline{a}_N\sigma\mu}{2}\int_D \frac{\mathrm{d}^2k}{(2\pi)^2}\left[\frac{\partial \mathcal{W}^{(h)}}{\partial \widehat{\varphi}^-_{k,\mu}}\widehat{\varphi}^-_{k+p,\mu} - \widehat{\varphi}^+_{k,\mu}\frac{\partial \mathcal{W}^{(h)}}{\partial \widehat{\varphi}^+_{k+p,\mu}}\right] \\ -\sum_\mu \frac{a_N+\overline{a}_N\sigma\mu}{2}\frac{\partial \mathcal{W}_{\mathcal{A}}^{(h)}}{\partial \widehat{\alpha}_{p,\mu}}(0,\jmath,\varphi)\ .\end{aligned} \tag{4.1.6}$$

By taking suitable derivatives w.r.t. the field $\widehat{\varphi}$ for $\jmath = \varphi = 0$, (4.1.6) generates all the WTI involving one density insertion: for instance, by taking derivatives w.r.t. $\widehat{\varphi}^{+}_{k,\omega}$ and $\widehat{\varphi}^{-}_{k+p,\omega}$, (4.1.6) gives (1.1.8) and (1.1.11), for

$$\Delta \widehat{H}^{(1;2)}_{\sigma,\omega}(p;k) \stackrel{def}{=} \frac{\partial \mathcal{W}^{(h)}_{\mathcal{A}}}{\partial \widehat{\alpha}_{p,\mu} \partial \widehat{\varphi}^{+}_{k,\omega} \partial \widehat{\varphi}^{-}_{k+p,\omega}}(0,0,0) \ .$$

4.1.2 Flows of $\nu^{(+)}_N$ and $\nu^{(-)}_N$. The remainder of the above WTI are the Schwinger functions generated from the functional $\mathcal{W}^{(h)}_{\mathcal{A}}$ with one – and only one – derivation in the field $\widehat{\alpha}$, and various number of derivation in the fields φ's. Therefore it is necessary to study the renormalization of the contraction of the vertices $\{\mathcal{A}_a\}_{a=0,\pm}$, up to linear order in $\widehat{\alpha}$, which lead to the flows of $\nu^{(+)}_N$ and $\nu^{(-)}_N$.

By induction, having integrated the scale from the N-th below to the j-th, it is possible to prove that, up to the renormalization of the coupling constants already present in functional $\mathcal{W}^{(h)}$, the functional $\mathcal{W}^{(h)}_{\mathcal{A}}$ reads:

$$e^{\mathcal{W}^{(h)}_{\mathcal{A}}(\alpha,\jmath,\varphi)} \stackrel{def}{=} \int \mathrm{d}P^{[h,j]}\ (\psi) \exp\left\{\mathcal{W}^{(j)}\left(\varphi, \jmath, \sqrt{Z_j}\psi\right) + \mathcal{W}^{(j)}_{\mathcal{A},\mathrm{irr}}\left(\alpha, \varphi, \jmath, \sqrt{Z_j}\psi\right)\right\}$$
$$\exp\left\{\Big[\left(\frac{Z_N}{Z_j}\right)\mathcal{A}_0 + \sum_{\mu=\pm} \nu^{(\mu)}_j \mathcal{A}_\mu\Big]\left(\alpha, \sqrt{Z_j}\psi\right)\right\} \ ,$$

where $\mathcal{W}^{(j)}$ and $\mathcal{W}^{(j)}_{\mathcal{A},\mathrm{irr}}$ are defined as in formula (3.3.7), but with propagators and couplings obtained for the Euclidean massless Thirring model; besides in the monomials of the fields of $\mathcal{W}^{(j)}_{\mathcal{A},\mathrm{irr}}$ there is also one α-field and either $n^{\psi} + n^{\varphi} \geq 2$ or $n^{\jmath} \geq 1$.

From this section to the end, since all the developments will be about the Euclidean Massless Thirring model, let $\widehat{g}^{(\mathrm{E1},h)}_{\omega,\sigma}$ be called, with abuse of notation, $\widehat{g}^{(h)}_{\omega}$.

Lemma 4.2. *Let the kernel* $U^{(i,j)}_{\varepsilon;\omega}(k,p) \stackrel{def}{=} C^{\delta}_{\omega}(k,p)\widehat{g}^{(j)}_{\omega}(k)\widehat{g}^{(i)}_{\omega}(p)$ *be considered. It can be decomposed into*

$$U^{(i,j)}_{\varepsilon;\omega}(k,p) \stackrel{def}{=} \sum_{\sigma} D_{\sigma}(p-k) S^{(i,j)}_{\varepsilon;\omega,\sigma}(k,p) \ ,$$

and $S^{(i,j)}_{\omega,s}$, *the limit* $\varepsilon \to 0$ *of* $S^{(i,j)}_{\varepsilon;\omega,s}$, *satisfies the bound*

$$|\partial^{s_i}_k \partial^{s_j}_p S^{(i,j)}_{\omega,\sigma}(k,p)| \leq \begin{cases} C\gamma^{-i(1+s_i)-j(1+s_j)} & \text{if } i \text{ or } j = h, N \\ 0 & \text{otherwise} \ . \end{cases}$$

The proof of the bound is given in appendix A6. It means that *formally* C^{δ}_{ω} can be thought as a 1-dimensional kernel: since the monomial $\alpha\psi\psi$ has dimension 1, the power counting for the graphs with insertion of the vertex $\mathcal{A}_0$ will be found to be always satisfied.

4.1.3 Improved localization I. As for the effective potential, also the multiscale integration of $\mathcal{W}_{\mathcal{A}}$ is companied by a localization and absorption in the effective parameters the graphs which are divergent according to the dimensional analysis. At the $j-1$-th scale, with the inductive hypothesis the previous scales were integrated and the local terms were extracted, they holds the following cases.

1. One field $\widehat{\psi}$ of the interaction $\mathcal{A}_0$, contracted with a kernel $\widehat{W}^{(j)}_{2,\omega}(k)$, has vanishing local part since $\widehat{W}^{(j)}_{2,\omega}(0)=0$ by symmetries; furthermore, for compact support arguments, such a contraction can only occur at scale j:

$$\mathcal{L}\left[\int_D \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\ \widehat{\alpha}_{k-q,\omega}\widehat{\psi}^+_{q,\omega}\widehat{\psi}^-_{k,\omega}\ C^\delta_\omega(q,k)\widehat{g}^{(j)}_\omega(k)\widehat{W}^{(j)}_{2,\omega}(k)\right]=0\ ,$$

$$\begin{aligned}&\mathcal{R}\left[\int_D \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\ \widehat{\alpha}_{k-q,\omega}\widehat{\psi}^+_{q,\omega}\widehat{\psi}^-_{k,\omega}\ C^\delta_\omega(q,k)\widehat{g}^{(j)}_\omega(k)\widehat{W}^{(j)}_{2,\omega}(k)\right]\\ &=\sum_{\mu=\pm}\int_D \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\ \widehat{\alpha}_{k-q,\omega}\widehat{\psi}^+_{q,\omega}\widehat{\psi}^-_{k,\omega}\\ &\qquad\qquad D_\mu(k)\left[C^\delta_\omega(q,k)\widehat{g}^{(j)}_\omega(k)\int_0^1\mathrm{d}\tau\ \left(\partial_\sigma\widehat{W}^{(j)}_{2,\omega}\right)(\tau k)\right]\ ;\end{aligned}$$

the derivative clearly improves the bound on the kernel $\widehat{W}^{(j)}_{2,\omega}$ of one negative dimension, at a loss of the bound on the kernel that will be obtained contracting the field $\widehat{\psi}^-_{k,\omega}$ in a scale lower than $j-1$.
This automatic dimensional gain is due to the fact that this situation cannot occur in more than one node v in the tree expansion, and in its first preceding v'; hence an alternative way to cure it is to multiply by $\gamma^{-2}\gamma^2$: the former factor makes negative the dimension of such a graph, the latter worsen the bound of a constant.
2. As in the previous point, one $\widehat{\psi}$-field of the vertex $\sum_\sigma \nu^{(\sigma)}_j\mathcal{A}_\sigma$, contracted with a kernel $\widehat{W}^{(j)}_{2,\sigma\omega}(k)$ has vanishing local part; since $\widehat{\psi}^+_{k,\omega}$ has to be contracted on scale j:

$$\mathcal{L}\left[\int_D \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\ \widehat{\alpha}_{k-q,\omega}\widehat{\psi}^+_{q,\sigma\omega}\widehat{\psi}^-_{k,\sigma\omega}\ D_\omega(k-q)\widehat{g}^{(j)}_{\sigma\omega}(k)\widehat{W}^{(j)}_{2,\sigma\omega}(k)\right]=0\ ,$$

$$\begin{aligned}&\mathcal{R}\left[\int_D \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\ \widehat{\alpha}_{k-q,\omega}\widehat{\psi}^+_{q,\sigma\omega}\widehat{\psi}^-_{k,\sigma\omega}\ D_\omega(k-q)\widehat{g}^{(j)}_{\sigma\omega}(k)\widehat{W}^{(j)}_{2,\sigma\omega}(k)\right]\\ &=\sum_{\mu=\pm}\int_D \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\ \widehat{\alpha}_{k-q,\omega}\widehat{\psi}^+_{q,\sigma\omega}\widehat{\psi}^-_{k,\sigma\omega}\ D_\mu(k)\\ &\qquad\qquad\cdot\left[D_\omega(k-q)\widehat{g}^{(j)}_{\sigma\omega}(k)\int_0^1\mathrm{d}\tau\ \left(\partial_\sigma\widehat{W}^{(j)}_{2,\sigma\omega}\right)(\tau k)\right]\ .\end{aligned}$$

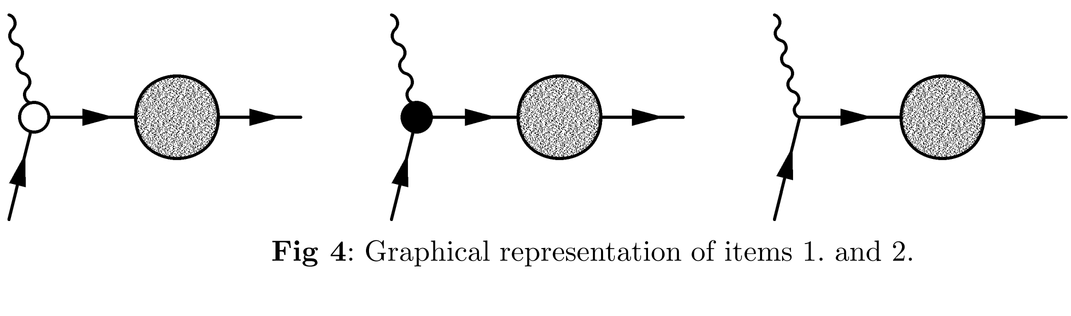

Fig 4: Graphical representation of items 1. and 2.

3. Both $\widehat{\psi}$-field of the interaction $\mathcal{A}_0$, contracted with a graph $\widehat{W}^{(j)}_{4,\omega,\mu}$, is identically vanishing, except if at least one of the two propagators is on scale N, or h. It is convenient to define:

$$\widehat{M}^{(r,s),(4)}_{\omega,\sigma\omega,\mu}(p,q) \stackrel{def}{=} \int_D \frac{\mathrm{d}^2 k}{(2\pi)^2}\; S^{(r,s)}_{\omega,\sigma\omega}(q+k,p+k)\widehat{W}^{(j)}_{4,\omega,\mu}(q,p,k)\ .$$

By symmetry under rotation and under space reflection (A4.2 and A4.3), it holds:

$$\widehat{M}^{(r,s),(4)}_{\omega,\sigma\omega,\mu}(0,0)\begin{cases} = 0 & \text{for } \mu = -\sigma\omega \\ \stackrel{def}{=} \Delta n_j^{(0,\sigma)} & \text{for } \mu = \sigma\omega\ . \end{cases}$$

Hence the localization of such graphs gives:

$$\begin{aligned}
&\mathcal{L}\left[\sum_\sigma \int_D \frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{\mathrm{d}^2 p}{(2\pi)^2}\; D_{\sigma\omega}(p-q)\widehat{\alpha}_{p-q,\omega}\widehat{\psi}^+_{q,\mu}\widehat{\psi}^-_{p,\mu}\widehat{M}^{(r,s),(4)}_{\omega,\sigma\omega,\mu}(p,q)\right] \\
&\quad = \sum_\sigma \Delta n_j^{(0,\sigma)} \int_D \frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{\mathrm{d}^2 p}{(2\pi)^2}\; D_{\sigma\omega}(p-q)\widehat{\alpha}_{p-q,\omega}\widehat{\psi}^+_{q,\sigma\omega}\widehat{\psi}^-_{p,\sigma\omega}\ , \\
&\mathcal{R}\left[\sum_\sigma \int_D \frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{\mathrm{d}^2 p}{(2\pi)^2}\; D_{\sigma\omega}(p-q)\widehat{\alpha}_{p-q,\omega}\widehat{\psi}^+_{q,\mu}\widehat{\psi}^-_{p,\mu}\widehat{M}^{(r,s),(4)}_{\omega,\sigma\omega,\mu}(p,q)\right] \\
&\quad = \sum_{k=p,q}\sum_{\sigma,\nu=\pm} \int_D \frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{\mathrm{d}^2 p}{(2\pi)^2}\; D_{\sigma\omega}(p-q)\widehat{\alpha}_{p-q,\omega}\widehat{\psi}^+_{q,\mu}\widehat{\psi}^-_{p,\mu}D_\nu(k) \\
&\qquad\qquad \cdot \int_0^1 \mathrm{d}\tau \left(\partial^k_\nu \widehat{M}^{(r,s),(4)}_{\omega,\sigma\omega,\mu}\right)(\tau p,\tau q)\ .
\end{aligned}$$

4. For the contraction of both $\widehat{\psi}$-field of the vertex $\sum_\sigma \nu_i^{(\sigma)}\mathcal{A}_\sigma$ with a graph $\widehat{W}^{(j)}_{4,\omega\sigma,\mu}$ it is convenient to define:

$$\widehat{M}^{(i,r,s),(4)}_{\sigma\omega,\mu}(p,q) \stackrel{def}{=} \nu_i^{(\sigma)} \int_D \frac{\mathrm{d}^2 k}{(2\pi)^2}\; \widehat{g}^{(r)}_{\sigma\omega}(q+k)\widehat{g}^{(s)}_{\sigma\omega}(p+k)\widehat{W}^{(j)}_{4,\sigma\omega,\mu}(q,p,k)\ .$$

As in the previous item, by symmetries it holds:

$$\widehat{M}^{(i,r,s),(4)}_{\sigma\omega,\mu}(0,0)\begin{cases} = 0 & \text{for } \mu = -\sigma\omega \\ \stackrel{def}{=} \Delta n_j^{(\sigma)} & \text{for } \mu = \sigma\omega\ . \end{cases}$$

hence, the localization of such graphs gives:

$$\begin{aligned}
&\mathcal{L}\left[\int_D \frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{\mathrm{d}^2 p}{(2\pi)^2}\; D_{\sigma\omega}(p-q)\widehat{\alpha}_{p-q,\omega}\widehat{\psi}^+_{q,\mu}\widehat{\psi}^-_{p,\mu}\widehat{M}^{(i,r,s),(4)}_{\sigma\omega,\mu}(p,q)\right] \\
&\quad = \Delta n_j^{(\sigma)} \int_D \frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{\mathrm{d}^2 p}{(2\pi)^2}\; D_{\sigma\omega}(p-q)\widehat{\alpha}_{p-q,\omega}\widehat{\psi}^+_{q,\sigma\omega}\widehat{\psi}^-_{p,\sigma\omega}\ , \\
&\mathcal{R}\left[\int_D \frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{\mathrm{d}^2 p}{(2\pi)^2}\; D_{\sigma\omega}(p-q)\widehat{\alpha}_{p-q,\omega}\widehat{\psi}^+_{q,\mu}\widehat{\psi}^-_{p,\mu}\widehat{M}^{(i,r,s),(4)}_{\omega,\sigma\omega,\mu}(p,q)\right] \\
&\quad = \sum_{k=p,q}\sum_{\nu=\pm} \int_D \frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{\mathrm{d}^2 p}{(2\pi)^2}\; D_{\sigma\omega}(p-q)\widehat{\alpha}_{p-q,\omega}\widehat{\psi}^+_{q,\mu}\widehat{\psi}^-_{p,\mu}D_\nu(k) \\
&\qquad\qquad \cdot \int_0^1 \mathrm{d}\tau \left(\partial^k_\nu \widehat{M}^{(i,r,s),(4)}_{\sigma\omega,\mu}\right)(\tau p,\tau q),
\end{aligned}$$

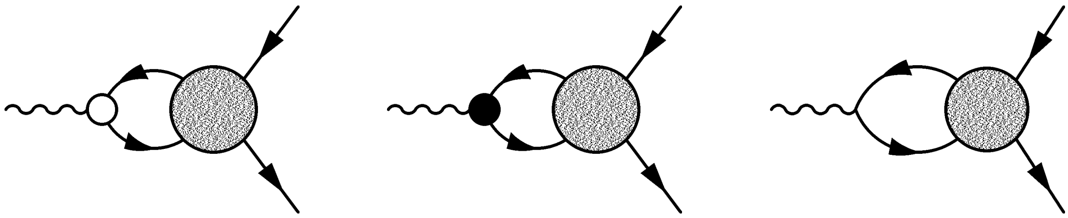

Fig 13: Graphical representation of items 3. and 4.

5. The self-contraction of the interactions $\mathcal{A}_0$ would give divergences because of C^ε_ω. Anyway, such a self-contraction, either for $\mathcal{A}_0$, and for $\{\mathcal{A}_\sigma\}_\sigma$, cannot occur in the expansion of the Schwinger function: in such expansions they cannot occur subgraphs with no external fields of type ψ or φ.

The local parts are absorbed into the effective parameter on scale $h-1$:

$$\nu^{(\sigma)}_{j-1} \stackrel{def}{=} \frac{Z_j}{Z_{j-1}} \left(\nu^{(\sigma)}_j + \Delta n^{(\sigma)}_j + \Delta n^{(0,\sigma)}_j \right) \ .$$

Theorem 4.1. *Fixed any $\vartheta : 0 < \vartheta < 1/16$, there exists $\varepsilon > 0$, a positive constant c_4 and two counterterms $\nu^{(+)}$ and $\nu^{(-)}$, analytically dependent on λ, such that, for any fixed cutoff scale, N, and choosing $\nu^{(\sigma)}_N = \nu^{(\sigma)}$, it holds*

$$\left| \nu^{(\sigma)}_j \right| \leq c_4 \varepsilon \gamma^{-(\vartheta/2)(N-j)} \ . \tag{4.1.7}$$

The proof is in appendix A7. It is a simple application of the fixed point theorem; once two counterterms $\{\nu^{(\sigma)}_N\}_{\sigma=\pm}$ with the required property are found, it is easy to verify they are sum of scaling invariant graphs, and therefore they are independent on the scale of the cutoff, N. Accordingly, it is natural to define:

$$a \stackrel{def}{=} \frac{1}{1 - \left(\nu^{(-)} + \nu^{(+)}\right)} \ , \qquad \overline{a} \stackrel{def}{=} \frac{1}{1 + \left(\nu^{(-)} - \nu^{(+)}\right)} \ .$$

Now it is possible to prove that, even removing the cutoff, the WTi are not equal to the formal one because of the non-vanishing anomaly $a - \overline{a}$.

Theorem 4.2. *In the same hypothesis of theorem 4.1, all the anomalous* WTi *for Schwinger functions, with only one density insertion and calculated at fixed momenta w.r.t. the cutoff scales, h and N, in the limit $-h, N \to \infty$ are generated by suitable derivatives of the following identity:*

$$D_\sigma(p) \frac{\partial \mathcal{W}}{\partial \widehat{\jmath}_{p,\sigma}}(\jmath, \varphi) = \zeta^{(2)}_b \sum_\mu \frac{a + \overline{a}\sigma\mu}{2} \int_D \frac{\mathrm{d}^2 k}{(2\pi)^2} \left[\frac{\partial \mathcal{W}}{\partial \widehat{\varphi}^-_{k,\mu}} \widehat{\varphi}^-_{k+p,\mu} - \widehat{\varphi}^+_{k,\mu} \frac{\partial \mathcal{W}}{\partial \widehat{\varphi}^+_{k+p,\mu}} \right] \ . \tag{4.1.8}$$

In particular, $(a+\overline{a}\sigma\mu)/2 = \delta_{\sigma,\mu} + \delta_{\sigma,-\mu}\lambda/4\pi + \mathrm{O}(\lambda^2)$.

The essence of the anomaly is that $(a+\overline{a}\sigma\mu)/2 \neq \delta_{\sigma,\mu}$, which implies, in spite of the formal result, the non-vanishing of $\widehat{S}^{(1;2)}_{-\omega;\omega}$. A celebrated consequence of the WTI, not wasted by the anomaly, is the following.

Theorem 4.3. *The anomalous exponent of the field strength and the anomalous exponent of the density strength coincide:* $\eta^{(2)}_\lambda = \eta_\lambda$.

This is what in formal language is stated as $Z^{(2)} = Z$.

Proof of Theorem 4.2. With reference to (4.1.6), it is only required to prove that the derivatives of $\mathcal{W}^{(h)}_{\mathcal{A}}$, made w.r.t. one field α and various fermionic fields at fixed momenta, fulfil the same bound of the derivatives of $\mathcal{W}^{(h)}$, with α replaced by $\jmath$, with a more factor which is vanishing in the limit of removed cutoff. Hence, let any integer $n \in \mathbb{N}$, any set of labels $\varepsilon_1, \ldots, \varepsilon_n$ and $\omega_1, \ldots, \omega_n$, and any momenta $p, k_1, \ldots, k_n$, chosen independently from h, N, be considered. It holds the bound

$$\frac{1}{|p|}\left|\frac{\partial^{1+n}\mathcal{W}^{(h)}_{\mathcal{A}}}{\partial\widehat{\alpha}_{p,\mu}\partial\widehat{\varphi}^{\varepsilon_1}_{k_1,\omega_1}\ldots\partial\widehat{\varphi}^{\varepsilon_n}_{k_n,\omega_n}}\right|_{\jmath\equiv\varphi\equiv 0} \leq \frac{C_{n;p,h_1,\ldots,h_n}}{\prod_{j=1}^n\sqrt{Z_{h_j}}}\left(\gamma^{-(\vartheta/2)(N-h_1)} + \gamma^{-(\vartheta/2)(h_1-h)}\right) , \quad (4.1.9)$$

where $\{h_j\}_{j=1}^n$ are the scales of $\{k_j\}_j$: $\kappa\gamma^{h_j-1} \leq |k_j| \leq \kappa\gamma^{h_j}$ and $C_{n;p,h_1,\ldots,h_n}/\prod_{j=1}^n\sqrt{Z_{h_j}}$ is the bound for the same derivatives of the functional $\mathcal{W}^{(h)}$. Such a bound can be obtained by the following argument. The graphs in the expansion of the l.h.s. member of (4.1.9) has to have an external propagator on scale h_1 – besides external propagators on scales $h_2, \ldots, h_n$; and they fall in one of the following cases.

1. An interaction $\mathcal{A}_0$ is contracted: this can happen only on scale $m = N, h$. By the short memory property (see A3.5), the sum of all such graphs is bounded, up to a constant, with $\gamma^{-(\vartheta/2)(N-h_1)}$ or $\gamma^{-(\vartheta/2)(h_1-h)}$.
2. An interaction $\mathcal{A}_\sigma$ is first contracted on scale m, and hence brings a coupling $\nu^{(\sigma)}_m$. By the short memory property and the bound in theorem 4.1, the sum of such graphs is bounded, up to a constant, with $\gamma^{-\vartheta|m-h_1|}\,\gamma^{-(\vartheta/2)(N-m)} \leq \gamma^{-(\vartheta/2)(N-h_1)}\gamma^{-(\vartheta/2)|m-h_1|}$.

Hence it is possible to take the sum over m, obtaining (4.1.9). ∎

Proof of Theorem 4.3. It simply follows from lowest order expansion of (1.1.8), and from the proof of Theorem 3.2 – in particular from the features of the anomalous exponents depicted in A5.3. Indeed, since

$$|\eta_\lambda - \Gamma_h| \leq c\varepsilon\gamma^{-\vartheta(N-h)} , \qquad \left|\eta^{(2)}_\lambda - \Gamma^{(2)}_h\right| \leq c\varepsilon\gamma^{-\vartheta(N-h)} ,$$

then

$$\log_\gamma \left(\frac{\zeta_h^{(2)}}{\zeta_N^{(2)}} \right) = (N-h) \left(\eta_\lambda - \eta_\lambda^{(2)} \right) + \mathrm{O}(\lambda^2) \; , \tag{4.1.10}$$

where $\mathrm{O}(\lambda^2)$ is a term of the order of λ^2 and bounded for every h. Calling $\overline{k}$ any momentum $\kappa\gamma^h \leq |\overline{k}| \leq \gamma^{h+1}$, by the lowest order graph expansion in Appendix A3, it holds,

$$\widehat{S}^{(1;2)}_{\omega,\omega}(2\overline{k};\overline{k}) = \frac{\zeta_h^{(2)}}{Z_h} \frac{1+\mathrm{O}(\lambda^2)}{D^2_\omega(\overline{k})} \; , \qquad \widehat{S}^{(2)}_\omega(\overline{k}) = \frac{1}{Z_h} \frac{1+\mathrm{O}(\lambda^2)}{D_\omega(\overline{k})} \; ,$$

$$\Delta \widehat{H}^{(1;2)}_{\omega,\omega}(2\overline{k};\overline{k}) = \frac{1}{Z_h} \frac{\mathrm{O}(\lambda^2)}{D_\omega(\overline{k})} \; , \qquad \frac{a_N + \overline{a}_N}{2} = 1 + \mathrm{O}(\lambda^2) \; .$$

Replacing the above identities into (1.1.8) and (1.1.11), the bound $\log_\gamma(\zeta_h^{(2)}/\zeta_N^{(2)}) = \mathrm{O}(\lambda^2)$ holds for any $h \leq N$: to be consistent with (4.1.10), it cannot be but $\eta_\lambda - \eta_\lambda^{(2)} = 0$. ■

4.1.4 Remark: anomaly and anomalous exponents. Formally, by the phase and chiral symmetry, it is possible to prove the identity of the field and density strength, $Z_N = Z_N^{(2)}$, so that the renormalization $\zeta_N^{(2)} \equiv 1$. But in a rigorous setting, WTi are seen to break this identity. Anyway, since the anomaly only changes a factor in front of the current, the identity between the exponents with which Z_N and $Z_N^{(2)}$ diverge *remains true*; therefore $\zeta_N^{(2)}$, although no longer constant, is anyway bounded.

4.2 Closed Equations

4.2.1 Schwinger-Dyson equation. The fermionic fields satisfy an evolution equation which can be turned into a set of equations for the Schwinger functions: see Appendix A8. Such equations relate the n-points Schwinger functions to the m-points Schwinger function with $m \leq n$ and one density insertion. Using the WTi to write the latter in terms of m-point Schwinger functions, the CE's arise.

4.2.2 Closed equations. In Appendix A8, the following equation, generator of all the SDe, is proved for any $k : \gamma^h\kappa \leq |k| \leq \gamma^N\kappa$ – where the cutoff $\chi_{h,N}(k) \equiv 1$:

$$D_\omega(k) \frac{\partial e^{\mathcal{W}^{(h)}}}{\partial \widehat{\varphi}^+_{k,\omega}} = \frac{\widehat{\varphi}^-_{k,\omega} e^{\mathcal{W}^{(h)}}}{Z_N} - \frac{\lambda_N}{\zeta_N^{(2)}} \int_D \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\partial^2 e^{\mathcal{W}^{(h)}}}{\partial \widehat{\jmath}_{p,-\omega} \partial \widehat{\varphi}^+_{k-p,\omega}} \; , \tag{4.2.1}$$

for $\jmath \equiv 0$ – since here only the CE for Schwinger function *without* density insertion are studied. Since it is possible to prove the convergence of the last integral for small p; and since $|p| \leq 2\gamma^N\kappa$, it is convenient make in the argument of the integral the following replacement:

$$1 \equiv \chi_{N+2}(p) \equiv \chi_{h+2,N+2}(p) + \chi_{h+2}(p) \stackrel{def}{=} \overline{\chi}_{h,N}(p) + \overline{\chi}_h(p) \; .$$

where $\chi_{h+2,N+2}(p) \stackrel{def}{=} \chi_{N+2}(p) - \chi_{h+2}(p)$. Then, from the generator of the WTi, (4.1.6), it holds the following integral identity:

$$\begin{aligned}
&\frac{1}{\zeta_N^{(2)}} \int_D \frac{\mathrm{d}^2 p}{(2\pi)^2}\, \overline{\chi}_{h,N}(p) \frac{\partial^2 e^{\mathcal{W}^{(h)}}}{\partial \widehat{\jmath}_{p,-\omega} \partial \widehat{\varphi}^+_{k-p,\omega}} \\
&= \sum_\mu \frac{a_N + \overline{a}_N \sigma\mu}{2} \int_D \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{\overline{\chi}_{h,N}(p)}{D_{-\omega}(p)} \left[\frac{\partial^2 e^{\mathcal{W}^{(h)}}}{\partial \widehat{\varphi}^+_{k-p,\omega} \partial \widehat{\varphi}^-_{q,\mu}} \widehat{\varphi}^-_{q+p,\mu} - \widehat{\varphi}^+_{q,\mu} \frac{\partial^2 e^{\mathcal{W}^{(h)}}}{\partial \widehat{\varphi}^+_{q+p,\mu} \partial \widehat{\varphi}^+_{p-k,\omega}} \right] \\
&\quad - \sum_\mu \frac{a_N + \overline{a}_N \sigma\mu}{2} \int_D \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\overline{\chi}_{h,N}(p)}{D_{-\omega}(p)} \frac{\partial^2 e^{\mathcal{W}^{(h)}_{\mathcal{A}}}}{\partial \widehat{\alpha}_{p,\mu} \partial \widehat{\varphi}^+_{k-p,\omega}} (0, \jmath, \varphi)\ .
\end{aligned} \tag{4.2.2}$$

Taking a derivative in $\widehat{\varphi}^-_{k,\omega}$, and putting $\varphi = 0$, gives (1.1.13) – apart from the function $\overline{\chi}_{h,N}(p)$ that had been skipped for reproducing the Johnson's argument. By the general analysis of the previous section, the term proportional to the derivatives of the functional $\mathcal{W}_{\mathcal{A}}$ would have been vanishing in the limit of removed cutoff *if the external momenta had been fixed*. But in this case the external momenta are integrated over, and there is no reason that this term is vanishing – differently from what implicitly stated in [J61].

4.2.3 Flows of $\widetilde{z}^{(\mu)}_N$ and $\widetilde{\lambda}^{(\mu)}_N$. To overcome the problem of not having, neither in the limit, a real closed equation, it is possible to write such a rest as addends that are already present in the SDe. To this purpose, let the functionals $\mathcal{W}^{(h)}_{\mathcal{T},\mu}$, for $\mu = \pm$ be defined as

$$\begin{aligned}
e^{\mathcal{W}^{(h)}_{\mathcal{T},\mu}(\beta,\jmath,\varphi)} &\stackrel{def}{=} \int \mathrm{d}P^{[h,N]}(\psi)\ \exp\left\{ -\lambda_N \mathcal{V}\left(\sqrt{Z_N}\psi\right) + \zeta^{(2)}_N \mathcal{J}\left(\jmath, \sqrt{Z_N}\psi\right) + \mathcal{F}(\varphi,\psi) \right\} \\
&\qquad \exp\left\{ \left[\mathcal{T}^{(\mu)}_0 + \sum_{\sigma=\pm} \nu^{(\sigma)}_N \mathcal{T}^{(\mu)}_\sigma - \alpha^{(\mu\omega)} \lambda_N \mathcal{B}^{(3)} - \sigma^{(\mu\omega)} \mathcal{B}^{(1)} \right] \left(\sqrt{Z_N}\psi, \sqrt{Z_N}\beta \right) \right\}\ ,
\end{aligned}$$

with $\{\alpha^\mu\}_{\mu=\pm}$ and $\{\sigma^\mu\}_{\mu=\pm}$ four real parameters later fixed; and

$$\begin{aligned}
\mathcal{T}^{(\mu)}_0(\psi,\beta) &\stackrel{def}{=} \sum_{\omega=\pm} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\mathrm{d}^2 q}{(2\pi)^2}\, \overline{\chi}_{h,N}(p) \frac{C_\mu(q, p+q)}{D_{-\omega}(p)} \widehat{\beta}_{k,\omega} \widehat{\psi}^-_{k-p,\omega} \widehat{\psi}^+_{q,\mu} \widehat{\psi}^-_{p+q,\mu}\ , \\
\mathcal{T}^{(\mu)}_\sigma(\psi,\beta) &\stackrel{def}{=} \sum_{\omega=\pm} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\mathrm{d}^2 q}{(2\pi)^2}\, \overline{\chi}_{h,N}(p) \frac{D_{\sigma\mu}(p)}{D_{-\omega}(p)}\ \widehat{\beta}_{k,\omega} \widehat{\psi}^-_{k-p,\omega} \widehat{\psi}^+_{q,\sigma\mu} \widehat{\psi}^-_{p+q,\sigma\mu}\ ;
\end{aligned}$$

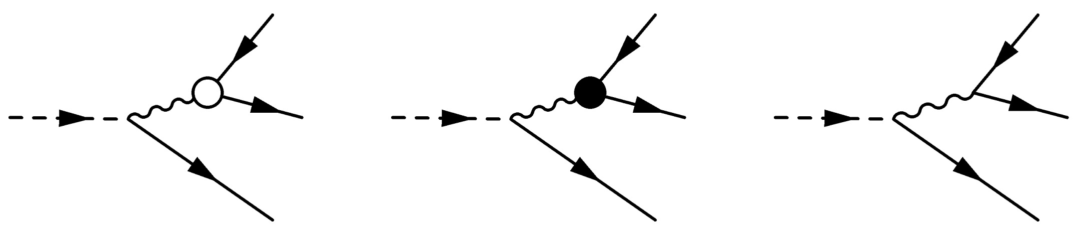

Fig 6: Graphical representation of the interactions $\mathcal{T}^{(\mu)}_0$, $T^{(\mu)}_-$ and $T^{(\mu)}_+$

$$\mathcal{B}^{(3)}(\psi,\beta)\stackrel{def}{=}\sum_{\omega=\pm}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\ \widehat{\beta}_{p+k-q,\omega}\widehat{\psi}^-_{p,\omega}\widehat{\psi}^+_{q,-\omega}\widehat{\psi}^-_{k,-\omega}\ ,$$

$$\mathcal{B}^{(1)}(\beta,\psi)\stackrel{def}{=}\sum_{\omega=\pm}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\ \widehat{\beta}_{k,\omega}D_\omega(k)\widehat{\psi}^-_{k,\omega}\ .$$

Because of the identity

$$\begin{aligned}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\ \frac{\overline{\chi}_{h,N}(p)}{D_{-\omega}(p)}\frac{\partial^2 e^{\mathcal{W}_\mathcal{A}}}{\partial\widehat{\alpha}_{p,\mu}\partial\widehat{\varphi}^+_{k-p,\omega}}=\frac{1}{Z_N}\frac{\partial e^{\mathcal{W}^{(h)}_{\mathcal{T},\mu}}}{\partial\widehat{\beta}_{k,\omega}}+\alpha^{(\mu\omega)}\frac{\lambda_N}{\zeta_N^{(2)}}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\ \frac{\partial^2 e^{\mathcal{W}}}{\partial\widehat{\jmath}_{p,-\omega}\partial\widehat{\varphi}^+_{k-p,\omega}}\\ +\sigma^{(\mu\omega)}D_\omega(k)\frac{\partial e^{\mathcal{W}}}{\partial\widehat{\varphi}^+_{k,\omega}}\ ,\end{aligned}\tag{4.2.3}$$

it is possible to turn equation (4.2.2) into:

$$\begin{aligned}&\left(1-\sum_\mu\frac{a_N-\overline{a}_N\mu}{2}\alpha^{(\mu)}\lambda_N\right)\frac{1}{\zeta_N^{(2)}}\int_D\frac{\mathrm{d}^2p}{(2\pi)^2}\ \overline{\chi}_{h,N}(p)\frac{\partial e^{\mathcal{W}^{(h)}}}{\partial\widehat{\jmath}_{p,-\omega}\partial\widehat{\varphi}^+_{k-p,\omega}}\\ &=\left(\sum_\mu\frac{a_N-\overline{a}_N\mu}{2}\sigma^{(\mu)}\right)D_\omega(k)\frac{\partial e^{\mathcal{W}^{(h)}}}{\partial\widehat{\varphi}^+_{k,\omega}}\\ &+\sum_\mu\frac{a_N-\overline{a}_N\omega\mu}{2}\int_D\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\ \frac{\overline{\chi}_{h,N}(p)}{D_{-\omega}(p)}\left[\frac{\partial e^{\mathcal{W}^{(h)}}}{\partial\widehat{\varphi}^+_{k-p,\omega}\partial\widehat{\varphi}^-_{q,\mu}}\widehat{\varphi}^-_{q+p,\mu}-\widehat{\varphi}^+_{q,\mu}\frac{\partial e^{\mathcal{W}^{(h)}}}{\partial\widehat{\varphi}^+_{q+p,\mu}\partial\widehat{\varphi}^+_{p-k,\omega}}\right]\\ &-\frac{1}{Z_N}\sum_\mu\frac{a_N-\overline{a}_N\omega\mu}{2}\frac{\partial e^{\mathcal{W}^{(h)}_{\mathcal{T},\mu}}}{\partial\widehat{\beta}_{k,\omega}}\\ &-\left(\sum_\mu\frac{a_N-\overline{a}_N\mu}{2}\alpha^{(\mu)}\lambda_N\right)\frac{1}{\zeta_N^{(2)}}\int_D\frac{\mathrm{d}^2p}{(2\pi)^2}\ \overline{\chi}_h(p)\frac{\partial e^{\mathcal{W}^{(h)}}}{\partial\widehat{\jmath}_{p,-\omega}\partial\widehat{\varphi}^+_{k-p,\omega}}\ .\end{aligned}\tag{4.2.4}$$

The term proportional to the derivatives of $\mathcal{W}^{(h)}_{T,\mu}$ *does vanish* for a suitable choice of the counterterms; as well as the second term in the last line vanishes, at least in some important cases – the CE for $S^{(2)}$ and for $S^{(4)}$. As consequence, it is suitable to replace (4.2.4) in (4.2.1), obtaining:

$$\begin{aligned}D_\omega(k)\frac{\partial e^{\mathcal{W}^{(h)}}}{\partial\widehat{\varphi}^+_{k,\omega}}&=\frac{B_N}{Z_N}\widehat{\varphi}^-_{k,\omega}e^{\mathcal{W}^{(h)}}\\ &-\lambda_N A_N\sum_\mu\frac{a_N-\overline{a}_N\omega\mu}{2}\int_D\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\ \frac{\overline{\chi}_{h,N}(p)}{D_{-\omega}(p)}\left[\frac{\partial e^{\mathcal{W}^{(h)}}}{\partial\widehat{\varphi}^+_{k-p,\omega}\partial\widehat{\varphi}^-_{q,\mu}}\widehat{\varphi}^-_{q+p,\mu}\right.\\ &\qquad\left.-\widehat{\varphi}^+_{q,\mu}\frac{\partial e^{\mathcal{W}^{(h)}}}{\partial\widehat{\varphi}^+_{q+p,\mu}\partial\widehat{\varphi}^+_{p-k,\omega}}\right]\\ &-\frac{\lambda_N A_N}{Z_N}\sum_\mu\frac{a_N-\overline{a}_N\omega\mu}{2}\frac{\partial e^{\mathcal{W}^{(h)}_{\mathcal{T},\mu}}}{\partial\widehat{\beta}_{k,\omega}}-\frac{\lambda_N A_N}{\zeta_N^{(2)}}\int_D\frac{\mathrm{d}^2p}{(2\pi)^2}\ \overline{\chi}_h(p)\frac{\partial e^{\mathcal{W}^{(h)}}}{\partial\widehat{\jmath}_{p,-\omega}\partial\widehat{\varphi}^+_{k-p,\omega}}\ ,\end{aligned}\tag{4.2.5}$$

where it was set

$$A_N \stackrel{def}{=} \frac{1}{1-(\lambda_N/2)\sum_\mu (a_N - \overline{a}_N\mu)\left(\alpha^{(\mu)}-\sigma^{(\mu)}\right)} ,$$
$$B_N \stackrel{def}{=} \frac{1-(\lambda_N/2)\sum_\mu (a_N - \overline{a}_N\mu)\alpha^{(\mu)}}{1-(1/2)\sum_\mu (a_N - \overline{a}_N\mu)\left(\alpha^{(\mu)}-\sigma^{(\mu)}\right)} . \tag{4.2.6}$$

Deriving (4.2.5) w.r.t. $\widehat{\varphi}^-_{k,\omega}$, for $\varphi\equiv 0$; since by the tree expansion, see A3,

$$\left|\int_D \frac{d^2p}{(2\pi)^2}\ \overline{\chi}_h(p)\widehat{S}^{(1;2)}_{-\omega;\omega}(p;k)\right| \le C\gamma^{(1-\vartheta)(h_1-h)} , \tag{4.2.7}$$

for any $\vartheta : 0<\vartheta<1$ and for h_1 the scale of the momentum k; and supposing the derivatives of $\mathcal{W}_{\mathcal{T},\mu}$ are vanishing, in the limit of removed cutoff, it holds the asymptotic formula (1.1.16).

More in general, in order to prove the derivatives of $\mathcal{W}_{\mathcal{T},\mu}$ are vanishing in the limit of removed cutoff, it is necessary a multiscale expansion.

4.2.4 Improved localization II. After the multiscale integration, down to the j-th scale, it holds:

$$\begin{aligned} e^{\mathcal{W}^{(h)}_{\mathcal{T},\mu}(\beta,\jmath,\varphi)} &\stackrel{def}{=} \int dP^{[h,j]}(\psi)\ \exp\left\{\mathcal{W}^{(j)}\left(\varphi,\jmath,\sqrt{Z_j}\psi\right)+\mathcal{W}^{(j)}_{\mathcal{T},\mathrm{irr}}\left(\beta,\varphi,\jmath,\sqrt{Z_j}\psi\right)\right\}\\ &\cdot\exp\left\{\left[\left(\frac{Z_N}{Z_j}\right)^2\mathcal{T}^{(\mu)}_0+\frac{Z_N}{Z_j}\sum_{\sigma=\pm}\nu^{(\sigma)}_j\mathcal{T}^{(\mu)}_\sigma\right]\left(\sqrt{Z_j}\psi,\sqrt{Z_j}\beta\right)\right\}\\ &\cdot\exp\left\{\left[\widetilde{\zeta}^{(3,\mu\omega)}_j\mathcal{B}^{(3)}+\sum_{k=j}^N\frac{Z_k}{Z_j}\widetilde{\zeta}^{(1,\mu\omega)}_k\mathcal{B}^{(1)}\right]\left(\sqrt{Z_j}\psi,\sqrt{Z_j}\beta\right)\right\} ,\end{aligned} \tag{4.2.8}$$

where $\widetilde{\zeta}^{(3,\mu)}_N \stackrel{def}{=} -\alpha^{(\mu)}\lambda_N$, while, for $j\le N-1$, $\widetilde{\zeta}^{(3,\mu)}_j \stackrel{def}{=} \left(\widetilde{\lambda}^{(\mu)}_j-\alpha^{(\mu)}\lambda_j\right)$; and, $\widetilde{\zeta}^{(1,\mu)}_N \stackrel{def}{=} \sigma^{(\mu)}$, while, for $j\le N-1$, $\widetilde{\zeta}^{(1,\mu)}_j \stackrel{def}{=} \left(\widetilde{z}^{(\mu)}_j-\alpha^{(\mu)}z_j\right)$. Indeed, these are the following possible contractions of the interactions in $\mathcal{W}^{(h)}_{\mathcal{T},\mu}$.

1. The contraction of the interactions $\mathcal{T}^{(\mu)}_0$, $\mathcal{T}^{(\mu)}_\sigma$, $\mathcal{B}^{(3)}$ and $\mathcal{B}^{(1)}$ through only one external field $\widehat{\psi}^-_{k,\omega}$ with a kernel $\widehat{W}^{(j)}_{2,\omega}$ are apparently marginal; instead the localization is proportional to $\widehat{W}^{(j)}_{2,\omega}(0)$, vanishing by symmetries; for instance, in the case of the occurring of the interaction $\mathcal{T}^{(\mu)}_0$, it holds:

$$\mathcal{L}\left[\int\frac{d^2k}{(2\pi)^2}\frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\ \overline{\chi}_{h,N}(p-q)\widehat{\beta}_{k+p-q}\widehat{\psi}^-_{k,\omega}\widehat{\psi}^+_{q,\mu}\widehat{\psi}^-_{p,\mu}\frac{C_\omega(q,p)}{D_{-\omega}(p-q)}\widehat{g}^{(s)}_\omega(k)\widehat{W}^{(j)}_{2,\omega}(k)\right]=0 ,$$

$$\begin{aligned}&\mathcal{R}\left[\int\frac{d^2k}{(2\pi)^2}\frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\ \overline{\chi}_{h,N}(p-q)\widehat{\beta}_{k+p-q}\widehat{\psi}^-_{k,\omega}\widehat{\psi}^+_{q,\mu}\widehat{\psi}^-_{p,\mu}\frac{C_\omega(q,p)}{D_{-\omega}(p-q)}\widehat{g}^{(s)}_\omega(k)\widehat{W}^{(j)}_{2,\omega}(k)\right]\\ &=\sum_\sigma\int\frac{d^2k}{(2\pi)^2}\frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\ \overline{\chi}_{h,N}(p-q)\widehat{\beta}_{k+p-q}\widehat{\psi}^-_{k,\omega}D_\sigma(k)\widehat{\psi}^+_{q,\mu}\widehat{\psi}^-_{p,\mu}\frac{C_\omega(q,p)}{D_{-\omega}(p-q)}\widehat{g}^{(s)}_\omega(k)\\ &\qquad\cdot\int_0^1 d\tau\ \left(\partial_\sigma\widehat{W}^{(j)}_{2,\omega}\right)(\tau k) .\end{aligned}$$

The above case is given by one-particle reducible graphs, therefore an alternative argument is the one similar to item 1. and 2. of the previous section.

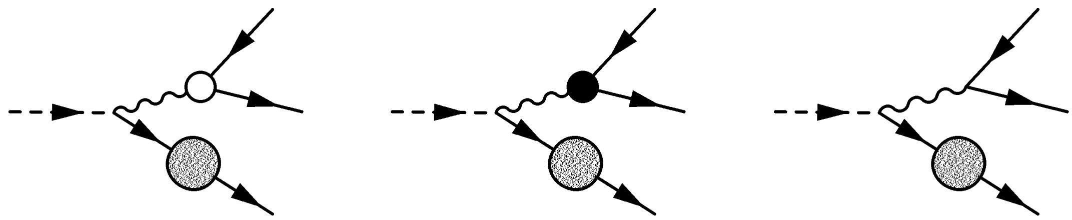

Fig 7: Graphical representation of items 1.

2. The fields $\widehat{\psi}^-_{k,\omega}$ and $\widehat{\psi}^-_{p,\mu}$ of the vertex $\mathcal{T}_0^{(\mu)}$ with the kernel $\widehat{W}^{(j)}_{4,\mu,\omega}$, is non-irrelevant; by the explicit expression of C^δ_ω, it holds:

$$\begin{aligned}
&\int \frac{\mathrm{d}^2 k'}{(2\pi)^2}\, \overline{\chi}_{h,N}(p+k'-q)\widehat{g}^{(r)}_\omega(k-k')\frac{C^\delta_\mu(q,p+k')}{D_{-\omega}(p+k'-q)}\widehat{g}^{(s)}_\mu(p+k')\widehat{W}^{(j)}_{4,\mu,\omega}(k',p,k)\\
&= \int \frac{\mathrm{d}^2 k'}{(2\pi)^2}\, \overline{\chi}_{h,N}(p+k'-q)\left[\widehat{g}^{(r)}_\omega(k-k')\frac{D_\mu(q)\left(1-(\chi^\delta_{h,N})^{-1}(q)\right)}{D_{-\omega}(p+k'-q)}\frac{f_s(p+k')}{D_\mu(p+k')}\right.\\
&\qquad\left. + \widehat{g}^{(r)}_\omega(k-k')\frac{\left(\delta_{r,N}+\delta_{s,h}\right)u_s(p+k')}{D_{-\omega}(p-q+k')}\right]\widehat{W}^{(j)}_{4,\mu,\omega}(k',p,k)\ ;
\end{aligned}$$

only the second term has a non-irrelevant part; indeed, for $j \geq h+2$, because of $f_s(p+k')$, with $s \geq j$, and because of $\left(1-(\chi^\delta_{h,N})^{-1}(q)\right)$, which, for $q \to 0$ compels q to be contracted on scale h,

$$\begin{aligned}
|D_\mu(p+k'-q)| &\geq |D_\mu(p+k')| - |D_\mu(q)| \geq \gamma^{j-1} - \gamma^{h+1}\\
&\geq (1-\gamma^{-1})\gamma^{j-1}\ ;
\end{aligned}$$

this means that the bound of such a kernel, w.r.t. the standard bound, has a more factor $\gamma^{-(j-h)}$ which gives a gain of one unity in the dimension of the kernel, making it strictly negative down to scale h, where the third field of the interaction, $\widehat{\psi}_{q,\mu}$, *is compelled to be contracted* by $\left(1-(\chi^\delta_{h,N})^{-1}(q)\right)$. On the contrary, the other term

$$\begin{aligned}
&\widehat{M}^{(r,s),(4)}_{\mu,\omega}(p,k,q)\\
&\stackrel{def}{=} \int \frac{\mathrm{d}^2 k'}{(2\pi)^2}\, \overline{\chi}_{h,N}(p-q+k')\widehat{g}^{(r)}_\omega(k-k')\frac{\left(\delta_{r,N}+\delta_{s,h}\right)u_s(p+k')}{D_{-\omega}(p-q+k')}\widehat{W}^{(4)}_{\mu,\omega}(k',p,k)\ ,
\end{aligned}$$

can occur only if r in on scale N, or s is on scale h; and requires the extraction of the coefficient:

$$\widehat{M}^{(r,s),(4)}_{\mu,\omega}(0,0,0)\begin{cases} = 0 & \text{if } \omega\mu = 1\\ \stackrel{def}{=} \Delta\widetilde{l}^{\,(-,0)}_j & \text{if } \omega\mu = -1\ ,\end{cases}$$

so that the above contraction is equal to

$$\mathcal{L}\Bigg[\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\widehat{\beta}_{k+p-q,\omega}\widehat{\psi}^-_{k,\omega}\widehat{\psi}^+_{q,\mu}\widehat{\psi}^-_{p,\mu} \cdot \int \frac{\mathrm{d}^2k'}{(2\pi)^2}\,\overline{\chi}_{h,N}(p+k'-q)\widehat{g}^{(r)}_\omega(k-k')\frac{C^\delta_\mu(q,p+k')}{D_{-\omega}(p+k'-q)}\widehat{g}^{(s)}_\mu(p+k')\widehat{W}^{(j)}_{4,\mu,\omega}(k',p,k)\Bigg]$$

$$=\Delta\widetilde{l}^{(-,0)}_j\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\widehat{\beta}_{k+p-q,\omega}\widehat{\psi}^-_{k,\omega}\widehat{\psi}^+_{q,-\omega}\widehat{\psi}^-_{p,-\omega}\ ,$$

$$\mathcal{R}\Bigg[\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\widehat{\beta}_{k+p-q,\omega}\widehat{\psi}^-_{k,\omega}\widehat{\psi}^+_{q,\mu}\widehat{\psi}^-_{p,\mu} \cdot \int \frac{\mathrm{d}^2k'}{(2\pi)^2}\,\overline{\chi}_{h,N}(p+k'-q)\widehat{g}^{(r)}_\omega(k-k')\frac{C^\delta_\mu(q,p+k')}{D_{-\omega}(p+k'-q)}\widehat{g}^{(s)}_\mu(p+k')\widehat{W}^{(j)}_{4,\mu,\omega}(k',p,k)\Bigg]$$

$$=\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\widehat{\beta}_{k+p-q,\omega}\widehat{\psi}^-_{k,\omega}\widehat{\psi}^+_{q,\mu}\widehat{\psi}^-_{p,\mu} \cdot \int \frac{\mathrm{d}^2k'}{(2\pi)^2}\left[\overline{\chi}_{h,N}(p+k'-q)\widehat{g}^{(r)}_\omega(k-k')\frac{D_\mu(q)\left(1-(\chi^\delta_{h,N})^{-1}(q)\right)}{D_{-\omega}(p+k'-q)}\frac{f_s(p+k')}{D_\mu(p+k')}\right]$$

$$+\sum_{p'=k,p}\sum_\nu \widehat{\beta}_{k+p-q,\omega}\widehat{\psi}^-_{k,\omega}\widehat{\psi}^+_{q,\mu}\widehat{\psi}^-_{p,\mu}D_\nu(p')\int_0^1 d\tau\ \left(\partial^{p'}_\nu\widehat{M}^{(r,s),(4)}_{\mu,\omega}\right)(\tau p,\tau k)\ .$$

With similar developments it is extracted $\Delta\widetilde{\lambda}^{(+,0)}_j$, the local part of the graphs with the fields $\widehat{\psi}^-_{k,\omega}$ and $\widehat{\psi}^+_{q,\mu}$ of the interaction $\mathcal{T}^{(\mu)}_0$ contracted with the kernel $\widehat{W}^{(j)}_{4,\mu,\omega}$.

3. The contraction of the fields $\widehat{\psi}^-_{k,\omega}$ and $\widehat{\psi}^-_{p,\sigma\mu}$ of the interaction $\mathcal{T}^{(\mu)}_\sigma$ with the kernel $\widehat{W}^{(j)}_{4,\mu,\omega}$ is non-irrelevant. Setting:

$$\widehat{N}^{(r,s),(4)}_{\mu,\sigma,\omega}(p,k,q) \stackrel{def}{=} \int \frac{\mathrm{d}^2k'}{(2\pi)^2}\,\widehat{g}^{(r)}_\omega(k-k')\overline{\chi}_{h,N}(p+k'-q)\frac{D_{\sigma\mu}(p+k'-q)}{D_{-\omega}(p+k'-q)}\widehat{g}^{(s)}_{\sigma\mu}(p+k')\widehat{W}^{(j)}_{4,\mu,\omega}(k',p,k)\ ,$$

such a contraction requires the extraction of the coefficient:

$$\widehat{N}^{(r,s),(4)}_{\mu,\sigma,\omega}(0,0,0)\begin{cases}=0 & \text{if } \omega\mu=0\\ \stackrel{def}{=}\Delta\widetilde{l}^{(-,\sigma)}_j & \text{if } \omega\mu=-1\ ,\end{cases}$$

so that the above contraction is equal to

$$\mathcal{L}\left[\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\widehat{\beta}_{k+p-q,\omega}\widehat{\psi}^-_{k,\omega}\widehat{\psi}^+_{q,\mu}\widehat{\psi}^-_{p,\mu}\widehat{N}^{(r,s),(4)}_{\mu,\sigma,\omega}(p,k,q)\right]$$

$$=\Delta\widetilde{l}^{(-,\sigma)}_j\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\widehat{\beta}_{k+p-q,\omega}\widehat{\psi}^-_{k,\omega}\widehat{\psi}^+_{q,-\omega}\widehat{\psi}^-_{p,-\omega}\ ,$$

$$\mathcal{R}\left[\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\widehat{\beta}_{k+p-q,\omega}\widehat{\psi}^-_{k,\omega}\widehat{\psi}^+_{q,\mu}\widehat{\psi}^-_{p,\mu}\widehat{N}^{(r,s),(4)}_{\mu,\sigma,\omega}(p,k,q)\right]$$

$$=\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\sum_{p'=k,p,q}\sum_\nu \widehat{\beta}_{k+p-q,\omega}\widehat{\psi}^-_{k,\omega}\widehat{\psi}^+_{q,\mu}\widehat{\psi}^-_{p,\mu}D_\nu(p') \cdot\int_0^1 d\tau\ \left(\partial^{p'}_\nu\widehat{N}^{(r,s),(4)}_{\mu,\sigma,\omega}\right)(\tau p,\tau k,\tau q)\ .$$

With similar developments it is extracted $\Delta\widetilde{\lambda}_j^{(+,\sigma)}$, the local part of the graphs with the fields $\widehat{\psi}^-_{k,\omega}$ and $\widehat{\psi}^+_{q,\mu}$ of the interaction $\mathcal{T}_\sigma^{(\mu)}$ contracted with the kernel $\widehat{W}^{(j)}_{4,\mu,\omega}$.

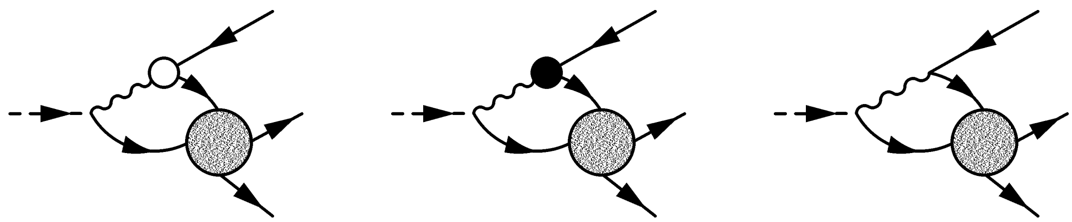

Fig 8: Graphical representation of items 2. and 3.

4. The contraction of all and three ψ-field of $\mathcal{T}_0^{(\mu)}$ with the kernel $\widehat{W}^{(j)}_{6,\mu,\omega,\nu}$ is non-vanishing if at least one between the two propagators g_μ, has scale N or h, otherwise the product $C_\mu(p,k)\widehat{g}_\mu(k)\widehat{g}_\mu(p+k)$ vanish; it generates non-irrelevant operators. Let the contraction be:

$$\widehat{M}^{(r,s,t),(6)}_{\omega,\sigma,\mu,\rho}(k,q,p) \stackrel{def}{=} \int \frac{\mathrm{d}^2k'}{(2\pi)^2}\frac{\mathrm{d}^2q'}{(2\pi)^2}\,\widehat{g}^{(r)}_\omega(k-k')\overline{\chi}_{h,N}(p+k'-q)\frac{D_{\sigma\mu}(p+k'-q)}{D_{-\omega}(p+k'-q)} \cdot S^{(s,t)}_{\mu,\sigma\mu}(q+q',p+k'+q')\widehat{W}^{(6)}_{\mu,\omega,\rho}(q,p,k,k',q')\ ,$$

and let the following coefficient be considered:

$$\sum_\sigma \widehat{M}^{(r,s,t),(6)}_{\omega,\sigma,\mu,\rho}(0,0,0)\begin{cases} = 0 & \text{if } \rho=\omega \\ \stackrel{def}{=} \Delta\widetilde{\lambda}_j^{(0,0,\mu\omega)} & \text{if } \rho=-\omega\ . \end{cases}$$

Then, the decomposition into marginal operator plus irrelevant one is:

$$\mathcal{L}\left[\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\widehat{\beta}_{p+k-q,\omega}\widehat{\psi}^-_{k,\omega}\widehat{\psi}^+_{q,\rho}\widehat{\psi}^-_{p,\rho}\sum_\sigma \widehat{M}^{(r,s,t),(6)}_{\omega,\sigma,\mu,\rho}(k,q,p)\right]$$
$$= \Delta\widetilde{\lambda}_j^{(0,0,\mu\omega)}\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\ \widehat{\beta}_{p+k-q,\omega}\widehat{\psi}^-_{k,\omega}\widehat{\psi}^+_{q,-\omega}\widehat{\psi}^-_{p,-\omega}\ ,$$
$$\mathcal{R}\left[\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\widehat{\beta}_{p+k-q,\omega}\widehat{\psi}^-_{k,\omega}\widehat{\psi}^+_{q,\rho}\widehat{\psi}^-_{p,\rho}\sum_\sigma \widehat{M}^{(r,s,t),(6)}_{\omega,\sigma,\mu,\rho}(k,q,p)\right]$$
$$= \sum_{p'=q,p,k}\sum_{\sigma,\sigma'}\widehat{\beta}_{p+q-k,\omega}\widehat{\psi}^-_{q,\omega}\widehat{\psi}^+_{k,\nu}\widehat{\psi}^-_{p,\nu}D_{\sigma'}(p')$$
$$\cdot\int \frac{\mathrm{d}^2k'}{(2\pi)^2}\frac{\mathrm{d}^2q'}{(2\pi)^2}\int_0^1 \mathrm{d}\tau\ \left(\partial^{p'}_{\sigma'}\widehat{M}^{(r,s,t),(6)}_{\omega,\sigma,\mu,\rho}\right)(\tau k,\tau q,\tau p)\ .$$

Besides, similar decomposition is done when $\mathcal{T}_0^{(\mu)}$ is replaced by $\mathcal{T}_\sigma^{(\mu)}$, with the replacements of $S^{(s,t)}_{\mu,\sigma\mu}$ with 1, and of $\Delta\widetilde{\lambda}_j^{(0,0,\mu\omega)}$ with $\Delta\widetilde{\lambda}_j^{(0,\sigma,\mu\omega)}$.

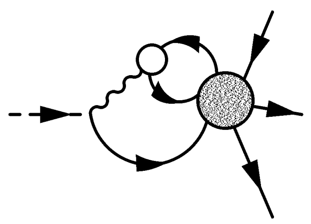

Fig 9: Graphical representation of item 4

5. The contraction of all and three ψ-fields of $\mathcal{T}_0^{(\mu)}$ with the kernel $\widehat{W}_{4,\mu,\nu}^{(j)}$ is non-vanishing if at least one between the two above propagators g_μ, has scale N or h. Let the contraction

$$\widehat{M}_{\omega,\rho,\mu}^{(r,s,t),(4)}(p) \stackrel{def}{=} \int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\; \widehat{g}_\omega^{(r)}(k)\overline{\chi}_{h,N}(p+k'-q)\frac{D_{\rho\mu}(p-k)}{D_{-\omega}(p-k)}S_{\mu,\rho\mu}^{(s,t)}(q,p-k+q)\widehat{W}_{4,\mu,\omega}^{(j)}(q,p,k)\;;$$

then $\widehat{M}_{\omega,\rho,\mu}^{(r,s,t),(4)}(0)=0$ by transformation under rotation; while

$$\sum_\rho \left(\partial_\sigma \widehat{M}_{\omega,\rho,\mu}^{(r,s,t),(4)}\right)(0)\begin{cases} =0 & \text{if } \sigma=-\omega \\ \stackrel{def}{=}\Delta\widetilde{z}_j^{(0,\mu\omega)} & \text{if } \sigma=\omega\;. \end{cases}$$

Finally:

$$\mathcal{L}\left[\sum_\rho \int \frac{\mathrm{d}^2p}{(2\pi)^2}\widehat{\beta}_{p,\omega}\widehat{\psi}_{p,\rho}^-\widehat{M}_{\omega,\rho,\mu}^{(r,s,t),(4)}(p)\right] = \Delta\widetilde{z}_j^{(0,\mu\omega)}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\;\widehat{\beta}_{p,\omega}D_\omega(p)\widehat{\psi}_{p,\omega}^-\;,$$

$$\mathcal{R}\left[\sum_\rho \int \frac{\mathrm{d}^2p}{(2\pi)^2}\widehat{\beta}_{p,\omega}\widehat{\psi}_{p,\rho}^-\widehat{M}_{\omega,\mu,\rho}^{(r,s,t),(4)}(p)\right] = \sum_{\sigma,\sigma'}\widehat{\beta}_{p,\omega}\widehat{\psi}_{p,\rho}^-D_\sigma(p)D_{\sigma'}(p)\int_0^1 d\tau\;(1-\tau)\left(\partial_{\sigma'}^p\partial_{\sigma'}^p\widehat{M}_{\omega,\rho,\mu}^{(r,s,t),(4}\right)(\tau p)\;.$$

Besides, similar decomposition is done when $\mathcal{T}_0^{(\mu)}$ is replaced by $\mathcal{T}_\sigma^{(\mu)}$, with the replacements of $S_{\mu,\rho\mu}^{(s,t)}$ with 1, and of $\Delta\widetilde{z}_j^{(0,\mu\omega)}$ with $\Delta\widetilde{z}_j^{(\sigma,\mu\omega)}$.

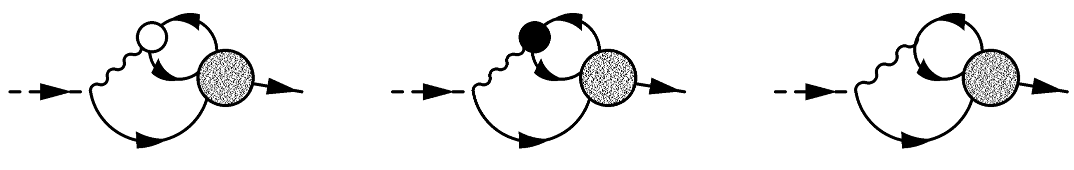

Fig 10: Graphical representation of item 5

6. The self-contraction of the fields $\widehat{\psi}^+_{q,\mu}$ and $\widehat{\psi}^-_{k-p,\omega}$ of the interactions $\mathcal{T}_0^{(\mu)}$, is non-vanishing for $\omega = \mu$ and $q = k - p$. The kernel is

$$\begin{aligned}
&\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\ \widehat{g}^{(s)}_\omega(k-p)\overline{\chi}_{h,N}(p)\frac{C^\delta_\omega(k-p,k)}{D_{-\omega}(p)}\\
&= \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\ \Bigg[\overline{\chi}_{h,N}(p)\frac{(\delta_{s,N}+\delta_{s,h})u_s(k-p)}{D_{-\omega}(p)D_\omega(k-p)}\\
&\qquad\qquad - \frac{f_s(k-p)}{D_\omega(k-p)}\overline{\chi}_{h,N}(p)\frac{D_\omega(k)\Big(1-(\chi^\delta_{h,N})^{-1}(k)\Big)}{D_{-\omega}(p)}\Bigg]\ ;
\end{aligned}$$

only the former addend has non-irrelevant part. Indeed, in the latter one, for $j \geq h+2$, because of $f_s(k-p)$, with $s \geq j$, and because of $\Big(1-(\chi^\delta_{h,N})^{-1}(k)\Big)$, which compels the momentum k to stay on scale h,

$$\begin{aligned}
&|D_{-\omega}(p)| \geq |D_{-\omega}(k-p)| - |D_{-\omega}(k)|\\
&\geq \gamma^{j-1} - \gamma^{h+1} \geq \left(1-\gamma^{-1}\right)\gamma^{j-1}\ ;
\end{aligned}$$

hence, as in item 2, there is a more factor $\gamma^{-(j-h)}$ in the bound of such a kernel, which gives it negative dimension down to scale h, where the field $\widehat{\psi}^-_{k,\mu}$ is compelled to be contracted by $\Big(1-(\chi^\delta_{h,N})^{-1}(k)\Big)$ in the limit $\delta \to 0$. Then, let the former addend be

$$\widehat{T}^{(s),(0)}_\omega(k) \stackrel{def}{=} \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\ \overline{\chi}_{h,N}(p)\frac{(\delta_{s,N}+\delta_{s,h})u_s(k-p)}{D_{-\omega}(p)D_\omega(k-p)}\ .$$

It is $\widehat{T}^{(s),(0)}_\omega(0) = 0$ by transformation under rotation; while

$$\left(\partial_\sigma \widehat{T}^{(s),(0)}_\omega\right)(0) \begin{cases} = 0 & \text{if } \sigma = -\omega \\ \stackrel{def}{=} \Delta\widetilde{z}^{(T,0)}_j & \text{if } \sigma = \omega\ . \end{cases}$$

Finally:

$$\begin{aligned}
&\mathcal{L}\left[\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\widehat{\beta}_{p,\omega}\widehat{\psi}^-_{p,\omega}\widehat{T}^{(s),(0)}_\omega(p)\right] = \Delta\widetilde{z}^{(T,0)}_j \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\ \widehat{\beta}_{p,\omega}D_\omega(p)\widehat{\psi}^-_{p,\omega}\ ,\\
&\mathcal{R}\left[\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\widehat{\beta}_{p,\omega}\widehat{\psi}^-_{p,\omega}\widehat{T}^{(s),(0)}_\omega(p)\right]\\
&= \sum_{\sigma,\sigma'}\widehat{\beta}_{p,\omega}\widehat{\psi}^-_{p,\rho}D_\sigma(p)D_{\sigma'}(p)\int_0^1 d\tau\ (1-\tau)\left(\partial^p_{\sigma'}\partial^p_{\sigma'}\widehat{T}^{(s),(0)}_\omega\right)(\tau p)\ .
\end{aligned}$$

7. The self-contraction of the fields $\widehat{\psi}^-_{k-p,\omega}$ and $\widehat{\psi}^+_{q,\sigma\mu}$ of the interaction $\mathcal{T}_\sigma^{(\mu)}$ is non-irrelevant. Setting:

$$\widehat{T}^{(s)}_\omega(k) \stackrel{def}{=} \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\ \widehat{g}^{(s)}_\omega(k-p)\overline{\chi}_{h,N}(p)\frac{D_\omega(p)}{D_{-\omega}(p)}\ ,$$

since $\widehat{T}^{(s),(\sigma)}_\omega(0)$, such a contraction requires the extraction of the coefficient:

$$\left(\partial^p_\nu \widehat{T}^{(s)}_\omega\right)(0)\begin{cases} =0 & \text{if } \omega\nu=-1 \\ \stackrel{def}{=}\Delta\widetilde{z}^{(T)}_j & \text{if } \omega\nu=1\ , \end{cases}$$

so that the above contraction is equal to

$$\mathcal{L}\left[\int\frac{\mathrm{d}^2k}{(2\pi)^2}\widehat{\beta}_{k,\omega}\widehat{\psi}^-_{k,\omega}\widehat{T}^{(s)}_\omega(k)\right]=\Delta\widetilde{z}^{(T)}_j\int\frac{\mathrm{d}^2k}{(2\pi)^2}\widehat{\beta}_{k,\omega}\widehat{\psi}^-_{k,\omega}D_\omega(k)\ ,$$
$$\mathcal{R}\left[\int\frac{\mathrm{d}^2k}{(2\pi)^2}\widehat{\beta}_{k,\omega}\widehat{\psi}^-_{k,\omega}\widehat{T}^{(s)}_\omega(k)\right]$$
$$=\int\frac{\mathrm{d}^2k}{(2\pi)^2}\sum_{\nu,\nu'}\widehat{\beta}_{k,\omega}\widehat{\psi}^-_{k,\omega}D_\nu(k)D_{\nu'}(k)\int_0^1 d\tau\ \left(\partial^k_\nu\partial^k_{\nu'}\widehat{T}^{(s)}_\omega\right)(\tau k)\ .$$

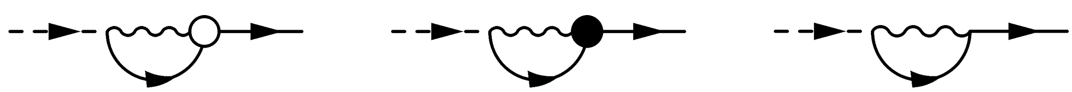

Fig 11: Graphical representation of items 6. and 7.

8. The self-contraction of the fields $\widehat{\psi}^+_{q,\mu}$ and $\widehat{\psi}^-_{p+q,\mu}$ of the interaction $\mathcal{T}^{(\mu)}_0$, or the fields $\widehat{\psi}^+_{q,\sigma\mu}$ and $\widehat{\psi}^-_{p+q,\sigma\mu}$ of the interactions $\{\mathcal{T}^{(\mu)}_\sigma\}_{\sigma=\pm}$, would give problems; but it arises only for $p=0$ and it is forbidden by the cutoff function $\overline{\chi}_{h,N}(p)$.
9. The contraction of one of or both the fields $\widehat{\psi}^+_{q,\mu}$ and $\widehat{\psi}^-_{p,\mu}$ was already discussed in the previous section, and give rise to the flow of $\{\nu^{(\sigma)}_j\}^{\sigma=\pm}_{j=h,\dots N}$.

Finally, the same above developments can be done for the contractions of the interactions $\mathcal{B}^{(3)}$: the localization containing the couplings $\widetilde{\lambda}^{(\mu\omega)}_j$ and $\widetilde{z}^{(\mu\omega)}_j$ are $\Delta\widetilde{\lambda}^{(\mu\omega)}_{j-1}$ and $\Delta\widetilde{z}^{(\mu\omega)}_{j-1}$; while the localization containing $\alpha^{(\mu\omega)}$ are exactly the same of the flows of λ_N and Z_N. Then:

$$\widetilde{\lambda}^{(\mu\omega)}_{j-1}\stackrel{def}{=}\left(\frac{Z_j}{Z_{j-1}}\right)^2\Bigg(\widetilde{\lambda}^{(\mu\omega)}_j+\Delta\widetilde{\lambda}^{(\mu\omega)}_{j-1}+\delta_{\omega\mu,-1}\sum_{a=\pm}\Delta\widetilde{l}^{(a,0)}_{j-1}$$
$$+\delta_{\omega\mu,-1}\sum_{\sigma,a=\pm}\frac{Z_N}{Z_j}\nu^{(\sigma)}_j\Delta\widetilde{l}^{(a,\sigma)}_{j-1}+\Delta\widetilde{l}^{(0,0,\mu\omega)}_{j-1}+\sum_{\sigma=\pm}\frac{Z_N}{Z_j}\nu^{(\sigma)}_j\Delta\widetilde{l}^{(0,\sigma,\mu\omega)}_{j-1}\Bigg)\ ,$$

$$\widetilde{z}^{(\mu\omega)}_{j-1}\stackrel{def}{=}\left(\Delta\widetilde{z}^{(\mu\omega)}_{j-1}+\Delta\widetilde{z}^{(0,\mu\omega)}_{j-1}+\delta_{\omega\mu,1}\Delta\widetilde{z}^{(T,0)}_{j-1}+\sum_{\sigma=\pm}\frac{Z_N}{Z_j}\nu^{(\sigma)}_j\left(\Delta\widetilde{z}^{(\sigma,\mu\omega)}_{j-1}+\delta_{\omega\mu,1}\Delta\widetilde{z}^{(T)}_{j-1}\right)\right)\ .$$

The remarkable point is that the following theorem holds.

Theorem 4.4. *For any fixed $\vartheta : 0<\vartheta<1$, there exist $\varepsilon>0$, a constant c and two counterterms $\{\alpha^{(\mu)}\}$, analytically dependent on λ, such that, for any fixed cutoff scale, N, and choosing*

$\alpha_N^{(\mu)} = \alpha^{(\mu)}$, *it holds:*

$$\left|\widetilde{\zeta}_j^{(3,\mu)}\right| \leq c\varepsilon\gamma^{-(\vartheta/2)(N-j)} \qquad \left|\widetilde{\zeta}_j^{(1,\mu)}\right| \leq c\varepsilon^2\gamma^{-(\vartheta/2)(N-j)} \ . \tag{4.2.9}$$

The proof is given in appendix A7. It is a simple application of the fixed point theorem in Banach spaces. Once two $\{\alpha_N^{(\mu)}\}_\mu$ are found with the required properties, it is simply to verify that they are actually independent from N.

Theorem 4.5. *In the same hypothesis of theorem 4.4 and choosing*

$$\sigma^{(\mu)} = -\sum_{k\leq N-1} \frac{Z_k}{Z_N}\widetilde{\zeta}_k^{(1,\mu)} \ ;$$

in the limit of removed cutoff, the following asymptotic identity

$$\begin{aligned} D_\omega(k)\frac{\partial e^{\mathcal{W}}}{\partial\widehat{\varphi}^+_{k,\omega}} &= \frac{B_N}{Z_N}\widehat{\varphi}^-_{k,\omega}e^{\mathcal{W}} \\ &- \lambda_N A_N \sum_\mu \frac{a_N - \overline{a}_N\omega\mu}{2}\int_D \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\ \frac{1}{D_{-\omega}(p)}\left[\frac{\partial e^{\mathcal{W}}}{\partial\widehat{\varphi}^+_{k-p,\omega}\partial\widehat{\varphi}^-_{q,\mu}}\widehat{\varphi}^-_{q+p,\mu}\right. \\ &\qquad\qquad\left. -\widehat{\varphi}^+_{q,\mu}\frac{\partial e^{\mathcal{W}}}{\partial\widehat{\varphi}^+_{q+p,\mu}\partial\widehat{\varphi}^+_{p-k,\omega}}\right] , \end{aligned} \tag{4.2.10}$$

generates the anomalous CE *of those Schwinger functions which have no density insertion and the addend relative to which generated by* $\int_D \frac{\mathrm{d}^2p}{(2\pi)^2}\ \overline{\chi}_h(p)\frac{\partial e^{\mathcal{W}^{(h)}}}{\partial\widehat{\jmath}_{p,-\omega}\partial\widehat{\varphi}^+_{k-p,\omega}}$ *is vanishing.*

The last requirement is fulfilled, as already stated, for the $S^{(2)}$ Schwinger function, see (4.2.7). A similar bound is valid also for $S^{(4)}$.

Theorem 4.6. *For* ε *small enough, for* $\vartheta : 0 < \vartheta < 1/16$, *and for any scale* $h \leq N$, *the effective coupling is almost constant:*

$$\lambda_h - \lambda_N = \mathrm{O}(\lambda^2) \ . \tag{4.2.11}$$

where $\mathrm{O}(\lambda^2)$ *is bounded uniformly in* h.

Proof of Theorem 4.5. The choice of $\sigma^{(\mu)}$ makes sense: by (4.2.9) and (3.3.14), for $c_0\varepsilon^2 \leq \vartheta/4$ it is finite:

$$\left|\sum_{k\leq N-1}\frac{Z_k}{Z_N}\widetilde{\zeta}_k^{(1,\mu)}\right| \leq c\varepsilon^2\left(1-\gamma^{-(\vartheta/4)}\right)^{-1} \ .$$

With reference to (4.2.5), the theorem is proved once it is shown the bound for the derivatives of $\mathcal{W}^{(h)}_{\mathcal{T},\mu}$ has a vanishing factor more than the bound of the derivatives of $\mathcal{W}^{(h)}$. Hence, let any integer $n \in \mathbb{N}$, any choice of the label $\underline{\varepsilon} \stackrel{def}{=} (\varepsilon_1, \ldots, \varepsilon_n)$ and $\underline{\omega} \stackrel{def}{=} (\omega, \omega_1, \ldots, \omega_n)$, and any momenta $\underline{k} \stackrel{def}{=} (k, k_1, \ldots, k_n)$ be considered. The CE equation for the Schwinger function $\widehat{S}^{(0;n+1)(\underline{\varepsilon})}_{\underline{\omega}}(\underline{q};\underline{k})$ is obtained by suitable derivatives of the above functional, plus the limit $-h, N \to \infty$ of the following rest:

$$\begin{aligned} &\frac{1}{Z_N}\left|\frac{\partial^{1+n}\mathcal{W}^{(h)}_{\mathcal{T},\mu}}{\partial\widehat{\beta}_{k,\omega}\partial\widehat{\varphi}^{\varepsilon_1}_{k_1,\omega_1}\ldots\partial\widehat{\varphi}^{\varepsilon_n}_{k_n,\omega_n}}\right|_{J\equiv\varphi\equiv 0} \\ &\quad\le \frac{C_{n;h_0,h_1\ldots,h_n}}{\sqrt{Z_{h_0}}\prod_{j=1}^n\sqrt{Z_{h_j}}}\left(\gamma^{-(\vartheta/4)(N-h_0)} + \gamma^{(\vartheta/4)(h_0-h)}\right)\,, \end{aligned} \tag{4.2.12}$$

where $\{h_j\}_{j=0}^n$ are the scales of the momenta $(k, \underline{k})$: $\kappa\gamma^{h_j-1} \le |k_j| \le \kappa\gamma^{h_j}$, with $k \stackrel{def}{=} k_0$; and $C_{n;h_0,h_1\ldots,h_n}/\prod_{j=0}^n\sqrt{Z_{h_j}}$ is the bound for the derivatives of $\mathcal{W}^{(h)}$. The bound derives from the following arguments. By the explicit choice of $\sigma^{(\mu)}$, and by (4.2.9), (3.3.14), for $c_0\varepsilon^2$ smaller than $\vartheta/4$, it holds:

$$\begin{aligned} &\left|\sum_{k=m}^N \frac{Z_k}{Z_m}\widetilde{\zeta}^{(1,\mu)}_k\right| = \left|\sum_{k\le m-1}\frac{Z_k}{Z_m}\widetilde{\zeta}^{(1,\mu)}_k\right| \\ &\le c\varepsilon^2\sum_{k\le m-1}\gamma^{c_0\varepsilon^2(m-k)}\gamma^{-(\vartheta/2)(N-k)} \le \widetilde{c}\varepsilon^2\gamma^{-(\vartheta/2)(N-m)}\,. \end{aligned} \tag{4.2.13}$$

for $\widetilde{c} \ge c\left(1-\gamma^{-(\vartheta/4)}\right)^{-1}$. Then, the graphs in the expansion of the r.h.s. member of (4.2.12) has one external propagator on scale h, and fall in one of the following classes.

1. An interaction $\mathcal{T}^{(\mu)}_0$ is contracted: there has to be one propagator on scale $m = h, N$. The factor $1/Z_N$ in the r.h.s. member of (4.2.12), times factors coming form the multiscale integration (see (4.2.8)) gives $(Z_N/Z_m)^2(\sqrt{Z_m}/Z_N) \le (Z_N/Z_m)(1/\sqrt{Z_{h_0}})\gamma^{\varepsilon^2 c_0|m-h_0|}$. And $Z_N/Z_m < 1$ since $\eta_\lambda < 0$; while $\gamma^{\varepsilon^2 c_0|m-h_0|}$ is transformed into $\gamma^{-(\vartheta/2)|m-h_0|}$ by a short memory factor.
2. An interaction $\mathcal{T}^{(\mu)}_\sigma$ is first contracted on scale m. The factor to be studied is now $(Z_N/Z_m)(\sqrt{Z_m}/Z_N)\left|\nu^{(\sigma)}_m\right| \le (1/\sqrt{Z_{h_0}})\gamma^{\varepsilon^2 c_0|h_0-m|}\gamma^{-(\vartheta/2)(N-h_0)}$; and, as in the previous item, extracting a short memory factor, $\gamma^{\varepsilon^2 c_0|h_0-m|}$ is turned into $\gamma^{-(\vartheta/2)|h_0-m|}$.
3. An interaction $\mathcal{B}^{(3)}$ first contracted on scale m. In this case the factor to be studied is $(\sqrt{Z_m}/Z_N)\left|\zeta^{(3,\omega\mu)}_m\right| \le (1/\sqrt{Z_{h_0}})\gamma^{\varepsilon^2 c_0(N-m)}\gamma^{\varepsilon^2 c_0|m-h_0|}\gamma^{-(\vartheta/2)(N-h_0)}$; then $\gamma^{\varepsilon^2 c_0|m-h_0|}$ is changed by the short memory factor into$\gamma^{-(\vartheta/2)|m-h_0|}$; while, for ε small, it holds $\gamma^{\varepsilon^2 c_0(N-m)}\gamma^{-(\vartheta/2)(N-h_0)} \le \gamma^{\varepsilon^2 c_0(N-h_0)}$.
4. The contraction of the interaction $\mathcal{B}^{(1)}$ can only occur in a scale compatible with the momentum k (hence two possible contiguous scales): let it be h_0. Then there is a factor $\sqrt{Z_{h_0}}/Z_N\left|\sum_{m=h_1}^N(Z_m/Z_{h_0})\widetilde{\zeta}^{(1)}_m\right| \le (1/Z_{h_0})\gamma^{-(\vartheta/4)(N-h_0)}$.

Besides the decay factor, in the first three items there is also $\gamma^{-(\vartheta/2)|m-h_0|}$, controlling the summation over m. This proves (4.2.12). Hence, keeping k fixed and non-zero, in the limit of removed cutoff, such derivatives are vanishing. ■

Proof of Theorem 4.6. Taking in (4.2.5) the derivatives $\partial\varphi^-_{k+q-s,\omega}\partial\varphi^+_{q,-\omega}\partial\varphi^-_{s,-\omega}$, for $\varphi\equiv 0$, it holds the following CE for $S^{(4)}$

$$
\begin{aligned}
&\frac{\widehat S^{(4)}_{\omega,-\omega}(k,q,s)}{\widehat g_\omega(k)}\\
&= -\lambda_N A_N \frac{a_N+\overline a_N}{2}\,\frac{\widehat S^{(2)}_{-\omega}(s)-\widehat S^{(2)}_{-\omega}(q)}{D_{-\omega}(s-q)}\widehat S^{(2)}_\omega(k+q-s)\\
&+\lambda_N A_N \frac{a_N-\overline a_N}{2}\int_D \frac{\mathrm d^2p}{(2\pi)^2}\,\frac{\overline\chi_{h,N}(p)}{D_{-\omega}(p)}\widehat S^{(4)}_{\omega,-\omega}(k-p,q,s)\\
&+\lambda_N A_N \frac{a_N+\overline a_N}{2}\int_D \frac{\mathrm d^2p}{(2\pi)^2}\,\frac{\overline\chi_{h,N}(p)}{D_{-\omega}(p)}\left[\widehat S^{(4)}_{\omega,-\omega}(k-p,q,s-p)-\widehat S^{(4)}_{\omega,-\omega}(k-p,q+p,s)\right]\\
&-\lambda_N A_N\sum_\mu \frac{a_N-\overline a_N\omega\mu}{2}\,\frac{1}{Z_N}\,\frac{\partial^4\mathcal W^{(\mu)}_{\mathcal T}}{\partial\widehat\beta_{k,\omega}\partial\widehat\varphi^-_{k+q-s,\omega}\partial\widehat\varphi^+_{q,-\omega}\partial\widehat\varphi^-_{s,-\omega}}\\
&-\frac{\lambda_N A_N}{\zeta^{(2)}_N}\int_D\frac{\mathrm d^2p}{(2\pi)^2}\,\overline\chi_h(p)\left[\widehat S^{(1;4)}_{-\omega;\omega,-\omega}(p;k-p,q,s)-\delta(q-s)\widehat S^{(2)}_{-\omega}(q)\widehat S^{(1;2)}_{-\omega;\omega}(p;k-p)\right]\ ,
\end{aligned}
\tag{4.2.14}
$$

where A_N was defined in (4.2.6). Now, fixing $-q=s=k=\overline k$, for any $\overline k:\kappa\gamma^h\le|\overline k|\le\kappa\gamma^{h+1}$, by lowest order computation it holds:

$$
\frac{\widehat S^{(4)}_{\omega,-\omega}(\overline k,-\overline k,\overline k)}{\widehat g_\omega(\overline k)}=\frac{1}{Z_h^2}\,\frac{\lambda_h+\mathrm O(\lambda^2)}{\overline k^2 D_{-\omega}(\overline k)}\ ,
$$

$$
\lambda_N A_N\frac{a_N+\overline a_N}{2}\,\frac{\widehat S^{(2)}_{-\omega}(\overline k)}{D_{-\omega}(\overline k)}\widehat S^{(2)}_\omega(\overline k)=\frac{1}{Z_h^2}\,\frac{-\lambda_N+\mathrm O(\lambda^2)}{\overline k^2 D_{-\omega}(\overline k)}\ ;
$$

while (see also [BM04] for more details)

$$
\left|\lambda_N A_N\frac{a_N-\overline a_N}{2}\int_D\frac{\mathrm d^2p}{(2\pi)^2}\,\frac{\overline\chi_{h,N}(p)}{D_{-\omega}(p)}\widehat S^{(4)}_{\omega,-\omega}(\overline k-p,-\overline k,\overline k)\right|\le\frac{\gamma^{-3h}}{Z_h^2}\mathrm O(\lambda^2)\ ,
$$

and identical bound for

$$
\left|\lambda_N A_N\frac{a_N-\overline a_N}{2}\int\frac{\mathrm d^2p}{(2\pi)^2}\,\frac{\overline\chi_{h,N}(p)}{D_{-\omega}(p)}\left[\widehat S^{(4)}_{\omega,-\omega}(\overline k-p,-\overline k,\overline k-p)-\widehat S^{(4)}_{\omega,-\omega}(\overline k-p,p-\overline k,\overline k)\right]\right|\ ,
$$

and

$$
\left|\frac{\lambda_N A_N}{\zeta^{(2)}_N}\int_D\frac{\mathrm d^2p}{(2\pi)^2}\,\overline\chi_h(p)\widehat S^{(1;4)}_{-\omega;\omega,-\omega}(p;\overline k-p,-\overline k,\overline k)\right|\ .
$$

Finally, by the study of the flow of $\mathcal W_{\mathcal T}$, it also hold

$$
\left|\lambda_N A_N\sum_\mu\frac{a_N-\overline a_N\omega\mu}{2}\,\frac{1}{Z_N}\,\frac{\partial^4\mathcal W^{(\mu)}_{\mathcal T}}{\partial\widehat\beta_{\overline k,\omega}\partial\widehat\varphi^-_{-\overline k,\omega}\partial\widehat\varphi^+_{-\overline k,-\omega}\partial\widehat\varphi^-_{\overline k,-\omega}}\right|\le\frac{\gamma^{-3h}}{Z_h^2}\mathrm O(\lambda^2)
$$

(namely, in this case, since k is on the infrared cutoff scale, the rest is not vanishing; but it diverges in $h \to -\infty$ with the same exponent, $3-2\eta_\lambda$, of the other terms in (4.2.14)). Considering together the above bound with (4.2.14), it holds the theorem. ■

4.2.5 Vanishing of the Beta function. In the end, it is remarkable how (4.2.11) is read in terms of the *Beta function* for the effective couplings. In agreement with (A5.2), the Beta function for the massless Thirring model, in Euclidean regularization is such that

$$\lambda_{h-1} - \lambda_h \stackrel{def}{=} \beta_h^{(\mathrm{T})}(\lambda_h) + \sum_{m=h}^{N} \beta_{h,m}^{(\mathrm{T},\lambda)}(\lambda_m - \lambda_h) \tag{4.2.15}$$

(see A5.2 for the explanation of the addends). As done for the anomalous exponent, by scaling invariance of the graphs in the expansion of $\{\beta_h^{(\mathrm{T})}\}_h$, it is possible to prove that there exist a real function B such that

$$|\beta_h^{(\mathrm{T})}(\lambda_h) - B(\lambda_h)| \le c\varepsilon^2 \gamma^{-\vartheta(N-h)} \ . \tag{4.2.16}$$

Well then, as consequence of (4.2.11), $B \equiv 0$. Otherwise, if the coefficient of the m-th order expansion of $B(\lambda)$, $B^{(m)}$, where non-zero, then replacing the expansion $\lambda_h \stackrel{def}{=} \sum_{n>0} c_h^{(n)} \lambda^n$ in (4.2.16), it would be possible to prove – by an iterative procedure similar to the one in A5.2 – that for any h, and for any $n < m$:

$$\left| \sum_{m=h}^{N} \beta_{h,m}^{(\mathrm{T},\lambda)(n)}(\lambda_m - \lambda_h) \right| \le C^n \gamma^{-(\vartheta/2)(N-h)} \ , \qquad \left| c_{h-1}^{(n)} - c_h^{(n)} \right| \le C^n \gamma^{-(\vartheta/2)(N-h)} \ ;$$

while, for $n = m$,

$$c_{h-1}^{(m)} = c_h^{(m)} + B^{(m)} + \mathrm{O}(\gamma^{-(\vartheta/2)(N-h)}) \ .$$

Therefore $\{c_h^{(m)}\}_{h\le N}$ would be a diverging sequence, in contradiction with (4.2.11).

4.3 Solution of the closed equation

With simple symmetry considerations and multiscale integration, it possible to prove the following general expression for the two point Schwinger function:

$$\widehat{S}_\omega^{(2)}(k) = \frac{1}{D_\omega(k)} \left(\frac{|k|}{\kappa}\right)^{\eta_\lambda} F_{h,N}\left(\frac{|k|}{\kappa}\right) \ , \tag{4.3.1}$$

where $F_{h,N}$ is finite, uniformly in h, N, and such that, for a suitable real constant F,

$$\sup_{\gamma^{(h/2)}\kappa \le |p| \le \gamma^{(N/2)}\kappa} \left| F_{h,N}\left(\frac{|p|}{\kappa}\right) - F \right| = C\left(\gamma^{-(\vartheta/4)N} + \gamma^{(\vartheta/4)h}\right) \ . \tag{4.3.2}$$

Indeed, once the factor $1/(D_\omega(k)Z_{h_0})$ is extracted (with h_0 the scale of k), the expansion of $\widehat{S}^{(2)}_{h,N;\omega}(k)$ is given by scaling invariant graphs. Calling F the limit of $F_{h,N}$, with all the couplings $\{\lambda_j\}_j$ replaced by λ, all the ratios $\{Z_{j-1}/Z_j\}_j$ replaced by γ^{η_λ} and the factor $(|k|/\kappa)^{\eta_\lambda}(1/Z_{h_0})$ with 1, the difference between $F_{h,N}$ and F is the sum of all the graphs with an external propagator on scale h_0 and falling in one of the following cases.

1. There is an interaction on scale $m > N$ or $m < h$. By the short memory property, given any $\vartheta : 0 < \vartheta < 1/16$, the sum of all such graphs is bounded with $\gamma^{-\vartheta(N-h_0)} + \gamma^{-\vartheta(h_0-h)}$, up to a constant.
2. There is a coupling $[(|k|/\kappa)^{\eta_\lambda}(1/Z_{h_0}) - 1]$. By the feature of the flow of the field strength – namely the analogous for the Euclidean regularization of (3.4.2) – the sum of all such graphs is bounded with $\gamma^{-(\vartheta/2)(N-h_0)}$, up to a constant.
3. There is an interaction $\lambda_m - \lambda$, or $(Z_{m-1}/Z_m) - \gamma^{\eta_\lambda}$ on scale $m : h \le m \le N$. By the short memory factor an features of the flows – analogous for the Euclidean regularization of (3.4.1) and (3.4.2) – the sum of such graphs is bounded by $\gamma^{-\vartheta|m-h_0|}\,\gamma^{-(\vartheta/2)(N-m)} \le \gamma^{-(\vartheta/2)(N-h_0)}\,\gamma^{-(\vartheta/2)|m-h_0|}$, up to a constant.

Hence, after summing over m, (4.3.2) holds.

Now, replacing (4.3.1) in the CE for the two point Schwinger function, and taking the limit $h \to -\infty$, it holds:

$$\left|\frac{k}{\kappa}\right|^{\eta_\lambda} F_N(k) = \frac{B_N}{Z_N} - \lambda_N A_N \frac{a_N - \overline{a}_N}{2} \int_D \frac{d^2p}{(2\pi)^2} \left|\frac{p}{\kappa}\right|^{\eta_\lambda} \frac{F_N(p)}{D_{-\omega}(k-p)D_\omega(p)} \\ + \Delta\widehat{K}_{N,\omega}(k) \,,$$

where, by (4.2.12),

$$\sup_{|k|\le\gamma^{(N/2)}\kappa} \left|\Delta\widehat{K}_{N,\omega}(k)\right| \le \frac{C}{Z_{h_0}}\gamma^{-(\vartheta/8)N} \,.$$

The equation for $k = 0$ – then $Z_{h_0} = +\infty$ – gives

$$\frac{B_N}{Z_N} = \lambda_N A_N \frac{a_N - \overline{a}_N}{2} \int_D \frac{d^2p}{(2\pi)^2} \left|\frac{p}{\kappa}\right|^{\eta_\lambda} \frac{F_N(p)}{p^2} \;;$$

therefore:

$$\left|\frac{k}{\kappa}\right|^{\eta_\lambda} F_{h,N}(k) = \lambda_N A_N \frac{a_N - \overline{a}_N}{2} \int_D \frac{d^2p}{(2\pi)^2} \left|\frac{p}{\kappa}\right|^{\eta_\lambda} F_N(p) \frac{k^2 + D_{-\omega}(p)D_\omega(k)}{(k-p)^2p^2} \\ + \Delta\widehat{K}_{N;\omega}(k) \,.$$

Now it is possible to take the limit $N \to +\infty$, for k fixed: since the rest is vanishing, by finiteness of F_N uniformly in N and by (4.3.2), the limit can be exchanged with the integral in d^2p, it holds:

$$|k|^{\eta_\lambda} = \lambda_b A \frac{a - \overline{a}}{2} \int \frac{d^2p}{(2\pi)^2} \, |p|^{\eta_\lambda} \, \frac{k^2 + D_{-\omega}(p)D_\omega(k)}{(k-p)^2p^2} \,.$$

The integral can be elementarily computed: the pure imaginary part is zero by symmetries, while for the real one it holds, for ϑ the angle between the vector p and the vector k, for $t \stackrel{def}{=} \tan(\vartheta/2)$, and calling, with abuse of notation, k and p the moduli of the vectors k and p themselves,

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}p\ p^{\eta_\lambda-1}\int_{-\pi}^{\pi}\mathrm{d}\vartheta\ \frac{k^2-pk\cos(\vartheta)}{k^2+p^2-2pk\cos(\vartheta)}\\
=&\frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}p\ p^{\eta_\lambda-1}\int_{-\infty}^{\infty}\mathrm{d}t\ \frac{2k}{1+t^2}\frac{\big(k+p\big)t^2+\big(k-p\big)}{\big(k+p\big)^2t^2+\big(k-p\big)^2}\\
=&\frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}p\ p^{\eta_\lambda-1}\int_{-\infty}^{\infty}\mathrm{d}t\ \frac{2k}{k+p}\left[\frac{1}{1+t^2}-\frac{k-p}{2k}\left(\frac{1}{1+t^2}-\frac{(k+p)^2}{(k+p)^2t^2+(k-p)^2}\right)\right]\\
=&\frac{2}{(2\pi)^2}\int_0^k \mathrm{d}p\ p^{\eta_\lambda-1}\int_{-\infty}^{\infty}\mathrm{d}t\ \frac{1}{1+t^2}=\frac{1}{2\pi\eta_\lambda}k^{\eta_\lambda}\ .
\end{aligned}$$

This gives the following expression for the critical index η_λ:

$$\eta_\lambda = A\frac{\lambda_b}{2\pi}\frac{a-\overline{a}}{2}\ ,$$

to be compared with the formula for the half value of η_λ given in [J61] just after (36) – with the following identification: Johnson's α is here $\eta_\lambda/2$; Johnson's λ is $\lambda_b/2$; while $a-\overline{a}$ is, according to Johnson, equal to $2\frac{\lambda/2\pi}{1-(\lambda/2\pi)^2}$.

Appendix 1:

Simple Analytical Properties

A1.1 Partial-fraction expansion. The functions

$$f_L^-(z) \overset{def}{=} \frac{e^{-(x_0+L)z}}{1+e^{-Lz}} \quad \text{for } -L < x_0 < 0\,, \qquad f_L^+(z) \overset{def}{=} \frac{e^{-x_0 z}}{1+e^{-Lz}} \quad \text{for } 0 < x_0 < L\,,$$

are both meromorphic, since in any circles, $\mathcal{C}_R$, of radius R and centre the origin, their only singularities are a finite number of poles. In particular, setting $D_0 \overset{def}{=} \left\{\frac{2\pi}{L}(m+\frac{1}{2})\right\}_{m\in\mathbb{Z}}$, they are on the imaginary axis, in $\{ik_0 : k_0 \in D_0\}$. Therefore, by the Cauchy theorem, for any $e \in \mathbb{R}$, $R > |e|$ and $\sigma = \pm$,

$$f_L^\sigma(e) = \oint_{\mathcal{C}_R} \frac{\mathrm{d}z}{2\pi i}\, \frac{f_L^\sigma(z)}{z-e} + \sigma \frac{1}{L} \sum_{k_0\in D_0}^{|k_0|\le R} \frac{e^{-ix_0k_0}}{-ik_0+e}\,. \tag{A1.1}$$

Since, for $0 \le \vartheta \le \pi/2$, $\cos\vartheta \ge 1 - 2\vartheta/\pi$, then it holds the following bound:

$$\left|\oint_{\mathcal{C}_R} \frac{\mathrm{d}z}{2\pi i}\, \frac{f_L^+(z)}{z-e}\right| \le \frac{2R}{R-|e|} \int_0^\pi \mathrm{d}\vartheta \frac{e^{-x_0 R\cos\vartheta}}{1+e^{-LR\cos\vartheta}} \le \frac{2}{R-|e|}\left[\frac{\pi}{2x_0} + \frac{\pi}{2(L-x_0)}\right]\,,$$

and similarly for f_L^-. Hence the first addend in the r.h.s. member of (A1.1) vanish for $R \to \infty$, and than, for any $x_0 \neq 0 : |x_0| < L$,

$$f_L^+(e)\chi(x_0 > 0) - f_L^-(e)\chi(x_0 < 0) = \lim_{R\to+\infty} \frac{1}{L} \sum_{k_0\in D_0}^{|k_0|\le R} \frac{e^{-ix_0k_0}}{-ik_0+e}\,.$$

Such a series, not absolutely convergent, can be written as $\sin^{-1}(\pi x_0/L)$, times an absolutely convergent series – and border terms vanishing for large R – so that it is clear the possibility of replacing the sharp constraint $|k_0| \le R$ with a smooth cutoff function.

A1.2 Gevrey compact-support functions It is easy to construct a compact support-function which also fulfil the Gevrey constraint on the derivatives.

Indeed, let the following C^∞ function be considered for any number $p > 0$:

$$\vartheta(t) \overset{def}{=} \begin{cases} 0 & \text{for } t < 0 \\ e^{1-(1/t^p)} & \text{for } 0 \le t \le 1 \\ 1 & \text{for } t > 1 \, . \end{cases}$$

For $t \le 0$ and $t \ge 1$ all the derivatives are identically zero. For $t : 0 < t < 1$, it is possible to find a bound for the derivatives using the analyticity of $\vartheta(t)$ in the half-plane $\mathbb{C}_+ \overset{def}{=} \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. For any $t : 0 < t < 1$, let the disc $D_t \overset{def}{=} \{z \in \mathbb{C} : |z - t| \le t \sin(\pi/4p)\}$ be considered. By the Cauchy theorem:

$$|\vartheta^{(n)}(t)| \le \frac{n!}{2\pi \Big(t \sin(\pi/4p) \Big)^n} \max_{z \in D_t} |\vartheta(z)| \, .$$

For any $z \overset{def}{=} re^{i\varphi} \in D_t$, since the lines passing through $z = 0$ and tangent to D_t have angular parameter $\pm\pi/4p$, then $\mathrm{Re}\,(z^{-p}) \ge r^{-p} \cos(\varphi p) \ge (2t)^{-p} \cos(\pi/4)$. Hence, since for any $x \ge 0$, and any constant $c > 0$, it holds $x^n e^{-cx^p} \le C^n (n!)^{(1/p)}$, then for a certain constant $C > 1$,

$$|\vartheta^{(n)}(t)| \le C^n (n!)^{1+(1/p)} \; ;$$

namely $\vartheta(t)$ is a Gevrey function of order $\alpha = 1 + (1/p)$. Finally, if $\widehat{\chi}_0(t) \overset{def}{=} 1 - \vartheta\left(\frac{t-1}{\gamma-1}\right)$, then $\widehat{f}_j(t) \overset{def}{=} \widehat{\chi}_0(t\gamma^{-j}) - \widehat{\chi}_0(t\gamma^{-j+1})$ is a compact-support Gevrey function for any integer j.

A1.3 Bounds for the propagators. If K is the compact support of $f_0(k)$, the n-th derivatives of $1/D_\omega(k)$ are bounded in K by $C_K c_K^n n!$, for suitable K-dependent constants C_K and c_K. Therefore, by Leibniz formula it follows that it $f(k)$ is a Gevrey, compact-support function of class $\alpha \ge 1$, also $f(k)/D_\omega(k)$ is. Therefore, for any $n_0, n_1 \in \mathbb{N}$, by partial derivation and Stirling formula,

$$\begin{aligned} |g_\omega^{(0)}(x)| &\le \frac{1}{|x_0|^{n_0} |x_1|^{n_1}} \sup_{k \in K} \left| \partial_0^{n_0} \partial_1^{n_1} \frac{f_0(k)}{D_\omega(k)} \right| \\ &\le C \left(\left| \frac{c}{x_0} \right|^{1/\alpha} \frac{n_0}{e} \right)^{\alpha n_0} \left(\left| \frac{c}{x_1} \right|^{1/\alpha} \frac{n_1}{e} \right)^{\alpha n_1} \, . \end{aligned}$$

Therefore, choosing for n_j such that $(|x_j|/c)^{1/\alpha} - 1 \le n_j \le (|x_j|/c)^{1/\alpha}$, it holds:

$$|g_\omega^{(0)}(x)| \le C e^{-\alpha(|x_0|/c)^{1/\alpha}} e^{-\alpha(|x_1|/c)^{1/\alpha}} \, .$$

Finally, with similar argument, it is possible to obtain the same bounds for lattice-spacetime propagators.

Appendix 2:

OS axioms

A2.1 Test functions. For any $n \in \mathbb{N}$, setting $\underline{x} \stackrel{def}{=} (x^{(1)}, \ldots, x^{(n)})$, let $\mathcal{S}\big(\mathbb{R}^{2n}\big)$ be the space of the complex functions on $\mathbb{R}^{2n}$, with labels, $\underline{\omega} \stackrel{def}{=} (\omega_1, \ldots, \omega_n)$, $\underline{\varepsilon} \stackrel{def}{=} (\varepsilon_1, \ldots, \varepsilon_n)$, s.t., for any integer m, and any $f^{(\underline{\varepsilon})}_{n,\underline{\omega}}(\underline{x}) \in \mathcal{S}\big(\mathbb{R}^{2n}\big)$, the Schwartz norm

$$||f^{(\underline{\varepsilon})}_{n,\underline{\omega}}||_m \stackrel{def}{=} \max_{\underline{r}:\sum_j r_j \le m} \sup_{x^{(j)} \in \mathbb{R}^4} \left| \left(1 + \sum_{i=1}^{n} |x^{(i)}|^m\right) \partial_1^{r_1} \cdots \partial_n^{r_n} f^{(\underline{\varepsilon})}_{n,\underline{\omega}}(\underline{x}) \right|$$

is finite. Let $\mathcal{S}_{\neq}\big(\mathbb{R}^{2n}\big)$ be the space of the functions in $\mathcal{S}\big(\mathbb{R}^{2n}\big)$ which vanish, together with all their partial derivatives, if $x^{(i)} = x^{(j)}$ for some $1 \le i < j \le n$; and let $\mathcal{S}_{<}\big(\mathbb{R}^{2n}\big)$ be the space of the functions in $\mathcal{S}_{\neq}\big(\mathbb{R}^{2n}\big)$ which vanish, together with all their partial derivatives, if the ordering of the times $x_0^{(1)}, \ldots, x_0^{(n)}$ is different from $0 < x_0^{(1)} < x_0^{(2)} < \ldots < x_0^{(n)}$.

Let the "space translation", τ_y, for $y = (0, y_1)$, be defined as

$$\big(\tau_y f\big)^{(\underline{\varepsilon})}_{n,\underline{\omega}}(\underline{x}) \stackrel{def}{=} f^{(\underline{\varepsilon})}_{n,\underline{\omega}}(\tau_y \underline{x}) \ ,$$

with $\tau_y \underline{x} \stackrel{def}{=} (x^{(1)} + y, \ldots, x^{(n)} + y)$.

Let the "time reflection" be defined as

$$(\Theta f)^{(\underline{\varepsilon})}_{n,\underline{\omega}}\,(\underline{x}) \stackrel{def}{=} \left(f^{(\underline{\varepsilon}^*)}_{n,\underline{\omega}^*}\right)^* (\vartheta_0 \underline{x}) \ ,$$

with $\vartheta_0 \underline{x} \stackrel{def}{=} (\vartheta_0 x^{(1)}, \ldots, \vartheta_0 x^{(n)})$, where $\vartheta_0(x_0, x_1) \stackrel{def}{=} (-x_0, x_1)$; $f^*(x^{(1)}, \ldots, x^{(n)})$ is the complex conjugate of $f(x^{(n)}, \ldots, x^{(1)})$; and the labels $\underline{\omega}^*$ and $\underline{\varepsilon}^*$ are defined respectively to be $\omega_n, \ldots, \omega_1$ and $-\varepsilon_n, \ldots, -\varepsilon_1$ (see [OS72], formula (6.2)).

In the end, it has to be noticed the following fact: for $\mathcal{W}$ being the generating functional of the Schwinger functions, then $e^{\mathcal{W}}$ is the generating functional of the correlations. Hence, each Schwinger function – also called "truncated correlation" – can be written as finite linear combination of correlations, in term of which the OSA are now listed – with the simplification in the notation that $G^{(0,n)(\underline{\varepsilon})}_{\underline{\sigma};\underline{\omega}}(\underline{z},\underline{x}) \stackrel{def}{=} G^{(n)(\underline{\varepsilon})}_{\underline{\omega}}(\underline{x})$.

Lemma A.2.1. *Given ε small enough, for any $\lambda : |\lambda| < \varepsilon$ and $\mu : 0 \le \mu \le \kappa\gamma^{-1}$, the correlations satisfy the Osterwalder-Schrader axioms:*

E1. $G^{(n)(\underline{\varepsilon})}_{\underline{\omega}}(\underline{x})$ *is a distribution on $\mathcal{S}_<(\mathbb{R}^{(2n)})$; indeed, for any integer m, there exist two constants $c_m, C_m > 0$ s.t.*

$$\left\|G^{(n)(\underline{\varepsilon})}_{\underline{\omega}}\right\|_m \stackrel{def}{=} \sup_{f\in\mathcal{S}_<(\mathbb{R}^{(2n)})} \frac{\left(G^{(n)(\underline{\varepsilon})}_{\underline{\omega}}, f\right)}{\|f\|_m} \le C_m (n!)^{c_m} .$$

E2. $G^{(n)(\underline{\varepsilon})}_{\underline{\omega}}$ *is invariant under the Euclidean group of translation and rotation of all the coordinates.*

E3. $G^{(n)(\underline{\varepsilon})}_{\underline{\omega}}$ *is antisymmetric under the exchange of the $x^{(i)}, \omega_i, \varepsilon_i$ respectively with $x^{(j)}, \omega_j, \varepsilon_j$, for any $1 \le i < j \le n$.*

E4. *For any finite sequence of "time ordered" test functions, $\left\{f^{(\underline{\varepsilon})}_{n,\underline{\omega}}(\underline{x}) \in \mathcal{S}_<(\mathbb{R}^{(2n)})\right\}_{n\ge 0,\underline{\omega},\underline{\varepsilon}}$, the correlations are "reflection invariant":*

$$G^{(n)(\underline{\varepsilon})}_{\underline{\omega}}\left((\Theta f)^{(\underline{\varepsilon})}_{n,\underline{\omega}}\right) = G^{(n)(\underline{\varepsilon})}_{\underline{\omega}}(f^{(\underline{\varepsilon})}_{n,\underline{\omega}})$$

and "reflection positive":

$$\sum_{m,\underline{\omega}',\underline{\varepsilon}'} \sum_{n,\underline{\omega},\underline{\varepsilon}} G^{(m+n)(\underline{\varepsilon}',\underline{\varepsilon})}_{\underline{\omega}',\underline{\omega}}\left((\Theta f)^{(\underline{\varepsilon}')}_{m,\underline{\omega}'} \otimes f^{(\underline{\varepsilon})}_{n,\underline{\omega}}\right) \ge 0 . \qquad (A2.1)$$

E5. *For any $f^{(\underline{\varepsilon})}_{n,\underline{\omega}} \in \mathcal{S}_<(\mathbb{R}^{(2n)})$ and $g^{(\underline{\varepsilon}')}_{m,\underline{\omega}'} \in \mathcal{S}_<(\mathbb{R}^{(2m)})$, decorrelation holds:*

$$\lim_{|y|\to\infty} G^{(m+n)(\underline{\varepsilon}',\underline{\varepsilon})}_{\underline{\omega}',\underline{\omega}}\left((\Theta g)^{(\underline{\varepsilon}')}_{m,\underline{\omega}'} \otimes (\tau_y f)^{(\underline{\varepsilon})}_{n,\underline{\omega}}\right) = G^{(m)(\underline{\varepsilon}')}_{\underline{\omega}'}\left((\Theta g)^{(\underline{\varepsilon}')}_{m,\underline{\omega}'}\right) G^{(n)(\underline{\varepsilon})}_{\underline{\omega}}\left(f^{(\underline{\varepsilon}')}_{n,\underline{\omega}}\right) .$$

The last property, called *cluster decomposition*, in terms of the Schwinger function reads:

$$\lim_{|y|\to\infty} S^{(m+n)(\underline{\varepsilon}',\underline{\varepsilon})}_{\underline{\omega}',\underline{\omega}}\left((\Theta g)^{(\underline{\varepsilon}')}_{m,\underline{\omega}'} \otimes (\tau_y f)^{(\underline{\varepsilon})}_{n,\underline{\omega}}\right) = 0 . \qquad (A2.2)$$

From the OSA, it is possible to derive the theory in Minkowskian spacetime, from the Euclidean one. The main difficulty, here, is to prove the validity of E2 and E4: a regularization that makes clear the one, usually makes obscure the other.

A2.2 Reflection Positivity for the Hamiltonian Regularization

The Euclidean fields operator in Heisemberg picture are:

$$\psi^{\sigma}_{x,\omega} \stackrel{def}{=} e^{-x_0 H} \left(\frac{1}{L} \sum_{k \in D} e^{\sigma i k x_1} a^{\sigma}_{k,\omega} \right) e^{x_0 H} \ , \qquad x \stackrel{def}{=} (x_0, x_1) \in \mathbb{R} \times \mathbb{T} \ ;$$

therefore $\psi^{\sigma}_{x,\omega}$ is *not* the Hermitian conjugate of $\psi^{-\sigma}_{x,\omega}$ – as it were in the Minkowskian picture: it is therefore suitable to define the operator ϑ "time reflection" s.t. $\vartheta x = (-x_0, x_1)$, so that $\psi^{\sigma}_{x,\omega}$ is the Hermitian of $\psi^{-\sigma}_{\vartheta x,\omega}$.

Let now the space $\mathcal{F}$ of the linear functionals of the operator-valued fields: namely the operators on the Fock space of the form:

$$F(\psi) = \sum_{n \geq 0} \sum_{\underline{\omega},\underline{\sigma}} \int d^2x^{(1)} \cdots d^2x^{(n)} \ f_{n,\underline{\omega},\underline{\sigma}} \left(x^{(1)}, \ \ldots, x^{(n)} \right) \psi^{\sigma_1}_{x^{(1)},\omega_1} \cdots \psi^{\sigma_n}_{x^{(n)},\omega_n}$$

for any choice of the test functions $f_{\underline{\omega},\underline{\sigma}} \in \mathcal{S}_{<}\Big((\mathbb{R} \times \mathbb{T})^n \Big)$. Then, it is simply to verify that Θ on the space $\mathcal{F}$ is the Hermitian conjugation. Hence, for any real L, the following quantity is non-negative:

$$\mathrm{Tr} \left[e^{-LH} (\Theta F) F \right] \geq 0 \ .$$

Such an inequality, by the definition of the correlations, reads as in (A2.1).

Appendix 3:

Tree Expansion and Convergence of the Schwinger functions

The renormalization procedure used here is slightly different from the classical one, the BPHZ scheme.

As noticed in the early works on the renormalization, the localization is necessary and effective in extracting the divergent contribution of the subgraphs whenever the momenta flowing in the internal propagators of the subgraphs are in some sense higher than the momenta flowing on the external ones (*Hepp's sectors*). Anyway, the localization has a further complication in the massless case: while it improves the convergence at large momenta, it worsen consequently the convergence at small ones.

Accordingly, in the BPHZ scheme, the propagators of the graphs are decomposed *a posteriori* in scales, and the subgraphs, selected by the Hepp procedure, are localized: this is done by extracting the first orders of the Taylor expansion around zero external momenta, if the theory is massive; around any fixed non-zero value, if the theory is massless: in the latter case some discrete symmetries are broken, and more "relevant" and "marginal" terms, even a mass term, are generated.

In the scheme here depicted, instead, the multiscale integration not only produces directly only subgraphs satisfying the Hepp's property; but it makes clear the possibility of localizing at zero external momenta *even the subgraphs with massless propagators*, since such a localization is naturally stopped below the scales of the momenta of the Schwinger function at hand.

A3.1 Tree structure. By expanding iteratively the truncated expectations (3.1.6) and (3.3.6), starting from $\mathcal{W}^{(M)}$, it is possible to write the effective potential on scale $\mathcal{W}^{(h)}$, for $h \leq M$, in terms of a *tree expansion*, quite similar to that described, for example, in [BGPS].

1. Let a tree, τ, be a tree graph with the following features: if there are $n+1$ points with incidence number equal to 1, one of such points is the *root*; the other n points are the *endpoints*; the integer n is the *order* of the tree. All the points of the tree graphs, except the root and the endpoint, are called *nodes*. The only node paired to the root by the tree graph is the *first node*: it is required not to be an endpoint.
2. The nodes, the root and the endpoints are partially ordered in the natural way by the tree structure, so that the root is lower than the endpoints: $v < v'$ means v is lower that v'. In correspondence of any node v, the integer s_v is the number of minimal nodes or endpoints greater than v: such nodes or endpoints are also said to be *first followers of* v, and are denoted $v_1, \ldots, v_{s_v}$. If $s_v > 1$, then v is a *branching node*. In correspondence of a node or an end point v, the unique maximal node lower than it is the *first preceding of* v, and is denoted v'.
3. Let the *topological trees* be the quotient set of the above depicted trees, in which any two of them are identified if, by a suitable continuous deformation of the length of the links and of the angled between them, – included permutation of the links coming out of the same branching node – they can be superposed. It is then easy to verify that, since the number of the branching nodes of a tree with n endpoints is not larger than $n-1$, then the number of all the topological tree with n endpoints is bounded by $4^{2n-1} < 16^n$.
4. With each node v of the tree, a scale $h_v : h \leq h_v \leq M$ is assigned, with the compatibility condition that $v' < v$ imply $h_{v'} < h_v$: therefore it is possible to draw the trees as lying vertically along a family of horizontal parallel lines, each one marking a scale $j : h-1 \leq j \leq M+1$, so that the each node v is contained in the horizontal line with index h_v. The scale h_u of the endpoint u ranges from $h+1$ to $M+1$; if v is the first preceding of such an endpoint, $h_u = h_v + 1$. The scale of the first node is h: because of the distinction that will be done between the nodes in correspondence of the hard fermion regime and the soft fermion regime, h is allowed to be $\leq N+1$; the scale of the root is $h_r = h-1$.
5. There are two kinds of endpoints, *normal* and *special*. With each normal endpoint u, it is associated one of the three self-interactions $\lambda_{h_u-1}\mathcal{V}$, $\gamma^{h_u-1}\nu_{h_u-1}\mathcal{N}$ or $\delta_{h_v-1}\mathcal{D}$, if $h_u - 1 \leq N$; otherwise the interactions $\lambda_N\mathcal{V}$, $\gamma^N\nu_N\mathcal{N}$ or $\delta_N\mathcal{D}$. They are called the endpoints of type λ, ν, δ, with an obvious correspondence. With each special endpoint u it is associated one of the three interactions with the external sources, $\zeta^{(2,+)}_{h_u-1}\mathcal{J}_+$, $\zeta^{(2,-)}_{h_u-1}\mathcal{J}_-$ or $\mathcal{F}$, if $h_u - 1 \leq N$; otherwise the interactions $\zeta^{(2,+)}_N\mathcal{J}_+$, $\zeta^{(2,-)}_N\mathcal{J}_-$ or $\mathcal{F}$. They are called the endpoints of type φ, $\jmath_+$ and $\jmath_-$. The endpoints of type $\jmath$ are the union of the ones of type $\jmath_+$ and $\jmath_-$.
6. Given a node v, n^φ_v and $n^\jmath_v$ are respectively the number of endpoints of type φ, and of type $\jmath$ greater than v; $n^{(4)}_v, n^{(2)}_v$ are respectively the number of normal endpoint of type λ and of type ν or δ greater than v; $n_v \stackrel{def}{=} n^{(4)}_v + n^{(2)}_v$. Analogously, given a tree τ, the integers $n^\varphi_\tau, n^\jmath_\tau, n^{(4)}_\tau, n^{(2)}_\tau$ and n_τ are respectively the number of endpoints of type φ, of type $\jmath$, of type λ, of type ν or δ and the total number of normal endpoints of the tree.

7. For any node v, the *cluster* L_v with frequency h_v is the set of endpoints greater than the node v; if v is an endpoint, it is itself a (*trivial*) cluster. The tree provides an organization of endpoints into a hierarchy of clusters: $L_w < L_v$ if $L_w \subset L_v$

8. A *field label* f distinguishes a field involved in the interactions. If v is an endpoint, I_v is the the set of all the fields ψ, φ and $\jmath$ involved in the interaction in v. If v is a node, I_v is defined as the union of the sets I_u, for any endpoint $u : u > v$; $x(f)$, $\sigma(f)$ and $\omega(f)$ denote the spacetime point, the (eventual) σ index and the ω index, respectively, of the field f. If $h_v < N$, one of the field variables belonging to I_v may also carry a derivative. It is associated with each field label f an integer $m(f) \in \{0, 1, 2\}$, denoting the order of the derivative.

9. In correspondence of any node or endpoint v, let $P_v \subset I_v$, the *external fields* of v, be constructed as follows. In each endpoint u all the fields are external: $P_u \stackrel{def}{=} I_v$. If v is a node, and $v_1, \ldots, v_{s_v}$ are its first followers, then P_v can be any set s.t. $P_v \subset (\cup_i P_{v_i})$. Let $Q_{v_i} \stackrel{def}{=} P_v \cap P_{v_i}$: the union of the complementary ones, $\cup_i P_{v_i} \backslash Q_{v_i}$, is the set of the *internal fields* of v – or the fields *contracted* in correspondence of the node v – and have not to be an empty at least

- in the first node, except if its scale is $h = N + 1$.
- in the branching points;
- in the first preceding nodes of the endpoints.

Hence, the endpoints are attached to nodes where some of their external fields are actually contracted; while the first point is the lowest node in correspondence of which some contraction actually occur, except in the case of trees lying only on the scales $\geq N + 1$, for which the first point has been set to be on scale $N+1$. Among the fields in P_v, the set of all the fields of type φ and $\jmath$ will be called S_v, the set of the "special fields". Finally, $|P_v| = n_v^\psi + n_v^\varphi + n_v^\jmath$, where n_v^ψ is the number of external fields of type ψ, while n_v^φ, $n_v^\jmath$, as already defined, are the the number of external fields φ and $\jmath$ – indeed there is only one source field in the special endpoint.

10. Let $\mathcal{T}_{w;h;n}^{n^\psi, n^\varphi, n^\jmath}$ be the set of all topological trees, with all the above depicted constraints, with root on scale h, first node w on scale $h+1$, and with n normal endpoints, n^ψ external fields of type ψ, n^φ endpoints of type φ and $n^\jmath$ endpoints of type $\jmath$. To each such tree it corresponds a sequence of instructions to built a class of Feynman graphs.

11. Let $\mathcal{G}$ one of the Feynman graphs corresponding to the tree $\tau \in \mathcal{T}_{w;h;n}^{n^\psi, n^\varphi, n^\jmath}$. The endpoints of τ represents the vertices of $\mathcal{G}$, with the specified couplings. Any node v is in correspondence with a subgraph $\mathcal{G}_v \subset \mathcal{G} \equiv \mathcal{G}_w$, in which the external legs are the external fields of v. Specifically, if $v_1, \ldots, v_{s_v}$ $(s_v \geq 1)$ are the first followers of v, the Feynman graph $\mathcal{G}_v$ is constructed by pairing the internal fields of v with propagators $g^{(h)}$, in a way that the subgraphs $\mathcal{G}(v_1), \ldots, \mathcal{G}(v_{s_v})$ remains connected. There are many possible way to chose $\{P_v\}_v$, or equivalently many possible ways of selecting the internal fields to be involved in the contractions; and there are many possible connecting contractions: that is why to each τ is associated a family of many different Feynman graphs.

12. Let the set of the nodes of τ – hence considering neither the root, nor the endpoints – be denoted, with abuse of notation, τ as well. For each node v, the integer l_v is the number of lines of the Feynman graph $\mathcal{G}_v$; while $l_{o,v}$ is the number of lines in $\mathcal{G}_v$, which are not in $\cup_{i=1}^{s_v}\mathcal{G}_{v_i}$. Similarly, $l_v^{\rm anti}$ and $l_{o,v}^{\rm anti}$ count the number of lines of the graph which correspond to antidiagonal propagators. Two fundamental relations are

$$\begin{aligned}&\sum_{u\in\tau}^{u\geq v}(s_u-1)=n_v+n_v^\varphi+n_v^\jmath-1\ ,\\&\sum_{u\in\tau}^{u\geq v}l_{o,v}=l_v=2n_v^{(4)}+n_v^{(2)}+(1/2)n_v^\varphi+n_v^\jmath-(1/2)n_v^\psi\ .\end{aligned}\qquad (A3.1)$$

For instance, from them, by telescopic decomposition of the differences of the scales, $h_u-h_v=\sum_{w\in\tau}^{v<w\leq u}h_w-h_{w'}$, other two identities descend:

$$\begin{aligned}\sum_{u\in\tau}^{u\geq v}(h_u-h_v)(s_u-1)&=\sum_{u\in\tau}^{u\geq v}(h_u-h_{u'})(n_v+n_v^\varphi+n_v^\jmath-1)\ ,\\ \sum_{u\in\tau}^{u\geq v}(h_u-h_v)l_{o,v}&=\sum_{u\in\tau}^{u\geq v}(h_u-h_{u'})l_u\ .\end{aligned}\qquad (A3.2)$$

The above formulas are stated as they are for shake of clarity; but sometimes it will be used that, by definition, $h_w-h_{w'}=1$.

13. It is natural to consider the following decomposition. Given any $\tau\in\mathcal{T}_{w;h;n}^{n^\psi,n^\varphi,n^\jmath}$, let the "auxiliary tree", $\tau^a\subset\tau$, be the union of the paths in τ which connects the special endpoint with the root r; for any $w\in\tau^a$, let s_w^*, the number of the nodes first followers of v and in τ^a. Besides, if w is one of the maximal nodes in τ^a, let the integers $n_{*,w}^\jmath$, $n_{*,w}^\varphi$, be the number of the external fields of type $\jmath$ or of type φ which are in the cluster L_w; otherwise, for $w\in\tau^a$ but not maximal, let them be the number of the external fields of type $\jmath$ or of type φ which are in the cluster L_w, but not in the following clusters $L_{w_1},\ldots,L_{w_{s_w}}$. Finally, the "main tree", $\tau^*\subset\tau^a$, is given by the auxiliary tree, deprived of the nodes above the maximal nodes with $s_w^*\geq 2$; for $w\in\tau^*$, let the integer b_w^* be the number of nodes of τ^* first followers of w: hence $s_w^*=b_w^*+n_{*,w}^\varphi+n_{*,w}^\jmath$.

14. Given any set of fields M, let $x(M)\stackrel{def}{=}\cup_{f\in M}x(f)$. Let D_v be the *tree distance* among $x\left(I_{v_1}\right),\ldots,x\left(I_{v_{s_v}}\right)$ the sets of the spacetime points of the clusters $L_{v_1}\ldots L_{v_{s_v}}$: namely $D_v\stackrel{def}{=}\min_{g\in\mathcal{C}}\sum_{l\in g}|l|$, where $\mathcal{C}$ the set of all the possible tree graphs g connecting the spacetime points in $x\left(I_{v_1}\right),\ldots,x\left(I_{v_{s_v}}\right)$, and l are the links. Similarly, $D_{0,w}$ and $D_{1,w}$ are respectively the "time" and "space" tree distance and are defined as the tree distance among the time component and the space component of the spacetime points in $x\left(I_{v_1}\right),\ldots,x\left(I_{v_{s_v}}\right)$.

A3.2 Cluster expansion. A standard tool in the fermionic Renormalization Group – first introduced in [Le87] – is the cluster expansion of the truncated expectations (see [B84]). It explains why in the bounds it is better to consider altogether all Feynman graphs corresponding to one tree, rather than one Feynman graph singly.

Let $P_1, \dots, P_s$ be disjoint sets of ψ fields s.t. $|\cup_i P_i| = 2n$; and let $P_j^\sigma \stackrel{def}{=} \{f \in P_j : \sigma(f) = \sigma\}$. A pairing l is the couple of a field f_l^+ in $\cup_j P_j^+$ and a field f_l^- in $\cup_j P_j^-$: let $x(f_l^+) - x(f_l^-) \stackrel{def}{=} x_l$; and $\big(\omega(f_l^+), \omega(f_l^-)\big) \stackrel{def}{=} \underline{\omega}_l$. Then, the truncated expectation w.r.t. the Gaussian measure of propagator $g^{(h)}$ is given, up to a global sign, by:

$$\mathbb{E}_h^T\big[\psi(P_1), \dots, \psi(P_s)\big] = \sum_T \left(\prod_{l \in T} g_{\underline{\omega}_l}^{(h)}(x_l)\right) \int \mathrm{d}P_T(t)\ \det G^{h,T}(t)\ , \qquad (A3.3)$$

where T is a set of pairings of elements of $\cup_i P_i$, which would be a connected tree graph if all the points in the same set P_i where identified; the parameters $t = \big\{t_{i,j} \in [0,1] : i, j = 1, \dots, s\big\}$ have a certain normalized distribution $\mathrm{d}P_T(t)$; finally $G^{h,T}(t)$ is a $(n - s + 1) \times (n - s + 1)$ matrix, the entries of which are given by $G^{h,T}_{f_l^-, f_l^+} = g^{(h)}_{\underline{\omega}_l}(\underline{x}_l) t_{\underline{i}_l}$, where $\underline{i} \stackrel{def}{=} (i_l^+, i_l^-)$ s.t. $f_l^- \in P^-_{i_l^-}$ and $f_l^+ \in P^+_{i_l^+}$, for any possible pair l of elements of $\cup_i P_i$, s.t. $l \notin T$.

The importance of this formula is that, if all the entries $M_{i,j}$ of an $n \times n$ matrix M are give by scalar products, $M_{i,j} = (v^{(i)}, w^{(j)})$, where $v^{(1)}, \dots, v^{(n)}$ and $w^{(1)}, \dots, w^{(n)}$ are vectors, bounded in norm by a constant C_0, the sum of $n!$ monomials that gives the determinant of M can be bounded with C_0^n, by a simple application of the volume inequality. In this way factorial bounds are avoided.

A3.3 Bounds for the kernels. Setting $(h \wedge N) \stackrel{def}{=} \min\{h, N\}$, the effective potential on scale h is a polynomial of the fields with coefficients given by the kernels:

$$\begin{aligned}&\mathcal{W}^{(h)}(\varphi, \jmath, \psi) \\ &= \sum_{n>0} \sum_{n^\psi, n^\varphi, n^\jmath \geq 0} \sum_{\tau_v \in \mathcal{T}^{n^\psi, n^\varphi, n^\jmath}_{v;(h \wedge N);n}} \sum_{P_v \subset I_v}^{|P_v| = n^\psi + n^\varphi + n^\jmath} \int \mathrm{d}^2 x(P_v)\ f(P_v) W^{(h)}\Big(\mathbf{x}(P_v); \tau_v; P_v\Big)\ ,\end{aligned}$$

where, $f(P_v)$ denotes the product of every external field in P_v. In its turn, the kernel is a sum over the Feynman graphs of the product of a propagator for each line of the graphs, $K^{(h)}\Big(\mathbf{x}(I_v); \tau_v; P_v\Big)$ integrated w.r.t. all the internal points of the cluster L_v:

$$W^{(h)}\Big(\mathbf{x}(P_v); \tau_v; P_v\Big) = \int \mathrm{d}^2 x(I_v \backslash P_v)\ K^{(h)}\Big(\mathbf{x}(I_v); \tau_v; P_v\Big)\ .$$

A useful norm to bound the kernels is obtained by integrating the product of the propagators w.r.t. all the spacetime points $x(I_v)$, except the "fixed points", $x(F_v)$: they are, if S_v is not empty, the points in $F_v \stackrel{def}{=} S_v$; otherwise the point in $F_v \stackrel{def}{=} \{x_v\}$, for any choice of $x_v \in P_v$. It holds the following lemma.

Lemma A.3.1. *If $h > N$, there exists a constant $C_2 \geq C$ such that, for any choice of the tree*

$\tau_v \in \mathcal{T}^{n^\psi,n^\varphi,n^\jmath}_{v;N;n}$, *with root* r,

$$\begin{aligned}&\int \mathrm{d}^2 x\big(I_v\backslash F_v\big)\ \Big|K^{(h)}\Big(x(I_v);\tau_v;P_v\Big)\Big|\\ \le&(C_2\varepsilon)^n C_2^{n^\varphi+n^\jmath}\gamma^{Nd_r}\left(\sum_{\{P_w\}_{w>r}}\prod_{w\in\tau_v}\gamma^{d_w+r_w}\right)\\ &\cdot\left(\prod_{w\in\tau_v^*}^{s_v^*\ge 2}\frac{\gamma^{(N+h_w)(s_w^*-1)}}{e^{\frac{c}{2(n^\varphi+n^\jmath)}\left(\sqrt{\gamma^N D_w}+\sqrt{\gamma^{h_w}D_{0,w}}\right)}}\right)\left(\prod_{i=1}^{n^\varphi}\frac{1}{\sqrt{Z_N}}\right)\left(\prod_{i=1}^{n^\jmath}\frac{Z_N^{(2)}}{Z_N}\right)\ ,\end{aligned}\tag{A3.4}$$

with

$$d_w \stackrel{def}{=} \begin{cases} 1-n_w-n_w^\jmath-n_w^\varphi & \text{for } h_w\ge N+1\\ 2-(1/2)n^\psi-(3/2)n^\varphi-n^\jmath & \text{for } w=r\ ,\end{cases}$$

and r_w *such that* $d_w+r_w\le -1/2-(1/8)n_w^\psi$.

Proof. Let $\tau_{v_1},\dots,\tau_{v_{s_v}}$ be the subtrees of τ_v branching from v – namely with root in v, and first nodes $v_1,\dots,v_{s_v}$; the product of propagators $K^{(h_v)}\Big(\mathbf{x}(I_v);\tau_v;P_v\Big)$ is obtained as

$$\begin{aligned}K^{(h_v)}\Big(\mathbf{x}(I_v);\tau_v;P_v\Big)=&\frac{1}{s_v!}\sum_{P_{v_1},\dots,P_{v_{s_v}}}\left(\prod_{i=1}^{s_v}K^{(h_v+1)}\Big(\mathbf{x}(I_{v_i});\tau_{v_i};P_{v_i}\Big)\right)\cdot\\ &\cdot\mathbb{E}_h^T\left[\psi(P_{v_1}\backslash Q_{v_1}),\dots,\psi(P_{v_{s_v}}\backslash Q_{v_{s_v}})\right]\ .\end{aligned}\tag{A3.5}$$

Applying (A3.3), and iterating till the endpoints, it holds:

$$\begin{aligned}K^{(h_v)}\Big(\mathbf{x}(I_v);\tau_v;P_v\Big)=&\left(\prod_u^{\text{e.p.}}\rho_u\right)\cdot\\ &\cdot\prod_{w\in\tau_v}\sum_{P_w}\sum_{T_w}\frac{1}{s_w!}\left(\prod_{l\in T_w}g^{(h_w)}_{\underline{\omega}_l}(x_l)\right)\int\mathrm{d}P_{T_w}(\mathbf{t})\ \det G^{h_w,T_w}(\mathbf{t})\ ,\end{aligned}\tag{A3.6}$$

where ρ_u denotes the coupling in the endpoints: λ_N, $\gamma^N\nu_N$ or δ_N, if u is a normal endpoint; $\zeta_N^{(2,\sigma)}$ if u is an endpoint of type $\jmath^{(\sigma)}$; 1 if u is an endpoint of type φ. Then, a bound for the integral of (A3.6) can be obtained as follows.

1. Calling $b_h(x-y)\stackrel{def}{=}e^{-(c/2)\left(\sqrt{\gamma^N|x_l|}+\sqrt{\gamma^h|x_{0,l}|}\right)}$, by (3.1.8) each of the s_w-1 propagators in a tree T_w is bounded with $C\gamma^N b^2_{h_w}(x-y)$; while $\left|\det G^{h_w,T_w}(\mathbf{t})\right|$ is bounded with a factor $C_0C\gamma^N$ for each of the $l_{o,w}-(s_w-1)$ rows of the matrix $G^{h_w,T_w}(\mathbf{t})$: globally, the product of the propagators can be bounded with

$$\left(C_0C\gamma^N\right)^{l_{o,w}}\prod_{l\in T_w}b^2_{h_w}(x_l)\ .$$

2. Collecting the products over $b_{h_w}(x_l)$ for any node of the tree τ_v, since the branching nodes of the main tree are not more than the special endpoints $n^\varphi + n^\jmath$,

$$\prod_{w\in\tau_v}\prod_{l\in T_w} b^2_{h_w}(x_l) \le \prod_{w\in\tau_v}\prod_{l\in T_w} b_{h_w}(x_l) \prod_{w\in\tau_v^*} e^{-\frac{c}{2(n^\varphi+n^\jmath)}\left(\sqrt{\gamma^N D_w}+\sqrt{\gamma^{h_w} D_{0,w}}\right)} . \tag{A3.7}$$

3. The integrations in $\mathrm{d}^2x(I_v/F_v)$ are performed, the left integrand being the product of the b_{h_w}'s, increased by replacing in them $\gamma^N|x_l|$ with $\gamma^N|x_{1,l}|$, times constant factors. It holds

$$\int \mathrm{d}^2x(I_v/F_v)\prod_{w\in\tau_v}\prod_{l\in T_w} b_{h_w}(x_l) \le \prod_{w\in\tau}\left(C_1\gamma^{-(N+h_w)}\right)^{(s_w-s^*_w)} . \tag{A3.8}$$

   Indeed, the above formula is obtained iteratively starting from the first node, v. Let the labels $w_1,\dots,w_{s_w}$ be assigned to the nodes following w so that: for $j=1,\dots,s^*_w$ the cluster L_{w_j} contains at least a special endpoint, $S_{w_i}\neq\emptyset$, and is called "special cluster"; for $j=s^*_w+1,\dots,s_w$, the cluster L_{w_j} contains no special endpoints, $S_{w_i}=\emptyset$ – eventually it may be $s^*_w=0,s_w$. Now, the graph T_w can be thought as a tree graph: the cluster L_{w_1} is its root, $L_{w_2},\dots,L_{w_{s_w}}$ are its nodes, while the factors b_{h_w}'s are its links. Then, considering the first node v, and starting from the endpoints of T_v, let L_{v_j} be the first followers of $L_{v_{j'}}$, and let b_{h_v} be the link connecting them. If L_{v_j} is a special cluster, than b_{h_v} is simply bounded with its maximum, $||b_{h_v}||_\infty$; whereas, if L_{v_j} is a normal cluster, the link b_{h_v} is bounded with $||b_{h_v}||_1$, the integral being taken w.r.t. the point in F_{v_j}. Since $||b_{h_v}||_\infty\le 1$, while $||b_{h_v}||_1\le C_1\gamma^{-(N+h_v)}$, this gives the factor in (A3.8) for $w=v$. Iterating to all the nodes following the first, the complete bound is found.
4. The sum over T_w is bounded by the number of the topological graphs with s_w nodes, 4^{s_w}, times the number of the possible permutations of such nodes, $s_w!$.
5. Each factor ρ_u are bounded, by (3.3.15), with 2ε if u is a normal endpoint; otherwise $\rho_u = 1/\sqrt{Z_N}$ or $Z^{(2)}_N/Z_N$ if respectively u is of type φ or $\jmath$.

In the end, the factorial in item 4. is compensated by the one in the denominator of (A3.6); while the powers of 2ε, C, C_0, C_1 and 4^{s_w} is all together bounded with

$$\prod_{w\in\tau_v}(4C_1)^{s_w}(C_0C)^{l_{o,w}}(2\varepsilon)^{n_{o,w}} \le (C_2\varepsilon)^n C_2^{n^\varphi+n^\jmath} ,$$

for $C_2\ge(4CC_0C_1)^2$. And the rest of the bound is reduced to simple dimensional analysis. For each of the $l_{o,w}$ propagators there is a factor γ^N; for each of the $s_w-s^*_w$ integrals there is a factor $\gamma^{-(N+h_w)}$ more. Furthermore, not yet counted in the above items, by (3.1.8) there is a factor $\gamma^{-(h_w-N)}$ more for any antidiagonal propagator. Finally, in correspondence of each endpoint of type δ and ν there is a factor γ^N. Therefore the collection of all such factors gives

$$\begin{aligned}&\left(\prod_{w\in\tau_w^*}\gamma^{(N+h_w)(s_w^*-1)}\right)\prod_{w\in\tau_v}\gamma^{h_w\left(1-s_w-l_{o,w}^{\rm anti}\right)}\gamma^{N\left(l_{o,w}-(s_w-1)+l_{o,w}^{\rm anti}+n_{o,w}^{(2)}\right)}\\ \le&\left(\prod_{w\in\tau_w^*}\gamma^{(N+h_w)(s_w^*-1)}\right)\gamma^{Nd_r}\prod_{w\in\tau_v}\gamma^{d_w+r_w}\ ,\end{aligned}\tag{A3.9}$$

where $r_w\stackrel{def}{=}-l_w^{\rm anti}$ for $n_w=1$, $n_w^\psi=n_w^\jmath=0$, and $r_w\stackrel{def}{=}0$ otherwise. Now it is possible to prove that $d_w+r_w\le-(1/2)-(1/16)n_w^\psi$. Indeed, there are the following possibilities.

1. The number of normal endpoints is zero. Then, since in the nodes of the tree there has to be at least a contraction, and since the self-contraction of the fields in the endpoint of type $\jmath$ is zero by oddness of the diagonal propagator, $n_w^\varphi+n_w^\jmath\ge2$. Then, since in such graphs the external fields of type ψ cannot be more than $2(n_w^\varphi+n_w^\jmath)$, it holds $d_w\le-(1/2)(n_w^\varphi+n_w^\jmath)\le-(1/2)-(1/8)n_w^\psi$.
2. The number of the normal endpoints is 1, while $n_w^\varphi+n_w^\jmath=0$. Then $d_w+r_w\le-l_w^{\rm anti}$. By explicit inspection, such graphs, made of self-contractions, either are zero by oddness of the diagonal propagator, or have at least one antidiagonal propagator; furthermore the number of external ψ fields cannot be more than two. Therefore $d_w+r_w\le-(1/2)-(1/4)n_w^\psi$.
3. The number of the total endpoints, $n_w+n_w^\varphi+n_w^\jmath$, is greater or equal to 2. Since in such graphs the external fields ψ cannot be more than $4(n_w+n_w^\varphi+n_w^\jmath)$, and $r_w=0$, then $d_w+r_w\le-(1/2)(n_w+n_w^\varphi+n_w^\jmath)\le-(1/2)-(1/16)n_w^\psi$.

The proof is complete. ∎

Lemma A.3.2. *If $h\le N-1$, and for ε small enough, there exists a constant $C_2\ge C$ such that*

$$\begin{aligned}&\int d^2\mathbf{x}(I_v\backslash F_v)\ \left|K^{(h)}\Big(\mathbf{x}(I_v);\tau_v;P_v\Big)\right|\\ \le&(C_3\varepsilon)^nC_3^{n^\varphi+n^\jmath}\gamma^{hd_r}\left(\sum_{\{P_w\}_{w>r}}\prod_{w\in\tau_v}\gamma^{d_w+r_w}\right)\cdot\\ &\left(\prod_{w\in\tau_v^*}^{s_v^*\ge2}\frac{\gamma^{\left((h_w\wedge N)+h_w\right)(s_w^*-1)}}{e^{\frac{c}{2(n^\varphi+n^\jmath)}\left(\sqrt{\gamma^{(h_w\wedge N)}D_w}+\sqrt{\gamma^{h_w}D_{0,w}}\right)}}\right)\left(\prod_{i=1}^{n^\varphi}\frac{1}{\sqrt{Z_{(h_i\wedge N)}}}\right)\left(\prod_{i=1}^{n^\jmath}\frac{Z^{(2)}_{(k_i\wedge N)}}{Z_{(k_i\wedge N)}}\right)\ ,\end{aligned}\tag{A3.10}$$

where

$$d_w\stackrel{def}{=}\begin{cases}1-n_w-n_w^\varphi-n_w^\jmath & \text{for } h_w\ge N+1\\ 2-(1/2)n_w^\psi-(3/2)n_w^\varphi-n_w^\jmath & \text{for } h_w\le N\end{cases}\ ,$$

and r_w is such that $d_w+r_w\le-1/4-(1/12)n_w^\psi$.

Proof. Neglecting the effects of the localization, with argument similar to the proof of the previous lemma, the bound is reduced to simple dimensional analysis: for each of the $l_{o,w}$

propagators there is a factor γ^{h_w}; for each of the the $s_w - s^*$ integrals there is a factor γ^{-2h_w}. Finally, regarding the endpoints, there is a factor 2ε for each endpoint of type λ; $2\varepsilon\gamma^{h_w}$ for each endpoint of type δ or ν. Therefore, collecting only the factors coming from the dimensional analysis,

$$\begin{aligned}&\left(\prod_{w\in\tau_w^*}\gamma^{2h_w(s_w^*-1)}\right)\prod_{w\in\tau_v}\gamma^{h_w\left(l_{o,w}-2(s_w-1)+n_{o,w}^{(2)}\right)}\\&=\left(\prod_{w\in\tau_w^*}\gamma^{2h_w(s_w^*-1)}\right)\gamma^{hd_v}\prod_{w\in\tau_v}\gamma^{d_w}\ ,\end{aligned}\tag{A3.11}$$

with $d_w \stackrel{def}{=} 2-(1/2)n_w^\psi-(3/2)n_w^\varphi-n_w^J$. Now the point is that they can occur nodes with non-negative dimension: here comes the role of the localization, which improves their dimension by absorbing the localized part of the graphs into the coupling constants. Indeed, for the kernel bringing an $\mathcal{R}$-operator, with reference to the items at point 3.3.4, the following facts have to be considered.

1. The local part $z_{h_w}D_\sigma$, occurring in a certain node w, is bounded, up to a constant, by $\gamma^{h_w}\gamma^{-(h_w-h_{w_0})}$, if w_0 is the node, lower than w, in correspondence of which one of the field of momenta k is contracted. While the local part $z_{h_w}\big|-ik_0+\omega e(k_1)-D_\omega(k)\big|$ is instead bounded, up to a constant, with $\gamma^{h_w}\gamma^{-(h_w-h_{w_0})}\gamma^{-(N-h_{w_0})}\le\gamma^{h_w}\gamma^{-(N-h_{w_0})}\gamma^{-2(h_w-h_{w_0})}$: the standard power counting, as it were using only the factor γ^{h_w}, because of $\gamma^{-2(h_w-h_{w_0})}$, is improved in all the nodes u along the path connecting w with w_0 by $r_u=2$. Furthermore, with reference to the proof of the equivalence of the Euclidean and the Hamiltonian regularization, the factor $\gamma^{-(N-h_w)}$ makes such a kernel – generated only in the latter regularization – vanishing in the limit of removed cutoff.
2. One or two increments D_ω, and respectively one or two derivatives in the companying kernels – the kernel occurring at node w, the increment having the same momenta of a ψ-field contracted on a lower node, w_0 – gives a gain w.r.t. the standard power counting: each derivative gives a factor γ^{-h_w} more, while each increment gives a factor $\gamma^{h_{w_0}}$ more. Since
$$\gamma^{-(h_w-h_{w_0})r}=\prod_u^{w_0\le u\le w}\gamma^{-r}\ ,\qquad \text{for } r=1,2\ ,$$
all the nodes u in the path connecting the node w with the node w_0 have a gain $r_u=1$ or 2.
3. The local terms which are linear or quadratic in the factors $\{\mu_k/\gamma^k\}_k$ gives a gain in the bounds since, if they occur in the node w on scale h, k has to be greater or equal to h, and, by (3.3.14) and the definition of h^*:
$$\left(\frac{\mu_k}{\kappa\gamma^k}\right)^r\le\left(\frac{\mu_{h^*}}{\kappa\gamma^{h^*}}\right)^r\gamma^{-r(1-2c_0\varepsilon)(k-h^*)}\le\prod_{u\le w}\gamma^{-r(1-2c_0\varepsilon)}\ ,$$
and therefore, for ε small enough, the dimension of every node u occurring along the path connecting the node w with the root is improved by $r_u=r3/4$.

4. In the kernels corresponding to nodes w with $n^{\jmath}_w = 0$, and $n^{\psi}_w = n^{\varphi}_w = 1$, the dimension is zero. It is possible to obtain a gain $r_w = 1$ at the price of worsening the final constant C_3 of a factor γ^2. Indeed, because of the compact support of the propagators, it is clear that such nodes can be both among the preceding ones of the n^{φ} special endpoints of type φ, let them be $w_1, \ldots, w_q$, and among the ones preceding $w_1, \ldots, w_q$ themselves: namely no more than $2n^{\varphi}$ nodes.

Therefore, with developments similar to the ones in the previous proof, it is possible to prove that $d_w + r_w \le -(1/4) - (1/12)n^{\psi}_w$. But since the localization produces the flows of the field and densities strengths, (A3.11) has to be replaced with

$$\gamma^{hd_v}\left(\prod_{i=1}^{n^{\varphi}} \frac{1}{\sqrt{Z_{h_i}}}\right)\left(\prod_{i=1}^{n^{\jmath}} \frac{Z^{(2)}_{k_i}}{Z_{k_i}}\right)\left(\prod_{w\in\tau^*_w} \gamma^{2h_w(s^*_w-1)}\right)\left(\prod_{w\in\tau_v}\left(\frac{Z_{h_w}}{Z_{h'_w}}\right)^{(n^{\psi}_w/2)}\gamma^{d_w+r_w}\right).$$

This completes the proof. ∎

A3.4 Remark. The argument in the last item does not apply in the case $n^{\jmath} = 1$ and $n^{\psi} = 2$. This is the main difference of the external sources $\jmath$ and φ: while the former requires a coupling constant for absorbing divergences due to interaction with the source, the latter need not, since it in interacts only by *one particle reducible graphs*.

Lemma A.3.3. *For ε small enough, the perturbative expansion for the $(n^{\jmath}; n^{\varphi})$-Schwinger functions is absolutely convergent to a distribution fulfilling property E1 and E5. of the* OSA.

Proof. The expansion for the Schwinger function is given by the expansion for the effective potential in the case $P_v = S_v$ and for any scale of the first node $h : h^* - 1 \le h \le N + 1$.

Since the case h^* finite is much more easier of the case $h^* = -\infty$, the following development will concern only the latter.

Calling $\mathcal{T}^{0,n^{\varphi},n^{\jmath}}_{v,h;\underline{k};\underline{h};n}$ the set of trees $\tau \in \mathcal{T}^{0,n^{\varphi},n^{\jmath}}_{v,h,n}$ having the n^{φ} external fields of type φ on scales $h_1, \ldots, h_{n^{\varphi}}$, and the $n^{\jmath}$ external fields of type $\jmath$ on scales $k_1, \ldots, k_{n^{\jmath}}$, it holds

$$S^{(n^{\jmath};n^{\varphi})(\underline{\varepsilon})}_{\underline{\sigma};\underline{\omega}}(\underline{z};\underline{x}) \stackrel{def}{=} \sum_{n\le 0}\sum_{h\le M}\sum_{\underline{k}}^{h<k_j\le M}\sum_{\underline{h}}^{h<h_j\le M}\sum_{\tau_v\in\mathcal{T}^{0,n^{\varphi},n^{\jmath}}_{v,(h\wedge N);\underline{h};\underline{k};n}} W^{(h)}\Big(\mathbf{x}(S_v);\tau_v;S_v\Big) \qquad (A3.12)$$

and, by the just proved bound on the kernels,

$$\Big|W^{(h)}\Big(\mathbf{x}(S_v);\tau_v;S_v\Big)\Big| \leq (C_2\varepsilon)^n C_2^{n^\varphi+n^\jmath}\gamma^{hd_r}\prod_{w\in\tau_v^*}^{h_w\geq N+1} e^{-\frac{c}{2(n^\varphi+n^\jmath)}\sqrt{\gamma^{h_w}D_{0,w}}}$$
$$\cdot\left(\prod_{w\in\tau_v^*}^{s_v^*\geq 2}\frac{\gamma^{\left((h_w\wedge N)+h_w\right)(s_w^*-1)}}{e^{\frac{c}{2(n^\varphi+n^\jmath)}\sqrt{\gamma^{(h_w\wedge N)}D_w}}}\right)\left(\prod_{i=1}^{n^\varphi}\frac{1}{\sqrt{Z_{(h_i\wedge N)}}}\right)\left(\prod_{i=1}^{n^\jmath}\frac{Z^{(2)}_{(k_i\wedge N)}}{Z_{(k_i\wedge N)}}\right)\cdot \qquad (A3.13)$$
$$\cdot\left(\sum_{\{P_w\}_{w>r}}\prod_{w\in\tau_v}\left(\frac{Z_{(h_w\wedge N)}}{Z_{(h_{w'})\wedge N}}\right)^{\frac{n^\psi_w}{2}}\gamma^{d_w+r_w}\right)\ .$$

Let the following facts be considered.

1. For the main tree it holds an identity similar to (A3.2), with s_v replaced by s_v^*, and with n_v removed from the r.h.s. member; so that:

$$\sum_{w\in\tau_v^*}\Big((h_w\wedge N)+h_w\Big)(s_w^*-1) = 2h(n^\varphi+n^\jmath-1)$$
$$+\sum_{w\in\tau_v^*}^{h_w\leq N}(h_w-h_{w'})2(n_w^\varphi+n_w^\jmath-1)+\sum_{w\in\tau_v^*}^{h_w\geq N+1}(h_w-h_{w'})(n_w^\varphi+n_w^\jmath-1)$$
$$\stackrel{def}{=} h\Delta d_v+\sum_{w\in\tau_v^*}(h_w-h_{w'})\Delta d_w\ .$$

These factors can be absorbed into the dimension of any node w of the main tree, changing it from d_w to

$$d_w+\Delta d_w = \begin{cases} n_w^\jmath+(1/2)n_w^\varphi-(1/2)n_w^\psi & \text{for } h_w\leq N\\ -n_w & \text{otherwise.}\end{cases}$$

2. Since $n_w^f=\sum_{v\geq w}n^f_{*,v}$ for $f=\varphi,\jmath$, then

$$\sum_{w\geq v}^{h_w<N}(h_w-h)\big(n^\jmath_{*,w}+(1/2)n^\varphi_{*,w}\big)+\sum_{w\geq v}^{h_w=N+1}(N-h)\big(n^\jmath_w+(1/2)n^\varphi_w\big)$$
$$=\sum_{w\geq v}^{h_w\leq N}(h_w-h_{w'})\big(n^\jmath_w+(1/2)n^\varphi_w\big)\ ,$$

which formula gives:

$$\begin{aligned}
&\gamma^{h\left(n^\jmath+(1/2)n^\varphi\right)} \prod_{w\in\tau_v^*}^{h_w\le N} \gamma^{(h_w-h_{w'})\left(n_w^\jmath+(1/2)n_w^\varphi-(1/2)n_w^\psi\right)}\\
&\qquad\cdot \prod_{w\in\tau_v^*}^{h_w\ge N+1} \gamma^{-(h_w-h_{w'})n_w}\\
&= \prod_{w\in\tau_v^*}^{h_w\le N} \gamma^{h_w\left(n_{*,w}^\jmath+(1/2)n_{*,w}^\varphi\right)} \prod_{w\in\tau_v^*}^{h_w\le N} \gamma^{-(h_w-h_{w'})(1/2)n_w^\psi}\\
&\quad\cdot \prod_{w\in\tau_v^*}^{h_w=N+1} \gamma^{N\left(n_w^\jmath+(1/2)n_w^\varphi\right)} \prod_{w\in\tau_v^*}^{h_w\ge N+1} \gamma^{-(h_w-h_{w'})n_w}\ .
\end{aligned} \tag{A3.14}$$

3. In view of the proof of cluster decomposition, since it can be, for $w = v_0^*$, the lowest branching point of τ^*, $n_{*,w}^\jmath + (1/2)n_{*,w}^\varphi = 0$, a further modification of the above decomposition is performed. Setting $m \stackrel{def}{=} n^\jmath + (1/2)n^\varphi$, $m_w \stackrel{def}{=} n_w^\jmath + (1/2)n_w^\varphi$ and $m_{*,w} \stackrel{def}{=} n_{*,w}^\jmath + (1/2)n_{*,w}^\varphi$; and letting h_0 be the scale of v_0^*, the following identity

$$1 = \gamma^{-\left(h_w-h_0\right)\frac{1}{8}\frac{m_{*,w}}{m}} \gamma^{\left(h_w-h_0\right)\frac{1}{8}\frac{m_{*,w}}{m}}\ ,$$

for each node $w \in \tau^* : h_w \le N$ turns (A3.14) into

$$\begin{aligned}
&\gamma^{h_0(1/8)} \prod_{w\in\tau_v^*}^{h_w\le N} \gamma^{h_w m_{*,w}\left(1-(1/8m)\right)} \prod_{w\in\tau_v}^{h_w\le N} \gamma^{(h_w-h_{w'})\left((m_w/8m)-(1/2)n_w^\psi\right)}\\
&\cdot \prod_{w\in\tau_v^*}^{h_w=N+1} \gamma^{Nm_w\left(1-(1/8m)\right)} \prod_{w\in\tau_v}^{h_w\ge N+1} \gamma^{-(h_w-h_{w'})n_w}\ .
\end{aligned} \tag{A3.15}$$

4. Let each factor $1/\sqrt{Z_{h_i\wedge N}}$ be considered for $h_i \le N$: if the $w \in t_v^*$, is the highest branching point in τ lower than the i-th endpoint of type φ, u_i, by (3.3.14), such a factor can be moved to the node w, $1/\sqrt{Z_{h_i}} \le 1/\sqrt{Z_{h_w}}\gamma^{(c_0/2)\varepsilon^2(h_i-h_w)}$, at the price of the factor $\gamma^{(c_0/2)\varepsilon^2(h_i-h_w)} = \prod_{w'}^{w\le w'\le u_i} \gamma^{(c_0/2)\varepsilon^2}$: it is absorbed in the dimension of the nodes along the path connecting u_i with the node w – by definition such nodes are not in the main tree – changing it, for ε small enough, from $d_w + r_w \le -1/4 - (1/12)n_w^\psi$ to the new dimension $\widehat{d}_w \le -1/8 - (1/12)n_v^\psi$. Similar decomposition is done in case $h_i \ge N+1$: the lost in the dimension is only in the nodes on scales $h_w \le N$.

5. Similar procedure is executed for each factor $Z_{k_i\wedge N}^{(2)}/Z_{k_i\wedge N}$, for $h_w \le N$: if $w \in \tau^*$, is the highest branching point in τ lower than the i–th endpoint of type $\jmath$, u_i by (3.3.14), $Z_{k_i}^{(2)}/Z_{k_i} \le Z_{h_w}^{(2)}/Z_{h_w}\gamma^{2c_0\varepsilon(k_i-h_w)}$; the factor $\gamma^{2c_0\varepsilon(k_i-h_w)}$ is absorbed in the dimension of the nodes along the path connecting u_i with w, again changing it from $d_w + r_w = -1/4 - (1/12)n_w^\psi$ to the new dimension $\widehat{d}_w \le -1/8 - (1/12)n_v^\psi$, for ε small enough. Similar decomposition is done in case $k_i \ge N+1$.

6. The exponent $(m_w/8m)-(1/2)n^\psi_w$ of the factors in the second product in formula (A3.15) can be bounded with $-1/8-(1/12)n^\psi_w$.
7. Since in every node $w : h_w \le N$, both in the main tree and in the rest of the tree, the dimension has been left to be $\widehat{d}_w = -1/8-(1/12)n^\psi_w$, and since for ε small enough, $(Z_{h_w}/Z_{h'_w}) \le \gamma^{c_0\varepsilon^2} \le \gamma^{(1/12)}$, it is possible to absorb all the factors $(Z_{h_w}/Z_{h'_w})^{(1/2)n^\psi_w}$ into the dimension $\widehat{d}_w$, turning it into $d'_w \le -1/8-(1/24)n^\psi_w$.
8. Regarding the nodes $w : h_w \ge N$, if $n_w > 0$, by inspection of the graphs – eventually involving the interaction of type $\jmath$ and φ – it can be $n_w \neq 0$, and than $-n_w \le -(1/4)n^\psi_w \le -1/8-(1/24)n^\psi_w$; otherwise $n_w = 0$: this can happen only on the highest node, in the sense that a node with $n_w = 0$ cannot be lower than any node v with $n_v \neq 0$ – since n_w is a cumulative counter – then the graphs corresponding to this latter case are contractions of special vertices only, and $n^\psi_w \le 2$. Hence in the region of the tree where $n_w = 0$ there can be no more than $n^\psi + n^\jmath$ branching points: it is in any case possible, multiplying C_2 by a factor $\gamma^{2/24}$, to extract a factor $\gamma^{-(1/24)n^\psi_w}$ for every node $w : h_w \ge N$ such that $P_w \neq P_{w'}$, namely where some contraction really occur.
9. The product over the nodes where at least a contraction of internal fields does occur, $\prod^{\text{b.p.}}_{w\in\tau_v} \gamma^{-(\vartheta/24)n^\psi_w}$, allows to control the summation in P_w – which, fixed the tree τ_v, is actually only a summation in $P_w \backslash S_w$:

$$\prod^{\text{b.p.}}_{w\in\tau_v} \sum_{P_w} \gamma^{-(1/24)n^\psi_w} \le \prod^{\text{b.p.}}_{w\in\tau_v} \sum_{n^\psi_w} \gamma^{-(1/24)n^\psi_w} \binom{n^\psi_{w_1} + \cdots + n^\psi_{w_{s_w}}}{n^\psi_w}$$
$$\le \prod^{\text{e.p.}}_{u\in\tau_v} \left(1-\gamma^{-(1/24)}\right)^{-n^\psi_u} \le \left(1-\gamma^{-(1/24)}\right)^{-4(n+n^\varphi+n^\jmath)} ,$$

where the last-but-one inequality can be easily proved by induction by thinking the endpoints u as the node at which are attached one or more further branches; while the last simply follows from the fact that $n^\psi_u \le 4$.

Finally, once C_3 is taken greater or equal to $C_2\gamma^{2/24}(1-\gamma^{-(1/24)})^{-4}$, the bound for the Schwinger function has become

$$\begin{aligned}
&\left|\mathcal{W}^{(h)}\left(\mathbf{x}(S_v);\tau_v;S_v\right)\right| \\
\le&(C_3\varepsilon)^n C_3^{n^\varphi+n^\jmath} \frac{\gamma^{h_0(m+1/8)}}{e^{\frac{c}{2(n^\varphi+n^\jmath)}\sqrt{\gamma^{h_0}D_{v_0^*}}}} \left(\prod^{h_w=N}_{w\in\tau^*_v} \frac{\gamma^{h_w m_w(1-(1/8m))}}{e^{\frac{c}{2(n^\varphi+n^\jmath)}\sqrt{\gamma^N D_w}}}\right) \\
&\cdot\left(\prod^{h_w\le N}_{w\in\tau^*_v} \frac{\gamma^{h_w m_{*,w}(1-(1/8m))}}{e^{\frac{c}{2(n^\varphi+n^\jmath)}\sqrt{\gamma^{h_w}D_w}}} \left(\frac{Z^{(2)}_{h_w}}{Z_{h_w}}\right)^{n^\jmath_{*,w}} \left(\frac{1}{Z_{h_w}}\right)^{(1/2)n^\varphi_{*,w}}\right) \qquad (A3.16) \\
&\cdot\left(\prod^{**}_{w\in\tau_v} \gamma^{-1/8}\right)\left(\prod^{***}_{w\in\tau_v} \gamma^{\frac{h_w}{2(n^\varphi+n^\jmath)}} e^{-\frac{c}{2(n^\varphi+n^\jmath)}\sqrt{\gamma^{h_w}D_{0,w}}}\right) .
\end{aligned}$$

The product $\prod_{w\in\tau_v}^{**}$ is over all the nodes in the tree, except the ones higher than the branching points w with $h_w \geq N+1$ and $n_w = 0$. The product $\prod_{w\in\tau_v}^{***}$ is over all the branching points w with $h_w \geq N+1$ and $n_w = 0$; and the factors $\gamma^{\frac{h_w}{2(n^\varphi+n^J)}}$ – strictly greater than 1 – are added for later purposes.

This bound is enough to prove the convergence of the Schwinger function. Indeed, for any $m > (1/4)$ and $d, \beta, z > 0$, the two inequalities hold:

$$z^m e^{-(\beta/m)\sqrt{z}} \leq C_\beta^{2m}(4m)!\,, \tag{A3.17}$$

$$\begin{aligned}\sum_{h=-\infty}^{+\infty} \left(\gamma^h d\right)^m e^{-(\beta/m)\sqrt{\gamma^h d}} &\leq \sum_{h\leq 0}\gamma^{hm} + \sum_{h>0}\gamma^{hm}e^{-(\beta/m)\sqrt{\gamma^h}} \\ &\leq C_\beta^{4m}(8m)!(1-\gamma^{-(1/8)})^{-1}\,.\end{aligned} \tag{A3.18}$$

Then (A3.17) allows to bound each factor of the product $\prod_{w\in\tau_v^*}^{h_w\leq N}$, as:

$$\begin{aligned}&\frac{\gamma^{h_w m_{*,w}\left(1-(1/8m)\right)}}{e^{\frac{c}{2(n^\varphi+n^J)}\sqrt{\gamma^{h_w}D_w}}}\left(\frac{Z_{h_w}^{(2)}}{Z_{h_w}}\right)^{n^J_{*,w}}\left(\frac{1}{\sqrt{Z_{h_w}}}\right)^{n^\varphi_{*,w}} \\ &\qquad\leq C_w\left(\frac{1}{D_w}\right)^{m_{*,w}(1-(1/8m)-\eta_\lambda)+n^J_{*,w}\eta_\lambda^{(2)}}\,,\end{aligned}$$

for $C_w \sim (m!)^p$, for some positive integer p; and $\prod_{w\in\tau_v^*}^{h_w\leq N} D_w^{-m_{*,w}(1-(1/8m)-\eta_\lambda)-n^J_{*,w}\eta_\lambda^{(2)}}$ is integrable against test functions which vanish with all their derivatives for each $D_w = 0$. Furthermore, (A3.18) allows in a similar manner to control the summation over the scales of the branching points with $n_w = 0$ of the factors in the product $\prod_{w\in\tau_v}^{***}$: apart a constant, it gives a factor $\prod_{w\in\tau_v}^{***} D_{0,w}^{-[1/2(n^\varphi+n^J)]}$, which is integrable against a test function, even if it does not vanish for $D_{0,w}$, since the number of the factor is not larger than $n^\varphi + n^J$.

The summation over the scales $\underline{h}, \underline{k}$, taking fixed the lowest, h, and also over the scales of all the remaining branching point in the tree τ_v is clearly controlled by the factors $\prod_{w\in\tau_v}\gamma^{-1/8}$ and, since the number of the branches in a tree is no more than twice the number of the endpoints, it is bounded by $(1-\gamma^{-1/8})^{-2(n+n^\varphi+n^J)}$.

Then it is possible to take the summation also over $-\infty < h_0 \leq N$, which is convergent by (A3.18), and gives a further factor $D_v^{-(m+1/8)}$, which, besides not to waste the integrability against the test function, guarantees the cluster decomposition, namely that the Schwinger function vanish if the distance of *any two points* is sent to infinity.

The summation over the topology of the trees, is bounded by $16^{(n+n^\varphi+n^J)}$. Finally the summation over n is convergent for any $\varepsilon \leq \left(16C_3(1-\gamma^{-1/8})^{-2}\right)^{-1}$.

The lemma is proved. ■

A3.5 Short memory property. Before performing the summation over the scales in the product $\prod_{w\in\tau_v}\gamma^{-1/8}$, it is possible to extract a factor $\gamma^{-(1/16)\left((h_{\max}\wedge N)-\eta_{\min}\right)}$, for $h_{\max}$ and

$h_{\min}$ respectively the scale of the one of the maximal nodes and of the minimal node of the tree, leaving $\prod_{w\in\tau_v}\gamma^{-1/16}$ to control such a summation.

Many consequences derives from such a factor. An example is the following lemma.

Lemma A.3.4. *In the limit of removed cutoff, the trees with unbounded maximal scale gives vanishing contribution to the integration of the Schwinger function against the test functions.*

Proof . Before removing the cutoff, let $M_N \stackrel{def}{=} h_{\max}\wedge N$; then $M_N \to +\infty$. With reference to the summation over $-\infty < h \le N$ of the factor $\gamma^{h(m+1/8)}e^{-\frac{c}{2(n^{\varphi}+n^{J})}\sqrt{\gamma^{h_0}D_{v_0^*}}}$, the following facts hold.

1. Since the integration against test functions over all the space time is finite, the integration in the region $\kappa|D_{v_0^*}|\le\gamma^{-(M_N/4)}$ is vanishing.
2. In the domain $\kappa|D_{v_0^*}|\le\gamma^{-(M_N/4)}$, the summation for $h\ge(M_N/2)$ is vanishing faster than $e^{-\frac{c}{4(n^{\varphi}+n^{J})}\gamma^{M_n/8}}$.
3. Trees with first node on scale $h\le(M_N/2)$ have a short memory factor $\le\gamma^{-(1/16)(M_N/2)}$, which is vanishing too.

A3.6 Completion of the proof of Theorem 1.1 The bound for the two point Schwinger function is, accordingly to (A3.16), for ε small enough,

$$\left|S_{\omega}^{(2)}(x-y)\right|\le C\sum_{h=h^*}^{N}\frac{\gamma^h}{e^{(c/4)\sqrt{\gamma^h\kappa|x-y|}}}\frac{1}{Z_h}\ . \tag{4.3.19}$$

Setting h_o s.t. $\gamma^{-h_o}\le k|x-y|<\gamma^{-h_o+1}$, if $h_o<h^*$, then

$$\sum_{h=h^*}^{N}\frac{\gamma^h}{e^{(c/4)\sqrt{\gamma^h\kappa|x-y|}}}\frac{1}{Z_h}\le K\frac{\gamma^{h^*}}{e^{(c/8)\sqrt{\gamma^{h^*}\kappa|x-y|}}}\frac{1}{Z_{h^*}}\ ;$$

while, if $h_o>h^*$, then

$$\sum_{h=h^*}^{N}\frac{\gamma^h}{e^{(c/4)\sqrt{\gamma^h\kappa|x-y|}}}\frac{1}{Z_h}\le K\gamma^{h_o}\frac{1}{Z_{h_o}}\ .$$

Since μ_{h^*} is proportional to $\kappa\gamma^{h^*}$, then: μ_{h^*} is proportional to $\kappa(\mu/\kappa)^{(1/1+\overline{\eta}_\lambda)}$; Z_{h^*} is proportional to $(\mu/\kappa)^{-(\eta_\lambda/1+\overline{\eta}_\lambda)}$; for $h_o\le N/2$, in the limit $N\to+\infty$, Z_{h_o} is proportional to $(\kappa|x-y|)^{\eta_\lambda}$. Hence the item is proved for $1+\overline{\tau}_\lambda\stackrel{def}{=}(1/1+\overline{\eta}_\lambda)$. ■

A3.7 Completion of the proof of Theorem 1.2 The bound for the current-current Schwinger function is the same of (4.3.19), with the replacement of γ^h/Z_h with $\gamma^{2h}(Z_h^{(2)}/Z_h)^2$. Therefore, with the same developments of Proof A3.6, using also the identity $\eta_\lambda=\eta_\lambda^{(2)}$, also this item is verified. ■

Appendix 4:

Exact symmetries

The following symmetries will be useful to prove some kernels are less divergent than what seems from dimensional bounds:

A4.1 Reflection. Let the "reflection" be $\vartheta(k_0, k_1) \stackrel{def}{=} (-k_0, -k_1)$. It is easy to verify the interactions $\mathcal{V}$, $\mathcal{N}$ and $\mathcal{D}$, as well as the free action, are all invariant under the transformation of the fields

$$\psi^{\sigma}_{k,\omega} \to i\omega\psi^{\sigma}_{\vartheta k,\omega} . \qquad (A4.1)$$

In terms of graphs, under reflection the propagator $\widehat{g}^{(j)}_{\mu,\omega}(k)$ transforms as follows

$$\widehat{g}^{(j)}_{\mu,\omega}(\vartheta k) = -\mu\omega\widehat{g}^{(j)}_{\mu,\omega}(k) ; \qquad (A4.2)$$

while the interactions are all invariant, except the ones corresponding to the interactions $\mathcal{D}$, which is odd. Specifically, let any graph contributing to the kernel $\widehat{W}^{(j)}_{2,\omega,\omega}(k)$ be considered: calling $m_2(\omega)$ and $m_2(-\omega)$ respectively the number of vertices with interaction linear in $\psi_\omega\psi_\omega$ and $\psi_{-\omega}\psi_{-\omega}$, after the contraction of only the off-diagonal propagators, they are left $2(l + m_2(\omega) - 1)$ half lines of kind ω and $2(l + m_2(-\omega))$ half lines of kind $-\omega$ to be contracted with diagonal (*odd*) propagators. As the number of odd vertices is $m_2(\omega)+m_2(-\omega)$, and the number of odd propagators is $2l+m_2(\omega)+m_2(-\omega)-1$, then $\widehat{W}^{(j)}_{2,\omega,\omega}(k)$ is odd. With a similar argument it is possible to prove $\widehat{W}^{(j)}_{2,\omega,-\omega}(k)$ is even. Therefore

$$\widehat{W}^{(j)}_{2,\alpha,\beta}(\vartheta k) = -\alpha\beta\widehat{W}^{(j)}_{2,\alpha,\beta}(k) , \qquad \left(\partial_\sigma\widehat{W}^{(j)}_{2,\alpha,\beta}\right)(\vartheta k) = \alpha\beta\left(\partial_\sigma\widehat{W}^{(j)}_{2,\alpha,\beta}\right)(k) . \qquad (A4.3)$$

A4.2 Space reflection. Let the "space reflection" be $\vartheta_1(k_0,k_1)\stackrel{def}{=}(k_0,-k_1)$. It is easy to verify the interactions $\mathcal{V}$, $\mathcal{N}$ and $\mathcal{D}$, as well as the free action, are all invariant under the transformation of the fields

$$\psi^\sigma_{k,\omega}\to\psi^\sigma_{\vartheta_1 k,-\omega}\ .$$

In terms of graphs, under space reflection the propagator $\widehat{g}^{(j)}_{\alpha,\beta}(k)$ transforms as follows

$$\widehat{g}^{(j)}_{\alpha,\beta}(\vartheta_1 k)=\widehat{g}^{(j)}_{-\alpha,-\beta}(k)\ ;$$

while the vertices are invariant; therefore,

$$\widehat{W}^{(j)}_{2,\alpha,\beta}(\vartheta_1 k)=\widehat{W}^{(j)}_{2,-\alpha,-\beta}(k)\ ,\qquad\left(\partial_\sigma\widehat{W}^{(j)}_{2,\alpha,\beta}\right)(\vartheta_1 k)=\left(\partial_{-\sigma}\widehat{W}^{(j)}_{2,-\alpha,-\beta}\right)(k)\ . \tag{A4.4}$$

Furthermore, with similar arguments, it is easy to prove

$$\widehat{W}^{(j)}_{1;2,\alpha;\beta}(\vartheta_1 p;\vartheta_1 k)=\widehat{W}^{(j)}_{1;2,-\alpha;-\beta}(p;k)\ . \tag{A4.5}$$

A4.3 Rotation. Let the "rotation" of $\pi/2$ be $(k_0,k_1)^*\stackrel{def}{=}(-k_1,k_0)$. It is easy to verify the interactions $\mathcal{V}$ and $\mathcal{N}$, as well as the free action of the massive Thirring model, are invariant under the transformation of the fields:

$$\psi^\sigma_{k,\omega}\to e^{i\omega\frac{\pi}{4}}\psi^\sigma_{k^*,\omega}\ .$$

In terms of graphs, under rotation the propagator $\widehat{g}^{(\mathrm{E},k)}_{\alpha,\beta}(k)$ transforms as follows

$$\begin{aligned}\widehat{g}^{(\mathrm{E},k)}_{\alpha,\beta}(k^*)&=-i\omega\widehat{g}^{(\mathrm{E},j)}_{\omega,\omega}(k)\ ,\\ \widehat{g}^{(\mathrm{E},j)}_{\omega,-\omega}(k^*)&=\widehat{g}^{(\mathrm{E},j)}_{\omega,-\omega}(k)\ .\end{aligned}$$

Let $\widehat{W}^{(\mathrm{E},j)}_{2,\mu,\nu}(k)$ be defined as the sum of the graphs of $\widehat{W}^{(j)}_{2,\mu,\nu}(k)$ which are made only with propagators $\widehat{g}^{(\mathrm{E},j)}_{\mu,\mu}(k)$ and only with vertices $\mathcal{V}$.

Then, each graph of $\widehat{W}^{(\mathrm{E},j)}_{2,\omega,\omega}(k)$ is made of l diagonal propagators $\widehat{g}^{(\mathrm{E},j)}_{\omega,\omega}$ and $l+1$ diagonal propagators $\widehat{g}^{(\mathrm{E},j)}_{-\omega,-\omega}$; whereas each graph of $\widehat{W}^{(\mathrm{E},j)}_{2,\omega,-\omega}(k)$ is made of l diagonal propagators $\widehat{g}^{(\mathrm{E},j)}_{\omega,\omega}$ and l diagonal propagators $\widehat{g}^{(\mathrm{E},j)}_{-\omega,-\omega}$ (and also at least one off-diagonal propagator). Therefore it holds

$$\begin{aligned}\widehat{W}^{(\mathrm{E},j)}_{2,\omega,\omega}(k^*)&=i\omega\widehat{W}^{(\mathrm{E},j)}_{2,\omega,\omega}(k)\ ,&\left(\partial_\sigma\widehat{W}^{(\mathrm{E},j)}_{2,\omega,\omega}\right)(k^*)&=\sigma\omega\left(\partial_\sigma\widehat{W}^{(\mathrm{E},j)}_{2,\omega,\omega}\right)(k)\ ,\\ \widehat{W}^{(\mathrm{E},j)}_{2,\omega,-\omega}(k^*)&=\widehat{W}^{(\mathrm{E},j)}_{2,\omega,-\omega}(k)\ ,&\left(\partial_\sigma\widehat{W}^{(\mathrm{E},j)}_{2,\omega,-\omega}\right)(k^*)&=-i\sigma\left(\partial_\sigma\widehat{W}^{(\mathrm{E},j)}_{2,\omega,-\omega}\right)(k)\ ,\end{aligned} \tag{A4.6}$$

and, with similar definitions and arguments:

$$\widehat{W}^{(\mathrm{E},j)}_{1;2,\mu;\nu}(p^*;k^*)=\mu\nu\widehat{W}^{(\mathrm{E},j)}_{1;2,\mu;\nu}(p;k)\ . \tag{A4.7}$$

Appendix 5:

Proof of Theorem 3.2

A5.1 Beta and Gamma functions. Let $x_N \stackrel{def}{=} (\nu_N, \delta_N)$, $\mu_h \stackrel{def}{=} \mu \overline{Z}_h$ and $\Delta\lambda_h \stackrel{def}{=} \lambda_h - \lambda$; a conventional way of writing the relation (3.3.10), (3.3.13) and (3.3.11), (3.3.12) is in terms of the *Gamma functions*:

$$\begin{aligned}
&\log_\gamma \frac{Z_{h-1}}{Z_h} = \Gamma_h(\lambda_h, x_h; \ldots; \lambda_N, x_N) , \\
&\log_\gamma \frac{\overline{Z}_{h-1}}{\overline{Z}_h} = \overline{\Gamma}_h(\lambda_h, \mu_h, x_h; \ldots; \lambda_N, \mu_N, x_N) , \qquad (A5.1) \\
&\log_\gamma \frac{Z^{(2,\sigma)}_{h-1}}{Z^{(2,\sigma)}_h} = \Gamma^{(2,\sigma)}_h(\lambda_h, x_h; \ldots; \lambda_N, x_N) ;
\end{aligned}$$

and *Beta functions*:

$$\begin{aligned}
&\nu_{h-1} - \gamma\nu_h = \beta^{(\nu)}_h(\lambda_h, x_h; \ldots; \lambda_N, x_N) , \\
&\delta_{h-1} - \delta_h = \beta^{(\delta)}_h(\lambda_h, x_h; \ldots; \lambda_N, x_N) , \qquad (A5.2) \\
&\Delta\lambda_{h-1} - \Delta\lambda_h = \beta^{(\lambda)}_h(\lambda_h, x_h; \ldots; \lambda_N, x_N) .
\end{aligned}$$

Furthermore, such Gamma and Beta function are given by convergent graph expansion.

Lemma A.5.1. *In the domain of the effective parameters given by (3.3.15), if (3.3.14) are satisfied, the Gamma and Beta function in (A5.1) and (A5.2) are well defined and analytic in* $\{\lambda_k, \delta_k, \nu_k\}_{k\le N}$.

Proof . Like the proof of the convergence of the Schwinger function, it is a consequence of the Lemmas A.3.1 and A.3.2, for the set of fixed points, F_v, given by only one point. ∎

The evolution of the effective parameters is determined by the equations (A5.1) and (A5.2), and by fixing the "initial data"; they are chosen to be:

$$\begin{gathered}\Delta\lambda_{-\infty} = 0\ , \qquad \delta_{-\infty} = 0\ , \qquad \nu_{-\infty} = 0\ , \\ \log_\gamma(Z_0) = 0\ , \qquad \log_\gamma(\overline{Z}_0) = 0\ , \\ \log_\gamma(Z_0^{(2,+)}) = 0\ , \qquad \log_\gamma(Z_0^{(2,-)}) = 0\ .\end{gathered} \tag{A5.3}$$

Well then, the strategy to find the solution of the evolution problem is first to skip the flow of the mass, and to find the solution of the other flows by a fixed point theorem in a suitable linear space; then to solve also the flow of the mass with the other flow already fixed.

A5.2 Flows of the couplings. Let $\mathfrak{M}$ be the linear space of sequences y,

$$y \stackrel{def}{=} \left\{ \left(\Delta\lambda_k, \delta_k, \nu_k, \log_\gamma(Z_k), \log_\gamma\left(Z_k^{(2,+)}\right), \log_\gamma\left(Z_k^{(2,-)}\right)\right) \in \mathbb{R}^6 : k \le N \right\}\ ,$$

such that, for any $\vartheta < 1/16$, the following properties hold.

- **i.** The initial data are as in (A5.3).
- **ii.** The increments of the effective coupling satisfy (3.4.1), for any $h : h \le N$.

Then, let such a space be endowed with the norm $||y||_\vartheta$, which is the smallest real number such that all the following inequalities hold.

- **iii.** There exist two positive constants, c_0 and c_1, such that, for every $k \le N$,

$$\begin{gathered} |\Delta\lambda_k| \le c_1 \varepsilon^2 \gamma^{-(\vartheta/2)(N-k)} ||y||_\vartheta\ , \\ |\delta_k| \le 2\varepsilon\gamma^{-(\vartheta/2)(N-k)}||y||_\vartheta\ , \qquad |\nu_k| \le 2\varepsilon\gamma^{-(\vartheta/2)(N-k)}||y||_\vartheta\ , \\ \left|\log_\gamma(Z_{k-1}/Z_k)\right| \le c_0\varepsilon^2 ||y||_\vartheta\ , \\ \left|\log_\gamma(Z_{k-1}^{(2,+)}/Z_k^{(2,+)})\right| \le 2c_0\varepsilon^2||y||_\vartheta\ , \qquad \left|\log_\gamma(Z_{k-1}^{(2,-)}/Z_k^{(2,-)})\right| \le 2c_0\varepsilon^2||y||_\vartheta\ . \end{gathered} \tag{A5.4}$$

The space $\mathfrak{M}_\vartheta$ is defined as $\{y \in \mathfrak{M} : ||y||_\vartheta \le 1\}$ and is clearly complete. Let the equation $y = Ty$ read in $\mathfrak{M}_\vartheta$:

$$\begin{gathered} \Delta\lambda_h = -\sum_{k \le h} \beta_k^{(\lambda)}\ , \qquad \delta_h = -\sum_{k\le h}\beta_k^{(\delta)}\ , \qquad \nu_h = -\sum_{k \le h}\gamma^{-(h-k+1)}\beta_k^{(\nu)}\ , \\ \log_\gamma(Z_h) = \sum_{k=0}^h \Gamma_k\ , \qquad \log_\gamma(Z_h^{(2,\sigma)}) = \sum_{k=0}^h \Gamma_k^{(2,\sigma)}\ , \end{gathered} \tag{A5.5}$$

where, for $h < 0$, let $\sum_{k=0}^h \stackrel{def}{=} -\sum_{k=h}^0$.

Lemma A.5.2. *There exist $\varepsilon > 0$, and $c, c_0, c_1 > 0$ such that there exists a (unique) solution to (A5.5) in the space $\mathfrak{M}_\vartheta$, for c_0 and c_1 the constants in (A5.4), and c the constant in (3.4.1). Furthermore, such a solution is analytic in λ.*

Proof. The equation makes sense since $\|y\|_\vartheta \leq 1$ and $|\lambda| \leq \varepsilon$, together to the first of (A5.4), for ε small enough, imply (3.3.14) and (3.3.15), and hence Lemma A.5.1.

The existence of a solution is consequence of the fact that T is a contraction from $\mathfrak{M}_\vartheta$ into itself. Indeed, because of the following arguments, if $y \in \mathfrak{M}_\vartheta$, then $Ty \in \mathfrak{M}_\vartheta$.

1. By inductive hypothesis and convergence of the graph expansion, there exists a constant $c_2 \geq 0$, such that $|z_{h-1}| \leq c_2\varepsilon^2$; hence, for ε small enough and $c_0 \geq 2c_2$, it holds the statement in (A5.4) regarding the field strength flow.

2. For the density strengths, by definitions (3.3.13), it is more convenient to define two new strengths, $\zeta_k^{(\mathrm{u})} \stackrel{def}{=} (\zeta_k^{(2,+)} + \zeta_k^{(2,-)})/2$ and $\zeta_k^{(\mathrm{d})} \stackrel{def}{=} (\zeta_k^{(2,+)} - \zeta_k^{(2,-)})/2$, so that the their flows are given by

$$\frac{\zeta_{h-1}^{(\mathrm{u})}}{\zeta_h^{(\mathrm{u})}} = \frac{Z_h}{Z_{h-1}} \left(1 + z_{h-1}^{(2)} + \Delta z_{h-1}^{(2,+)} + \Delta z_{h-1}^{(2,-)}\right) ,$$
$$\frac{\zeta_{h-1}^{(\mathrm{d})}}{Z_h^{(\mathrm{d})}} = \frac{Z_h}{Z_{h-1}} \left(1 + z_{h-1}^{(2)} + \Delta z_{h-1}^{(2,+)} - \Delta z_{h-1}^{(2,-)}\right) .$$

Then, an argument similar to the one of the previous item proves statement in (3.3.14) regarding the density strengths.

3. For the flow of the effective coupling, the argument is more involved: it is based on a cancellation, the vanishing of the Beta function, which exactly holds only in the limit of removed cutoff. Let $\beta_k^{(\mathrm{T})}(\lambda_k, \dots, \lambda_N)$ be the sum of the graphs of $\beta_k^{(\lambda)}$ which are made only with diagonal propagators $\{g_{\omega,\omega}^{(\mathrm{E1}),\mathrm{k}}\}_k$ and interactions $\mathcal{V}$; then, setting all the arguments equal, let $\beta_k^{(\mathrm{T})}(\lambda_k) \stackrel{def}{=} \beta_k^{(\mathrm{T})}(\lambda_k, \dots, \lambda_k)$. As proved in 4.2.5, there exists a constant $c_2 \geq 0$ such that $|\beta_k^{(\mathrm{T})}(\lambda_k)| \leq c_2\varepsilon^2\gamma^{-\vartheta(N-k)}$. Accordingly, it is convenient to expand each coupling λ_m in the function $\beta_k^{(\mathrm{T})}(\lambda_k, \dots, \lambda_N)$ as $\lambda_m = \lambda_k + (\lambda_m - \lambda_k)$, so that the following decomposition of the whole Beta function holds:

$$\beta_k^{(\lambda)} = \beta_k^{(\mathrm{T})}(\lambda_k) + \sum_{m=k}^{N} \beta_{k,m}^{(\mathrm{T},\lambda)}(\lambda_m - \lambda_k) + \sum_{m=k}^{N} \beta_{k,m}^{(\mathrm{R},\lambda)} + \sum_{a=\delta,\nu} \sum_{m=k}^{N} \beta_{k,m}^{(\lambda,a)} a_m ,$$

where $\beta_{k,m}^{(\mathrm{T},\lambda)}$ is the sum of the graphs in $\beta_k^{(\mathrm{T})}(\lambda_k, \dots, \lambda_N)$, with the replacement of the all the couplings $\lambda_n : k \leq n < m$ with λ_k, and a coupling $\lambda_m - \lambda_k$ on scale m put apart from it; $\beta_{k,m}^{(\mathrm{R},\lambda,)}$ is the sum of the graphs made with interactions $\mathcal{V}$ and with at least one propagator $g_{\omega,\omega}^{(\mathrm{R1},\mathrm{m})}$ on scale m; $\beta_{k,m}^{(\lambda,a)}$ is the sum of the graphs with at least one coupling a_m on scale m and only diagonal propagators $g_{\omega,\omega}^{(\mathrm{E1},\mathrm{m})}$ – if a graph falls in more than one category the assignment is arbitrary. By the convergence of power expansion in λ, as stated in A.5.1, and the short memory property of the tree ordering, the following bounds holds for the same constant c_2 – if it is chosen large enough:

$$|\beta_{k,m}^{(\mathrm{T},\lambda)}| \leq c_2\varepsilon\gamma^{-\vartheta(m-k)} , \qquad |\beta_{k,m}^{(\mathrm{R},\lambda)}| \leq \gamma^{-(3/4)(N-m)} c_2\varepsilon^2\gamma^{-\vartheta(m-k)} ,$$

$$|\beta_{k,m}^{(\lambda,a)}| \leq c_2\varepsilon^2\gamma^{-\vartheta(m-k)} .$$

It is straightforward to conclude that, to obtain (A5.4) and (3.4.1), as far as the flow $\{\lambda_h\}_h$ is regarded, c_1 and c have to be chosen $c \geq 4c_2(1-\gamma^{-(\vartheta/2)})^{-1}$ and $c_1 \geq c(1-\gamma^{-(\vartheta/2)})^{-1}$.

4. Similarly, it is possible to decompose the Beta function for the couplings $a = \delta, \nu$:

$$\beta_k^{(a)} \stackrel{def}{=} \sum_{m=k}^{N} \beta_{k,m}^{(a,\mathrm{R})} + \sum_{b=\nu,\delta} \sum_{m=k}^{N} \beta_{k,m}^{(a,b)} b_m \ ,$$

where $\beta_{k,m}^{(a,\mathrm{R})}$ contains all the graphs made only with interactions $\mathcal{V}$ and with at least one diagonal propagator $g_{\omega,\omega}^{(\mathrm{R1,m})}$ on scale m; whereas $\beta_{k,m}^{(a,b)}$ is made with all the graphs with an interaction b on scale m and only diagonal propagators $g_{\omega,\omega}^{(\mathrm{E1,m})}$ – in ambiguous cases the assignment is arbitrary. Again, by convergence of the power expansion in λ, and by the short memory property of the tree ordering,

$$|\beta_{k,m}^{(a,2)}| \leq \gamma^{-\vartheta(N-m)} c_2 \varepsilon^2 \gamma^{-\vartheta(m-k)} \ , \qquad |\beta_{k,m}^{(a,b)}| \leq c_2 \varepsilon^2 \gamma^{-\vartheta(m-k)} \ ;$$

and since for ε small enough $5c_2\varepsilon^2\left(1-\gamma^{-(\vartheta/2)}\right)^{-1} \leq 2\varepsilon$, then (A5.4) holds also for what concerns $\{\delta_k\}_k$ and $\{\nu_k\}_k$.

Therefore Ty is in $\mathfrak{M}_\vartheta$ for ε small enough; and, by Lemma A.5.1, if y is analytic in $\lambda : |\lambda| \leq \varepsilon$, then also Ty does. The next step is to prove that, taken any two $y, y' \in \mathfrak{M}_\vartheta$, it holds $||Ty - Ty'||_\vartheta \leq \rho ||y - y'||_\vartheta$, for a constant $\rho < 1$.

1. The variation of the Beta function $\beta^{(\lambda)}$ due to the variation of the y is given by:

$$\begin{aligned}\beta_k^{(\lambda)} - {\beta'}_k^{(\lambda)} = &\sum_{m=k}^{N} \Delta\beta_{k,m}^{(\lambda)}(\lambda_m - \lambda'_m) + \sum_{m=k}^{N} \beta_{k,m}^{(\mathrm{T},\lambda)}\left[(\lambda_m - \lambda_k) - (\lambda'_m - \lambda'_k)\right] \\ &+ \sum_{m=k}^{N} \Delta\beta_{k,m}^{(\lambda,Z)}\left(\frac{Z_{m-1}}{Z_m} - \frac{Z'_{m-1}}{Z'_m}\right) + \sum_{a=\delta,\nu}\sum_{m=k}^{N} \Delta\beta_{k,m}^{(\lambda,a)}(a_m - {a'}_m) \ ,\end{aligned}$$

where $\Delta\beta_{k,m}^{(\lambda)}$ corresponds to a variation of the coupling λ_m in one of the two previously defined $\beta_{k,m}^{(\mathrm{T})}$ and $\beta_{k,m}^{(\mathrm{T},\lambda)}$; the term $\Delta\beta_{k,m}^{(\lambda,Z)}$ is due to a variation one factor Z_{m-1}/Z_m; and $\Delta\beta_{k,m}^{(\lambda,a)}$ to a variation of a_m. Since the power series of the variation has the same domain of convergence of the Beta function itself, and since the vanishing of the Beta function holds for each order of the power series, using also the short memory property, the following bounds holds for a suitable constant $c_3 \geq 0$:

$$|\Delta\beta_{k,m}^{(\lambda)}| \leq \gamma^{-(\vartheta/2)(N-k)} c_3 \varepsilon \gamma^{-\vartheta(m-k)} \ , \qquad |\Delta\beta_{k,m}^{(\lambda,a)}| \leq c_3 \varepsilon^2 \gamma^{-\vartheta(m-k)} \ ,$$

$$|\Delta\beta_{k,m}^{(\lambda,Z)}| \leq \gamma^{-(\vartheta/2)(N-k)} c_3 \varepsilon^2 \gamma^{-\vartheta(m-k)} \ ,$$

where the factors $\gamma^{-(\vartheta/2)(N-k)}$ in the first and third bound come from the bound on the Beta function on its own, which has been made previously.

2. The variation of the Beta functions $\{\beta^{(a)}\}_{a=\nu,\delta}$ is given by:

$$\beta_k^{(a)} - {\beta'}_k^{(a)} \stackrel{def}{=} \sum_{m=k}^{N} \Delta\beta_{k,m}^{(a,\lambda)}(\lambda_m - \lambda'_m) + \sum_{m=k}^{N} \Delta\beta_{k,m}^{(a,Z)} \left(\frac{Z_{m-1}}{Z_m} - \frac{Z'_{m-1}}{Z'_m}\right) + \sum_{b=\nu,\delta} \sum_{m=k}^{N} \Delta\beta_{h,m}^{(a,b)} (b_m - {b'}_m) \ ,$$

where $\Delta\beta_{k,m}^{(a,\lambda)}$ is due to the variation of the coupling λ_m; $\Delta\beta_{k,m}^{(a,Z)}$ to the variation of the ratio Z_{m-1}/Z_m; $\Delta\beta_{k,m}^{(a,\lambda)}$ to the variation of the coupling β_m. And they holds the bounds:

$$|\Delta\beta_{k,m}^{(a,\lambda)}| \le c_3 \varepsilon \gamma^{-\vartheta(m-k)} \gamma^{-(\vartheta/2)(N-k)} \ , \qquad |\Delta\beta_{h,k}^{(a,b)}| \le c_3 \varepsilon^2 \gamma^{-\vartheta(m-k)} \ ,$$

$$|\Delta\beta_{k,m}^{(a,Z)}| \le \gamma^{-(\vartheta/2)(N-k)} c_3 \varepsilon^2 \gamma^{-\vartheta(m-k)} \ .$$

3. The variation of the Gamma function of the field strength is

$$\Gamma_k - {\Gamma'}_k \stackrel{def}{=} \sum_{m=k}^{N} \Delta\Gamma_{k,m}^{(\lambda)}(\lambda_m - \lambda'_m) + \sum_{m=k}^{N} \Delta\Gamma_{k,m}^{(Z)} \left(\frac{Z_{m-1}}{Z_m} - \frac{Z'_{m-1}}{Z'_m}\right) + \sum_{b=\nu,\delta} \sum_{m=k}^{N} \Delta\Gamma_{h,m}^{(b)} (b_m - {b'}_m) \ .$$

with clear justification of the various addends. Now, by the short memory property,

$$|\Delta\Gamma_{k,m}^{(\lambda)}| \le c_3 \varepsilon \gamma^{-\vartheta(m-k)} \ , \qquad |\Delta\Gamma_{k,m}^{(Z)}| \le c_3 \varepsilon^2 \gamma^{-\vartheta(m-k)} \ ,$$

$$|\Delta\Gamma_{k,m}^{(b)}| \le c_3 \varepsilon \gamma^{-\vartheta(m-k)} \ .$$

4. Similar arguments hold for the field strengths.

By such bounds, the operator T is a contraction with rate $\rho \stackrel{def}{=} e^2(c_3c_1 + 2c_2c_1 + c_3c_0 + 2c_3)$: for ε small enough, $\rho < 1$. The proof of the Lemma is obtained by the fixed point theorem with analytic parameterization. ■

Once the flows y has been found, it is possible to consider the flow for the mass:

$$\log_\gamma(\overline{Z}_h) = \sum_{k=0}^{h} \overline{\Gamma}_k \ , \qquad (A5.6)$$

restricted to the range $0 \le k \le N$. In the remaining scales, $h^* \le k < 0$, in fact, the flow is determined directly, and not by an equation; and since h^*, in its turn, depends on the flow, it is more convenient to exclude it from the fixed point theorem.

As for the other flow, it is defined the linear space $\overline{\mathfrak{M}}$ of the sequences

$$x \stackrel{def}{=} \left\{\log_\gamma(\overline{Z}_k) \in \mathbb{R} : 0 \le k \le N\right\}$$

such that

i. the initial datum is as in (A5.3).

Furthermore, such a space is endowed with the norm $||x||$, the lowest real number such that

ii. for the same constant c_0 in (A5.4), and for $0 \le k \le N$,

$$\left|\log_\gamma(\overline{Z}_{k-1}/\overline{Z}_k)\right| \le 2c_0\varepsilon^2||x|| \,. \tag{A5.7}$$

The equation $x = Tx$, which is defined to be (A5.6), can be solved in $\overline{\mathfrak{M}}_\vartheta$, the subspace of $\overline{\mathfrak{M}}$ of the sequences x with $||x|| \le 1$, with the fixed point theorem.

Lemma A.5.3. *There exists $\varepsilon > 0$ and the positive constant c_0 such that there exists a (unique) solution of (A5.6) in the space $\overline{\mathfrak{M}}_\vartheta$, for c_0 the constant in (A5.7).*

1. If $x \in \overline{\mathfrak{M}}_\vartheta$, then also $Tx \in \overline{\mathfrak{M}}_\vartheta$ by the following argument. The local part s_{h-1} is the sum of the graphs with one antidiagonal propagator $g^{(\mathrm{E1},k)}_{\omega,-\omega}$ or $g^{(\mathrm{R1},k)}_{\omega,-\omega}$. As consequence of the convergence of the graphs expansion and of the dimensional bounds of s_{h-1}, calling $s_{h-1,k}$ the sum of all the graphs of s_{h-1} with $g^{(\mathrm{E1},k)}_{\omega,-\omega}$ or $g^{(\mathrm{R1},k)}_{\omega,-\omega}$ on scale k and divided by $\mu_k/\kappa\gamma^k$,

$$s_{h-1} \stackrel{def}{=} \sum_{k=h}^{N} s_{h-1,k}\frac{\mu_k}{\kappa\gamma^k} \,, \qquad \text{with } |s_{h-1,k}| \le \gamma^{h-1}c_2\varepsilon \,.$$

By (A5.7), for ε small enough, it holds $(\mu_k/\mu_h) \le \gamma^{2c_0\varepsilon(k-h)} < \gamma^{(1/2)(k-h)}$, and hence $m_{h-1} = (s_{h-1}/\mu_h) \le c_1(1-\gamma^{-(1/2)})^{-1}\varepsilon$: since by (A5.4) $\gamma^{-c_0\varepsilon^2}(Z_{h-1}/Z_h) \le \gamma^{c_0\varepsilon^2}$ and $\log_\gamma(1+m_{h-1}) \le \left|m_{h-1}\ln(\gamma)\int_0^1 \mathrm{d}t(1+tm_{h-1})^{-1}\right|$, it is straightforward to obtain that $\gamma^{-2c_0\varepsilon} \le (\mu_{h-1}/\mu_h) \le \gamma^{2c_0\varepsilon}$ for ε small enough and $c_0 \ge 2c_2(1-\gamma^{-(1/2)})^{-1}$.

2. If $x, x' \in \overline{\mathfrak{M}}_\vartheta$, then $||Tx - Tx'|| \le \rho||x - x'||$, for $\rho < 1$. Indeed, under variation of the mass flow, – having fixed all the other flows –

$$\overline{\Gamma}_k - \overline{\Gamma}'_k \stackrel{def}{=} \sum_{m=k}^{N} \Delta\overline{\Gamma}^{(\mu)}_{k,m}\left(\frac{\mu_m}{\mu_k} - \frac{\mu'_m}{\mu'_k}\right) \,.$$

Now, by the short memory property, and by (A5.7),

$$|\Delta\overline{\Gamma}^{(\mu)}_{k,m}| \le c_3\varepsilon\gamma^{-\vartheta(m-k)} \,, \qquad \left|\frac{\mu_m}{\mu_k} - \frac{\mu'_m}{\mu'_k}\right| \le c_4\gamma^{(\vartheta/2)(m-k)}\sup_{n\ge0}\left|\overline{\Gamma}_n - \overline{\Gamma}'_n\right| \;;$$

– indeed, $|(\mu_m/\mu_k) - (\mu'_m/\mu'_k)| \le \max\left\{(\mu_m/\mu_k), (\mu'_m/\mu'_k)\right\}\ln(\gamma)\sum_{n=k}^{m}\left|\overline{\Gamma}_n - \overline{\Gamma}'_n\right|$, which, by (A5.7), is less or equal to $(4/\vartheta)\ln(\gamma)\gamma^{\left(2c_0\varepsilon+(\vartheta/4)\right)(m-k)}\sup_n\left|\overline{\Gamma}_n - \overline{\Gamma}'_n\right|$. Then the assertion follows enlarging c_0 chosen for the field strength to $c_0 \ge c_3c_4(1-\gamma^{-(\vartheta/2)})^{-1}$.

This proves the Lemma. ∎

A5.3 Further properties of the Gamma functions. In order to complete the proof of the Theorem 3.2, it is left to prove the existence of the critical indexes η_λ, $\eta_\lambda^{(2)}$ and $\overline{\eta}_\lambda$, which only depends on the choice of λ and on the graphs that can be obtained using the diagonal propagator $\{g_{\omega,\omega}^{(\mathrm{E1},h)}\}$ and the interaction $\mathcal{V}$, and not from the mass, or from the regularization of the model. Indeed, let it be inductively supposed that there exists a positive constant c_2 such that, for any $k : h \le k \le N$,

$$\frac{Z_{k-1}}{Z_k} = \gamma^{\Gamma_k^{(0)}+\Gamma_k^{(1)}} , \qquad \text{with } |\Gamma_k^{(1)}| \le c_4 \varepsilon^2 \gamma^{-(\vartheta/2)(N-k)} , \tag{A5.8}$$

while $\Gamma_k^{(0)}$ is given in terms of graphs made only with the diagonal propagator $\{g_{\omega,\omega}^{(\mathrm{E1},h)}\}$ and the interaction $\lambda\mathcal{V}$, and bounded, $|\Gamma_h^{(0)}| \le c_2\varepsilon^2$. Then, let the following decomposition be considered:

$$\begin{aligned} z_{h-1} =& z_{h-1}^{(0)} + \sum_{k=h}^{N} \Delta z_{h-1,k}^{(\lambda)} \Delta\lambda_k + \sum_{k=h}^{N} \Delta z_{h-1,k}^{(Z)} \left(\frac{Z_{k-1}}{Z_k} - \gamma^{\Gamma_k^{(0)}} \right) \\ &+ \sum_{k=h}^{N} \Delta z_{h-1,k}^{(2)} + \sum_{a=\delta,\nu} \sum_{k=h}^{N} \Delta z_{h-1,k}^{(a)} a_k , \end{aligned}$$

where $z_{h-1}^{(0)}$ is the sum of the graphs contributing to z_{h-1} which are made only with propagators $\{g^{(\mathrm{E1},k)}\}$ and interactions $\mathcal{V}$, with all the coupling $\{\lambda_k\}_k$ replaced by coupling λ and all the ratios (Z_{k-1}/Z_k) replaced by $\gamma^{\Gamma_k^{(0)}}$; $\Delta z_{h-1,k}^{(\lambda)}$ is due to the replacement of λ_k with $\Delta\lambda_k$; $\Delta z_{h-1,k}^{(Z)} \left[(Z_{k-1}/Z_k) - \gamma^{\Gamma_k^{(0)}} \right]$ is the sum of the same graphs, but with at least a factor $(Z_{k-1}/Z_k) - \gamma^{\Gamma_k^{(0)}}$ in place of the ratio (Z_{k-1}/Z_k); $\Delta z_{h-1}^{(2)}$ is the sum of the graphs which do not contain interactions $\mathcal{N}$ or $\mathcal{D}$, and have a propagator $g^{(\mathrm{R1},k)}$ on scale k; $\Delta z_{h-1,k}^{(a)}$ is the sum of the graphs with an interaction $a = \delta, \nu$ on scale k – whenever a graph falls in more than one of the above categories, the assignment is made in arbitrary way. Because of the following bound

$$|z_{h-1}^{(0)}| \le c_3\varepsilon^2 , \qquad |\Delta z_{h-1,k}^{(\lambda)}| \le c_3\varepsilon^2 \gamma^{-\vartheta(k-h+1)} , \qquad |\Delta z_{h-1,k}^{(Z)}| \le c_3\varepsilon^2 \gamma^{-\vartheta(k-h+1)} ,$$

$$|\Delta z_{h-1,k}^{(2)}| \le \gamma^{-\vartheta(N-k)} c_3\varepsilon^2 \gamma^{-\vartheta(k-h+1)} , \qquad |\Delta z_{h-1,k}^{(a)}| \le c_3\varepsilon^2 \gamma^{-\vartheta(k-h+1)} ,$$

$$|\Delta\lambda_k| \le c_1\varepsilon^2 \gamma^{-(\vartheta/2)(N-k)} , \qquad |(Z_{k-1}/Z_k) - \gamma^{\Gamma_k^{(0)}}| \le 2c_4\varepsilon^2\gamma^{-(\vartheta/2)(N-k)} ,$$

$$|a_k| \le 2\varepsilon\gamma^{-(\vartheta/2)(N-k)} ,$$

the property (A5.8) follows straightforwardly for $c_4 \ge 5c_3(1+c_1)(1-\gamma^{-(\vartheta/2)})^{-1}$ and

$$\Gamma_h^{(0)} \stackrel{def}{=} \log_\gamma \left(1 + z_{h-1}^{(0)} \right) .$$

By construction, $\Gamma_h^{(0)}$ is the sum of scaling invariant graphs: again using the fixed point theorem theorem with analytic parameterization, it is possible to prove the existence of η_λ, limit for

$N \to \infty$ of $\Gamma_h^{(0)}$, analytic in λ and such that there exists a constant c_5 for which $\left|\Gamma_h^{(0)} - \eta_\lambda\right| \leq c_5\varepsilon^2\gamma^{-(\vartheta/2)(N-h)}$, and then the statements in (3.4.2) referring to the field strength flow holds for $c_2 \geq (c_4 + c_5)(1 - \gamma^{-(\vartheta/2)})^{-1}$.

For the Gamma function of the mass a similar argument can be applied. Let it be inductively supposed for any $k : h \leq k \leq N$ that

$$\frac{\mu_k}{\mu_{k+1}} = \gamma^{\overline{\Gamma}^{(0)}_{k+1} + \overline{\Gamma}^{(1)}_{k+1}} , \qquad \text{with } |\overline{\Gamma}_k^{(0)}| \leq c_2\varepsilon , \quad |\overline{\Gamma}_k^{(1)}| \leq c_4\varepsilon\gamma^{-(\vartheta/2)(N-k)} , \tag{A5.9}$$

and $\overline{\Gamma}_k^{(0)}$ only made with the propagator $\{g^{(\mathrm{E1},k)}\}_k$ and interactions $\lambda\mathcal{V}$. It follows that $(\mu_k/\mu_h) = \gamma^{-\sum_{m=h}^{k-1}\overline{\Gamma}_m^{(0)}} + \overline{\Delta}_{k,h}$ with $|\overline{\Delta}_{k,h}| \leq c_6\varepsilon\gamma^{-(\vartheta/2)(N-k)}$, for $c_6 \geq 2c_4 2(1-\gamma^{-\vartheta})^{-1}$ and ε small enough. Then, with a decomposition similar to the case of the field strength:

$$\begin{aligned} m_{h-1} =& m_{h-1}^{(0)} + \sum_{k=h}^{N} \Delta m_{h-1,k}^{(\lambda)}(\lambda_k - \lambda) + \sum_{k=h}^{N} \Delta m_{h-1,k}^{(Z)} \left(\frac{Z_{k-1}}{Z_k} - \gamma^{\Gamma_k^{(0)}}\right) \\ &+ \sum_{k=h}^{N} \Delta m_{h-1,k}^{(1)}\overline{\Delta}_{k,h} + \sum_{k=h}^{N} m_{h-1,k}^{(2)} + \sum_{a=\delta,\nu}\sum_{k=h}^{N} m_{h-1,k}^{(a)} a_k ; \end{aligned}$$

where $m_{h-1}^{(0)}$ is the sum of the graphs made only with interactions $\lambda\mathcal{V}$, all the ratios $\{Z_{m-1}/Z_m\}$ replaced with $\gamma^{\Gamma_k^{(0)}}$, all the ratios $\{\mu_m/\mu_h\}$ replaced with $\gamma^{-\sum_{n=h}^{m-1}\Gamma_n^{(0)}}$ and all diagonal propagators $g_{\omega,\omega}^{(\mathrm{E1},k)}$ on scale $k \geq h$, except one, which is antidiagonal, $g_{\omega,-\omega}^{(\mathrm{E1},k)}$; $\Delta m_{h-1,k}^{(\lambda)}$ is the sum of the graphs of m_{h-1} with all the couplings $\{\lambda_m\}_m$ replaced, for $m < k$, by λ, and at a coupling λ_k neglected; $\Delta m_{h-1,k}^{(1)}$ is the sum of the graphs in which one ratio μ_k/μ_h neglected. Then equation (A5.9) holds true also in the case $k = h - 1$ for c_4 large enough and

$$\overline{\Gamma}_h^{(0)} \stackrel{def}{=} \Gamma_h^{(0)} + \log_\gamma\left(1 + m_{h-1}^{(0)}\right) .$$

Finally, since $\overline{\Gamma}_k^{(0)}$ is given by scale invariant graphs, using the fixed point theorem with analytic parameterization, it would be possible to prove the existence of an $\overline{\eta}_\lambda$ analytic in λ and such that $\left|\overline{\Gamma}_k^{(0)} - \overline{\eta}_\lambda\right| \leq c_5\varepsilon\gamma^{-\vartheta(N-k)}$ and the statements about the mass flow in (3.4.2) holds for $c_2 \geq (c_5 + c_4)(1 - \gamma^{-\vartheta})^{-1}$.

Finally, with similar arguments, it is straightforward to prove (3.4.3).

Appendix 6:

Proof of Lemma 4.2

By definition

$$U_\omega^{(i,j)}(k,p) \stackrel{def}{=} C_\omega(k,p)\widehat{g}_\omega^{(i)}(k)\widehat{g}_\omega^{(j)}(p)$$
$$= f_i(k)\left(1-\chi_{h,N}^{-1}(k)\right)\frac{f_j(p)}{D_\omega(p)} - f_j(p)\left(1-\chi_{h,N}^{-1}(k)\right)\frac{f_i(k)}{D_\omega(k)}\ .$$

Setting:

$$u_N(k) \stackrel{def}{=} \begin{cases} 0 & \text{for } |k| < \kappa\gamma^N \\ 1-f_N(k) & \text{for } |k| \geq \kappa\gamma^N \end{cases},$$
$$u_h(k) \stackrel{def}{=} \begin{cases} 0 & \text{for } |k| \geq \kappa\gamma^h \\ 1-f_h(k) & \text{for } |k| < \kappa\gamma^h \end{cases},$$

the expansion of $U_\omega^{(i,j)}(k,p)$ in terms of $\left\{S_{\omega,\sigma}^{(i,j)}(k,p)\right\}_{\sigma=\pm}$ can be explicitly given in each of the possible case.

1. For $i=j=N$,

$$U_\omega^{(N,N)}(k,p) = \frac{u_N(p)f_N(k)}{D_\omega(k)} - \frac{u_N(k)f_N(p)}{D_\omega(p)}$$
$$= \sum_{\sigma=\pm} D_\sigma(p-k)\left[\delta_{\omega,\sigma}\frac{u_N(k)f_N(p)}{D_\omega(p)D_\omega(k)} + \frac{f_N(p)}{D_\omega(k)}\int_0^1 d\tau\ \left(\partial_\sigma u_N\right)\left(p+\tau(k-p)\right)\right.$$
$$\left. -\frac{u_N(p)}{D_\omega(k)}\int_0^1 d\tau\ \left(\partial_\sigma f_N\right)\left(k+\tau(p-k)\right)\right]$$
$$\stackrel{def}{=} \sum_{\sigma=\pm} D_\sigma(p-k)S_{\omega,\sigma}^{(N,N)}(k,p)\ .$$

2. For $i = N$ and $h < j < N$:

$$U_\omega^{(N,j)}(k,p) = -\frac{u_N(k) f_j(p)}{D_\omega(p)} \; .$$

Being that $u_N(p) f_j(p) \equiv 0$, it holds

$$\begin{aligned} U_\omega^{(N,j)}(k,p) &= \sum_\sigma D_\sigma(p-k) \frac{f_j(p)}{D_\omega(p)} \int_0^1 d\tau \; \big(\partial_s u_N\big)\big(p + \tau(k-p)\big) \\ &\stackrel{def}{=} \sum_\sigma D_\sigma(p-k) S_{\omega,\sigma}^{(N,j)}(k,p) \; . \end{aligned}$$

3. For $i = N$ and $j = h$

$$U_\omega^{(N,h)}(k,p) = -\frac{u_N(k) f_h(p)}{D_\omega(p)} + \frac{u_h(p) f_N(k)}{D_\omega(k)} \; .$$

The first addend was already studied in point 2. For the second, the expansion is similar to the first since $u_h(k) f_n(k) \equiv 0$; finally:

$$\begin{aligned} U_\omega^{(N,j)}(k,p) = \sum_\sigma D_\sigma(p-k) \Bigg[&\frac{f_j(p)}{D_\omega(p)} \int_0^1 d\tau \; \big(\partial_s u_N\big)\big(p + \tau(k-p)\big) \\ &- \frac{f_N(k)}{D_\omega(k)} \int_0^1 d\tau \; \big(\partial_s u_N\big)\big(k + \tau(p-k)\big) \Bigg] \\ \stackrel{def}{=} \sum_\sigma D_\sigma(p-k) S_{\omega,\sigma}^{(N,h)}(k,p) \; . \end{aligned}$$

4. For $h < i < N$ and $j = h$:

$$U_\omega^{(i,h)}(k,p) = \frac{u_h(p) f_i(k)}{D_\omega(k)} \; .$$

Being that $u_h(k) f_i(k) \equiv 0$ it holds

$$\begin{aligned} U_\omega^{(N,j)}(k,p) &= \sum_\sigma D_\sigma(p-k) \frac{f_i(p)}{D_\omega(p)} \int_0^1 d\tau \; \big(\partial_s u_h\big)\big(k + \tau(p-k)\big) \\ &\stackrel{def}{=} \sum_\sigma D_\sigma(p-k) S_{\omega,\sigma}^{(N,j)}(k,p) \; . \end{aligned}$$

For $i = j = h$, expanding like in point 1

$$\begin{aligned} U_\omega^{(h,h)}(k,p) = \sum_{\sigma=\pm} D_\sigma(p-k) \Bigg[&\delta_{\omega,\sigma} \frac{u_h(k) f_h(p)}{D_\omega(p) D_\omega(k)} \\ &+ \frac{f_h(p)}{D_\omega(k)} \int_0^1 d\tau \; \big(\partial_\sigma u_h\big)\big(p + \tau(k-p)\big) \\ &- \frac{u_h(p)}{D_\omega(k)} \int_0^1 d\tau \; \big(\partial_\sigma f_h\big)\big(k + \tau(p-k)\big) \Bigg] \\ \stackrel{def}{=} \sum_{\sigma=\pm} D_\sigma(p-k) S_{\omega,\sigma}^{(h,h)}(k,p) \; . \end{aligned}$$

By inspection in each case, since for $n = N, h$ it holds $\left|\big(\partial_\sigma f_n\big)(k)\right|, \left|\big(\partial_\sigma u_n\big)(k)\right| \leq c\gamma^{-n}$, it is simply to get the following bound

$$\left|\big(\partial_k^{s_i}\partial_p^{s_j}S_{\omega,\sigma}^{(i,j)}\big)(k,p)\right| \leq c\gamma^{-i(1+s_i)-j(1+s_j)} \ .$$

Appendix 7:

Proof of Theorems 4.1 and 4.4

It is natural to introduce the Beta functions also for the flow of the counterterms $\{\nu_N^{(\sigma)}\}_{\sigma=\pm}$, and the coupling $\widetilde{\lambda}^{\mu}_{N-1}$, generated in the multiscale integration of the generating functional $\mathcal{W}^{(h)}_{\mathcal{T},\mu}$:

$$\begin{aligned}
&\nu_{j-1}^{(\sigma)} - \nu_j^{(\sigma)} = \beta_j^{(\sigma)}\left(\lambda_j, \nu_j; \ldots, \lambda_N, \nu_N\right) \ , \\
&\widetilde{\lambda}_{j-1}^{(\mu)} - \widetilde{\lambda}_j^{(\mu)} = \widetilde{\beta}_j^{(\mu)}\left(\lambda_j, \nu_j, \widetilde{\lambda}_j^{(\mu)}, \widetilde{z}_j^{(\mu)}; \ldots, \lambda_N, \nu_N\right) \ .
\end{aligned}$$

It has to be remarked that the above Beta function are defined for the generating functionals $\mathcal{W}^{(h)}_{\mathcal{A}}$ and $\mathcal{W}^{(h)}_{\mathcal{T},\mu}$ with infrared cutoff $h = -\infty$: this is not restrictive, since, by inspection of the properties of the kernel $U_\omega^{(i,j)}$, the flows obtained have the property that $\widetilde{\lambda}_k^{(\mu)}$ and $\nu_k^{(\sigma)}$, are, in the range $k : h+1 \le k \le N$, *exactly equal to the effective coupling of such generating functionals with infrared cutoff on scale h finite.*

Proof of Theorem 4.1. Let $\mathcal{B}_\vartheta$ be the Banach space of all the finite sequences of vectors $x \stackrel{def}{=} \left\{(\nu_j^{(+)}, \nu_j^{(-)}) : j \le N\right\}$ s.t.

$$\|x\|_\vartheta \stackrel{def}{=} \max_{\sigma=\pm, j\le N} |\nu_j^{(\sigma)}|\gamma^{(\vartheta/2)(N-j)} \le c_1\varepsilon \ .$$

In this space, it is possible to find a solution for the fixed point equation $x = Tx$, which explicitly reads

$$\nu_j^{(\sigma)} = -\sum_{m=-\infty}^{j} \beta_m^{(\sigma)}(x) \qquad (A7.1)$$

(where the argument of the Beta function has been abridged); such a solution gives a choice of $\{\nu_N^{(\sigma)}\}_{\sigma\pm}$, such that their flows $\{\nu_N^{(\sigma)}\}_{h+1\le j\le N}^{\sigma\pm}$ have the required decay property. Indeed, given $x, x' \in \mathcal{B}_\vartheta$:

$$\beta_m^{(\sigma)}(x) \stackrel{def}{=} \beta_{m,N}^{(\sigma,0)} + \sum_{n=m}^{N} \beta_{m,n}^{(\sigma)} \nu_n^{(\sigma)} \ , \qquad \beta_m^{(\sigma)}(x) - \beta_m^{(\sigma)}(x') \stackrel{def}{=} \sum_{n=m}^{N} \beta_{m,n}^{(\sigma)} \left(\nu_n^{(\sigma)} - {\nu'}_n^{(\sigma)}\right) \ ,$$

where $\beta_{m,N}^{(\sigma,0)}$ is the localization of the sum of the graphs made with no interaction $\{\nu_k^{(\sigma)}\mathcal{A}_\sigma\}_k$ and one propagator connecting the interaction $\mathcal{A}_0$ contracted on scale N; whereas $\beta_{m,n}^{(\sigma)}$ is the localization of the sum of the graphs made with an interaction $\nu_n^{(\sigma)}\mathcal{A}_\sigma$, and deprived of $\nu_n^{(\sigma)}$. The following bounds hold:

$$\left|\beta_{m,N}^{(\sigma,0)}(x)\right| \le c_2 \varepsilon \gamma^{-\vartheta(N-m)}, \qquad \left|\beta_{m,n}^{(\sigma)}\right| \le c_2 \varepsilon \gamma^{-\vartheta(n-m)}, \tag{4.3.2}$$

Therefore, if $x \in \mathcal{B}_\vartheta$, then also $Tx \in \mathcal{B}_\vartheta$ for ε small enough and if $c_1 \ge 2c_2(1-\gamma^{-(\vartheta/2)})^{-1}$; and $||x - x'|| \le C\varepsilon ||Tx - Tx'||$ for $C > c_2(1-\gamma^{-(\vartheta/2)})^{-2}$, so that, for ε small enough, T is a contraction in a Banach space; therefore there exists $x \in \mathcal{B}_\vartheta$, solution of the fixed point equation, with analytic parameterization in $\lambda : |\lambda| \le \varepsilon$.

Finally, since all the graphs contributing to $\beta_m^{(\sigma)}$, are scale invariant, by (A7.1) for $j = N$ it is easy to realize that $\{\nu_N^{(\sigma)}\}_{\sigma=\pm}$ are constant in the scale of the cutoff, N: hence

$$\nu_N^{(\sigma)} = \nu_{N+1}^{(\sigma)} = \nu^{(\sigma)} \ .$$

The proof of the theorem is completed. ∎

Proof of Theorem 4.4. The strategy is based on the fixed point theorem as the previous proof. Let $x \stackrel{def}{=} \left\{ \left(\widetilde{\lambda}_j^{(+)} - \alpha_N^{(+)}\lambda_j, \widetilde{\lambda}_j^{(-)} - \alpha_N^{(-)}\lambda_j\right) : j \le N \right\}$ (with $\lambda_N^{(\mu)} = 0$): the fixed point equation to be solved in $\mathcal{B}_{\vartheta/2}$ is $x = Tx$, which explicitly reads:

$$\widetilde{\lambda}_j^{(\mu)} - \alpha_N^{(\mu)}\lambda_j = - \sum_{m=-\infty}^{j} \left(\widetilde{\beta}_m^{(\mu)} - \alpha_N^{(\mu)}\beta_m\right) \ .$$

Given $\alpha_N^{(\mu)}$ and ${\alpha'}_N^{(\mu)}$ such that both $\widetilde{\lambda}_j^{(\mu)} - \alpha_N^{(\mu)}\lambda_j$ and $\widetilde{\lambda}_j^{(\mu)} - {\alpha'}_N^{(\mu)}\lambda_j$ are in $\mathcal{B}_{\vartheta/2}$, it holds:

$$\widetilde{\beta}_m^{(\mu)} - \alpha_N^{(\mu)}\beta_m \stackrel{def}{=} \widetilde{\beta}_{m,N}^{(\mu,o)} - \alpha_N^{(\mu)}\lambda_N \beta_{m,N}^{(\lambda)} + \sum_{\sigma=\pm}\sum_{n=m}^{N} \widetilde{\beta}_{m,n}^{(\mu,\sigma)} \nu_n^{(\sigma)} \frac{Z_N}{Z_n}$$
$$+ \sum_{n=m}^{N-1} \beta_{m,n} \left(\widetilde{\lambda}_n^{(\mu)} - \alpha_N^{(\mu)}\lambda_n\right) \ ;$$

while

$$\left({\alpha'}_N^{(\mu)} - \alpha_N^{(\mu)}\right)\beta_m \stackrel{def}{=} \left({\alpha'}_N^{(\mu)} - \alpha_N^{(\mu)}\right)\lambda_N \beta_{m,N}^{(\lambda)} + \sum_{n=m}^{N-1} \beta_{m,n} \left({\alpha'}_N^{(\mu)} - \alpha_N^{(\mu)}\right)\lambda_n \ ;$$

where $\widetilde{\beta}^{(\mu,o)}_{m,N}$ is the sum of the graphs made with an interaction $\mathcal{A}_o$, contracted on scale N; $\widetilde{\beta}^{(\mu,\sigma)}_{m,n}$ is the sum of the graphs with an interaction $\mathcal{T}^{(\mu)}_\sigma$ on scale n, deprived of the coupling $\nu^{(\sigma)}_n(Z_N/Z_n)$; $b^{(\lambda)}_{m,N}$ is the sum of the graphs contributing to he flow of $\alpha^{(\mu)}_N\lambda_m$ which have an interaction $\mathcal{B}^{(3)}$ on scale N, deprived of the coupling $\alpha^{(\mu)}_N\lambda_N$; $\beta_{m,n}$ is the sum of the graphs contributing to the flow of $\alpha^{(\mu)}_N\lambda_m$ with an interaction $\mathcal{B}^{(3)}$ on scale n, deprived of the coupling $\alpha^{(\mu)}_N\lambda_n$. Since the following bounds hold,

$$|\widetilde{\beta}^{(\mu,o)}_{m,N}|, |\beta^{(\lambda)}_{m,N}| \le c_2\varepsilon\gamma^{-\vartheta(N-m)} , \qquad |\widetilde{\beta}^{(\mu,\sigma)}_{m,n}| \le c_2\varepsilon^2\gamma^{-\vartheta(n-m)} , \qquad |\beta_{m,n}| \le c_2\varepsilon\gamma^{-\vartheta(n-m)} ,$$

if $x \in \mathcal{B}_{\vartheta/2}$, also $Tx \in \mathcal{B}_{\vartheta/2}$, for ε small enough and $c_1 \ge 2c_2(1-\gamma^{-(\vartheta/4)})^{-1}$; moreover,m for $C > 2c_2(1-\gamma^{-(\vartheta/4)})^{-2}$, $\|x-x'\|_{\vartheta/2} \le C\varepsilon\|Tx-Tx'\|_{\vartheta/2}$ so that, for ε small enough, T is a contraction: by the fixed point theorem, the solution of such an equation exists and is in $\mathcal{B}_{\vartheta/2}$. As consequence, since

$$\widetilde{z}^{(\mu)}_j - \alpha^{(\mu)}_N z_j = \widetilde{z}^{(\mu,o)}_{j,N} - \alpha^{(\mu)}_N\lambda_N z^{(\lambda)}_{j,N} + \sum_{\sigma=\pm}\sum_{n=j}^{N} \widetilde{z}^{(\mu,\sigma)}_{j,n}\nu^{(\sigma)}_n\frac{Z_N}{Z_n} + \sum_{n=j}^{N-1} z_{j,n}\left(\widetilde{\lambda}^{(\mu)}_n - \alpha^{(\mu)}_N\lambda_n\right) ,$$

where $\widetilde{z}^{(\mu,o)}_{j,N}$ is the sum of the graphs made with an interaction $\mathcal{A}_o$, contracted on scale N; $\widetilde{z}^{(\mu\sigma)}_{j,n}$ is the sum of the graphs with an interaction $\mathcal{T}^{(\mu)}_\sigma$ on scale n, deprived of the coupling $\nu^{(\sigma)}_n(Z_N/Z_n)$; $z^{(\lambda)}_{j,N}$ is the sum of the graphs contributing to the flow of $\alpha^{(\mu)}_N z_j$ which have an interaction $\mathcal{B}^{(3)}$ on scale N, deprived of the coupling $\alpha^{(\mu)}_N\lambda_N$; $z_{m,n}$ is the sum of the graphs contributing to the flow of $\alpha^{(\mu)}_N z_j$ with an interaction $\mathcal{B}^{(3)}$ on scale n, deprived of the coupling $\alpha^{(\mu)}_N\lambda_n$. Since the following bounds hold,

$$|\widetilde{z}^{(\mu,o)}_{j,N}|, |z^{(\lambda)}_{j,N}| \le c_2\varepsilon\gamma^{-\vartheta(N-j)} , \qquad |\widetilde{z}^{(\mu,\sigma)}_{j,n}| \le c_2\varepsilon^2\gamma^{-\vartheta(n-j)} , \qquad |z_{j,n}| \le c_2\varepsilon\gamma^{-\vartheta(n-j)} ,$$

also $\left\{(\widetilde{z}^{(+)}_j - \alpha^{(+)}_N z_j, \widetilde{z}^{(-)}_j - \alpha^{(-)}_N z_j)\right\}_j \in \mathcal{B}_{\vartheta/2}$. Finally, since all the graphs contributing to $\{\widetilde{\lambda}^{(\mu)}_m\}_m$ and to $\{\lambda_m\}_m$ are scale invariant,

$$\alpha^{(\mu)}_N = \alpha^{(\mu)}_{N+1} = \alpha^{(\mu)} .$$

The proof of the theorem is completed. ∎

Appendix 8:

Schwinger-Dyson equation

A8.1 Functional derivation. By decomposing the fermionic fields $\psi^+_{k,\omega} \longrightarrow \psi^+_{k,\omega} + \widehat{\beta}_{k,\omega}$, it holds:

$$\begin{aligned}\mathcal{W}^{(h)}(\jmath,\varphi) =& \mathcal{W}^{(h)}_{\mathcal{B}}(\beta,\jmath,\varphi) + \sum_{\omega=\pm}\int_D \frac{\mathrm{d}^2k}{(2\pi)^2}\ \widehat{\beta}_{k,\omega}\widehat{\varphi}^-_{k,\omega} \\ &- \sum_{\omega=\pm}\int_D \frac{\mathrm{d}^2k}{(2\pi)^2}\ \widehat{\beta}_{k,\omega}D_\omega(k)\left[1 + Z_N\left(\chi^{-1}_{h,N}(k)-1\right)\right]\frac{\partial\mathcal{W}}{\partial\widehat{\varphi}^+_{k,\omega}}(\jmath,\varphi) + \mathrm{O}(\beta^2)\ ,\end{aligned} \tag{A8.1}$$

where $\mathcal{W}^{(h)}_{\mathcal{B}}$ is the following functional with the further source field β:

$$\begin{aligned}e^{\mathcal{W}^{(h)}_{\mathcal{B}}(\beta,\jmath,\varphi)} \stackrel{def}{=} \int \mathrm{d}P^{[h,N]}(\psi)\exp\left\{-l_N\mathcal{V}(\psi) + Z^{(2)}_N\mathcal{J}(\jmath,\psi) + \mathcal{F}(\varphi,\psi)\right\} \\ \exp\left\{-l_N\mathcal{B}^{(3)}(\beta,\psi) + Z^{(2)}_N\mathcal{B}^{(2)}(\beta,\jmath,\psi) - z_N\mathcal{B}^{(1)}(\beta,\psi)\right\}\ ,\end{aligned}$$

with:

$$\begin{aligned}\mathcal{B}^{(3)}(\beta,\psi) &\stackrel{def}{=} \sum_{\omega=\pm}\int_D \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2q}{(2\pi)^2}\ \widehat{\beta}_{p+k-q,\omega}\widehat{\psi}^-_{p,\omega}\widehat{\psi}^+_{q,-\omega}\widehat{\psi}^-_{k,-\omega}\ , \\ \mathcal{B}^{(2)}(\beta,\jmath,\psi) &\stackrel{def}{=} \sum_{\omega=\pm}\int_D \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\ \widehat{\beta}_{k,\omega}\widehat{\jmath}_{p-k,\omega}\widehat{\psi}^-_{p,\omega}\ , \\ \mathcal{B}^{(1)}(\beta,\psi) &\stackrel{def}{=} \sum_{\omega=\pm}\int \frac{\mathrm{d}^2k}{(2\pi)^2}\ \widehat{\beta}_{k,\omega}D_\omega(k)\widehat{\psi}^-_{k,\omega}\ .\end{aligned}$$

Therefore, extracting the linear part of (A8.1), for $k:\gamma^h\kappa\le|k|\le\gamma^N\kappa$ (so that $\chi^{-1}_{h,N}(k)-1=0$), it yield the SDE:

$$\widehat{g}_\omega^{-1}(k)\frac{\partial\mathcal{W}_\mathcal{B}}{\partial\widehat{\varphi}^+_{k,\omega}}(0,\jmath,\varphi)=\widehat{\varphi}^-_{k,\omega}+\frac{\partial\mathcal{W}_\mathcal{B}}{\partial\widehat{\beta}_{k,\omega}}(0,\jmath,\varphi)\ . \tag{A8.2}$$

Now, writing the last derivative in terms of the derivative of $\mathcal{W}$ – but loosing in this way the evidence of the renormalization of composite operators – and multiplying both members by $e^{\mathcal{W}^{(h)}}$ in order to shorten the equations, it simply holds (4.3.3) . By derivatives in the sources $\widehat{\jmath}$ and $\widehat{\varphi}$, for $\widehat{\jmath}=\widehat{\varphi}=0$, such an equation generates all the SDE: for instance, taking a derivative in $\widehat{\varphi}^-_{k,\omega}$ gives (1.1.12).

Appendix 9:

Lowest Order Computations

It is interesting to calculate the lowest order expansion of the anomalies. The computation of the anomaly of the WTI shows *a violation of the Adler-Bardeen theorem*: the correction to the classical identity is not linear in the coupling, but has at least also a non-vanishing second order term. Then, the computation of the anomaly of the CE – made in a quite approximate way – would imply *the incorrectness of the Johnson solution*.

A9.1 WTI **anomaly** Simplifying the notations, let $\chi(k) \stackrel{def}{=} \chi_0(k)$ and $u(k) \stackrel{def}{=} u_0(k)$. A useful identity is

$$\begin{aligned} U_\omega(k, k+p) &= \left\{ u(k+p) \frac{\chi(k)}{D_\omega(k)} - u(k) \frac{\chi(k+p)}{D_\omega(k+p)} \right\} \\ &= D_\omega(p) \left\{ \frac{u(k+p)\chi(k)}{D_\omega(k+p)D_\omega(k)} - \int_0^1 \mathrm{d}\tau \; \frac{(\partial_\omega \chi)(k+\tau p)}{D_\omega(k+p)} \right\} \\ &- D_{-\omega}(p) \int_0^1 \mathrm{d}\tau \; \frac{(\partial_{-\omega}\chi)(k+\tau p)}{D_\omega(k+p)} \; . \end{aligned}$$

To simplify the computations, it is performed the following modification to the shape of the cutoff which, as can be easily checked, it completely harmless to the development done in the previous Chapters. Let $\chi(k) \stackrel{def}{=} \widehat{\chi}(|k|)$, and $\widehat{\chi}(t)$ is a Gevrey function with compact support $\{t : |t| \leq \kappa\gamma^N\gamma_0\}$, for $\gamma_0 : 1 < \gamma_0 < \gamma$, and equal to 1 in $\{t : |t| \leq \kappa\gamma^N\}$.

1. **Computation of $\nu^{(-)}$.** The lowest order expansion of $\nu^{(-)}$ is given by only one Feynman

graph, which can be computed exactly:

$$\nu^{(-)} = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{(\partial_{-\omega}\chi)(k)}{D_\omega(k)} = -\frac{1}{4\pi}\int_0^\infty \mathrm{d}t\ \widehat{\chi}'(t) = \frac{1}{4\pi}\ .$$

where it was used that $(\partial_{-\omega}\chi)(k)/D_\omega(k) = -(1/2|k|)\widehat{\chi}'(k)$.

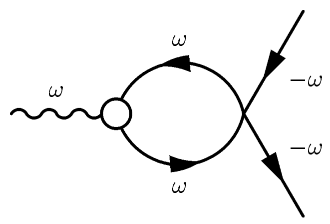

Fig 12: Graphical representation of the lowest order contribution to $\nu^{(-)}$

2. Computation of $\nu^{(+)}$. Also the lowest order contribution to $\nu^{(+)}$ is given by only one Feynman graph:

$$\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\left\{\frac{u(p)\chi(p)}{D_\omega(p)D_\omega(p)} - \frac{\big(\partial_\omega\chi\big)(p)}{D_\omega(p)}\right\}\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\hat{g}_{-\omega}(k)\hat{g}_{-\omega}(p+k)$$
$$= \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\left\{\frac{u(p)\chi(p) - D_\omega(p)\big(\partial_\omega\chi\big)(p)}{p^4}\right\}\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\frac{\chi(k)\chi(p+k)}{k^2(k+p)^2}D^2_{-\omega}(p)D_\omega(k)D_\omega(k+p)\ . \tag{A9.1}$$

The explicit computation is not so simple as the previous; anyway it is possible to prove it is strictly non-zero. Since $-D_\omega(p)\big(\partial_\omega\chi\big)(p) = -(|p|/2)\widehat{\chi}'(|p|) \geq 0$, as well as $u(p)\chi(p) \geq 0$, while, calling ϑ the angle between p and k and $\xi \stackrel{def}{=} (|k|/|p|)$,

$$D^2_{-\omega}(p)D_\omega(k)D_\omega(k+p) = |k||p|^3\Big[\cos(\vartheta) + \xi\cos(2\vartheta)\Big] \stackrel{def}{=} |k||p|^3 J_\xi(\vartheta)\ ,$$

up to a pure imaginary contribution which integrated gives zero by symmetries. Now, since by support of the cutoff functions $|k| \leq \gamma_0$ and $1 \leq |p| \leq \gamma_0$, then $\cos(\vartheta) < 1/2$ if γ_0 is chosen $\leq 3/2$. Hence, $J_1(\vartheta) = \big[\cos(\vartheta) - (1/2)\big]\big[\cos(\vartheta) + 1\big] < 0$, except for $\vartheta = \pm(\pi/3), \pi$, where it vanishes. Then, the integral over ϑ of $J_\xi(\vartheta)$ is continuous in ξ, and strictly negative for $\xi = 1$; therefore it remains strictly negative also for $\xi = |k|/|p|$, if $\gamma_0 - 1 \geq |k|/|p| - 1$ is small enough. Therefore, for such values of γ_0, the lowest order contribution to $\nu^{(+)}$ is strictly negative.

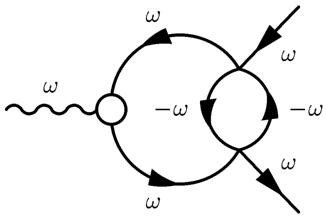

Fig 13: Graphical representation of the lowest order contribution to $\nu^{(+)}$

A9.2 CE **anomaly.** From (1.1.18), and since $a - \overline{a} = \mathrm{O}(\lambda)$, while $a + \overline{a} = 1 + \mathrm{O}(\lambda^2)$, the contribution $\mathrm{O}(\lambda)$ to A is proportional to the terms O(1) of $\alpha^{(-)} - \sigma^{(-)}$.

1. The 0-th order of $\alpha^{(-)}$ is given by two graphs with values cancelling each other.
2. There is no possible graph for $\sigma^{(-)}$ at the 0-th order, since there are no possible tadpoles.

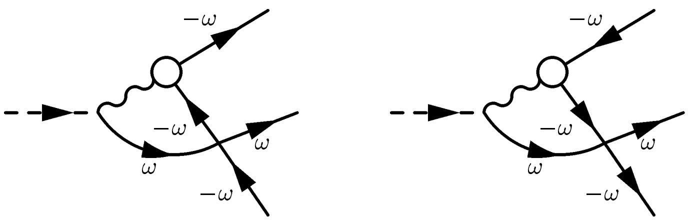

Fig 14: Graphical of item 1

Well then, $A = 1 + \mathrm{O}(\lambda^2)$. Then, the quadratic order in λ comes from the linear order of $\alpha^{(-)} - \sigma^{(-)}$, and the O(1) order of $\alpha^{(+)} - \sigma^{(+)}$.

1. There are more than one Feynman graphs contributing to the linear order of $\alpha^{(-)}$.
   - **First graph.** A first contribution are the two graphs with all and three external leg of T involved: they are two, with the same value. Furthermore, the factor 1/2! of the expansion of the interaction is compensated by multiplicity obtained by exchanging the labels to the two vertices $\mathcal{V}$ of each graph. Therefore the sum of them gives the *first graph*:

$$
\begin{aligned}
&2\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{U_{-\omega}(k,k+p)}{D_{-\omega}(p)} g_\omega(p+k) g_\omega(k) \\
=&-2\int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\chi(p)}{p^2}\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{u(k)\chi^2(k+p)}{(p+k)^2} \\
&-2\int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\chi(p)}{p^2}\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{u(k+p)\chi(k+p)\chi(k)}{D_{-\omega}(k)D_\omega(p+k)}\ .
\end{aligned}
$$

The latter addend is vanishing in the limit $\gamma_0 \to 1$. The former is convergent. Indeed:

$$\int_{|p|\leq 1/2} \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\chi(p)}{p^2} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{u(k)\chi^2(k+p)}{(p+k)^2}$$
$$= \int_{|p|\leq 1/2} \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\chi(p)}{p^2} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{u(k)(\chi^2(k+p) - \chi^2(k))}{(p+k)^2} ,$$

and $|p+k| \geq |k| - |p| \geq 1/2$; while

$$\int_{|p|>1/2} \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\chi(p)}{p^2} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{u(k)\chi^2(k+p)}{(p+k)^2}$$
$$= \int_{|p|>1/2} \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{(\chi(p) - \chi(k))}{p^2} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{u(k)\chi^2(k+p)}{(p+k)^2} .$$

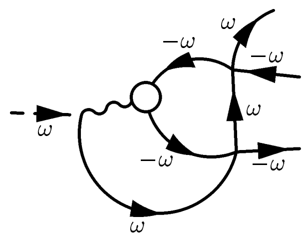

Fig 15: First graph

- **Second graphs.** The second contribution is given by the graph

$$- \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\mathrm{d}^2 k}{(2\pi)^2} \, g_\omega(p) g_\omega(p) \frac{U_{-\omega}(k, k+p)}{D_{-\omega}(p)}$$
$$= \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\mathrm{d}^2 k}{(2\pi)^2} \, \frac{\chi^2(p)}{D_\omega(p) D_\omega(p)} \left\{ - \frac{u(k+p)\chi(k)}{D_{-\omega}(k+p) D_{-\omega}(k)} + \int_0^1 \mathrm{d}\tau \, \frac{\big(\partial_{-\omega}\chi\big)(k+\tau p)}{D_{-\omega}(k+p)} \right\}$$
$$+ \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\mathrm{d}^2 k}{(2\pi)^2} \, \frac{\chi^2(p)}{D_\omega(p) D_{-\omega}(p)} \int_0^1 \mathrm{d}\tau \, \frac{\big(\partial_\omega \chi\big)(k+\tau p)}{D_{-\omega}(k+p)} \; ;$$

and, subtracting the graph containing the counterterm $\nu_N^{(-)}$,

$$\int \frac{\mathrm{d}^2 p}{(2\pi)^2} \, \frac{\chi^2(p)}{D_\omega(p) D_{-\omega}(p)} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \, \frac{\big(\partial_\omega \chi\big)(k)}{D_{-\omega}(k)}$$

the last addend is convergent; while the first two terms are convergent automatically:

$$\int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\mathrm{d}^2 k}{(2\pi)^2} \, \frac{\chi^2(p)}{D_\omega(p) D_\omega(p)} \frac{u(k+p)\chi(k)}{D_{-\omega}(k+p) D_{-\omega}(k)}$$
$$= \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\mathrm{d}^2 k}{(2\pi)^2} \, \frac{\chi^2(p)}{D_\omega(p) D_\omega(p)} \left\{ \frac{u(k+p)\chi(k)}{D_{-\omega}(k+p) D_{-\omega}(k)} - \frac{u(k)\chi(k)}{D_{-\omega}(k) D_{-\omega}(k)} \right\}$$

$$\int \frac{d^2p}{(2\pi)^2}\frac{d^2k}{(2\pi)^2}\ \frac{\chi^2(p)}{D_\omega(p)D_\omega(p)}\int_0^1 d\tau\ \frac{\big(\partial_{-\omega}\chi\big)(k+\tau p)}{D_{-\omega}(k+p)}$$
$$= \int \frac{d^2p}{(2\pi)^2}\frac{d^2k}{(2\pi)^2}\ \frac{\chi^2(p)}{D_\omega(p)D_\omega(p)}\int_0^1 d\tau\ \left\{\frac{\big(\partial_{-\omega}\chi\big)(k+\tau p)}{D_{-\omega}(k+p)} - \frac{\big(\partial_{-\omega}\chi\big)(k)}{D_{-\omega}(k)}\right\}$$

since the subtracted terms are zero by transformation under rotation.

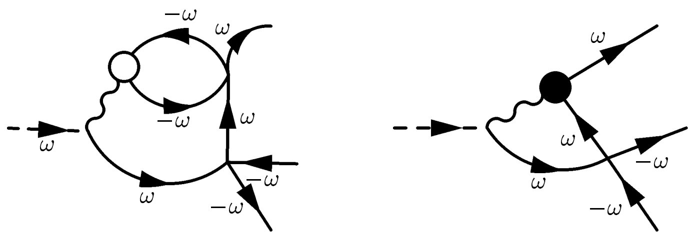

Fig 16: Second graphs

- **Vanishing graphs.** There are four graphs subleading in the limit $\gamma_0 \to 1$: their total value is the double of the two *vanishing graphs*

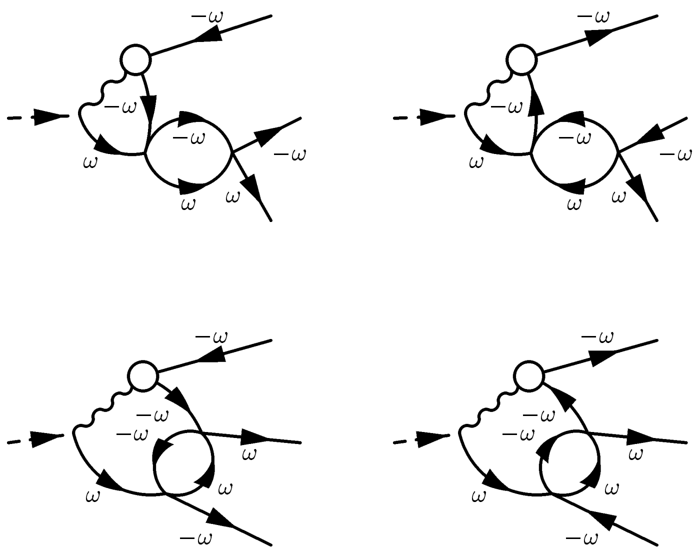

Fig 17: Vanishing graphs

2. The linear order of $\sigma^{(-)}$ is given by only one graph.

- **Third graph.** Such graph is very similar to the previous: it is given by the the second graph, with the replacement of $g_\omega^2(p)$ with $(\partial_\omega g_\omega)(p)$:

$$
\begin{aligned}
&-\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\frac{\mathrm{d}^2 k}{(2\pi)^2}\,(\partial_\omega g_\omega)(p)\frac{U_{-\omega}(k,k+p)}{D_{-\omega}(p)}\\
&=\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\frac{\mathrm{d}^2 k}{(2\pi)^2}\left[\frac{(\partial_\omega\chi)(p)}{D_\omega(p)}-\frac{\chi(p)}{D_\omega(p)D_\omega(p)}\right]\left\{-\frac{u(k+p)\chi(k)}{D_{-\omega}(k+p)D_{-\omega}(k)}\right.\\
&\qquad\qquad\left.+\int_0^1 \mathrm{d}\tau\,\frac{\big(\partial_{-\omega}\chi\big)(k+\tau p)}{D_{-\omega}(k+p)}\right\}\\
&+\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\frac{\mathrm{d}^2 k}{(2\pi)^2}\left[\frac{(\partial_\omega\chi)(p)}{D_{-\omega}(p)}-\frac{\chi(p)}{D_\omega(p)D_{-\omega}(p)}\right]\int_0^1 \mathrm{d}\tau\,\frac{\big(\partial_\omega\chi\big)(k+\tau p)}{D_{-\omega}(k+p)}\ ;
\end{aligned}
$$

and, subtracting the graph containing the counterterm $\nu^{(-)}$,

$$
\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\left[\frac{(\partial_\omega\chi)(p)}{D_{-\omega}(p)}-\frac{\chi(p)}{D_\omega(p)D_{-\omega}(p)}\right]\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\,\frac{\big(\partial_\omega\chi\big)(k)}{D_{-\omega}(k)}
$$

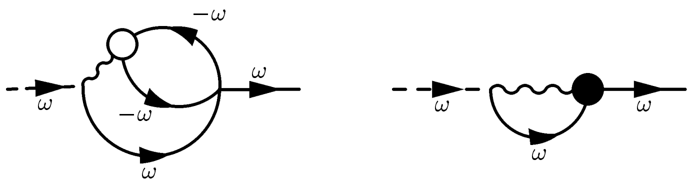

Fig 19: Third graph

the last addend is convergent.

3. The 0-th order of $\alpha^{(+)}$ is given by only one graph, which is subleading: it vanishes in the limit $\gamma_0 \to 1$.

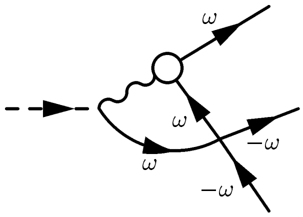

Fig 20: Graph in item 3.

4. The 0-th order of $\sigma^{(+)}$ is only given by a tadpole.

- **Fourth graph.** It derives from the tadpole of $T_0^{(+)}$: for any $N' \geq 2$

$$
\int \frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{u_N(q)\chi_{N+N'}(k-q)}{D_{-\omega}(k-q)}\ ;
$$

the localization of this term is the extraction of the zeroth and first order Taylor expansion in the external momentum k: the former is clearly summable and zero by symmetries; the latter is:

$$\int \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{u(q)\big(\partial_\omega \chi_{N'}\big)(q)}{D_{-\omega}(q)} = \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{u_{-N'}(q)(\partial_\omega \chi)(q)}{D_{-\omega}(q)} \ .$$

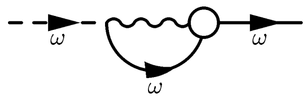

Fig 21: Fourth graph

A9.3 Explicit computation. To make the computation easier, the cutoff is chosen to be a distribution

$$\chi(k) \stackrel{def}{=} f(k_0) f(k_1) \ , \qquad \text{for } f(x) \stackrel{def}{=} \vartheta(x+1) - \vartheta(x-1) \ .$$

Then $f'(x) = \delta(x+1) - \delta(x-1)$. Since, by definition of $D_\omega(k)$, it holds $k_0 = (i/2)\big[D_\omega(k) + D_{-\omega}(k)\big]$ while $k_1 = (\omega/2)\big[D_\omega(k) - D_{-\omega}(k)\big]$, then:

$$\Big(\partial_\omega \chi\Big)(k) = \frac{i}{2} f'(k_0) f(k_1) + \frac{\omega}{2} f(k_0) f'(k_1) \ .$$

It is suitable to remark that the above choice of the cutoff function, in contrast with what done for the anomaly of the WTI, *is not allowed* in the developments of the previous Chapter. Furthermore the computation of the following integrals is not exact, but rather is performed with a simple Montecarlo simulation. That is way the incorrectness is not proved, but it has to be considered as a *conjecture*, enforced by such a calculation.

- **F.** For the first graph it holds:

$$\begin{aligned} \mathbf{F} \stackrel{def}{=} & -2 \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\chi(p)}{p^2} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{u(k)\chi^2(k+p)}{(p+k)^2} \\ = & -\frac{2}{(2\pi)^4} \int_{-1}^{1} \mathrm{d}p_0 \mathrm{d}p_1 \ \frac{1}{p^2} \int_{-1}^{1} \mathrm{d}k_0 \mathrm{d}k_1 \frac{1 - f(k_0 - p_0) f(k_1 - p_1)}{k^2} = 52.64 \frac{1}{(2\pi)^4} \ . \end{aligned}$$

- **S.** Calling $p^\tau = (1-\tau)p$, for the second graph it holds:

$$\begin{aligned} \mathbf{Sa} \stackrel{def}{=} & \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\mathrm{d}^2 k}{(2\pi)^2} \ \frac{\chi^2(p)}{D_\omega(p) D_\omega(p)} \frac{u(k+p)\chi(k)}{D_{-\omega}(k+p) D_{-\omega}(k)} \\ = & \frac{2}{(2\pi)^4} \int_{-1}^{1} \mathrm{d}p_0 \mathrm{d}p_1 \ \frac{(p_0^2 - p_1^2)}{p^4} \int_{-1}^{1} \mathrm{d}k_0 \mathrm{d}k_1 \ \frac{1 - f(k_0 + p_0) f(k_1 + p_1)}{(k+p)^2} (k_0 + p_0) \frac{k_0}{k^2} \\ & + \frac{4}{(2\pi)^4} \int_{-1}^{1} \mathrm{d}p_0 \mathrm{d}p_1 \ \frac{p_0 p_1}{p^4} \int_{-1}^{1} \mathrm{d}k_0 \mathrm{d}k_1 \ \frac{1 - f(k_0 + p_0) f(k_1 + p_1)}{(k+p)^2} (k_1 + p_1) \frac{k_0}{k^2} \ . \end{aligned}$$

Finally:

$$\mathbf{Sa1} \stackrel{def}{=} \frac{2}{(2\pi)^4} \int_{-1}^{1} \mathrm{d}p_0 \mathrm{d}p_1 \ \frac{(p_0^2 - p_1^2)}{p^4} \int_{-1}^{1} \mathrm{d}k_0 \mathrm{d}k_1 \ \frac{1 - f(k_0 + p_0) f(k_1 + p_1)}{(k+p)^2} (k_0 + p_0) \frac{k_0}{k^2}$$
$$= 2.69 \frac{1}{(2\pi)^4} \ ,$$

$$\mathbf{Sa2} \stackrel{def}{=} \frac{4}{(2\pi)^4} \int_{-1}^{1} \mathrm{d}p_0 \mathrm{d}p_1 \ \frac{p_0 p_1}{p^4} \int_{-1}^{1} \mathrm{d}k_0 \mathrm{d}k_1 \ \frac{1 - f(k_0 + p_0) f(k_1 + p_1)}{(k+p)^2} (k_1 + p_1) \frac{k_0}{k^2}$$
$$= 0.29 \frac{1}{(2\pi)^4} \ .$$

The second addend of the second graph is:

$$\begin{aligned}
\mathbf{Sb} \stackrel{def}{=} & \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\mathrm{d}^2 k}{(2\pi)^2} \ \frac{\chi^2(p)}{D_\omega(p) D_\omega(p)} \int_0^1 \mathrm{d}\tau \ \frac{\big(\partial_{-\omega}\chi\big)(k + \tau p)}{D_{-\omega}(k+p)} \\
= & - \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \ \frac{\chi^2(p)}{p^4} D_{-\omega}^2(p) \int_0^1 \mathrm{d}\tau \ \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \ \frac{\big(\partial_{-\omega}\chi\big)(k)}{\left(k + p^\tau\right)^2} D_\omega \left(k + p^\tau\right) \\
= & \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \ \frac{\chi^2(p)}{p^4} (p_0^2 - p_1^2) \int_0^1 \mathrm{d}\tau \ \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \ \frac{(k_0 + p_0^\tau) f'(k_0) f(k_1)}{\left(k + p^\tau\right)^2} \\
& + 2 \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \ \frac{\chi^2(p)}{p^4} p_0 p_1 \int_0^1 \mathrm{d}\tau \ \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \ \frac{(k_1 + p_1^\tau) f'(k_0) f(k_1)}{\left(k + p^\tau\right)^2} \\
= & \frac{2}{(2\pi)^4} \int_{-1}^{1} \mathrm{d}p_0 \mathrm{d}p_1 \ \frac{p_0^2 - p_1^2}{p^4} \int_0^1 \mathrm{d}\tau \ \int_{-1}^{1} \mathrm{d}k_1 \ \frac{p_0^\tau - 1}{\left(p_0^\tau - 1\right)^2 + \left(p_1^\tau + k_1\right)^2} \\
& + \frac{4}{(2\pi)^4} \int_{-1}^{1} \mathrm{d}p_0 \mathrm{d}p_1 \ \frac{p_0 p_1}{p^4} \int_0^1 \mathrm{d}\tau \ \int_{-1}^{1} \mathrm{d}k_1 \ \frac{p_1^\tau + k_1}{\left(p_0^\tau - 1\right)^2 + \left(p_1^\tau + k_1\right)^2} \ .
\end{aligned}$$

The third addend of the second graph is

$$\begin{aligned}
\mathbf{Sc} \stackrel{def}{=} & \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\mathrm{d}^2 k}{(2\pi)^2} \ \frac{\chi^2(p)}{D_\omega(p) D_{-\omega}(p)} \int_0^1 \mathrm{d}\tau \ \frac{\big(\partial_{\omega}\chi\big)(k + \tau p)}{D_{-\omega}(k+p)} \\
= & \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \ \frac{\chi^2(p)}{p^2} \int_0^1 \mathrm{d}\tau \ \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \ \frac{\big(\partial_{\omega}\chi\big)(k)}{(p^\tau + k)^2} D_\omega(p^\tau + k) \\
= & \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \ \frac{\chi^2(p)}{p^2} \int_0^1 \mathrm{d}\tau \ \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \ \frac{(p_0^\tau + k_0) f'(k_0) f(k_1)}{(p^\tau + k)^2} \\
= & \frac{2}{(2\pi)^4} \int_{-1}^{1} \mathrm{d}p_0 \mathrm{d}p_1 \ \frac{1}{p^2} \int_0^1 \mathrm{d}\tau \ \int_{-1}^{1} \mathrm{d}k_1 \ \frac{(p_0^\tau - 1)}{\left(p_0^\tau - 1\right)^2 + \left(p_1^\tau + k_1\right)^2} \ ;
\end{aligned}$$

and its regularization is obtained by subtracting the ∞ term

$$- \frac{2}{(2\pi)^4} \int_{-1}^{1} \mathrm{d}p_0 \mathrm{d}p_1 \ \frac{1}{p^2} \int_{-1}^{1} \mathrm{d}k_1 \ \frac{1}{1 + k_1^2} \ .$$

Therefore:

$$\mathbf{Sc} \stackrel{def}{=} \frac{2}{(2\pi)^4} \int_{-1}^{1} \mathrm{d}p_0 \mathrm{d}p_1 \ \frac{1}{p^2} \int_0^1 \mathrm{d}\tau \ \int_{-1}^{1} \mathrm{d}k_1 \ \left[\frac{(p_0^\tau - 1)}{\left(p_0^\tau - 1\right)^2 + \left(p_1^\tau + k_1\right)^2} + \frac{1}{1 + k_1^2} \right] \ .$$

Setting $\mathbf{Sd} \stackrel{def}{=} \mathbf{Sb} + \mathbf{Sc}$ finally:

$$\begin{aligned}\mathbf{Sd1} \stackrel{def}{=} & -\frac{4}{(2\pi)^4}\int_{-1}^{1} \mathrm{d}p_0 \mathrm{d}p_1 \, \frac{p_1^2}{p^4}\int_0^1 \mathrm{d}\tau \, \int_{-1}^{1} \mathrm{d}k_1 \, \left[\frac{p_0^\tau - 1}{(p_0^\tau-1)^2 + (p_1^\tau+k_1)^2} + \frac{1}{1+k_1^2}\right] \\ &= -0.49\frac{1}{(2\pi)^4} \, , \\ \mathbf{Sd2} \stackrel{def}{=} & \frac{4}{(2\pi)^4}\int_{-1}^{1} \mathrm{d}p_0 \mathrm{d}p_1 \, \frac{p_0 p_1}{p^4}\int_0^1 \mathrm{d}\tau \, \int_{-1}^{1} \mathrm{d}k_1 \, \left[\frac{p_1^\tau + k_1}{(p_0^\tau-1)^2 + (p_1^\tau+k_1)^2} - \frac{k_1}{1+k_1^2}\right] \\ &= 0.0056\frac{1}{(2\pi)^4} \, .\end{aligned}$$

- **T.** For the third graph it holds:

$$\begin{aligned}\mathbf{Ta} \stackrel{def}{=} & \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\frac{\mathrm{d}^2 k}{(2\pi)^2} \, \frac{(\partial_\omega \chi)(p)}{D_\omega(p)}\frac{u(k+p)\chi(k)}{D_{-\omega}(k+p)D_{-\omega}(k)} \\ = & \frac{2}{(2\pi)^4}\int_{-1}^{1} \mathrm{d}p_0 \, \frac{1}{p_0^2+1}\int_{-1}^{1} \mathrm{d}k_0 \mathrm{d}k_1 \, \frac{1-f(k_0+p_0)f(k_1-1)}{(k_0+p_0)^2+(k_1-1)^2}(k_0+p_0)\frac{k_0}{k^2} \\ & -\frac{2}{(2\pi)^4}\int_{-1}^{1} \mathrm{d}p_1 \, \frac{1}{1+p_1^2}\int_{-1}^{1} \mathrm{d}k_0 \mathrm{d}k_1 \, \frac{1-f(k_0-1)f(k_1+p_1)}{(k_0-1)^2+(k_1+p_1)^2}(k_0-1)\frac{k_0}{k^2} \\ & +\frac{2}{(2\pi)^4}\int_{-1}^{1} \mathrm{d}p_1 \, \frac{p_1}{1+p_1^2}\int_{-1}^{1} \mathrm{d}k_0 \mathrm{d}k_1 \, \frac{1-f(k_0-1)f(k_1+p_1)}{(k_0-1)^2+(k_1+p_1)^2}(k_1+p_1)\frac{k_0}{k^2} \\ & +\frac{2}{(2\pi)^4}\int_{-1}^{1} \mathrm{d}p_0 \, \frac{p_0}{p_0^2+1}\int_{-1}^{1} \mathrm{d}k_0 \mathrm{d}k_1 \, \frac{1-f(k_0+p_0)f(k_1-1)}{(k_0+p_0)^2+(k_1-1)^2}(k_1-1)\frac{k_0}{k^2} \, .\end{aligned}$$

Finally

$$\begin{aligned}\mathbf{Ta1} \stackrel{def}{=} & \frac{2}{(2\pi)^4}\int_{-1}^{1} \mathrm{d}p_0 \, \frac{1}{p_0^2+1}\int_{-1}^{1} \mathrm{d}k_0 \mathrm{d}k_1 \, \frac{1-f(k_0+p_0)f(k_1-1)}{(k_0+p_0)^2+(k_1-1)^2}(k_0+p_0)\frac{k_0}{k^2} \\ &= 1.96\frac{1}{(2\pi)^4} \, , \\ \mathbf{Ta2} \stackrel{def}{=} & -\frac{2}{(2\pi)^4}\int_{-1}^{1} \mathrm{d}p_1 \, \frac{1}{1+p_1^2}\int_{-1}^{1} \mathrm{d}k_0 \mathrm{d}k_1 \, \frac{1-f(k_0-1)f(k_1+p_1)}{(k_0-1)^2+(k_1+p_1)^2}(k_0-1)\frac{k_0}{k^2} \\ &= -4.1\frac{1}{(2\pi)^4} \, , \\ \mathbf{Ta3} \stackrel{def}{=} & +\frac{2}{(2\pi)^4}\int_{-1}^{1} \mathrm{d}p_1 \, \frac{p_1}{1+p_1^2}\int_{-1}^{1} \mathrm{d}k_0 \mathrm{d}k_1 \, \frac{1-f(k_0-1)f(k_1+p_1)}{(k_0-1)^2+(k_1+p_1)^2}(k_1+p_1)\frac{k_0}{k^2} \\ &= -0.28\frac{1}{(2\pi)^4} \, , \\ \mathbf{Ta4} \stackrel{def}{=} & +\frac{2}{(2\pi)^4}\int_{-1}^{1} \mathrm{d}p_0 \, \frac{p_0}{p_0^2+1}\int_{-1}^{1} \mathrm{d}k_0 \mathrm{d}k_1 \, \frac{1-f(k_0+p_0)f(k_1-1)}{(k_0+p_0)^2+(k_1-1)^2}(k_1-1)\frac{k_0}{k^2} \\ &= 0.11\frac{1}{(2\pi)^4} \, .\end{aligned}$$

The second addend of the third graph is

$$\begin{aligned}
\mathbf{Tb} &\stackrel{def}{=} \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\frac{\mathrm{d}^2 k}{(2\pi)^2}\ \frac{(\partial_\omega \chi)(p)}{D_\omega(p)} \int_0^1 \mathrm{d}\tau\ \frac{\big(\partial_{-\omega}\chi\big)(k+\tau p)}{D_{-\omega}(k+p)} \\
&= \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\ \frac{(\partial_\omega \chi)(p)}{p^2} D_{-\omega}(p) \int_0^1 \mathrm{d}\tau\ \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\ \frac{\big(\partial_{-\omega}\chi\big)(k)}{(k+p^\tau)^2} D_\omega\left(k+p^\tau\right) \\
&= \frac{1}{2}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\ \frac{f'(p_0)f(p_1)p_0 - f(p_0)f'(p_1)p_1}{p^2} \int_0^1 \mathrm{d}\tau\ \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\ \frac{(k_0+p_0^\tau)f'(k_0)f(k_1)}{(k+p^\tau)^2} \\
&+ \frac{1}{2}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\ \frac{f'(p_0)f(p_1)p_1 + f(p_0)f'(p_1)p_0}{p^2} \int_0^1 \mathrm{d}\tau\ \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\ \frac{(k_1+p_1^\tau)f'(k_0)f(k_1)}{(k+p^\tau)^2}\ ;
\end{aligned}$$

$$\begin{aligned}
\mathbf{Tc} &\stackrel{def}{=} \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\frac{\mathrm{d}^2 k}{(2\pi)^2}\ \frac{(\partial_\omega \chi)(p)}{D_{-\omega}(p)} \int_0^1 \mathrm{d}\tau\ \frac{\big(\partial_{\omega}\chi\big)(k+\tau p)}{D_{-\omega}(k+p)} \\
&= \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\ \frac{(\partial_\omega \chi)(p) D_\omega(p)}{p^2} \int_0^1 \mathrm{d}\tau\ \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\ \frac{\big(\partial_{\omega}\chi\big)(k)}{(p^\tau+k)^2} D_\omega(p^\tau+k) \\
&= \frac{1}{2}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\ \frac{f'(p_0)f(p_1)p_0 + f(p_0)f'(p_1)p_1}{p^2} \int_0^1 \mathrm{d}\tau\ \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\ \frac{(p_0^\tau+k_0)f'(k_0)f(k_1)}{(p^\tau+k)^2} \\
&+ \frac{1}{2}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\ \frac{f'(p_0)f(p_1)p_1 - f(p_0)f'(p_1)p_0}{p^2} \int_0^1 \mathrm{d}\tau\ \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\ \frac{(p_0^\tau+k_0)f(k_0)f'(k_1)}{(p^\tau+k)^2}\ .
\end{aligned}$$

Setting $\mathbf{Td} \stackrel{def}{=} \mathbf{Tb} + \mathbf{Tc}$, some cancellation occurs:

$$\begin{aligned}
\mathbf{Td} &\stackrel{def}{=} \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\frac{\mathrm{d}^2 k}{(2\pi)^2}\ \frac{(\partial_\omega \chi)(p)}{D_{-\omega}(p)} \int_0^1 \mathrm{d}\tau\ \frac{\big(\partial_{\omega}\chi\big)(k+\tau p)}{D_{-\omega}(k+p)} \\
&= \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\ \frac{(\partial_\omega \chi)(p) D_\omega(p)}{p^2} \int_0^1 \mathrm{d}\tau\ \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\ \frac{\big(\partial_{\omega}\chi\big)(k)}{(p^\tau+k)^2} D_\omega(p^\tau+k) \\
&= \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\ \frac{f'(p_0)f(p_1)p_0}{p^2} \int_0^1 \mathrm{d}\tau\ \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\ \frac{(p_0^\tau+k_0)f'(k_0)f(k_1)}{(p^\tau+k)^2} \\
&+ \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\ \frac{f(p_0)f'(p_1)p_0}{p^2} \int_0^1 \mathrm{d}\tau\ \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\ \frac{(p_1^\tau+k_1)f'(k_0)f(k_1)}{(p^\tau+k)^2}
\end{aligned}$$

Therefore:

$$\begin{aligned}
\mathbf{Td1} &\stackrel{def}{=} \frac{2}{(2\pi)^4}\int_{-1}^1 \mathrm{d}p_1\ \frac{1}{1+p_1^2}\int_0^1 \mathrm{d}\tau\ \int_{-1}^1 \mathrm{d}k_1\ \Bigg[\frac{\tau}{\tau^2+(p_1^\tau+k_1)^2} \\
&\qquad - \frac{\tau-2}{(\tau-2)^2+(p_1^\tau+k_1)^2} - \frac{2}{1+k_1^2}\Bigg] = 0.86\frac{1}{(2\pi)^4}\ , \\
\mathbf{Td2} &\stackrel{def}{=} \frac{4}{(2\pi)^4}\int_{-1}^1 \mathrm{d}p_0\ \frac{p_0}{p_0^2+1}\int_0^1 \mathrm{d}\tau\ \int_{-1}^1 \mathrm{d}k_1\ \left[\frac{1-\tau+k_1}{(p_0^\tau+1)^2+(1-\tau+k_1)^2} - \frac{k_1}{1+k_1^2}\right] \\
&= -0.62\frac{1}{(2\pi)^4}\ .
\end{aligned}$$

- **Q.**Regarding the fourth graph, since $(\partial_\omega \chi_0)(q) = -\big(1/2|q|\big)\chi_0'(q) D_{-\omega}(q)$, and since when $\chi_0'(q) \neq 0$, $u_{-N'}(q) \equiv 1$, the last integral is equal to

$$-\frac{1}{2}\int \frac{d^2 q}{(2\pi)^2}\frac{\chi_0'(q)}{|q|} = -\frac{1}{2}\,\frac{1}{4\pi}\chi_0(q)\Big|_{q=1}^{q=\gamma} = \frac{1}{8\pi}\ ,$$

independently on the scale N' and on the shape of the function χ. Such a contribution has to be multiplied times $(a-\overline{a})/2 = \nu^{(-)} + O(\lambda^2) = \frac{\pi}{(2\pi)^2}$, obtaining

$$6.18\frac{1}{(2\pi)^2}\ .$$

In the end, the quadratic coefficient of the second anomaly, A, is non zero, and in particular $\geq 18/(2\pi)^4$.

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