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NUMBER OF MODULI OF FAMILIES OF PLANE CURVES WITH NODES AND CUSPS

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Introduction

In this thesis we compute the number of moduli of certain families of plane curves with nodes and cusps as singularities. Let $\Sigma^n_{k,d} \subset \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n))) := \mathbb{P}^N$, with $N = \frac{n(n+3)}{2}$, be the closure, in the Zariski's topology, of the locally closed set of reduced and irreducible plane curves of degree n with k cusps and d nodes. We recall that, for $k = 0$, the varieties $V_{n,g} = \Sigma^n_{0,d}$ are called the Severi varieties of irreducible plane curves of degree n and geometric genus $g = \binom{n-1}{2} - d$. Let $\Sigma \subset \Sigma^n_{k,d}$ be an irreducible component of $\Sigma^n_{k,d}$ and let $g = \binom{n-1}{2} - d - k$ be the geometric genus of the plane curve corresponding to the general point of Σ. It is naturally defined a rational map

$$\Pi_\Sigma : \Sigma \dashrightarrow \mathcal{M}_g,$$

sending the general point $[\Gamma] \in \Sigma$ to the isomorphism class of the normalization of the plane curve Γ corresponding to the point $[\Gamma]$. We set

$$\textit{number of moduli of } \Sigma := dim(\Pi_\Sigma(\Sigma)).$$

We say that Σ has *general moduli* if Π_Σ is dominant. Otherwise, we say that Σ has *special moduli.* It is possible to prove that, if $k < 3n$, then

$$dim(\Pi_\Sigma(\Sigma)) \leq min(dim(\mathcal{M}_g), dim(\mathcal{M}_g) + \rho - k), \tag{1}$$

where $\rho := \rho(2, g, n) = 3n - 2g - 6$ is the Brill-Neother number of the linear series of degree n and dimension 2 on a smooth curve of genus g. We say that Σ has the expected number of moduli if the equality holds in (1). By classical Brill-Neother theory when ρ is positive and by a well know result of Sernesi when $\rho \leq 0$ (see [**37**]), we have that $\Sigma^n_{0,d}$, (which is irreducible), has the expected number of moduli for every $d \leq \binom{n-1}{2}$. Working out the main ideas and techniques that Sernesi uses in [**37**], under the hypothesis $0 \leq k < 3n$, we find sufficient conditions in order that an irreducible component $\Sigma \subset \Sigma^n_{k,d}$ has the expected number of moduli. If Σ verifies these conditions, then $\rho \leq 0$. In particular, we prove the following two results.

Proposition 1 (See proposition 3.1 of chapter 3.). *Let $\Sigma \subset \Sigma^n_{k,d}$, with $0 \leq k < 3n$, be an irreducible component of $\Sigma^n_{k,d}$ and let $[\Gamma] \in \Sigma$ be a general element, corresponding to a plane curve Γ with normalization map $\phi : C \to \Gamma$. Let $H \subset \Gamma$ be the divisor cut out on Γ from the general line of $\mathbb{P}^2$ and K_C the canonical divisor of C. Suppose that:*

(1) *Γ is geometrically linearly normal, i.e. $h^0(C, \phi^*(H)) = 3$,*

(2) *the Brill-Noether map*

$$\mu_{o,C} : H^0(C, \phi^*(H)) \otimes H^0(C, K_C - \phi^*(H)) \to H^0(C, K_C)$$

of the pair (C,H), is surjective.

Then Σ has the expected number of moduli equal to $3g - 3 + \rho - k$.

Lemma 1 (See lemma 3.8 and corollary 3.9 of chapter 3.). *Let $\Sigma \subset \Sigma^n_{k,d}$ be an irreducible component of $\Sigma^n_{k,d}$, with $n \geq 5$ and $0 \leq k < 3n$. Suppose that Σ has the expected number of moduli and that the general element $[\Gamma] \in \Sigma$ corresponds to a geometrically linearly normal plane curve Γ of geometric genus g such that, if $C \to \Gamma$ is the normalization of Γ, then the map $\mu_{o,C}$ is surjective. Then, for every $k' \leq k$ and $d' \leq d + k - k'$, there is at least an irreducible component $\Sigma' \subset \Sigma^n_{k',d'}$, such that $\Sigma \subset \Sigma'$, the general element $[D] \in \Sigma'$ corresponds to a g.l.n. plane curve D of geometric genus g' with normalization $D^\nu \to D$ and the Brill-Noether map μ_{0,D^ν} surjective. In particular, also Σ' has the expected number of moduli.*

In the following theorem, by using induction on the degree n and on the genus g of the general curve of the family, we construct examples of families of plane curves with nodes and cusps verifying the hypotheses of proposition 1 and so having the expected number of moduli.

Theorem 1 (See theorem 3.11 of chapter 3.11.). *Let $\Sigma^n_{k,d}$ be the algebraic system of irreducible plane curves of degree $n \geq 4$ with k cusps, d nodes and geometric genus $g = \binom{n-1}{2} - k - d$. Suppose that:*

$$n - 2 \leq g \text{ equivalently } k + d \leq h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-4)) \tag{2}$$

and

$$k \leq 6 + \left[\frac{n-8}{3}\right] \text{ if } 3n - 9 \leq g \text{ and } n \geq 6, \tag{3}$$

$$k \leq 6 \quad \text{otherwise.} \tag{4}$$

Then $\Sigma^n_{k,d}$ has at least one irreducible component Σ which is not empty and whose general element $[\Gamma] \in \Sigma$ parametrizes a geometrically linearly normal curve Γ such that the Brill-Noether map of the pair (C, H), where C is the normalization of Γ and H denote the pull-back to C of the hyperplane section of $\mathbb{P}^2$, has maximal rank. In particular, when $\rho \leq 0$, the algebraic system Σ has the expected number of moduli equal to $3g - 3 + \rho - k$.

The previous result may be improved, see remark 3.12 of chapter 3. By theorem 1, it follows that $\Sigma^n_{1,d}$, (which is irreducible), has the expected number of moduli if $\rho \leq 0$. Moreover, from a result of Eisembud and Harris, it follows that $\Sigma^n_{1,d}$ has general moduli if $\rho \geq 2$. In theorem 3.13 of chapter 3, by using induction on n we find that $\Sigma^n_{1,d}$ has general moduli also when $\rho = 1$, concluding that $\Sigma^n_{1,d}$ has the expected number of moduli for every $d \leq \binom{n-1}{2} - 1$. We are extending this result to the case $k \leq 3$. Finally, we consider the variety $\Sigma^6_{6,0}$ of irreducible sextics with six cusps. It is classically known that $\Sigma^6_{6,0}$ is reducible. One of the irreducible components of $\Sigma^6_{6,0}$ is the parameter space Σ_1 of the family of plane curves of equation $f_2^3(x_0, x_1, x_2) + f_3^2(x_0, x_1, x_2) = 0$, where f_2 and f_3 are homogeneous polynomials of degree two and three respectively. The general point of Σ_1 corresponds to an irreducible sextic with six cusps on a conic as singularities. Moreover, $\Sigma^6_{6,0}$ contains at least an irreducible component Σ_2 whose general element corresponds to a sextic with six cusps not on a conic as singularities. The results above can't be useful in order to compute the number of moduli the irreducible components of $\Sigma^6_{6,0}$, because in this case $\rho = 4$. In section 4 we prove that Σ_1 has the expected number of moduli equal to $dim(\mathcal{M}_4) + 4 - 6 = 7$ and that $\Sigma^6_{6,0}$ contains at least an irreducible component having the expected number of moduli and whose general element corresponds to a sextic with six cusps not on a conic. We don't still know examples of irreducible complete families of plane curves with nodes and cusps having number of moduli smaller that the expected.

The previous results are contained in chapter three of this thesis. The first chapter of this thesis is devoted to very basic notions of algebraic geometry. In chapter two we recall some standard results of deformation theory of plane curves which will need in chapter three and we prove some little results of deformation of plane singularities. In particular, section 6 of chapter 2 is devoted to the equigeneric locus of the étale versal deformation space of an ordinary plane singularity, (see section 5 of chapter 2 for the étale versal deformation space of a plane singularity). We prove the following result.

Proposition 2. *[See proposition 6.1 of chapter 2.] There exists an étale neighborhood U of $\underline{0} \in EG \subset B$ in the equigeneric locus EG of the étale versal deformation space B of Γ, such that every point $y \in U$ corresponds to a plane curve with only ordinary multiple points.*

This result is very 'natural' and in all probability it is known, but we have not found in literature a proof of it. In order to prove the previous proposition, we show the following lemma.

Lemma 2. *[See lemma 6.7 of chapter 2.] Let $C_{r+1} \subset \mathbb{P}^{r+1}$ be a rational plane curve of degree $r+1$ and let Λ be a $(r-2)$-plane with no intersections with C_{r+1} and having finitely many intersections with the secant variety $S(C_{r+1})$ of C_{r+1}. Then, the projection plane curve $\pi_\Lambda(C_{r+1}) := C$ of C_{r+1} from Λ has only ordinary multiple points as singularities if and only if Λ transversally intersects $S(C_{r+1})$ at $r(r-1)/2$ points each of which lies on a proper secant line to the rational normal curve.*

Then, by a result of Franchetta and a result of Morelli (see lemma 6.14 and 6.11 of chapter 2) we deduce the following proposition.

Proposition 3. *For every integer r there exists an integer $R > r$ such that for every ordinary plane singularity of multiplicity r of analytic equation $g(x,y) = 0$, there exists an irreducible rational plane curve of degree R with an ordinary r-fold point analytically equivalent to $g(x,y) = 0$.*

By using proposition 3 and the properties of the étale versal deformation space of a plane singularity, we deduce proposition 2 by lemma 2.

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CHAPTER 1

Preliminaries

The standard results on algebraic systems of plane curves of a given degree with prescribed singularities, which we shall use in this paper, are collected in chapter 2. In this chapter we recall some basic results, which we shall use later. First of all we fix some notation and terminology. Through all this paper a *curve* X will be a projective separated scheme of finite type over $\mathbb{C}$, of pure dimension one. We shall say that X is reduced (smooth) if every local ring of X has no nilpotent element (is regular). We define the *geometric genus* $g(X)$ of a reduced curve X to be the arithmetic genus of its normalization. In particular if X has irreducible components $X_1, \dots, X_q$ then $g(X) = \sum_i g(X_i) - q + 1$. A plane curve will be a projective curve X contained in the projective plane $\mathbb{P}^2(\mathbb{C}) := \mathbb{P}^2$. We shall assume as known the correspondence between base point free linear series on a smooth curve C and morphisms $\phi : C \to \mathbb{P}^r$ from C to a projective space. Usually we shall work with projective singular curves. If $X \subset \mathbb{P}^r$ is such a curve and σ is a linear system on $\mathbb{P}^r$, then *the linear series cut out by* σ *on the normalization curve* $\phi : \tilde{X} \to X \subset \mathbb{P}^r$ *of* X will be the linear series cut out on $\tilde{X}$ by the linear system which is the pullback to $\tilde{X}$, with respect to ϕ, of σ.

1. Adjoint curves to a plane curve

Our definition of adjoint plane curve follows that of [**6**], Appendix A of chapter 1. Let $f(x, y) = 0$ be the affine equation of a reduced plane curve $\Gamma \subset \mathbb{P}^2$ of degree n with normalization $\phi : C \to \Gamma$. Let $p \in \Gamma$ be a singular point of Γ and let $p_1, \dots, p_s$ be the points of C which lie over p. *The adjoint divisor* Δ_p *of* p is the divisor on C defined by

$$\Delta_p = \sum_i \alpha_i p_i$$

where

$$\alpha_i = -mult_{p_i}(\phi^* \frac{dx}{\partial f / \partial y}).$$

We say that a plane curve of affine equation $g(x, y) = 0$ is adjoint to $f(x, y) = 0$ at p if, denoting by (-) the divisor associated to -, we have that

$$(\phi^* g) \geq \Delta_p.$$

The plane curves adjoint to $f(x, y) = 0$ at p form an ideal which we denote by $A_p \subset \mathcal{O}_{\mathbb{P}^2, p}$, while the adjoint curves to Γ at p of a given degree form a linear system. Notice that, by definition, a plane curve $g(x, y) = 0$ is adjoint to Γ at p if and only if the local form

$$\phi^* \frac{g(x, y) dx}{\partial f / \partial y}$$

is holomorphic at each of the points $p_1, \dots, p_s \in C$ mapping to p. Moreover, we define *the number of adjoint conditions* δ_p *at* p as the index of the ideal A_p in $\mathcal{O}_{\mathbb{P}^2, p}$, i.e. $\delta_p = dim(\mathcal{O}_{\mathbb{P}^2, p}/A_p)$ as vector space over $\mathbb{C}$. Denoting by $Sing(\Gamma)$ the set of singular points of Γ, the *adjoint divisor*

Δ *of* $\phi : C \to \Gamma$ is the divisor $\Delta = \sum_{p \in Sing(\Gamma)} \Delta_p$ and the *number of adjoint conditions* δ of Γ is defined by $\delta = \sum_{p \in Sing(\Gamma)} \delta_p$. A reduced plane curve is adjoint to Γ if it is adjoint to Γ at every point $p \in Sing(\Gamma)$. The plane curves adjoint to Γ form an ideal, which we shall denote by $A \subset \mathcal{O}_{\mathbb{P}^2}$. The notion of adjoint curve is very important in the theory of plane curves for several reasons, first of all because of the relation with the canonical sheaf ω_C which we explain below. Let (U, z) be an holomorphic chart of C. If we set $\phi_{|U}(z) = (x(z), y(z))$, then the local form of ω in U is

$$\omega_{|U} = \frac{\partial x(z)}{\partial z} \frac{dz}{\partial f(x(z), y(z))/\partial y}$$

and we find that

$$\Delta_{|U} = (\partial x(z)/\partial z) - (\partial f(x(z), y(z))/\partial y).$$

We observe that, if $\pi : C \to \mathbb{P}^1$ is the composition of the normalization morphism of Γ with the first projection map, then the divisor $(\partial x(z)/\partial z)$ is the restriction to U of the ramification divisor R_π of π, that is the zero divisor of the differential map of π. Denoting by η a meromorphic one form on $\mathbb{P}^1$, we recall that

$$(\pi^*\eta) = R_\pi + \pi^*(\eta).$$

From the former two equalities, denoting by $K_C = (\pi^*\eta)$ a canonical divisor on C and by H the pullback to C of the divisor cut out on Γ by a general line, we deduce that

$$\Delta \equiv R_\pi - (n-1)H \equiv K_C + 2H - (n-1)H,$$

or equivalently,

$$\omega_C = \phi^*(\mathcal{O}_C(n-3)(-\Delta)), \tag{5}$$

where we set $\mathcal{O}_C(n-3) = \mathcal{O}_C((n-3)H)$.

Lemma 1.1. *The linear system of plane curves of degree $n-3$ adjoint to Γ cuts out on C the complete canonical series, that is*

$$h^0(\omega_C) = H^0(\mathbb{P}^2, A \otimes \mathcal{O}_{\mathbb{P}^2}(n-3)) = \binom{n-1}{2} - \delta.$$

In particular, for every $r \geq n-3$ the linear system of plane curves of degree r adjoint to Γ has the expected dimension equal to $h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r)) - \delta$.

Proof. By the equality (5) it follows that, if $g(C)$ is the geometric genus of C, then

$$g(C) = h^0(\omega_C) = \frac{(n-1)(n-2)}{2} - 1/2deg(\Delta)$$

and

$$\frac{(n-1)(n-2)}{2} - 1/2deg(\Delta) \geq h^0(O_C(n-3)(-\Delta)) \geq \frac{(n-1)(n-2)}{2} - \delta,$$

that is

$$deg(\Delta) \leq 2\delta. \tag{6}$$

We want to show that in (6) the equality holds. As before, let $p \in \Gamma$ be a singular point of Γ and let $p_1, \dots, p_k$ be the points of C which lie over p. Let V be the $(deg\Delta_p)$-dimensional vector space

$$V = H^0(C, \mathcal{O}_C/\mathcal{O}_C(-\Delta_p)).$$

We consider the bilinear pairing

$$\psi : V \times V \to \mathbb{C}$$

defined by

$$\psi(g,h) = \sum_i Res_{p_i}(gh\omega).$$

First of all we observe that this pairing is not degenerate. To see this, let $g \in V$ be a not zero element. We still denote by g a meromorphic function on C whose image in V is equal to $g \in V$. If $r_i = mult_{p_i}(g\omega)$, then there exist a meromorphic function h on C having order $-r_2, \dots, -r_k$ at $p_2, \dots, p_k$ respectively and order $-r_1 - 1$ at p_1. The one form $gh\omega$ is holomorphic at $p_2, \dots, p_k$ and have a simple pole at p_1. This proves that ψ is not degenerate. Now we claim that

$$\sum_{i=1}^{k} Res_{p_i}\phi^* g\omega = 0, \tag{7}$$

for every polynomial $g(x,y)$. We prove the claim by induction on the minimal number blowing-ups necessary to resolve the singularity of Γ at p. If this number is zero, that is if p is a smooth point of Γ, thus $\phi^* g\omega$ is holomorphic at p and the claim is true. Now suppose that Γ has a singular point of multiplicity s at p and that, blowing up the plane at p, the singular points of the strict transform $\tilde{\Gamma}$ of Γ which map to p verify (7). Let $g(x,y)$ be a polynomial which vanishes with multiplicity r at p. If we assume, as we may, that $p = (0,0)$ and that the line $x = 0$ is not tangent to Γ at $(0,0)$, then, by taking analytic coordinates $(x, \tilde{y})$ on the blowing-up of the plane $b : \mathbb{B}l_p\mathbb{P}^2 \to \mathbb{P}^2$, where $\tilde{y} = y/x$, we have that the strict transforms of Γ and $g(x,y) = 0$ have equations $\tilde{f}(x,\tilde{y}) = 0$ and $\tilde{g}(x,\tilde{y}) = 0$ respectively, with

$$f(x,y) = x^s \tilde{f}(x,\tilde{y}) = 0$$

and

$$g(x,y) = x^r \tilde{g}(x,\tilde{y}) = 0.$$

Moreover, all the points of the strict transform of $\tilde{f}(x,\tilde{y}) = 0$ which lie over p have finite coordinates. Differentiating the previous equalities with respect to y we find that

$$\frac{g(x,y)dx}{\partial f/\partial y} = x^{r-s+1}\frac{\tilde{g}(x,\tilde{y})dx}{\partial f/\partial \tilde{y}}.$$

Denoting by $\tilde{\phi} : C \to \mathbb{B}l_p\mathbb{P}^2$ the map such that $\phi = b\tilde{\phi}$, we find that

$$\sum_{i=1}^{k} Res_{p_i}\phi^* g\omega = \sum_{q\in\tilde{\Gamma}|b(q)=p}\ \sum_{p_i|\tilde{\phi}(p_i)=q} Res_{p_i}\tilde{\phi}^*(x^{r-s+1}\frac{\tilde{g}(x,y)dx}{\partial f/\partial \tilde{y}}) = 0.$$

The equality (7) follows by inductive hypothesis on $\tilde{\Gamma}$. This proves that, if $W = H^0(\Gamma, \mathcal{O}_\Gamma/A_p)$ and $\tilde{W}$ is the image of W under the injection

$$H^0(\Gamma, \mathcal{O}_\Gamma/A_p) \to H^0(C, \mathcal{O}_C/\mathcal{O}_C(-\Delta_p)),$$

then $\psi(\tilde{W}, \tilde{W}) = 0$ and hence $\tilde{W} \subset Ann(\tilde{W})$. We deduce that

$$\delta_p = dim\tilde{W} \le deg(\Delta_p) - \delta_p,$$

for every $p \in Sing(\Gamma)$. Hence the equality holds in (6) and

$$h^0(\omega_C) = H^0(\mathbb{P}^2, A \otimes \mathcal{O}_{\mathbb{P}^2}(n-3)) = \binom{n-1}{2} - \delta.$$

In particular, it follows that the evaluation map

$$H^0(\mathbb{P}^2,\, \mathcal{O}_{\mathbb{P}^2}(n-3)) \to H^0(\mathbb{P}^2,\, \mathcal{O}_{\mathbb{P}^2}(n-3)/A) = \mathbb{C}^\delta,$$

which we deduce from the following exact sequence

$$0 \to A \otimes \mathcal{O}_{\mathbb{P}^2}(n-3) \to \mathcal{O}_{\mathbb{P}^2}(n-3) \to \mathcal{O}_{\mathbb{P}^2}(n-3)/A \to 0$$

is surjective. Since for every $r \geq n-3$, we have that $H^0(\mathbb{P}^2,\, \mathcal{O}_{\mathbb{P}^2}(r)) \supsetneqq H^0(\mathbb{P}^2,\, \mathcal{O}_{\mathbb{P}^2}(n-3))$, the evaluation map

$$H^0(\mathbb{P}^2,\, \mathcal{O}_{\mathbb{P}^2}(r)) \to H^0(\mathbb{P}^2,\, \mathcal{O}_{\mathbb{P}^2}(r)/A) = \mathbb{C}^\delta$$

is surjective too and hence $h^0(A \otimes \mathcal{O}_{\mathbb{P}^2}(r)) = h^0(\mathcal{O}_{\mathbb{P}^2}(r) - \delta$. □

We conclude this section by observing that, *if the plane curve* Γ *has only nodes and cusps as singularities*, then the rational one-form ω defined above has simple poles at the points of C mapping to the singular locus of Γ and hence *a plane curve* D *is adjoint to* Γ *if and only if* D *passes through every singular point of* Γ. In this case it is immediate that $2\delta = deg(\Delta)$.

2. Dualizing sheaf of a curve with nodes and cusps

We refer to [**22**] for the properties of the dualizing sheaf of a projective variety. In this paper it will be convenient for us to give the following definition of the dualizing sheaf of a curve.

Definition 2.1. *Let* C *be a curve with normalization* $\nu : C^\nu \to C$. *The dualizing sheaf* ω_C *associates to each* $U \subset C$ *the set of rational one-forms* η *on* $\nu^{-1}(U) \subset C^\nu$ *such that*

$$\sum_{q \in \nu^{-1}(p)} Res_q(\nu^* f \eta) = 0 \qquad (8)$$

for every $p \in C$ *and for each* $f \in \mathcal{O}_{C,p}$.

We compute the dualizing sheaf of a curve C with at most nodes and cusps as singularities. Let C be such a curve, $U \subset C$ an open set of C and $\eta \in \omega_C(U)$ a local section of the dualizing sheaf of C on U. If $q \in U$ is a smooth point, then, by (8), η has to be holomorphic at q, and hence, if C is smooth, the dualizing sheaf of C coincides with its canonical sheaf. Suppose that q is a node of C and let q_1 and q_2 be the points C^ν which lie over q. In this case, by (8), the one-form η can have at most simple poles at each of the points q_1 and q_2. In order to see this, let $(U_1,\, z_1)$ and $(U_2,\, z_2)$ be disjoint holomorphic charts of q_1 and q_2 respectively. We may assume that $\nu_{|U_1}(z_1) = (z_1, 0)$ and $\nu_{|U_2}(z_2) = (0, z_2)$. Now suppose that $\eta_{|U_2} = \sum_{i \geq r} a_i z_2^i$, with $r \leq -2$ and $a_r \neq 0$. If $y : \nu(U_1) \cap \nu(U_2) \to \mathbb{C}$ is the second projection map, then $\nu^* y^{-r-1} \eta_{|U_2}(z_2) = a_i z_2^{-1} + f(z_2)$, where f is an holomorphic function, while $\nu^* y_{|U_1}^{-r-1} = 0$ and hence

$$Res_{q_2} \nu^* y^{-r-1} \eta + Res_{q_1} \nu^* y^{-r-1} \eta = a_r \neq 0.$$

That proves that η, as rational form on C^ν has at most simple poles at q_1 and q_2. Moreover, applying (8) to the case that f is a not zero constant function, we find that

$$Res_{q_1} \eta + Res_{q_2} \eta = 0. \qquad (9)$$

Viceversa, every rational one-form $\eta \in \omega_{C^\nu}(\nu^{-1}(U))$ having at most simple poles at q_1 and q_2 and verifying (9) is a local form of $\omega_{C'}$ on U. Indeed, if η is such a local rational one form and $f \in \mathcal{O}_{C,p}$, then, by using the same notation as before, we may set $\eta_{|U_1} = (\sum_{i\geq -1} c_i z_1^i)dz_1$, $\eta_{|U_2} = -c_{-1}z_2^{-1}dz_2 + \sum_{i\geq 0}(d_i z_2^i)dz_2$, $\nu^* f_{|U_1} = \sum_{i\geq 0} a_i z_1^i$ and $\nu^* f_{|U_2} = a_0 + \sum_{i\geq 0} b_i z_2$. Thus

$$Res_{q_1}\nu^* f\eta + Res_{q_2}\nu^* f\eta = c_{-1}a_0 - c_{-1}a_0 = 0,$$

as we wanted. Finally, suppose that C has a cusp at q. Let p be the point of C^ν which lies over q and let (V, z) be a holomorphic chart containing p such that $\nu_{|V}(z) = (z^2, z^3)$. First of all we notice that if a local rational form on V satisfies (8), then η may have at p a pole of multiplicity at most equal to two. Indeed, if $\eta_{|V} = \sum_{i\geq r} a_i z^i$, with $r \leq -3$, and $x : \nu(V) \to \mathbb{C}$ is the first projection map, then $Res_p(\nu^* x^{-r-1}\eta) = a_r \neq 0$. Moreover, a rational one form with at most a double pole at p satisfies (8) if and only if $res_p\eta = 0$. Indeed, if $f \in \mathcal{O}_{C,q}$ is a not zero constant function then $Res_p\nu^* f\eta = 0$ if and only if $Res_p\eta = 0$. On the other hand, if $Res_p\eta = 0$, that is $\eta_{|V} = a_{-2}z^{-2}dz + \sum_{i\geq 0} a_i z^i dz$, then, for every $f \in \mathcal{O}_{C,q}$ such that $\nu^* f_{|V} = \sum_{i\geq 0} b_i z^i$, we have that $\nu^*(f)\eta_{|V}(z) = f(z^2, z^3)\eta(z) = b_0 a_{-2} z^{-2}dz + \eta'(z)dz$, where η' is an holomorphic function, and hence $Res_p\nu^* f\eta = 0$. This proves the following result.

Lemma 2.2. *Let C be a curve with nodes and cusps as singularities. Then, denoting by $Sing(C)$ the singular locus of C and by setting $\nu^*(q) = \nu^{-1}(q)$ if q is a node and $\nu^*(q) = 2\nu^{-1}(q)$ if q is a cusp, the dualizing sheaf ω_C of C is the subsheaf of $\nu_*(\omega_{C^\nu}(\sum_{q\in Sing(C)} \nu^*(q)))$ which associates to each $U \subset C$ the set of local section η on $\nu^{-1}(U) \subset C^\nu$ such that*

$$\sum_{q\in\nu^{-1}(p)} Res_q(\eta) = 0$$

for every $p \in C$.

3. Geometric genus in a flat family of curves.

In this section we prove that the geometric genus is a lower semicontinuous function on a flat family of curves. This is a very basic fact in the theory of curves. All the proofs of this section have been taken from [**13**]. From now on, for a flat family family of curves we mean a proper morphism $X \to Y$ which is flat and whose fibres have all dimension one.

Let

$$\begin{array}{ccc} X & \subset & \mathbb{P}^m \times Y \\ \pi\downarrow & & \\ Y & & \end{array} \tag{10}$$

be a flat family of projective curves with all fibres reduced and such that X and Y are reduced separated scheme of finite type over $\mathbb{C}$. Let $\phi_\pi(y)$ be the geometric genus of $\pi^{-1}(y)$, for all $y \in Y$.

Lemma 3.1. *Suppose that Y is a regular curve and let $\pi : X' \to X$ be the normalization map. Then $\pi \cdot f : X' \to Y$ is a flat family of projective curves with all fibres reduced.*

Proof. By proposition III.9.7 of [**22**], a family of curves $X \to Y$ parametrized by a smooth curve Y is flat if and only if every component of X dominates Y. We deduce that X and X' dominate Y and the family $\pi \cdot f : X' \to Y$ is flat. In order to prove that a fibre of $\pi \cdot f$ cannot have multiple components, we recall that π is a birational finite map which is invertible at every smooth point of X. Since all fibres of f are reduced and X is smooth at every smooth point of any fibre of f, every fibre of $\pi \cdot f$ is birational to the corresponding fibre of f, and so it cannot have multiple components. To prove that a fibre of $\pi \cdot f$ has not nonreduced points, it is enough to prove that on each irreducible component of X'. Since Y is smooth, every fibre of $\pi \cdot f$ is principal. Recalling that in a normal Noetherian domain principal ideals are unmixed, the lemma is proved. □

Proposition 3.2. *Let $X \to Y$ be a family like* (10). *Then ϕ_π is a lower semicontinuous function in the Zariski topology.*

Proof. Suppose that Y is a smooth curve. By using lemma 3.1, we see that the fibres of $\pi \cdot f$ are the normalizations or partial normalizations of the corresponding fibres of f. Actually, since X' is normal, it has at most finitely many singular points. Applying the generic smoothness theorem to the restriction of $\pi \cdot f$ to the open set of smooth points of X', we find that the general fibre of $\pi \cdot f$ is smooth. We conclude that $\phi_\pi = \phi_{\pi \cdot f}$ and $\phi_{\pi \cdot f}$ is constant on the Zariski open set of Y where all fibres are nonsingular, because the arithmetic genus remains constant in a flat family. Finally, recalling that the geometric genus of a reduced singular curve is always strictly less than the arithmetic genus, we have that $\phi_{\pi \cdot f}$ decreases at singular fibres. Suppose now that Y is a reduced separated scheme of finite type over $\mathbb{C}$. First of all we prove that there is an open set U of Y on which the function ϕ_π is constant. Let U_1 be the open set of regular points of Y. Let V be the normalization of $\pi^{-1}(U_1)$ and $g : V \to U_1$ the induced family of curves. since V is normal, its singularities points form a closed set A of codimension at least two in V. Since g is proper, $U_1 - g(A)$ is a dense open set of U_1. By generic smoothness theorem, we find that there is an open set $U \subset U_1 - g(A)$ on which the morphism $g^{-1}(U) \to U$ is smooth. The fibre of g on U are just the normalizations of the corresponding fibers of π. Thus ϕ_π is constant on U. Now if $y \in Y - U$, then $\phi_\pi(y) \leq g := \phi_\pi(U)$. To see this, let Z be a general curve in Y through y whose general points belongs to U. Let Z' be the normalization of Z and let $h : C \to Z'$ be the pullback family of $X \to Y$ to Z'. From what we proved before, ϕ_h is a lower semicontinuous function. Thus $\phi_\pi(y) \leq g$. Finally, let $n \in \mathbb{Z}$ and let $B(n) = \{y \in Y | \, \phi_\pi(y) \leq n\}$. We have to show that $B(n)$ is Zariski closed. Fix n. If $B(n) = Y$ we are done. If not, from what we proved before, there is a Zariski closed set $Y_1 \subsetneqq Y$, $(Y_1 = Y - U)$ such that $B(n) \subseteqq Y_1$. If $B(n) = Y_1$ we are done. If not, arguing as before on the family $\pi^{-1}(Y_1) \to Y_1$, we find that there is a Zariski closed $Y_2 \subsetneqq Y_1$ with $B(n) \subseteqq Y_2$. Since Y is assumed to be of finite type over $\mathbb{C}$ and hence Noetherian, this process must terminate. So we find a k such that $Y_k = B(n)$. □

Finally we need the following result.

Theorem 3.3 ([**44**], p. 80). *Under the general assumptions of proposition 3.2 assume further that Y is normal and ϕ_π is constant. Let $f : X' \to X$ be the normalization map. Then $\pi \cdot f : X' \to Y$ is a smooth family of curves and each fiber of $\pi \cdot f$ is the normalization of the corresponding fiber of π.*

Example (2.6) of [**13**] shows that there are families of reduced projective curves $\pi : X \to Y$ with Y non normal, such that ϕ_π is constant and which don't admit a simultaneous desingularization.

Remark 3.4. *Finally, we observe that, if $\pi : X \to Y$ is a flat family like* (10) *whose fibres of maximal genus have genus g, then, applying generic smoothness theorem to the regular locus of the normalization X' of X, by functorial properties of the moduli space $\mathcal{M}_g$ of curves of genus g, we get a rational map*

$$\Pi : Y \dashrightarrow \mathcal{M}_g$$

which sends the general point $y \in Y$ to the isomorphism class of the normalization curve of $\pi^{-1}(y)$. In general, even if ϕ_π is constantly equal to g, the map Π doesn't extends to a regular map on all Y. By theorem 3.3, under the hypothesis that ϕ_π is constant and denoting by Y' the normalization of Y we have that

$$\begin{array}{ccc} Y' & & \\ \downarrow & \searrow & \\ Y & \dashrightarrow & \mathcal{M}_g \end{array}$$

the map Π is defined at the regular locus of Y and it extends to a regular map on Y'.

4. Deformations of morphisms on a smooth curve

In this section we give the definition of deformation of a morphism on a smooth curve and we state the two main theorems of Horikawa deformation theory. In section 7 of chapter 2 we will show some applications of Horikawa deformation theory to the study of families of plane curves with nodes and cusps. In this section, we shall denote by C a smooth curve of genus g and by Y a smooth projective algebraic scheme over $\mathbb{C}$. Given two smooth curves C and C', we say that two holomorphic maps $\phi : C \to Y$ and $\psi : C' \to Y$ are *equivalent* if there exists an isomorphism $f : C \to C'$ such that $\phi = \psi \cdot f$. We recall the following definition.

Definition 4.1. *A deformation $(\mathcal{C}, \tilde{\phi}, \pi, B)$ of an holomorphic map $\phi : C \to Y$ is given by a flat deformation $\pi : \mathcal{C} \to B$, where $\mathcal{C}$ and B are separated schemes of finite type over $\mathbb{C}$, a morphism $\tilde{\phi} : \mathcal{C} \to Y$ and a closed point $0 \in B$ such that the restriction morphism $\mathcal{C}_0 := \pi^{-1}(0) \to Y$ is equivalent to ϕ.*

A deformation of ϕ is said to be *infinitesimal* if $B = Spec(\mathbb{C}[\epsilon]) = Spec(\mathbb{C}[t]/(t^2))$ and it is said to be *effective* if $dim(B) = 1$. If we denote by Θ_C and Θ_Y the tangent sheaf to C and Y respectively, we have the following exact sequence of sheaf on C

$$0 \to \Theta_C \to \phi^*\Theta_Y \to \mathcal{N}_\phi \to 0$$

where the map $\phi_* : \Theta_C \to \phi^*\Theta_Y$ is the differential map of ϕ. The cokernel $\mathcal{N}_\phi$ of ϕ_* is called the *normal sheaf to* ϕ. Let $(\mathcal{C}, \tilde{\phi}, \pi, B)$ be a deformation of ϕ and let $0 \in Y$ be a closed point such that the induced morphism $\mathcal{C}_0 := \pi^{-1}(0) \to Y$ is equivalent to ϕ. From what has been proved in [**26**], there exists a *characteristic map*

$$\rho : T_0B \to H^0(C, \mathcal{N}_\phi),$$

from the tangent space to B at 0 to the global sections space of the normal sheaf to ϕ. We have the following two results.

Theorem 4.2 (Horikawa, theorem 2.1 of [**26**]). *If the characteristic map ρ defined above is surjective, then the family of morphisms $(\mathcal{C}, \tilde{\phi}, \pi, B)$ is complete, i.e. for every other deformation $(\mathcal{D}, F, G, B')$ of ϕ and for every closed point $0' \in B'$ such that the induced morphism $\mathcal{D}_0 \to Y$ is equivalent to ϕ, there exist an open neighborhood U of $0'$ in B' and a morphism $h : U' \to B$ such that $h(0') = 0$ and such that the restriction of $(\mathcal{D}, F, G, B')$ to U is equivalent to the pullback family of $(\mathcal{C}, \tilde{\phi}, \pi, B)$ with respect to h.*

Theorem 4.3 (Horikawa, theorem 3.1 of [**26**]). *If $H^1(C, \mathcal{N}_\phi) = 0$, then there exists a deformation $(\mathcal{C}, \tilde{\phi}, \pi, B)$ of C such that the characteristic map is an isomorphism.*

A family of morphisms $(\mathcal{C}, \tilde{\phi}, \pi, B)$ satisfying (4.3) is said to be a *universal deformation family of* ϕ, and we will call the parameter space B *a universal deformation space of* ϕ.

CHAPTER 2

Families of plane curves with nodes and cusps

1. Introduction

In this chapter we recall some standard results of deformation theory of plane curves which will need in the next chapter. In particular, we are interested in the scheme $\Sigma^n_{k,d} \subset \mathbb{P}^{n(n+3)/2}$ parametrizing irreducible plane curves of degree n and geometric genus $g = \binom{n-1}{2} - k - d$, with k cusps and d nodes as singularities.

Section 2 is devoted to Severi varieties. We define the Severi variety $V_{n,g} \subset \mathbb{P}^{n(n+3)/2}$ as the Zariski closure in the Hilbert scheme of plane curve of degree n of the locally closed subset parametrizing irreducible plane curves of degree n and genus g. It is well know that $V_{n,g}$ is irreducible and the general element of $V_{n,g}$ corresponds to a plane curve with $d = \binom{n-1}{2} - g$ nodes as singularities, (see [**19**], [**2**] and [**47**]).

In section 3, by following the Zariski's papers [**47**], [**48**] and [**49**], we introduce classical techniques used to study algebraic systems. An algebraic system of plane curves of degree n is a Zariski closed subset of the Hilbert scheme $\mathbb{P}^{n(n+3)/2}$ of plane curves of degree n. We give the definition of equivalence between plane singularities and we prove the Dimension Characterization Theorem of Severi varieties of Zariski, (see theorem 3.12). As corollary of this theorem, we deduce that every irreducible component Σ of $\Sigma^n_{k,d} \subset \mathbb{P}^{n(n+3)/2}$ has dimension at least equal to $3n+g-1-k$ and that the equality holds if $k < 3n$, see corollary 3.13. Moreover, we prove that, if $k < 3n$, then nodes and cusps of an irreducible plane curve of degree n with k cusps and d nodes as singularities, may be smoothed independently, (see section 3 of section 5 for the meaning of this statement). In particular, we prove that for every k' and d' such that $k' \leq k$ and $d' \leq d + k - k'$, there exists at least an irreducible component Σ' of $\Sigma^n_{k',d'}$ such that $\Sigma \subset \Sigma'$, see lemma 3.17. As corollary of lemma 3.17 we prove that $\Sigma^n_{k,d}$ is not empty for every $k \leq 4$. In lemma 3.22 we study the local geometry of the varieties $\Sigma^n_{1,0}$ and $\Sigma^n_{0,1}$ at a neighborhood of a point parametrizing a plane curve with d nodes and k cusps. Finally, in the examples 3.15 and 3.20 we give examples of families of plane curves with nodes and cusps as singularities with dimension bigger that the expected one and we prove that the algebraic system $\Sigma^6_{6,0}$ of sextics with six cusps as singularities is reducible, see [**49**].

In section 4, by following modern literature and, in particular, the Wahl's paper [**45**], we describe the moduli scheme of irreducible plane curves of degree n with k cusps and d nodes, as a scheme representing a suitable deformation functor F. When $k < 3n$, the scheme representing F is reduced, if it is not empty. When $k \geq 3n$ are known examples of non reduced moduli schemes of plane curve with nodes and cusps as singularities.

In section 5, by following essentially [**13**], we introduce the étale versal deformation family of a plane curve. We introduce the notions of equigeneric deformation and equisingular deformation of a plane curve and we recall the main properties of the étale versal deformation space of a plane singularity. Finally, in this section we give a second proof of lemma 3.17.

Section 6 is devoted to the étale versal deformation space of an ordinary plane singularity. In particular, we prove that, if Γ is an irreducible plane curve with an ordinary r-fold point and no further singularities, then there exists an étale neighborhood U of $\underline{0} \in EG \subset B$ in the equigeneric locus of the étale versal deformation space B of Γ, such that every point $y \in U$ corresponds to a plane curve with only ordinary multiple points, see lemma 6.1. In order to get this result, in lemma 6.7 we prove that a rational plane curve Γ of degree n has only ordinary multiple points if and only if it is projection of the rational normal plane curve $C_n \subset \mathbb{P}^n$ from an $(n-3)$-plane intersecting the secant $S(C_n)$ transversally. By this lemma, by using lemmas 6.11 and 6.14, we deduce lemma 6.1.

Finally, section 7 is devoted to Horikawa deformation theory and to its applications to the study of families of plane curves with nodes and cusps as singularities. In particular, in this section we identify the tangent space to an irreducible component Σ of $\Sigma^n_{k,d}$ at its general element $[\Gamma] \in \Sigma$ with a suitable subspace of the infinitesimal deformations space of the normalization map $\phi : C \to \Gamma$ of Γ. This identification will be very useful in proposition 3.1 of chapter 3. By using this identification and lemma 7.1, we prove a very special case of lemma 3.17, see lemma 7.5 and remark 7.7. Moreover, we recall some known results on the local geometry of $\Sigma^n_{k,d}$ at a point corresponding to an irreducible plane curve of genus $g = \binom{n-1}{2} - k - d$ with singularities worst than nodes and cusps, see proposition 7.8 and theorem 7.12. Theorem 7.11 gives a sufficient condition in order that an irreducible plane curve Γ of degree n, genus g and class c may be obtained as limit of curves of genus g and class c with only nodes and cusps as singularities. (We recall that the class of a plane curve is equal to the degree of its dual curve). Finally, our proposition 7.13 is a simple application of Horikawa deformation theory to the study of the deformations preserving genus and class of a curve with prescribed singularities.

2. Severi varieties

Let $\mathbb{P}^2$ be the complex projective plane. Since a plane curve of degree n is defined by an homogeneous polynomial up to multiply by a scalar, the set of plane curves of degree n can be identified with the projective space $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n))) = \mathbb{P}^{\frac{n(n+3)}{2}} := \mathbb{P}^N$, whose coordinates are the coefficients of a general homogeneous polynomial. Notice that $\mathbb{P}^N$ is the Hilbert scheme of complex plane curves of degree n and every flat family $\eta : X \to Y$ of projective plane curves of degree n is the pullback of the tautological family

$$\begin{array}{ccc} \mathcal{G} := \{(P, [C]) \textit{such that } P \in C\} & \subset & \mathbb{P}^2 \times \mathbb{P}^N \\ & & \downarrow \\ & & \mathbb{P}^N \end{array}$$

with respect to the natural morphism $\pi : Y \to \mathbb{P}^N$, sending every point y of Y to the point corresponding to the curve $\eta^{-1}(y)$. From now on we will denote by $[\Gamma]$ the point of $\mathbb{P}^N$ associated to a curve $\Gamma \subset \mathbb{P}^2$ and by $Sing(\Gamma)$ the singular locus of Γ. By using elimination theory, we see that the set of smooth plane curves of degree n is parametrized by a Zarisky open subset of $\mathbb{P}^N$. More in general, the set of reduced plane curves of degree n corresponds to a Zariski open set $R \subset \mathbb{P}^N$. Indeed, every such curve is defined by an homogeneous polynomial without multiple components. Then R is the complement of the image in $\mathbb{P}^N$ of the proper closed set

$$NR = \cup_{2a+b=n} \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2a))) \times \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(b))) \subset \mathbb{P}^N.$$

Moreover, R properly contains the open set I of reduced and irreducible plane curves of degree n, which is the complement of the image in $\mathbb{P}^N$ of the closed subset

$$\cup_{a+b=n}\mathbb{P}(H^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(a)))\times\mathbb{P}(H^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(b)))\supsetneq NR.$$

Recalling that the arithmetic genus of a plane curve of degree n is $\binom{n-1}{2}$, by proposition 3.2 of chapter 1, for every $0\leq g\leq\binom{n-1}{2}$ the locus of points corresponding to curves of genus at most g is Zariski closed in R.

Definition 2.1. *The Severi variety $V_{n,g}\subset\mathbb{P}^N$ of plane curves of degree n and genus g is the closure in $\mathbb{P}^N$ of the locally closed set of reduced and irreducible plane curves of degree n and geometric genus g.*

To compute the expected dimension of $V_{n,g}$, we essentially follow [**2**]. Let V be an irreducible component of $V_{n,g}$ and let $[\Gamma]\in V$ be a general element, corresponding to a curve $\Gamma\subset\mathbb{P}^2$ with normalization $C\to\Gamma$. Let $[C]$ be the point of $\mathcal{M}_g$ corresponding to C. There exists a smooth family of curves of genus g

$$p:\mathcal{C}\to S$$

parametrized by a smooth connected algebraic variety S, such that the canonical morphism $\pi:S\to\mathcal{M}_g$ is finite and dominant and such that $[C]\in\pi(S)$ (see [**33**]). Let us denote by Pic^n the relative Picard variety of the family p and by $\mathcal{W}^2_n$ the subvariety of Pic^n whose points are the pairs (s,γ), with $s\in S$ and where γ is a complete linear series of dimension at least two on the curve $p^{-1}(s)$. Let $\delta=0$ if $g\geq 2$, $\delta=1$ if $g=1$ and $\delta=3$ if $g=0$. Then Pic^n is a smooth variety of dimension $4g-3+\delta$, whereas $\mathcal{W}^2_n$ is a locally determinantal variety, locally defined by the vanishing of the minors of order $m-2$ of an holomorphic matrix of type $m\times l$ where $m-l=n-g+1$, (see [**3**] or [**6**]). It follows that, when $2\geq n-g$, the dimension of $\mathcal{W}^2_n$ is at least equal to

$$4g-3+\delta-[m-(m-3)][l-(m-3)]=4g-3+\delta-3(g-n+2).$$

Let $\mathcal{G}^2_n$ the variety whose points are the pairs (s,g^2_n) where $s\in S$ and g^2_n is a linear series of dimension two and degree n on $p^{-1}(s)$. In [**3**] or [**6**] one can find a construction of $\mathcal{G}^2_n$. Since is naturally defined a surjective map $\mathcal{G}^2_n\to\mathcal{W}^2_n$, always under the hypothesis $2\geq n-g$, we have that

$$dim(\mathcal{G}^2_n)\geq 4g-3+\delta-3(g-n+2)=3n+g-9+\delta. \tag{11}$$

If $2\leq n-g$, then $\mathcal{W}^2_n\equiv Pic^n$, the general fibre of the map $\mathcal{G}^2_n\to\mathcal{W}^2_n$ is the grassmannian of subspaces of dimension three of a space of dimension $n-g+1$ and (11) still holds. Let now $\mathcal{F}^2_n$ be the variety whose points are the triples $(s,g^2_n,\{s_0,s_1,s_2\})$ where $(s,g^2_n)\in\mathcal{G}^2_n$ and $\{s_0,s_1,s_2\}$ is a frame of the three dimensional space associate to the linear series g^2_n. Since all the fibres of the projection map $\mathcal{F}^2_n\to\mathcal{G}^2_n$ are isomorphic to $Aut(\mathbb{P}^2)$, we have that $dim(\mathcal{F}^2_n)\geq 3n+g-1+\delta$. Notice that every point $(s,g^2_n,\{s_0,s_1,s_2\})\in\mathcal{F}^2_n$ determines a morphism $p^{-1}(s)\to\mathbb{P}^2$. Let $\mathcal{F}$ be the irreducible component of $\mathcal{F}^2_n$ containing the point corresponding to the morphism $C\to\Gamma$. Sending every point of $\mathcal{F}$ to the point of $\mathbb{P}^N$ parametrizing the image curve of the associates morphism, we get a rational map

$$\mathcal{F}\to V\subset\mathbb{P}^N.$$

Since two morphisms $\psi,\phi:C\to\mathbb{P}^2$ from a smooth curve C to $\mathbb{P}^2$ have the same image if and only if there is an automorphism $A:C\to C$ such that $\phi\cdot A=\psi$, we deduce that

$$dim(V)\geq 3n+g-1.$$

Actually, every irreducible component of the Severi variety has the above expected dimension.

Theorem 2.2 (Zariski [**47**], Arbarello-Cornalba [**2**]). *Let V be an irreducible component of $V_{n,g}$. Then*

$$dim(V) = 3n + g - 1 = \frac{n(n+3)}{2} - d = N - d$$

where $d = \binom{n-1}{2} - g$ and the general point of V corresponds to an irreducible plane curve of degree n with d nodes and no further singularities.

From now on we will denote by $V^o_{n,g}$ *the locally closed set of $V_{n,g}$ parametrizing irreducible nodal curves of genus g.* Notice that the dimension of $V^o_{n,g}$ has been computed first by Severi, by showing that the tangent space to every irreducible component $V^o_{n,g}$ at the general point $[\Gamma]$ is the linear space parametrizing the linear system of the adjoint curves of degree n to Γ, (see section 1 of chapter 1 and section 3.1). An attempt of proof of the fact that every irreducible plane curve of degree n and genus g is limit of nodal plane curve of the same degree and genus, can be found in the paper by Albanese [**1**]. But, this paper contains a gap which we have not been able to fill-up. Albanese does not prove that, if $\Gamma \subset \mathbb{P}^2$ is an irreducible plane curve of degree $2n$ and genus g, with $n > g - 1$, having three ordinary triple points at three general points P_1, P_2 and P_3 of the plane and other singularities, then there exists a one parameter family of plane curves $\mathcal{G} \to \Delta$ of degree $2n$ and genus g, whose special fibre $\mathcal{G}_0$ is equal to Γ and whose general fibre has an ordinary triple point at a neighborhood of every P_i, for $i = 1, 2, 3$, and nodes as further singularities. As we will see in the next section, in order to prove the theorem 2.2, Zariski starts from Severi ideas, putting them in a more general and formal context, see theorem 3.12. Arbarello and Cornalba approach is very different. They look at the Severi variety as the locus of pairs (C, ϕ), where C is a smooth curve of genus g and $\phi : C \to \mathbb{P}^2$ is a morphism from C to $\mathbb{P}^2$, and they use Horikawa deformation theory to prove theorem 2.2, ([**2**] and [**3**]). This paper doesn't contain Arbarello and Cornalba proof of the theorem 2.2, but we started on their ideas to prove our proposition 7.13. Now we want to show the following more elementary fact.

Lemma 2.3. *The set $S_{n,d}$ which parametrizes reduced plane curves with at least d singular points is Zariski closed in $\mathbb{P}^N$ and every its irreducible component has dimension at least equal to $N - d$, if it is not empty. Moreover, let $U_{n,d} \subset S_{n,d}$ be the locus of reduced d-nodal plane curves of degree n. Every not empty irreducible component U of $U_{n,d}$ is Zariski dense in an irreducible component S of $S_{n,d}$.*

Before proving the lemma, we remark that $V^o_{n,g}$ is contained in $U_{n,d}$ and it is the union of irreducible components of $U_{n,d}$ parametrizing irreducible plane curves. Then, by the former lemma and by theorem 2.2, we have that if $1 \leq d \leq \binom{n-1}{2}$ and $g = \binom{n-1}{2} - d$ then, the union of irreducible components of $S_{n,d}$, whose general element corresponds to an irreducible curve, coincides with $V_{n,g}$.

Proof of lemma 2.3. Let R be the open set of $\mathbb{P}^N$ parametrizing reduced curves. Let $\widetilde{S_{n,d}} \subset \mathbb{P}^N \times (\mathbb{P}^2)^d$ be the closure in $\mathbb{P}^N \times (\mathbb{P}^2)^d$ of the incidence family

$$\{([C], p_1, \dots, p_d) | p_i \neq p_j \, and \, p_i \in Sing(C), \, for \, 1 \leq i, j \leq d\} \subset R \times (\mathbb{P}^2)^d.$$

Denoting by $F(x_0, x_1, x_2) = \sum_{i+j+k=n} a_{ijk} x_0^i x_1^j x_2^k$ the general homogeneous polynomial of degree n, then, by Euler's equality, a point $([F = 0], p)$ belongs to $\widetilde{S_{n,d}}$ if and only if it verifies the following bihomogeneous equations in $\mathbb{P}^N \times \mathbb{P}^2$

$$\begin{aligned} F(p_i) &= 0, \ \text{i=1,...,d} \\ \frac{\partial F}{\partial x_0}|_{p_i} &= 0, \ \text{i=1,...,d} \\ \frac{\partial F}{\partial x_1}|_{p_i} &= 0, \ \text{i=1,...,d.} \end{aligned}$$

Since the projection map $p_1 : \widetilde{S_{n,d}} \to \mathbb{P}^N$ is a closed morphism we get that $S_{n,d} = p_1(\widetilde{S_{n,d}})$ is a Zariski closed subset of $\mathbb{P}^N$, for every $0 \leq d \leq \frac{n(n-1)}{2}$. Moreover, for every irreducible component $S \subset S_{n,d}$, we have that the general fibre of p_1 on S is finite and hence we find that

$$dim(S) \geq N + 2d - 3d = N - d = \frac{(n+3)n}{2} - d = 3n + g - 1,$$

where $g = \binom{n-1}{2} - d$. Moreover, let $\Gamma \subset \mathbb{P}^2$ be a plane curve with exactly d singular points $p_1, \ldots, p_d$. If $U_2 = \{[x_0 : x_1 : x_2)]|x_2 \neq 0\} \subset \mathbb{P}^2$, up to projective transformations, we can suppose that $p_i \in U_2$ for every i. Then, denoting by $F(x_0, x_1, x_2) = \sum_{i+j+k=n} a_{ijk} x_0^i x_1^j x_2^k$ the homogeneous polynomial defining Γ, we have that Γ has a node at every point p_i if and only if

$$(\frac{\partial F}{\partial x_0 \partial x_1}|_{p_i})^2 - \frac{\partial F}{\partial^2 x_0}|_{p_i} \frac{\partial F}{\partial^2 x_1}|_{p_i} \neq 0.$$

for every $1 \leq i \leq d$. This proves the lemma. □

Remark 2.4. *Now, let us consider the second projection map*

$$\begin{array}{ccc} \{([C], p_1, \ldots, p_d) | p_i \neq p_j \, and \, p_i \in Sing(C), \, for \, 1 \leq i, j \leq d\} & \subset & \mathbb{P}^N \times (\mathbb{P}^2)^d \\ & p_2 \downarrow & \\ & (\mathbb{P}^2)^d & \end{array}$$

We want to remark that,in general, is not simple to compute the dimension of the image of this map. Has been proved in [**5**] *that p_2 is dominant when $3d \leq \frac{n(n+3)}{2}$. Given $p_1, \ldots, p_d$ different points of $\mathbb{P}^2$, then $p_2^{-1}(p_1, \ldots, p_d)$ parametrizes the linear system of plane curves of degree n singular at $p_1, \ldots, p_d$. The problem to compute the dimension of $p_2^{-1}(p_1, \ldots, p_d)$ is a classical problem of interpolation theory. We will go back on this topics in section 2 of chapter 3.11. Anyway, when $d = 1$ it is immediate that p_2 is surjective and all its fibres are linear spaces of dimension exactly $N - 3$. It follows that $S_{n,1} = V_{n,\binom{n-1}{2}-1}$ is a not empty irreducible hypersurface parametrizing singular curves of degree n.*

Actually, Severi varieties are always not empty and irreducible.

Theorem 2.5 (Severi, [**38**]). *For any fixed d, such that $0 \leq d \leq \binom{n-1}{2}$, there exist irreducible plane curves of degree n with d nodes and no further singularities. In particular the Severi varieties are nonempty.*

Theorem 2.6 (Harris, [**19**]). *$V_{n,g}$ is irreducible for every $0 \leq g \leq \binom{n-1}{2}$.*

We shall prove Severi's theorem 2.5 in a more general form, see corollary 3.18. As it is well known, theorem 2.6 was stated from Severi in [**38**], but its proof contains a mistake, (see [**47**] for a discussion about it). First of all, Severi shows that $V_{n,0}$ is irreducible. By using a deformation argument, which we will explain in the next section (see lemma 3.17), he proves that for every $0 \leq g \leq \binom{n-1}{2} - 1$, there exists at least one irreducible component of $V_{n,g}$ containing $V_{n,0}$. Moreover, by using a suitable monodromy argument, he shows that there is only one irreducible component of $V_{n,g}$ having this property, see lemma 1.1 of [**19**]. Then Severi would prove that every irreducible component of $V_{n,g}$ contains $V_{n,0}$, deducing that $V_{n,g}$ is irreducible. But this part of the proof is not complete. Several mathematicians attempted to prove the irreducibility of Severi varieties, obtaining sometime a partial result, see for example [**47**] or [**4**]. Finally, in [**19**], Harris gave a complete proof, by showing that every irreducible component of $V_{n,g}$ contains at least one irreducible component of $V_{n,g-1}$, for every $1 \leq g \leq \binom{n-1}{2}$. We shall not include the Harris proof of this fact. We prove theorem 2.6 only for the variety $V_{n,0}$ of rational plane curves of degree n. In this case the result follows by elementary projective geometry.

Lemma 2.7. *The Severi variety $V_{n,0}$ of rational plane curves of degree n is irreducible and not empty.*

Proof. Let $\Gamma \subset \mathbb{P}^2$ be a rational plane curve of degree n. Let $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}^1}(n)|$ be the linear series associated to the normalization morphism $\mathbb{P}^1 \to \Gamma$. The complete linear series $|\mathcal{O}_{\mathbb{P}^1}(n)|$ embeds $\mathbb{P}^1$ in $\mathbb{P}^n$ as a rational normal curve $C_n \subset \mathbb{P}^n$. The previous linear series $\mathcal{L}$ corresponds in $\mathbb{P}^n$ to a two dimensional space of hyperplanes whose base locus is a linear space Λ of dimension $n-3$ with no intersections with C_n. Since the hyperplanes through Λ cut out on C_n just the linear series $\mathcal{L}$ and since a linear series defines a morphism only up to projective motion, projecting C_n to $\mathbb{P}^2$ from Λ we shall get the curve Γ or one projectively equivalent to Γ. If $U \subset \mathbb{G}(n-3,n)$ is the open set parametrizing the hyperplanes which have not intersection with C_n, all the rational plane curves of degree n can be obtained from C_n by projecting from an $(n-3)$-plane corresponding to a point in U. It follows that there is a rational dominant map

$$\mathbb{G}(n-3,n) \times Aut(\mathbb{P}^2) \dashrightarrow V_{n,0}$$

from an irreducible variety to $V_{n,0}$, from which we deduce that $V_{n,0}$ is not empty and irreducible. We also notice that the singularities of the projections of the rational normal curve C_n, arise from the intersections of the $(n-3)$-space, which is the center of the projection, with the secant variety $S(C_n)$ of C_n, and by proposition 6.6 of section 6, it follows that the general projection of C_n is a nodal curve. □

3. Algebraic systems of plane curves of degree n

Definition 3.1. *An algebraic system Σ is a set of plane curves of degree n parametrized by a Zariski closed subset of $\mathbb{P}^N = \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n)))$.*

We will say that an algebraic system $\Sigma \subset \mathbb{P}^N$ is irreducible if it is parametrized by an irreducible Zariski closed subset of $\mathbb{P}^N$. Linear systems are examples of algebraic systems parametrized by linear subspaces of $\mathbb{P}^N$. Another example of algebraic system is the hypersurface $S_{n,1} \subset \mathbb{P}^N$ of singular plane curves of degree n. Consider for example the case $n = 3$. Since there exist cubics with only a node as singularity, (take for example $x^2 + y^2 = x^3$), by lemma 2.3, the general element of $S_{3,1}$ corresponds to a cubic with only a node as singularity. In particular, two general points $[\Gamma]$ and $[D]$ of $S_{3,1}$ correspond to two plane curves Γ and D with the same number of singularities and the singularities of D are analytically equivalent to the singularities of Γ. The same is not true for all algebraic systems. We recall the following definition.

Definition 3.2. *Two points $p \in \Gamma$ and $q \in D$ of two plane curves Γ and D are analytically equivalent if the completion $\hat{\mathcal{O}}_{\Gamma,p}$ of the local ring of Γ at p is isomorphic to the completion $\hat{\mathcal{O}}_{D,q}$ of the local ring of D at q.*

If p is an ordinary double (triple) point of a plane curve Γ , then the complete local ring $\hat{\mathcal{O}}_{\Gamma,p}$ is isomorphic to $\mathbb{C}[[x,y]]/(xy)$ $(\mathbb{C}[[x,y]]/(xy(x-y)))$, (see [**22**], ex. I.5.14). On the contrary, ordinary plane curve singularities of multiplicity ≥ 4 are not all analytically equivalent. Indeed, it is well known that two four-fold points of affine equation

$$xy(x-y)(x-\lambda y) + g_5(x,y) = 0$$

and

$$xy(x-y)(x-\mu y) + h_5(x,y) = 0,$$

where g_5 and h_5 are two polynomials in x and y of degree at least five, are analytically equivalent if and only if there exist an automorphism of $\mathbb{P}^1$ sending the points 0, ∞, 1, λ to the points 0, ∞, 1, μ, (see, for example, [**45**]). This happens if and only if

$$\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2} = \frac{(\mu^2-\mu+1)^3}{\mu^2(\mu-1)^2},$$

(see [**20**], p. 121). It follows that, denoting by $\Sigma_4 \subset \mathbb{P}^N$ the irreducible algebraic system of plane curves of degree n with a four-fold point, for every given class of analytical equivalence of four-fold points, the locus of points corresponding to a plane curve with a four-fold point of that equivalence class, is Zariski closed in Σ_4. We expect that it is possible to define a right notion of equivalence between plane curve singularities in such a way that the property of parametrizing curves with singularities equivalent to those of the curve corresponding to the general point of an irreducible algebraic system, is open in such algebraic system. That has been done by Zariski in [**47**] and [**49**]. Let Γ be a reduced plane curve of affine equation $f(x,y) = 0$ at a point $p = (0,0) \in \Gamma$. By theorems I.5.4A and I.5.5A of [**22**], the completion of the local ring $\mathcal{O}_{\Gamma,p}$ of Γ at p is isomorphic to the power series ring

$$\begin{aligned}(\mathbb{C}[x,y]/(f))_{(x,y)} \otimes_{\mathbb{C}[x,y]_{(x,y)}} \mathbb{C}[[x,y]] &\simeq \big((\mathbb{C}[x,y])_{(x,y)} \otimes_{\mathbb{C}[x,y]} \mathbb{C}[x,y]/(f)\big) \otimes_{\mathbb{C}[x,y]_{(x,y)}} \mathbb{C}[[x,y]] \\ &\simeq \mathbb{C}[[x,y]]/(f).\end{aligned}$$

The scheme $Spec(\mathbb{C}[[x,y]]/(f))$ is called the *algebroid plane curve associated to* Γ *at* p. The irreducible components of $Spec(\mathbb{C}[[x,y]]/(f))$ are called the *branches of* Γ *at* p. Let Γ and D be two reduced plane curves with the same number of branches at two points p, $q \in \Gamma$. We denote by $\gamma_1, \dots, \gamma_k$ the branches of Γ at a point $p \in \Gamma$ and by $\nu_1, \dots, \nu_k$ the branches of D at $q \in D$.

Definition 3.3 ([**49**], p.510). *A $(1,1)$ mapping π of the set of branches of Γ onto the set of branches of D is said to be a tangentially stable pairing $\pi : \Gamma \to D$ between the branches of Γ and those of D if the following condition is satisfied: given any two branches γ_i and γ_j of Γ, the corresponding branches $\pi(\gamma_i)$ and $\pi(\gamma_j)$ of D have the same tangent line if and only if γ_i and γ_j have the same tangent line.*

Now blow-up the plane $Spec(\mathbb{C}[[x,y]])$ at the origin. The proper transforms Γ' and D' of Γ and D will have a certain number of connected components. Now, the proper transforms of two branches of Γ will be in the same connected component of Γ' if and only if they have the same tangent line at p. Since $\pi : \Gamma \to D$ is tangentially stable, the number of connected components of Γ' will be the same as the number of connected component D'. We denote by

Γ'_j and $j = 1, 2, \ldots$, the connected components of Γ' and by D'_j and $j = 1, 2, \ldots$, the connected components of D'. We can suppose to number them in such a way that the pairing $\pi : \Gamma \to D$ between the branches of Γ and the branches of D, induces a pairing $\pi_j : \Gamma'_j \to D'_j$ between the branches of Γ'_j and the branches of D'_j, for every $j = 1, 2, \ldots$. We now define *equivalence of algebroid plane curves* by induction on the number of blow-ups required to resolve the singularity. If Γ and D are smooth at p and q respectively, we say that a pairing of the unique branch of Γ with the unique branch of D is an equivalence.

Definition 3.4 ([**49**], p.511). *An equivalence $\pi : \Gamma \to D$ is a pairing π between the branches of Γ and the branches of D having the following properties:*
(1) π is tangentially stable,
(2) if $\nu_j = \pi(\gamma_i)$, then $mult_p(\gamma_i) = mult_q(\nu_j)$,
(3) the pairing $\pi_j : \Gamma'_j \to D'_j$ is an equivalence, for every $j = 1, 2, \ldots$.

We say that two singularities of plane curves are equivalent if the associated algebroid plane curves are equivalent.

Remark 3.5. *Notice that if two plane singularities are analytically equivalent, then they are equivalent. Moreover, by remark V.3.9.4 and example V.3.9.5 of* [**22**]*, it follows that two plane singularities of multiplicity two are equivalent if and only if they are analytically equivalent. But equivalence relation is weaker than analytical equivalence. For instance, two ordinary plane singularities of the same multiplicity are equivalent. But we know that there exist ordinary plane singularities of multiplicity $n \geq 4$ which are not analytically equivalent.*

Theorem 3.6 ([**47**], p. 213-214). *Let $\Sigma \subset \mathbb{P}^N$ be an irreducible algebraic system of reduced plane curves of degree n. Then there exists a Zariski open set $U \subset \Sigma$ such that, for every equivalence class of singularity, every plane curve $\Gamma \subset \mathbb{P}^2$, corresponding to a point $[\Gamma] \in U$, has the same number of singularities of that equivalence class. Moreover, for every point $[\Gamma] \in U$ and for every closed curve $Y \subset U$ with $[\Gamma] \in Y$, denoting by $\pi : \mathcal{C} \to Y$ the tautological family, we have that for every singular point p of the special fibre $\mathcal{C}_0 = \Gamma$, there is a analytic neighborhood $V \subset \mathcal{C}$ of p, such that every fibre of the family $V \to \pi(V)$ has singular point equivalent to p and no further singularity, and the locus of singular points of the fibres is a section of $V \to \pi(V)$.*

We can now introduce the algebraic systems which are the subject of this thesis. We recall that a cusp, or more precisely an ordinary cusp is a plane singularity which is analytically equivalent to the plane singularity of equation $y^2 = x^3$ while a nonordinary cusp is a plane singularity of analytic equation $y^2 = x^{2s+1}$, with $s \geq 2$.

Definition 3.7. *Let $\Sigma^n_{k,d}$ be the Zariski closure in $\mathbb{P}^N$ of the locally closed set of points corresponding to a reduced and irreducible plane curve of degree n and geometric genus equal to $g = \binom{n-1}{2} - k - d$ with d nodes and k cusps as singularities. More in general, we shall denote by $\mathcal{S}^n_{k,d}$ the Zariski closure of the locus of the reduced plane curves of degree n with k cusps and d nodes as singularities.*

Notice that $\mathcal{S}^n_{k,d}$ and $\Sigma^n_{k,d} \subset \mathcal{S}^n_{k,d}$ are well defined by the theorem 3.6. Our object of studies is $\Sigma^n_{k,d}$. Observe that, by theorem 2.2, we have that $\Sigma^n_{0,d} \simeq V_{n,g}$, where $g = \binom{n-1}{2} - d$ and $0 \leq d \leq \binom{n-1}{2}$, while, for $k > 0$, $\Sigma^n_{k,d}$ is a proper closed subvariety of $V_{n,g}$. We shall see later that, when $k > 0$, these algebraic systems may be reducible or empty.

Example 3.8. *The simplest example of algebraic system of plane curves with nodes and cusps which is irreducible and not empty is $\Sigma^n_{1,0}$. With the tools we have available at this point, we can't prove that $\Sigma_{1,0}$ is not empty. We shall prove this in corollary 3.18. In order to see that $\Sigma^n_{1,0}$ is irreducible, let $\widehat{\Sigma^n_{1,0}}$ be the incidence family of plane curves with at least a cusp with assigned tangent line. If $U_2 = \{[x_0 : x_1 : x_2] | x_2 \neq 0\} \subset \mathbb{P}^2$ then $\widehat{\Sigma^n_{1,0}}$ is locally defined in $\mathbb{P}^N \times U_2 \times U_2^*$ by the equations*

$$\begin{aligned} F(q) = \frac{\partial F}{\partial x_0}|_q = \frac{\partial F}{\partial x_1}|_q &= 0, \\ A_1 \frac{\partial F}{\partial^2 x_0}|_q - A_0 \frac{\partial F}{\partial x_1 \partial x_0}|_q &= 0, \\ A_0 \frac{\partial F}{\partial^2 x_1}|_q - A_1 \frac{\partial F}{\partial x_0 \partial x_1}|_q &= 0, \end{aligned}$$

where $F(x_0, x_1, x_2) = 0$ is the equation of the generic homogeneous polynomial of degree n, the point q lies in U_2 and A_0 and A_1 are the coefficients of a line passing through q. If $\mathcal{L}$ is the incidence family $\mathcal{L} = \{(p, l) | p \in l\} \subset \mathbb{P}^2 \times (\mathbb{P}^2)^*$, then the second projection map $p_2 : \widehat{\Sigma^n_{1,0}} \to \mathcal{L}$ is surjective and all its fibres are linear space of dimension $N - 5$. It follows that $\widehat{\Sigma^n_{1,0}}$ and $\Sigma^n_{1,0}$ are irreducible of dimension $N - 2$.

In order to compute the dimension of $\Sigma^n_{k,d}$, we need some further terminology.

Definition 3.9. *An algebraic system $\Sigma \subset \mathbb{P}^N$ of plane curves of degree n is said to be complete if there does not exist an algebraic system $\Sigma' \subset \mathbb{P}^N$ containing Σ such that, for every equivalence class of singularity, the curve Γ corresponding to the general point $[\Gamma]$ of Σ has the same number of singular points of that equivalence type as the curve $D \subset \mathbb{P}^2$ corresponding to the general point $[D]$ of Σ'.*

By definition 3.7, every irreducible component of $\Sigma^n_{k,d}$ is complete. We give an example of algebraic system which is not complete.

Example 3.10. *Let $\Sigma^o \subset \mathbb{P}^{27}$ be the locally closed set of irreducible plane curves of degree six with six nodes on a conic. It is the intersection of the locally closed set $V^o_{6,4}$ of irreducible plane curves of degree six with six nodes and the closed set which is the projection of the tautological family*

$$\{([\Gamma], p_1, \ldots, p_6, [C]) | p_i \in Sing(\Gamma) \text{ and } p_i \in C \text{ for } 1 \leq i \leq 6\} \subset \mathbb{P}^{27} \times (\mathbb{P}^2)^6 \times \mathbb{P}^5$$

where $\mathbb{P}^5$ is the parameter space of the conics. In order to see that Σ^o is not empty, let $C \subset \mathbb{P}^3$ be a canonical curve of genus 4. By proposition 6.6 of section 6, a general projection $\Gamma \subset \mathbb{P}^2$ of C to $\mathbb{P}^2$ is a nodal plane curve of degree six. Moreover, the lines of $\mathbb{P}^2$ cut out on C a subseries of dimension two of the canonical series. On the other hand, the complete canonical series is cut out on C by the plane curves of degree 3 passing through the node of Γ, see section 1. By Bezout theorem, we conclude that the nodes of Γ lie on a conic. Likewise every irreducible plane curve Γ of degree six with six nodes on a conic is a projection of a canonical curve of genus four, because, denoting by $C \to \Gamma$ the normalization of $\mathbb{P}^2$, the lines of $\mathbb{P}^2$ cut out on C a subseries of dimension two of the canonical series. It follows that

$$dim(\Sigma^o) = dim(\mathcal{M}_4) + dim(Aut(\mathbb{P}^2)) + dim(\mathbb{G}(0, 3)) = 20$$

and Σ^o is irreducible because it is dominated by the irreducible variety $\mathcal{M}_4 \times Aut(\mathbb{P}^2) \times \mathbb{P}^3$. The Zariski closure Σ of Σ^o is an example of irreducible algebraic system which is not complete. Indeed, we have that $\Sigma^o \subset V^o_{6,4}$ and we proved at the and of the previous section that

$$dim(V^o_{6,4}) \geq 27 - 6 = 21.$$

Actually, we shall see in this section that the equality holds.

A very important result in the theory of algebraic systems is the theorem 3.12 which is called the *dimensional characterization theorem of Severi varieties* by Zariski.

Definition 3.11. *The characteristic linear system of an algebraic system Σ at a point $[\Gamma]$ is the linear system parametrized by the tangent space to Σ at $[\Gamma]$. The linear series cut out on the normalization curve C of Γ by the pull-back, with respect to the normalization morphism, of the characteristic linear system, is called the characteristic linear series of Σ at $[\Gamma]$.*

Theorem 3.12 ([**47**], p. 215-226). *Let Σ be an irreducible algebraic system whose general element $[\Gamma]$ parametrizes a reduced plane curve Γ of degree n and geometric genus g. Then, the characteristic linear system of Σ at $[\Gamma]$ is contained in the linear system of adjoint curves to Γ of degree n. In particular,*

$$dim(\Sigma) \leq 3n + g - 1$$

and equality holds if and only if Σ is complete and Γ is a nodal curve.

Proof. If $dim(\Sigma) = 0$ the theorem is true, hence we assume that $dim(\Sigma) > 0$. Let Γ be the plane curve corresponding to the general element $[\Gamma]$ of $\Sigma^n_{k,d}$. By generality, we can suppose that Σ is smooth at $[\Gamma]$. Under this hypothesis for every line $l \subset T_{[\Gamma]}(\Sigma)$ passing trough $[\Gamma]$, there is a holomorphic arc $\gamma : \mathbb{C} \to \Sigma$ such that l is tangent to the image of γ at $[\Gamma]$. We want to show that all the plane curves of the pencil corresponding to l are adjoint to Γ. Let p be a fixed point of Γ. Denoting by $f(x, y) = 0$ the local affine equation of Γ at p, by

$$f(x, y) + tf_1(x, y) = 0$$

the equation of the pencil of curves corresponding to l and by $\phi : C \to \Gamma$ the normalization of Γ, we have to prove the following

claim: the pull-back to C of the local form

$$\omega = \frac{f_1(x, y)dx}{\frac{\partial f(x,y)}{\partial y}} \tag{12}$$

is regular at each of the finitely many points of C which lie over p.

If Γ is regular at p the claim is true. Let r_p be the minimal number of blowing-ups necessary to resolve the singularity of Γ at p. By induction, we suppose that the claim is true for every r_p such that $r_p \leq n$ and we prove it for $r_p = n$. Let Γ be a holomorphic arc $\gamma : \mathbb{C} \to \Sigma$ passing through $[\Gamma]$ with tangent line at $[\Gamma]$ equal to l. It corresponds to a one parameter family of plane curves Γ_t of degree n of local equation

$$F(x, y; t) = f(x, y) + f_1(x, y)t + \frac{\partial F}{\partial^2 t}|_{t=0}t^2 + ...$$

Since $[\Gamma]$ is general in Σ, by theorem 3.6, for t small, the curve C_t has a singular point p_t which specializes to p, as t specializes to 0, and the singularity that Γ and Γ_t have at p and p_t respectively are equivalent. Let $\xi(t)$ and $\eta(t)$ be the x and y coordinates of p_t. They are power series in t with coefficients in $\mathbb{C}$. By blowing-up $\mathbb{A}^2 \times \mathbb{A}^1$ along the holomorphic arc

$(\xi(t), \eta(t), t)$, we get a monoidal transformation $T : \widetilde{\mathbb{A}^2 \times \mathbb{A}^1} \to \mathbb{A}^2 \times \mathbb{A}^1$ such that for every t, the restriction map $T^{-1}(\mathbb{A}^2 \times \{t\}) \to A^2 \times \{t\}$ is the blowing-up of the plane at the point $(\xi(t), \eta(t))$. Assuming, as we may, that the line $x = 0$ is not tangent to Γ at $p = (0, 0)$, it follows that the line $x = \xi(t)$ is not tangent to Γ_t and hence, by taking on $T^{-1}(\mathbb{A}^2 \times \{t\})$ the analytic coordinates x and $\tilde{y}$, where

$$\tilde{y} = \tilde{y}(t) = \frac{y - \eta(t)}{x - \xi(t)}, \tag{13}$$

we have that all the points of the strict transform of Γ_t in $T^{-1}(\mathbb{A}^2 \times \{t\})$ have finite $\tilde{y}$ coordinate. Assuming that p_t is an s-fold point of Γ_t, we will have that

$$F(x, y; t) = (x - \xi(t))^s \tilde{F}(x, \tilde{y}; t), \tag{14}$$

where $\tilde{F}(x, \tilde{y}; t) = 0$ is the equation of the strict transform $\widetilde{\Gamma_t}$ of Γ_t with respect to the blowing-up of the plane at p_t. Applying our inductive hypothesis to every singular point of the affine plane curve of equation $\tilde{F}(x, \tilde{y}; 0) := \tilde{f}(x, \tilde{y}) = 0$ mapping to p, we find that the pullback to C of the local one form

$$\tilde{\omega} = \frac{\tilde{f}_1(x, \tilde{y})dx}{\frac{\partial \tilde{f}(x,\tilde{y})}{\partial \tilde{y}}}$$

is regular at each of the finitely many points of C over p, where we set $\tilde{f}_1(x, \tilde{y}) = \frac{\partial \tilde{F}(x,\tilde{y};t)}{\partial t}|_0$. Now, by using (14), we get that

$$\begin{aligned} \frac{\partial F(x, y; t)}{\partial t} &= [x - \xi(t)]^s [\frac{\partial \tilde{F}(x, \tilde{y}; t)}{\partial t} + \frac{\partial \tilde{F}(x, \tilde{y}; t)}{\partial \tilde{y}} \frac{\partial \tilde{y}}{\partial t}] \\ & \quad -s[x - \xi(t)]^{s-1} \frac{\partial \xi(t)}{\partial t} \tilde{F}(x, \tilde{y}; t). \end{aligned}$$

Moreover, by using (13), we have that

$$\frac{\partial \tilde{y}}{\partial t} = -\frac{\frac{\partial \eta(t)}{\partial t}}{x - \xi(t)} + \frac{\partial \xi(t)}{\partial t} \frac{\tilde{y}}{(x - \xi(t))}$$

and hence

$$\begin{aligned} \frac{\partial F(x, y; t)}{\partial t} &= [x - \xi(t)]^{s-1} \\ & \quad \{(x - \xi(t)) \frac{\partial \tilde{F}(x, \tilde{y}; t)}{\partial t} + \frac{\partial \tilde{F}(x, \tilde{y}; t)}{\partial \tilde{y}} [\frac{\partial \xi(t)}{\partial t} \tilde{y} - \frac{\partial \eta(t)}{\partial t}] \\ & \quad s \frac{\partial \xi(t)}{\partial t} \tilde{F}(x, \tilde{y}; t)\}. \end{aligned}$$

By dividing the former equality by

$$\frac{\partial F(x, y; t)}{\partial y} = (x - \xi(t))^{s-1} \frac{\partial \tilde{F}(x, \tilde{y}; t)}{\partial \tilde{y}},$$

by setting $t = 0$ and by using that the pull-back to C of the local form

$$\frac{\tilde{F}(x, \tilde{y}; 0)dx}{\frac{\tilde{F}(x,\tilde{y};0)}{\partial \tilde{y}}}$$

is zero, we deduce the relation

$$\omega = x\tilde{\omega} + [\frac{\partial \xi(0)}{\partial t}\tilde{y} - \frac{\partial \eta(0)}{\partial t}]dx. \tag{15}$$

Since every term of the right hand of this equality is a local one form whose pull-back to C is regular at every point over p, we conclude that the pull-back of ω to C is regular at every point of C over p too. This proves that the characteristic linear system $L_\Sigma([\Gamma])$ of Σ at $[\Gamma]$ is contained in the linear system $\mathcal{A}_{[\Gamma]}$ of the adjoint plane curves of degree n to Γ. We prove the second part of theorem only when Γ is irreducible and we refer to [**47**] p. 222-226 for the general case. Then assume that $\Gamma \subset \mathbb{P}^2$ is irreducible. By section 1 of chapter 1, we have that

$$dim(\mathcal{A}_{[\Gamma]}) = \frac{n(n+3)}{2} - \binom{n-1}{2} + g = 3n + g - 1,$$

and hence we find that

$$dim(\Sigma) \leq dim(\mathcal{A}_{[\Gamma]}) = 3n + g - 1. \tag{16}$$

Now we recall that in the previous section we proved that the locus $U_{n,d}$ of reduced d-nodal plane curves is locally closed in $\mathbb{P}^N$ and every its irreducible component has dimension at least equal to $N - d = 3n + g - 1$, where $g = \binom{n-1}{2} - d$. It follows that the equality holds in (16) if Σ is a complete algebraic system whose general element is a nodal irreducible plane curve. Suppose now that Γ has a singular point p which is not a node. We consider separately the following three cases.

(1) At least one branch γ of Γ at p has multiplicity $s > 1$, that is a branch γ of analytic equation $y^s = x^{s+1}$.
(2) All branches of γ at p are smooth and there exist at least two branches γ_1 and γ_2 which have distinct tangent lines at p.
(3) All branches of Γ have the same tangent line.

Suppose that (1) holds. Then, denoting by $\tilde{p}$ the point of C over p which corresponds to the branch γ, we have that $\phi^* x$ and $d(\phi^* x)$ vanish at $\tilde{p}$ with order s and $s - 1$ respectively. Thus, from the equality (15), the pull-back to C of the one form (12) vanishes with order at least $s - 1$ at $\tilde{p}$. It follows that the point $(s - 1)p$ is contained in the base locus of the characteristic linear series $g^{r-1}_{3n+2g-2}$ of Σ at $[\Gamma]$. Then, outside of this point, the characteristic linear system $L_\Sigma([\Gamma])$ cuts out on C a linear series of degree $3n+2g-2-s+1 = 3n+2g-s-1$ and dimension equal to $dim(L_\Sigma([\Gamma])) - 1 = dim(\Sigma) - 1$, since $\Gamma \in L_\Sigma([\Gamma])$. This is a non special series, since $s - 1 < n$. Hence, by using Riemann-Roch theorem, we find that $dim(\Sigma) \leq 3n + g - s < 3n + g - 1$, because $s > 1$.

Suppose now that (2) holds. Let $l_i : y = a_i x$ be the equation of the tangent line to the branch γ_i at p, for $i = 1, 2$, and let $\tilde{p_1}$ and $\tilde{p_2}$ be the points of C over p corresponding to the branches γ_1 and γ_2. If we impose to the curves of the characteristic linear system $L_\Sigma([\Gamma])$ of Σ at $[\Gamma]$ to be tangent to l_1 and l_2 at p, we get a subspace $S \subset L_\Sigma([\Gamma])$ of dimension $\rho \geq dim(\Sigma) - 2$, such that for every plane curve $D \in S$ the associated one form $\phi^*\omega$ on C vanishes on $\tilde{p_1}$ and $\tilde{p_2}$. Now, by using that $\phi^* x$ vanishes at $\tilde{p_1}$ and $\tilde{p_2}$, $d\phi^* x$ is not zero at these points because γ_1 and γ_2 are smooth branches of Γ and $\phi^*(\tilde{y})(\tilde{p_i}) = a_i$, for $i = 1, 2$, by looking at the relation (15), we get that

$$a_1 \frac{\partial \xi}{\partial t}|_{t=0} - \frac{\partial \eta}{\partial t}|_{t=0} = a_2 \frac{\partial \xi}{\partial t}|_{t=0} - \frac{\partial \eta}{\partial t}|_{t=0} = 0.$$

Since $a_1 \neq a_2$, it follows that $\frac{\partial \xi}{\partial t}|_{t=0} = \frac{\partial \eta}{\partial t}|_{t=0} = 0$ and, by (15), the one form $\phi^*\omega$ vanishes at each of the points $\tilde{p}_1, \ldots, \tilde{p}_s$ of C which lie over p. It follows that the linear system S cuts out on C a $g^{\rho-1}_{3n+2g-2-s}$. By Riemann-Roch theorem, we find that $\rho \leq 3n+g-1-s$ and hence $dim(\Sigma) \leq \rho + 2 \leq 3n+g-s+1 < 3n+g-1$ since $s \geq 3$.

Finally, suppose that (3) holds. Let $t : y = ax$ be the common tangent line of the s branches $\gamma_1, \ldots, \gamma_s$ of Γ at p and let $\tilde{p}_1, \ldots, \tilde{p}_s$ be the s points of C which lie over p. If we impose to the curves of the characteristic linear system $L_\Sigma([\Gamma])$ to be tangent to t, we get a subspace $S \subset L_\Sigma([\Gamma])$ of codimension at most equal to one, such that for every adjoint curve $D \in S$, the divisor $\phi^*(D)$ on C contains all the point $\tilde{p}_1, \ldots, \tilde{p}_s$. It follows that S cuts out on C a linear series of degree $3n+2g-2-s$ and dimension $dim(S)-1$. Again by using the Riemann-Roch theorem, we find that $dim(\Sigma) \leq dim(S)+1 \leq 3n+g-s < 3n+g-1$ since $s \geq 2$. □

Proof of theorem 2.2. In the previous section, we proved that every irreducible component $V \subset V_{n,g}$ of the Severi variety of reduced and irreducible plane curves of degree n and geometric genus g has dimension at least equal to $3n+g-1$. On the other hand, by theorem 3.12 we have that $dim(V) \leq 3n+g-1$. Hence $dim(V) = 3n+g-1$ and, again using theorem 3.12 and theorem 3.6, the general element of V corresponds to an irreducible nodal curve of genus g and the tangent space to $V_{n,g}$ at $[\Gamma]$ parametrizes the linear system of plane curves of degree n adjoint to Γ, that is the linear system of plane curves of degree n passing through the nodes of Γ. □

Corollary 3.13. *For every irreducible not empty component $\mathcal{S}$ of the complete algebraic system $\mathcal{S}^n_{k,d}$ of reduced plane curves of degree n with k cusps and d nodes, we have that*

$$dim(\mathcal{S}) \geq N - d - 2k = 3n+g-1-k.$$

and, if $k < 3n$ then $dim(\mathcal{S}) = 3n+g-1-k$.

Proof. Let $\widetilde{\mathcal{S}^n_{k,d}} \subset \mathbb{P}^N \times (\mathbb{P}^2)^d \times (\mathbb{P}^2)^k$ be the Zariski closure of the locally closed set

$$\{([\Gamma], p_1, \ldots, p_d, q_1, \ldots, q_k) | \Gamma \text{ is reduced, } p_i,\ q_j \in Sing(\Gamma) \text{ for } 1 \leq i \leq d \text{ and } 1 \leq j \leq k \text{ and there exists at least a singular branch of } \Gamma \text{ passing through every } q_j\}.$$

In order to see that every irreducible component $\widetilde{\mathcal{S}}$ of $\widetilde{\mathcal{S}^n_{k,d}}$ has codimension at most $3d+4k$ and the locus

$$\{([C], p_1, \ldots, p_d, q_1, \ldots, q_k) | p_i \text{ is a node of } \Gamma \text{ and } q_j \text{ is a cusp of } \Gamma\}$$

is open in $\widetilde{\mathcal{S}}$, it is enough to prove it locally. Let $F(x_0, x_1, x_2) = \sum_{i+j+k=n} a_{ijk} x_0^i x_1^j x_2^k$ be the equation of the generic homogeneous polynomial of degree n and let U_2 be the open set of the plane $U_2 = \{[x_0 : x_1 : x_2] | x_2 \neq 0\} \subset \mathbb{P}^2$. Thus $\widetilde{\mathcal{S}^n_{k,d}}$ is defined in $\mathbb{P}^N \times (U_2)^d \times (U_2)^k$ by the following $3d+4k$ equations

$$\begin{aligned} F(p_i) = \frac{\partial F}{\partial x_0}|_{p_i} = \frac{\partial F}{\partial x_1}|_{p_i} &= 0, \quad \text{i=1,\ldots,d} \\ F(q_j) = \frac{\partial F}{\partial x_0}|_{q_j} = \frac{\partial F}{\partial x_1}|_{q_j} &= 0, \quad \text{j=1,\ldots,k} \\ (\frac{\partial F}{\partial x_0 \partial x_1}|_{q_j})^2 - 4\frac{\partial F}{\partial^2 x_0}|_{q_j}\frac{\partial F}{\partial^2 x_1}|_{q_j} &= 0,, \quad \text{j=1,\ldots,k.} \end{aligned}$$

In the previous section we proved that having a node a each of the points $p_1, \dots, p_d$ is an open condition in $\widetilde{\mathcal{S}^n_{d,k}}$. Moreover, a reduced plane curve of degree n of equation

$$F(x_0, x_1, x_2) = \sum_{i=2}^{n} f_i(x_0, x_1) x_2^{n-i}$$

with a double branch at the point $[0:0:1]$, has a cusp at $[0:0:1]$ if and only if $f_2(x_0, x_1) = (ax_0 + bx_1)^2 \neq 0 \neq f_3(x_0, x_1)$ and $ax_0 + bx_1$ does not divide $f_3(x_0, x_1)$. It follows that having a cusp at each of the points $q_1, \dots, q_k$ is an open condition in $\widetilde{\mathcal{S}^n_{d,k}}$. We get that, if $p_1 : \widetilde{\mathcal{S}^n_{d,k}} \to \mathbb{P}^N$ is the first projection map, then $p_1(\widetilde{\mathcal{S}^n_{d,k}}) = \mathcal{S}^n_{d,k}$ and hence, for every irreducible component $\mathcal{S}$, we have that

$$dim(\mathcal{S}) \geq N - d - 2k = 3n + g - 1 - k.$$

Moreover, if $[\Gamma]$ is the general element of $\mathcal{S}$ corresponding to a plane curve $\Gamma \subset \mathbb{P}^2$ with nodes at $p_1, \dots, p_d$ and cusps at $q_1, \dots, q_k$, it follows from the case (1) of the proof of the theorem 3.12, that the characteristic linear system $L_{\mathcal{S}}([\Gamma])$ of $\mathcal{S}$ at $[\Gamma]$ is contained in the linear system $\mathcal{A}_{[\Gamma]}$ of plane curves of degree n adjoint to Γ and tangent at every point q_j to the cuspidal tangent line to Γ at q_j. Therefore the adjoint curves of degree n cut out on the normalization curve C of Γ a linear series of dimension $dim(\mathcal{A}_{[\Gamma]}) - 1$ and degree $n^2 - 2d - 3k = 3n + 2g - 2 - k$, where $g = \binom{n-1}{2} - d - k$. If $k < 3n$ this series is not special and applying the Riemann-Roch theorem, we get that

$$dim(\mathcal{S}) \leq dim(\mathcal{A}_{[\Gamma]}) \leq 3n + 2g - 2 - k - g + 1 = 3n + g - 1 - k.$$

□

Remark 3.14. *Notice that, from the previous corollary it follows that there does not exist reduced plane curves of degree n with $k = 3n - 1$ cusps as singularities with geometric genus $g \leq 7$. Indeed, by setting $k = 3n - 1$ in the statement of the former corollary and by using that every irreducible not empty algebraic system has dimension at least equal to $8 = dim(Aut(\mathbb{P}^2))$, we find that*

$$dim(\mathcal{S}) = g \geq 8.$$

Example 3.15. *We give an example of algebraic systems of irreducible plane curves with nodes and cusps of dimension greater than the expected one. This example has been found by Segre (see [**48**]). Let Σ_m be the parameter space of plane curves of degree $6m$ of equation*

$$f^3_{2m}(x_0, x_1, x_2) + f^2_{3m}(x_0, x_1, x_2) = 0, \tag{17}$$

where f_{2m} and f_{3m} are homogeneous polynomials of degree $2m$ and $3m$ respectively. As the reader may verify, the general element of Σ_m corresponds to an irreducible plane curve with a cusp at every intersection points of $f_{2m} = 0$ and $f_{3m} = 0$ and no further singularities. In particular, in order that a plane curve $\Gamma : f^3_{2m}(x_0, x_1, x_2) + f^2_{3m}(x_0, x_1, x_2) = 0$ has only $6m$ ordinary cusps at the intersections points of $f_{2m} = 0$ and $f_{3m} = 0$, it is enough that $f_{2m} = 0$ and $f_{3m} = 0$ are smooth and they intersect transversally, (see lemma 4.5 of chapter 3). It follows that Σ_m is contained in an irreducible component of $\Sigma^{6m}_{6m^2,0}$. On the other hand, Σ_m is an irreducible algebraic system of dimension

$$\frac{(2m+1)(2m+2)}{2} + \frac{(3m+1)(3m+2)}{2} - 1 = \frac{13m^2 + 15m}{2} + 1.$$

Now, $\frac{13m^2+15m}{2}+1 \geq \frac{6m(6m+3)}{2}-12m^2 = 6m^2+9m$ *and the equality holds if and only if* $m=1,2$. *We deduce that* Σ_m *is an irreducible component of* $\Sigma^{6m}_{6m^2,0}$ *if* $m=1,2$ *and it there exists at least an irreducible component of* $\Sigma^{6m}_{6m^2,0}$, *containing* Σ_m *and having dimension bigger than the expected one, if* $m>2$. *Actually, as it follows for instance from* [**42**], *we have that* Σ_m *is complete even if* $m>2$. *Finally, we observe that, as we shall see later,* $\Sigma^6_{6,0}$ *contains at least an irreducible component different from* Σ_1.

Another consequence of theorem 3.12 is lemma 3.17. In order to show it we need the following intermediate lemma, whose proof is very elementary.

Lemma 3.16. *Let* $\mathcal{G}\to B$ *be a one parameter flat family of plane curves. Suppose that the special fibre* $\Gamma := \mathcal{G}_0$ *of the family has only singular points of multiplicity two. Then the general fibre of the family has at most singular points of multiplicity two. Moreover, if* Γ *has only nodes and cusps as singularities, the general fibre of the family has at most nodes and cusps as singularities.*

Proof. Let $f(x,y;t)=0$ be the equation of a one parameter family of plane curves of degree n. Suppose that $f(x,y;0)=0$ has only singular points of multiplicity two. Then also the general curve of the family has at most singular points of multiplicity at most two. Indeed, if it is not true, then there are power series $\xi(t),\zeta(t)$, with coefficients in $\mathbb{C}$, such that $\frac{\partial f(\xi(t),\zeta(t);t)}{\partial x^i\partial y^j}=0$, for every i,j such that $i+j=s$ for some $s>1$. By specializing t to 0 we get a contradiction. Similarly, if $f(x,y;0)=0$ has only nodes and ordinary cusps as singularities, the same is true for the general fibre of the family. Indeed, the plane curve $f(x,y;0)=0$ has a node at $(0,0)$ if and only if $f(x,y)=\sum_{i\geq 2} f_i(x,y)$, where f_i is an homogeneous polynomial of degree i, and f_2 is reduced. This properties is locally closed. Then, if the general fibre of the family is not smooth at a neighborhood of $(0,0)$, i.e. if there exist power series $\xi(t),\zeta(t)$ with coefficients in $\mathbb{C}$ such that $(\xi(0),\zeta(0))=(0,0)$ and such that $f(x,y;t)=0$ is singular at $(\xi(t),\zeta(t))$, the degree two homogeneous part of the polynomial $f(x-\xi(t),y-\zeta(t))$ is reduced, for every t sufficiently small. Finally, the plane curve $f(x,y)=0$ has a cusp at the point $(0,0)$ if and only if $f(x,y)=\sum_{i\geq 2} f_i(x,y)$, $f_2(x,y)=(ax+by)^2\neq 0\neq f_3(x,y)$ and $ax+by$ doesn't divide $f_3(x,y)$. Also this property is locally closed. Then, if the general fibre of the family is not smooth or it has not a node at a neighborhood of $(0,0)$, it must have a cusp at a neighborhood of $(0,0)$. $\square$

Lemma 3.17. *Let* $\mathcal{S}$ *be an irreducible not empty component of* $\mathcal{S}^n_{k,d}$ $(\Sigma^n_{k,d})$ *with* $k<3n$. *Then, for every* k' *and* d' *such that* $k'\leq k$ *and* $d'\leq d+k-k'$, *there exists at least an irreducible component* $\mathcal{S}'$ *of* $\mathcal{S}^n_{k',d'}$ $(\Sigma^n_{k',d'})$ *such that* $\mathcal{S}\subset\mathcal{S}'$.

Proof. Let Γ be a reduced (and irreducible) plane curve Γ with k cusps and d nodes as singularities. Let

$$\begin{array}{ccc} \widetilde{\Sigma^n_{0,1}} = \{([C],p) | p\in Sing(C)\} & \subset & \mathbb{P}^N\times\mathbb{P}^2 \\ & \pi_1 \swarrow \quad \searrow \pi_2 & \\ & \mathbb{P}^N \qquad\quad \mathbb{P}^2 & \end{array}$$

be the incidence family of the hypersurface $\Sigma^n_{0,1}\simeq S_1\simeq V_{n,\binom{n-1}{2}-1}$ of singular plane curves of degree n. Let

$$\widetilde{\Sigma^n_{1,0}} = p_1^{-1}(\Sigma^n_{1,0}) \subset \widetilde{\Sigma^n_{0,1}}$$

the incidence family of $\Sigma^n_{1,0} \subset \Sigma^n_{0,1}$. We denote by $p_1, \dots, p_d$ the nodes of Γ and by $q_1, \dots, q_k$ the cusps of Γ. Let $U_1, \dots, U_d$ and $V_1, \dots, V_k$ be analytic neighborhoods of $p_1, \dots, p_d$ and $q_1, \dots, q_k$ respectively, such that $U_i \cap U_j = \emptyset = V_i \cap V_j$ if $i \neq j$ and $U_l \cap V_m = \emptyset$, for every $1 \leq l \leq d$ and $1 \leq m \leq k$. We set $N_{p_i} := \pi_1(\pi_2^{-1}(U_i))$, $N_{q_j} := \pi_1(\pi_2^{-1}(V_j))$ and $C_{q_j} := \pi_1(\pi_2^{-1}(V_j) \cap \widetilde{\Sigma^n_{1,0}})$, for every $1 \leq i \leq d$ and $1 \leq j \leq k$. Notice that $N_{p_i} := \pi_1(\pi_2^{-1}(U_i))$ and $N_{q_j} := \pi_1(\pi_2^{-1}(V_j))$ are analytic neighborhoods of $[\Gamma]$ in $\Sigma^n_{0,1}$ while $C_{q_j} := \pi_1(\pi_2^{-1}(V_j) \cap \widetilde{\Sigma^n_{1,0}})$ is an analytic open neighborhood of $[\Gamma]$ in $\Sigma^n_{1,0}$, for every i and j. Now, choose at pleasure k' cusp points of Γ, say $q_1, \dots, q_{k'}$. Then choose at pleasure $\tilde{k}$ points among the other cusp points of Γ, say $q_{k'+1}, \dots, q_{k'+\tilde{k}}$ and $\tilde{d}$ points among the nodes of Γ, say $p_1, \dots, p_{\tilde{d}}$. If we set $d' = \tilde{d} + \tilde{k}$, for every irreducible component $\mathcal{V}$ of

$$\cap_{i=1}^{\tilde{d}} N_{p_i} \bigcap \cap_{j=1}^{k'} C_{q_j} \bigcap \cap_{l=k'+1}^{k'+\tilde{k}} N_{q_l}$$

we have that

$$dim(\mathcal{V}) \geq N - d' - 2k'.$$

Moreover, since $[\Gamma] \in \mathcal{V}$ and Γ is reduced (reduced and irreducible) with only double points as singularities, we deduce that the general element $[D]$ of $\mathcal{V}$, (which we assume to be smooth), corresponds to a reduced (and irreducible) plane curve with at most singular points of multiplicity two. On the other hand, since the analytic open set U_i and V_j are pairwise disjoint, the point $[D] \in \mathcal{V}$ corresponds to a plane curve D with at least $d' + k'$ distinct singular points. In particular, by construction, the curve D has k' cusps $r_1, \dots, r_{k'}$ specializing to $q_1, \dots, q_{k'}$, as D specializes to Γ, and d' double points specializing to the other marked singular points of Γ. It follows that the geometric genus $g(D)$ of D is at most equal to $\binom{n-1}{2} - k' - d'$. Moreover, denoting by $\tilde{r_1}, \dots, \tilde{r_{k'}}$ the points of the normalization curve $\tilde{D}$ of D which lie over $r_1, \dots, r_{k'}$ respectively, by the proof of theorem 3.12, the points $\tilde{r_1}, \dots, \tilde{r_{k'}}$ are contained in the base locus of the characteristic linear series γ of $\mathcal{V}$ at $[D]$. Out of these points, γ has degree equal to $3n + 2g(D) - 2 - k'$ and hence

$$dim(\mathcal{V}) \leq 3n + g(D) - 1 - k' \leq 3n + \binom{n-1}{2} - 2k' - d' = N - 2k' - d'.$$

It follows that $dim(\mathcal{V}) = N - 2k' - d'$ and $g(D) = \binom{n-1}{2} - k' - d'$. In particular, the curve D has an ordinary cusp at each point of the points $r_1, \dots, r_{k'}$, d' nodes and no further singularities. Finally $\mathcal{V}$ is an analytic neighborhood $[\Gamma]$ in an irreducible $\mathcal{S}$ of $\mathcal{S}^n_{k,d}$ $(\Sigma^n_{k,d})$. □

Corollary 3.18. *For every $k \leq 4$ and $0 \leq d \leq \binom{n-1}{2} - k$, the algebraic system $\Sigma^n_{k,d}$ is not empty.*

Proof. When $d = k = 0$ the lemma is trivially true. When $n = 3$ we may have $(k, d) = (0, 1)$ or $(k, d) = (1, 0)$. The plane curve of equation $xy = x^3 + y^3$ is an example of rational cubic with a node, while $y^2 = x^3$ is an example of rational cubic with a cusp. An irreducible quartic may have at most three double points. In order to show the statement for $n = 4$, by lemma 3.17, it is enough to prove that there exist rational quartics with three cusps. By Plücker formulas, (see [**17**], p.280), a rational plane curve of degree 4 with three cusps is the dual curve of a cubic with a node. Since there exist cubics with a node, there exist too quartics with three cusps. When $n \geq 5$, by lemma 3.17, it is enough to show that there exist rational plane curves with four cusps and nodes as other singularities. We shall prove this statement by induction on n. A rational plane curve of degree 5 with four cusps and two nodes is the dual curve of a quartic with two nodes and one cusp. Since there exist quartics with two nodes and one cusp, there exist too quintics with four cusps and two nodes. Let Γ be a rational plane curve with four cusps and nodes as other singularities. Let R be a line intersecting transversally the curve Γ and let Γ' be the union curve of Γ and R. By lemma 3.17, we have that $[\Gamma' = \Gamma \cup R] \in \Sigma^{n+1}_{3,\binom{n-1}{2}+n-4}$. In particular, for every fixed node p of Γ' staying of $\Gamma \cap R$, there exists a family of plane curves $\mathcal{C} \to Y$ of degree $n+1$, with special fibre $\mathcal{C}_0 \sim \Gamma'$ and whose general fibre $\mathcal{C}_t$ has four cusps and $\binom{n-1}{2} - 3 + n - 1 = \binom{n+1}{2} - 3$ nodes as singularities, in such a way the cusps of $\mathcal{C}_t$ specialize to the cusps of Γ, the nodes of $\mathcal{C}_t$ specialize to the nodes of Γ' different from p. The curve $\mathcal{C}_t$ can't be reducible. Indeed, if $\mathcal{C}_t$ is reducible then $\mathcal{C}_t = \mathcal{C}'_t \cup R_t$, where $\mathcal{C}'_t$ is a plane curve of degree n specializing to Γ, as t specializes to 0, and R_t is a line specializing to R as t specializes to 0. In particular, R_t intersects $\mathcal{C}'_t$ at n points specializing to the intersection points of R and Γ, as t specializes to 0. This is not possible, because, by construction, the point $p \in \Gamma \cap R$ is not limit of any singular point of $\mathcal{C}_t$. We deduce that $\mathcal{C}_t$ is irreducible and rational. □

Remark 3.19. *The bound $k \leq 4$ in the statement of the former corollary, is sharp. Indeed, it is known the there exist irreducible quintics with five cusps as singularities, (see* [**25**]*, example 6.4.4). But, by Plücker formulas (see* [**17**]*, p.280), the dual curve of a rational quintic with five cusps and a node is a cubic with two nodes. Since there are not irreducible cubics with two nodes, we have that $\Sigma^5_{5,1}$ is empty. As far as we know, the existence problem of the varieties $\Sigma^n_{k,d}$ is still open, i.e. there does not exist a complete list of value of d and k such that $\Sigma^n_{k,d}$ is not empty.*

Example 3.20. *By theorem 2.6, the Severi variety $\Sigma^n_{0,d}$ of irreducible plane curves with d nodes are irreducible. On the contrary, for $k > 0$, we may found examples of reducible algebraic systems of irreducible plane curves of degree n with d nodes and k cusps. We proved that $\Sigma^6_{6,0}$ contains an irreducible component Σ_1, whose general point corresponds to an irreducible sextic with six cusps on a conic. In order to prove that $\Sigma^6_{6,0}$ is reducible, it is enough to prove that there exists irreducible sextics with six cusps not on a conic as singularities. To do this, let D be a smooth cubic. By using Plücker formula, we see that the dual curve Γ of D is a sextic with nine cusps $p_1, \dots, p_9$. These points can't lie on a conic by Bezout's theorem. If we choose five points $p_1, \dots, p_5$ among $p_1, \dots, p_9$, then, by still using Bezout theorem, we find that no four of these points are aligned, and in particular, there exists an unique conic C_2 passing through $p_1, \dots, p_5$. At least one cusp, say p_6, among $p_6, \dots, p_9$ does not lie on C_2. By the proof of lemma 3.17, we have that there exists a one parameter family $\mathcal{G} \to \Delta$ of sextics such that $\mathcal{G}_0 = \Gamma$ and the general curve $\mathcal{G}_t$ of the family has a cusp at a neighborhood of p_i, for $i = 1, \dots, 6$, and no further singularities. Since $p_1, \dots, p_6$ don't lie on a conic, also the six cusps of Γ_t don't lie on a conic. It follows that there exists at least one irreducible component Σ_2 of $\Sigma^6_{6,0}$ whose general point corresponds to a sextic with six cusps not on a conic. This example of reducible complete algebraic system has been given by Zariski in* [**48**].

The irreducibility problem of $\Sigma^n_{k,d}$ is still open. Enough conditions for the irreducibility or emptiness of $\Sigma^n_{k,d}$, under the hypothesis $n \geq 8$ and a list of examples of values of n, k and d, such that $\Sigma^n_{k,d}$ is reducible are in [**39**], [**40**] and [**28**]. The irreducibility if $\Sigma^n_{1,d}$, for every $d \leq \binom{n-1}{2} - 1$, has been proved by Ziv Ran in [**50**]. Finally, Kang has proved the irreducibility of $\Sigma^n_{k,d}$ for $k \leq 3$, generalizing the Harris proof of the irreducibility of Severi varieties of irreducible plane curves of degree n and genus g.

Theorem 3.21 (Kang, [**29**]). *The variety $\Sigma^n_{k,d}$ of irreducible plane curves of degree n with d nodes and k cusps is irreducible if $k \leq 3$.*

Let now $C \subset \mathbb{P}^2$ be a reduced plane curve with d nodes, k cusps and possibly other singularities. By the proof of the lemma 3.17, for every node $p \in C$, the point $[C] \in \mathbb{P}^N$ is origin of an analytic branch N_p of $\Sigma^n_{0,1}$, whose general element corresponds to an irreducible plane curve of degree n with only a node at a neighborhood of p. Similarly, for every cusp $q \in C$, the point $[C] \in \mathbb{P}^N$ is origin of an analytic branch C_q of $\Sigma^n_{1,0}$, whose general element corresponds to an irreducible plane curve of degree n with only a cusp at a neighborhood of q. Arguing as in [**21**] on p. 30, we want to prove the following lemma.

Lemma 3.22. *For every node $p \in C$, we have that N_p is smooth at the point $[C] \in \mathbb{P}^N$ and the tangent space $T_{[C]}N_p$ to N_p at $[C]$ corresponds to the linear system of plane curves of degree n passing through the point $p \in \mathbb{P}^2$. Similarly, for every cusp $q \in C$, we have that C_q is smooth at the point $[C] \in \mathbb{P}^N$ and the tangent space $T_{[C]}C_q$ to C_q at $[C]$ parametrizes the plane curves of degree n passing trough the point q and tangent to q to the cuspidal tangent line to C at q.*

Proof. Let $p \in C$ be a node of C and let $([C], p) \in \mathbb{P}^N \times \mathbb{P}^2$ be the corresponding point of the incidence family $\widetilde{\Sigma^n_{0,1}}$ of $\Sigma^n_{0,1}$. Since N_p is the image with respect to the projection map $\widetilde{\Sigma^n_{0,1}} \to \mathbb{P}^N$ of an analytic neighborhood of the point $([C], p)$, in order to see that $\Sigma^n_{0,1}$ is smooth at $[C]$, it is enough to prove that $\widetilde{\Sigma^n_{0,1}}$ is smooth at $([C], p)$ and the projection map $\widetilde{\Sigma^n_{0,1}} \to \mathbb{P}^N$ is a local immersion at $([C], p)$. To prove this, we can assume that $p = [0 : 0 : 1]$. Thus, if we fix affine coordinates x and y on $U_2 = \{[x_0 : x_1 : x_2] | x_2 \neq 0\} \subset \mathbb{P}^2$, the local equations of $\widetilde{\Sigma^n_{0,1}}$ in $\mathbb{P}^N \times U_2$ are

$$\begin{aligned} F(x,y) = a_{00} + a_{10}x + a_{01}y + a_{11}xy + \dots &= 0, \\ G(x,y) = \frac{\partial F}{\partial x}(x,y) = a_{10} + a_{11}y + 2a_{20}x + 3a_{30}x^2 + 2a_{21}xy + a_{12}y^2 + \dots &= 0, \\ H(x,y) = \frac{\partial F}{\partial y}(x,y) = a_{01} + a_{11}x + 2a_{02}y + 3a_{03}y^2 + 2a_{12}xy + a_{21}x^2 + \dots &= 0, \end{aligned}$$

where $F(x,y) = \sum_{i+j \leq n} a_{ij}x^iy^j$ is the affine equation of the generic plane curve of degree n. Denoting by A_{ij} the coefficients of the affine equation of C at p, the first three rows of jacobian matrix of $\widetilde{\Sigma^n_{0,1}}$ at a point $([C], [0 : 0 : 1])$ are

	F	G	H
$\frac{\partial}{\partial x}$	0	$2A_{20}$	A_{11}
$\frac{\partial}{\partial y}$	0	A_{11}	$2A_{02}$
$\frac{\partial}{\partial a_{00}}$	1	0	0

Since $p = [0 : 0 : 1]$ is a node of C, we have that $4A_{2,0}A_{0,2} - A_{11}^2 \neq 0$ and hence $\widetilde{\Sigma^n_{0,1}}$ is smooth at $([C], [0 : 0 : 1])$. Moreover the tangent space to $\widetilde{\Sigma^n_{0,1}}$ at $([C], [0 : 0 : 1])$ is defined by the equations

$$\begin{aligned} a_{00} &= 0, \\ 2A_{20}x + A_{11}y + a_{10} &= 0, \\ A_{11}x + 2A_{02}y + a_{01} &= 0, \end{aligned}$$

and hence, using again that $4A_{20}A_{02} - A_{11}^2 \neq 0$, we find that the map $\mathbb{P}^N \times \mathbb{P}^2 \to \mathbb{P}^N$ is a local immersion mapping isomorphically the tangent space to $\widetilde{\Sigma_{0,1}^n}$ at $([C], [0:0:1])$ to the hyperplane of $\mathbb{P}^N$ of equation $a_{00} = 0$. This proves the first part of the lemma. Let now $q \in C$ be a cusp of C. As before, since C_q is the image, with respect to the projection map $\widetilde{\Sigma_{1,0}^n} \to \mathbb{P}^N$, of an analytic neighborhood of the point $([C], q)$, in order to see that $\Sigma_{1,0}^n$ is smooth at $[C]$ it is enough to see that $\widetilde{\Sigma_{1,0}^n}$ is smooth at $([C], q)$. We may assume that $q = [0:0:1]$. By using the same notation as before, $\widetilde{\Sigma_{1,0}^n}$ is locally defined in $\widetilde{\Sigma_{0,1}^n} \cap (\mathbb{P}^N \times U_2)$ by the equation

$$K(x,y) = (\frac{\partial F}{\partial x \partial y}(x,y))^2 - \frac{\partial F}{\partial^2 x}(x,y)\frac{\partial F}{\partial^2 y}(x,y) = 0.$$

If we assume, as we may, that $y = 0$ is the cuspidal tangent line to C at q, thus the first rows of the jacobian matrix of $\widetilde{\Sigma_{1,0}^n}$ at $(C, [0:0:1])$ are

	F	G	H	K
$\frac{\partial}{\partial x}$	0	0	0	$-12A_{30}A_{02}$
$\frac{\partial}{\partial y}$	0	0	$2A_{02}$	$-4A_{21}A_{02}$
$\frac{\partial}{\partial a_{00}}$	1	0	0	0
$\frac{\partial}{\partial a_{10}}$	0	1	0	0
$\frac{\partial}{\partial a_{01}}$	0	0	1	0
$\frac{\partial}{\partial a_{20}}$	0	0	0	$-4A_{02}$

It follows that the jacobian matrix of $\widetilde{\Sigma_{1,0}^n}$ at $([C], q)$ has maximal rank and $\widetilde{\Sigma_{1,0}^n}$ is smooth at $([C], q)$. Moreover, all the missing entries in the matrix above are zero and hence the tangent space to $\widetilde{\Sigma_{1,0}^n}$ at $([C], q)$ is defined by the equations

$$\begin{aligned} a_{00} &= 0, \\ a_{10} &= 0, \\ 2A_{02}y + a_{01} &= 0, \\ 3A_{03}x - 2A_{21}y - 2a_{20} &= 0. \end{aligned}$$

Since C has an ordinary cusp at $q = [0:0:1]$ with cuspidal tangent line equal to $y = 0$, we have that $A_{03} \neq 0$ and the map $\mathbb{P}^N \times \mathbb{P}^2 \to \mathbb{P}^N$ is a local immersion mapping isomorphically the tangent space to $\widetilde{\Sigma_{1,0}^n}$ at $([C], [0:0:1])$ to the $(N-2)$-space of $\mathbb{P}^N$ of equations $a_{00} = a_{10} = 0$. The lemma is proved. □

Corollary 3.23. *Every reduced plane curve $\Gamma \subset \mathbb{P}^2$ of degree n with nodes and $k < 3n$ cusps as singularities, corresponds to a smooth point $[\Gamma]$ of $\mathcal{S}_{k,d}^n$.*

PROOF. By using lemma 3.22, for every node (resp. cusp) p_i (resp. q_j), the point $[\Gamma] \in \mathbb{P}^N$ is origin of a smooth analytic branch N_{p_i} (resp. C_{p_i}) of the variety $\Sigma_{0,1}^n$ (resp. $\Sigma_{1,0}^n$) whose general point corresponds to a plane curve with only a node (resp. a cusp) at a neighborhood of p_i (resp. q_j). Moreover, the tangent space $T_{[[\Gamma]]}N_{p_i}$ to N_{p_i} at $[\Gamma]$ is the linear space parametrizing the plane curves of degree n passing through p_i. Similarly, the tangent space $T_{[[\Gamma]]}C_{q_j}$ to C_{q_j} at $[\Gamma]$ is the linear space parametrizing the plane curves of degree n passing through q_j and tangent at q_j to the cuspidal tangent line to Γ at q_j, for every $0 \leq j \leq k$. By the proof of lemma 3.17, we already know that every irreducible component of

$$\cap_{i=1}^d N_{p_i} \bigcap \cap_{j=1}^k C_{q_j},$$

containing $[\Gamma]$, is an analytic open subset of $\mathcal{S}^n_{k,d}$. On the other hand the linear system $\mathcal{L}$ corresponding to

$$\cap_{i=1}^{d'} T_{[[\Gamma]]} N_{p_i} \bigcap \cap_{j=1}^{k'} T_{[[\Gamma]]} C_{q_j}$$

cuts out on the normalization curve of Γ a linear series of degree $n^2 - 3k - 2d = 3n + 2g - 2 - k$, where g is the geometric genus of Γ. Since $k < 3n$, this linear series is not special and $dim(\mathcal{L}) = 3n + g - 1 - k$. Since

$$\cap_{i=1}^{d} T_{[[\Gamma]]} N_{p_i} \bigcap \cap_{j=1}^{k} T_{[[\Gamma]]} C_{q_j} \supsetneqq T_{[[\Gamma]]}(\cap_{i=1}^{d} N_{p_i} \bigcap \cap_{j=1}^{k} C_{q_j}),$$

we conclude that $\cap_{i=1}^{d'} N_{p_i} \bigcap \cap_{j=1}^{k'} C_{q_j}$ is smooth at $[\Gamma]$ and in particular it is locally irreducible at $[\Gamma]$. □

We conclude this section by proving the so called Enriques' conjecture. Notice that it is an easy consequence of Zariski's Main Theorem, (corollary III.11.4 of [**22**]). But, following Albanese in [**1**], we show it by using the result of this section.

Lemma 3.24 (Enriques' conjecture). *Let Γ be a reduced plane curve of degree n with irreducible components $\Gamma_1, \dots, \Gamma_r$ such that Γ_i intersects transversally Γ_j, if $i \neq j$. Let $\mathcal{G} \to Y$ be a one-parameter family of plane curves of degree n with special fibre $\mathcal{G}_0 \simeq \Gamma$ and such that the general fibre $\mathcal{G}_t$ is irreducible. Then there exist at least $r-1$ singular points $p_1, \dots, p_{r-1}$ of Γ such that*

- *each of them is not limit of singular points of the general curve $\mathcal{G}_t$ of the family;*
- *for every $1 \leq i \leq r-1$ there exist two irreducible components Γ_{i_1} and Γ_{i_2} of Γ such that $p_i \in \Gamma_{i_1} \cap \Gamma_{i_2}$;*
- *the partial normalization $C \to \Gamma$ obtained by smoothing all the singular points except $p_1, \dots, p_{r-1}$, is connected.*

Proof. Let us choose $s \leq r$ irreducible components of Γ, say $\Gamma_1, \dots, \Gamma_s$ and let $\Gamma' = \Gamma_1 \cup \dots \cup \Gamma_s$ and $\Gamma'' = \Gamma_{s+1} \cup \dots \cup \Gamma_r$. If every irreducible curve Γ_i has degree n_i, we set $n' = \sum_{i=1}^{s} n_i$ and $n'' = \sum_{i=s+1}^{r} n_i$. Thus, the curves $\Gamma' = \Gamma_1 \cup \dots \cup \Gamma_s$ and Γ'' meet transversally at $n'n''$ points $p_1, \dots, p_{n'n''}$. By using lemma 3.22, there exist $n'n''$ different smooth analytic open sets $U_1, \dots, U_{n'n''}$ of $\Sigma^n_{0,1}$ passing through $[\Gamma]$, such that the general point of every U_i corresponds to an irreducible plane curve of degree n with a node at in neighborhood of p_i and no further singularities. Moreover, for every $1 \leq i \leq n'n''$, the tangent space $T_{[\Gamma]} U_i$ to U_i at the point $[\Gamma] \in \mathbb{P}^N$ parametrizes plane curves of degree n passing through the point $p_i \in \mathbb{P}^2$. Since the nodes of a reduced plane curve of degree n impose independent linear conditions to plane curves of degree n, (see section 1 of chapter 1), we have that the linear spaces $T_{[\Gamma]} U_i$ for $1 \leq i \leq n'n''$ intersects transversally. It follows that $U_1, \dots, U_{n'n''}$ intersect in an unique analytic open set of $\mathcal{S}^n_{0,n'n''}$, which is smooth at the point $[\Gamma]$ and whose general element corresponds to a reduced plane curve D of degree n with $n'n''$ nodes specializing to $p_1, \dots, p_{n'n''}$ as D specializes to Γ. In particular, we have that

$$dim(U_1 \cap \dots \cap U_{n'n''}) = \frac{n(n+3)}{2} - n'n''.$$

We claim that all the points of $U := U_1 \cap \dots \cap U_{n'n''}$ correspond to reducible curves. Indeed, U contains the locus $\mathcal{R}$ of plane curves of degree n with two irreducible components of degree n' and n'' respectively, which specialize to Γ. But this family is irreducible of

$$\frac{n'(n'+3)}{2} + \frac{n''(n''+3)}{2} = \frac{n(n+3)}{2} - n'n'$$

and hence $\mathcal{R} \simeq U$. This proves the claim. Let now $\mathcal{G} \to Y$ be a one-parameter family of plane curves of degree n whose special fibre is $\mathcal{G}_0 = \Gamma$ and whose general fibre $\mathcal{G}_t$ is irreducible. Applying the previous result to $\Gamma' = \Gamma_1$ and $\Gamma'' = \Gamma_2 \cup \cdots \cup \Gamma_r$, we find that there is at least a point p_1 of connection between Γ_1 and Γ'' which is not limit of any singular point of the general fibre. Let Γ_2 be the other component with contains p_1. Repeating the same argument for $\Gamma' = \Gamma_1 \cup \Gamma_2$ and $\Gamma'' = \Gamma_3 \cup \cdots \cup \Gamma_r$, we find that there exists at least a point p_2 of connection between $\Gamma' = \Gamma_1 \cup \Gamma_2$ and $\Gamma'' = \Gamma_3 \cup \cdots \cup \Gamma_r$ which is not limit of any singular point of the general fibre of $\mathcal{G} \to Y$. Up to rename the irreducible component of Γ we can suppose that $p_2 \in \Gamma_2 \cap \Gamma_3$. Repeating the same argument finitely many times, we get the statement. □

4. Universal family of plane curves of a degree n with d nodes and k cusps

In the previous sections we essentially followed Zariski's paper [**47**] and we introduced classical techniques used to study and describe the geometry of a family of plane curves with assigned singularities. Modern literature about families of plane curves differs from the classical literature not only for the techniques but especially for the way to formulate the problems. By the definition 3.1, an algebraic system of plane curves of degree n is a Zariski closed subset of $\mathbb{P}^N$ and, in particular, it is always reduced. After the work of Kuranishi and Kodaira-Spencer in the analytic case and Grothendieck, Murford and others in the algebraic case, every moduli object is defined as the object representing a suitable deformation functor F. In particular, if F is defined on the category of $\mathbb{C}$-schemes, the scheme representing F, if it there exists, may be not reduced. In order to define the deformation functor of plane curves with nodes and cusps, we recall the following definition.

Definition 4.1. *A flat morphism of finite type $X \to S$, where S is an algebraic $\mathbb{C}$-scheme, is a formally locally trivial family at $s \in S$ if, denoting by $\mathcal{O}_{S,s}$ the local ring of S at s and by m_s the maximal ideal of $\mathcal{O}_{S,s}$, for every $n > 0$, the induced family*

$$\begin{array}{ccccc} X & \leftarrow & X_n & \leftarrow & X_1 \\ \downarrow & & \downarrow & & \downarrow \\ S & \leftarrow & Spec\mathcal{O}_{S,s}/m_s^n & \leftarrow & Spec\mathbb{C} \end{array}$$

is such that X_n is a locally trivial deformation of X_1 in the Zariski topology.

From the following proposition, formal locally trivial families of plane curves are families of plane curves with analytically equivalent singularities.

Proposition 4.2 (Prop. 3.23 of [**13**]). *Let*

$$\begin{array}{ccc} X & \subset & Y \times \mathbb{A}^2 \\ \downarrow p & = & projection \\ Y & & \end{array}$$

be a flat family of reduced affine plane curves. Assume that X and Y are reduced, separated and of finite type over $\mathbb{C}$. Let $y \in Y$ be a closed point and let x be a singular point of $p^{-1}(y)$. Then the following two conditions are equivalent:

(1) *There exists a Zariski open set $U \subset Y$ with $y \in U$ such that the restricted family $p : U \to p(U)$ is formally locally trivial.*

(2) *For each Zariski open neighborhood U of x in X there exists a Zariski open neighborhood $U' \subset U$ of x in X such that for all closed points $z \in p(U')$, we have that $p^{-1}(z) \cap U'$ has only one singularity which is analytically isomorphic to the singularity of $p^{-1}(y)$ at x.*

Recalling that plane singularities of multiplicity two are analytically equivalent if and only if they are equivalent (see definition 3.4 and remark 3.5), if J is the functor on the category of $\mathbb{C}$-schemes defined by

$$J(S) = \{\text{Relative effective Cartier divisor } \mathcal{C} \subset \mathbb{P}^2 \times S \text{ which, as a family of curves over } S, \text{ is formally locally trivial at all } s \in S, \text{ and which is such that the geometric fibres of } \mathcal{C} \to S \text{ are } \textit{reduced} \text{ plane curves of degree } n \text{ with exactly } d \text{ nodes and } k \text{ cusps as singularities}\},$$

we expect that there exists a *scheme* $X \subset \mathbb{P}^N$ representing this functor and that the associated reduced scheme X_{red} is open in the algebraic system $\mathcal{S}^n_{k,d}$ defined in the previous sections. This has been showed by Wahl in [**46**]. We briefly summarize his results.

Let $\underline{C}$ be the category of finite local artinian $\mathbb{C}$-algebras.

Definition 4.3. *We say that a covariant functor*

$$F : \underline{C} \to Sets$$

from $\underline{C}$ to the category of sets is pro-representable if there exists a complete local $\mathbb{C}$-algebra R with maximal ideal m, such that $R/m^n \in \underline{C}$ for every $n > 0$ and such that there exists an isomorphism of functors Hom(R,-) $\simeq F$.

Definition 4.4. *Given a covariant functor $F : \underline{C} \to Set$, a vector space $0(F)$, is said to be an obstruction space of F, if for any $A \in \underline{C}$ and for any element $\Delta \in F(A)$ there is a linear map*

$$\xi_{\Delta,A} : Ext(A, \mathbb{C}) \to 0(F)$$

from the vector space $Ext(A, \mathbb{C})$ of the extensions

$$0 \to \mathbb{C} \to B \to A \to 0$$

in $\underline{C}$, to $0(F)$, such that B goes to 0 if and only if $\Delta \in F(A)$ lifts to an element of $F(B)$, i.e. Δ lie in the image of the map $H(B) \to H(A)$. If the linear map $\xi_{\Delta,A}$ is zero, we say that the element Δ is not obstructed.

Under quite mild conditions, a functor F on $\underline{C}$, has an obstruction space, see [**15**]. All the functors which we shall define in this section have an obstruction space.

Given a smooth projective variety Y and a divisor $\Gamma \subset Y$, we consider the functors H_Γ and H'_Γ on $\underline{C}$ defined by

$$H_\Gamma(A) = \{\text{Subschemes of } Y \times_{Spec\mathbb{C}} SpecA, \text{ flat over } A, \text{ inducing } \Gamma \text{ on } Y\}$$

and

$$H'_\Gamma(A) = \{\text{Subschemes } \mathcal{C} \subset Y \times_{Spec\mathbb{C}} SpecA, \text{ flat over } A, \text{ inducing } \Gamma \text{ on } Y \text{ and such that the family } \mathcal{C} \to SpecA \text{ is formally locally trivial at the closed point of } SpecA\},$$

for every $A \in \underline{C}$. We recall that, for every $A \in \underline{C}$, the elements of H_Γ are called *infinitesimal deformation of* Γ. By using the Lichtenbaum-Schlessinger cotangent complex bundle [**30**], we define the sheaf $\mathcal{N}'_{\Gamma|Y}$ on Γ as the kernel of the map $\mathcal{N}_{\Gamma|Y} \to T^1 \to 0$.

Proposition 4.5 (Proposition 3.2.5. of [**46**]). *The functors H_Γ and H'_Γ are pro-representable. Moreover, denoting by $\mathbb{C}[\epsilon]$ the artinian algebra $\mathbb{C}[\epsilon]/(\epsilon^2)$, we have that*

(1) $H_\Gamma(\mathbb{C}[\epsilon]) = H^0(C, \mathcal{N}_{\Gamma|Y})$ *and the obstruction space of H_Γ is contained in the image of the natural map $H^1(\Gamma, \mathcal{O}_Y(\Gamma)) \to H^1(\Gamma, \mathcal{N}_\Gamma)$.*
(2) $H'_\Gamma(\mathbb{C}[\epsilon]) = H^0(\Gamma, \mathcal{N}'_{\Gamma|Y})$ *and the obstruction space of H'_Γ is contained in $H^1(\Gamma, \mathcal{N}'_\Gamma)$.*

Let now $Y = \mathbb{P}^2$, let $\Gamma \subset \mathbb{P}^2$ be a reduced plane curve of degree n with k cusp and d nodes as singularities and let $J = J_{n,k,d}$ be the functor defined before.

Theorem 4.6 (Wahl, theorem 3.3.5 of [**46**]). *There exists a scheme X, which is a disjoint union of locally closed subschemes of $\mathbb{P}^N$, and a family of curves $\mathcal{C} \in J(X)$, which represents the functor J. Moreover,*

(1) *if x is the point corresponding to Γ in X and if R is the complete local ring pro-representing $H' = H'_\Gamma$, then $R = \mathcal{O}_{X,x}$.*
(2) *Finally, denoting by X_{red} the reduced scheme associated to X, we have that*

$$\begin{aligned} r := dim(\mathcal{O}_{X_{red},x}) = dim(R/\sqrt{0}) \quad &\leq \quad dim(m/m^2) \\ &= dim(H'(\mathbb{C}[\epsilon])) = dim(H^0(C, \mathcal{N}'_{\Gamma|\mathbb{P}^2})), \end{aligned}$$

where m is the maximal ideal of R, and the equality holds if and only if X is reduced and smooth at x.

The difference $w = h^0(\Gamma, \mathcal{N}'_{\Gamma|\mathbb{P}^2}) - dim(\mathcal{O}_{X_{red},x})$ is said to be *the deficiency* of X at x. In order to compute explicitly $H^0(\Gamma, \mathcal{N}'_{\Gamma|\mathbb{P}^2})$, we recall that, denoting by $\mathcal{I}_\Gamma$ the ideal sheaf of Γ, the normal sheaf of Γ in $\mathbb{P}^2$ is defined by $\mathcal{N}_{\Gamma|\mathbb{P}^2} := \mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{I}_\Gamma/\mathcal{I}_\Gamma^2, \mathcal{O}_\Gamma) \simeq \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{I}_\Gamma, \mathcal{O}_\Gamma) \simeq \mathcal{O}_\Gamma(\Gamma)$. Moreover in this case, all the infinitesimal deformations of Γ are not obstructed because $H^1(\Gamma, \mathcal{N}_{\Gamma|\mathbb{P}^2}) = H^1(\Gamma, \mathcal{O}_\Gamma(\Gamma)) = 0$. Denoting by f the local equation of Γ, by Θ_Γ the sheaf of derivations of $\mathcal{O}_\Gamma$ and by $\Theta_{\mathbb{P}^2}$ the tangent sheaf of $\mathbb{P}^2$, in the exact sequence

$$0 \to \Theta_\Gamma \to \phi^* \Theta_{\mathbb{P}^2} \xrightarrow{\alpha} \mathcal{N}_{\Gamma|\mathbb{P}^2} \to T^1_\Gamma \to 0$$

the locally free sheaf map $\Theta_{\mathbb{P}^2}|_\Gamma \xrightarrow{\alpha} \mathcal{N}_{\Gamma|\mathbb{P}^2}$ is locally given by

$$\begin{aligned} \frac{\partial}{\partial x} \quad &\mapsto \quad (f \mapsto \frac{\partial f}{\partial x}) \\ \frac{\partial}{\partial y} \quad &\mapsto \quad (f \mapsto \frac{\partial f}{\partial y}), \end{aligned}$$

and T^1_Γ is a skyscraper sheaf supported at the singular points of Γ, where the map α is zero. In particular, in a neighborhood of any node $P_1, \ldots, P_d$, the curve Γ is analytically equivalent to $xy = 0$ and the map α_{P_i}, induced by α on the stalks, is defined by

$$\begin{aligned} \frac{\partial}{\partial x} \quad &\mapsto \quad (xy \mapsto y) \\ \frac{\partial}{\partial y} \quad &\mapsto \quad (xy \mapsto x), \end{aligned}$$

In a neighborhood of any cusp $Q_1, \ldots, Q_t$ the curve Γ is analytically equivalent to $y^2 = x^3$ and the map α_{Q_j} is given by

$$\begin{aligned} \frac{\partial}{\partial x} &\mapsto (y^2 - x^3 \mapsto 3x^2) \\ \frac{\partial}{\partial y} &\mapsto (y^2 - x^3 \mapsto 2y). \end{aligned}$$

So, at any node $P_1, \ldots, P_d$ the stalk of T^1_Γ in P_i is

$$(\mathcal{N}_{\Gamma|\mathbb{P}^2})_{P_i}/Im(\alpha_{P_i}) \simeq \mathcal{O}_{\Gamma,P_i}/(x,y)_{P_i} \simeq \mathbb{C}$$

and at any cusp $Q_1, \ldots, Q_t$ is

$$(\mathcal{N}_{\Gamma|\mathbb{P}^2})_{Q_i}/Im(\alpha_{Q_i}) \simeq \mathcal{O}_{\Gamma,Q_i}/(x^2,y)_{Q_i} \simeq \mathbb{C}^2.$$

Finally, denoting by $\mathcal{I}$ the ideal sheaf of plane curves passing through any singular point of Γ and tangent at every cusp Q_i, for $i = 1, ..., t$, to the cuspidal tangent line to Γ at Q_i, we have that the image sheaf of α is given by

$$\mathcal{I} \otimes \mathcal{N}_{\Gamma|\mathbb{P}^2} = \mathcal{I} \otimes \mathcal{O}_\Gamma(\Gamma) := \mathcal{N}'_{\Gamma|\mathbb{P}^2}, \tag{18}$$

as we would expect. Moreover, by the long exact sequence

$$0 \to H^0(\Gamma, \mathcal{N}'_{\Gamma|\mathbb{P}^2}) \to H^0(\Gamma, \mathcal{N}_{\Gamma|\mathbb{P}^2}) \to H^0(\Gamma, T^1_\Gamma) \to H^1(\Gamma, \mathcal{N}'_{\Gamma|\mathbb{P}^2}) \to 0 \tag{19}$$

which we deduce from

$$0 \to \mathcal{N}'_{\Gamma|\mathbb{P}^2} \to \mathcal{N}_{\Gamma|\mathbb{P}^2} \to T^1_\Gamma \to 0$$

we find that

$$\chi(\mathcal{N}'_{\Gamma|\mathbb{P}^2}) = h^0(\mathcal{N}'_{\Gamma|\mathbb{P}^2}) - h^1(\mathcal{N}'_{\Gamma|\mathbb{P}^2}) = N - 2k - d.$$

Since $r = dim(\mathcal{O}_{X_{red},x}) \geq N - 2d - k$, we deduce the upper-bound on the deficiency of X at x

$$w \leq h^1(\mathcal{N}'_{\Gamma|\mathbb{P}^2}).$$

Notice that if $k < 3n$, thus $h^1(\Gamma, \mathcal{N}'_{\Gamma|\mathbb{P}^2}) = w = 0$ and so $H^0(\Gamma, \mathcal{N}'_{\Gamma|\mathbb{P}^2})$ has the expected dimension and X is smooth at the point corresponding to Γ. It follows that when $k < 3n$, the scheme X is reduced and smooth at every its point. Always in [**46**], Wahl proves that when $k = 900$, $d = 3636$ and $n = 104$ the scheme X representing J is not reduced but X_{red} is smooth. Tannenbaum proves in [**42**] that for every reduced plane curve of degree $n = 6m$ with $m > 2$, with $k = 6m^2$ cusps and $d = 0$ nodes as the only singularities, we have that $h^1(\Gamma, \mathcal{N}'_{\Gamma|\mathbb{P}^2}) = \frac{(m-2)(m-1)}{2}$ but the universal scheme X is smooth at every its point, (see also p. 25). Recently, it has been given an example of universal scheme X of plane curves with nodes and cusps which is not reduced and such that the singular locus of X_{red} is not empty, see [**18**].

5. Étale versal deformation family of a plane curve

This section is devoted to the étale versal deformation family of a reduced plane curve. The existence of such a deformation is a very strong result which follows from [**35**], [**36**] and [**8**]. We briefly summarize main results and properties of étale versal deformation spaces, following essentially [**13**].

Definition 5.1 (Versality). *Let*

$$\begin{array}{ccc} \mathcal{C} & \subset & \mathbb{A}^2 \times B \\ \pi \downarrow & & \\ B & & \end{array}$$

be a flat family of reduced affine plane curves and let $\Gamma = \pi^{-1}(0)$ *be a fibre of* π *over a closed point* $0 \in B$. *Suppose that* $\mathcal{C}$ *and* B *are of finite type over* $\mathbb{C}$. *Then the family* π *is said to be a versal deformation of* Γ *if, given the following data: a flat family of reduced curves* $f : X \to Y$, *where* X *and* Y *are of finite type over* $\mathbb{C}$, *a closed point* $y \in Y$, *a finite number of closed points* $x_1, \dots, x_n \in f^{-1}(y)$ *and an isomorphism of an étale neighborhood of* $\{x_1, \dots, x_n\}$ *in* $f^{-1}(y)$ *with an étale neighborhood of a finite set of points* $p_1, \dots, p_n$ *in* Γ, *then there exist étale neighborhoods* V *of* 0 *in* B, V' *of* y *in* Y, W *of* $\{p_1, \dots, p_n\}$ *in* $V \times_B \mathcal{C}$ *and* W' *of* $\{x_1, \dots, x_n\}$ *in* $V' \times_Y X$, *a morphism* $g : V' \to V$ *and an isomorphism* $\phi : W' \to W \times_V V'$ *such that the following diagram commutes.*

$$\begin{array}{ccccccccc} \mathcal{C} & \leftarrow & W & \stackrel{p_1}{\leftarrow} & W \times_V V' & \stackrel{\phi}{\to} & W' & \to & X \\ \downarrow & & \downarrow & & & \stackrel{p_2}{\searrow} & \downarrow & & \downarrow \\ B & \leftarrow & V & & \leftarrow & & V' & \to & Y \end{array} \tag{20}$$

Definition 5.2 (Miniversality). *A flat family like the family* $\mathcal{C} \to B$ *in definition 5.1 is said to be a miniversal deformation of* Γ *if, for every other étale versal deformation* $p : \mathcal{E} \to \mathcal{F}$ *of* Γ, *the dimension of* $\mathcal{F}$ *is greater than or equal to the dimension of* B.

Suppose Γ of affine equation $f(x, y) = 0$. Let $J = (f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ be the jacobian ideal of Γ. Choose $g_1, \dots, g_m \in \mathbb{C}[x, y]$ so that their images in $\mathbb{C}[x, y]/J$ form a base for this complex vector space.

Theorem 5.3 ([**8**], [**35**] and [**36**]). *The following family of affine complex plane curves*

$$\begin{array}{rcl} \mathcal{C} =: \{f + \sum t_i g_i = 0\} & \subset & Spec\mathbb{C}[x, y] \times Spec\mathbb{C}[t_1, \dots, t_m] \\ \downarrow \pi & & \swarrow \\ B & := & Spec\mathbb{C}[t_1, \dots, t_m] \end{array} \tag{21}$$

is a miniversal deformation of Γ. *Moreover, there exists a Zariski open subset* $U \subset B$ *containing* $(0, \dots, 0)$ *such that* $\pi : \mathcal{C} \to B$ *is an étale versal deformation of all fibres over closed points of* U.

From now on we shall call the family $\mathcal{C} \to B$ constructed as before the *étale versal deformation of* Γ while B will be the *étale versal deformation space of* Γ.

Example 5.4. *Let* Γ *be the affine plane curve of affine equation* $xy = 0$. *The curve* Γ *has only a node and the étale versal deformation family is given by*

$$\mathcal{C} = \{(x, y, a) | xy + a = 0\} \subset \mathbb{A}^2 \times \mathbb{A}^1$$

We see that the étale versal deformation space of Γ *is* $\mathbb{A}^1 := B$ *and, for every* $a \in \mathbb{A}^1 - (0, 0)$, *the corresponding fibre of* $\mathcal{C}$ *is smooth.*

Example 5.5. *Let* Γ *be the affine plane curve of equation* $y^2 + x^3 = 0$. *It has a cusp as singularity. The étale versal deformation space of* Γ *is given by* $B := Spec(\mathbb{C}[xy]/(x^2, y)) \simeq Spec(\mathbb{C}[a, b]) = \mathbb{A}^2$ *and the étale versal deformation family* $\mathcal{C} \subset \mathbb{A}^2 \times \mathbb{A}^2$ *of* Γ *has equation*

$$y^2 + x^3 + ax + b = 0.$$

By computing the discriminant of the previous equation, we see that the locus $S \subset \mathbb{A}^2$, parametrizing singular curves, has equation $4a^3 - 27b^2 = 0$. Moreover, the reader can verify that $(0,0)$ is the only point of S corresponding to a cuspidal curve and every point $(a,b) \in S - (0,0)$ corresponds to a plane curve with a node as singularity.

By using versality, from the examples 5.4 and 5.5, we find again lemma 3.16. In particular we deduce that, if Γ corresponds to a plane curve with only nodes and cusps as singularities and if $x \in B$ is a point of the étale versal deformation space B of Γ corresponding to a singular curve D, then D has nodes and cusps as singularities. More generally, as it follows by the following two propositions, in order to describe the singular fibres of the étal versal deformation family of a reduced affine plane curve Γ with singularities at the points $p_1, \dots, p_r$, it is enough to describe singular fibres of the étale versal deformation family of a reduced affine plane curve Γ_i with only a singular point analytically equivalent to that of Γ at p_i, for every $1 \le i \le r$.

Proposition 5.6 (Corollary 3.20 of [**13**]). *Let $\Gamma_1, \dots, \Gamma_r$ be reduced curves in $\mathbb{P}^2$ and $p_i \in \Gamma_i$ singular points. Then, for any sufficiently large e, there exists a reduced irreducible curve $E \subset \mathbb{P}^2$ of degree e such that E has exactly r singular points $q_1, \dots, q_r$ and, for each i, an étale neighborhood of q_i in E is isomorphic to an étale neighborhood of p_i in Γ_i.*

Let $\mathcal{C} \to B$ be the étale versal deformation family of an affine plane curve Γ. By the previous proposition, for every finite set of singular points $x_1, \dots, x_r$ of $\pi^{-1}(y)$, there exist affine reduced plane curves $\Gamma_1, \dots, \Gamma_r$ so that Γ_i has an unique singular point at $(0,0)$ and there is an étale neighborhood of $(0,0)$ in Γ_i which is isomorphic to an étale neighborhood of x_i in $\pi^{-1}(y)$, for every $i \le n$. If $\pi_i : \mathcal{C}_i \to B_i$ is the étale versal deformation of Γ_i constructed as (21), for every $1 \le i \le r$, by versality, there are étale neighborhoods V_i' of $(0, \dots, 0)$ in B and V_i of $(0, \dots, 0)$ in B_i and morphisms $g_i : V_i' \to V_i$ making a diagram like (20) commute. This give a morphism

$$g : \bigcap_{i=1}^{r} V_i' \to \prod_i V_{i=1}^r \subset \prod_{i=1}^{r} B_i$$

from the intersection $\bigcap_{i=1}^r V_i' \subset B$ to the product $\prod_{i=1}^r V_i \subset \prod_{i=1}^r B_i$.

Proposition 5.7 (See p.439 of [**13**]). *There exists an étale open neighborhood of the point $y \in B$ corresponding to Γ such that, for every $y \in U$, the morphism $g : \bigcap_{i=1}^r V_i' \to \prod_i V_{i=1}^r$ constructed before is surjective.*

In order to describe the étale versal deformation of a plane singularity, we need the following definition.

Definition 5.8. *Let $X \to Y$ be a flat family of affine plane curves parametrized by a separated scheme Y over C. We say that this family is equisingular if it satisfies the following properties.*

(1) *There exists a finite number of disjoint sections of the family, the union of whose contains the locus of singular points of the fibres, and X is equimultiple along these sections.*
(2) *If, in addition, all the singular points of the fibres are ordinary double points, we say that the family is equisingular. If not, we blow up the sections.*
(3) *Now we require that in the family of reduced total transforms there exist sections lying over the former section (at least one over each former sections) satisfying* (1). *Then return to* (2).

Moreover, if X and Y are as before, we say that the family $p : X \to Y$ is *locally equisingular in the étale topology* if for each closed point $y \in Y$ and each closed point $x \in p^{-1}(y)$ there exist étale open neighborhoods U of y in Y and V of x in $p^{-1}(U)$ such that the induced family $V \to U$ is equisingular.

Proposition 5.9 (Proposition 3.32 of [**13**]). *Let*

$$\begin{array}{rcl} X & \subset & Y \times S \\ p \downarrow & & \\ Y & & \end{array}$$

be a flat family of reduced curves on a smooth surface S. Assume that X, Y and S are all reduced, separated and of finite type over $\mathbb{C}$. Let Δ be the locus of the singular points of the fibres of p. Assume that Δ is proper over Y. Then the following two conditions are equivalent.

(1) *The family is locally equisingular in the étale topology.*
(2) *For each equivalence class of singularity all the fibres over the closed points of Y have the same number of singularities of that equivalence class, (see definition 3.4).*

Now, let $\Gamma \subset \mathbb{A}^2$ be an affine plane curve of equation $f(x, y) = 0$. Assume that $(0, 0) = p$ is the only singular point of Γ. Let $J = (f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ be the jacobian ideal of Γ and let $\mathcal{C} \to B$ be the étale versal deformation of Γ.

Theorem 5.10. *The étale versal deformation space B of Γ satisfies the following properties:*

(1) *B is smooth, with tangent space T_0B at $(0, \ldots, 0)$ naturally identified with the quotient $\mathcal{O}_{\Gamma,p}/J$. (See [**8**], [**35**] and [**36**].)*
(2) *The dimension of B is equal to $2\delta - r + 1$, where δ are the number of adjoint conditions of $\Gamma : f(x, y) = 0$ at $p = (0, 0)$ and r is the number of irreducible branches of Γ at p. This number is said to be the Milnor number of Γ at p. (See Corollary 6.4.3 of [**10**].)*
(3) *There exists a Zariski open subset U of B containing $(0, \ldots, 0)$ on which $(0, \ldots, 0)$ is the only point whose fibre has a singularity analytically equivalent to that of Γ. (See lemma 3.21 of [**13**]).*
(4) *There exists a Zariski closed subset $ES \subset B$, which is called the equisingular locus of B, parametrizing equisingular deformations of Γ. Moreover, ES is smooth at the point $(0, \ldots, 0)$ and there exists an ideal $I \subset \mathcal{O}_{\Gamma,p}$ such that $I \supseteq J$ and such that the tangent space T_0ES at $(0, \ldots, 0)$ is naturally identified with the quotient I/J. (See [**45**].)*

Of course, the equisingular locus $ES \subset B$ is contained in the locus $EG \subset B$ of points corresponding to curves having the same geometric genus as Γ. We say that EG is the *equigeneric locus* of B. By proposition 3.2 of chapter 1, we deduce that, for every singular plane curve Γ, the equigeneric locus of the étale versal deformation space B of Γ is Zariski closed in B. Moreover, we have the following result.

Theorem 5.11 (Lemma 4.4 and Theorem 4.15 of [**13**]). *Let $\Gamma \subset \mathbb{P}^2$ be a plane curve, let J be the jacobian ideal of Γ and let B be the étale versal deformation space of Γ. Then, denoting by A the adjoint ideal of Γ, the tangent cone to EG at $\underline{0}$ is supported on a linear subspace of B which, under the identification $B = Spec\mathbb{C}[x, y]/J$, is identified with the quotient A/J.*

Remark 5.12. *Notice that, by our definition of adjoint ideal (see section 1 of chapter 1), the jacobian ideal of a reduced plane curve* Γ *is contained in the adjoint ideal of* Γ*. Then, the quotient* A/J *is well defined. Moreover, we remark that, as we have seen in the example 5.5, the equigeneric locus of the étale versal deformation space of a plane curve* Γ *may be singular at the point* $\underline{0} = (0, \dots, 0) \in B$*. More about the equigeneric locus of a plane singularity can be found in* [**13**]*, section 4 and 5. An easy consequence of theorem 2.2 is that the general element of the equigeneric locus of every plane singularity is a plane curve with only nodes as singularity. In the next section we shall consider the equigeneric locus of an ordinary plane singularity, proving that it contains only points parametrizing plane curves with ordinary singularities.*

Finally we remark that the local result of theorem 5.11 corresponds to the following global result, proved by Albanese in [**1**]

Theorem 5.13 (Albanese, section II of [**1**]). *Let* $V_{n,g}$ *be the Severi variety of irreducible plane curves of degree* n *and genus* g*. Let* $[\Gamma] \in V_{n,g}$ *be a point corresponding to an irreducible plane curve of genus* g*. Then the tangent cone to* $V_{n,g}$ *at* $[\Gamma]$ *is supported on the linear space parametrizing the linear system of plane curves of degree* n *adjoint to* Γ*.*

Another proof of the locally irreducibility of the Severi variety $V_{n,g}$ at the points parametrizing irreducible plane curves of genus g is contained in the more recent paper [**4**]. We shall go back on the étale versal deformation B of a plane curve in the last section of this chapter, defining the equiclassical locus of B. Now we want to use the results of this section to give another proof of lemma 3.17.

Let $\Gamma \subset \mathbb{P}^2$ be a reduced plane curve of degree n with singular points $p_1, \dots, p_r$. For every $1 \leq i \leq r$, let Γ_i be an affine reduced plane curve with only a singular point at $(0,0)$ analytically equivalent to that of Γ at p_i. We denote by B_i the étale versal deformation space of $\Gamma_i \subset \mathbb{P}^2$, for every $1 \leq i \leq r$. By versality there exist open étale neighborhoods U of $[\Gamma]$ in $\mathbb{P}^N$ and U_i of $\underline{0} \in B_i$ and a map

$$g : U \to \prod_{i=1}^{r} U_i$$

such that the tautological family parametrized by U is the pull-back, with respect to g, of the product of the étale versal families parametrized by the open sets U_i. By recalling the versality property of every U_i and the property (3) of theorem 5.10, we give the following definition.

Definition 5.14. *We say that the singular points of* Γ *can be smoothed independently if the map* g *is surjective.*

Lemma 5.15 (Corollary 6.3 of [**25**]). *By using the same notation as before, let* J *be the jacobian ideal of* Γ*. Then, we have that* $H^1(\Gamma, \mathcal{O}_\Gamma(n) \otimes J) = 0$ *if and only if the singular points of* Γ *can be smoothed independently.*

Proof. Consider the following standard exact sequence

$$0 \to \Theta_\Gamma \to \phi^*\Theta_{\mathbb{P}^2} \stackrel{\alpha}{\to} \mathcal{N}_{\Gamma|\mathbb{P}^2} \to T^1_\Gamma \to 0$$

Exactly as we did in the previous section for a curve with only nodes and cusps, we deduce the exact sequence

$$0 \to \mathcal{O}_\Gamma(n) \otimes J \to \mathcal{N}_{\Gamma|\mathbb{P}^2} \to T^1_\Gamma \to 0$$

from which it follows the long exact sequence

$$0 \to H^0(\Gamma, \mathcal{O}_\Gamma(n) \otimes J) \to H^0(\Gamma, \mathcal{N}_{\Gamma|\mathbb{P}^2}) \to H^0(\Gamma, T^1_\Gamma) \to H^1(\Gamma, \mathcal{O}_\Gamma(n) \otimes J) \to 0 \tag{22}$$

The map $H^0(\Gamma, \mathcal{N}_{\Gamma|\mathbb{P}^2}) \to H^0(\Gamma, T^1_\Gamma)$ is the differential map of the map g constructed before and $H^0(\Gamma, \mathcal{O}_\Gamma(n) \otimes J)$ is the tangent space to the fibre of g over the point $(\underline{0} \times \cdots \times \underline{0})$. If we assume that $H^1(\Gamma, \mathcal{O}_\Gamma(n) \otimes J) = 0$, then

$$h^0(\Gamma, \mathcal{O}_\Gamma(n) \otimes J) = \frac{n(n+3)}{2} - h^0(\Gamma, T^1_\Gamma) = \frac{n(n+3)}{2} - \sum_i dim(U_i)$$

and the map g is surjective. For the proof of the second part of the lemma see [**25**], corollary 6.3. □

Another proof of lemma 3.17. If Γ is a reduced curve of degree n with d nodes and k cusps as singularities, then, as we have seen in the previous section, the condition $k < 3n$ is a sufficient condition to the vanishing $h^1(\Gamma, \mathcal{O}_\Gamma(n) \otimes J) = 0$ and hence nodes and cusps of Γ can be smoothed independently. In particular we have that $\mathcal{S}^n_{k,d} \subset \mathcal{S}^n_{k',d'}$, for every k' and d' such that $k' \le k$ and $d' \le d + k - k'$. □

6. On the equigeneric deformations of an ordinary plane singularity

Let $\Gamma \subset \mathbb{A}^2$ be a plane curve with an ordinary singular point of multiplicity r at the point $p = (0,0)$ and no further singularities. Let B be the étale versal deformation space of Γ and let $EG \subset B$ be the equigeneric locus of B. In [**13**], lemma 3.6, it has been proved that $dim(EG) = r - 2$. In particular, there exist subvarieties $X_i \subset EG$, with $0 \le i \le r-2$, such that $dim(X_i) = dim(X_{i+1}) - 1$ and

$$\{\underline{0}\} = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_{r-2} = EG$$

and such that the general element of X_i corresponds to an affine plane curve with an ordinary singular point of multiplicity $r - i$ and $ir - \frac{i^2+i}{2}$ nodes as singularities. In this section we want to prove that following result.

Proposition 6.1. *There exists an étale neighborhood U of $\underline{0} \in EG \subset B$ in the equigeneric locus of the étale versal deformation space B of Γ, such that every point $y \in U$ corresponds to a plane curve with only ordinary multiple points.*

To make an example, proposition 6.1 implies that if $\mathcal{C} \to Y$ is a flat equigeneric family of plane curves whose general fibre has a tacnode and four nodes as singularities, then, there are no special fibres of $\mathcal{C} \to Y$ having a four-fold ordinary point as singularity. More generally, we give the following definition.

Definition 6.2. *Let $\pi : \mathcal{C} \to Y$ be a flat family of curves, let $y \in Y$ and let $p \in \pi^{-1}(y)$ be a singular point of $\pi^{-1}(y)$. We say that $\mathcal{C} \to Y$ is locally equigeneric at p if there is an analytic neighborhoods U of p in $\mathcal{C}$ such that p is the only singular point of $\pi^{-1}(y)$ in U and, for every $y' \in \pi(U)$, we have that*

$$\sum_{x \in \pi^{-1}(y') \cap U} \delta(x) = \delta(p),$$

where we denoted by $\delta(x)$ the number of adjoint conditions of $\pi^{-1}(y')$ at x, (see section 1 of chapter 1).

By using versality, proposition 6.1 is equivalent to the following proposition.

Proposition 6.3. *Let $\mathcal{C} \to Y$ be a flat family of plane curves. Let $y \in Y$ be a point such that $\pi^{-1}(y)$ has an ordinary singularity at x. Suppose that $\mathcal{C} \to Y$ is equigeneric at a point x. Then there exists an analytic neighborhood U of x in $\mathcal{C}$ such that, for every $y' \in \pi(U)$, the curve $\pi^{-1}(y')$ has only ordinary singularities in U.*

In order to show proposition 6.1, we will show proposition 6.3 in the special case of a family of rational plane curves and then we shall deduce the general case. To get started, we recall the basic properties of the secant variety of a non degenerated curve. Let $X \subset \mathbb{P}^m$, with $m \geq 3$, be a non degenerate projective curve. Let $S(X)$ and $T(X)$ be the secant variety and the tangent variety of X.

Lemma 6.4 (Terracini Lemma for curves.). *Let x and y be two points of X and let $p \in S(X)$ be a point lying on the secant line $< x, y >$ generated by x and y. Then, denoting by T_xX the tangent line to X at x and by $T_pS(X)$ the tangent space to $S(X)$ at p, we have that*

$$< T_xX, T_yX > \subseteq T_pS(X), \tag{23}$$

where we denoted by $< T_xX, T_yX >$ the subspace generated by T_xX and T_yX. Moreover, if x and y are general in X and z is general in $< x, y >$, the equality holds in (23)*. In particular, we have that $dim(S(X)) = 3$.*

We recall that a secant line of X is said to be proper if it schematically intersects X only at two points. A multisecant is a secant line which is not proper.

Lemma 6.5 (Trisecant lines lemma for curves.). *A general point $p \in S(X)$ lies on a proper secant line of X.*

By Bertini theorem, it follows that a generic $(m-3)$-plane $\Lambda \subset \mathbb{P}^m$ transversally intersects $S(X)$ at $deg(S(X))$ points each one lying on a proper secant line of X. Projecting X from Λ, we obtain a plane curve whose singularities arise from the intersection points of Λ with $S(X)$.

Proposition 6.6 ([**32**]). *Let X be a not degenerate curve in $\mathbb{P}^m$. Let $\Lambda \subset \mathbb{P}^m$ be an $(m-3)$-plane with no intersections with X and intersecting the secant variety in finitely many points, and let $\pi : X \to \mathbb{P}^2$ be the projection morphism from Λ. Then π is birational onto its image and $\pi(X)$ has only $deg(S(X))$ nodes as singularities, if and only if*

(1) *Λ doesn't meet any tangent line of X,*
(2) *Λ doesn't meet any multisecant of X, and*
(3) *every $(m-2)$-plane passing through Λ contains at most a proper secant line to X. Moreover, every hyperplane passing through Λ and containing a proper secant line to X at points P and Q, contains at most one of the tangent lines to X at P and Q.*

In particular the general projection of X to $\mathbb{P}^2$ is a nodal curve.

We don't show the former proposition but its proof is rather elementary. The reader can verify that condition (1) is equivalent to require that the projection from Λ of X is a local embedding, whereas the conditions (2) and (3) are verified if and only if the projection curve of X from Λ does not contain a multiple point of order bigger that two, or a multiple point with at least two branches with the same tangent line. Moreover, the sets of points of $\mathbb{G}(m-3, m)$ parametrizing the $(m-3)$-planes of $\mathbb{P}^m$ verifying one of the conditions (1), (2) or (3) are locally closed of codimension at least one.

Lemma 6.7. *Let $C_{r+1} \subset \mathbb{P}^{r+1}$ be a rational plane curve of degree $r+1$ and let Λ be a $(r-2)$-plane with no intersections with C_{r+1} and having finitely many intersections with the* *secant variety $S(C_{r+1})$ of C_{r+1}. Then, the projection plane curve $\pi_\Lambda(C_{r+1}) := C$ of C_{r+1} from Λ has only ordinary multiple points as singularities if and only if Λ transversally intersects $S(C_{r+1})$ at $r(r-1)/2$ points each of which lies on a proper secant line to the rational normal curve.*

PROOF. Before proving the lemma, we recall that the secant variety $S(C_{r+1}) \subset \mathbb{P}^{r+1}$ is smooth at every point p which does not lie on C_{r+1} and which is contained in a proper secant line of C_{r+1}. Indeed, if $T(C_{r+1})$ is the tangent variety of C_{r+1}, the open subset $S(C_{r+1}) - T(C_{r+1})$ is an orbit of the action of $PGL_2\mathbb{C} = Aut(\mathbb{P}^1)$ on $\mathbb{P}^{r+1}$, (see for example section 10 of [**20**]). Now, let Λ be a $(r-2)$-plane such that the projection $\pi_\Lambda(C_{r+1}) := C$ to the plane is a rational plane curve with an ordinary multiple point of order $s \leq r$ at a point $p \in \mathbb{P}^2$. Let $p_1, \dots, p_s$ be the points of C_{r+1} lying over the point p and let $l_1, \dots, l_s$ be the tangent lines to C at p. Denoting by $< - >$ the linear space of $\mathbb{P}^{r+1}$ generated by $-$, we have that

$$< \Lambda, p_i > \cap \mathbb{P}^2 = p,$$

and

$$< \Lambda, T_{p_i}C_{r+1} > \cap \mathbb{P}^2 = l_i,$$

for every $i = 1, \dots, s$. In other words, Λ is contained in an $(r-1)$-plane $\Omega =< \Lambda, p >=< \Lambda, p_i >$, for every $i = 1, \dots, s$, containing $\frac{s(s-1)}{2}$ proper secant lines $\overline{p_ip_j}$ to C_{r+1}, for $1 \leq i < j \leq s$, in such a way that neither of the tangent lines $T_{p_i}C_{r+1}$ is contained in Ω and the hyperplanes $< \Lambda, T_{p_i}C_{r+1} >$ are all different, for $i = 1, \dots, s$. In particular, Λ intersects every secant line $\overline{p_ip_j}$ at a point P_{ij}, for $1 \leq i < j \leq s$. Moreover, since $r+2$ points over the rational normal curve C_{r+1} generate $\mathbb{P}^{r+1}$, we have that $P_{i,j} \neq P_{l,m}$ if $(i,j) \neq (l,m)$ and

$$dim(< T_{p_i}C_{r+1}, T_{p_j}C_{r+1} >) = 3.$$

By Terracini lemma,

$$< T_{p_i}C_{r+1}, T_{p_j}C_{r+1} >= T_{P_{i,j}}S(C_{r+1}). \tag{24}$$

Moreover,

$$< T_{P_{i,j}}S(C_{r+1}), \Lambda >= \mathbb{P}^{r+1}$$

and, hence, every point $P_{i,j}$ is a transversal intersection of Λ with $S(C_{r+1})$. The first part of the lemma is proved. Let, now, Λ be a $(r-2)$-plane such that the projection $\pi_\Lambda(C_{r+1})$ to the plane is a rational plane curve with a singular point at $p \in \mathbb{P}^2$ containing at least an analytic branch C_1 which is singular at p. Then, denoting by p_1 the point of C_{r+1} mapping to C_1, the dimension of the linear space $< T_{p_1}C_{r+1}, \Lambda >$ is $r-1$, because

$$< T_{p_1}C_{r+1}, \Lambda > \cap \mathbb{P}^2 = p,$$

and hence Λ intersects the tangent line $T_{p_1}C_{r+1}$. Therefore, in this case, Λ has not empty intersection with the tangent secant variety $T(C_{r+1})$ of C_{r+1}. Finally, let Λ be a $(r-2)$-plane such that the plane projection $\pi_\Lambda(C_{r+1})$ has a multiple point at a point $p \in \mathbb{P}^2$ with at least two smooth branches C_1 and C_2 having the same tangent line $l \subset \mathbb{P}^2$. We denote by p_1 and p_2 the points of C_{r+1} which lie over p. By using that the linear space

$$< T_{p_1}C_{r+1}, \Lambda >=< T_{p_2}C_{r+1}, \Lambda >=< l, \Lambda >$$

is an hyperplane, Λ doesn't intersect the tangent lines $T_{p_2}C_{r+1}$ and $T_{p_1}C_{r+1}$. Moreover, since $< \overline{p_1p_2}, \Lambda >=< p, \Lambda >$ has dimension $r-2$, the linear space Λ intersects the proper secant $\overline{p_1p_2}$

to C_{r+1} at a point P. Now, since $< T_{p_1}C_{r+1}, T_{p_2}C_{r+1} >= T_P S(C_{r+1})$ has dimension three, P is a smooth point of the secant variety of C_{r+1}. But, since

$$< T_P S(C_{r+1}), \Lambda >= \mathbb{P}^r$$

in this case P will not be a transversal intersection point of Λ with $S(C_{r+1})$. □

Lemma 6.8. *Let* $W_r \subset \mathbb{P}^{\frac{n(n+3)}{2}}$ *be the closure, in the Zariski topology of the locus of rational plane curves of degree* n *with an ordinary plane singularity of multiplicity* r*. Then, if* W_r *is not empty, we have that*

$$dim(W_r) = dim(V_{n,0}) - r + 2 = 3n - r + 1.$$

Proof. Let $C_n \subset \mathbb{P}^n$ be a rational normal curve of degree n. Since, up to projective transformations, every rational plane curve is a plane projection of C_n, in order to compute the dimension of W_r, we compute the dimension of the locally closed subset $\mathcal{R} \subset \mathbb{G}(n-3,n)$ of $(n-3)$-planes Λ such that the plane projection $\pi_\Lambda(C_n)$ has a ordinary r-fold point. Then, the dimension of W_r will be equal to $dim(W_r) = dim(\mathcal{R}) + dim(Aut(\mathbb{P}^2)) - dim(Aut(\mathbb{P}^1))$. To see that $\mathcal{R}$ is locally closed and to compute its dimension, let $Sym^r(C_n)$ be the r-th symmetric product of C_n and let U be the open set of $\mathbb{G}(n-3,n)$ of $(n-3)$-planes which have not intersections with C_n. Consider the incidence family $\widetilde{\mathcal{R}} \subset Sym^r(C_n) \times U$ defined by

$$\begin{aligned}\widetilde{\mathcal{R}} \;=\; & \{(q_1,\ldots,q_r;[\Lambda])|\text{there is } q \in \mathbb{P}^2 \text{ such that } \pi_\Lambda(q_i) = q \text{ for } i = 1,\ldots,r \text{ and} \\ & \text{the plane projection } \pi_\Lambda(C_n) \text{ has an ordinary } r\text{-fold point}\}.\end{aligned}$$

In order to prove that $\mathcal{R}$ is locally closed and to compute its dimension, it is sufficient to do this for $\widetilde{\mathcal{R}}$. Let $\widetilde{\mathcal{R}}(q_1,\ldots,q_r) \subset \mathbb{G}(n-3,n)$ be the set of $(n-3)$-planes $\Lambda \subset \mathbb{P}^n$ such that $\pi_\Lambda(C_n)$ has an ordinary r-fold point at a point $q \in \mathbb{P}^2$ and $q_1,\ldots,q_r$ are the points of C_n over q. If $[\Lambda] \in \widetilde{\mathcal{R}}(q_1,\ldots,q_r)$, then the linear space $< \Lambda, p_1,\ldots,p_r >$ has dimension $n-2$. In particular, if $\widetilde{\mathcal{R}}(q_1,\ldots,q_r)$ is not empty, then it is an open set in the irreducible subvariety of $\mathbb{G}(n-3,n)$ parametrizing the $(n-3)$-planes Λ, such that there exists a $(n-2)$-plane Θ such that $\Lambda \subset \Theta$ and $< p_1,\ldots,p_r >\subset \Theta$. Then

$$dim(\widetilde{\mathcal{R}}(q_1,\ldots,q_r)) = dim(\mathbb{G}(n-r-2,n-r)) + dim(\mathbb{G}(n-2,n-3)) = dim(\mathbb{G}(n-3,n)) - 2(r-1).$$

Finally, $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are locally closed of dimension

$$dim(\widetilde{\mathcal{R}}(q_1,\ldots,q_r)) + r = dim(\mathbb{G}(n-3,n)) - r + 2$$

and

$$dim(W_r) = dim(\mathcal{R}) + dim(Aut(\mathbb{P}^2)) - dim(Aut(\mathbb{P}^1)) = 3(n-2) - r + 2 + 8 - 3 = 3n - r + 1,$$

if they are not empty. □

Proposition 6.9. *For every integer* r *there exists an integer* $R > r$ *such that for every ordinary plane singularity of multiplicity* r *of analytic equation* $g(x,y) = 0$*, there exists an irreducible rational plane curve of degree* R *with an ordinary* r*-fold point analytically equivalent to* $g(x,y) = 0$*.*

In order to show the previous proposition we need the following intermediate results.

Lemma 6.10. *Let $z_1, \dots, z_k$ be k distinct complex numbers. Then, however we choose positive integers $h_1, \dots, h_k$ and constants c_{il}, for $1 \leq i \leq k$ and $0 \leq l \leq h_i$, there exists a polynomial $p(z)$ of degree equal to $\sum_i h_i - 1$ such that*

$$p^{(l)}(z_i) = c_{il}, \textit{for } 1 \leq i \leq k \textit{ and } 0 \leq l \leq h_i - 1.$$

Proof. For every fixed i, let p_i be the polynomial defined by

$$p_i(z) = \prod_{j \neq i} (z - z_j)^{h_j} f_i(z),$$

where $f_i(z) = (1/\prod_{j \neq i}(z_i - z_j)^{h_j})(c_{i0} + \sum_{s=1}^{h_i - 1} c_{il} \frac{(z-z_i)^l}{l!})$. The polynomial p defined by

$$p(z) = \sum_{i=1}^{k} p_i(z)$$

verifies the desired properties. □

Let Δ be the open complex disc $\Delta = \{z \in \mathbb{C} \text{ t.c. } \mid z \mid < 1\}$ and let

$$\begin{cases} x = t \\ y = \sum_{l=1}^{\infty} \alpha_{i_l} t^l, \quad t \in \Delta \text{ and } i = 1, \dots, r, \end{cases} \tag{25}$$

be the parameterizations of r smooth analytic branches $\tilde{\gamma}_1, \dots, \tilde{\gamma}_r$ of plane curve passing through the origin $(0,0)$ of the plane and having pairwise distinct tangent lines $l_1, \dots, l_r$ at $(0,0)$.

Lemma 6.11 (Morelli, [**31**]). *However we choose positive integers $h_1, \dots, h_r$ and g, there exists an irreducible plane curve Γ of genus g having at $(0,0)$ an ordinary multiple point of order r with tangent lines $l_1, \dots, l_r$ in such a way that the branch of Γ, passing through $(0,0)$ and tangent to l_i, approximates the branch $\tilde{\gamma}_i$ up to order h_i, for every $1 \leq i \leq r$.*

Proof. Fixed positive integers $h_1, \dots, h_r$ and g, we want to prove that there exists a plane curve Γ of genus g with an ordinary plane singularity at $(0,0)$ and such that, if $p_1 \dots, p_r$ are the points of the normalization curve C of Γ mapping to $(0,0)$, then the local expression of the normalization morphism at every point p_i is given by

$$\begin{cases} x = t \\ y = \sum_{l=1}^{\infty} \beta_{i_l} t^l, \quad t \in \Delta \end{cases}$$

with

$$\beta_{i_l} = \alpha_{i_l} \text{for every } l \text{ such that } i_l \leq h_i - 1 \text{ and } \beta_{h_i} = \overline{\alpha_{h_i}},$$

where $\overline{\alpha_{h_i}}$ is any constant different from α_{h_i}. In order to prove this, let Γ' be an irreducible plane curve of genus g. Let (ξ, η) be affine coordinates of the plane. We will find a birational transformation of the plane of equation

$$\begin{cases} x = \phi(\xi) & = a_0 \xi^n + a_1 \xi^{n-1} + \dots + a_n \\ y = \psi(\eta) & = b_0 \eta^m + b_1 \eta^{m-1} + \dots + b_m, \end{cases} \tag{26}$$

sending the plane curve Γ' to a plane curve Γ with the desired properties.

Step 1. Let $q_i' \equiv (\xi_i, \eta_i)$, for $i = 1, \dots, r$, be r smooth points of Γ' such that the tangent line to Γ' at every point (ξ_i, η_i) is not parallel to the ξ-axis or to the η-axis. Suppose, moreover, that the constants $\xi_1, \dots, \xi_r$ and $\eta_1, \dots, \eta_r$ are all different. Let $\phi(\xi)$ be a polynomial such that

$$\phi(\xi_i) = 0, \text{ for every } \ 1 \leq i \leq r. \tag{27}$$

Let q_i be the image point of q_i' with respect to (26), for every i. Let $n' : C' \to \Gamma'$ be the normalization of Γ' and let

$$t \to (t, \eta_i(t))$$

be the local expression of n' in a neighborhood of q_i, for every $i = 1, \dots, r$. Then, if we set $\Psi_i(\xi) = \psi(\eta_i(\xi))$, then, the branch γ_i of Γ passing through q_i which is image, with respect to (26), of the branch of Γ' passing through q_i, has equation

$$\gamma_i \;:\; \begin{cases} x & = \overline{\phi}_i^{(1)}(\xi - \xi_i) + \dots + \overline{\phi}_i^{(n)} \frac{(\xi-\xi_i)^n}{n!} \\ y - \Psi_i(\xi_i) & = \overline{\Psi}_i^{(1)}(\xi - \xi_i) + \overline{\Psi}_i^{(2)} \frac{(\xi-\xi_i)^2}{2!} + \dots \end{cases} \tag{28}$$

where $\overline{\phi}_i^{(l)} = \frac{\partial \phi(\xi)}{\partial^l \xi}|_{\xi=\xi_i}$ and $\overline{\Psi}_i^{(l)} = \frac{\partial \Psi_i(\xi)}{\partial^l \xi}|_{\xi=\xi_i}$. Setting $x = \tau$ in the first equality of (28), let $\xi - \xi_i = \sum_{j \geq 1} \lambda_{ij} \tau^j$ be the inverse function of the holomorphic function $\tau = \tau(\xi) = \sum_l \overline{\phi}_i^{(l)} \frac{(\xi-\xi_i)^l}{l!}$. By substituting in the second equality of (28), we find the local expression

$$\begin{cases} x & = \tau \\ y - \Psi_i(\xi_i) & = \sum_{i \geq 1} \beta_{i_l} \tau^l \end{cases}$$

of the normalization morphism of γ_i at q_i, for every $i = 1, \dots, r$. Now we impose the following conditions

$$\begin{cases} \Psi_i(\xi_i) & = 0 \\ \beta_{i_l} & = \alpha_{i_l} \text{ for every } i_l \leq h_i - 1, \\ \beta_{h_i} & = \overline{\alpha_{h_i}}, \end{cases}$$

for every $i = 1, \dots, r$. By these relations, by using that every β_{i_l} is a linear combination of the constants $\overline{\Psi}_i^{(s)}$, with $s = 1, \dots, l$, we find the value of $\overline{\Psi}_i^{(s)}$, for every s. Then, by the equality $\overline{\Psi}_i^{(1)}(\xi) = \psi^{(1)}(\eta_i)\eta_i^{(1)}(\xi)$, by setting $\xi = \xi_i$, we find the value $\psi^{(1)}(\eta_i) := \delta_1$. In a similar way we go on determining the values $\psi^{(s)}(\eta_i) := \delta_s$, for $s = 1, \dots, h_i$. Finally we deduce the following conditions:

$$\begin{cases} \psi(\eta_i) & = 0, \\ \psi^{(s)}(\eta_i) & = \delta_s, \text{ with } s \leq h_i \text{ and } 1 \leq i \leq r. \end{cases} \tag{29}$$

For every pair of polynomials $\phi(\xi)$ and $\psi(\xi)$ verifying the conditions (27) *and* (29)*, the image curve Γ of Γ', with respect to the transformation* (26)*, has r smooth branches $\gamma_1, \dots, \gamma_r$ passing through* $(0,0)$*, such that every branch γ_i approximates the branch $\tilde{\gamma}_i$ defined by* (25)*, up to order h_i, for every $1 \leq i \leq r$.* Notice that the set of polynomials verifying (27) and (29) are not empty by lemma 6.10.

Step 2. Now we want to find conditions on $\psi(\xi)$ in such a way that the only branches of Γ passing through the point $(0,0)$ are the smooth branches $\gamma_1, \dots, \gamma_r$, constructed before. Let $f'(\xi, \eta) = 0$ be the equation of the plane curve Γ'. Let $\xi_{r+1}, \dots, \xi_{n'}$, with $n' \leq n$, be the roots of the polynomial $\phi(\xi)$ different from $\xi_1, \dots, \xi_r$. Moreover, let $\eta_{r+1}, \dots, \eta_\sigma$, be the roots, different from $\eta_1, \dots, \eta_r$, of the polynomials $f'(\xi_i, \eta)$, for $i = r+1, \dots, n'$. If we choose constants $\rho_i \neq 0$, with $i = r+1, \dots, \sigma$, and if we assume that $\psi(\eta)$ verifies the further conditions

$$\psi(\eta_i) = \rho_i \text{ for every } i = r+1, \dots, \sigma, \tag{30}$$

then the plane curve Γ has the desired property. As before, notice that the set of polynomials verifying conditions (30), (27) and (29) are not empty by lemma 6.10.

Step 3. Finally, we want to find conditions on $\psi(\eta)$ in such a way the transformation (26) of the plane defines a birational transformation between Γ' and Γ. First of all, we notice that if $\overline{\psi}(\eta)$ is a polynomial which verifies the conditions (29) and (30), then, for every polynomial $\chi(\eta)$, also the polynomial

$$\psi(\eta) = \overline{\psi}(\eta) + (\eta - \eta_1)^{\mu_1} \dots (\eta - \eta_\sigma)^{\mu_\sigma} \chi(\eta), \tag{31}$$

with $\mu_1, \dots, \mu_\sigma$ large enough, satisfies the conditions (29) and (30). Let (β_0, θ_0) be a general point of Γ', whose image with respect to (26) is the point $(x_0, y_0) \in \Gamma$. Let $\beta_1, \dots, \beta_{n'}$, with $n' \leq n - 1$, be the other roots of the polynomial $\phi(\eta) - x_0$, different from β_0. We denote by $\theta_1, \dots, \theta_\pi$ the roots, different from θ_0, of the polynomials $f'(\beta_i, \eta)$, for $i = 0, \dots, n'$. If $d_1, \dots, d_\pi$ are constants different from zero and if the polynomial $\chi(\eta)$ in (31) satisfies the properties

$$\begin{aligned} \chi(\theta_0) &= 0 \\ \chi(\theta_i) &= d_i, \text{ for every } i = 1, \dots, \pi, \end{aligned} \tag{32}$$

then the point (β_0, θ_0) is the only point of Γ' sent by (26) to the point $(x_0, y_0) \in \Gamma$. As before, notice that, by lemma 6.10, the set of polynomials verifying the condition (32) is not empty.

We have found conditions on $\phi(\xi)$ and $\psi(\eta)$ in such a way that the image curve Γ of Γ', with respect to (26), has at $(0, 0)$ an ordinary r-fold point with the desired properties. Applying lemma 6.10, we conclude. □

Remark 6.12. *By using the same notation as in the proof of lemma 6.11, notice that, by the proof of lemma 6.10, the degree of the polynomials* $\phi(\xi)$ *and* $\psi(\eta)$ *verifying conditions* (27), (29), (30), (31) *and* (32), *in such a way that the regular transformation of equations* (26) *has the desired property, does not depend on the analytic class of the ordinary* r*-fold defined by* (25), *but it depends only on* r, *on the degree of the curve* Γ' *and on the constants* $h_1, \dots, h_r$.

Remark 6.13. *For our convenience, we stated and proved lemma 6.11 for an ordinary plane singularity, but in* [**31**] *it is proved that, given any set of plane singularities* $g_1(x, y) = 0$, $g_2(x, y) = 0, \dots, g_k(x, y) = 0$, *for any reduced plane curve* Γ', *there exists a plane birational model* Γ *of* Γ' *having* k *singularities equivalent, but not necessarily analytically equivalent, to the given plane singularities. (For the definition of equivalent plane singularities see definition 3.4).*

Lemma 6.14 (Franchetta, [**16**]). *Let* Γ *and* Γ' *be two reduced plane curves with an ordinary* r*-fold point at* $(0, 0)$ *and let* $\gamma_1, \dots, \gamma_r$ *and* $\gamma'_1, \dots, \gamma'_r$ *be the branches of* Γ *and* Γ', *respectively, passing through* $(0, 0)$. *Then, if every branch* γ_i *of* Γ *intersects the corresponding branch* γ'_i *of* Γ' *with multiplicity at least equal to* $r - 1$, *then the* r*-fold point of* Γ *at* $(0, 0)$ *is analytically equivalent to the* r*-fold point of* Γ' *at* $(0, 0)$.

Proof. Let (x, y) be analytic coordinates of the plane. We want to construct an analytic transformation of the plane of equations

$$\begin{cases} x' &= x \\ y' &= a_0(x) y^{r-1} + \dots + a_{r-1}(x), \end{cases} \tag{33}$$

where $a_i(x)$ are holomorphic functions in a neighborhood of zero, defining an isomorphism between an analytic neighborhood of $(0, 0)$ in Γ and a neighborhood of $(0, 0)$ in Γ'. Since we work locally in the analytic topology, we may suppose that the equations of Γ and Γ' are given by $\prod_{i=1}^r f_i(x, y) = 0$ and $\prod_{i=1}^r g_i(x, y) = 0$. Moreover, we may assume that the x-axis and the y-axis are not tangent to everyone of the branches γ_i and γ'_i at the point $(0, 0)$.

Then, by the Implicit Function Theorem, there exist holomorphic functions $y_i(x)$ and $y_i'(x)$ on $\{x \in \mathbb{C} \mid 0 <\mid x \mid< \sigma\}$, with σ small enough, such that $(x, y_i(x)) \in \gamma_i$ and $(x, y_i'(x)) \in \gamma_i'$, for every $i = 1, \ldots, r$. We want to determine the functions $a_i(x)$ in such a way that the transformation (33) sends every point $(x, y_i(x))$ to the corresponding point $(x, y_i'(x))$, for every $i = 1, \ldots, r$. Now, for every fixed x such that $0 <\mid x \mid< \sigma$, we get a linear system of r equations in r variables $a_0(x), \ldots, a_{r-1}(x)$

$$\begin{cases} y_1'(x) & = y_1(x)^{r-1}a_0(x) + y_1(x)^{r-2}a_1(x) + \cdots + a_{r-1}(x) \\ \vdots & \quad \vdots \\ y_r'(x) & = y_r(x)^{r-1}a_0(x) + y_r(x)^{r-2}a_1(x) + \cdots + a_{r-1}(x). \end{cases} \tag{34}$$

Since $y_i(x) \neq y_j(x)$, if $i \neq j$, and the determinant of the matrix $A(x)$ of the coefficients of the former linear system is the Vandermonde's determinant relative to $y_1(x), \ldots, y_r(x)$, the previous system has an unique solution, given by

$$\begin{cases} a_0(x) & = \frac{\det(A(x)_1}{\det(A(x))} \\ \vdots & \quad \vdots \\ a_{r-1}(x) & = \frac{\det(A(x)_r)}{\det(A(x))}, \end{cases} \tag{35}$$

where $A(x)_i$ is the matrix obtained from $A(x)$ by substituting the $i-th$ column of $A(x)$ with the vector $(y_1'(x), \cdots, y_r'(x))$. Moreover, if $A(x)_i'$ is the matrix which we get by substituting the $i-th$ column of $A(x)$ with the vector $(y_1'(x) - y_1(x), \cdots, y_r'(x) - y_r(x))$, then we have that

$$\begin{cases} a_0(x) & = \frac{\det(A(x)_1)}{\det(A(x))} = \frac{\det(A(x)_1')}{\det(A(x))} \\ \vdots & \quad \vdots \\ a_{r-2}(x) & = 1 + \frac{\det(A(x)_{r-1}')}{\det(A(x))} \\ a_{r-1}(x) & = \frac{\det(A(x)_r')}{\det(A(x))}. \end{cases} \tag{36}$$

Now we notice that the determinant $\det(A(x)) = \prod_{i \neq j}(y_i(x) - y_j(x))$ is an holomorphic function at a neighborhood of $x = 0$. Moreover, by using that $f_i(x, y_i(x)) = 0$ and the hypothesis that the tangent line to γ_i at $(0, 0)$ does not coincides with the x-axis or the y-axis, we find that every holomorphic function $y_i(x)$ vanishes at zero with order equal to one and, in particular we find that $\det(A(x)) = \sum_{i \geq \binom{r}{2}} \alpha(x)_i x^i$, where $\alpha(x)_i := \frac{\partial \det(A(x))}{i! \partial^i x}$ and $\alpha(x)_{\binom{r}{2}} \neq 0$. Similarly, if we assume that the multiplicity of intersection between γ_i and γ_i' is at least equal to $r - 1$, then the Taylor expansion of the holomorphic function $\det(A(x)_i')$ at $x = 0$, begins with a term of order $\geq \binom{r}{2}$ if $i = 1$ and with a term of order $> \binom{r}{2}$ if $i \neq 1$. Then, under the hypothesis $ord_0(y_i(x) - y_i'(x)) \geq r - 1$, the functions $a_i(x)$, for $i = 0, \ldots, r-1$ are defined and holomorphic at zero and the equations (33) define an analytic transformation of a neighborhood $U \subset \mathbb{A}^2$ of the origin, which restricts, by construction, to an analytic isomorphism between $\Gamma \cap U$ and $\Gamma' \cap U$. □

From the previous lemma we may deduce the following well-known result.

Corollary 6.15. *Ordinary triple points of plane curves are all analytically equivalent.*

Proof. Let Γ be a reduced plane curve with an ordinary singularity of multiplicity three at a point $P \in \mathbb{P}^2$ and let Γ' be a reduced plane curve with an ordinary singularity of multiplicity three at a point $P' \in \mathbb{P}^2$. There exists a projective motion of the plane sending the point P to the point P' and the tangent cone to Γ at P to the tangent cone to Γ' at P'. The statement follows by lemma 6.14. $\square$

Remark 6.16. *Unfortunately, the result of lemma 6.14 is not sharp. Indeed, it is well know that two ordinary four-point of plane curve are analytically equivalent if and only if they have projectively equivalent tangent cones, (see* [**45**]*). Then, for $r = 4$, lemma 6.14, gives only a sufficient conditions in order that Γ and Γ' have analytically equivalent singularities at $(0,0)$. Finally, notice that there exist examples of ordinary plane singularities with projectively equivalent tangent cones which are not analytically equivalent. For instance, in* [**34**] *it is proved that the plane singularities of equation $(x^4 - y^4)x = 0$ and $(x^4 - y^4)(x - y^2) = 0$ are not analytically equivalent. Notice that the branch $x = 0$ intersects the branch $x - y^2 = 0$ with multiplicity two.*

Proof of proposition 6.9. Let $g(x,y) = 0$ be the analytic equation of an ordinary plane singularity of multiplicity r and let Γ' be a rational cubic. By the proof of lemma 6.11, we may construct a rational transformation of the plane $A : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ of affine equations

$$\begin{cases} x = \phi(\xi) & = a_0\xi^n + a_1\xi^{n-1} + \cdots + a_n \\ y = \psi(\eta) & = b_0\eta^m + b_1\eta^{m-1} + \cdots + b_m, \end{cases}$$

which restricts to a birational map on Γ' and such that the strict transform Γ of Γ', with respect to A, has an ordinary r-fold point at $(0,0)$, with the same tangent cone as $g(x,y) = 0$ and such that every branch of Γ' at $(0,0)$ intersects the branch of $g(x,y) = 0$, having the same tangent line, with multiplicity $r - 1$. By remark 6.12, the degree of the polynomials ψ and ϕ does not depend on the analytic equivalence class of singularity of $g(x,y) = 0$, but it depends only on r and the on degree of Γ'. We conclude by using lemma 6.14. $\square$

Proof of proposition 6.1. Let $g(x,y) = 0$ be the analytic equation of an ordinary plane singularity of multiplicity r. By the property (2) of the theorem 5.10, the dimension of the étale versal deformation space B of $g(x,y) = 0$ is given by

$$dim(B) = dim(\mathbb{C}[x,y]/(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, g) = (r-1)^2.$$

Moreover, if $ES \subset B$ is the equisingular locus of B, then

$$dim(ES) = (r-1)^2 - \frac{r(r+1)}{2} = \frac{(r-2)(r-3)}{2}.$$

To see this we recall that, by the proposition 5.6, for any n sufficiently large, there exist plane curves of degree n with an ordinary r-fold point analytically equivalent to $g(x,y) = 0$ and no further singularities. If Γ is such a plane curve of degree n, by versality there exists a morphism $F : U \to V$ from an étale neighborhood U of $[\Gamma]$ in $\mathbb{P}^N$ to an étale neighborhood of $\underline{0} \in B$. By the proof of lemma 5.15 the tangent space to the fibre over $\underline{0}$ of F can be identified with $H^0(\Gamma, \mathcal{O}_\Gamma(n) \otimes J)$, where J is the jacobian ideal of Γ. Up to consider plane curves of higher degree, we may assume that $H^1(\Gamma, \mathcal{O}_\Gamma(n) \otimes J) = 0$. Under this hypothesis F is surjective and, in particular, $F^{-1}(ES)$ is the locus in U of plane curves with an ordinary r-fold point. We deduce that

$$dim(ES) = \frac{n(n+3)}{2} - \frac{r(r+1)}{2} + 2 - \frac{n(n+3)}{2} + (r-1)^2 = \frac{(r-2)(r-3)}{2}.$$

Now, let $EG \subset B$ be the equigeneric locus of B. By theorem 5.11, we now that

$$dim(EG) = (r-1)^2 - \frac{(r-1)r}{2} = \frac{(r-1)(r-2)}{2}.$$

We want to prove that there exists an étale neighborhood U of $\underline{0} \in EG \subset B$ such that every point $y \in U$ corresponds to a plane curve with only ordinary multiple points. In order to see this, we recall that, by proposition 6.9, there exists an integer R such that, for every ordinary plane singularity of multiplicity r of analytic equation $g'(x,y) = 0$, there exist irreducible rational plane curves of degree R with an ordinary R-fold point analytically equivalent to $g'(x,y) = 0$. Then, if Γ is a rational plane curve of degree R with an ordinary r-fold point analytic equivalent to $g(x,y) = 0$, then, by versality we have a morphism $F : U \to V$ from an étale neighborhood U of $[\Gamma]$ in $\mathbb{P}^{\frac{R(R+3)}{2}}$ to an étale neighborhood V of $\underline{0}$ in B. By construction, if $W_r \subset \mathbb{P}^{\frac{R(R+3)}{2}}$ is the locus in U of rational plane curves with an ordinary plane singularity of multiplicity r, then F maps W_r surjectively on ES. By lemma 6.8, we know that $dim(W_r) = 3R-1-r+2 = 3R-r+1$. In particular, the general fibre of F on ES has dimension equal to

$$3R - 1 - r + 2 - \frac{(r-2)(r-3)}{2} = 3R - 1 - \frac{r^2 - 3r + 2}{2} = dim(V_{R,0}) - dim(EG).$$

We deduce that F maps surjectively the Severi variety $V_{R,0}$ of rational plane curves of degree R on the locus in V of equigeneric deformations of $g(x,y) = 0$. Thus, in order to show the proposition, it is enough to prove that if $\mathcal{G} \to Y$ is a one-parameter family of rational plane curves with special fibre $\mathcal{G}_0 = \Gamma$, then the general fibre $\mathcal{G}_t$ of the family has only ordinary multiple points as singularities. To prove this, we remember that every irreducible rational plane curve of degree R, is projection of a rational normal curve $C_R \subset \mathbb{P}^R$ of degree R from a linear space Λ of dimension $R-3$. Thus there exists a one parameter family Λ_t, $t \in Y$ of $(R-3)$-planes of $\mathbb{P}^R$ such that for every t the projection curve of C_R from Λ_t is equal to $\mathcal{G}_t$, up to projective motions. The statement follows by lemma 6.7 by using that transversally intersecting the secant variety $S(C_R))$ is an open condition in $\mathbb{G}(R-3, R)$. □

7. Families of plane curves with nodes and cusps and Horikawa deformation theory

In this section we shall assume as known the notion of deformation of a morphism, for which we refer to section 4 of chapter 1. Let $\Sigma \subset \Sigma^n_{k,d}$ be an irreducible component of $\Sigma^n_{k,d}$ and let Γ be the irreducible plane curve of geometric genus g with k cusps and d nodes, corresponding to the general element $[\Gamma]$ of Σ. In the next chapter will be convenient for us to identify the tangent space to Σ at $[\Gamma]$ with a suitable subspace of the infinitesimal deformations space of the normalization map $\phi : C \to \Gamma$ of Γ. We recall that, if Θ_C and $\Theta_{\mathbb{P}^2}$ are the tangent sheaf of C and of $\mathbb{P}^2$ respectively and

$$\phi_* : \Theta_C \to \phi^* \Theta_{\mathbb{P}^2}$$

is the differential map of ϕ, then the cokernel $\mathcal{N}_\phi$ of ϕ_* is called the *normal sheaf to* ϕ. If we denote by Z the *ramification divisor* of ϕ, i.e. the zero divisor of ϕ_*, the normal sheaf to ϕ is invertible if and only if ϕ is not ramified, i.e. $Z = 0$. On the other hand, ϕ_* naturally extends to a sheaves map $\Theta_C(Z) \to \phi^* \Theta_{\mathbb{P}^2}$, which we still denote by ϕ_* and whose cokernel $\mathcal{N}_\phi{}'$ is an invertible sheaf.

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \mathcal{K}_\phi & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \rightarrow & \Theta_C & \stackrel{\phi_*}{\rightarrow} & \phi^*\Theta_{\mathbb{P}^2} & \rightarrow & \mathcal{N}_\phi & \rightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \rightarrow & \Theta_C(Z) & \stackrel{\phi_*}{\rightarrow} & \phi^*\Theta_{\mathbb{P}^2} & \rightarrow & \mathcal{N}_\phi{}' & \rightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

The normal sheaf $\mathcal{N}_\phi$ to ϕ maps surjectively on $\mathcal{N}'_\phi$ with kernel $\mathcal{K}_\phi$ supported on Z. By the Horikawa deformation theory, (see [**26**], [**27**] and page 15 of this paper), the vector space $H^0(C, \mathcal{N}_\phi)$ parametrizes the infinitesimal deformations of C. Moreover, if $H^1(C, \mathcal{N}_\phi) = 0$, every infinitesimal deformation of ϕ, corresponding to an element $s \in H^0(C, \mathcal{N}_\phi)$, is not obstructed, i.e. it extends to an effective deformation of ϕ, which we say to have *Horikawa class* equal to s. Now, by the exact sequence which defines $\mathcal{N}'_\phi$, we have that

$$\wedge^2(\phi^*\Theta_{\mathbb{P}^2}) \simeq \Theta_C(Z) \otimes \mathcal{N}'_\phi$$

and then

$$\mathcal{N}'_\phi \simeq \mathcal{O}_C(\phi^*(-K_{\mathbb{P}^2})) \otimes \mathcal{O}_C(K_C) \otimes \mathcal{O}_C(-Z), \tag{37}$$

where $K_{\mathbb{P}^2}$ and K_C are the canonical divisors of $\mathbb{P}^2$ and C. From what we proved in the section 1 of chapter 1, denoting by H the pullback to C of the divisor cut out on Γ by the general line and by Δ the adjoint divisor of the map $\phi : C \to \Gamma$, we have that $\mathcal{O}_C(K_C) = \mathcal{O}_C((n-3)H)(-\Delta)$. Hence, by (37) and (18) we find that

$$\mathcal{N}'_\phi \simeq \mathcal{O}_C(nH)(-\Delta - Z) \simeq \phi^*\mathcal{N}'_{\Gamma|\mathbb{P}^2}.$$

It follows that, if the number $k = deg(Z)$ of cusps of Γ is smaller that $3n$, we have that $\mathcal{N}'_\phi$ has degree $deg(\mathcal{N}'_\phi) = 3d - k + 2g - 2 > 2g - 2$, so $H^1(C, \mathcal{N}'_\phi) = H^1(C, \mathcal{N}_\phi) = 0$ and, by Riemann-Roch theorem, $h^0(C, \mathcal{N}'_\phi) = 3n + g - 1 - k$. In particular, we find that, if $k < 3n$, then

$$H^0(C, \mathcal{N}'_\phi) \simeq H^0(\Gamma, N'_{\Gamma|\mathbb{P}^2}) \simeq T_{[\Gamma]}(\Sigma), \tag{38}$$

the space $H^0(C, \mathcal{N}_\phi)$ has the expected dimension equal to $3n + g - 1$ and all the infinitesimal deformations of ϕ are not obstructed. Going back to the exact sequence

$$0 \to \mathcal{K}_\phi \to \mathcal{N}_\phi \to \mathcal{N}'_\phi \to 0$$

we find the exact sequence

$$0 \to H^0(C, \mathcal{K}_\phi) \to H^0(C, \mathcal{N}_\phi) \to H^0(C, \mathcal{N}'_\phi) \to 0$$

from what we deduce that, if $k < 3n$, the tangent space to Σ at $[\Gamma]$ may be identified with a subspace of dimension $3n + g - 1 - k$ of $H^0(C, \mathcal{N}_\phi)$ intersecting $H^0(C, \mathcal{K}_\phi)$ only at zero. In order to say more about the global sections of the torsion sheaf $\mathcal{K}_\phi$, we recall the following standard definitions. Let D be a smooth curve of genus g, let $\psi : D \to \mathbb{P}^r$ be a holomorphic map and let $p \in D$ be a point. We may always choose a holomorphic chart (U, z) of D at p and affine coordinates of $\mathbb{P}^r$ at $\psi(p)$, in such a way the local expression of ψ on U is given by

$$\psi(z) = (z^{k_1} + g_{k_1}(z), \dots, z^{k_r} + g_{k_r}(z))$$

with $1 \leq k_1 < k_2 < \cdots < k_r$ and $k_1 \nmid k_2$ and where $g_i(z)$ is a holomorphic function with vanishing order $\geq k_i + 1$ at $z = 0$, for every $i = 1, \dots, r$. These integers depend only on ψ and p. The integer $k_1 - 1$ is said to be the *ramification index* of ψ at p, while the integer $k_2 - 2$ is called the *ramification type* of ψ at p. The point p is said to be a *ramification point* of ψ if $k_1 \geq 2$ and it is said to be *simple* if $k_1 = 2$ and $k_2 = 3$. The normalization map of a double branch of a plane curve of analytic equation $y^2 - x^3 = 0$ has a simple ramification point at the point over $(0, 0)$.

Lemma 7.1 (Corollary 6.11 of [**3**]). *Let $\pi : \mathcal{C} \to B = \{t \in \mathbb{C} :\mid t \mid< 1\}$ be a family of smooth curves of genus g and let $\psi : \mathcal{C} \to \mathbb{P}^r$, with $r \geq 2$, be a holomorphic morphism. Let us suppose that, for every fixed t, the morphism $\psi_t : \mathcal{C}_t = \pi^{-1}(t) \to \mathbb{P}^r$ is birational onto its image and that number, index and type of ramification points of ψ_t don't depend on t. Thus, the Horikawa class s of the family of morphism $(\mathcal{C}, \psi, \pi, \mathbb{P}^2)$ at $t = 0$ is not contained the space $H^0(C_0, \mathcal{K}_{\psi_0})$, unless $s = 0$.*

Remark 7.2. *Actually, it is evident by its proof (see page 27 and 28 of* [**3**]*) that the previous result is local. Therefore it can be generalized as follows. Let $\mathcal{C} \to B$ be a deformation of a smooth curve $C = \mathcal{C}_0$ parametrized by the open disc $B = \{t \in \mathbb{C} :\mid t \mid< 1\}$. Suppose that there is a holomorphic map $\psi : \mathcal{C} \to \mathbb{P}^r$ such that the restriction to ψ to each fibre is birational onto its image. Moreover, suppose that there exists a section $\rho : B \to \mathcal{C}$ such that the ramification index and type of the restriction morphism $\psi_t : \mathcal{C}_t = \pi^{-1}(t) \to \mathbb{P}^r$ at the point $\rho(t) \in \mathcal{C}_t$ don't depend on t. Thus the localization s_p at the point $p = \rho(0) \in C$ of the Horikawa class s of $(\mathcal{C}, \psi, \pi, \mathbb{P}^2)$ at $t = 0$ is not contained in $\mathcal{K}_{\psi_0, p}$, unless $s_p = 0$.*

Lemma 7.1 and remark 7.2 allow us to give another proof of lemma 3.17 in a very special case, (see lemma 7.5 and remark 7.7). In order to do this, *let $\Gamma \subset \mathbb{P}^2$ be an irreducible plane curve of degree n and genus g whose normalization map $\phi : C \to \Gamma$ has only k simple ramification points $p_1, \dots, p_k$, with $k < 3n$.* Notice that Γ could have singularities worst than nodes and cusps. By using the same notation as before, we find that $deg(\mathcal{N}'_\phi) = 3d - k + 2g - 2 > 2g - 2$ and hence $H^1(C, \mathcal{N}'_\phi) = H^1(C, \mathcal{N}_\phi) = 0$. By theorem 4.3 of chapter 1 , this vanishing is a sufficient condition for the existence of a *universal deformation*

(39)
$$\begin{array}{ccc} \mathcal{C} & \stackrel{\tilde{\phi}}{\to} & \mathbb{P}^2 \\ \pi \downarrow & & \\ B & & \end{array}$$

of the normalization map ϕ, which we denote by $(\mathcal{C}, \tilde{\phi}, \pi, B)$. The tangent space to B at the point 0 corresponding to the morphism ϕ is naturally identified with $H^0(C, \mathcal{N}_\phi)$ and B is smooth at 0 of dimension $3n + g - 1$. If $\underline{\tau}$ is a holomorphic coordinate of B centered at 0, we set $\mathcal{C}_{\underline{\tau}} = \pi^{-1}(\underline{\tau})$ and $\tilde{\phi}_{\underline{\tau}} = \tilde{\phi}|_{\mathcal{C}_{\underline{\tau}}}$. With this notation, it is naturally defined a $1 : 1$ map

$$n : B \to V_{n,g}$$

from B to an analytic open neighborhood U of $[\Gamma]$ in $V_{n,g}$, sending a point $\underline{\tau} \in B$ to the point of $V_{n,g}$ corresponding to the plane curve $\tilde{\phi}_{\underline{\tau}}(\mathcal{C}_{\underline{\tau}})$. Now let B_s be the locus of B parametrizing the morphisms with at least s ramification points. Notice that, if we assume that there exists a plane curve Γ as before, then B_s is not empty, for every $s \leq k < 3n$, since the point $0 \in B$ corresponding to ϕ belongs to B_s.

Lemma 7.3. *Under the hypotheses and with the notation introduced before, we have that B_k is an analytic closed subset of B which is smooth at 0 of codimension equal to k. Similarly, for every $s = 1, \dots, k-1$, we have that B_s has codimension equal to s in B. The general element of B_s corresponds to a morphism with only s ramification points and B_s is smooth at every point corresponding to a morphism with only s ramification points.*

Proof. Let $\mathcal{C}^k$ be the k-th symmetric fibre product over B of $\mathcal{C}$ and $\tilde{B}_k \subset B \times_B \mathcal{C}^k$ the set

$\tilde{B}_k = \{(\underline{\tau}; p_1^{\underline{\tau}}, \dots, p_k^{\underline{\tau}}) \mid \underline{\tau} \in B \text{ and } p_1^{\underline{\tau}}, \dots, p_k^{\underline{\tau}} \in \pi^{-1}(\underline{\tau}) := \mathcal{C}_{\underline{\tau}} \in \text{ are ramification points of } \tilde{\phi}|_{\mathcal{C}_{\underline{\tau}}}\}.$

Since the natural projection $\tilde{B}_k \to B_k$ is a finite map, in order to show that B_k is closed of dimension $3n+g-1-k$, it's enough to show it for $\tilde{B}_k$. We may work in a neighborhood U of the point $x = (0; p_1, \dots, p_k) \in \tilde{B}_k$ corresponding to ϕ. We choose local complex parameters z_i on C centered respectively at p_i, for $i = 1, ..., k$, in such a way that the pair $(\underline{\tau},\, z_i)$ is a local parameter of $\mathcal{C}$ at the point p_i, for every i. Moreover, always working locally, we may suppose that $\tilde{\phi}$ maps in to the complex affine plane and we may set $\tilde{\phi}(\underline{\tau}, z_i) = (\tilde{\phi}_1(\underline{\tau}, z_i), \tilde{\phi}_2(\underline{\tau}, z_i))$. Under these assumptions, $\tilde{B}_k$ is defined by the $2k$ equations

$$\frac{\partial \tilde{\phi}(\underline{\tau}, z_i)}{\partial z_i}|_{z_i} = 0, \quad \text{for} \quad i = 1, \dots, k.$$

This proves that $\tilde{B}_k$ (and hence B_k) is closed and that every its irreducible component has dimension at least equal to

$$dim(B \times_B \mathcal{C}^k) - 2k = dim(B) - k.$$

Moreover, by lemma 7.1, the tangent space $T_0 B_k \subset H^0(C, \mathcal{N}_\phi)$ to B_k at 0 intersects $H^0(C, \mathcal{K}_\phi)$ only at zero. Since $h^0(C, \mathcal{K}_\phi) = k$, it follows that B_k is smooth at 0 of codimension equal to k in B. Similarly, let $\underline{\tau}$ be a general point B_s, with $s < k$. By arguing as before, we have that every irreducible component of B_s has codimension $\leq s$ in B. Moreover, if $\tilde{\phi}_{\underline{\tau}} : \mathcal{C}_{\underline{\tau}} \to \mathbb{P}^2$ is the morphism corresponding to the point $\underline{\tau} \in B_s$ and, if $q_1, \dots, q_s \in \mathcal{C}_{\underline{\tau}}$ are ramifications points of $\tilde{\phi}_{\underline{\tau}}$, then, by remark 7.2, the tangent space $T_{\underline{\tau}} B_s$ to B_s at $\underline{\tau}$ intersects the linear space W generated by $H^0(\mathcal{C}_{\underline{\tau}}, \mathcal{K}_{\tilde{\phi}_{\underline{\tau}}, q_i})$, for $i = 1, \dots, s$, only at zero. To see that $dim(W) = s$ and hence that every irreducible component of B_s has dimension $dim(B) - s$, observe that, however we choose s sections $s_1, \dots, s_s$, with $s_i \in H^0(\mathcal{C}_{\underline{\tau}}, \mathcal{K}_{\tilde{\phi}_{\underline{\tau}}, q_i})$, we have that they are linearly independent, because every s_i has support at q_i. This proves that there is a stratification

$$B_k \subsetneq B_{k-1} \subsetneq \cdots \subsetneq B_1 \subsetneq B,$$

from which we deduce that the general point of B_s corresponds to a morphism with only s ramification points. Finally, by using again lemma 7.1, we see that every point corresponding to a morphism with only s ramification points, is a smooth point of B_s, for every $s = 1, \dots, k$. □

Remark 7.4. *Notice that, since ϕ has only simple ramification points, for every $s = 1, \dots, k$, the general element of B_s corresponds to a morphism with only simple ramification points.*

Lemma 7.5. *Let $\Gamma \subset \mathbb{P}^2$ be an irreducible plane curve of degree n and genus g whose normalization map $\phi : C \to \Gamma$ has only k simple ramification points $p_1, \dots, p_k$, with $k < 3n$.*

We denote by $\pi : \mathcal{C} \to B$ the universal deformation of ϕ. Then, for any subset of $s \leq k$ ramification points of ϕ, say $p_1, \dots, p_s$, there exists a one parameter deformation of ϕ,

$$\begin{array}{ccccc} i^*(\mathcal{C}) & \subset & \mathcal{C} & \stackrel{\tilde{\phi}}{\to} & \mathbb{P}^2 \\ \downarrow & & \pi\downarrow & & \\ \Delta & \stackrel{i}{\to} & B & & \end{array}$$

parametrized by a curve $\Delta \subset B$, such that the general element z of Δ corresponds to a morphism $\tilde{\phi}_z : \mathcal{C}_z \to \mathbb{P}^2$ with only s ramifications points which specialize to $p_1, \dots, p_s$, as z specializes to the point $0 \in \Delta$ corresponding to $\phi = \tilde{\phi}_0$.

Proof. Consider the incidence family

$$\widetilde{B_1} = \{(\underline{\tau}; p^{\underline{\tau}}) \mid \underline{\tau} \in B_1 \text{ and } p^{\underline{\tau}} \in \pi^{-1}(\underline{\tau}) := \mathcal{C}_{\underline{\tau}} \text{ is a ramification point of } \tilde{\phi}|_{\mathcal{C}_{\underline{\tau}}}\} \subset \mathcal{C} \times_{B_1} B_1$$

of B_1. Let $\pi_1 : \widetilde{B_1} \to \mathcal{C}$ and $\pi_2 : \widetilde{B_1} \to B_1$ be the projection maps. We choose pairwise disjoint analytic open neighborhoods $U_1, \dots, U_k$ of the points $p_1, \dots, p_s$ in $\mathcal{C}$. If $V_i = \pi_2(\pi_1(U_i))$, then, by construction, the general element $\underline{\tau}$ of $\cap_i V_i$ corresponds to a morphism with at least s different ramification points, specializing to $p_1, \dots, p_s$, as $\underline{\tau}$ specializes to 0. In particular $\cap_i V_i \subset B_s$. On the other hand, every irreducible component of $\cap_i V_i$ has dimension $\geq dim(B) - s$ and hence $\cap_i V_i$ is an analytic neighborhood of 0 in B_s. This proves the lemma. Moreover, we observe that, by remark 7.2, the tangent space $T_0(\cap_i V_i)$ to $\cap_i V_i$ at 0 intersects the linear space W generated by $H^0(C, \mathcal{K}_{\phi,p_i})$, for every $i = 1, \dots, s$, only at zero. Since $dim(W) = s$, we deduce that $\cap_i V_i$ is smooth at the point $0 \in B$ corresponding to ϕ. □

Corollary 7.6. *For every $s = 1, \dots, k$, the variety B_s has an ordinary $\binom{k}{s}$-fold at the point $0 \in B_s$ corresponding to ϕ.*

Proof. The statement follows from the proof of the previous lemma. □

Remark 7.7. *Notice that, if the above plane curve Γ has d nodes and $k < 3n$ cusps as singularities, then the former lemma says that, for any subset of cusps $x_1, \dots, x_s$ of Γ, there exists a one parameter family of plane curves $\mathcal{G} \to \Delta$, such that $\mathcal{G}_0 = \Gamma$ and the general curve $\mathcal{G}_t$ of the family has a cusp in a neighborhood of every point $x_1, \dots, x_s$ and a node in a neighborhood of every cusp of Γ, different from $x_1, \dots, x_s$. In particular, this proves the existence of the following stratification*

$$\Sigma^n_{k,d} \subsetneq \Sigma_{k-1,d+1} \subsetneq \cdots \subsetneq \Sigma_{0,d+k},$$

which we already know by lemma 3.17 or lemma 5.15.

We have seen how Horikawa deformation theory may be useful to study equigeneric deformations of an irreducible plane curve $\Gamma \subset \mathbb{P}^2$ of genus g and degree n. Looking at the universal deformation space B of the normalization map $\phi : C \to \Gamma$ of Γ, (when it there exists), instead of the Severi variety $V_{n,g}$, is very convenient if we are interested in the local geometry of $V_{n,g}$ at the point $[\Gamma]$, corresponding to Γ. Without further restrictions on index and type of ramification points of ϕ, if we assume that the degree of the ramification divisor of ϕ is smaller than $3n$, the existence of a universal deformation family of ϕ like (39) follows from theorem 4.3 of chapter 1. As before, the universal deformation space B of ϕ is smooth at the point 0 corresponding to ϕ and it is naturally defined a map

$$n : B \to V_{n,g}$$

from B to $V_{n,g}$, sending every point $x \in B$ to the image plane curve of the morphism corresponding to x. But, in this case, $H^0(C, \mathcal{K}_\phi)$ is the 'normal space at 0' to the locus of deformations of ϕ preserving number, type and index of the ramification points of ϕ.

Proposition 7.8 (Arbarello-Cornalba, [**4**], p. 487). *Let* $[\Gamma] \in V_{n,g}$ *be a point corresponding to an irreducible plane curve* $\Gamma \subset \mathbb{P}^2$ *of genus* g *and degree* n*. Let* $Z \subset C$ *be the ramification divisor of the normalization map* $\phi : C \to \Gamma$ *of* Γ*. Assume that* $deg(Z) < 3n$*. Then, if we denote by* B *the universal deformation space of* ϕ*, by* $n : B \to V_{n,g}$ *the natural map from* B *to* $V_{n,g}$ *and if we identify the tangent space to* B *at the point* 0*, corresponding to* ϕ*, with* $H^0(C, \mathcal{N}_\phi)$*, then the kernel of the differential map*

$$dn : T_0 B \to T_\Gamma V_{n,g}$$

of n *at* 0 *coincides with* $H^0(C, \mathcal{K}_\phi)$*. In particular,* $V_{n,g}$ *is smooth at* $[\Gamma]$ *if and only if the normalization map* $C \to \Gamma$ *is not ramified.*

Remark 7.9. *If in the previous theorem we have that* $0 < k = deg(Z) < 3n$*, then the Severi variety* $V_{n,g}$ *is singular at* $[\Gamma]$ *and, from what we proved before, the map* n *is a desingularization of an analytic neighborhood* U *at* $[\Gamma]$ *in* $V_{n,g}$*.*

Proposition 7.8 has been generalized to the variety $\Sigma^n_{k,d}$ of irreducible plane curves of degree n with d nodes and k cusps by Diaz in [**12**]. But, before stating his result, we have to say something about the boundary of $\Sigma^n_{k,d}$. In section 3 of chapter 1 and in sections 2 and 3.1 of this chapter, we proved that the genus is a lower semicontinuous function on a flat family of curves and that d-nodal plane curves of degree n are general in the locally closed family $V^o_{n,g}$ of irreducible plane curves of degree n and geometric genus equal to $g = \binom{n-1}{2} - d$. Of course, the degree of the ramification divisor is an upper semicontinuous function on a flat family of morphisms to $\mathbb{P}^2$ on a smooth curve of genus g, and then the set $V^o_{n,g,k} \subset V^o_{n,g}$ of irreducible plane curves Γ of genus $g = \binom{n-1}{2} - d - k$, whose ramification divisor of the normalization map $C \to \Gamma$ has degree k, is locally closed in the Severi variety $V_{n,g}$. Under suitable hypotheses on k, reduced plane curves with d nodes and k cusps as singularities are general in the Zariski closure $V_{n,g,k}$ of $V^o_{n,g,k}$.

Theorem 7.10 (Diaz-Harris, Theorem (1.2) of [**13**].). *Suppose that* $k \leq 2n - 1$*, then the general element of every irreducible component of* $V^o_{n,g,k}$ *corresponds to an irreducible plane curve with* k *cusp and* $d = \binom{n-1}{2} - g - k$ *nodes as singularities, i.e.* $V_{n,g,k} = \Sigma^n_{k,d}$*.*

Notice that in [**13**] $V^o_{n,g,k}$ is defined as the locus of irreducible plane curves of degree n and genus g with class equal to $c = n(n-1) - \delta - 2k$, where $\delta = \binom{n-1}{2} - g$, and it is denoted by $V^{n,g,c}$. We recall that the class of a plane curve Γ is defined as the degree of the dual curve of Γ. For a proof of the Plücker formula

$$c = n(n-1) - \delta - 2k, \tag{40}$$

see [**13**]or [**10**]. By (40) we see that c remain constant if δ and k remain constant and hence the variety $V^{n,g,c}$ coincides with $V_{n,g,k}$. The Diaz and Harris result has been improved by Shustin in [**41**].

Theorem 7.11 (Shustin, Theorem 1.1 of [**41**].). *Suppose that* $k \leq 3n - 4$*, then the general element of every irreducible component of* $V^o_{n,g,k}$ *corresponds to an irreducible plane curve with* k *cusp and* $d = \binom{n-1}{2} - g - k$ *nodes as singularities, i.e.* $V_{n,g,k} = \Sigma^n_{k,d}$*.*

We can now state the following Diaz's result.

Theorem 7.12 (Diaz, [**12**].). *Suppose that $k < 3n$ and let $[\Gamma] \in V_{n,g,k}$ be a point corresponding to an irreducible plane curve of degree n and genus g such that the ramification divisor of the normalization map $\phi : C \to \Gamma$ has degree equal to k. Then, the variety $V_{n,g,k}$ is smooth at $[\Gamma]$ if and only if the local expression of ϕ at each ramification point is $\phi(z) = (z^r, z^{r+1})$, with $r \geq 2$.*

Outline of the proof. In order to prove the theorem, Diaz looks at the universal deformation space B of ϕ. If we denote by B_k the locus in B of morphisms with at least k ramification points and by $n : B \to V_{n,g}$ the natural map from B to $V_{n,g}$, then $n(B_k) \subset V_{n,g,k}$. In [**12**], Diaz writes explicitly the equations of the tangent space $T_0 B_k$ to B_k at the point $0 \in B_k$, corresponding to ϕ, proving that B_k is smooth at 0. Moreover, in [**12**] it is proved that, if ϕ has local expression $\phi(z) = (z^r, z^{r+1})$ at every its singular point, then $H^0(C, \mathcal{K}_\phi)$ intersects $T_0 B_k$ only at zero. On the contrary, if $p \in C$ is a ramification point such that the local expression is given by $\phi(z) = (z^k, z^r)$, with $r > k+1$ then $H^0(C, \mathcal{K}_{\phi,p}) \subset T_0 B_k$. The theorem follows from proposition 7.8. □

If V is an irreducible component of $V_{n,g,k}$ and $[\Gamma]$ is the general point of V, then theorems 7.10 and 7.11 are obtained by studying the tangent space to the locus of equisingular deformations of Γ and by proving that if Γ has singularities different from nodes and cusps, then the dimension of $V_{n,g,k}$ is smaller than the expected. We mean that, by using Diaz's results contained in [**12**] and working out the ideas and techniques used by Arbarello and Cornalba to prove theorem 2.2 (see theorem 3.1 of [**2**]), it is possible to get an alternative proof of the Shustin's result. Unfortunately, we don't know how to weaken the bound $k \leq 3n-4$ in theorem 7.11. Naturally, the hypothesis $k \leq 3n-4$ may be weakened under more restrictive hypotheses on the singularities of the plane curves which we want to prove to be limit of plane curves with nodes and cusps. In the following proposition, where we use the same techniques of 2.2, we consider the case of plane curves with nodes, cusps and triple points of analytic equation $(x-y)(y^2-x^3) = 0$ as singularities.

Proposition 7.13. *Let $\Gamma \subset \mathbb{P}^2$ be an irreducible and reduced plane curve of degree n of genus g with d nodes, k cusps and k′ triple points of analytic equation $(y-x)(y^2-x^3) = 0$ as singularities. If $k + k' < 3n-2$ the curve Γ' is limit of plane curves of degree n with $d + 2k'$ nodes and $k + k'$ cusps.*

Proof. Let $\phi : C \to \Gamma$ be the normalization of Γ. By the hypothesis about the singularities of Γ, the map ϕ has $k + k'$ simple ramification points and no further ramification points. Moreover, by the hypothesis $k + k' < 3n - 2 < 3n$, there exists an universal deformation family

$$\begin{array}{ccc} \mathcal{C} & \stackrel{\tilde{\phi}}{\to} & \mathbb{P}^2 \\ \pi \downarrow & & \\ B & & \end{array}$$

of ϕ and the universal deformation space B of ϕ which is smooth at the point 0 corresponding to ϕ. From what we proved before, the locus $B_{k+k'}$, parametrizing the morphisms with $k + k'$ ramification points, is smooth at 0 of the expected codimension equal to $k + k'$. In particular, every infinitesimal deformation of ϕ, corresponding to a tangent vector in $T_0(B_{k+k'})$, extends to an effective deformation of ϕ parametrized by a curve in $B_{k+k'}$, passing through 0. Moreover, by lemma 7.1, the tangent space to $T_0 B_{k+k'} \subset H^0(C, \mathcal{N}_\phi)$ to $B_{k+k'}$ at 0 intersects $H^0(C, \mathcal{K}_\phi)$ only at zero. Let now $p \in \Gamma$ be a triple point of Γ and let p_1, p_2 be the points of the normalization C of Γ which lie over p. Choose two disjoint holomorphic charts (U_1, h_1) and (U_2, h_2) of C biholomorphic to the open disc $\Delta = \{z \in \mathbb{C} \text{ t.c. } \mid z \mid < 1\}$, with $p_1 \in U_1$, $p_2 \in U_2$ and $h_1(p_1) = 0 = h_2(p_2)$. Suppose that ϕ maps U_1 to the singular branch of Γ through p. Denote by

$$f^i : \Delta \to \mathbb{C}^2$$

the holomorphic maps defined on Δ by $f^i(z) = \phi(h_i^{-1}(z))$, for $i = 1, 2$. Note that, from the hypothesis that $\phi(U_1)$ is singular at p, we have $\frac{\partial f^1}{\partial z}(0) = 0 \neq \frac{\partial f^2}{\partial z}(0)$. Now, let $s \in H^0(C, \mathcal{N}_\phi)$ be an infinitesimal deformation of ϕ such that $s(p_1) = 0$ and $s(p_2) \neq 0$. Let

$$\begin{array}{ccccc} \mathcal{C} \times_B E & \to & \mathcal{C} & \to & \mathbb{P}^2 \\ \downarrow & & \downarrow & & \\ E & \to & B & & \end{array}$$

be an effective deformation of ϕ parametrized by a curve $E \subset B$ passing trough the point 0 corresponding to ϕ. Suppose that it has Horikawa class equal to s at $t = 0$. We claim that the associated plane deformation of Γ is not equisingular at p. Let us suppose by contradiction that the previous statement is not true. Thus, assume that for every t there exists a point $z(t) \in \Delta$ such that there exist holomorphic maps $f_t^i(z) = f^i(t, z)$ such that $f_0^i(z) = f^i(z)$, for $i = 1, 2$, $\frac{\partial f_t^1}{\partial z}(z(t)) = 0 \neq \frac{\partial f_t^2}{\partial z}(z(t))$ and $f_t^1(z(t)) = f_t^2(z(t))$. Since $\frac{\partial f_t^2}{\partial z}(z(t)) \neq 0$, by using the Implicit Function theorem, we see that the function $t \to z(t)$ is holomorphic. Differentiating the last equality with respect to t, we find that

$$\frac{\partial f_t^1}{\partial t} \mid_t (z(t)) + \frac{\partial f_t^1}{\partial z} \mid_{z(t)} (\frac{\partial z(t)}{\partial t}) - \frac{\partial f_t^2}{\partial t} \mid_t (z(t)) - \frac{\partial f_t^2}{\partial z} \mid_{z(t)} (\frac{\partial z(t)}{\partial t}) = 0.$$

Recalling that $\frac{\partial f^1}{\partial z}(z(t)) = 0$ and $\frac{\partial f_t^1}{\partial t} \mid_{t=0} (0) = s(p_1) = 0$, evaluating the previous equality at $t = 0$, we find that

$$\frac{\partial f_t^2}{\partial t} \mid_{t=0} (0) + \frac{\partial f_0^2}{\partial z}(0) \frac{\partial z(t)}{\partial t} \mid_{t=0} = 0. \tag{41}$$

But, by construction $\frac{\partial f_t^2}{\partial t} \mid_{t=0} (0) = s(p_2) \neq 0$, while, by (41), s vanishes at p_2 as global section of the sheaf $\mathcal{N}_\phi = \phi^*(\Theta_{\mathbb{P}^2})/\phi_*(\Theta_C)$, because $\frac{\partial f_0^2}{\partial z}$ is the zero section of $\mathcal{N}_\phi$. This proves the claim. Now, via the isomorphism

$$T_0 B_{k+k'} \xrightarrow{\sim} H^0(C, \mathcal{N}'_\phi)$$

the vector space $T_0 B_{k+k'}(-p_1 - p_2)$ maps injectively to $H^0(C, \mathcal{N}'_\phi(-p_2 - p_1))$. On the other and, by the hypothesis $k + k' + 2 < 3n$, we find that $h^0(C', \mathcal{N}'_\phi(-p_2 - p_1)) = h^0(C', \mathcal{N}'_\phi) - 2$ and $T_0 B_{k+k'}(-p_1 - p_2) \backsimeq H^0(C, \mathcal{N}'_\phi(-p_2 - p_1))$. It follows that there exist a section $s \in T_0 B_{k+k'}$ such that $s(p_1) = 0$ and $s(p_2) \neq 0$. This section extends to an effective deformation of ϕ parametrized by a curve contained in $B_{k+k'}$. This proves that the general element of the image locus $V_{k+k'}$ of $B_{k+k'}$ in $V_{n,g}$ via the natural map

$$B \to V_{n,g},$$

is not a plane curve D with a triple point of analytic equation $(x - y)(x^3 - y^2) = 0$. Of course D as a node in a neighborhood of every node of Γ and a cusp in a neighborhood of every cusp and triple point of Γ. Finally, D could have a tacnode and a cusp at a neighborhood of a triple point of Γ. By arguing exactly as in [**2**], p. 97-98, and by using the hypothesis $3n - 2 > k + k'$, we can exclude this possibility and we may conclude that D has $k + k'$ cusps and $d + 2k'$ nodes. $\square$

CHAPTER 3

On the number of moduli of algebraic systems of plane curves with nodes and cusps

1. Definitions and known results

By using the notation introduced in section 2 and section 3.1 of chapter 2, we denote by $\Sigma^n_{k,d} \subset \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n))) := \mathbb{P}^N$, with $N = \frac{n(n+3)}{2}$, the closure, in the Zariski's topology, of the locally closed set of reduced and irreducible plane curves of degree n with k cusps and d nodes. Moreover, we still denote by $V_{n,g} = \Sigma^n_{0,d}$ the Severi variety of irreducible plane curves of degree n and geometric genus $g = \binom{n-1}{2} - d$. In this chapter we are interested in the number of moduli of complete irreducible families of reduced and irreducible plane curves with nodes and cusps. Let $\Sigma \subset \Sigma^n_{k,d}$ be an irreducible component of the variety $\Sigma^n_{k,d}$ of irreducible plane curves of degree n with d nodes and k cusps. We denote by Σ_0 the open set of Σ of points $[\Gamma] \in \Sigma$ such that Σ is smooth at $[\Gamma]$ and such that $[\Gamma]$ corresponds to a reduced and irreducible plane curve of degree n with d nodes, k cusps and no further singularities. Since the tautological family $\mathcal{S}_0 \to \Sigma_0$, parametrized by Σ_0, is an equigeneric family of curves, by normalizing the total space and by using the theorem 3.3 on page 14, we get a family

$$\begin{array}{ccc} \mathcal{S}'_0 & \to \mathcal{S}_0 & \subset \mathbb{P}^2 \times \Sigma_0 \\ \searrow & \swarrow & \\ & \Sigma_0 & \end{array}$$

of smooth curves of genus $g = \binom{n-1}{2} - t - d$. Because of the functorial properties of the moduli space $\mathcal{M}_g$ of smooth curves of genus g, we get a regular map $\Sigma_0 \to \mathcal{M}_g$, sending every point $[\Gamma] \in \Sigma_0$ to the isomorphism class of the normalization of the plane curve Γ corresponding to the point $[\Gamma]$. This map extends to a rational map

$$\Pi_\Sigma : \Sigma \dashrightarrow \mathcal{M}_g.$$

We set

$$\textit{number of moduli of } \Sigma := dim(\Pi_\Sigma(\Sigma)).$$

Notice that, when $\Sigma^n_{k,d}$ is reducible, two different irreducible components of $\Sigma^n_{k,d}$ can have different number of moduli. We say that Σ has *general moduli* if Π_Σ is dominant. Otherwise, we say that Σ has *special moduli.* In general, to calculate the number of moduli of these families of curves, is not simple. But, when $k < 3n$ and $g \geq 2$, we may always find an upper-bound for $dim(\Pi_\Sigma(\Sigma))$.

Definition 1.1. *When $0 \leq k < 3n$ and $g \geq 2$, we say that Σ has the expected number of moduli if*

$$dim(\Pi_\Sigma(\Sigma)) = min(dim(\mathcal{M}_g), dim(\mathcal{M}_g) + \rho - k),$$

where $\rho := \rho(2, g, n) = 3n - 2g - 6$ is the number of Brill-Noether of the linear series of degree n and dimension 2 on a smooth curve of genus g.

In order to understand the previous definition, we have to recall some elements of Brill-Noether theory. Given a smooth curve C of genus g, the set $G^2_n(C)$ of the linear series g^2_n on C of dimension 2 and degree n, is a projective variety which is not empty of dimension at least ρ, if $\rho(2,n,g) = 3n - 2g - 6 \geq 0$, (see theorem V.1.1 and proposition IV.4.1 of [**6**]). More precisely, let g^2_n be a given linear series, let $H \in g^2_n$ be a divisor and let $W \in H^0(C,H)$ be the three dimensional space corresponding to g^2_n. Denoting by K_C the canonical divisor of C, let

$$\mu_{o,C} : W \otimes H^0(C, K_C - H) \to H^0(C, K_C)$$

be the natural multiplication map, also called the Brill-Noether map of the pair (C,W). The dimension of the tangent space to $G^2_n(C)$ at the point $[g^2_n]$,corresponding to g^2_n, is equal to

$$dim(T_{[g^2_n]}G^2_n(C)) = \rho + dim(ker(\mu_{0,C})),$$

(see [**3**] or [**6**] for a proof). Moreover, if C is a curve with general moduli (i.e. if $[C]$ varies in an open set of $\mathcal{M}_g$), the variety $G^2_n(C)$ is empty if $\rho < 0$, it consists of a finite number of points if $\rho = 0$ and it is reduced, irreducible, smooth and not empty variety of dimension exactly ρ, when $\rho \geq 1$. In the latter case, by theorem 2.2 of chapter 2, the general g^2_n on C defines a local embedding on C and it maps C to $\mathbb{P}^2$ as a nodal curve. We deduce that, the Severi variety $\Sigma^n_{0,d} = V_{n,g}$ of irreducible plane curves of genus $g = \binom{n-1}{2} - d$, (which we recall to be not empty and irreducible for $0 \leq g \leq \binom{n-1}{2}$), has general moduli when $\rho \geq 1$ and has special moduli when $\rho < 0$. When $\rho \leq 0$, and then $g \geq 2$, we expect that the image of $V_{n,g}$ to $\mathcal{M}_g$ has codimension exactly $-\rho$. Equivalently, recalling that, in this case,

$$dim(V_{n,g}) = 3n + g - 1 = 3g - 3 + \rho + 8 = dim(\mathcal{M}_g) + \rho + dim(Aut(\mathbb{P}^2)),$$

we expect that on the smooth curve C, obtained normalizing the plane curve corresponding to the general element of $V_{n,g}$, there are only a finite number of g^2_n mapping C to the plane as a nodal curve. This is a well known result proved by Sernesi in [**37**].

Theorem 1.2 (Sernesi, [**37**]). *The Severi variety $V_{n,g}$ of irreducible plane curves of degree n and genus g has number of moduli equal to*

$$min(dim(\mathcal{M}_g), dim(\mathcal{M}_g) + \rho).$$

What happens when $k > 0$? By section 7 of chapter 2, we know that an ordinary cusp P of a plane curve Γ corresponds to a simple ramification point p of the normalization map $\phi : C \to \Gamma$, i.e. to a simple zero of the differential map $d\phi$. We denote by $G^2_{n,k}(C) \subset G^2_n(C)$ the set of g^2_n on C defining a birational morphism with k ramification points.

Lemma 1.3. *$G^2_{n,k}(C)$ is a locally closed subset of $G^2_n(C)$ and every irreducible component G of $G^2_{n,k}(C)$ has dimension at least equal to $\rho - k$, if it is not empty.*

Proof. First of all, we observe that linear series defining a birational morphism on C, are an open set of $G^2_n(C)$. Let $F^2_n(C)$ be the variety whose points correspond to the pairs $([g^2_n], \{s_0, s_1, s_2\})$ where $[g^2_n] \in G^2_n(C)$ and $\{s_0, s_1, s_2\}$ is a frame of the three dimensional space associate to the linear series g^2_n. Since all the fibres of the natural map $F^2_n(C) \to G^2_n(C)$ are isomorphic to $Aut(\mathbb{P}^2)$, we have that $dim(F^2_n(C)) \geq \rho + 8$. Let $p_1, \ldots, p_k$ be fixed points of C. Let F be the irreducible component of $F^2_n(C)$ mapping to G. We choose a local parameter z_i centered at p_i, for every $1 \leq i \leq k$. The points $p_1, \ldots, p_k$ are ramification points for a g^2_n on C

if, given a base $\{s_0, s_1, s_2\}$ of the three dimensional vector space associated to g^2_n, we have that $s_0(p_i) \neq 0$, $ord_{p_i} s_1 \leq ord_{p_i} s_2 - 1$ and

$$\begin{aligned} \frac{\partial s_1}{\partial z_i}(0) &= 0, \\ \frac{\partial s_2}{\partial z_i}(0) &= 0, \end{aligned}$$

for every $1 \leq i \leq k$. It follows that the set of g^2_n on C with k ramification points at $p_1, \ldots, p_k$ is locally closed of codimension $\leq 2k$ in $C^k \times F$, where C^k is the k-symmetric product of C. This proves the lemma. □

Now we have the tools to understand the definition 1.1. As before, let $\Sigma \subset \Sigma^n_{k,d}$ be an irreducible component, let $[\Gamma]$ be a smooth point of Σ corresponding to a plane curve $\Gamma \subset \mathbb{P}^2$ of genus $g \geq 2$ with d nodes and k cusps as singularities, and let $C \to \Gamma$ be the normalization curve of Γ. By considering the natural map $G^2_{n,k}(C) \times Aut(\mathbb{P}^2) \to \Sigma^n_{k,d}$ and using that the image of an irreducible variety is irreducible, we see that, if g^2_n is the linear series associated to the normalization map $C \to \Gamma$, then, every linear series lying in one of the irreducible components of $G^2_{n,k}(C)$ containing g^2_n, has to map C to a plane curve lying in Σ. It follows that *the dimension of the general fibre of the moduli of Σ is always at least equal to eight if $\rho - k \leq 0$ and it is at least equal to $\rho - k + 8$, if $\rho - k \geq 0$.* Furthermore, if we assume that $k < 3n$, then

$$dim(\Sigma) = 3n + g - 1 - k = 3g - 3 + 8 + \rho - k = dim(\mathcal{M}_g) + \rho - k + dim(Aut(\mathbb{P}^2)),$$

and, by definition, Σ has the expected number of moduli if the general fibre of the map of moduli of Σ has the expected dimension. Hence, *if $k < 3n$ then*

$$dim(\Pi_\Sigma(\Sigma)) \leq min(dim(\mathcal{M}_g), dim(\mathcal{M}_g) + \rho - k).$$

On the contrary, when $k \geq 3n$, we have not a bound for $dim(\Pi_\Sigma(\Sigma))$, since the dimension of the fibre of the moduli map of Σ is always at least equal to $\rho - k + 8$, but Σ may have dimension bigger than $3n + g - 1 - k$. However, from the following result, we may deduce that every not empty irreducible component of $\Sigma^n_{k,d}$ has special moduli if $k > 3n$.

Proposition 1.4 (Arbarello-Cornalba, [**2**]). *Let C be a general curve of genus $g \geq 2$ and $\phi : C \to \mathbb{P}^2$ be a birational morphism, then the ramification divisor degree of ϕ is smaller than ρ. In particular, every irreducible component of $\Sigma^n_{k,d}$ has special moduli if $\rho = 3n - 2g - 6 < k$.*

A sufficient condition for the existence of complete irreducible algebraic systems of plane curves with nodes and cusps with general moduli is given by the following result.

Proposition 1.5 (Kang, [**28**]). *$\Sigma^n_{k,d}$ is irreducible, not empty and with general moduli if $n > 2g - 1 + 2k$, where $g = \binom{n-1}{2} - d - k$.*

Actually, in [**28**], Kang proves that if $n > 2g-1+2k$, then $\Sigma^n_{k,d}$ is not empty and irreducible. But from his proof it follows that, under the hypothesis of proposition 1.5, $\Sigma^n_{k,d}$ has general moduli because the general element of $\Sigma^n_{k,d}$ corresponds to a curve which is a projection of an arbitrary smooth curve C of genus g in $\mathbb{P}^{n-g}$, from a general $(n-3)$-plane intersecting the tangent variety of C in k different points. Another result which may be used to find examples of families of plane curves with nodes and cusps having general moduli is the following.

Theorem 1.6 (Corollary 5.43 of [**21**]). *Let C be a general curve of genus g, let p be a general point on C and let $b = (b_0, \dots, b_r)$ be any ramification sequence. There exists a g^r_n on C having ramification at least b at p if and only if*

$$\sum_{i=0}^{r} (b_i + g - n + r)_+ \leq g,$$

where $(-)_+ := max(-, 0)$.

We recall that, if g^r_n is a linear series on C associated to a $(r+1)$-space $W \subset H^0(C, \mathcal{L})$, where $\mathcal{L}$ is an invertible sheaf on C, and if $\{s_0, \dots, s_r\}$ is a basis of W, then the ramification sequence of the g^r_n at p is the sequence $b = (b_0, \dots, b_r)$ where $b_i = ord_p s_i - i$. Choosing another basis of W, the ramification sequence of g^r_n at p doesn't change. We say that the ramification sequence of the g^r_n at p is at least equal to $b = (b_0, \dots, b_r)$ if $b_i \leq ord_p s_i - i$, for every i, and we write $(ord_p s_0, \dots, ord_p s_r - r) \geq (b_0, \dots, b_r)$. From theorem 1.6, we easily deduce the following result.

Corollary 1.7. *$\Sigma^n_{1,d}$ has general moduli if $\rho = 3n - 2g - 6 \geq 2$, where $g = \binom{n-1}{2} - 1 - d$.*

PROOF. It has been proved in [**29**] that $\Sigma^n_{1,d}$ is irreducible for every $d \leq \binom{n-1}{2} - 1$. Moreover, by using the terminology of theorem 1.6, the variety $\Sigma^n_{1,d}$ contains every point of $\mathbb{P}^N$ corresponding to a plane curve Γ of genus g such that the normalization morphism of Γ has at least a ramification point with ramification sequence $(b_0, b_1, b_2) \geq (0, 1, 1)$. Then, by theorem 1.6, if $\rho \geq 2$, then moduli map of $\Sigma^n_{1,d}$ is surjective. □

In this chapter we construct examples of complete irreducible families of plane curves with nodes and cusps with the expected number of moduli. Theorems 2.4 and 3.11 and technical results of sections 2 and 3 are obtained using and working out the main ideas and techniques that Sernesi uses in [**37**]. In theorem 2.4 we prove the existence of complete irreducible families of plane curves with nodes and cusps in sufficiently general position. In proposition 3.1 and corollary 3.7, we give sufficient conditions in order that a complete irreducible family of plane curves with nodes and cusps has the expected number of moduli, when $\rho \leq 0$. In corollary 3.9 we prove that, if an irreducible component Σ of $\Sigma^n_{k,d}$ verifies the conditions of proposition 3.1, then, for every $k' \leq k$ and $d' \leq d$ there exists an irreducible component Σ' of $\Sigma^n_{k',d'}$ containing Σ and verifying the conditions of proposition 3.1. In theorem 3.11, by using also theorem 2.4, we prove that if $k \leq 6$ and $\rho = 3n - 2g - 6 \leq 0$, where $g = \binom{n-1}{2} - k - d$, then there exists a non empty irreducible component Σ of $\Sigma^n_{k,d}$ verifying the conditions of proposition 3.1 and hence having the expected number of moduli. In theorem 3.13 we prove that if $\rho = 1$, then $\Sigma^n_{1,d}$ has general moduli. By theorem 3.11 and corollary 1.7, we deduce that $\Sigma^n_{1,d}$ has the expected number of moduli for every $d \leq \binom{n-1}{2} - 1$. Finally, lemma 4.2 proves that, under the less restrictive hypotheses $\rho - k \leq 0$ and $g \geq 2$, if Σ is an irreducible component of $\Sigma^n_{k,d}$ with expected number of moduli, then there exist irreducible components Σ' of $\Sigma^n_{k-1,d}$ and Σ'' of $\Sigma^n_{k,d-1}$, having the expected number of moduli and containing Σ. By using this lemma, we prove that $\Sigma^6_{6,0}$ has at least an irreducible component with expected number of moduli equal to seven and whose general element corresponds to a sextic with six cusps not on a conic. Finally, in corollary 4.7 we find that also the irreducible component of $\Sigma^6_{6,0}$ parametrizing sextics with six cusps on a conic has the expected number of moduli. We don't know examples of complete irreducible families of plane curves with nodes and cusps with number of moduli smaller than the expected.

2. On the existence of certain families of plane curves with nodes and cusps in sufficiently general position

It is well know that there exist values of n, d and k such that there are no irreducible plane curves of degree n with d nodes and k cusps, (see for instance remark 3.19 of chapter 2). As far as we know, the existence problem of $\Sigma^n_{k,d}$ is still open. We known that $\Sigma^n_{k,d}$ is irreducible and not empty for $k \leq 3$ and $k + d \leq \binom{n-1}{2}$, (see corollary 3.18 and theorem 3.21 of chapter 2). In this section we are interested in a little more specific existence problem. We shall prove the existence of plane curves with nodes and cusps as singularities whose singular points are in sufficiently general position to impose independent linear conditions to a linear system of plane curves of a given degree. Notice that, if we fix a set S of general points of the plane, it is not always true that there is an irreducible and reduced plane curve with nodes and cusps only at S. For example, it is classically known that the only sextic with nine nodes $P_1, ..., P_9$ in general position in the plane is the double cubic determined by $P_1, ..., P_9$, (see [**1**] and [**5**]). On the other hand, it has been proved in [**5**], that if $3d \leq \frac{n(n+3)}{2}$, $n \neq 6$ and $d \neq 9$, however we choose d points in general position in the plane, there exists an irreducible plane curve of degree n with nodes at these points and no further singularities.

Definition 2.1. *A projective curve $C \subset \mathbb{P}^r$ is said to be* geometrically t-normal *if the linear series cut out on the normalization curve $\tilde{C}$ of C by the pull-back to $\tilde{C}$ of the linear system of hypersurfaces of $\mathbb{P}^r$ of degree t is complete.*

From a geometric point of view, a projective curve $C \subset \mathbb{P}^r$ is geometrically t- normal if and only if the image $\nu_{t,r}(C)$ of C by the Veronese embedding $\nu_{t,r} : \mathbb{P}^r \to \mathbb{P}^{\binom{r+t}{t}}$ of degree t, is not a projection of a not degenerate curve living in a higher dimensional projective space. We shall say that a curve is *geometrically linearly normal* (g.l.n. for short) if it is geometrically 1-normal. Every such a curve C is not a projection of a curve lying in a projective space of bigger dimension.

Lemma 2.2. *Let $\Gamma \subset \mathbb{P}^2$ be an irreducible and reduced plane curve of degree n and genus g with at most nodes and cusps as singularities. Let t be an integer such that $n - 3 - t < 0$, then Γ is geometrically t-normal if and only if it is smooth. On the contrary, if $n - 3 - t \geq 0$, the plane curve Γ is geometrically t-normal if and only if its singular points impose independent linear conditions to plane curves of degree $n - 3 - t$.*

Proof. Let $\phi : C \to \Gamma$ be the normalization of Γ and let Δ be the adjoint divisor of ϕ. We denote by H the divisor on C which is the pullback of the divisor cut out on Γ by a general line. The plane curve Γ is geometrically t-normal if and only if, by definition,

$$h^0(C, \mathcal{O}_C(tH)) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(t)) - h^0(\mathbb{P}^2, \mathcal{I}_\Gamma(t))$$

where $\mathcal{I}_\Gamma$ is the ideal sheaf of Γ in $\mathbb{P}^2$. By Riemann-Roch theorem, Γ is geometrically t-normal if and only if

$$h^0(C, \mathcal{O}_C(K_C - tH)) = -nt + g - 1 + \frac{(t+1)(t+2)}{2} - h^0(\mathbb{P}^2, \mathcal{I}_\Gamma(t)), \tag{42}$$

where g is the geometric genus of Γ and K_C is the canonical divisor of C. On the other hand, from section 1 of chapter 1, we have that $H^0(C, \mathcal{O}_C(K_C - tH)) = H^0(C, \mathcal{O}_C((n-3-t)H)(-\Delta))$, where Δ is the adjoint divisor of ϕ. If $n - 3 - t < 0$ then $h^0(C, \mathcal{O}_C((n-3-t)H) = 0$ and Γ is geometrically t-normal if and only if

$$h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(t)) - h^0(\mathbb{P}^2, \mathcal{I}_\Gamma(t)) = nt - \frac{n^2-3n}{2} - \delta,$$

where $\delta = \binom{n-1}{2} - g = deg(\Delta)/2$. This equality is verified if and only if $\delta = 0$, i.e. Γ is smooth. If $n-3 \geq t$, $h^0(\mathbb{P}^2, \mathcal{I}_\Gamma(t)) = 0$ and (42) is verified if and only if

$$h^0(C, \mathcal{O}_C((n-3-t)H)(-\Delta)) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-3-t)) - \delta.$$

On the other hand, if $\tilde{\phi} : S \to \mathbb{P}^2$ is the blowing-up of the plane at the singular locus of Γ, then the strict transform of Γ with respect to $\tilde{\phi}$ coincides with the normalization curve C of Γ. Moreover, still denoting by H the pullback to S of the general line of $\mathbb{P}^2$ and denoting by $\sum_i E_i$ the pullback of the singular locus of Γ with respect to $\tilde{\phi}$, we have the following exact sequence

$$0 \to \mathcal{O}_S(-(3+t)H) \to \mathcal{O}_S(n-3-t)(-\sum_i E_i) \to \mathcal{O}_C(n-3-t)(-\Delta) \to 0$$

from which we deduce that

$$h^0(C, \mathcal{O}_C((n-3-t)H)(-\Delta)) = h^0(S, \mathcal{O}_S(n-3-t)(-\sum_i E_i)) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-3-t) \otimes A)$$

where A is the ideal sheaf of adjoint plane curves to Γ, i.e. the ideal sheaf of the plane curves passing through the singular points of Γ. □

Remark 2.3. *Notice that, if an irreducible and reduced plane curve Γ of degree n with only nodes and cusps as singularities is geometrically t-normal, with $t \leq n-3$, then it is geometrically r-normal for every $r \leq t$. Indeed, by denoting with A the ideal sheaf of adjoint plane curves to Γ, if we consider the following exact sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^2}(n-t-3) \otimes A \to \mathcal{O}_{\mathbb{P}^2}(n-t-3) \to \mathcal{O}_{\mathbb{P}^2}/A \to 0,$$

the plane curve Γ is geometrically t-normal if and only if the induced map

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-t-3)) \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}/A) = \mathbb{C}^{d+k},$$

where $k+d = \binom{n-1}{2} - g$, is surjective. Since $|\mathcal{O}_{\mathbb{P}^2}(n-r-3)| \supseteqq |\mathcal{O}_{\mathbb{P}^2}(n-t-3)|$ for every $r \leq t$, the valuation map

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-r-3)) \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}/A)$$

is surjective too, and hence Γ is geometrically r-normal for every $r \leq t$.

Theorem 2.4. *Let $\Sigma^n_{k,d}$ be the variety of irreducible and reduced plane curves of degree n with d nodes and k cusps. Suppose that d, k, n and t are such that*

$$d+k \leq \frac{n^2-(3+2t)n+2+t^2+3t}{2} = h^0(\mathcal{O}_{\mathbb{P}^2}(n-t-3)) \tag{43}$$

$$t \leq n-3 \text{ if } k=0, \tag{44}$$

$$k \leq min\big(6, h^0(\mathcal{O}_{\mathbb{P}^2}(n-t-3))\big) \text{ if } t = 1,\ 2 \text{ and} \tag{45}$$

$$k \leq min\big(6 + [\frac{n-8}{3}], h^0(\mathcal{O}_{\mathbb{P}^2}(n-6))\big) \text{ if } t = 3, \tag{46}$$

where $[-]$ is the integer part of $-$. Then the variety $\Sigma^n_{k,d}$ is not empty and there is at least one irreducible component $W \subset \Sigma^n_{k,d}$ whose general element parametrizes a geometrically t-normal plane curve.

Before proving the previous theorem, we want to consider the example of the algebraic system of reduced and irreducible plane curves $\Sigma^6_{6,0}$ of degree six with six cusps. By example 3.15 and remark 3.20 of chapter 2 we know that it contains at least two irreducible components Σ_1 and Σ_2. The general point of Σ_1 parametrizes a sextic with six cusps on a conic, whereas the general element of Σ_2 corresponds to a sextic with six cusps not on a conic. Note that, by the previous lemma the general element of Σ_2 parametrizes a geometric linearly normal sextic, unlike the general element of Σ_1. Then, theorem 2.4 could be useful to show the reducibility of some variety $\Sigma^n_{k,d}$, with $k \leq 6$, if one knows the existence of a component whose general element parametrizes a non geometrically t-normal curve, with $t \leq 3$.

In the case of $k = 0$ and $t = 1$, theorem 2.4 has been proved by Sernesi in [**37**], section 4. The case $k = 0$ and $t \leq n - 3$ is already contained in [**7**]. To show theorem 2.4, we proceed by induction on the degree n and on the number of nodes and cusps of the curve. The geometric idea at the base of the induction on the degree of the curve is, mutatis mutandis, the same as that of Sernesi.

Proof of the theorem 2.4. First of all, we observe that, if t is an arbitrary positive integer such that $n - 3 - t \geq 0$, and if $W \subset \Sigma^n_{k,d}$ is an irreducible component of $\Sigma^n_{k,d}$ such that there exists a point $[C] \in W$ which parametrizes a geometrically t-normal curve with only k cusps and d nodes as singularities, then this is true for the general element of W. In fact, under the hypotheses (43), (45) and (46), every component of $\Sigma^n_{k,d}$ has the expected dimension equal to $3n - k - 1 + g$, where $g = \binom{n-1}{2} - k - d$, and every point which parametrizes an irreducible curve with only k cusps and d nodes as singularities, is a smooth point for $\Sigma^n_{k,d}$, (see corollary 3.13 and corollary 3.23 of chapter 2). Then, let $\Delta \subset \Sigma^n_{k,d}$ be a general complete curve through $[C]$. Consider a local parametrization of Δ in $[C]$, which we will still denote by Δ. Taking the restriction to Δ of the tautological family

$$\{(P, [C]) \;\; \textit{such that} \;\; P \in C\} \subset \mathbb{P}^2 \times \Sigma^n_{k,d},$$

we obtain a family of irreducible plane curves with k cusps and d nodes

$$\phi : \mathcal{C} \to \Delta$$

parameterized by a smooth curve, whose special fibre is $\mathcal{C}_0 = C$. By theorem 3.3 of chapter 1, normalizing $\mathcal{C}$ we obtain a family of smooth curves

$$\tilde{\phi} : \tilde{\mathcal{C}} \to \Delta$$

of geometric genus g,

$$\begin{array}{ccc} \tilde{\mathcal{C}} & \rightarrow & \mathcal{C} \subset \mathbb{P}^2 \times \Delta \\ & \searrow \quad \swarrow & \\ & \Delta & \end{array}$$

whose fibres are the normalizations the curves of family $\mathcal{C}$. If H_z is the pullback to $\tilde{\mathcal{C}}_z := \tilde{\phi}^{-1}(z)$ of a general line $H \subset \mathbb{P}^2$, by using semicontinuity theorem, we have

$$\begin{aligned} h^0(\tilde{\mathcal{C}}_z, \mathcal{O}_{\tilde{\mathcal{C}}_z}(tH_z)) &\leq h^0(\tilde{\mathcal{C}}_0, \mathcal{O}_{\tilde{\mathcal{C}}_0}(tH_0)) \\ &= h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(t)). \end{aligned}$$

Then $\mathcal{C}_z := \phi^{-1}(z)$ is geometrically t-normal. Similarly, if the theorem is true for fixed n, $t \leq n-3$, k as in (45) or in (46) and $k+d$ as in (43), then the theorem is true for n, t and any $k' \leq k$ and $d' \leq d+k-k'$. Indeed, by the proof of lemma 3.17 of the section 2, if $k \leq 6 < 3n$, for any subset S of $k' \leq k$ cusps of C, there exists a family of plane curves

$$\begin{array}{lcl} \mathcal{C} & \subset & \mathbb{P}^2 \times \Sigma^n_{k',d'} \\ \phi\downarrow & & \\ \Delta & \subset & \Sigma^n_{k',d'} \end{array}$$

of degree n and geometric genus g, parametrized by a complete curve $\Delta \subset \Sigma^n_{k',d'}$, with special fibre $\mathcal{C}_0 = \Gamma$ and whose general fibre $\mathcal{C}_z$ with a cusp in a neighborhood of any cusp of Γ in S, and at most a node in a neighborhood of the other singular points of Γ. The curve Δ may be not regular at $[C]$. But this is not a problem. In fact, normalizing Δ and considering the pull-back of $\mathcal{C}$ over the obtained smooth curve $\tilde{\Delta}$, and still normalizing this family, we obtain a family of curves

$$\tilde{\phi} : \tilde{\mathcal{C}} \to \Delta$$

whose general fibre is the normalization of the general fibre of $\mathcal{C}$, and whose special fibre $\tilde{C}_0$ is a partial normalization of the original curve C. By the geometric t-normality of C, with obvious notation, we have that $h^0(\tilde{\mathcal{C}}_0, \mathcal{O}_{\tilde{\mathcal{C}}_0}(t)) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(t))$. Applying the semicontinuity theorem to $\tilde{\mathcal{C}}$, we have that the general fibre of $\mathcal{C}$ is geometrically t-normal. Finally, it's enough to show the theorem when the equality holds in (45), (46) and (43).

First of all we consider the case $k = 0$. We will show the statement for any fixed t and by induction on n. Let, then $t \geq 1$ and $n = t+3$. In this case the equality holds in (43) if $d = 1 = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})$. Since one point imposes independent linear conditions to regular functions, by using lemma 2.2, we find that every irreducible plane curve of degree $n = t+3$ with one node and no more singularities is geometrically t-normal. So, the first step of the induction is proved. Suppose, now, the theorem is true for $n = t+3+a$ and let $[\Gamma] \in V_{n,g}$ be a point corresponding to a geometrically t-normal curve with $\frac{a^2+3a+2}{2}$ nodes. Let D be a line which intersects transversally Γ and let $P_1, ..., P_{t+1}$ be $t+1$ marked points of $\Gamma \cap D$. If $\Gamma' = \Gamma \cup D \subset \mathbb{P}^2$, then $P_1, ..., P_{t+1}$ are nodes for Γ'. Let $C \to \Gamma$ be the normalization of Γ and $C' \to \Gamma'$ the partial normalization of Γ', obtained by smoothing all singular points of Γ', except $P_1, ..., P_{t+1}$. We have the following exact sequence of sheaves on C',

$$0 \to \mathcal{O}_D(tH)(-P_1 - ... - P_{t+1}) \to \mathcal{O}_{C'}(tH) \to \mathcal{O}_C(tH) \to 0, \tag{47}$$

where H is the pullback by $C' \to \Gamma'$ of the generic line in $\mathbb{P}^2$. Since

$$deg(\mathcal{O}_D(tH)(-P_1 - ... - P_{t+1})) < 0,$$

we get that

$$h^0(D, \mathcal{O}_D(tH)(-P_1 - ... - P_{t+1})) = 0$$

and so

$$\begin{aligned} h^0(C', \mathcal{O}_{C'}(tH)) &\leq h^0(C, \mathcal{O}_C(tH)) \\ &= h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(tH)). \end{aligned}$$

But $h^0(C', \mathcal{O}_{C'}(tH)) \geq h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(tH))$ and hence $h^0(C', \mathcal{O}_{C'}(tH)) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(tH))$. Now, from the lemma 3.17 of chapter 2, we can obtain Γ' as the limit of a family of irreducible plane curves

$$\psi : \mathcal{C} \to \Delta$$

of degree $n+1=t+a+4$ with

$$\frac{a^2+3a+2}{2}+n-t-1=\frac{(a+1)^2+3(a+1)+2}{2}=h^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(n+1-t-3))$$

nodes specializing to nodes of Γ' different from the marked points $P_1,...,P_{t+1}$. Normalizing $\mathcal{C}$, we obtain a family whose special fibre is exactly C', and we conclude the induction on the degree via semicontinuity as before.

Now we consider the case $t=1,2$ or 3 and k as in (45) and in (46). Suppose the theorem is true for n and let $[\Gamma]\in\Sigma^n_{k,d}$ be a general point in one of the irreducible components of $\Sigma^n_{k,d}$. Then, let D be a smooth plane curve of degree t if $t=1,2$ or an irreducible cubic with a cusp if $t=3$. By the generality of Γ, we may suppose that D transversally intersects Γ. Let $P_1,...,P_{t^2+1}$ be t^2+1 fixed points of $\Gamma\cap D$. If $\Gamma'=\Gamma\cup D$, then $P_1,...,P_{t^2+1}$ are nodes for Γ'. Let $C\to\Gamma$ be the normalization of Γ and $C'\to\Gamma'$ the partial normalization of Γ', obtained by smoothing all singular points except $P_1,...,P_{t^2+1}$. We have the following exact sequence of sheaves on C',

$$0\to\mathcal{O}_D(tH)(-P_1-...-P_{t^2+1})\to\mathcal{O}_{C'}(tH)\to\mathcal{O}_C(tH)\to 0,$$

where tH is the pullback by $C'\to\Gamma'$ of the general line in $\mathbb{P}^2$. Since

$$deg(\mathcal{O}_D(tH)(-P_1-...-P_{t^2+1}))<0,$$

we have that

$$h^0(\mathcal{O}_D(tH)(-P_1-...-P_{t^2+1}))=0$$

and so

$$\begin{aligned}h^0(C',\mathcal{O}_{C'}(tH)) &\leq h^0(C,\mathcal{O}_C(tH))\\ &= h^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(tH)).\end{aligned}$$

But, $h^0(C',\mathcal{O}_{C'}(tH))\geq h^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(tH))$. So, $h^0(C',\mathcal{O}_{C'}(tH))=h^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(tH)$. Now, from what we showed in section 2, we have $\Gamma'\in\Sigma^{n+t}_{k+\frac{t^2-3t+2}{2},d+nt-t^2-1}$. In particular, we can obtain Γ' as limit of a family of irreducible plane curves

$$\phi:\mathcal{C}\to\Delta$$

of degree $n+t$ with $d+nt-t^2-1=\frac{(n+t)^2+(3+2t)(n+t)+t^2+3t+2}{2}$ nodes specializing to nodes of Γ' different to $P_1,...,P_{t^2+1}$, and $k+\frac{t^2-3t+2}{2}$ cusps specializing to cusps of Γ. Normalizing $\mathcal{C}$, we obtain a family whose special fibre is exactly C', and we conclude the inductive step by semicontinuity theorem, as before. Now we have to show the first step of the induction. For $t=1$ the induction begins with the cases $(n,k)=(4,1)$, $(5,3)$, $(6,6)$. Trivially, if $n=4$ and $k=1$ one point imposes independent conditions to the linear system of regular functions. If $n=5$ and $k=3$ we have to show that there are irreducible quintics with three cusps not on a line. We already observed that there exist quintics with three cusps as singularities, see corollary 3.18 of chapter 2. By Bezout theorem the three cusps of such a plane curve can't lie on a line. If $n=k=6$, we proved that there exist sextics with six cusps which do not lie on a conic, see example 3.20 of chapter 2. For $t=2$ we have to show the theorem for $(n,k)=(5,1)$, $(6,3)$, $(7,6)$, $(8,6)$. If $n=5$, then $k=1$ as before. When $n=6$ and $k=3$ we have that $n-3-t=1$. Now, three points impose independent linear conditions to the lines if and only if they are not collinear. To show that there exists an irreducible sextic with three cusps not on a line, consider a rational quartic C_4 with three cusps, (see corollary 3.18 of chapter 2 for the existence). By Bezout theorem, the three double points of C_4 can't be aligned. Then consider a sextic C_6 which is union of C_4 and a conic C_2 which intersects C_4 transversally. By lemma 3.17 of chapter 2, there exist a one-parameter family $\mathcal{G} \to \Delta$ of sextics, whose special fibre is equal to C_6 and whose general fibre $\mathcal{G}_t$ is an irreducible sextic with three cusps at a neighborhood of the cusps of C_4 and no further singularities. The three cusps of $\mathcal{G}_t$ are not aligned. For $n = 7$ and $k = 6$ we argue in the previous case, by using a sextic C_6 with six cusps not on a conic and a line R with intersects C_6 transversally. Similarly for $n = 8$ and $k = 6$. For $t = 3$ we have to show the theorem for $(n, k) = (6, 1)$, $(7, 3)$, $(8, 6)$, $(9, 6)$, $(10, 6)$. If $n = 6$ and $k = 1$ we argue as in the previous cases. If $n = 7$ and $k = 3$, we have to show the existence of an irreducible plane curve of degree 7 with three cusps not on a line. We can obtain such a plane curve by a reducible septic which is union of a geometrically linearly normal quintic with three cusps and an irreducible conic intersecting transversally. If $n = 8$ and $k = 6$, we have to show the existence of an irreducible plane curve of degree 8 with six cusps not on a conic. We can construct this curve as before by using a geometrically linearly normal sextic and a conic intersecting transversally. The cases $(n, k) = (9, 6)$, $(10, 6)$ can be showed similarly. □

Remark 2.5. *We observe that a result like theorem 2.4 can be found for every fixed* t*. But we are not able to find a result which works for every* $t \leq n-3$ *when* $k > 0$*. Moreover we notice that the inequality (43) of the previous theorem can't be improved. Indeed, if* $g = \binom{n-1}{2} - k - d$*, then* $k + d > h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-3-t))$ *if and only if* $g < \frac{2tn-t^2-3t}{2}$*. On the other hand, by using the same notation of theorem* (2.4)*, if* $g < \frac{2tn-t^2-3t}{2}$*, then, by Riemann-Roch theorem, we have that* $h^0(C, \mathcal{O}_C(t)) \geq tn - g + 1 > \frac{t^2+3t}{2} + 1 = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(t))$*. On the contrary, the inequalities (45) and* (46) *are not sharp. To see this, we can consider the example of curves of degree ten. Theorem 2.4 ensure the existence of geometrically linearly normal irreducible plane curves of degree ten with* $k \leq 6$ *cusps and at most nodes as other singularities. But by using the same ideas of theorem 2.4 it is simple to prove the existence of geometrically linearly normal plane curves of degree ten with nodes and* $k \leq 9$ *cusps. It is enough to consider a sextic* Γ_6 *with six cusp not on a conic and a rational quartic* Γ_4 *with three cusps intersecting* Γ_6 *transversally. We choose five points* $P_1, \ldots, P_5$ *of* $\Gamma_4 \cap \Gamma_6$*. If* Γ_6' *and* Γ_4' *are the normalization curves of* Γ_6 *and* Γ_4 *respectively and* C' *is the partial normalization of* $\Gamma_6 \cup \Gamma_4$ *obtained by normalizing all its singular points except* $P_1, \ldots, P_5$*, by considering the following exact sequence*

$$0 \to \mathcal{O}_{\Gamma_4'}(1)(-P_1 - \cdots - P_5) \to \mathcal{O}_{C'}(1) \to \mathcal{O}_{\Gamma_6'}(1) \to 0$$

we find that $h^0(C', \mathcal{O}_{C'}(1)) = 3$*. The statement follows by the lemma 3.17 on page 31 and by semicontinuity theorem, as in the proof of theorem 2.4. The bound on the number of cusps of theorem 2.4 can be improved also for* $t = 2$ *or* $t = 3$*. For example, theorem 2.4 ensure the existence of geometrically three normal curve of degree* 12 *with* $k \leq 6$ *and nodes as further singularities. But, by considering a geometrically three normal curve of degree* 8 *with six cusps and a quartic with three cusps and arguing as before, we can find geometrically three normal irreducible plane curve of degree twelve with nodes and* $k \leq 9$ *cusps.*

3. Examples of families with expected number of moduli

First of all we find sufficient conditions for the existence of complete families of plane curves of degree n with d nodes and k cusps with the right number of moduli.

Proposition 3.1. *Let* $\Sigma \subset \Sigma^n_{k,d}$*, with* $0 \leq k < 3n$*, be an irreducible component of* $\Sigma^n_{k,d}$ *and let* $[\Gamma] \in \Sigma$ *be a general element, corresponding to a plane curve* Γ *with normalization map*

$\phi : C \to \Gamma$. Let $H \subset \Gamma$ be the divisor cut out on Γ from the general line of $\mathbb{P}^2$ and K_C the canonical divisor of C. Suppose that:

(1) *Γ is geometrically linearly normal, i.e. $h^0(C, \phi^*(H)) = 3$,*

(2) *the Brill-Noether map*

$$\mu_{o,C} : H^0(C, \phi^*(H)) \otimes H^0(C, K_C - \phi^*(H)) \to H^0(C, K_C)$$

of the pair (C,H), is surjective.

Then Σ has the expected number of moduli equal to $3g - 3 + \rho - k$.

Proof. Let Γ be a plane curve which verifies (1) and (2). We recall that, by the hypothesis $k < 3n$, we have that Σ is smooth at $[\Gamma]$ of dimension $3n+g-1-k$. Now, consider the following exact sequence of sheaves on C

$$0 \to \Theta_C \xrightarrow{\phi_*} \phi^*\Theta_{\mathbb{P}^2} \to \mathcal{N}_\phi \to 0 \tag{48}$$

where Θ_C and $\Theta_{\mathbb{P}^2}$ are respectively the tangent sheaf on C and on $\mathbb{P}^2$,

$$\phi_* : \Theta_C \to \phi^*\Theta_{\mathbb{P}^2}$$

is the differential map associated to ϕ and $\mathcal{N}_\phi$ is the normal sheaf to ϕ. Since $k < 3n$, we have that $H^1(C, \mathcal{N}_\phi) = 0$ and we deduce the following long exact sequence

$$0 \to H^0(C, \Theta_C) \to H^0(C, \phi^*\Theta_{\mathbb{P}^2}) \to H^0(C, \mathcal{N}_\phi) \xrightarrow{\delta_C} H^1(C, \Theta_C) \to H^1(C, \phi^*\Theta_{\mathbb{P}^2}) \to 0$$

By theorem 4.3 of chapter 1, the vanishing $H^1(C, \mathcal{N}_\phi) = 0$ is a sufficient condition for the existence of a universal deformation family

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\tilde{\phi}} & \mathbb{P}^2 \\ \pi \downarrow & & \\ B & & \end{array}$$

of the normalization map ϕ. Moreover, denoting by $V_{n,g} = \Sigma^n_{0,d+k}$ the Severi variety of irreducible plane curves of degree n and genus $g = \binom{n-1}{2} - d - k$, it is naturally defined a $1:1$ map

$$n : B \to V_{n,g}$$

sending every point $x \in B$ to the point of $V_{n,g}$ corresponding to the image curve of the morphism $\tilde{\phi}|_{\pi^{-1}(x)}$. By Horikawa deformation theory, the tangent space to B at the point 0, corresponding to the normalization map $\phi : C \to \Gamma$, is naturally identified with $H^0(C, \mathcal{N}_\phi)$. In particular B is smooth at 0. Moreover, if we denote by $B_k = n^{-1}(\Sigma)$ the locus of points of B corresponding to a morphism with k ramification points, then, by lemmas 7.1 and 7.3 of chapter 2, under the isomorphism $T_0B \simeq H^0(C, \mathcal{N}_\phi)$, the tangent space to $n^{-1}(\Sigma)$ at 0 is identified with a subspace W of $H^0(C, \mathcal{N}_\phi)$ of codimension k such that $W \cap H^0(C, \mathcal{K}_\phi) = 0$, where $\mathcal{K}_\phi$ is the torsion subsheaf of $\mathcal{N}_\phi$. By proposition 7.8 of chapter 2, we deduce that the Severi variety $V_{n,g}$ is singular at $[\Gamma]$, the universal deformation space B is a desingularization of $V_{n,g}$ at $[\Gamma]$ and, finally, the differential map $dn : T_0B \to T_0V_{n,g}$ induces an isomorphism between W and the tangent space $T_{[\Gamma]}\Sigma$ to Σ at $[\Gamma]$. Going back to the number of moduli of Σ, we recall that the space $H^1(C, \Theta_C)$ is canonically identified with the tangent space $T_{[C]}\mathcal{M}_g$ to $\mathcal{M}_g$ at the point associated to the normalization C of Γ. The coboundary map $\delta_C : H^0(C, \mathcal{N}_\phi) \to H^1(C, \Theta_C)$

maps the Horikawa class of an infinitesimal deformation of ϕ to the Kodaira- Spencer class of the corresponding infinitesimal deformation of C. So, $\delta_C|W$ is the differential map at the point $0 \in B$ of the moduli map $\Pi_\Sigma n : B_k = n^{-1}(\Sigma) \dashrightarrow \mathcal{M}_g$. Since the point $[\Gamma]$ is general in Σ, and recalling the isomorphism $dn : W \stackrel{\sim}{\to} T_{[\Gamma]}\Sigma$, the

$$\textit{number of moduli of } \Sigma = dim(\delta_C(W)).$$

Now, from the exact sequence (48), we have that

$$dim(\delta_C(H^0(C, \mathcal{N}_\phi)) = 3g - 3 - h^1(C, \phi^*\Theta_{\mathbb{P}^2}).$$

Moreover, from the pull-back to C of the Euler exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(\phi^*(H)) \otimes (H^0(C, \phi^*(H)))^* \to \phi^*\Theta_{\mathbb{P}^2} \to 0 \tag{49}$$

we get a map

$$\cdots \to H^1(C, \mathcal{O}_C) \stackrel{\mu^*_{o,C}}{\to} H^1(C, \phi^*(H)) \otimes (H^0(C, \phi^*(H)))^* \to H^1(C, \phi^*\Theta_{\mathbb{P}^2}) \to 0$$

which is the dual of the Brill-Noether map. In particular, we find an isomorphism

$$H^1(C, \phi^*\Theta_{\mathbb{P}^2}) \simeq coker(\mu^*_{0,C}) \simeq (ker(\mu_{0,C}))^*$$

and we conclude that

$$dim(\delta_C(H^0(C, \mathcal{N}_\phi))) = 3g - 3 - dim(ker(\mu_{0,C})). \tag{50}$$

Notice that the previous equality is always true, also if Γ doesn't verify (1) or (2) of the statement. Moreover, if Γ is geometrically linearly normal, i.e. $h^0(C, \phi^*(H)) = 3$, we have that

$$\rho = 3n - 2g - 6 = dim(coker(\mu_{o,C})) - dim(ker(\mu_{o,C})).$$

When $\mu_{o,C}$ is surjective, $\rho = -dim(ker(\mu_{o,C}))$ and

$$dim(\delta_C(H^0(C, \mathcal{N}_\phi)) = 3g - 3 + \rho = dim(B) - 8 = dim(V_{n,g}) - 8. \tag{51}$$

Since the dimension of the fibre of the moduli map

$$\Pi_\Sigma \circ n : B \to \mathcal{M}_g$$

has dimension at least equal to $8 = dim(Aut(\mathbb{P}^2))$, from (51) we deduce that the differential map of $\Pi_\Sigma \circ n$ has maximal rank at 0. It follows that B is mapped to $\mathcal{M}_g$ with general fibre of dimension exactly equal to 8. In particular, $dim((\Pi_\Sigma \circ n)^{-1}([C])) = 8$. Equivalently, there exist only finitely many g^2_n on C. It follows that there are only finitely many g^2_n on C mapping C to the plane as a curve with k cusps and d nodes. Equivalently,

$$dim(\delta_c(W)) = dim(\Pi_\Sigma(\Sigma)) = 3g - 3 + \rho - k.$$

□

Remark 3.2. *Arguing as in the proof of the previous proposition, it has been proved in* [**37**] *that, if Γ is a geometrically linearly normal plane curve with only d nodes as singularities and the Brill-Noether map $\mu_{o,C}$ of the normalization morphism of Γ is injective, then $\Sigma = \Sigma^n_{0,d}$ has general moduli. When Γ is an irreducible plane curve, with nodes and $0 < k < 3n$ cusps as singularities, verifying the hypothesis* (1) *of the proposition 3.1 but such that $\mu_{o,C}$ is injective, we may only conclude that $\Pi_{V_{n,g}} \circ n$ is dominant with surjective differential map at* $[\Gamma]$. *So*

$dim(\Pi^{-1}_{V_{n,g}}([C])) = \rho+8$. But this is not useful to compute the dimension of $\delta_C(W) = \delta_C(T_{[\Gamma]}\Sigma)$. However, in this case we get that

$$\delta_C(T_{[\Gamma]}\Sigma) + \delta_C(H^0(C, \mathcal{K}_\phi)) = \delta_C(H^0(C, \mathcal{N}_\phi)) = H^1(C, \Theta_C).$$

Then, by using that $dim(\delta_C(H^0(C, \mathcal{K}_\phi))) \leq k$ and by recalling that if $k < 3n$ the number of moduli of $\Sigma^n_{k,d}$ is at most the expected one (see section 1), we find that

$$3g - 3 - k \leq \textit{number of moduli of } \Sigma \leq 3g - 3 + \rho - k.$$

Remark 3.3. *Notice that, if a plane curve Γ of genus g verifies the hypotheses* (1) *and* (2) *of the previous proposition, then the Brill-Noether number $\rho(2, g, n)$ is not positive and, in particular, $g \geq 2$. We don't know examples of complete irreducible families $\Sigma \subset \Sigma^n_{k,d}$ with the expected number of moduli whose general element $[\Gamma]$ corresponds to a curve Γ of genus g, with $\rho(2, g, n) \leq 0$, which doesn't verify properties* (1) *and* (2). *But we mean that, also under the hypothesis $\rho \leq 0$, the properties* (1) *and* (2) *are not necessary conditions in order that Σ has the expected number of moduli. Indeed, if $h^0(C, \mathcal{O}_C(1)) = 3$, and Σ has the expected number of moduli, then, it may happen the $dim(G^2_n(C)) = dim(coker(\mu_{o,C})) > 0$, but on C there are only a finite number of g^2_n mapping C to the plane as a curve with d nodes and k cusps.*

Definition 3.4. *A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r$ is said to be m-*regular *if and only if*

$$h^i(\mathbb{P}^r, \mathcal{F}(m - i)) = 0, \textit{ for any } i > 0.$$

Proposition 3.5 (Castelnuovo, Mumford). *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^r$. Then, if $\mathcal{F}$ is m-regular, we have that:*
1) the maps

$$H^0(\mathbb{P}^r, \mathcal{F}(l - 1)) \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \to H^0(\mathbb{P}^r, \mathcal{F}(l))$$

are surjective for any $l > m$; and
2) $H^i(\mathbb{P}^r, \mathcal{F}(l)) = 0$, for $i > 0$, $l + i \geq m$.

Lemma 3.6 ([**7**], Corollary 3.4). *Let Γ be an irreducible plane curve of degree n with only nodes and cusps as singularities and let $\phi : C \to \Gamma$ be the normalization morphism of Γ. Suppose that Γ is geometrically* 2*-normal, i.e. $h^0(C, \mathcal{O}_C(2)) = 6$, where $\mathcal{O}_C(1)$ is the sheaf on C associated to the pull-back of the hyperplane section of Γ. Then the Brill-Noether map*

$$\mu_{o,C} : H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, \omega_C(-1)) \to H^0(C, \omega_C)$$

is surjective, where ω_C is the canonical sheaf of C.

Proof. By lemma 2.2, the curve Γ is geometrically 2-normal if and only if the scheme N of the singular points of Γ imposes independent linear conditions to the linear system $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n - 5))$ of plane curves of degree $n - 5$, i.e. the evaluation map

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n - 5)) \to \mathbb{C}^{d+k}$$

is surjective. Since $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n - 5)) \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n - 4))$, also the evaluation map

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n - 4)) \to \mathbb{C}^{d+k}$$

is surjective, and, still using lemma 2.2, we get that $h^0(C, \mathcal{O}_C(1)) = 3$, i.e. Γ is geometrically linearly normal. Now, denote by $\mathcal{I}_{N|\mathbb{P}^2}$ the ideal sheaf of N. By using the terminology introduced in the definition 3.4, we have that the curve Γ is geometrically 2-normal if and only if the ideal sheaf $\mathcal{I}_{N|\mathbb{P}^2}(n - 4)$ is 0-regular. Indeed, since $h^2(\mathbb{P}^2, \mathcal{I}_{N|\mathbb{P}^2}(n - 6)) = 0$, the ideal sheaf

$\mathcal{I}_{N|\mathbb{P}^2}(n-4)$ is 0-regular if and only if $h^1(\mathbb{P}^2, \mathcal{I}_{N|\mathbb{P}^2}(n-5)) = 0$. On the other hand, by the standard exact sequence

$$0 \to \mathcal{I}_{N|\mathbb{P}^2}(n-5) \to \mathcal{O}_{\mathbb{P}^2}(n-5) \to \mathcal{O}_N \to 0$$

we have that $h^1(\mathbb{P}^2, \mathcal{I}_{N|\mathbb{P}^2}(n-5)) = 0$ if and only if Γ is geometrically 2-normal. Therefore $\mathcal{I}_{N|\mathbb{P}^2}(n-4)$ is 0-regular and, by setting $l = 1$ and $r = 2$ in the proposition 3.5, we deduce that the natural map

$$H^0(\mathbb{P}^2, \mathcal{I}_{N|\mathbb{P}^2}(n-4)) \otimes H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(\mathbb{P}^2, \mathcal{I}_{N|\mathbb{P}^2}(n-3))$$

is surjective. On the other hand, by section 1 of chapter 1, there is the following commutative diagram

$$\begin{array}{ccccc} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) & \otimes & H^0(\mathbb{P}^2, \mathcal{I}_{N|\mathbb{P}^2}(n-4)) & \to & H^0(\mathbb{P}^2, \mathcal{I}_{N|\mathbb{P}^2}(n-3)) \\ \downarrow & & \downarrow & & \downarrow \\ H^0(C, \mathcal{O}_C(1)) & \otimes & H^0(C, \omega_C(-1)) & \stackrel{\mu_{o,C}}{\to} & H^0(C, \omega_C) \end{array}$$

where the left-hand vertical arrow and the middle vertical arrow are surjective. Since Γ is geometrically linearly normal, the left-hand vertical arrow is bijective and hence the Brill-Noether map $\mu_{o,C}$ is surjective too. $\square$

Corollary 3.7. *Let $\Sigma \subset \Sigma^n_{k,d}$, with $0 \le k < 3n$, be an irreducible component of $\Sigma^n_{k,d}$, such that the general point $[\Gamma] \in \Sigma$ corresponds to a geometrically 2-normal plane curve. Then Σ has the expected number of moduli equal to $3g - 3 + \rho - k$.*

Proof. It follows from proposition 3.1 and lemma 3.6. $\square$

In order to produce examples of families of plane curves with nodes and cusps with the expected number of moduli, we study how increases the rank of the Brill-Noether map by smoothing a node or a cusp of the general curve of the family.

Let $\Sigma \subset \Sigma^n_{k,d}$, with $n \ge 5$, be an irreducible component of $\Sigma^n_{k,d}$, let $[\Gamma] \in \Sigma$ be a general point of Σ and let $\phi : C \to \Gamma$ be the normalization of Γ. Consider the multiplication map

$$\mu_{o,C} : H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, \omega_C(-1)) \to H^0(C, \omega_C),$$

where $\mathcal{O}_C(1)$ is the sheaf on C associated to the pull-back of the linear series cut out on Γ by the lines of $\mathbb{P}^2$, and ω_C is the canonical sheaf of C. Choose a singular point $P \in \Gamma$ and denote by $\phi' : C' \to \Gamma$ the partial normalization of Γ obtained by smoothing all singular points of Γ, except the point P. If $\omega_{C'}$ is the dualizing sheaf of C' and

$$\mu_{o,C'} : H^0(C', \mathcal{O}_{C'}(1)) \otimes H^0(C', \omega_{C'}(-1)) \to H^0(C', \omega_{C'}),$$

is the natural multiplication map, we have the following result.

Lemma 3.8. *If $h^0(C, \mathcal{O}_C(1)) = 3$, and the geometric genus g of C is such that $g > n-2$, with $n \ge 5$, then $rk(\mu_{o,C'}) \ge rk(\mu_{o,C}) + 1$. In particular, if $h^0(C, \mathcal{O}_C(1)) = 3$, $n \ge 5$ and $\mu_{o,C}$ is surjective, then also $\mu_{o,C'}$ is surjective.*

Proof of lemma 3.8. Let $\psi : C \to C'$ be the normalization map.

$$\begin{array}{ccc} C & \stackrel{\psi}{\to} & C' \\ \phi \searrow & & \swarrow \phi' \\ & \Gamma & \end{array}$$

We recall that, if we set $\phi^*(P) := p_1 + p_2$ when P is a node and $\phi^*(P) = 2\phi^{-1}(P)$ when P is a cusp, then the dualizing sheaf of C' is a subsheaf of $\psi_*(\omega_C(\phi^*(P)))$, (see page 12 of this paper or [**21**], p.80). More precisely, a local section η of $\psi_*(\omega_C(\phi^*(P)))$, as section of $\omega_C(\phi^*(P))$, is a section of $\omega_{C'}$, if and only if $Res_{\psi^{-1}(P)}\eta = 0$ if P is a cusp and $Res_{p_1}\eta + Res_{p_2}\eta = 0$, if P is a node of Γ. We deduce the following exact sequence

$$0 \to \omega_{C'} \to \psi_*\omega_C(\phi^*(P)) \to \mathbb{C}_P \to 0 \tag{52}$$

where $\mathbb{C}_P$ is the skyscraper sheaf on C with support at P. By the Residue Theorem, we have that

$$H^0(C', \omega_{C'}) \simeq H^0(C, \omega_C(\phi^*(P))).$$

Moreover, tensoring (52) by $\mathcal{O}_{C'}(-1)$, we find the exact sequence

$$0 \to \omega_{C'}(-1) \to \psi_*\omega_C(\phi^*(P))(-1) \to \mathbb{C}_P \to 0 \tag{53}$$

from which we get an injective map $H^0(C', \omega_{C'}(-1)) \to H^0(C, \omega_C(\phi^*(P))(-1))$. On the other and $h^0(C', \omega_{C'}(-1)) = h^0(C, \omega_C(\phi^*(P))(-1)) = g - n + 3$ and so

$$H^0(C', \omega_{C'}(-1)) \simeq H^0(C, \omega_C(\phi^*(P))(-1)).$$

Moreover, from the hypothesis $h^0(C, \mathcal{O}_C(1)) = 3$, we have that $H^0(C, \mathcal{O}_C(1)) \simeq H^0(C', \mathcal{O}_{C'}(1)) \simeq H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. Therefore, in the following commutative diagram

$$\begin{array}{ccccc} H^0(C', \mathcal{O}_{C'}(1)) & \otimes & H^0(C', \omega_{C'}(-1)) & \stackrel{\mu_{o,C'}}{\to} & H^0(C', \omega_{C'}) \\ & \downarrow & & & \downarrow \\ H^0(C, \mathcal{O}_C(1)) & \otimes & H^0(C, \omega_C(-1)(\phi^*(P))) & \stackrel{\mu'_{o,C}}{\to} & H^0(C, \omega_C(\phi^*(P))) \end{array}$$

where we denoted by $\mu'_{o,C}$ the natural multiplication map, the vertical maps are isomorphisms. In particular, $rk(\mu_{o,C'}) = rk(\mu'_{o,C})$. In order to compute the rank of $\mu'_{o,C}$, we consider the following commutative diagram

$$\begin{array}{ccccc} H^0(C, \mathcal{O}_C(1)) & \otimes & H^0(C, \omega_C(-1)) & \stackrel{\mu_{o,C}}{\to} & H^0(C, \omega_C) \\ & F\downarrow & & & \downarrow G \\ H^0(C, \mathcal{O}_C(1)) & \otimes & H^0(C, \omega_C(-1)(\phi^*(P))) & \stackrel{\mu'_{o,C}}{\to} & H^0(C, \omega_C(\phi^*(P))) \end{array}$$

where the vertical maps are injections. Notice that, since we supposed $n \geq 5$, $h^0(C, \mathcal{O}_C(1)) = 3$ and $g > n - 2 \geq 3$, the sheaf $\mathcal{O}_C(1)$ is special. We deduce that C is not hyperelliptic and, chosen a basis of $H^0(C, \omega_C)$, the associated map $C \to \mathbb{P}^{g-1}$ is an immersion. On the contrary, the sheaf $\omega_C(\phi^*(P))$ doesn't define an immersion on C. Choosing a basis of $H^0(C, \omega_C(\phi^*(P))$ and denoting by $\Phi : C \to \mathbb{P}^g$ the associated map, this will be an immersion outside $\phi^*(P)$. If P is a node of C and $\phi^*(P) = p_1 + p_2$, the image of C to $\mathbb{P}^g$, with respect to Φ, will have a node at the image point Q of P_1 and P_2. If $P \in \Gamma$ is a cusp, then $\Phi(C)$ will have a cusp at the image point Q of $\phi^{-1}(P)$. The hyperplanes of $\mathbb{P}^g$ passing through Q cut out on C the canonical linear series $|\omega_C|$. Moreover, if we denote by $B \subset \mathbb{P}^g$ the subspace which is the base locus of the hyperplanes of $\mathbb{P}^g$ corresponding to $Im(\mu'_{o,C})$, then $Q \notin B$. Indeed, B intersects the curve C in the image of the base locus of $|\mathcal{O}_C(1)| + |\omega_C(\phi^*(P))(-1)| := \mathbb{P}(Im(\mu'_{0,C}))$, which coincides with the base locus of $|\omega_C(\phi^*(P))(-1)|$, since $|\mathcal{O}_C(1)|$ is base point free. Now,

$$\begin{aligned} h^0(\omega_C(\phi^*(P))(-1)) &= h^0(C, \mathcal{O}_C(1)(-\phi^*(P))) + 2g - 2 + 2 - n - g + 1 \\ &= 3 + g - n \\ &= h^0(C, \omega_C(-1)) + 1. \end{aligned}$$

Then $\phi^*(P)$ is not in the base locus of $|\omega_C(\phi^*(P))(-1)|$, and so $dim(< Q, B >_{\mathbb{P}^g}) = dim(B)+1$. Finally, we find that

$$\begin{aligned} rk(\mu_{o,C}) = rk(G\mu_{o,C}) &\leq dim(Im(G) \cap Im(\mu'_{o,C})) \\ &\leq g + 1 - dim(< B, Q >_{\mathbb{P}^g}) - 1 \\ &= g - 1 - dim(B) \\ &= rk(\mu'_{o,C}) - 1. \end{aligned}$$

□

Corollary 3.9. *Let $\Sigma \subset \Sigma^n_{k,d}$ be an irreducible component of $\Sigma^n_{k,d}$, with $n \geq 5$ and $0 \leq k < 3n$. Suppose that Σ has the expected number of moduli and that the general element $[\Gamma] \in \Sigma$ corresponds to a g. l. n. plane curve Γ of geometric genus g such that, if $C \to \Gamma$ is the normalization of Γ, then the map $\mu_{o,C}$ is surjective. Then, for every $k' \leq k$ and $d' \leq d+k-k'$, there is at least an irreducible component $\Sigma' \subset \Sigma^n_{k',d'}$, such that $\Sigma \subset \Sigma'$, the general element $[D] \in \Sigma'$ corresponds to a g.l.n. plane curve D of geometric genus g' with normalization $D^\nu \to D$ and the Brill-Noether map μ_{0,D^ν} surjective. In particular, also Σ' has the expected number of moduli.*

Proof. Let Γ be the curve corresponding to the general element $[\Gamma]$ of $\Sigma \subset \Sigma^n_{k,d}$, with cusps at $q_1, \dots, q_k$ and nodes at $p_1, \dots, p_d$. Since $k < 3n$, we can "smooth independently nodes, cusps and cusps to nodes" of Γ, (see corollary 3.17 of chapter 2), i.e. chosen arbitrarily k_1 cusps, say $q_1, \dots, q_{k_1}$ among the k cusps of Γ, k_2 cusps among $q_{k_1+1}, \dots, q_k$ and d' nodes among the nodes of Γ, there exists a family of plane curves $\mathcal{G} \to \Delta$ of degree n, whose general element $\mathcal{G}_t$ has d' nodes, tending to the marked nodes of Γ, as $\mathcal{G}_t$ specializes to Γ, k_1 cusps tending to $q_1, \dots, q_{k_1}$, and k_2 nodes tending to the second group of marked cusps of Γ. Suppose now $k' \leq k$ and $d' = d + k - k'$. Let Σ' be an irreducible component of $\Sigma^n_{k',d'}$ containing Σ. The general element of Σ' corresponds to a plane curve with the same geometric genus g of the plane curve Γ, corresponding to the general element $[\Gamma]$ of Σ. Let $\Delta \subset \Sigma'$ be a general curve of Σ' passing through $[\Gamma]$ and let Δ' be the normalization of Δ. We denote by $\mathcal{C} \to \Delta'$ the pullback to Δ' of the tautological family of plane curves parametrized by Δ. Normalizing the total space $\mathcal{C}$, we get a family $\mathcal{C}' \to \Delta'$ of smooth curves of genus g whose fibres are the normalizations of the respective fibres of $\mathcal{C} \to \Delta'$. Since the special fibre $\mathcal{C}'_0 := C$ of $\mathcal{C}'$ is such that $h^0(C, \mathcal{O}_C(1)) = 3$ and the map $\mu_{o,C}$ has maximal rank, by semicontinuity, if $\mathcal{C}'_t$ is the general fibre of $\mathcal{C}'$, then $h^0(\mathcal{C}'_t, \mathcal{O}_{\mathcal{C}'_t}(1)) = 3$ and the Brill-Noether map

$$H^0(C'_t, \mathcal{O}_{C'_t}(1)) \otimes H^0(C'_t, \omega_{C'_t}(-1)) \stackrel{\mu_{o,C'_t}}{\to} H^0(C'_t, \omega_{C'_t})$$

is surjective. In order to prove the statement in the general case, it is enough to show it under the hypothesis $k = k'$ and $d = d' - 1$ or $k = k' - 1$ and $d = d'$. Let Σ' be an irreducible component of $\Sigma^n_{k',d'}$ containing Σ. In this case, if $\mathcal{C}' \to \Delta'$ is a family of curves constructed as before, the general fibre of it is a smooth curve of genus $g+1$ which is the normalization of the plane curve corresponding to the general element $[\Gamma']$ of $\Sigma^n_{k',d'}$. On the contrary, the special fibre $\mathcal{C}'_0 := C'$ is a partial normalization of Γ obtained by smoothing all the singular points of Γ, except for a node or a cusp. By using the geometrically linear normality of C, we find that $h^0(C', \mathcal{O}_{C'}(1))$=3. Moreover, since the map $\mu_{o,C}$ has maximal rank, by lemma 3.8 also the map $\mu_{o,C'}$ has maximal rank equal to $g+1$. The corollary follows by semicontinuity and by proposition 3.1. □

The following lemma has been stated and proved by Sernesi in [**37**]. Actually, Sernesi supposes that Γ has only nodes as singularities. But, since his proof works for plane curves Γ with any type of singularities and, since we need it for curves with nodes and cusps, we state the lemma in a more general form.

Lemma 3.10. *([**37**], lemma 2.3) Let Γ be an irreducible and reduced plane curve of degree $n \geq 5$ with any type of singularities. Denote by C the normalization of Γ. Suppose that $h^0(C, \mathcal{O}_C(1)) = 3$ and the Brill-Noether map*

$$\mu_{o,C} : H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, \omega_C(-1)) \to H^0(C, \omega_C),$$

has maximal rank. Let R be a general line and let P_1, P_2 and P_3 be three fixed points of $\Gamma \cap R$. We denote by C' the partial normalization of $\Gamma' = \Gamma \cup R$, obtained smoothing all the singular points, except P_1, P_2 and P_3. Then $h^0(C', \mathcal{O}_{C'}(1)) = 3$ and, denoting by $\omega_{C'}$ the dualizing sheaf of C', the multiplication map

$$\mu_{o,C'} : H^0(C', \mathcal{O}_{C'}(1)) \otimes H^0(C', \omega_{C'}(-1)) \to H^0(C', \omega_{C'}),$$

has maximal rank.

Proof. Consider the following exact sequence of sheafs on C'

$$0 \to \mathcal{O}_{\mathbb{P}^1}(1)(-P_1 - P_2 - P_3) \to \mathcal{O}_{C'}(1) \to \mathcal{O}_C(1) \to 0$$

By using that $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)(-P_1 - P_2 - P_3)) = 0$, we find that $h^0(C', \mathcal{O}_{C'}(1)) = h^0(C, \mathcal{O}_C(1)) = 3$. We have to prove that the map $\mu_{0,C'}$ has maximal rank. If $h^0(C', \omega_{C'}(-1)) \leq 1$, the map $\mu_{0,C'}$ is obviously injective. We assume that $h^0(C', \omega_{C'}(-1)) > 1$. Then, by using that the arithmetic genus g' of C' is equal to $g' = g+2$ and that, by Riemann-Roch theorem for singular curves, we have that

$$h^0(C', \omega_{C'}(-1)) = h^0(C', \mathcal{O}_{C'}(1)) + 2g' - 2 - n - 1 - g' + 1 = g - n + 3 = h^0(C, \omega_C(-1)) + 1,$$

we conclude that $h^0(C, \omega_C(-1)) \geq 1$ and C is a not hyperelliptic curve of genus $g > n-2 \geq 3$. Now, by the generality of R, we may assume that P_1, P_2 and P_3 are nodes of C'. Let, then, $n : C \cup R \to C'$ be the normalization map of C' and let p_1, p_2, p_3 and q_1, q_2, q_3 be respectively the points of C and R over P_1, P_2, P_3. We recall that the dualizing sheaf $\omega_{C'}$ of C' associates to every open set $U \subset C'$ the set of the rational one-form η on $n^{-1}(U)$ with at most simple poles at the points p_i and q_i and such that $Res_{p_i}\eta + Res_{q_i}\eta = 0$, for $i = 1, 2, 3$. Since, for every triple a_1, a_2, $a_3 \in \mathbb{C}$ such that $a_1 + a_2 + a_3 = 0$, there is a rational one-form η on $\mathbb{P}^1$ with at most simple poles at q_1, q_2, q_3 and such that $Res_{q_i}\eta = a_i$, for $i = 1, 2, 3$, we get that the natural restriction map $\omega_{C'} \to \omega_C(p_1 + p_2 + p_3)$ is surjective with kernel equal to $\omega_{\mathbb{P}^1}(q_1 + q_2 + q_3)(-q_1 - q_2 - q_3) = \omega_{\mathbb{P}^1}$. It follows the isomorphism

$$H^0(C', \omega_{C'}) \simeq H^0(C, \omega_C(p_1 + p_2 + p_3)).$$

Furthermore, by tensoring the exact sequence

$$o \to \omega_{\mathbb{P}^1} \to \omega_{C'} \to \omega_C(p_1+p_2+p_3) \to 0$$

by $\mathcal{O}_{C'}(-1)$, we find an isomorphism

$$H^0(C', \omega_{C'}(-1)) \simeq H^0(C, \omega_C(-1)(p_1+p_2+p_3)).$$

Finally, in the commutative diagram below

$$\begin{array}{ccccc} H^0(C', \mathcal{O}_{C'}(1)) & \otimes & H^0(C', \omega_{C'}(-1)) & \stackrel{\mu_{o,C'}}{\to} & H^0(C', \omega_{C'}) \\ & \downarrow & & & \downarrow \\ H^0(C, \mathcal{O}_C(1)) & \otimes & H^0(C, \omega_C(-1)(p_1+p_2+p_3)) & \stackrel{\mu'_{o,C}}{\to} & H^0(C, \omega_C(p_1+p_2+p_3)) \end{array}$$

the verticals maps are isomorphisms. Therefore, we conclude that $rk(\mu_{o,C'}) = rk(\mu'_{o,C})$. In order to prove that the map $\mu'_{o,C}$ has maximal rank, let $< g_0, \dots, g_{g+1} >$ be a basis of $H^0(C, \omega_C(p_1+p_2+p_3))$ and let $\Psi : C \to \mathbb{P}^{g+1}$ be the associated morphism. Notice that Ψ is an embedding. Let us denote by C the image of C in $\mathbb{P}^{g+1}$ with respect to Ψ. Let $B \subset \mathbb{P}^{g+1}$ be the base locus of the hyperplanes of $\mathbb{P}^{g+1}$ cutting out on C the linear series $|\mathcal{O}_C(1)| + |\omega_C(-1)(p_1+p_2+p_3)| := \mathbb{P}(Im(\mu'_{o,C}))$. Since $h^0(\omega_C) = g$, the linear span $< p_1, p_2, p_3 >_{\mathbb{P}^{g+1}} := r$ of the point p_i in $\mathbb{P}^{g+1}$ is a line. Moreover, since $|\omega_C|$ is base point free, we have that

$$r \cap C = \{p_1, p_2, p_3\}.$$

We claim that

$$B \cap r = \emptyset.$$

Notice that, since $|\mathcal{O}_C(1)|$ is base point free, the intersection of B with C coincides with the base locus of $|\omega(-1)(p_1+p_2+p_3)|$. Since p_1, p_2 and p_3 are not base points of $|\omega_C(-1)(p_1+p_2+p_3)|$, we deduce that $p_i \notin B$, for $i = 1, 2, 3$. Suppose that B intersects r at a point Q different from p_1, p_2 and p_3. Let $H \in |\mathcal{O}_C(1)|$ be the divisor associated to a line different from R passing through p_1. Moreover let $H' \in |\omega_C(-1)(p_1+p_2+p_3)|$ be a divisor such that $p_1, p_2, p_3 \notin Supp(H')$. The divisor $H + H' \in |\mathcal{O}_C(1)| + |\omega_C(-1)(p_1+p_2+p_3)|$ generates a hyperplane Λ in $\mathbb{P}^{g+1}$. By construction, $\Lambda \supset < B, p_1 >_{\mathbb{P}^{g+1}} = < B, r >_{\mathbb{P}^{g+1}}$. It follows that

$$p_2, p_3 \in \Lambda \cap C.$$

But this is not possible because

$$p_2, p_3 \notin Supp(H) \cup Supp(H') = \Lambda \cap C.$$

Thus $B \cap r = \emptyset$ and so, from the commutative diagram below

$$\begin{array}{ccccc} H^0(C, \mathcal{O}_C(1)) & \otimes & H^0(C, \omega_C(-1)) & \stackrel{\mu_{o,C}}{\to} & H^0(C, \omega_C) \\ & F \downarrow & & & \downarrow G \\ H^0(C, \mathcal{O}_C(1)) & \otimes & H^0(C, \omega_C(-1)(p_1+p_2+p_3)) & \stackrel{\mu'_{o,C}}{\to} & H^0(C, \omega_C(p_1+p_2+p_3)) \end{array}$$

where the vertical map are injections, we deduce that

$$\begin{aligned} rk(\mu_{o,C}) = rk(G\mu_{o,C}) &\leq dim(Im(G) \cap Im(\mu'_{o,C})) \\ &= g+1-dim(< B, p_1, p_2, p_3 >) \\ &= g-1-dim(B) \\ &= rk(\mu'_{o,C}) - 2. \end{aligned} \tag{54}$$

It follows that, if $\mu_{0,C}$ is surjective then $\mu'_{o,C}$ too is surjective. Now, suppose that $\mu_{o,C}$ is injective but not surjective. In this case, we have that $rk(\mu_{o,C}) = rk(G\mu_{o,C}) = 3(g-n+2)$. Since $h^0(\mathcal{O}_C(1))h^0(\omega_C(-1)(p_1+p_2+p_3)) = 3(g-n+3)$, the lemma follows if we show that $rk(\mu'_{o,C}) \geq rk(\mu_{o,C})+3$. By (54), it is enough to prove that

$$Im(G\mu_{o,C}) \subsetneq Im(G) \cap Im(\mu'_{o,C}).$$

In order to do this, we consider the following multiplication map

$$\mu''_{o,C} : H^0(C, \mathcal{O}_C(1)(-p_1-p_2-p_3)) \otimes H^0(C, \omega_C(-1)(p_1+p_2+p_3)) \to H^0(C, \omega_C).$$

Since $H^0(C, \mathcal{O}_C(1)(-p_1-p_2-p_3)) \subset H^0(C, \mathcal{O}_C(1))$, we have that

$$Im(G\mu''_{0,C}) \subset Im(\mu'_{o,C}) \cap Im(G).$$

We shall prove that

$$Im(\mu''_{0,C}) \nsubseteq Im(\mu_{o,C}).$$

Let $C \to \mathbb{P}^{g-1}$ be the canonical map of C. Let $A \subset P^{g-1}$ be the base locus of the hyperplanes cutting out on C the minimal sum $|\mathcal{O}_C(1)|+|\omega_C(-1)| := \mathbb{P}(Im(\mu_{o,C}))$. We denote by $p_1+\cdots+p_n$ the divisor cut out on C by the line R in $\mathbb{P}^2$. Notice that A is contained in every $(n-3)$-plane $\Lambda_{E'} :=< Supp(E') >_{\mathbb{P}^{g-1}}$, for $E' \in |\mathcal{O}_C(1)|$ and, in particular, we have that

$$A \subset < p_1, \dots, p_n > .$$

(Indeed, for every $H \in |\mathcal{O}_C(1)|$, we have that

$$< Supp(H) >_{\mathbb{P}^{g-1}} = \cap_{D \in |\omega_C(-H)|} < Supp(H) \cup Supp(D) >_{\mathbb{P}^{g-1}}$$

Observing that $H + |\omega_C(-1)| \subset |\mathcal{O}_C(1)| + |\omega_C(-1)|$, we conclude that

$$A \subset < Supp(H) \cup Supp(D) >_{\mathbb{P}^{g-1}}$$

for every $D \in |\omega_C(-H)|$, and so $A \subset < Supp(H) >_{\mathbb{P}^{g-1}}$). Now, by using that $h^0(p_3+\cdots+p_n) = h^0(\mathcal{O}_C(1)(-p_1-p_2))$, we find that

$$< p_3, \dots, p_n >_{\mathbb{P}^{g-1}} = < p_1, \dots, p_n >_{\mathbb{P}^{g-1}}$$

and $< p_3, \dots, \hat{p}_i, \dots, p_n >_{\mathbb{P}^{g-1}}$ is an hyperplane of $< p_1, \dots, p_n >_{\mathbb{P}^{g-1}}$, for $i = 3, \dots, n$. Moreover, since $\cap_i < p_3, \dots, \hat{p}_i, \dots, p_n >_{\mathbb{P}^{g-1}} = \emptyset$, there is some i such that

$$A \nsubseteq < p_3, \dots, \hat{p}_i, \dots, p_n >_{\mathbb{P}^{g-1}} .$$

On the other hand, by the generality of the divisor $p_1+\cdots+p_n$, by using general position theorem, we can interchange any two of the p_i's by moving $p_1, \dots, p_n$ in $|\mathcal{O}_C(1)|$. It follows that

$$A \nsubseteq < p_3, \dots, \hat{p}_i, \dots, p_n >_{\mathbb{P}^{g-1}}, \quad \text{for all } i.$$

In particular, A is not contained in $< p_4, \dots, p_n >$. Now, the $n-4$ plane $< p_4, \dots, p_n >$ is contained in the base locus Y of the family of hyperplanes in $\mathbb{P}^{g-1}$ cutting out on C the linear series

$$|Im(\mu''_{o,C})| = p_4+\cdots+p_n+|\omega_C(-p_4-\cdots-p_n)| = |\mathcal{O}_C(1)(-p_1-p_2-p_3))|+|\omega_C(-1)(p_1+p_2+p_3)|.$$

On the other hand, the dimension of the previous linear series is equal to

$$h^0(C, \omega_C(-1)(p_1+p_2+p_3)) = g-(n-3).$$

Then, we have that $dim(Y) = n-4$ and $Y =< p_4, \dots, p_n >$. It follows that the linear series $|Im(\mu''_{o,C})|$ is not contained in $|\mathcal{O}_C(1)| + |\omega_C(-1)|$. This completes the proof. $\square$

Theorem 3.11. *Let $\Sigma^n_{k,d}$ be the algebraic system of irreducible plane curves of degree $n \geq 4$ with k cusps, d nodes and geometric genus $g = \binom{n-1}{2} - k - d$. Suppose that:*

$$n - 2 \leq g \ \text{ equivalently } \ k + d \leq h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-4)) \tag{55}$$

and

$$k \leq 6 + \left[\frac{n-8}{3}\right] \ \text{ if } 3n - 9 \leq g \text{ and } n \geq 6, \tag{56}$$

$$k \leq 6 \quad \text{otherwise.} \tag{57}$$

Then $\Sigma^n_{k,d}$ has at least one irreducible component Σ which is not empty and whose general element $[\Gamma] \in \Sigma$ parametrizes a geometrically linearly normal curve Γ such that the Brill-Noether map of the pair (C, H), where C is the normalization of Γ and H denote the pull-back to C of the hyperplane section of $\mathbb{P}^2$, has maximal rank. In particular, when $\rho \leq 0$, the algebraic system Σ has the expected number of moduli equal to $3g - 3 + \rho - k$.

Proof. *Suppose that* (56) *holds.* Then, by observing that

$$g \geq 3n - 9 \ \text{ if and only if } \ k + d \leq h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-6))$$

and by using theorem 2.4 for $t = 3$, we have that there exists an irreducible component Σ of $\Sigma^n_{k,d}$ whose general element is a plane curve Γ geometrically 3-normal, i.e. a plane curve Γ with nodes and cusps in sufficiently general position to impose independent linear conditions to the linear system of plane curves of degree $n - 6$. In other words, the linear system cut out by the cubics on the normalization C of Γ is complete. By remark 2.3, it follows that also the linear systems cut out on C by the conics and the lines are complete. The statement follows from corollary 3.7.

In order to prove the theorem under the hypothesis (57)*, we consider the following subcases:*

(1) $2n - 5 \leq g \leq 3n - 9$, i.e. $h^0(\mathcal{O}_{\mathbb{P}^2}(n-6)) \leq k + d \leq h^0(\mathcal{O}_{\mathbb{P}^2}(n-5))$ and $n \geq 5$,

(2) $n - 2 \leq g \leq 2n - 7$ and $n \geq 5$,

(3) $g = 2n - 6$ and $n \geq 4$.

Suppose that (1) *holds.* By theorem 2.4 for $t = 2$, we know that, under this hypothesis, there exists a nonempty component $\Sigma \subset \Sigma^n_{k,d}$, whose general element is geometrically 2-normal. We conclude as in the previous case, by corollary 3.7.

Now, suppose that g and n verify (2)*. We shall prove the theorem by induction on n and g.* Set $g = 2n - 7 - a$, with $a \geq 0$ fixed. Suppose that the theorem is true for the pair (n, g), with $n \geq 7$. We shall prove the theorem for $(n + 1, g + 2)$, observing that $g + 2 = 2(n + 1) - 7 - a$. Notice also that, since $g \leq 2n - 7$ and $n \geq 7$, then $\binom{n-1}{2} - g \geq 6$. Let Γ be a g. l. n. irreducible plane curve of degree n and genus $g = 2n - 7 - a$ with $k \leq 6$ cusps, d nodes and no more singularities. Let C be the normalization of Γ. Suppose that the Brill-Noether map $\mu_{o,C}$ has maximal rank. Let $R \subset \mathbb{P}^2$ be a general line and let P_1, P_2 and P_3 be three fixed points of $\Gamma \cap R$. Since the number of cusps of Γ is less than $3n$, we can "smooth independently the nodes and cusps of $\Gamma \cup R$", see corollary 3.17 of chapter 2. In particular, $[\Gamma \cup R] \in \Sigma^{n+1}_{k,d+n-3}$ and there exists a family of irreducible plane curves

$$\mathcal{C} \to \Delta$$

of degree $n+1$, parametrized by a projective curve $\Delta \subset \Sigma^{n+1}_{k,d+n-3}$, whose special fibre $\mathcal{C}_0 = \Gamma \cup R$ and whose general fibre is irreducible of genus $g+2$ with $d+n-3$ nodes specializing to nodes of $\Gamma \cup R$ different to P_1, P_2 and P_3 and k cusps specializing to cusps of Γ. Normalizing Δ and $\mathcal{C}$, we get a family of curves $\mathcal{C}' \to \Delta'$, parametrized by a smooth projective curve Δ', whose general fibre is the normalization of the general fibre of $\mathcal{C} \to \Delta$ and whose special fibre is exactly the partial normalization C' of $\Gamma \cup R$ obtained by smoothing all the singular points, except P_1, P_2 and P_3. By lemma 3.10, $h^0(C', \mathcal{O}_{C'}(1)) = 3$ and, denoting by $\omega_{C'}$ the dualizing sheaf of C', the multiplication map,

$$\mu_{o,C'} : H^0(C', \mathcal{O}_{C'}(1)) \otimes H^0(C', \omega_{C'}(-1)) \to H^0(C', \omega_{C'}),$$

has maximal rank. By semicontinuity, we conclude that also the general curve $\mathcal{C}_t$ of the family $\mathcal{C} \to \Delta$ is geometrically linearly normal and the Brill-Noether map of the pair $(\mathcal{C}'_t, H)$, where H is the pull-back to $\mathcal{C}'_t$ of the general hyperplane section of $\mathcal{C}_t$, has maximal rank. The induction step is proved.

Now we prove the first step of induction for $n \geq 7$. If $n = 7$, we get $0 \leq a \leq 2$. Let $a = 0$, then $g = 2n - 7 - a = 7$. Let Γ be a geometrically linearly normal irreducible plane curve of degree $n = 7$, of genus $g = n = 7$ with $k \leq 6$ cusps and nodes as other singularities, such that no seven singular points of Γ lie on an irreducible conic. To prove that there exists such a plane curve, notice that, by applying theorem 2.4 for $t = 1$, we find that, for any fixed $k \leq 6$, there exists a geometrically linearly normal irreducible sextic D of genus four with k cusps and $d = 6 - k$ nodes. Let $R_1, \dots, R_6$ be the singular points of D. Since the points $R_1, \dots, R_6$ of D impose independent linear conditions to the conics, however we choose five singular points $R_{i_1}, \dots, R_{i_5}$, with $I = (i_1, \dots, i_5) \subset (1, \dots, 6)$, of D, there exists only one conic C_I, passing through these points. Let us set $S = \bigcup_I C_I \cap D$ and let R be a line intersecting D transversally at six points out of S. By Bezout theorem, no seven singular points of $\Gamma' = D \cup R$ belongs to an irreducible conic. Moreover, if $\tilde{D}$ is the normalization of D, if $Q_1, \dots, Q_4$ are four fixed points of $D \cap R$ and D' is the partial normalization of Γ' obtained by smoothing the singular points except $Q_1, \dots, Q_4$, then, by the following exact sequence

$$0 \to \mathcal{O}_R(1)(-Q_1 - \cdots - Q_4) \to \mathcal{O}_{D'}(1) \to \mathcal{O}_D(1) \to 0 \qquad (58)$$

we find that $h^0(D', O_{D'}(1)) = 3$. It follows that every irreducible septic Γ obtained from Γ' "by smoothing the nodes $Q_1, \dots, Q_4$", is like we need. Let now C be the normalization of such a plane curve Γ. We shall prove that $ker(\mu_{o,C}) = 0$. Let $\Delta \subset C$ be the adjoint divisor of the normalization map $\phi : C \to \Gamma$. We recall that, if Γ has a cusp at each of points $P_1, \dots, P_k$ and it has a node at every point $P_{k+1}, \dots, P_8$, then $\Delta = \sum_{i=1}^k 2p_i + \sum_{j=k+1}^8 (p^1_j + p^2_j)$, where $p_i = \phi^{-1}(P_i)$, for $1 \leq i \leq k$, and $\{p^1_j, p^2_j\} = \phi^{-1}(P_j)$, for $k+1 \leq j \leq 8$, (see section 1 of chapter 1). Since Γ is geometrically linearly normal, we have that

$$h^0(C, \omega_C(-1)) = h^0(C, \mathcal{O}_C(3)(-\Delta))) = g - n + 2 = 2.$$

Then, by the base point free pencil trick, we find that

$$ker(\mu_{o,C}) = H^0(C, \omega^*_C(B) \otimes \mathcal{O}_C(2)),$$

where B is the base locus of $|\omega_C(-1) = \mathcal{O}_C(3)(-\Delta)|$. Let $\mathcal{F}$ be the pencil of plane cubics passing through the eight double points $P_1, \dots, P_8$ of Γ and let $B_{\mathcal{F}}$ be the base locus of the pencil $\mathcal{F}$. Let Γ_3 be the general element of $\mathcal{F}$. Suppose that $B_{\mathcal{F}}$ has dimension one. If $B_{\mathcal{F}}$ contains a line l, then, by Bezout theorem, at most three points among $P_1, \dots, P_8$, say $P_1, \dots, P_3$ can lie on l and the other points have to be contained in the base locus of a pencil of conics $\mathcal{F}'$. Still

using Bezout theorem, we find that also the curves of $\mathcal{F}'$ are reducible and the base locus of $\mathcal{F}'$ contains a line l'. But also l' contains at most three points of $P_4, \ldots, P_6$. It follows that there is only one cubic through $P_1, \ldots, P_8$. This is not possible by construction. Suppose that $B_{\mathcal{F}}$ contains an irreducible conic Γ_2. By Bezout theorem, at most seven points of $P_1, \ldots, P_8$ may lie on Γ_2. On the other hand, since $dim(\mathcal{F}) = 1$, there are exactly seven points of $P_1, \ldots, P_8$, say $P_1, \ldots, P_7$, on Γ_2 and the general cubic Γ_3 of $\mathcal{F}$ is union of Γ_2 and a line passing through P_8. Since, by construction, no seven singular points of Γ lie on a conic, also in this case we get a contradiction. So the general element Γ_3 of $\mathcal{F}$ is irreducible. Still using Bezout theorem, we find that Γ_3 is smooth and $\mathcal{F}$ has only an other base point Q. We consider the following cases:
a) Q doesn't lie on Γ;
b) Q lies on Γ, but $Q \neq P_1, \ldots, P_8$;
c) Q is infinitely near to one of the points $P_1, \ldots, P_8$, say $P_{\hat{i}}$, i.e. the cubics of $\mathcal{F}$ have at $P_{\hat{i}}$ the same tangent line l, but l is not contained in the tangent cone to Γ at $P_{\hat{i}}$;
d) Q is like in the case c), but l is contained in the tangent cone to Γ at $P_{\hat{i}}$.
Suppose that the case a) or c) holds. Thus $B = 0$ and

$$ker(\mu_{o,C}) = H^0(C, \omega_C^* \otimes \mathcal{O}_C(2)) = H^0(C, \mathcal{O}_C(-2)(\Delta)).$$

By Riemann-Roch theorem, $h^0(C, \mathcal{O}_C(-2)(\Delta)) = h^0(C, \mathcal{O}_C(6)(-2\Delta)) - 4$. In order to compute $h^0(C, \mathcal{O}_C(6)(-2\Delta))$, let $\Phi : S \to \mathbb{P}^2$ be the blow-up of the plane at $P_1, \ldots, P_8$, and let $E_1, \ldots, E_8$ be the exceptional divisors of S. Since Γ has only nodes and cusps as singularities, the map Φ restricts on the strict transform $C := \Phi^*(\Gamma)$ to the normalization map of Γ and $\Delta = C.(\sum_i E_i)$. From the following exact sequence,

$$0 \to \mathcal{O}_S(-7)(2\sum_i E_i) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$$

by tensoring with $\mathcal{O}_S(6)(-2\sum_i E_i)$, we get the exact sequence

$$0 \to \mathcal{O}_S(-1) \to \mathcal{O}_S(6)(-2\sum_i E_i) \to \mathcal{O}_C(6)(-2\Delta) \to 0$$

from which we deduce that $h^0(C, \mathcal{O}_C(6)(-2\Delta)) = h^0(S, \mathcal{O}_S(6)(-2\sum_i E_i))$, because, by using Leray spectral sequence, we find that $H^i(\tilde{S}, \mathcal{O}_{\tilde{S}}(-1)) = H^i(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1)) = 0$, for $i = 0, 1$. Denoting by C_3 the strict transform with respect to Φ of the general cubic Γ_3 of $\mathcal{F}$, from the following exact sequence

$$0 \to \mathcal{O}_S(-3)(\sum_i E_i) \to \mathcal{O}_S \to \mathcal{O}_{C_3} \to 0$$

by tensoring with $\mathcal{O}_S(6)(-2\sum_i E_i)$, we obtain the exact sequence

$$0 \to \mathcal{O}_S(3)(-\sum_i E_i) \to \mathcal{O}_S(6)(-2\sum_i E_i) \to \mathcal{O}_{C_3}(6)(-2\sum_i E_i) \to 0 \tag{59}$$

By Riemann-Roch theorem,

$$H^0(\mathcal{O}_{C_3}(6)(-2\sum_i E_i)) = 18 - 16 - 1 + 1 = 2 \text{ and } H^1(\mathcal{O}_{C_3}(6)(-2\sum_i E_i)) = 0.$$

Moreover, from the following exact sequence

$$0 \to \mathcal{O}_S(-\sum_i E_i) \to \mathcal{O}_S \to \bigoplus_{i=1}^{8} \mathcal{O}_{E_i} \to 0$$

by tensoring with $\mathcal{O}_S(3)$, we get the exact sequence

$$0 \to \mathcal{O}_S(3)(-\sum_i E_i) \to \mathcal{O}_S(3) \to \bigoplus_{i=1}^{8} \mathcal{O}_{E_i} \to 0$$

By using that $H^1(S, \mathcal{O}_S(3)) = 0$ and $H^0(S, \mathcal{O}_S(3)((-\sum_i E_i))) = 2$, we find that

$$h^1(S, \mathcal{O}_S(3)((-\sum_i E_i))) = 2 - 10 + 8 = 0.$$

So, from the exact sequence (59), we find that

$$h^0(S, \mathcal{O}_S(6)(-2\sum_i E_i)) = 2 + 2 = 4 \quad \text{and} \quad ker(\mu_{o,C}) = 0.$$

Suppose now that the case b) holds. Thus $B = Q$ and

$$ker(\mu_{o,C}) = H^0(C, \mathcal{O}_C(-2)(\Delta + Q)) = H^0(C, \mathcal{O}_C(6)(-2\Delta - Q)) - 3.$$

Let $\tilde{\phi} : \tilde{S} \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at $P_1, \dots, P_8$ and Q. Let $E_1, \dots, E_8, C$ and C_3 as before and let E_Q be the exceptional divisor corresponding to the point Q. By the following exact sequence,

$$0 \to \mathcal{O}_{\tilde{S}}(-1) \to \mathcal{O}_{\tilde{S}}(6)(-2\sum_i E_i - E_Q) \to \mathcal{O}_C(6)(-2\Delta - Q) \to 0$$

we find that $h^0(C, \mathcal{O}_C(6)(-2\Delta - Q)) = h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(6)(-2\sum_i E_i - E_Q))$, because, by Leray spectral sequence $H^i(\tilde{S}, \mathcal{O}_{\tilde{S}}(-1)) = H^i(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1)) = 0$, for $i = 0, 1$. From the following exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}(-3)(\sum_i E_i + E_Q) \to \mathcal{O}_{\tilde{S}} \to \mathcal{O}_{C_3} \to 0$$

by tensoring with $\mathcal{O}_{\tilde{S}}(6)(-2\sum_i E_i - E_Q)$, we get the exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}(3)(-\sum_i E_i) \to \mathcal{O}_{\tilde{S}}(6)(-2\sum_i E_i - E_Q) \to \mathcal{O}_{C_3}(6)(-2\sum_i E_i - E_Q) \to 0 \tag{60}$$

Now, from the following exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}(3)(-\sum_i E_i) \to \mathcal{O}_{\tilde{S}}(3) \to \bigoplus_{i=1}^{8} \mathcal{O}_{E_i} \to 0$$

we find that $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(3)(-\sum_i E_i)) = 0$. Therefore, by (60), we conclude that

$$h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(6)(-2\sum_i E_i - E_Q)) = 2 + 1 = 3, \text{ and } ker(\mu_{o,C}) = 0.$$

Finally, suppose that d) holds. Let $\Phi : S \to \mathbb{P}^2$ be the blow-up of the plane at $P_1, \dots, P_8$ with exceptional divisors $E_1, \dots, E_8$. Let $Q \in E_{\hat{i}}$ be the intersection point of $E_{\hat{i}}$ and the strict transform C_3 of the general cubic Γ_3 of the pencil $\mathcal{F}$. We denote by $\tilde{\Phi} : \tilde{S} \to S$ the blow-up of S at Q and by $\Psi : \tilde{S} \to \mathbb{P}^2$ the composition map of the maps Φ and $\tilde{\Phi}$. We still denote by $E_1, \dots, E_8$ their strict transforms on $\tilde{S}$, by C and C_3 the strict transforms of Γ and Γ_3 and by E_Q the new exceptional divisor of $\tilde{S}$. In this case we have that $\Psi^{-1}(\Gamma) = C + 2\sum_i E_i + 3E_Q$, $\Psi^{-1}(\Gamma_3) = C_3 + \sum_i E_i + 2E_Q$. Moreover, the divisor Δ is cut out on C from $\sum_i E_i + E_Q$ and the base locus B of the linear series $|\omega_C(-1)|$ coincides with the intersection point of E_Q and C. So, we have that

$$dim(ker(\mu_{o,C})) = h^0(C, \mathcal{O}_C(-2)(\sum_i E_i + 2E_Q)) = h^0(C, \mathcal{O}_C(6)(-2\sum_i E_i - 3E_Q)) - 3.$$

Moreover, from the following exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}(-7)(2\sum_i E_i + 3E_Q) \to \mathcal{O}_{\tilde{S}} \to \mathcal{O}_C \to 0$$

by tensoring with $\mathcal{O}_{\tilde{S}}(6)(-2\sum_i E_i - 3E_q)$, we get the exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}(-1) \to \mathcal{O}_{\tilde{S}}(6)(-2\sum_i E_i - 3E_Q) \to \mathcal{O}_C(6)(-2\sum_i E_i - 3E_Q) \to 0$$

from which we find that $H^0(C, \mathcal{O}_C(6)(-2\sum_i E_i - 3E_Q)) = H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(6)(-2\sum_i E_i - 3E_Q))$ and $H^1(C, \mathcal{O}_C(6)(-2\sum_i E_i - 3E_Q)) = H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(6)(-2\sum_i E_i - 3E_Q))$. Now, from the following exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}(3)(-\sum_i E_i - E_Q) \to \mathcal{O}_{\tilde{S}}(6)(-2\sum_i E_i - 3E_Q) \to \mathcal{O}_{C_3}(6)(-2\sum_i E_i - 3E_Q) \to 0 \tag{61}$$

by using that, by Riemann-Roch theorem,

$$h^0(C_3, \mathcal{O}_{C_3}(6)(-2\sum_i E_i - 3E_Q)) = 1 \text{ and } h^1(C_3, \mathcal{O}_{C_3}(6)(-2\sum_i E_i - 3E_Q) = 0$$

we find that

$$\begin{aligned} h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(6)(-2\sum_i E_i - 3E_Q)) - h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(6)(-2\sum_i E_i - 3E_Q)) &= \\ h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(3)(-\sum_i E_i - E_Q)) - h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(3)(-\sum_i E_i - E_Q)) + 1. \end{aligned}$$

Moreover, from the exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}(3)(-\sum_i E_i - E_Q) \to \mathcal{O}_{\tilde{S}}(3) \to \mathcal{O}_{E_{\hat{i}} \cup E_Q} \bigoplus_{i \neq \hat{i}} \mathcal{O}_{E_i} \to 0 \tag{62}$$

we find that

$$\begin{aligned} h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(3)(-\sum_i E_i - E_Q)) - h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(3)(-\sum_i E_i - E_Q)) &= \\ h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(3)) - h^0(E_Q \cup_i E_i, \mathcal{O}_{E_{\hat{i}} \cup E_Q} \bigoplus_{i \neq \hat{i}} \mathcal{O}_{E_i}) &= 10 - 8 = 2. \end{aligned}$$

Now, by using Serre duality, we have that

$$H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(3)(-\sum_i E_i - E_Q))) = H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(-6)(2\sum_i E_i + 3E_Q))).$$

From the exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}(-6)(+2\sum_i E_i + 3E_Q)) \to \mathcal{O}_{\tilde{S}}(1) \to \mathcal{O}_C(1) \to 0 \tag{63}$$

by using that the map $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(1)) \to H^0(C, \mathcal{O}_C(1))$ is surjective and that $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(1)) = 0$, we find that

$$H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(-6)(+2\sum_i E_i + 3E_Q))) = H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(3)(-\sum_i E_i - E_Q))) = 0.$$

Finally, by the exact sequence (61), we deduce that $h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(6)(-2\sum_i E_i - 3E_Q)) = 0$,

$$h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(6)(-2\sum_i E_i - 3E_q)) = 3, \text{ and } ker(\mu_{o,C}) = 0.$$

The first step of induction for $g = n = 7$ and $k \leq 6$ is proved.

We complete the proof of the first step of the induction. When $n = 7$ and $a = 1$ or $a = 2$, the existence of a g. l. n. plane curve Γ always follows from theorem 2.4. Using the above notation, $h^0(C, \omega_C(-1)) = 1$ if $a = 1$ and $h^0(C, \omega_C(-1)) = 0$ if $a = 2$. In any case $\mu_{o,C}$ is injective. When $n \geq 8$ and $a \leq n - 6 = 2(n-1) - 7 - (n-1-2)$ the theorem follows by induction from the case $n = 7$. For $n \geq 8$ and $a = 2n-7-n+2 = n-5$, we find that $g = n-2$, or, equivalently, $k + d = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-4))$. Always in theorem 2.4, we proved the existence of geometrically linearly normal plane curves of degree $n \geq 8$ and genus $g = n - 2$, with nodes and $k \leq 6$ cusps. For every such plane curve Γ, using the notation above, the Brill-Noether map $\mu_{o,C}$ is injective since $h^0(C, \omega_C(-1)) = 0$. We still have to prove the first step of the induction for $n = 5$, 6. For $n = 5$ we find $g = n - 2$ and we argue as before. For $n = 6$ we find $g = 4 = n - 2$ or $g = 5 = n - 1$. By lemma 2.2, every quintic Γ of genus 4 with nodes and cusps as singularities is geometrically linearly normal. Moreover, if C is the normalization of Γ, then $\mu_{o,C}$ is injective since $h^0(C, \omega_C(-1)) = 1$. The cases $n = 6$ and $g = 4$ and $n = 6$ and $g = 5$ are similar.

Suppose now that n and g verify (3). First of all we prove the theorem for $(n, g) = (4, 2)$, $(5, 4)$, $(6, 6)$. For $n = 4$ and $g = 2$, we find $n = g + 2$ and we argue as in the case $n \geq 8$ and $g = n - 2$. Similarly, for $(n, g) = (5, 4)$. For $n = 6$ and $g = 6$ in theorem 2.4 we proved the existence of plane curves Γ with $k \leq 4$ cusps and at most nodes as singularities. For every such a plane curve Γ, denoting by C its normalization, we get that $h^0(C, \omega_C(-1)) = 2$, i.e. the linear system $\mathcal{F}$ of conics passing through the four singular points $P_1, \ldots, P_4$ of Γ is a pencil which cuts out on C the complete linear series $|\omega_C(-1)|$. Since, by Bezout theorem, two irreducible conics intersect in four points, the base locus of $\mathcal{F}$ consists of the points $P_1, \ldots, P_4$ and the linear series $|\omega_C(-1)|$ has no base points. Then, by the base point free pencil trick , we find that $ker(\mu_{o,C}) = H^0(C, \omega_C^* \otimes \mathcal{O}(2)) = H^0(C, \mathcal{O}_C(-1)(\Delta))$, where $\Delta \subset C$ is the adjoint divisor of the normalization map $C \to \Gamma$. By Riemann-Roch theorem, we have that $h^0(C, \mathcal{O}_C(-1)(\Delta)) = h^0(C, \mathcal{O}_C(4)(-2\Delta)) - 3$. If S is the blow-up of the plane at $P_1, \ldots, P_4$ and $E_1, \ldots, E_4$ are the exceptional divisors of S, then, by the following exact sequence

$$0 \to \mathcal{O}_S(-2) \to \mathcal{O}_S(4)(-2\sum_i E_i) \to \mathcal{O}_C(4)(-2\sum_i E_i) \to 0$$

we find that $h^0(C, \mathcal{O}_C(4)(-2\Delta)) = h^0(S, \mathcal{O}_S(4)(-2\sum_i E_i))$. Moreover, if $C_2 \subset S$ is the strict transform of the general conic of the pencil $\mathcal{F}$, then, by the following exact sequence

$$0 \to \mathcal{O}_S(2)(-\sum_i E_i) \to \mathcal{O}_S(2) \to \bigoplus_i \mathcal{O}_{E_i} \to 0$$

we have that $h^1(S, \mathcal{O}_S(2)(-\sum_i E_i)) = 0$. By the following exact sequence

$$0 \to \mathcal{O}_S(2)(-\sum_i E_i) \to \mathcal{O}_S(4)(-2\sum_i E_i) \to \mathcal{O}_{C_2}(4)(-2\sum_i E_i) \to 0$$

we deduce that

$$h^0(S, \mathcal{O}_S(4)(-2\sum_i E_i)) = h^0(S, \mathcal{O}_S(2)(-\sum_i E_i)) + h^0(C_2, \mathcal{O}_{C_2}(4)(-2\sum_i E_i)) = 3$$

and, in particular, $ker(\mu_{o,C}) = 0$, as we wanted.

Finally, we show the theorem under the hypothesis (3) for $n \geq 7$, by using induction on n. In order to prove the inductive step we may use lemma 3.10, exactly as we did in the case (2). We prove the first step of induction. If $n = 7$ we have that $g = 8$. On page 83 we proved the existence of geometrically linearly normal plane curves Γ of degree 7 and genus 7 with $k \leq 6$, such that, if $P_1, \dots, P_8$ are the singular points of Γ, then no seven points among $P_1, \dots, P_8$ lie on a conic. In particular, we proved that, for every such a plane curve Γ, the general element of the pencil of cubics passing through $P_1, \dots, P_8$ is irreducible and, if $\phi : C \to \Gamma$ is the normalization of Γ, then the Brill-Noether map $\mu_{o,C}$ is injective. Let C' be the partial normalization of Γ which we get by smoothing all the singular points of Γ except a node, say P_8. We recall that, if we denote by $\phi^*(P_8)$ the divisor of the points of C which lie over P_8 and by $\psi : C \to C'$ be the normalization morphism of C', then the dualizing sheaf $\omega_{C'}$ of C' is the subsheaf of $\psi_*(\omega_C(\phi^*(P_8)))$, whose local sections η are such that $\sum_{p \in \phi^{-1}(P_8)} res_p \eta = 0$. By using the same notation and by arguing exactly as in the proof of lemma 3.8, we get the following commutative diagram

$$\begin{array}{ccccc} H^0(C', \mathcal{O}_{C'}(1)) & \otimes & H^0(C', \omega_{C'}(-1)) & \stackrel{\mu_{o,C'}}{\to} & H^0(C', \omega_{C'}) \\ & \downarrow & & & \downarrow \\ H^0(C, \mathcal{O}_C(1)) & \otimes & H^0(C, \omega_C(-1)(\phi^*(P_8))) & \stackrel{\mu'_{o,C}}{\to} & H^0(C, \omega_C(\phi^*(P_8))) \end{array}$$

where $\mu'_{o,C}$ is the multiplication map and the vertical maps are isomorphisms. We want to prove that the map $\mu_{o,C'}$ is surjective. By the previous diagram it is enough to prove that $\mu'_{o,C}$ is surjective. Since $h^0(C, \omega_C(\phi^*(P_8))) = 8$ and $h^0(C, \mathcal{O}_C(1))h^0(C, \omega_C(-1)(\phi^*(P_8))) = 3(7-7+3) = 9$, we have that $dim(ker(\mu_{o,C'})) \geq 1$ and $\mu_{o,C'}$ is surjective if $dim(ker(\mu_{o,C'})) = 1$. By recalling that Γ is geometrically linearly normal, we have that, if Z is the scheme of the points $P_1, \dots, P_7$ and $\mathcal{I}_{Z|\mathbb{P}^2}$ is the ideal sheaf of Z in $\mathbb{P}^2$, then in the following commutative diagram

$$\begin{array}{ccccc} H^0(C, \mathcal{O}_C(1)) & \otimes & H^0(C, \omega_C(-1)(\phi^*(P_8))) & \stackrel{\mu'_{o,C}}{\to} & H^0(C, \omega_C(\phi^*(P_8))) \\ & \downarrow & & & \downarrow \\ H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) & \otimes & H^0(\mathbb{P}^2, \mathcal{I}_{Z|\mathbb{P}^2}(3)) & \stackrel{\mu}{\to} & H^0(\mathbb{P}^2, \mathcal{I}_{Z|\mathbb{P}^2}(4)) \end{array}$$

the vertical maps are isomorphisms. Hence, it is enough to prove that the kernel of the multiplication map μ has dimension equal to one. Let $\{f_0, f_1, f_2\}$ be a basis of the vector space $H^0(\mathbb{P}^2, \mathcal{I}_{Z|\mathbb{P}^2}(3))$. Since the general cubic passing through $P_1, \dots, P_8$ is irreducible, we may assume that f_0, f_1 and f_2 are irreducible. Suppose, by contradiction, that there exist at least two linearly independent vectors in the kernel of μ. Then, there exist sections u_0, u_1, u_2 and v_0, v_1, v_2 of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ such that the sections $\sum_i u_i \otimes f_i$ and $\sum_i v_i \otimes f_i$ are linearly independent in

$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \otimes H^0(\mathbb{P}^2, \mathcal{I}_{Z|\mathbb{P}^2}(3))$ and

$$\left\{ \begin{array}{rcl} \sum_{i=0}^3 u_i f_i & = & 0 \\ \sum_{i=0}^3 v_i f_i & = & 0. \end{array} \right. \tag{64}$$

The linear system (64) is a linear system of rank one in the variables f_0, f_1, f_2 and the space of solutions of (64) is generated by the vector $(u_1v_2-u_2v_1, u_3v_0-u_0v_3, u_0v_1-u_1v_0)$. In particular, if we set $q_i = (-1)^{1+i}u_iv_j - v_iu_j$, we find that $f_jq_i = f_iq_j$, for every $i \neq j$. But this is not possible since f_1, f_2 and f_3 are irreducible. We deduce that

$$dim(ker(\mu)) = dim(ker(\mu_{o,C'})) = 1$$

and $\mu_{o,C'}$ is surjective. Now let $\mathcal{G} \to \Delta$ be a family of plane curves of degree 7, parametrized by a smooth curve Δ, whose special fibre is equal to Γ and whose general fibre is a curve of genus 8 with a node at a neighborhood of every node of Γ different from P_8 and a cusp at a neighborhood of every cusp of Γ. Such a family there exists by the lemma 3.17 of chapter 2. If $\mathcal{C} \to \Delta$ is the family obtained by $\mathcal{G} \to \Delta$ by normalizing the total space, then the general fibre $\mathcal{C}_t$ of $\mathcal{C} \to \Delta$ is the normalization of the general fibre $\mathcal{G}_t$ of $\mathcal{G} \to \Delta$, whereas the special fibre of $\mathcal{C}$ coincides with C'. Since $h^0(C', \mathcal{O}_{C'}(1)) = 3$ and the map $\mu_{o,C'}$ is surjective, by semicontinuity, we conclude that $h^0(C_t, \mathcal{O}_{C_t}(1)) = 3$ and the Brill-Noether map

$$H^0(C_t, \mathcal{O}_{C_t}(1)) \otimes H^0(C_t, \omega_{C_t}(-1)) \stackrel{\mu_{o,C_t}}{\to} H^0(C_t, \omega_{C_t})$$

is surjective. This completes the proof of the theorem. □

Remark 3.12. *Notice that the conditions which we found in theorem 3.11 in order that $\Sigma_{k,d}^n$ has at least an irreducible component with the expected number of moduli, are not sharp, even if we suppose $\rho \leq 0$. To see this, notice that in remark 2.5 we proved the existence of an irreducible component Σ of $\Sigma_{9,0}^{12}$ whose general element corresponds to a 3-normal plane curve. By remark 2.3 and corollary 3.7, we have that Σ has the expected number of moduli.*

Theorem 3.13. *$\Sigma_{1,d}^n$ has the expected number of moduli, for every $d \leq \binom{n-1}{2} - 1$.*

Proof. Before proving the theorem, we recall that, by theorem 3.21 and corollary 3.18 of chapter 2, the variety $\Sigma_{1,d}^n$ is irreducible and not empty for every $d \leq \binom{n-1}{2} - 1$. Moreover, from theorem 3.11 and from corollary 1.7, we know that $\Sigma_{1,d}^n$ has the expected number of moduli if $\rho \leq 0$ or $\rho \geq 2$. Now we shall prove that, if $\rho = 1$, then the algebraic system

$$\Sigma_{1,d}^n = \Sigma_{1,\frac{(n-3)^2}{2}-1}^n$$

has general moduli. Equivalently, we will show that, if $[\Gamma] \in \Sigma_{1,d}^n$ is a general point and $g = \binom{n-1}{2} - 1 - d = \frac{3n-7}{2}$, then, on the normalization curve C of Γ there are only finitely many linear series g_n^2 with at least a ramification point. First of all we notice that, if $g = \binom{n-1}{2} - 1 - d = \frac{3n-7}{2}$, then n is odd and $n \geq 5$. We prove the statement by induction on n.

Let $n = 5$ and let $U \subset \Sigma_{1,1}^5$ be the open set of $\Sigma_{1,1}^5$ parametrizing irreducible plane curves Γ of degree 5 with a node and a cusp as singularities such that the cuspidal tangent line of Γ does not contain the node of Γ. In order to see that U is not empty, we recall that, by corollary 3.18 of chapter 2, there exists an irreducible plane quintic D with a cusp and two node s_1 and s_2 as singularities. By Bezout theorem, at least one, say s_1, among the nodes of D doesn't lie on the cuspidal tangent line of D. By corollary 3.17 of chapter 2, there exists a family of plane quintics $\mathcal{D} \to \Delta$ whose special fibre is equal to D and whose general fibre $\mathcal{D}_t$ has a node at a neighborhood of s_1, a cusp at a neighborhood of the cusp of D and no further singularities.

The node of $\mathcal{D}_t$ does not lie on the cuspidal tangent line of $\mathcal{D}_t$. Thus U is not empty. Now, in order to prove that $\Sigma^5_{1,1}$ has general moduli, it is enough to prove that U has general moduli. Let $\Gamma \subset \mathbb{P}^2$ be a plane quintic corresponding to a point $[\Gamma] \in U$ and let $\phi : C \to \Gamma$ be the normalization of Γ. If $p \in \Gamma$ is the cusp of Γ, we denote by $P \in C$ the point $P = \phi^{-1}(p)$. If we denote by $\mathcal{O}_C(1)$ the sheaf on C associated to the divisor which is the pull-back of the general hyperplane divisor on Γ, then $h^0(C, \mathcal{O}_C(1)) = 3$, the sheaf $\mathcal{O}_C(1)$ is special and C is a not hyperelliptic curve of genus four. Moreover, the linear series $|\omega_C(-1)|$ consists of a point $R \in C$. By the hypothesis that $[\Gamma] \in U$, we have that $R \neq P$. Indeed, if we denote by S_1 and S_2 the points of C over the node s of Γ, then $H^0(C, \omega_C(-1)) = H^0(C, \mathcal{O}_C(1)(-S_1 - S_2 - 2P))$ and $R = P$ if and only if the node of Γ lies on the cuspidal tangent line l to Γ at the point p. Thus $R \neq P$. Now, let $C \subset \mathbb{P}^3$ be a canonical model of C in $\mathbb{P}^3$. We recall that there exists an unique quadric S_2 in $\mathbb{P}^3$ containing C, and C is complete intersection of S_2 and a cubic S_3, (see for example [**22**] or [**6**]). If we still denote by P and R the image points of P and R in $\mathbb{P}^3$, then the linear series $|\mathcal{O}_C(1)|$ is cut out on C in $\mathbb{P}^3$ by the two dimension family of hyperplanes passing through the point $R \in C$. Projecting C from R we get a birational morphism from $C \subset \mathbb{P}^3$ and $\Gamma \subset \mathbb{P}^2$. Moreover, since Γ has a cusp at the point $p \in \mathbb{P}^2$, then the tangent line to C at P in $\mathbb{P}^3$ passes through $R \in C$. Now, suppose by contradiction that on C there exist infinitely many g^2_5 mapping C to the plane as a quintic parametrized by a point of $U \subset \Sigma^5_{1,1}$. Then, through the general point x of C passes a tangent line to C at a point $x' \neq x$. In particular, the general tangent lines to C cuts out on C a divisor of degree $r \geq 3$. By using Bezout theorem, it follows that the general tangent line to C is contained in S_2, i.e. S_2 is the tangent variety $T(C)$ of C. But, by using Hurwitz formula, we see that $T(C)$ has degree 8. Hence $S_2 \neq T(C)$ and, if

$$\Pi : \Sigma^5_{1,1} \dashrightarrow \mathcal{M}_4$$

is the moduli map of $\Sigma^5_{1,1}$, then $\Pi(U) = \mathcal{M}_4 = \Pi(\Sigma^5_{1,1})$.

Now we suppose that the theorem is true for n and we prove the theorem for $n+2$. Let $\Gamma \subset \mathbb{P}^2$ the plane curve with a cusps and $\frac{(n-3)^2}{2} - 1$ nodes corresponding to a general point $[\Gamma] \in \Sigma^n_{1,\frac{(n-3)^2}{2}-1}$ and let C_2 be an irreducible plane conic intersecting Γ transversally. By lemmas 3.17 of chapter 2 the point $[C_2 \cup \Gamma]$ belongs to $\Sigma^{n+2}_{1,\frac{(n+2-3)^2}{2}-1}$. In particular, however we choose four points $P_1, \dots, P_4$ of intersection between Γ and C_2, there exists an analytic branch $\mathcal{S}_{P_1,\dots,P_4}$ of $\Sigma^{n+2}_{1,\frac{(n-1)^2}{2}-1}$, passing through $[C_2 \cup \Gamma]$ and whose general point corresponds to an irreducible plane curve of degree $n+2$ with a cusp at a neighborhood of the cusp of Γ and a node at a neighborhood of every node of $C_2 \cup \Gamma$ different from $P_1, \dots, P_4$. Moreover, it follows by lemma 3.22 of chapter 2 that $\mathcal{S} := \mathcal{S}_{P_1,\dots,P_4}$ is smooth at the point $[C_2 \cup \Gamma]$. Let

$$\Pi : \Sigma^{n+2}_{1,\frac{(n-1)^2}{2}-1} \dashrightarrow \mathcal{M}_{\frac{3(n+2)-7}{2}}$$

be the moduli map of $\Sigma^{n+2}_{1,\frac{(n-1)^2}{2}-1}$. In order to prove that Π is dominant it is sufficient to show that $\Pi(\mathcal{S}) = \mathcal{M}_{\frac{3n-1}{2}}$. Always by lemma 3.17 of chapter 2, there exists an analytic branch $\mathcal{S}^i$ of $\Sigma^{n+2}_{1,\frac{(n-3)^2}{2}-1+2n-i}$, with $i = 1, 2, 3$, and such that

$$\mathcal{S}^0 := \mathcal{S} \cap (\mathbb{P}^5 \times \Sigma^n_{1,\frac{(n-3)^2}{2}-1}) \subset \mathcal{S}^1 \subset \mathcal{S}^2 \subset \mathcal{S}^3 \subset \mathcal{S}.$$

The general point of every irreducible component of $\mathcal{S}^i$, with $i = 1, 2, 3$, corresponds to an irreducible plane curve Γ_i of degree $n+2$ with a cusp at a neighborhood of the cusp of Γ, a node at a neighborhood of every node of $C_2 \cup \Gamma$ different from $P_1, \dots, P_4$ and $4-i$ nodes specializing to $4-i$ fixed points among $P_1, \dots, P_4$, as Γ_i specializes to $C_2 \cup \Gamma$. Moreover, it follows by lemma 3.22 of chapter 2 that every irreducible component of $\mathcal{S}^i$ is smooth at $[C_2 \cup \Gamma]$ and, hence, $\mathcal{S}^i$ has an ordinary multiple point at $[C_2 \cup \Gamma]$ of order $\binom{4}{i}$ for every $i = 1, 2, 3$. Now, notice that the moduli map Π is not defined at the point $[C_2 \cup \Gamma]$, but, if $\mathcal{S}$ is sufficiently small, then the restriction of Π to $\mathcal{S}$ extends to a regular function on $\mathcal{S}$. More precisely, let $\mathcal{C} \to \Delta$ be any family of curves, parametrized by a projective curve $\Delta \subset \mathcal{S}$, passing thought the point $[C_2 \cup \Gamma]$ and whose general point corresponds to an irreducible plane curve of degree $n+2$ of genus $\frac{3n-1}{2} = \frac{3(n+2)-7}{2}$ with a cusp and nodes as singularities. If we denote by $\mathcal{C}' \to \Delta$ the family of curves obtained from $\mathcal{C} \to \Delta$ by normalizing the total space, we have that the general fibre of $\mathcal{C}' \to \Delta$ is a smooth curve of genus $\frac{3n-1}{2}$, corresponding to the normalization of the general fibre of $\mathcal{C} \to \Delta$, whereas the special fibre $\mathcal{C}'_0$ is the partial normalization of $C_2 \cup \Gamma$, obtained by normalizing all the singular points, except $P_1, \dots, P_4$. Then, the map $\Pi_{|\mathcal{S}}$ is defined at $[C_2 \cup \Gamma]$ and it associates to the point $[C_2 \cup \Gamma]$ the isomorphism class of $\mathcal{C}'_0$. Similarly, if $[C'_2 \cup \Gamma']$ is a point of $\mathcal{S}^0$, corresponding to the union of an irreducible conic C'_2 and an irreducible plane curve Γ' of degree n with a cusps and nodes, then $\Pi_{|\mathcal{S}}([C'_2 \cup \Gamma'])$ is the isomorphism class of the partial normalization of $C'_2 \cup \Gamma'$, obtained by smoothing all the singular points of $C'_2 \cup \Gamma'$, except the four nodes tending to $P_1, \dots, P_4$ as $C'_2 \cup \Gamma'$ tends to $C_2 \cup \Gamma$. Finally, if $[\Gamma_i]$ is a general point in one of the irreducible components of $\mathcal{S}^i$, with $i = 1, 2, 3$, then $\Pi_{|\mathcal{S}}([\Gamma_i])$ is the partial normalization of Γ_i obtained by smoothing all the singular points except for the $4-i$ nodes of Γ_i tending to $P_1, \dots, P_4$ as Γ_i tends to $C_2 \cup \Gamma$. It follows that, if we denote by $\mathcal{M}^j_{\frac{3n-1}{2}}$ the locus of $\mathcal{M}_{\frac{3n-1}{2}}$ parametrizing j-nodal curves, then $\Pi_{\mathcal{S}}(\mathcal{S}^i) \subseteq \mathcal{M}^{4-i}_{\frac{3n-1}{2}}$, for every $i = 0, \dots, 4$, and $\Pi_{\mathcal{S}}(\mathcal{S}^i) \subsetneq \Pi_{\mathcal{S}}(\mathcal{S}^{i+1})$. In particular, we find that

$$dim(\Pi_{|\mathcal{S}}(\mathcal{S})) \geq dim(\Pi_{|\mathcal{S}}(\mathcal{S}^0)) + 4.$$

In order to compute the dimension of $\Pi_{|\mathcal{S}}(\mathcal{S}^0)$ we consider the rational map

$$F : \Pi_{|\mathcal{S}}(\mathcal{S}^0) \dashrightarrow \mathcal{M}_{\frac{3n-7}{2}}$$

forgetting the rational tail. Since, by the hypothesis that $\Sigma^n_{1, \frac{(n-3)^2}{2}-1}$ has general moduli, we have that F is dominant. Moreover, if C is the normalization curve of Γ, by the generality of $[\Gamma]$ in $\Sigma^n_{1, \frac{(n-3)^2}{2}-1}$, we may assume that C is general in $\mathcal{M}_{\frac{3n-7}{2}}$. We want to show that $dim(F^{-1}([C])) = 5$. In order to see this, we recall that, by the hypothesis that $\Sigma^n_{1, \frac{(n-3)^2}{2}-1}$ has general moduli, on C there exist only a finite number of linear series of degree n and dimension two, mapping C to the plane as curve with a cusp and nodes as singularities. If g^2_n is such a linear series, $\{s_0, s_1, s_2\}$ is a basis of g^2_n and $\phi' : C \to \Gamma' \subset \mathbb{P}^2$ is the associated morphism, then, if $Q_1, \dots, Q_4$ are four general points of Γ', the linear system of conics through $Q_1, \dots, Q_4$ is a pencil $\mathcal{F}(Q_1, \dots, Q_4)$. Let C'_2 and D'_2 be two general conics of $\mathcal{F}(Q_1, \dots, Q_4)$. If $\eta : \mathbb{P}^1 \to C'_2$ and $\beta : \mathbb{P}^1 \to D'_2$ are isomorphisms between $\mathbb{P}^1$ and C'_2 and D'_2 respectively, then the points $\eta^{-1}(Q_1), \dots, \eta^{-1}(Q_4)$ are not projectively equivalent to the points $\beta^{-1}(Q_1), \dots, \beta^{-1}(Q_4)$. In order to prove this, we may consider a conic $F \subset \mathbb{P}^2$ of the plane. If we choose two sets of points $p_1, \dots, p_4$ and $q_1, \dots, q_4$ of F not projectively equivalent, then there exist projective automorphisms $A : \mathbb{P}^2 \to \mathbb{P}^2$ and $A' : \mathbb{P}^2 \to \mathbb{P}^2$ such that $A(p_i) = Q_i$ and $A'(q_i) = (Q_i)$, for every i. This shows that, if C_2' and D_2' are general in $\mathcal{F}(Q_1,\dots,Q_4)$, then the points $\eta^{-1}(Q_1),\dots,\eta^{-1}(Q_4)$ are not projectively equivalent to the points $\beta^{-1}(Q_1),\dots,\beta^{-1}(Q_4)$. In particular, this implies that the partial normalizations C' and D' of $\Gamma'\cup C_2'$ and $\Gamma'\cup D_2'$, obtained by smoothing all the singular points except $Q_1,\dots,Q_4$, are not isomorphic. Now, let C_2' be a general conic of $\mathcal{F}(Q_1,\dots,Q_4)$ and let $R_1,\dots,R_4$ be four general points of Γ' and different from $Q_1,\dots,Q_4$. If D_2' is a general conic of the pencil $\mathcal{F}(R_1,\dots,R_4)$, then the partial normalization C' and D' of $\Gamma'\cup C_2'$ and $\Gamma'\cup D_2'$ obtained, respectively, by smoothing all the singular points except $Q_1,\dots,Q_4$ and $R_1,\dots,R_4$, are not isomorphic. Indeed, since C is a general curve of genus $\frac{3n-7}{2}\geq 7$, the only automorphism of C is the identity. We deduce that $dim(F^{-1}([C]))=5$. In particular, we have proved that

$$dim(\Pi_{|\mathcal{S}}(\mathcal{S}^0)) = 3\frac{3n-7}{2}-3+5$$

and

$$dim(\Pi_{|\mathcal{S}}(\mathcal{S})\geq 3\frac{3n-7}{2}-3+9 = 3\frac{3(n+2)-7}{2}-3.$$

□

By using theorem 7.11 of chapter 2, theorem 1.6 and theorem 3.11, we are extending the previous theorem to the case $k\leq 3$.

4. On the number of moduli of complete irreducible families of plane sextics with six cusps

Corollary 3.9 tells us how to compute the number of moduli of an irreducible component Σ' of $\Sigma^n_{k',d'}$ if we know the number of moduli of an irreducible component Σ of $\Sigma^n_{k,d}$ such that $\Sigma\subset\Sigma'$, under the hypothesis that the Brill-Noether number $\rho(2,n,g)$ of the linear series of dimension two and degree n on the normalization curve of the plane curve corresponding to the general element of Σ is not positive. In this section we shall prove a result like corollary 3.9 by assuming $\rho-k\leq 0$ but not necessarily $\rho\leq 0$. In order to do this we need the following result.

Fact 4.1. *Let*

$$\begin{array}{ccl}\mathcal{D} & = & \{(a,b,x,y)|\, y^2=x^3+ax+b\}\subset\mathbb{C}^2\times\mathbb{A}^2\\ \downarrow & & \\ \mathbb{C}^2 & & \end{array}$$

*be the versal deformation family of an ordinary cusp, (see section 5 of chapter 2). We recall that the general curve of this family is smooth. The locus Δ of $\mathbb{C}^2$ of the pairs (a,b), such that the corresponding curve is singular, has equation $27b^2=4a^3$. For $(a,b)\in\Delta$ and $(a,b)\neq(0,0)$, the corresponding curve has a node and no other singularities. Let $\mathcal{G}\to\mathbb{C}^2$ be a two parameter family of smooth curves of genus $g\geq 2$, locally given by $y^2=x^3+ax+b$, with $(a,b)\in\mathbb{C}^2$. It has been proved in [**21**], page 129, that, if $\Delta\subset\mathbb{C}^2$ is a curve passing through $(0,0)$ and not tangent to the axis $b=0$ at $(0,0)$, then the j-invariant of the elliptic tail of the curve which corresponds to the stable limit of $\mathcal{G}_{(0,0)}$, with respect to Δ, doesn't depend on Δ. Otherwise, for every $j_0\in\mathbb{C}$, there exists a curve $\Delta_{j_0}\subset\mathbb{C}^2$ passing through $(0,0)$ and tangent to the axis $b=0$ at this point, such that the elliptic tail of the stable reduction of $\mathcal{G}_{(0,0)}$ with respect to Δ_{j_0}, has j-invariant equal to j_0.*

Lemma 4.2. *Let $\Sigma \subset \Sigma^n_{k,d}$, with $k < 3n$, be an irreducible component of $\Sigma^n_{k,d}$. Let g be the geometric genus of the plane curve corresponding to the general element of Σ. Suppose that $g \geq 2$, $\rho(2,g,n) - k \leq 0$ and Σ has the expected number of moduli equal to $3g-3+\rho-k$. Then, for every $k' \leq k$ and $d' \leq d$ such that $g' = \binom{n-1}{2} - k' - d' > g = \binom{n-1}{2} - k - d$, there is at least a component Σ' of $\Sigma^n_{k',d'}$ such that $\Sigma \subset \Sigma'$ and such that the number of moduli of Σ' is equal to $3g' - 3 + \rho(2,g',n) - k'$.*

Proof. It is enough to prove the theorem in the following two cases:
(a) $k' = k-1$ and $d' = d$,
(b) $k' = k$ and $d' = d-1$.

Suppose that (a) holds. Let $q_1, \ldots, q_k$ be the cusps of Γ. Since $k < 3n$, by using lemmas 3.17 and 3.22 of chapter 2, $[\Gamma]$ is a k-fold ordinary point for $\Sigma^n_{k-1,d}$. In particular, for every fixed cusp p_i of Γ there exists an analytic smooth branch $\mathcal{S}_i$ of $\Sigma^n_{k-1,d}$ passing through the point $[\Gamma]$ and whose general point corresponds to a plane curve Γ' of degree n with d nodes and $k-1$ cusps specializing to the singular points of Γ different from p_i, as Γ' specializes to Γ. Let Σ' one of the irreducible components of $\Sigma^n_{k-1,d}$ containing Σ. Since $\rho(2,g',n) - k' = 3n - 2g - 2 - 6 - k + 1 = \rho(2,g,n) - k - 1 < 0$, in order to prove the theorem it is enough to show that the general fibre of the moduli map

$$\Pi_{\Sigma'} : \Sigma' \dashrightarrow \mathcal{M}_{g+1}$$

has dimension equal to eight. Let us notice that the map $\Pi_{\Sigma'}$ is not defined at the general element $[\Gamma]$ of Σ. More precisely, let $\gamma \subset \mathcal{S}_i \subset \Sigma'$ be a curve passing through $[\Gamma]$ and not contained in Σ. Let $\mathcal{C} \to \gamma$ be the tautological family of plane curves parametrized by γ. Let $\mathcal{C}' \to \gamma$ be the family obtained from $\mathcal{C} \to \gamma$ by normalizing the total space. The general fibre of $\mathcal{C}' \to \gamma$ is a smooth curve of genus $g+1$, while the special fibre $\mathcal{C}'_0 := \Gamma'$ is the partial normalization of Γ obtained by smoothing all the singular points of Γ, except the marked cusp p_i. If we restrict the moduli map $\Pi_{\Sigma'}$ to γ, we get a regular map which associates to $[\Gamma]$ the point corresponding to the stable reduction of Γ' with respect the family $\mathcal{C}' \to \gamma$, which is the union of the normalization curve C of Γ and an elliptic curve, intersecting at the point $p \in C$ which maps to the cusp $p_i \in \Gamma$. Now, let $\mathcal{G} \subset \Sigma' \times \mathcal{M}_{g+1}$ be the graph of $\Pi_{\Sigma'}$, let $\pi_1 : \mathcal{G} \to \Sigma'$ and $\pi_2 : \mathcal{G} \to \mathcal{M}_{g+1}$ be the natural projections and let $U \subset \Sigma$ be the open set parametrizing curves of degree n and genus g with exactly k cusps and d nodes as singularities. From what we observed before, if we denote by $\Pi_{\Sigma'}(\Sigma)$ the Zariski closure in $\mathcal{M}_{g+1}$ of $\pi_2\pi_1^{-1}(U)$, then $\Pi_{\Sigma'}(\Sigma)$ is contained in the divisor $\Delta_1 \subset \mathcal{M}_{g+1}$, whose points are the isomorphism class of the reducible curves which are union of a smooth curve of genus g and an elliptic curve, meeting at a point. Denoting by $\Pi_\Sigma : \Sigma \to \mathcal{M}_g$ the moduli map of Σ, the rational map

$$\Delta_1 \dashrightarrow \mathcal{M}_g$$

which forgets the elliptic tail, restricts to a rational dominant map

$$q : \Pi_{\Sigma'}(\Sigma) \dashrightarrow \Pi_\Sigma(\Sigma).$$

The dimension of the general fibre of q is at most two. Since, by hypothesis, the dimension of the fibre of the moduli map Π_Σ is eight, there exists only a finite number of g^2_n on C, ramified at k points, which maps C to a plane curve D such that the associated point $[D] \in \mathbb{P}^{\frac{n(n+3)}{2}}$ belongs to Σ. In particular, the set of points x of C such that there is a g^2_n with k simple ramification points, one of which at x, is finite. So, the dimension of the general fibre of q is at most one. In order to decide if the general fibre of q has dimension zero or one, we have to understand how the j-invariant of the elliptic tail of the stable reduction of Γ' with respect the family $\mathcal{C}' \to \gamma$, depends on γ. If $\mathcal{C} \to \mathbb{C}^2$ is the étale versal deformation family of the cusp, by versality, for every fixed cusp p_i of Γ, there exist étale neighborhoods $U \stackrel{u}{\to} \mathbb{P}^{\frac{n(n+3)}{2}}$ of $[\Gamma]$ in $\mathbb{P}^{\frac{n(n+3)}{2}}$, $V \stackrel{v}{\to} \mathbb{C}^2$ of $(0,0)$ in $\mathbb{C}^2$ and U_i of p_i in the tautological family $\mathcal{U} \to \mathbb{P}^{\frac{n(n+3)}{2}}$ with a morphism $f : U \to V$ such that the family $U_i \to U$ is the pullback, with respect to f, of the restriction to V of the versal family. By the property (3) of theorem 5.10 of chapter 2, we have that $f^{-1}((0,0))$ is an étale neighborhood of $[\Gamma]$ in the (smooth) analytic branch $\Sigma^n_{1,0}$ whose general element corresponds to an irreducible plane curves with only one cusp at a neighborhood of the cusp p_i of Γ. So, $dim(f^{-1}((0,0))) = \frac{n(n+3)}{2} - 2$ and the map f is surjective. Moreover, if g is the restriction of f at $u^{-1}(\Sigma')$, then also g is surjective. Indeed,

$$g^{-1}((0,0)) = f^{-1}((0,0)) \cap u^{-1}(\Sigma') = u^{-1}((\Sigma)).$$

Since $k < 3n$, then $dim(\Sigma) = 3n + g - 1 - k = dim(\Sigma') - 2$ and g is surjective. By using (4.1), it follows that the general fibre of the natural map $\Pi_{\Sigma'}(\Sigma) \to \Pi_\Sigma(\Sigma)$ has dimension exactly equal to one. Therefore,

$$dim(\Pi_{\Sigma'}(\Sigma)) = dim(\Pi_\Sigma(\Sigma)) + 1 = 3g - 3 + \rho(2, g, n) - k + 1 = 3(g+1) - 3 + \rho(2, g+1, n) - k,$$

and

$$dim(\Pi_{\Sigma'}(\Sigma')) \geq dim(\Pi_{\Sigma'}(\Sigma)) + 1 = 3(g+1) - 3 + \rho(2, g+1, n) - k + 1.$$

Since it is always true that $dim(\Pi_{\Sigma'}(\Sigma')) \leq 3(g+1) - 3 + \rho(2, g+1, n) - k + 1$, (see section 1), the statement has been proved in this case.

Suppose, now, that $k = k'$ and $d' = d - 1$. Also in this case Σ is not contained in the regularity domain of $\Pi_{\Sigma'}$. More precisely, if $[\Gamma] \in \Sigma$ is general, then $\Pi_{\Sigma'}([\Gamma])$ consists of a finite number of points, corresponding to the isomorphism class of the partial normalizations of Γ obtained by smoothing all the singular points of Γ, except for a node. Then $\Pi_{\Sigma'}(\Sigma)$ is contained in the divisor Δ_0 of $\mathcal{M}_{g+1}$ parametrizing the isomorphism classes of the analytic curves of arithmetic genus $g + 1$ with a node and no more singularities. The natural map $\Delta_0 \dashrightarrow \mathcal{M}_g$ sending the general point $[C']$ of Δ_0 to the isomorphism class of the normalization of C', restricts to a rational dominant map $q : \Pi_{\Sigma'}(\Sigma) \dashrightarrow \Pi_\Sigma(\Sigma)$. Since we suppose that Σ has the expected number of moduli and $\rho(2, g, n) - k \leq 0$, if C is the normalization of the plane curve corresponding to the general element of Σ, then the set S of the linear series of dimension 2 and degree n on C with k simple ramification points, mapping C to a plane curve D such that the associated point $[D]$ in the Hilbert Scheme belongs to Σ, is finite. We deduce that also the set S' of the pairs of points (p_1, p_2) of C, such that there is a $g^2_n \in S$ such that the associated morphism maps p_1 and p_2 to the same point of the plane, is finite. So, also $q^{-1}([C])$ is finite and $dim(\Pi_{\Sigma'}(\Sigma)) = dim(\Pi_\Sigma(\Sigma))$. It follows that

$$dim(\Pi_{\Sigma'}(\Sigma')) \geq 3g - 3 + 3n - 2g - 6 - k + 1 = 3(g+1) - 3 + 3n - 2(g+1) - 6 - k.$$

□

Remark 4.3. *Notice that, the arguments used before to prove lemma 4.2 don't work if the dimension of the general fibre of the moduli map of Σ has dimension bigger than eight. Indeed, in this case, the dimension of the general fibre of the map $\Pi_{\Sigma'}(\Sigma) \dashrightarrow \Pi_\Sigma(\Sigma)$ could be bigger than one if $k' = k - 1$ and $d = d'$, or than zero if $k' = k$ and $d = d' - 1$.*

Corollary 4.4. *There exists at least one irreducible component Σ_2 of $\Sigma^6_{6,0}$ having the expected number of moduli equal to $dim(\mathcal{M}_4) - 2$ and whose general element corresponds to a sextic with six cusps not on a conic.*

Proof. Let $\Sigma^6_{9,0}$ be the variety of elliptic plane curves of degree six with nine cusps and no more singularities. It is not empty and irreducible, because, by the Plücker formulas (see [**6**]), the family of dual curves is $\Sigma^3_{0,0} \simeq \mathbb{P}^9$, which is irreducible and not empty. Moreover, if we compose an holomorphic map $\phi : C \to \mathbb{P}^2$ from a complex torus C to a smooth plane cubic with the natural map $\phi(C) \to \phi(C)^*$, we get a morphism from C to a plane sextic with nine cusps, and viceversa. Therefore, the number of moduli of $\Sigma^6_{9,0}$ is equal of the number of moduli of $\Sigma^3_{0,0}$, equal to one. Since $6 < 3n = 18$, there is at least one irreducible component Σ' of $\Sigma^6_{8,0}$ containing $\Sigma^6_{9,0}$. Let $\Pi_{\Sigma'} : \Sigma' \dashrightarrow \mathcal{M}_2$ be the moduli map of Σ' and let $\mathcal{G} \subset \Sigma' \times \mathcal{M}_2$ be its graph. If we denote by $\pi_1 : \mathcal{G} \to \Sigma'$ and $\pi_2 : \mathcal{G} \to \mathcal{M}_2$ the natural projection, by U the open set of $\Sigma^6_{9,0}$ parametrizing cubics of genus one with nine cusps and by $\Pi_{\Sigma'}(\Sigma^6_{9,0})$ the Zariski closure in $\mathcal{M}_2$ of $\pi_2\pi_1^{-1}(U)$, then, by arguing as in the first part of the proof of the lemma 4.2, we have a dominant map $\Pi_{\Sigma'}(\Sigma^6_{9,0}) \dashrightarrow \mathcal{M}_1$, whose general fibre has dimension one. We conclude that

$$dim(\pi_{\Sigma'}(\Sigma')) \geq dim(\pi_{\Sigma'}(\Sigma^6_{9,0})) + 1 = 3$$

and so, the moduli map of Σ' is dominant, as one expects, because $\rho(2,2,6)-8 = 18-4-6-8 = 0$. Let D be the plane sextic corresponding to the general point of Σ'. By Bezout theorem, the height cusps $P_1, \dots, P_8$ of D don't belong to a conic and, for however we choose five cusps of D, no four of them lie on a line. Then, let C_2 be the unique conic containing $P_1, \dots, P_5$. There exists at least a cusp, say P_6, which does not belong to C_2. 'By smoothing the cusps P_7 and P_8' of Γ and by applying lemma 4.2, we get a family of irreducible sextics with six cusps not on a conic as singularities, parametrized by a curve Δ, contained in an irreducible component Σ_2 of $\Sigma^6_{6,0}$ with the expected number of moduli. □

Now we consider the irreducible component Σ_1 of $\Sigma^6_{6,0}$ parametrizing plane curves of equation $f_3^2(x_0,x_1,x_2) + f_2^3(x_0,x_1,x_2) = 0$, where f_2 is an homogeneous polynomial of degree two and f_3 is an homogeneous polynomial of degree three, (see remark 3.15 of chapter 2). The general element of Σ_1 corresponds to an irreducible plane curve of degree six with six cusps on a conic. We want to show that Σ_1 has the expected number of moduli equal to $12-3+\rho(2,4,6)-6 = 7 = dim(\mathcal{M}_4)-2$. Equivalently, we want to show that the general fibre of the moduli map

$$\Sigma_1 \dashrightarrow \mathcal{M}_4$$

has dimension equal to eight.

Lemma 4.5. *Let $\Gamma_2 : f_2(x_0,x_1,x_2) = 0$ and $\Gamma_3 : f_3(x_0,x_1,x_2) = 0$ be a smooth conic and a smooth cubic intersecting transversally. Then, the plane curve*

$$\Gamma : f_3^2(x_0,x_1,x_2) - f_2^3(x_0,x_1,x_2) = 0$$

is an irreducible sextic of genus four with six cusps at the intersection points of Γ_2 and Γ_3 as singularities. The curve Γ is projection of a canonical curve $C \subset \mathbb{P}^3$ from a point $p \in \mathbb{P}^3$ which is contained in six tangent lines to C. Moreover, for every point $q \in \mathbb{P}^3 - C$ such that the projection plane curve $\pi_q(C)$ of C from q is a sextic with six cusps on a conic of equation $g_3^2(x_0,x_1,x_2) - g_2^3(x_0,x_1,x_2) = 0$, where g_3 and g_2 are two homogeneous polynomials of degree three and two respectively, there exists a cubic surface $S_3 \in |\mathcal{I}_{C|\mathbb{P}^3}(3)|$, containing C, such that the plane curve $\pi_q(C)$ is the branch locus of the projection $\pi_q : S_3 \to \mathbb{P}^2$.

Remark 4.6. *It was pointed out to our attention by G. Pareschi, who provided a proof of this, that every irreducible sextic with six cusps on a conic as singularities has equation given* *by* $(f_2(x_0,x_1,x_2))^3 + (f_3(x_0,x_1,x_2))^2 = 0$*, with* f_2 *and* f_3 *homogeneous polynomials of degree two and three. In order words, all the sextics with six cusps on a conic as singularities are parametrized by a point of* Σ_1*.*

Proof of lemma 4.5. Let $f(x_0,x_1,x_2) = f_3^2(x_0,x_1,x_2) - f_2^3(x_0,x_1,x_2) = 0$ be the equation of Γ. From the relation $f_3(\underline{x}) = \pm f_2(\underline{x})\sqrt{f_2(\underline{x})}$, we deduce that $\frac{\partial f_3}{\partial x_i}(\underline{x}) = \pm 2\frac{\partial f_2}{\partial x_i}(\underline{x})\sqrt{f_2(\underline{x})}$ and hence

$$\frac{\partial f}{\partial x_i}(\underline{x}) = 2\frac{\partial f_3}{\partial x_i}(\underline{x})f_3(\underline{x}) - 3f_2(\underline{x})^2\frac{\partial f_2}{\partial x_i}(\underline{x}) = -f_2(\underline{x})^2\frac{\partial f_2}{\partial x_i}(\underline{x}). \tag{65}$$

By using that the conic $\Gamma_2 : f_2 = 0$ is smooth, it follows that, if a point $\underline{x} \in \Gamma$ is singular, then $\underline{x} \in \Gamma_2$ and hence $\underline{x} \in \Gamma_3 \cap \Gamma_2$. On the other hand, always from (65), if $\underline{x} \in \Gamma_2 \cap \Gamma_3$, then $\underline{x}$ is a singular point of Γ. Hence, the singular locus of Γ coincides with $\Gamma_3 \cap \Gamma_2$. Let $\underline{x}$ be a singular point of Γ. If

$$p_1(x,y) + \text{terms of degree two} = 0$$

and

$$q_1(x,y) + \text{terms of degree } \geq \text{ two} = 0$$

are respectively affine equations of Γ_2 and Γ_3 at $\underline{x}$, then, the affine equation of Γ at x is given by

$$q_1(x,y)^2 - p_1(x,y)^3 + \text{terms of degree } \geq \text{ four} = 0.$$

Since Γ_2 and Γ_3 intersect transversally, we have that $q_1(x,y)$ does not divide $p_1(x,y)$ and hence Γ has an ordinary cusp at x. Let now $\phi : C \to \Gamma$ be the normalization of Γ. Then, by section 1 of chapter 1, the cubics passing through the six cusps of Γ cut out on C the complete canonical series $|\omega_C|$. Since the cusps of Γ is contained in the conic $\Gamma_2 \subset \mathbb{P}^2$ of equation $f_2 = 0$, the lines of $\mathbb{P}^2$ cut out on C a subseries $g \subset |\omega_C|$ of dimension two of the canonical series. Moreover, if we still denote by C a canonical model of C in $\mathbb{P}^3$, then the linear series g is cut out on C in $\mathbb{P}^3$ from a two dimensional family of hyperplanes passing through a point $p \in \mathbb{P}^3 - C$. If we project C from p we get a plane curve projectively equivalent to Γ. Since Γ has six cusps as singularities, we deduce that there are six tangent lines to C passing through p. To see that Γ is the branch locus of a triple plane, let $S_3 \subset \mathbb{P}^3$ be the cubic surface of equation

$$F_3(x_0,\dots,x_3) = x_3^3 - 3f_2(x_0,x_1,x_2)x_3 + 2f_3(x_0,x_1,x_2) = 0.$$

If $p = [0,0,0,1]$, then, by using Implicit Function Theorem, the ramification locus of the projection $\pi_p : S_3 \to \mathbb{P}^2$, is given by the intersection of S_3 with the quadric S_2 of equation $\frac{\partial F_3}{\partial x_3} = x_3^2 - f_2(x_0,x_1,x_2) = 0$. Now, if $\underline{x} = [x_0,x_1,x_2] \in S_3 \cap S_2$, then $x_3 = \pm\sqrt{f_2(x_0,x_1,x_2)}$. By substituting in the equation of S_3, we find that the branch locus of the projection $\pi_p : S_3 \to \mathbb{P}^2$ coincides with the plane curve Γ. From what we proved before, it follows that the ramification locus of the projection map $\pi_p : S_3 \to \mathbb{P}^2$ is the normalization curve C of Γ. Finally, if $q \in \mathbb{P}^3 - C$ is an other point such that the plane projection $\pi_q(C)$ is an irreducible sextic with six cups on a conic parametrized by a point $x_q \in \Sigma_1 \subset \mathbb{P}^{27}$, then, up to projective motion, we may always assume that $q = [0:0:0:1]$ and hence, if $g_3^2(x_0,x_1,x_2) - g_2^3(x_0,x_1,x_2) = 0$ is the equation of the plane curve $\pi_q(C)$, then C is the locus of ramification of the projection from q to the plane of the cubic surface of equation

$$x_3^3 - 3g_2(x_0,x_1,x_2)x_3 + 2g_3(x_0,x_1,x_2) = 0.$$

□

Corollary 4.7. *The irreducible component* Σ_1 *of* $\Sigma^6_{6,0}$ *parametrizing plane curves of equation* $f_3^2(x_0,x_1,x_2)+f_2^3(x_0,x_1,x_2)=0$, *where* f_2 *is an homogeneous polynomial of degree* 2 *and* f_3 *is an homogeneous polynomial of degree three, has the expected number of moduli equal to* $7=dim(\mathcal{M}_4)+\rho(2,4,6)-6$.

Proof. Let $[\Gamma]\subset\mathbb{P}^2$ be a plane sextic of equation $f_3^2(x_0,x_1,x_2)-f_2^3(x_0,x_1,x_2)=0$, where the conic $f_2=0$ and the cubic $f_3=0$ are smooth and they intersect transversally. Let $C\subset\mathbb{P}^3$ be the normalization curve of Γ and let $\mathcal{S}_C$ be the set of points $\underline{v}=[v_0:\dots:v_3]\in\mathbb{P}^3$ such that there exists a cubic surface $S_3\in|\mathcal{I}_{C|\mathbb{P}^3}(3)|$, containing C, such that the plane curve C is the ramification locus of the projection $\pi_{\underline{v}}:S_3\to\mathbb{P}^2$. By the former lemma, in order to prove that Σ_1 has the expected number of moduli, it is enough to find a point $[\Gamma]$ of Σ_1 corresponding to an irreducible plane sextic $\Gamma\subset\mathbb{P}^2$ with six cusps of a conic such that the set $\mathcal{S}_C$ is finite. Let Γ_2 be the smooth conic of equation $f_2(x_0,x_1,x_2)=x_0^2+x_1^2-x_2^2=0$ and let Γ_3 be the smooth cubic of equation $f_3(x_0,x_1,x_2)=x_0^3+x_0x_2^2-x_1^2x_2=0$. If a_1,a_2 and a_3 are the three different solutions of the polynomial $x^3+x^2+x-1=0$, then Γ_2 and Γ_3 intersect transversally at the points $[a_i,\sqrt{a_i},1]$, $[a_i,-\sqrt{a_i},1]$, with $i=1,2,3$. By the former lemma, the plane sextic Γ of equation $f_2^3-f_3^2=0$ is irreducible and it has six cusps at the intersection points of Γ_2 and Γ_3 as singularities. Moreover, the normalization curve C of Γ is the canonical curve of genus 4 in $\mathbb{P}^3$ which is intersection of the cubic surface $S_3\subset\mathbb{P}^3$ of equation

$$F_3(x_0,x_1,x_2,x_3)=x_3^3+(x_0^2+x_1^2-x_2^2)x_3+x_0^3+x_0x_2^2-x_1^2x_2=0$$

and the quadric S_2 of equation

$$\frac{\partial F_3}{\partial x_3}=3x_3^2+x_0^2+x_1^2-x_2^2=0.$$

We want to show that $\mathcal{S}_C$ is finite. To see this we observe that, since $h^0(\mathbb{P}^3,\mathcal{I}_{C|\mathbb{P}^3}(2))=1$ and $h^0(\mathbb{P}^3,\mathcal{I}_{C|\mathbb{P}^3}(3))=6$, the equation of every cubic surface containing C and which is not the union of S_2 and an hyperplane is given by

$$G(x_0,\dots,x_3;\beta_0,\dots,\beta_3)=F_3(x_0,x_1,x_2,x_3)+\sum_{j=0}^{3}\beta_jx_j\frac{\partial F_3(x_0,x_1,x_2,x_3)}{\partial x_3}=0,$$

with $\beta_j\in\mathbb{C}$, for $i=0,\dots,3$. Now, a point $[\underline{v}]=[v_0,\dots,v_3]\in\mathcal{S}_C$ if and only if there exist $\beta_0,\dots,\beta_3$ such that C is contained in the intersection of $G(x_0,\dots,x_3;\beta_0,\dots,\beta_3)=0$ and $\frac{\partial G(x_0,\dots,x_3;\beta_0,\dots,\beta_3)}{\partial\underline{v}}=0$. Still using that $h^0(\mathbb{P}^3,\mathcal{I}_{C|\mathbb{P}^3}(2))=1$, a point $[\underline{v}]\in\mathbb{P}^3$ belongs to $\mathcal{S}_C$ if and only if

$$\frac{\partial G(x_0,\dots,x_3;\beta_0,\dots,\beta_3)}{\partial\underline{v}}=\lambda\frac{\partial F_3(x_0,\dots,x_3)}{\partial x_3}\tag{66}$$

for some $\lambda\in\mathbb{R}-0$, or, equivalently,

$$\sum_{i=0}^{2}v_i\frac{\partial F_3}{\partial x_i}+\sum_{i=0}^{3}v_i(\sum_{j=0}^{3}\beta_jx_j)\frac{\partial F_3}{\partial x_3\partial x_i}=(\lambda-\sum_{i=0}^{3}v_i\beta_i-v_3)\frac{\partial F_3}{\partial x_3}.\tag{67}$$

The previous equality of polynomials is equivalent to the following bilinear system of ten equations in the variables $v_0, \dots, v_3$ and $\beta_0, \dots, \beta_3$

$$
(68) \qquad \begin{cases}
(1+\beta_3)v_0 + 3\beta_0 v_3 = 0 & (x_0x_3) \\
(1+\beta_3)v_1 + 3\beta_1 v_3 = 0 & (x_1x_3) \\
(1+\beta_3)v_2 - 3\beta_2 v_3 = 0 & (x_2x_3) \\
\beta_1 v_0 + \beta_0 v_1 = 0 & (x_0x_1) \\
\beta_2 v_0 + (1-\beta_0)v_2 = 0 & (x_0x_2) \\
(1-\beta_2)v_1 + \beta_1 v_2 = 0 & (x_1x_2) \\
2\beta_1 v_1 - v_2 = \lambda - \sum_{j=0}^3 \beta_j v_j - v_3 & (x_1^2) \\
-v_0 + 2\beta_2 v_2 = \lambda - \sum_{j=0}^3 \beta_j v_j - v_3 & (x_2^2) \\
(3+2\beta_0)v_0 = \lambda - \sum_{j=0}^3 \beta_j v_j - v_3 & (x_0^2) \\
2\beta_3 v_3 = \lambda - \sum_{j=0}^3 \beta_j v_j - v_3 & (x_3^2)
\end{cases}
$$

The points of $\mathcal{S}_C$ are the solutions $\underline{v}$ of the previous linear system, as a linear system whose coefficients depend on $\beta_0, \dots, \beta_3$. In order to resolve this linear system we consider the matrix A of the coefficients of the equations (x_0x_3), (x_1x_3), (x_2x_3) and (x_0x_1). The determinant of A is equal to

$$
det(A) = det \begin{pmatrix} 1+\beta_3 & 0 & 0 & 3\beta_0 \\ 0 & 1+\beta_3 & 0 & 3\beta_1 \\ 0 & 0 & 1+\beta_3 & -3\beta_2 \\ \beta_1 & \beta_0 & 0 & 0 \end{pmatrix} = -6\beta_0\beta_1(1+\beta_3)^2.
$$

It follows that, if $\beta_0\beta_1(1+\beta_3)^2 \neq 0$, then the linear system (68) has not solutions because, under this hypothesis, the subsystem of (68) of equations (x_0x_3), (x_1x_3), (x_2x_3) and (x_0x_1) admits an unique solution equal to $v = (0, \dots, 0)$ but $(0, \dots, 0)$ is not a solution of (x_1^2). Let now $\beta_0\beta_1(1+\beta_3)^2 = 0$.

Suppose that $\beta_0 = 0$. Then, by the equation (x_0x_1), we deduce that $v_0\beta_1 = 0$. If $v_0 = \beta_0 = 0$, then, by the equation (x_0x_2), it follows that $v_2 = v_0 = \beta_0 = 0$. By substituting in the equation (x_2^2), we find that $\lambda - \sum_{j=0}^3 \beta_j v_j - v_3 = 0$, and hence, by substituting in (x_1^2) it follows that $v_1\beta_1 = 0$. If $v_2 = v_0 = \beta_0 = v_1 = 0$, then the system (68) admits solutions only for $\beta_0 = \dots = \beta_3 = 0$ and, in this case, (68) admits an unique solution equal to $(v_0, \dots, v_3) = (0, \dots, 0, \lambda)$. If $v_2 = v_0 = \beta_0 = \beta_1 = 0$ then, by recalling that $\lambda \neq 0$, we find that the equations (x_1^2) and (x_3^2) are compatible if and only if $\beta_3 = 0$ and $v_3 = \lambda$. By (x_2x_3) and by (x_1x_3) it follows that $\beta_2 = v_1 = 0$. Then, also in this case, the linear system (68) admits an unique solution equal to $(0, 0, 0, \lambda)$ if $\beta_0 = \beta_1 = \beta_2 = \beta_3 = 0$ and it has not solutions otherwise.

Now suppose that $\beta_1 = 0$ *and* $\beta_0 \neq 0$. Then, by the equation (x_0x_1) it follows that $v_1 = 0$ and the linear system (68) is equivalent to the following linear system

$$(69) \qquad \begin{cases} (1+\beta_3)v_0 + 3\beta_0 v_3 = 0 & (x_0x_3)' \\ (1+\beta_3)v_2 - 3\beta_2 v_3 = 0 & (x_2x_3)' \\ \beta_2 v_0 + (1-\beta_0)v_2 = 0 & (x_0x_2)' \\ \beta_0 v_0 + (\beta_2 - 1)v_2 + (\beta_3+1)v_3 = \lambda & (x_1^2)' \\ -v_0 + (1+2\beta_2)v_2 = 0 & (x_2^2)' \\ (3+2\beta_0)v_0 + v_2 = 0 & (x_0^2)' \\ v_2 + 2\beta_3 v_3 = 0 & (x_3^2)' \end{cases}$$

where the last three equations of (69) are obtained from the respective equations of (68) subtracting the equation (x_1^2). In order to resolve the previous linear system, we consider the matrix B of the coefficients of the subsystem of equations $(x_0x_3)'$, $(x_2x_3)'$ and $(x_3^2)'$. The determinant of B is equal to

$$det(B) = det\begin{pmatrix} 1+\beta_3 & 0 & 3\beta_0 \\ 0 & 1+\beta_3 & -3\beta_2 \\ 0 & 1 & 2\beta_3 \end{pmatrix} = (1+\beta_3)[2\beta_3(1+\beta_3) + 3\beta_2].$$

We deduce that, if $(1+\beta_3)[2\beta_3(1+\beta_3)+3\beta_2] \neq 0$, then the subsystem of (69) of equations $(x_0x_3)'$, $(x_2x_3)'$ and $(x_3^2)'$ admits an unique solution equal to $(v_0, v_2, v_3) = (0,0,0)$. Since $(0,0,0)$ is not a solution $(x_1^2)'$, we deduce that, in this case, the system (69) has not solutions. Now suppose that $(1+\beta_3)(3\beta_2 + 2\beta_3(1+\beta_3)) = 0$. If $\beta_3 = -1$, then, by $(x_0x_3)'$ it follows that $v_3 = 0$. Therefore, by $(x_3^2)'$ and $(x_2^2)'$ it follows that $v_2 = v_0 = 0$ and, as before, the linear system (69) has not solutions. Finally, suppose that $\beta_3 \neq -1$ and $\beta_2 = -\beta_3(1+\beta_3)$. If $\beta_2 = \beta_3 = 0$, then, by the equations $(x_3^2)'$, $(x_2^2)'$ and $(x_0x_3)'$, it follows that $v_2 = v_0 = v_3 = 0$ and by $(x_1^2)'$ the linear system (69) has not solutions. Finally, suppose that $\beta_2 = -\frac{2}{3}\beta_3(1+\beta_3)$ but $\beta_2\beta_3 \neq 0$. Then, by $(x_3^2)'$, it follows that $v_2 = -2\beta_3 v_3$. By substituting in $(x_0x_2)'$ and by using $(x_0x_3)'$, it follows that $(2\beta_0 - 1)v_3 = 0$. Now, if $v_3 = 0$, then, as before, the linear system (69) has not solutions. If $\beta_0 = 1/2$, then the subsystem of (69) of equations $(x_0x_2)'$, $(x_0^2)'$ and $(x_2^2)'$ admits an unique solution equal to $(v_0, v_2) = (0,0)$. As before, also in this case, the linear system (69) has not solutions.

Finally, suppose that $\beta_3 = -1$ *and* $\beta_0\beta_1 \neq 0$. Then, by the equation (x_0x_3), it follows that

$$(70) \qquad v_3 = 0$$

By (x_3^2) it follows that

$$(71) \qquad \lambda - \sum_i \beta_i v_i = 0$$

and by (x_1^2) it follows that

$$(72) \qquad v_2 = 2v_1\beta_1.$$

By substituting in $(x_0)^2$, we find that

$$(73) \qquad (3+2\beta_0)v_0 = 0.$$

If $v_0 = 0$, then, by (x_0x_1) and by (x_3^2), it follows that $v_1 = v_2 = 0$. Since $(v_0, v_1, v_2, v_3) = (0, \ldots, 0)$ is not a solution of (x_1^2), in this case, the linear system (68) has not solution. If

$\beta_0 = -3/2$, then the determinant of the matrix of the coefficients of the equations (x_0x_1), (x_0x_2) and (x_1x_2) is equal to

$$det \begin{pmatrix} \beta_1 & -3/2 & 0 \\ \beta_2 & 0 & 5/2 \\ 0 & 1-\beta_2 & \beta_1 \end{pmatrix} = (-5+8\beta_2)\beta_1/2.$$

If $\beta_2 \neq 5/8$, then the subsystem of (68) of equations (x_0x_1), (x_0x_2) and (x_1x_2) admits an unique solution equal to $(v_0, v_1, v_2) = (0,0,0)$ and the linear system (68) has not solution, because, by (71) and (70), we have find that $0 \neq \lambda = \sum_i \beta_i v_i = 0$. If $\beta_0 = -3/2$ and $\beta_2 = 5/8$, then, by using (71), we find that the determinant of the matrix of the coefficients of the equations (x_0x_1), (x_0x_2) and (x_2^2) is equal to

$$det \begin{pmatrix} \beta_1 & -3/2 & 0 \\ 5/8 & 0 & 5/2 \\ -1 & 0 & 2\beta_2 \end{pmatrix} = -\frac{3}{2}(\frac{5}{4}\beta_2 + \frac{5}{2}) = -\frac{15}{4}(\frac{\beta_2}{2}+1) = -\frac{15}{4}(\frac{5}{16}+1) \neq 0.$$

As before, also in this case, the linear system (68) has not solutions.

Finally, we have found that the linear system (68) has only a solution equal to $(v_0, v_1, v_2, v_3) = (0,0,0,\lambda)$ if $\beta_0 = \beta_1 = \beta_2 = \beta_3 = 0$ and it has not solutions otherwise. By the previous lemma, we conclude that the point $[0:0:0:1] \in \mathbb{P}^3$ is the only point which belongs to six tangent lines to the canonical curve $C \subset \mathbb{P}^3$ which is intersection of the cubic surface of equation

$$F_3(x_0, x_1, x_2, x_3) = x_3^3 + (x_0^2 + x_1^2 - x_2^2)x_3 + x_0^3 + x_0x_2^2 - x_1^2x_2 = 0$$

and the quadric of equation

$$\frac{\partial F_3}{\partial x_3} = 3x_3^2 + x_0^2 + x_1^2 - x_2^2 = 0.$$

It follows that, on the normalization curve D of the plane curve Γ' corresponding to the general point of $\Sigma_1 \subset \Sigma_{6,0}^6$ there exists only a finite number of linear series of dimension two with six ramification points. □

Remark 4.8. *By using the notation introduced in the proof of corollary 4.7, we observe that in this corollary we have proved that if C is a general canonical curve of genus four such that the set $\mathcal{S}_C$ is not empty, then $\mathcal{S}_C$ is finite. Actually, C. Ciliberto pointed out that it is possible to show, with a very simple argument, that for every canonical curve C of genus four such that $\mathcal{S}_C$ is not empty, we have that $\mathcal{S}_C$ is finite. Finally, we observe that, by remark 4.6, for every canonical curve C of genus four, the set $\mathcal{S}_C$ coincides with the set of points of $\mathbb{P}^3$ which are contained in six tangent lines to C.*

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