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## Twisted spin curves


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# TWISTED SPIN CURVES

TESI DI DOTTORATO

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# Contents

# Introduction

The problem of constructing a compactification for the Picard scheme (or generalized Jacobian) of a singular algebraic curve has been studied by several authors. More generally, the same problem can be considered for families of curves.

Several constructions have been carried out since Igusa's pioneering work [**I**], which gave a construction for nodal and irreducible curves. Constructions are known for families of geometrically integral curves, by Altman and Kleiman [**AK**], and geometrically connected, possibly reducible, nodal curves, by Oda and Seshadri [**OS**].

A common approach to the problem is the use of the Geometric Invariant Theory (GIT). We recall in particular Caporaso's [**C1**] and Pandharipande's [**P**] modular compactifications of the universal Picard variety over the moduli space of Deligne-Mumford stable curves.

A different method was employed by Esteves [**Es**] to produce a compactification (admitting also a universal object after an étale base change) for a family of geometrically reduced and connected curves.

On the other hand one may be interested in distinguished subschemes of the Picard scheme.

In [**Co**] Cornalba constructed a geometrically meaningful compactification $\overline{S_g}$ of the moduli space of theta characteristics of smooth curves of genus $g$. $\overline{S_g}$ is well-known as *moduli space of stable spin curves* and is endowed with a natural finite morphism $\varphi : \overline{S_g} \longrightarrow \overline{M_g}$ onto the moduli space of Deligne-Mumford stable curves.

As one can expect, the degree of $\varphi$ is $2^{2g}$ and $\overline{S_g}$ is a disjoint union of two irreducible components, $\overline{S_g^+}$ and $\overline{S_g^-}$ whose restrictions over $M_g$ parametrize respectively even and odd theta characteristics on smooth curves. In particular the degree of the restriction of $\varphi$ to $\overline{S_g^-}$ is $N_g := 2^{g-1}(2^g - 1)$.

The fibers of $\varphi$ over singular curves parameterize "generalized theta-characteristics" or *stable spin curves.* [**CC**] provides an explicit combinatorial description of the boundary, parametrizing certain line bundles on quasistable curves having degree 1 on exceptional components (that is rational components intersecting the rest of the curve in exactly 2 points).

More recently, in [**CCC**] the authors generalize the construction compactifying in the same spirit the moduli space of pairs $(C, L)$, $C$ a smooth curve and $L$ a $r$-th root of a fixed line bundle $N \in \operatorname{Pic} C$.

In this thesis we deal with families of line bundles, sometimes under the following set of assumptions

(1) we consider one-parameter projective families of local complete intersection (l.c.i.) canonical curves which are connected, Gorenstein and reduced
(2) we require that a singular curve is irreducible with at most nodal, cuspidal and tacnodal singularities
(3) we consider compactifications of families of odd theta characteristics on the smooth fibers of a family as in (1).

The above assumptions allow us to find rather explicit results. In particular we are able to give a geometric description of degenerations of odd theta characteristics.

Our method is very close in spirit to the well-known Stable Reduction Theorem for curves and gives the possibility to reduce ourselves to results on Deligne-Mumford stable curves.

Loosely speaking this approach can be viewed as a “Stable Reduction for polarized curves”.

Let us give more details.

We say that a one-parameter family $f : \mathcal{W} \to B$ with $B$ a smooth curve is a smoothing of a curve $W$ if its general fiber is smooth and the fiber over a special point $0 \in B$ is $W$.

Let $f : \mathcal{W} \to B$ be a smoothing of a singular curve $W$. Assume that $f$ satisfies (1). Set $B^* := B - 0$ and consider the restricted family $\mathcal{W}^* \to B^*$. It is well-known that there exists a curve $S^-_{\omega^*_f}$ finite over $B^*$ whose points parametrize odd theta characteristics of the smooth fibers of $\mathcal{W} \to B$.

Some natural questions arise

(i) how can one get a compactification of $S^-_{\omega^*_f}$ (over $B$) reflecting the geometry of $W$?
(ii) are the corresponding boundary points independent of the chosen family $f : \mathcal{W} \to B$?
(iii) if the answers to $(i)$ and $(ii)$ is positive, can we give a geometric description of the boundary points?

It is well-known that a smooth curve $C$ of genus $g$ has exactly $N_g$ odd theta characteristics (see the above definition of $N_g$). If $C$ is general, any such line bundle $L$ satisfies $h^0(C, L) = 1$ and hence the canonical model of $C$ admits exactly one hyperplane $H_L$ cutting the double of the effective divisor associated to the non-zero section of $L$. In this case we say that $C$ is *theta generic* and that $H_L$ is a *theta hyperplane* of $C$. Therefore if we collect the theta hyperplanes of a theta generic curve $C$, we get a configuration $\theta(C)$ which is a point of $Sym^{N_g}(\mathbb{P}^{g-1})^\vee$.

Let $H_g$ be the irreducible component of the Hilbert scheme $\mathrm{Hilb}^{p(x)}[\mathbb{P}^{g-1}]$ of curves in $\mathbb{P}^{g-1}$ having Hilbert polynomial $p(x) = (2g-2)x - g + 1$ and containing smooth canonical curves. In this way we get a rational map

$$\theta : H_g - - > Sym^{N_g}(\mathbb{P}^{g-1})^\vee$$

defined at least over the set of smooth theta generic canonical curves. If the smooth fibers of $f : \mathcal{W} \to B$ are theta generic, the family of theta hyperplanes associated to $\mathcal{W}^* \to B^*$ is isomorphic to $S^-_{\omega^*_f}$ and its projective closure provides a natural compactification, answering (i).

In this way we can also consider “limit theta hyperplanes” on singular canonical curves arising from smoothings to theta generic curves. We say that a singular curve is *theta generic* if it admits a finite number of theta hyperplanes.

Theorem 1 answers question $(ii)$ positively for some theta generic canonical curves or l.c.i. curves.

THEOREM 1.

- *Let $W$ be a theta generic canonical curve parameterized by a smooth point of $H_g$. There exists a unique natural configuration of theta hyperplanes $\theta(W)$ such that, when $W$ is smooth, $\theta(W)$ is the ordinary configuration of theta hyperplanes.*
- *Let $W$ be a canonical l.c.i. curve parametrized by $h \in H_g$. Then $H_g$ is smooth at $h$.*
- *Fix non negative integers $\tau, \gamma, \delta$. If $W$ is a general irreducible canonical l.c.i. curve with $\tau$ tacnodes, $\gamma$ cusps and $\delta$ nodes, then it is theta generic.*

See Lemma 3.2, Proposition 3.3 and Theorem 3.9.

We are able to give an explicit description of $\theta(W)$ as follows.

If $W$ is an irreducible canonical l.c.i. curve with tacnodes, cusps and nodes, we denote by $t^j_{ikh}$ for $j \leq i$, the number (when it is finite) of theta hyperplanes of $W$ containing $i$ tacnodes and $j$ tacnodal tangents of these $i$ tacnodes, $k$ cusps and $h$ nodes. We call such a hyperplane *a theta hyperplane of type* $(i, j, k, h)$.

We get the following Theorem, extending known results from [**C2**].

Theorem 2. *Let $g$ be a positive integer with $g \geq 3$. Let $W$ be an irreducible theta generic canonical l.c.i. curve with $\tau$ tacnodes, $\gamma$ cusps and $\delta$ nodes. Let $g$ be the genus of $W$ and $\tilde{g}$ be the genus of its normalization.*

*If $j < i$ or $h \neq \delta$*

$$t^j_{ikh} = 2^{\tau-j+\delta-h-1}\binom{\tau}{i}\binom{i}{j}\binom{\delta}{h}\binom{\gamma}{k}(N^+_{\tilde{g}} + N_{\tilde{g}}).$$

*If $i = j$ and $h = \delta$*

$$t^i_{ik\delta} = \begin{cases} 2^{\tau-i}\binom{\tau}{i}\binom{\gamma}{k}N^+_{\tilde{g}} & \text{if } \tau - i + \gamma - k \equiv 1\ (2) \\[2ex] 2^{\tau-i}\binom{\tau}{i}\binom{\gamma}{k}N_{\tilde{g}} & \text{if } \tau - i + \gamma - k \equiv 0\ (2) \end{cases}$$

See Theorem 3.9.

Notice that if $W$ is singular, then $\theta(W)$ contains multiple hyperplanes. We are able to find the multiplicity of a limit theta hyperplane as a multiplicative function of the singularities of $W$ as stated in the following

Theorem 3. *Let $W$ be an irreducible theta generic canonical l.c.i. curve of genus $g \geq 3$ with tacnodes and cusps. The multiplicity of a theta hyperplane of type $(i, j, k)$ is $4^{i-j}\, 6^j\, 3^k$.*

See Theorem 3.15, Theorem 3.16 and Theorem 3.17.

The techniques used to prove Theorem 3 also lead to answer question $(iii)$ above.

We explain the main idea, starting with an example.

Consider a projective irreducible canonical curve $W$ having exactly one cusp. Consider a general projective smoothing $\mathcal{W} \to B$ of $W$. Modulo a base change we can assume that it admits a stable reduction over $B$ which we denote by $f : \mathcal{C} \to B$. The central fiber $C$ of $\mathcal{C}$ is reducible. There exists a morphism from $\mathcal{C}$ to $\mathcal{W}$ given by $\mathcal{N} = \omega_f(D)$, a twist of the relative dualizing sheaf $\omega_f$ by a non-trivial Cartier divisor $D$ of $\mathcal{C}$ supported on irreducible components of $C$. This morphism encodes the stable reduction of the polarized curve $(W, \mathcal{O}_W(1))$.

This suggests a geometrically meaningful connection between limit theta characteristics on $W$ and square roots of the restriction of $\mathcal{N}$ to the central fiber.

A natural setup is provided by Caporaso's modular compactification $\overline{P_{g-1,g}}$ of

$$P_{g-1,g} = \{(X, L) : X \text{ smooth genus } g \text{ curve, } L \text{ line bundle on } X \text{ of degree } g-1\}/\text{iso}.$$

Recall that $\overline{P_{g-1,g}}$ was constructed via GIT as a quotient of a suitable Hilbert scheme $H_{g-1}$.

In [**F**], Fontanari showed that there exists a natural morphism

$$\chi : \overline{S_g} \longrightarrow \overline{P_{g-1,g}}.$$

The Hilbert points of $H_{g-1}$ parametrizing stable spin curves have a closed orbit (in the set of GIT-semistable points) and this yields the set-theoretic description of $\chi$.

Call $\hat{S}_g$ the image of $\chi$. In [**CCC**] the authors show that $\hat{S}_g$ parametrizes not only stable spin curves (i.e. limit square roots of the dualizing sheaf of a stable curve) but also "extra line bundles," which we shall call *twisted spin curves*. The twisted spin curves are square roots of suitable twists of the dualizing sheaf of quasistable curves (see Definition 2.18).

Recall that $\overline{P_{g-1,g}}$ is not a geometric quotient. The Hilbert point of $H_{g-1}$ parametrizing a twisted spin curves is identified in $\hat{S}_g$ with some stable spin curve.

Our key technical part is the comparison of curves of stable spin curves within $\overline{S_g}$, curves of twisted spin curves within $\hat{S}_g$ and curves of theta hyperplanes, allowing us to give the following geometric interpretation of our compactification.

- Let $W$ be as in Theorem 3 and fix a general projective smoothing of $W$. Then the hyperplanes of $\theta(W)$ correspond to suitable twisted spin curves of the curve which is the stable reduction of the fixed general smoothing of $W$.

See Theorem 3.22.

Below we discuss two applications of our techniques.

Consider a family $f : \mathcal{C} \to B$, with smooth total space and with $B \subset \overline{M_g}$, which is a smoothing of a stable curve without non-trivial automorphisms. Consider $S_{\omega_f} := B \times_{\overline{M_g}} \overline{S_g}$ and its restriction $S_{\omega_f^*}$ over $B^* = B - 0$.

PROPOSITION. *$S_{\omega_f^*}$ admits an étale completion over $B$ if and only if the dual graph of $C$ is ètale (see Definition 4.5). In particular the existence of such completion does not depend on the chosen family but only on the dual graph of $C$.*

See Proposition 4.6.

The second application involves the Geometric Invariant Theory. We find an approach to the stable reduction of curves based on the GIT-stability of configurations of theta hyperplanes.

We say that a canonical curve $W$ of $\mathbb{P}^{g-1}$ is *theta-stable* if it has a well-defined configuration $\theta(W)$ of theta hyperplanes which does not depend on smoothings to theta generic curves and $\theta(W)$ is GIT-stable (with respect to the natural action of $SL(g)$). Nevertheless if $f : \mathcal{W} \to B$ is a smoothing of $W$ to theta generic curves, we can always consider the configuration $\theta_f(W)$ obtained by taking the projective closure.

An easy argument of Geometric Invariant Theory gives the following Lemma.

LEMMA. *Let $W$ be a theta-stable canonical curve and $\mathcal{W} \to B$ be a projective smoothing of $W$ to theta generic curves. Let $C$ be a stable curve and $f : \mathcal{C} \to B$ be a smoothing of $C$ to theta generic curves. If $\theta_f(C)$ is GIT-stable and not conjugate to $\theta(W)$, then $\mathcal{C}$ is not the stable reduction of $\mathcal{W}$.*

See Lemma 4.14.

In order to apply this criterion we study configurations of theta hyperplanes of canonical stable curves. This is rather explicit for curves with at most two irreducible components (see Theorem 4.3). In Theorem 4.16 we give examples of theta-stable curves, as the well-known "split curves" and in Corollary 4.18 we show the typical result one can get from the above Lemma.

The thesis is organized as follows.

In Chapter 1 we shall recall some results about the theory of algebraic curves.

In Chapter 2 we shall recall basic facts from [**CCC**] and the construction of Caporaso's compactification $\overline{P_{d,g}}$ of the universal Picard variety. Moreover we shall prove Theorem 2.24, which will be an important tool in the proof of the above Theorem 3.

In Chapter 3 we shall describe our compactification, proving the above Theorem 1, Theorem 2 and Theorem 3 and giving the geometric interpretation of theta hyperplanes on non-stable curves.

In Chapter 4 we shall give two applications of our techniques, proving the above Proposition and Lemma.

Let us conclude pointing out an interesting open problem.

It is well-known that the above morphism

$$\chi : \overline{S_g} \longrightarrow \hat{S}_g.$$

is a bijection. Then $\overline{S_g}$ and $\hat{S}_g$ are the same topological space. A natural question arises: is $\chi$ an isomorphism?

There exists a more general setup dealing with $r$-spin curves and generalizing the above construction (which is the case $r = 2$). For $r > 2$ the corresponding spaces are not isomorphic, because they are not even the same topological space (see [**CCC**]).

There is evidence that $\chi$ itself is not an isomorphism because, as stressed also in this thesis, $\hat{S}_g$ parameterizes line bundles (twisted spin curves) which does not appear in $\overline{S_g}$. In the future we hope to use our techniques to prove the following

CONJECTURE. *$\chi$ is not an isomorphism.*

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CHAPTER 1

# Preliminary tools and basic results

**Notation and Terminology 1.**

(1) We work over the field of complex numbers. By a *curve* we will always mean a connected projective curve which is Gorenstein and reduced. If $W$ is a curve, we shall denote by $\omega_W$ its dualizing sheaf. The (arithmetic) genus of a curve is $g_W = h^0(W, \omega_W)$.

(2) Let $W$ be a curve. We shall denote by $W^{sm}$ the set of smooth points of $W$ and by $W^{sg}$ the set of its singular points. If $Z \subset W$ is a subcurve, we shall denote by $Z^c$ the complementary curve $Z^c := \overline{W - Z}$.

(3) A *family of curves* is a proper and flat morphism $f : \mathcal{W} \to B$ whose fibers are curves. By a *projective family of curves* we will mean a family $B \times \mathbb{P}^n \supset \mathcal{W} \stackrel{f}{\to} B$, where $f$ is the first projection. The fiber of a family $f : \mathcal{W} \to B$ over the point $b \in B$ will be denoted by $W_b$. A *smoothing* of a curve $W$ is a family $f : \mathcal{W} \to B$ where $B$ is a smooth, connected, affine curve of finite type with a distinguished point $0 \in B$ such that the fiber over $0$ is isomorphic to $W$ and smooth general fiber over $b \in B - 0$. A *general smoothing* is a smoothing with smooth total space.

(4) The *dual graph* $\Gamma_X$ of a nodal curve $X$ is the graph having the irreducible components of $X$ as vertices and where an edge connects two vertices if and only if the corresponding components meet in a node.

(5) A *stable* curve $C$ is a nodal curve such that every smooth rational component of $C$ meets the rest of the curve in at least three points. A *semistable* curve is a nodal curve such that every smooth rational component meets the rest of the curve in at least two points. Every smooth rational component of a semistable curve meeting its complementary curve in exactly two points is called *destabilizing*.

(6) Let $\mathbb{C}^N$ have coordinates $x_1, \ldots, x_N$ and let $f$ be a polynomial in the $x_i$. We shall denote by $\mathrm{v}(f) \subset \mathbb{C}^N$ the set of the zeroes of $f$.

## 1.1. Background in the theory of algebraic curves

We shall recall some basic facts about the theory of algebraic curves.

Throughout the other chapters we shall widely use both the language and the results of this section, in particular a smoothness criterion for the Hilbert scheme.

### 1.1.1. Geometric Invariant Theory (GIT).

The Geometric Invariant Theory gives an answer to the problem of constructing quotients in algebraic-geometry and provides the cornerstone of the construction of the moduli space of stable curves. The main properties are included in the so-called *Fundamental Theorem of the Geometric Invariant Theory* (see below). References for what follows will be [**MFK**] and [**N**].

Let $P$ be a projective scheme over $\mathbb{C}$ embedded in a projective space $\mathbb{P}(V)$. Assume that $P = \mathrm{Proj}R$, where $R$ is a graded ring finitely generated over $\mathbb{C}$. Consider a reductive algebraic group $G$ acting on $P$. A prototype of such a group is $\mathrm{SL}(N+1)$.

An interesting case is when the action of $G$ lifts to a linear action on $V$. In this case one says that *$G$ acts linearly on $P$ embedded in* $\mathbb{P}(V)$. It follows that $G$ acts also on $R$ and hence one can consider the subring $R^G$ of $R$ of the elements which are invariant under the action of $G$.

For every point $p \in P$, one denotes by $O_G(p) \subset P$ the orbit of $p$ under the action of $G$.

- A point $p \in P$ is said to be *GIT-semistable* if there exists a homogeneous non constant $f \in R^G$ such that $f(p) \neq 0$.
- A GIT-semistable point $p \in P$ is said to be *GIT-stable* if $O_G(p)$ is closed in the set of the GIT-semistable points and has maximal dimension among the dimensions of all the orbits in the set of GIT-semistable points
- A point of $P$ which is not GIT-semistable is called *unstable*.

One sets

$$P^{ss} := \{p \in P : p \text{ is GIT-semistable}\}$$

$$P^{s} := \{p \in P : p \text{ is GIT-stable}\}.$$

It is well-known that if $G$ is reductive, then $R^G$ is a graded algebra, finitely generated over $\mathbb{C}$. It follows that the natural rational map

$$\pi : P = \mathrm{Proj}R \dashrightarrow Q := \mathrm{Proj}R^G$$

induced by the inclusion $R^G \subset R$ is regular on $P^{ss}$. In general the fibers of $\pi$ are not equal to the orbits of $G$, and this happens whenever there are non-closed orbits.

One can view $Q$ as an algebraic-geometric quotient of $P^{ss}$. It is called *the GIT-quotient of $P$ under the action of $G$* and is usually denoted by by $Q = P/G$.

The most important properties of the quotient $Q$ are contained in the following Theorem, well-known as *Fundamental Theorem of GIT.*

THEOREM 1.1. *Let $P$ be a projective scheme endowed with a linear action of a reductive group $G$. The scheme $Q = \mathit{Proj}\, R^G$ is a projective scheme such that the natural morphism*

$$\pi : P^{ss} \longrightarrow Q$$

*satisfies*

(i) *for every $p, q \in P^{ss}$ we have*

$$\pi(p) = \pi(q) \Leftrightarrow \overline{O_G(p)} \cap \overline{O_G(q)} \cap P^{ss} \neq \emptyset$$

(ii) *For every pair $(Q', \pi')$, where $Q'$ is a scheme and $\pi' : P^{ss} \to Q'$ is a $G-$invariant morphism, there exists a unique morphism $\alpha : Q \to Q'$ such that $\pi' = \alpha \circ \pi$*

(iii) *for every $p, q \in P^{s}$ we have*

$$\pi(p) = \pi(q) \Leftrightarrow O_G(p) = O_G(q).$$

When all the GIT-semistable points are GIT-stable, one says that the GIT-quotient is a *geometric quotient.*

### 1.1.2. The Hilbert scheme.

We recall a smoothness criterion for the Hilbert scheme of curves at a point parametrizing a local complete intersection.

A detailed construction of the Hilbert scheme can be found in [**ACGH2**] and [**S**].

We shall restrict to the case of the Hilbert scheme of projective curves.

Consider the projective space $\mathbb{P}^N$ and the polynomial $p(x) = dx - g + 1$, where $d > 0$ and $g \geq 3$. The Hilbert scheme $\mathrm{Hilb}_N^{p(x)}$ parametrizes closed one-dimensional subschemes of $\mathbb{P}^N$ having $p(x)$ as Hilbert polynomial. For a given point $h \in \mathrm{Hilb}_N^{p(x)}$, one denotes by $X_h$ the curves parametrized by $h$.

A set-theoretic description of $\mathrm{Hilb}_N^{p(x)}$ is as follows. Let $X \subset \mathbb{P}^N$ be a projective curve having $p(x)$ as Hilbert polynomial. Set $\mathcal{O}_X(1) := \mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{O}_X$ and denote by $\mathcal{I}_X$ the ideal sheaf of $X$. It is well-known that by a theorem of Serre there exists an integer $n >> 0$ and an exact sequence

$$0 \to H^0(\mathbb{P}^N, \mathcal{I}_X(n)) \to H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n)) \to H^0(X, \mathcal{O}_X(n)) \to 0.$$

One can choose an integer $n_0$ such that for every $n \geq n_0$ and for all the curves $X$ of $\mathbb{P}^N$ having $p(x)$ as Hilbert polynomial, the sequence is exact. Moreover the space $H^0(\mathbb{P}^n, \mathcal{I}_X(n))$ characterizes $X$.

One associates to $X$ the corresponding point in the Grassmannian of $p(n)-$dimensional quotients of $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n))$. $\mathrm{Hilb}_N^{p(x)}$ is a closed subset of this Grassmannian.

The Hilbert scheme $\mathrm{Hilb}_N^{p(x)}$ finely represents the contravariant functor

$$\mathcal{H}ilb_N^{p(x)} : SCH \longrightarrow SETS$$

associating to a scheme $B$ the set $\mathcal{H}ilb_N^{p(x)}(B)$ of the projective families over $B$ of curves of $\mathbb{P}^N$ having $p(x)$ as Hilbert polynomial.

Recall that a scheme $H$ *finely represents* a functor $\mathcal{H}$ if

- $H$ *coarsely represents* $\mathcal{H}$, that is there exists a transformation of functors

  $$\Phi : \mathcal{H} \longrightarrow \mathcal{H}om(-, H)$$

  such that
  (1) for every field $k$ such that $\overline{k} = k$, then

  $$\Phi(\operatorname{Spec} k) : \mathcal{H}(\operatorname{Spec} k) \longrightarrow \mathcal{H}om(\operatorname{Spec} k, H)$$

  is an isomorphism;
  (2) for every scheme $H'$ and transformation of functors

  $$\Phi' : \mathcal{H} \longrightarrow \mathcal{H}om(-, H')$$

  there is a unique morphism $\chi : H \to H'$ such that $\Phi' = \chi \cdot \Phi$.
- The transformation of functors $\Phi$ is an isomorphism.

Consider the case of the Hilbert scheme of curves. It is easy to see that these conditions imply the existence of a universal family $\mathcal{U} \to \mathrm{Hilb}_N^{p(x)}$.

Now we shall recall a smoothness criterion for the Hilbert scheme at a point parametrizing a local complete intersection.

For every point $h \in \mathrm{Hilb}_N^{p(x)}$, the tangent space to $\mathrm{Hilb}_N^{p(x)}$ at $h$ is given by

$$T_h(\mathrm{Hilb}_N^{p(x)}) = H^0(X_h, \mathcal{N}_{X_h/\mathbb{P}^N})$$

where $\mathcal{N}_{X_h/\mathbb{P}^N} := \mathcal{H}om(\mathcal{I}_{X_h}/\mathcal{I}_{X_h}^2, \mathcal{O}_{X_h})$.

Moreover it is well-known (see [**ACGH2**, pag. 28]) that a lower bound for the dimension of $\mathrm{Hilb}_N^{p(x)}$ at a point $h$ parametrizing a local complete intersection is

$$h^0(X_h, \mathcal{N}_{X_h/\mathbb{P}^N}) - h^1(X_h, \mathcal{N}_{X_h/\mathbb{P}^N}).$$

Thus we get the following

*Smoothness criterion for the Hilbert scheme:* if $h \in \mathrm{Hilb}_N^{p(x)}$ parametrizes a local complete intersection and

$$h^1(X_h, \mathcal{N}_{X_h/\mathbb{P}^N}) = 0$$

then $\mathrm{Hilb}_N^{p(x)}$ is smooth at $h$.

**1.1.3. The moduli space of the stable curves.**
A classical application of the Geometric Invariant Theory and of the Hilbert scheme is the construction of the moduli space of the stable curves.

References for what follows will be [**G1**], [**G2**] and [**Mu**].

Let $g \geq 3$ and $d = m(2g-2)$ for $m >> 0$. Consider the Hilbert scheme $\mathrm{Hilb}_N^{p(x)}$, where $p(x) = dx - g + 1$ and $N = d - g$. Consider the reductive algebraic group $G = SL(N+1)$. For $n >> 0$ we have an embedding (depending on $n$) of $\mathrm{Hilb}_N^{p(x)}$ in the Grassmannian of $p(n)-$quotients of $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n))$ and hence also in

$$\mathbb{P}(\wedge^{p(n)} H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n)))$$

on which $G$ acts. This induces a linearization of the action of $G$ on $\mathrm{Hilb}_N^{p(x)}$.

Now consider the following subset of $\left(\mathrm{Hilb}_N^{p(x)}\right)^{ss}$

$$K = \{h \in \mathrm{Hilb}_N^{p(x)} \ : \ h \text{ is GIT-semistable and } X_h \text{ is connected with } \mathcal{O}_{X_h}(1) = \omega_{X_h}^{\otimes m} \ \}.$$

One can show that $K$ is a $G-$invariant closed subscheme of $\left(\mathrm{Hilb}_N^{p(x)}\right)^{ss}$ such that $K^s = K^{ss}$.

Moreover $K$ is not empty, because if $X_h \subset \mathbb{P}^N$ is a smooth, connected, nondegenerate curve of genus $g$ and degree $d$, then $h$ is GIT-stable (this is the so-called Mumford-Gieseker Theorem, see [**G2**] and [**Mu**]).

The points of $K$ parametrize stable curve. In fact if $h \in \mathrm{Hilb}_N^{p(x)}$ is GIT-stable, then the connected components of $X_h$ are semistable curves (this is so-called Gieseker Theorem, see [**G2**]) and if $X_h$ is semistable, then $|\omega_{X_h}^{\otimes m}|$ is base point free and contracts exactly the destabilizing components of $X_h$.

Conversely one can show that for any stable curve $X_h \subset \mathbb{P}^N$ such that $\mathcal{O}_{X_h}(1) = \omega_{X_h}^{\otimes m}$, then $h$ is GIT-semistable.

The moduli space of the stable curve is given by taking the GIT-quotient

$$\overline{M_g} := K/G.$$

### 1.1.4. The stable reduction of algebraic curves.

As a consequence of the existence of the moduli space of stable curves as a projective space we find that limits of one-parameter families of stable curves are again stable curves (up to birational transformations of the total space of the family).

This result is well-known as Theorem of stable reduction of algebraic curves. Its first proof was given in [**DM**], before the Gieseker construction of $\overline{M_g}$ via GIT and in this way the authors showed the properness of the moduli space of stable curves a-priori. Other good references are [**B**] and [**HM**].

THEOREM 1.2. *Let $B$ be a smooth curve, $0$ a point of $B$ and set $B^* := B - 0$. Let $\mathcal{X} \to B^*$ be a family of stable curves of genus $g \geq 2$. Then there exists a branched cover $B' \to B$ totally ramified over $0 \in B$ and a family $\mathcal{X}' \to B'$ of stable curves extending the fiber product $\mathcal{X} \times_{B^*} B'$. Moreover the central fiber of $\mathcal{X}'$ is uniquely determined in the sense that any two such extensions are dominated by a third and in particular their special fibers are isomorphic.*

## 1.2. Canonical desingularization of double covers

In this section we will follows [**BPV**]. All the complex surfaces will be reduced and connected.

Let $\mathcal{W}$ and $\mathcal{Z}$ be complex surfaces with $\mathcal{W}$ normal and $\mathcal{Z}$ smooth. Assume that $\mathcal{W} \to \mathcal{Z}$ is a double cover ramified along a curve $B \subset \mathcal{Z}$. If $b$ is a singular point in $B^{sg}$, then $\mu_b$ will be its multiplicity.

The following procedure of *canonical desingularization* produces a desingularization of $\mathcal{W}$ which is minimal if the singularities of $\mathcal{W}$ are $A$-$D$-$E$.

Consider the blow-up of $\pi_1 : \mathcal{Z}_1 \to \mathcal{Z}$ over the points $b \in B^{sg}$. Let $\overline{B_1}$ be the strict transform of $B$ and $E_b$ be the exceptional component corresponding to $b \in B^{sg}$. Consider the curve

$$B_1 := \bigcup_{\mu_b \text{ odd}} E_b \cup \overline{B_1}.$$

Define inductively $\mathcal{Z}_{k+1}$ and $B_{k+1}$ for $k \geq 1$ respectively as the blow-up $\pi_{k+1} : \mathcal{Z}_{k+1} \to \mathcal{Z}_k$ over the singular points of $B_k$ and, if $\overline{B_{k+1}}$ is the strict transform of $B_k$, $\mu_b$ the multiplicity of $b \in B_k^{sg}$ and $E_b$ the corresponding exceptional component,

$$B_{k+1} := \bigcup_{\substack{b \in B_k^{sg} \\ \mu_b \text{ odd}}} E_b \cup \overline{B_{k+1}}.$$

It is well-known that there exists $k_0$ such that $B_{k_0-1}$ has only nodal singularities and since its singular points have even multiplicities. Then $B_{k_0}$ is smooth.

The fiber product

$$\mathcal{W}^{can} := \mathcal{W} \times_{\mathcal{Z}} \mathcal{Z}_{k_0}$$

is the double cover of $\mathcal{W}_k$ ramified along the smooth curve $B_k$ and hence it is smooth. We call $\mathcal{W}^{can}$ the *canonical desingularization* of $\mathcal{W}$.

**1.2.1. Elliptic normal singularities.**
Let $v$ be an isolated normal singularity of a complex surface $\mathcal{W}$. If $\pi : \mathcal{W}' \to \mathcal{W}$ is a desingularization of $(\mathcal{W}, v)$ one defines the *arithmetic genus* $p_a(v)$ as

$$p_a(v) := \sup g_Z$$

where $Z$ runs over the set of the subcurves $Z$ in $\pi^{-1}(v)$. One can show that $p_a(v)$ does not depend on the chosen desingularization (see [**W**, Proposition 1.9]).

Definition 1.3. The singularity is said to be *elliptic* if $p_a(v) = 1$.

The classification of the elliptic normal singularities is contained in [**L**] and [**W**]. We recall the following result.

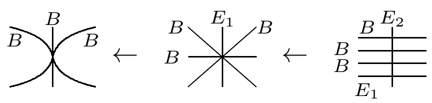
Lemma 1.4. *Let $v$ be an isolated normal elliptic singularity of a complex surface $\mathcal{W}$ and let $\pi : \mathcal{W}' \to \mathcal{W}$ be the minimal desingularization of $(\mathcal{W}, v)$.*

(1) *The curve $\pi^{-1}(v)$ is a reduced elliptic curve $F$ with $F^2 = -1$ if and only if $\mathcal{W}$ locally is given by $\mathrm{v}(y^2 - g(x,t)) \subset \mathbb{C}^3_{x,y,t}$ where $g = x^3 + cxt^4 + c't^6$ for $c, c' \in \mathbb{C}$.*
(2) *The curve $\pi^{-1}(v)$ is a reduced elliptic curve $F$ with $F^2 = -2$ if and only if $\mathcal{W}$ locally is given by $\mathrm{v}(y^2 - g(x,t))) \subset \mathbb{C}^3_{x,y,t}$, where $g$ is a homogeneous polynomial in $x$ and $t$ of degree 4 with 4 distinct roots.*

Proof. See [**W**, 6.2 and Corollary pag. 449]. □

Below we shall consider two special cases of elliptic singularities..

Example 1.5. $y^2 - x^3 + t^6 = 0$
Let $\mathcal{W}$ be the complex surface given by $\mathrm{v}(y^2 - x^3 + t^6) \subset \mathbb{A}^3_{x,y,t}$ which has a normal elliptic singularity in the origin. Notice that $\mathcal{W}$ is the double cover of $\mathcal{Z} := \mathbb{A}^2_{x,t}$ ramified along the plane curve $B := \mathrm{v}(x^3 - t^6) \subset \mathbb{A}^2_{x,t}$ which has a singular point of multiplicity 3. We can apply the canonical desingularization. We have to blow-up twice as said before and as shown below.



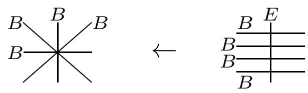

$\mathcal{Z}$ $\mathcal{Z}_1$ $\mathcal{Z}_2$

We have $B_1 = B \cup E_1 \subset \mathcal{Z}_1$ and $B_2 = B \cup E_1 \subset \mathcal{Z}_2$ and $B_2$ is smooth. The canonical desingularization $\mathcal{W}^{can}$ is the double cover of $\mathcal{Z}_2$ ramified along $B_2$. There are two curves over the elliptic singularity of $\mathcal{W}$ : an elliptic curve $F$ which is the double cover of $E_2$ ramified over the 4 points $E_2 \cap (B \cup E_1)$ and a $(-1)-$curve $E_1'$ over $E_1$.

Notice that in this case the canonical desingularization is not minimal.

Example 1.6. $y^2 - x^4 + t^4 = 0$
Let $\mathcal{W}$ be the complex surface given by $\mathrm{v}(y^2 - x^4 + t^4) \subset \mathbb{A}^3_{x,y,t}$ which has a normal elliptic singularity in the origin. It is the double cover of $\mathcal{Z} := \mathbb{A}^2_{x,t}$ ramified along the plane curve $B := \mathrm{v}(x^4 - t^4) \subset \mathbb{A}^2_{x,t}$ which has a singular point of multiplicity 4. We can apply the canonical

desingularization. We have to blow-up once as said before and as shown below.




$\mathcal{Z}$ $\mathcal{Z}_1$

We have $B_1 = B \subset \mathcal{Z}_1$ which is smooth. The canonical desingularization $\mathcal{W}^{can}$ is the double cover of $\mathcal{Z}_1$ ramified along $B_1$. Over the elliptic singularity of $\mathcal{W}$ there is an elliptic curve $F$ which is the double cover of $E$ ramified over the 4 points $E \cap B_1$. $F$ is an elliptic curve as in Lemma 1.4.

CHAPTER 2

# Square roots of line bundles on curves

In this chapter we shall recall known fact from [**Ha2**], [**Co**] and [**CCC**]. Moreover we shall find results which will be used in Chapter 3 and Chapter 4.

In Section 2.1 we shall recall how to get semicanonical line bundles on singular curves.

In Section 2.2 we shall recall basic facts of the construction of moduli spaces of square roots of line bundles on nodal curves and in 2.2.1 we shall prove a property of one-dimensional subvarieties of Cornalba's moduli space of stable spin curves.

In Section 2.3 we shall recall Caporaso's compactification of the universal Picard variety and we will introduce the notion of twisted spin curve of a quasistable curve.

In Section 2.4 we shall prove an interesting property of equivalence classes of line bundle, which will be used in Chapter 3.

**Notation and Terminology 2.**

(1) By a *l.c.i.* curve we will mean a curve which is a local complete intersection. By a *curve with cusps or tacnodes* we will always mean a curve on a smooth surface and whose singularities are double singularities of curves of type $A_2$ or $A_3$.

Notice that a curve with cusps and tacnodes is l.c.i..

(2) We say that a nodal curve $X$ is obtained from $C$ by *blowing-up* a subset $\Delta$ of the set of the nodes of $C$ if there exists a morphism $\pi : X \to C$ such that for every $n_i \in \Delta$, $\pi^{-1}(n_i) = E_i \simeq \mathbb{P}^1$ and $\pi : X - \cup_i E_i \to C - \Delta$ is an isomorphism. For every $n_i \in \Delta$ we call $E_i$ an *exceptional component* and $E_i \cap \overline{X - E_i}$ *exceptional nodes* of $X$. A *quasistable* curve is a semistable curve obtained by blowing-up a stable curve. A family of nodal curves $\mathcal{X} \to B$ is said to be a *blow-up* of a family $\mathcal{C} \to B$ if there exists a $B-$morphism $\pi : \mathcal{X} \to \mathcal{C}$ such that for every $b \in B$ the restriction $\pi|_{X_b} : X_b \to C_b$ is a blow-up of $C_b$.

(3) If 0 is a distinguished point of a $1-$dimensional scheme $B$, we shall denote by $B^* := B - 0$. In this case if $f : \mathcal{C} \to B$ is a family of stable curves over $B$, we shall denote by $\mathcal{C}^*$ the restriction of $\mathcal{C}$ over $B^*$. Similarly if $\mathcal{N} \in \operatorname{Pic}\mathcal{C}$ we denote by $\mathcal{N}^* := \mathcal{N}|_{\mathcal{C}^*}$.

(4) If $X$ is a quasistable curve, we set $\tilde{X} := \overline{X - \cup E}$ where $E$ runs over the set of the exceptional components of $X$. We denote by $\Sigma_X$ the graph having the connected components of $\tilde{X}$ as vertices and the exceptional components of $X$ as edges.

(5) Let $X = \cup_{1 \le i \le \gamma} X_i$ be the decomposition of a semistable curve into its irreducible components. If $Z$ is any subcurve of $X$, we denote by $g_Z$ its arithmetic genus and by $k_Z := |Z \cap Z^c|$. Moreover if $L \in \operatorname{Pic}(X)$ is a line bundle, we denote by $\underline{\deg}\, L$ the *multidegree* of $L$, which is the string of integers

$$\underline{\deg}\, L = (\deg_{X_1} L, \dots, \deg_{X_\gamma} L).$$

(6) For any graph $\Gamma$ and commutative group $G$, we denote by $\mathcal{C}^0(\Gamma, G)$ and $\mathcal{C}^1(\Gamma, G)$ the groups of formal linear combinations respectively of vertices and edges of $\Gamma$ with coefficients in $G$. When we fix an orientation for $\Gamma$, then $\mathcal{C}^0(\Gamma, G) \to \mathcal{C}^1(\Gamma, G)$ denotes the usual coboundary operator. We denote by $\mu_2 = \{1, -1\}$ the multiplicative group of square roots of 1.

## 2.1. The spin gluing data

In [**Ha2**] one can find a description of line bundles which are square roots of the dualizing sheaf of a curve, well-known also as *semicanonical line bundles* (recall that a curve is always Gorenstein and reduced). At the end of this Section we shall recall some results from [**Ha2**] for semicanonical line bundles of a tacnodal and cuspidal curve.

Let $W$ be a curves with double points. Consider its normalization $\nu : W^\nu \to W$ and the standard exact sequence

$$0 \to \mathcal{O}_W^* \to \nu_*\mathcal{O}_{W^\nu}^* \to \mathcal{F} \to 0$$

where $\mathcal{F}$ is a torsion sheaf supported on the singularities of $W$. Passing in cohomology we get for a suitable positive integer $b_1(W)$

$$0 \to H^0(\mathcal{F})/\mathrm{Im}(H^0(\nu_*\mathcal{O}_{W^\nu}^*)) \to \mathrm{Pic}(W) \xrightarrow{\nu^*} \mathrm{Pic}(W^\nu) \to 0.$$

Let $W = \cup_{1 \le i \le \gamma} W_i$ be the decomposition of $W$ into irreducible components. We shall denote a line bundle on $W^\nu$ by a string of $\gamma$ line bundles on the normalizations $W_i^\nu$ of the $W_i$.

- We say that a line bundle $N \in \mathrm{Pic}\, W$ is *divisible by* 2 or *even* if the degree of $\nu^*N$ is even on each connected component of $W^\nu$.

Fix $N \in \mathrm{Pic}\, W$ divisible by 2 and set

$$S(N) := \{L \in \mathrm{Pic}\, W : L^{\otimes 2} = N\}.$$

Assume that $W$ has nodal singularities. In this case we have $H^0(\mathcal{F})/\mathrm{Im}(H^0(\nu_*\mathcal{O}_{W^\nu}^*)) \simeq (\mathbb{C}^*)^{b_1(W)}$ where $b_1 = b_1(\Gamma_W)$ and $\Gamma_W$ is the dual graph of $W$. By the hypothesis on the divisibility there exists $\underline{L} = (L_1, \dots, L_\gamma) \in \mathrm{Pic}\, W^\nu$ such that $\underline{L}^{\otimes 2} = \nu^*N$. Pick any lifting $L$ of $\underline{L}$, that is $\nu^*L = \underline{L}$. Thus there is $\underline{c} = (c_1, \dots, c_{b_1(W)}) \in (\mathbb{C}^*)^{b_1(W)}$ such that

$$\underline{c} \otimes L^{\otimes 2} = N$$

and if we set $\underline{c}' := (\sqrt{c_1}, \dots, \sqrt{c_{b_1(W)}})$, then $\underline{c}' \otimes L \in S(N)$.

We see that the kernel of $S(N) \to \mathrm{Pic}(W^\nu)$ is the $\mu_2-$module

$$D(W) := (\mu_2)^{b_1(W)} \subset (\mathbb{C}^*)^{b_1(W)}$$

of the square roots of unity which we shall call *the module of the spin gluing data of* $W$.

This subgroup is described in [**Ha2**]. We shall describe it in a slightly different way.

From now on $L$ will be a fixed line bundle in $S(N)$ (in particular we are fixing its pull-back $\nu^*L$. Let $s_1, \dots, s_M$ be the singular points of $W$ such that $\nu^{-1}(s_k) = \{p_k, q_k\}$. The given $\nu^*L$

yields identifications $\psi_k : \mathbb{C} \simeq (\nu^* L)_{p_k} \to (\nu^* L)_{q_k} \simeq \mathbb{C}$. All the line bundles in $S(N)$ whose pull-back is $\nu^* L$ are obtained by choosing for each $s_k$ either the identification $\psi_k$ or the identification $-\psi_k$. Consider the free $\mu_2-$module $\mu_2^M$ generated by $d_{s_k} := (1, \ldots, \underset{k}{-1}, \ldots, 1)$ for $k = 1, \ldots, M$. We get an exact sequence

$$0 \longrightarrow \mathrm{Ker}\beta \longrightarrow \mu_2^M \xrightarrow{\beta} D(W) \longrightarrow 0. \tag{2.1}$$

where for every $d = (\epsilon_1, \ldots, \epsilon_N) \in \mu_2^M$, $\beta(d)$ is the gluing datum of $D(W)$ corresponding to the line bundle $L_d$ obtained from $\nu^* L$ using the identification $\epsilon_k \psi_k$ at $s_k$.

- CLAIM: there exists a geometric description of (2.1) describing $\mathrm{Ker}\beta \simeq (\mu_2)^{\gamma-1}$.

Notice that the claim implies the well-know result that $b_1(W) = b_1(\Gamma_W)$ ($W$ is nodal).

Let us show the claim. First of all observe that $d \in \mathrm{Ker}\beta$ if and only if one can construct an isomorphism $L \simeq L_d$.

For every subcurve $Z \subset W$ we shall denote by $d_Z := \prod_{s \in Z \cap Z^c} d_s$.

We can construct an isomorphism

$$L \xrightarrow{\sim} L_{d_Z}$$

by fiber multiplication in $L|_Z$ by $-1$ and hence $d_Z \in \mathrm{Ker}\beta$ for every subcurve $Z \subset W$.

Conversely let $\tau : L \simeq L_d$. Since for every component $W_i$ we have $\tau|_{W_i} : L|_{W_i} \xrightarrow{\sim} L_d|_{W_i}$ and $L^{\otimes 2}|_{W_i} = N|_{W_i} = L_d^{\otimes 2}|_{W_i}$, then $\tau|_{W_i}^{\otimes 2} = id$ and hence $\tau|_{W_i}$ is either the identity or the fiber multiplication by $-1$. Hence the elements of $\ker\beta$ are only of type $d_Z$ for $Z$ running over the subcurves of $W$.

We want to show that $\mathrm{Ker}\beta$ is generated by the $\gamma$ elements $d_{W_i}$, $1 \le i \le \gamma$ (recall that $W_1, \ldots, W_\gamma$ are the irreducible components of $W$). In fact for every subcurve $Z \subset W$ (obviously $d_s^2 = 1$ for every $s \in \{s_1, \ldots, s_M\}$)

$$d_Z = \prod_{s \in Z \cap Z^c} d_s = \left( \prod_{s \in Z \cap Z^c} d_s \right) \left( \prod_{\substack{s \in W_i \cap W_i^c \subset Z \\ s \notin Z \cap Z^c}} d_s^2 \right) = \prod_i \left( \prod_{\substack{s \in W_i \\ W_i \subset Z}} d_s \right) = \prod_{W_i \subset Z} d_{W_i}.$$

We show that a minimal set of generators of $\ker\beta$ is given by $d_{W_1}, \ldots, d_{W_{\gamma-1}}$. In fact

$$d_{W_\gamma} = \left( \prod_{s \in W_\gamma \cap W_\gamma^c} d_s \right) = \left( \prod_{\substack{s \in W_i \cap W_i^c \\ i \neq \gamma}} d_s \right) = \prod_{i \neq \gamma} d_{W_i}.$$

Moreover for every $\{i_1, \ldots, i_R\} \subseteq \{1, \ldots, \gamma - 1\}$ it is easy to see that if $d_{W_{i_1}} = d_{W_{i_2}} \cdots d_{W_{i_R}}$, then $W_{i_1} \cup \cdots \cup W_{i_R}$ is a connected component of $W$ yielding a contradiction.

In the last part of this Section we shall recall some results on $S(N)$, when $N = \omega_W$. In this case a natural partition of this set is given by

$$S^-(W) = \{L \in \mathrm{Pic}\, W : L^{\otimes 2} = \omega_W \, ; \, L \text{ odd, that is } h^0(L) \equiv 1 \; mod\,(2)\}$$

$$S^+(W) = \{L \in \mathrm{Pic}\, W : L^{\otimes 2} = \omega_W \, ; \, L \text{ even, that is } h^0(L) \equiv 0 \; mod\,(2)\}.$$

It is well-known that if $W$ is smooth of genus $g$, then $S^-(W)$ and $S^+(W)$ have respectively cardinality $N_g = 2^{g-1}(2^g - 1)$ and $N_g^+ = 2^{g-1}(2^g + 1)$. In the singular case we have

PROPOSITION 2.1. *Let $W$ be an irreducible curve with $\tau$ tacnodes, $\gamma$ cusps and $\delta$ nodes. Let $\tilde{g}$ be the genus of its normalization.*

- *If $\delta \neq 0$, then*
$$|S^-(W)| = 2^{\tau+\delta-1}(N_{\tilde{g}}^+ + N_{\tilde{g}}).$$
- *If $\delta = 0$, then*
$$|S^-(W)| = \begin{cases} 2^\tau N_{\tilde{g}}^+ & \text{if } \tau+\gamma \equiv 1 \ mod(2) \\ 2^\tau N_{\tilde{g}} & \text{if } \tau+\gamma \equiv 0 \ mod(2). \end{cases}$$

PROOF. See [**Ha2**, Corollary 2.7, Corollary 2.8]. □

Recall that if $t$ is a tacnode (respectively $c$ is a cusp) of a curve $W$ and if $W^\nu$ is the normalization of $W$ at $t$ (respectively at $c$) with $\nu^{-1}(t) = \{p, q\}$ (respectively $\nu^{-1}(c) = \{p\}$), then $\nu^*(\omega_C) = \omega_{W^\nu}(2p + 2q)$ (respectively $\nu^*(\omega_W) = \omega_{W^\nu}(2p)$).

PROPOSITION 2.2. *Let $W$ be an irreducible curve with tacnodes $t_1, \dots, t_s$ (respectively cusps $c_1, \dots, c_s$). Consider the normalization $\nu : W^\nu \to W$ of $W$ at $t_1, \dots, t_s$ (respectively at $c_1, \dots, c_s$) and set $\nu^{-1}(t_i) = \{p_i, q_i\}$ (respectively $\nu^{-1}(c_i) = \{p_i\}$).*

(i) *The map $S(W) \to S(W^\nu)$ sending $L \in S(W)$ to $\nu^*L(-\sum_{1\leq i\leq s}(p_i + q_i))$ (respectively to $\nu^*L(-\sum_{1\leq i\leq s} p_i)$) is a $2^s$-to one map (respectively one to one).*

(ii) *Let $L \in S(W)$ and $M := \nu^*L(-\sum_{1\leq i\leq s}(p_i + q_i))$ (respectively $M = \nu^*L(-\sum_{1\leq i\leq s} p_i)$). Then*
$$h^0(L) \equiv h^0(M) + s \ mod(2).$$

PROOF. See the proof of [**Ha2**, Theorem 2.22]. □

## 2.2. Moduli of roots of line bundles of curves

In the recent paper [**CCC**] the authors focused on the compactification of the moduli space of roots of line bundles on smooth curves. In particular for any fixed family of nodal curves and a line bundle $\mathcal{N}$ on the total space of the family, a moduli space compactifying the isomorphism classes of fiberwise $r$-th roots of $\mathcal{N}$ was constructed in the spirit of the paper [**Co**], where one can find a compactification $\overline{S_g}$ of the moduli space of theta characteristics of smooth curves.

We want to recall known facts about this construction in the case of square roots.

Let $C$ be a nodal curve and $N \in \mathrm{Pic}(C)$ be a line bundle on $C$ divisible by 2.

DEFINITION 2.3. Consider a triple $(X, L, \alpha)$ where $\pi : X \to C$ is a blow-up of the nodal curve $C$, $L$ is a line bundle on $X$ and $\alpha$ is a homomorphism $\alpha : L^{\otimes 2} \to \pi^*(N)$. The triple is said to be a *limit square root* of $(C, N)$ if the following properties are satisfied

- the restriction of $L$ to every exceptional component of $X$ has degree 1;
- the map $\alpha$ is an isomorphism at the points of $X$ not belonging to an exceptional component;
- for every exceptional component $E_i$ of $X$ such that $E_i \cap E_i^c = \{p_i, q_i\}$ the orders of vanishing of $\alpha$ at $p_i$ and $q_i$ add up to 2.

If $C$ is stable and $N = \omega_C$, a triple $(X, L, \alpha)$ as above is said to be a *stable spin curve.*

Notice that a pair $(C, L)$ where $C$ is a smooth curve and $L$ a theta characteristic of $C$ is a stable spin curve.

A similar definition works for families $\mathcal{C} \to B$ of nodal curves.

If $\mathcal{N} \in \operatorname{Pic}\mathcal{C}$ is a line bundle of even relative degree, a *limit square root* $(\mathcal{X}, \mathcal{L}, \alpha)$ of $(\mathcal{C}, \mathcal{N})$ is the datum of a blow-up $\pi : \mathcal{X} \to \mathcal{C}$ of $\mathcal{C}$, a line bundle $\mathcal{L} \in \operatorname{Pic}\mathcal{X}$ and a homomorphism $\alpha : \mathcal{L}^{\otimes 2} \to \pi^*(\mathcal{N})$ such that for every $b \in B$, $(X_b, \mathcal{L}_b, \alpha_b)$ is a limit square root of $(C_b, \mathcal{N}_b)$.

Definition 2.4. An isomorphism of limit square roots of $(\mathcal{C}, \mathcal{N})$ between $(\mathcal{X} \to B, \mathcal{L}, \alpha)$ and $(\mathcal{X}' \to B, \mathcal{L}', \alpha')$ is the datum of

- an isomorphism $\sigma : \mathcal{X} \to \mathcal{X}'$ over $\mathcal{C}$
- an isomorphism $\tau : \sigma^*\mathcal{L}' \to \mathcal{L}$ that makes the following diagram commute

$$\begin{tikzcd}
\sigma^*(\mathcal{L}')^{\otimes 2} \arrow[r, "\tau^{\otimes 2}"] \arrow[d, "\sigma^*(\alpha')"'] & \mathcal{L}^{\otimes 2} \arrow[d, "\alpha"] \\
\sigma^*(\pi')^*(\mathcal{N}) \arrow[r, "\sim"] & \pi^*\mathcal{N}
\end{tikzcd}$$

Given a limit square root $(\mathcal{X} \to B, \mathcal{L}, \alpha)$ of $(\mathcal{C}, \mathcal{N})$ we denote by $\operatorname{Aut}(\mathcal{X} \to B, \mathcal{L}, \alpha)$ the group of its automorphisms. Moreover we denote by $\operatorname{Aut}_{\mathcal{C}}(\mathcal{X})$ the group of automorphisms of $\mathcal{X}$ over $\mathcal{C}$, the so-called *inessential automorphisms.* Notice that $\operatorname{Aut}(\mathcal{X}, \mathcal{L}, \alpha)$ maps to $\operatorname{Aut}_{\mathcal{C}}(\mathcal{X})$.

When no confusion may arise we shall abuse notation denoting by $\xi = (X, L, \alpha)$ both a limit square root and its isomorphism class.

Recall the definition of the graph $\Sigma_X$ in Not.Ter. 2 (4). The description of the isomorphisms of a limit square root is given by the following Lemma.

Lemma 2.5. *Let $\xi = (X, L, \alpha)$ be a limit square root of $(C, N)$ and fix an orientation of the graph $\Sigma_X$. Then*

(i) *There are natural identifications $Aut_C(X) \simeq \mathcal{C}^1(\Sigma_X, \mathbb{C}^*)$ and $Aut(\xi) \simeq \mathcal{C}^0(\Sigma_X, \mu_2)$.*

(ii) *The natural homomorphism $Aut(\xi) \to Aut_C(X)$ corresponds to the composition of the coboundary map*

$$\mathcal{C}^0(\Sigma_X, \mu_2) \longrightarrow \mathcal{C}^1(\Sigma_X, \mu_2)$$

*with the inclusion $\mathcal{C}^1(\Sigma_X, \mu_2) \hookrightarrow \mathcal{C}^1(\Sigma_X, \mathbb{C}^*)$.*

(iii) *Let $\xi = (X, L, \alpha)$, $\xi' = (X, L', \alpha')$ be two limit square roots of $(N, C)$. If the restrictions of $L$ and $L'$ to $\tilde{X}$ are equal, then $\xi$ and $\xi'$ are isomorphic.*

The first isomorphism is clear. In fact let $E_1, \ldots, E_m$ be the exceptional components of $X$ and set $E_i \cap E_i^c = \{0, \infty\}$. Then any inessential automorphism in $\operatorname{Aut}_C(X)$ acts on each $E_i$ as multiplication by a non-zero constant and conversely any $m-$tuple of non-zero constants yields an inessential automorphism in $\operatorname{Aut}_C(X)$.

For the other statements see [**CCC**, Lemma 2.3.2.] and [**Co**, Lemma 2.1].

There exists a moduli space parametrizing isomorphism classes of limit square roots of a line bundle on the total space of a given family of nodal curves. Let us recall the related moduli problem.

In the sequel $f : \mathcal{C} \to B$ will be a fixed family of nodal curves and $\mathcal{N} \in \operatorname{Pic}(\mathcal{C})$ a line bundle of even relative degree. For a given $B-$scheme $P$ consider the fiber product

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "p"] \arrow[d] & \mathcal{C} \arrow[d, "f"] \\
P \arrow[r] & B
\end{tikzcd}$$

and the contravariant functor

$$\overline{\mathcal{S}}_f(\mathcal{N}) : \{B\text{-schemes}\} \longrightarrow \{\text{sets}\}$$

associating to a $B-$scheme $P$ the set $\overline{\mathcal{S}}_f(\mathcal{N})(P)$ of all limit square roots of $p^*\mathcal{N}$ modulo isomorphisms of limit square roots.

THEOREM 2.6. *Let $f : \mathcal{C} \to B$ be a family of nodal curves over a quasi-projective scheme $B$. Let $\mathcal{N}$ be a line bundle on $\mathcal{C}$ of even relative degree. The functor $\overline{\mathcal{S}}_f(\mathcal{N})$ is coarsely represented by a quasi-projective scheme $\overline{S}_f(\mathcal{N})$, finite over $B$. If $B$ is projective, then $\overline{S}_f(\mathcal{N})$ is projective.*

Below we shall recall the local structure of $\overline{S}_f(\mathcal{N})$, where $f$ is a family of stable curves. We shall widely use this construction in the sequel. We refer to [**CCC**, Theorem 2.4.1.] for details and proofs.

Fix a stable fiber $C$ of $f : \mathcal{C} \to B$ and a limit square root $\xi = (X, L, \alpha)$ of $(C, \mathcal{N}|_C)$. Let $E_1, \dots, E_m$ be the exceptional components of $X$ and $n_1, \dots, n_m$ the corresponding nodes of $C$.

First of all the *base of the universal deformation* $U_\xi$ of $\xi$ is obtained as follows. Let $D_C$ be the base of the universal deformation of $C$, where $D_C$ is the unit polydisc in $\mathbb{C}^{3g-3}$. We can write $D_C = D_t \times D'_t$, where $D_t$ is the unit polydisc with coordinates $t_1, \dots, t_m$ such that $\{t_i = 0\}$ is the locus where the node $n_i$ persists and $D'_t$ corresponds to the remaining coordinates $t_{m+1}, \dots, t_{3g-3}$. Consider a copy $D_\xi$ of $D_C$ and write $D_\xi = D_s \times D'_s$, where $D_s$ has coordinates $s_1, \dots, s_m$ and $D'_s$ has coordinates $s_{m+1}, \dots, s_{3g-3}$.

Consider the morphism

$$\rho : D_\xi \to D_C \qquad (s_1 \dots s_m, s_{m+1}, \dots s_{3g-3}) \xrightarrow{\rho} (s_1^2, \dots, s_m^2, s_{m+1}, \dots, s_{3g-3})$$

sending $D_s$ to $D_t$ and (up to restrict $B$) the modular morphism $B \to D_C$ induced by the family $f$. The base $U_\xi$ of the universal deformation of $\xi$ is the fiber product

$$\begin{tikzcd} U_\xi \arrow[r] \arrow[d] & B \arrow[d] \\ D_\xi \arrow[r, "\rho"] & D_C \end{tikzcd}$$

In order to complete the local description of $\overline{S}_f(\mathcal{N})$ we recall how $\mathrm{Aut}(\xi)$ acts on $U_\xi$. This is given by the following Lemma. If $W \to Z$ is a morphism of schemes, then $\mathrm{Aut}_Z(W)$ denotes the group of automorphism of $W$ over $Z$.

LEMMA 2.7. *There is a natural isomorphism $Aut_{D_C} D_\xi \simeq \mathcal{C}^1(\Sigma_X, \mu_2)$. Moreover the action of $Aut_{D_C} D_\xi$ on $D_\xi$ lifts to a natural action $Aut_{D_C} D_\xi \to Aut_B(U_\xi)$.*

In fact observe that the automorphisms of $D_\xi$ over $D_C$ are the automorphisms of $D_s$ over $D_t$ and hence they are generated by $\beta_h : (s_1, \dots, s_h, \dots, s_m) \to (s_1, \dots, -s_h, \dots, s_m)$ for every $h = 1, \dots, m$. Thus $\mathrm{Aut}_{D_C} D_\xi \simeq \mu_2^m$. Indeed one can show that there exists a homomorphism $\mathrm{Aut}_{D_C} D_\xi \to \mathrm{Aut}_C(X)$ inducing an isomorphism $\mathrm{Aut}_{D_C} D_\xi \simeq \mathcal{C}^1(\Sigma_X, \mu_2)$ (see [**CCC**, Lemma 3.3.1.]). The universal property of the fiber product implies the second statement of the Lemma.

It follows from Lemma 2.5 and Lemma 2.7 that we can write the coboundary operator as

$$\mathrm{Aut}(\xi) \simeq \mathcal{C}^0(\Sigma_X, \mu_2) \longrightarrow \mathcal{C}^1(\Sigma_X, \mu_2) \simeq \mathrm{Aut}_{D_C} D_\xi$$

and $\mathrm{Aut}(\xi)$ acts on $U_\xi$ via this homomorphism. The local picture of $\overline{S}_f(\mathcal{N})$ at $\xi$ is given by an injective map

$$U_\xi/\mathrm{Aut}(\xi) \hookrightarrow \overline{S}_f(\mathcal{N}). \tag{2.2}$$

Let $B'$ be any $B-$scheme. Set $f' : \mathcal{C}' = \mathcal{C} \times_{B'} B \longrightarrow B'$ and consider the pull-back $\mathcal{N}'$ of $\mathcal{N}$ to $\mathcal{C}'$. Then

$$\overline{S}_{f'}(\mathcal{N}') = \overline{S}_f(\mathcal{N}) \times_{B'} B.$$

In the sequel we shall denote by $\overline{S}_C(N)$ the zero dimensional scheme $\overline{S}_{f_C}(N)$, where $f_C : C \to \{pt\}$ is the trivial family. Obviously the fiber of $\overline{S}_f(\mathcal{N}) \to B$ over $b \in B$ is given by $\overline{S}_{C_b}(\mathcal{N}|_{C_b})$.

Below we recall the structure of the zero dimensional scheme $\overline{S}_C(N)$, where $C$ is a stable curve and $N \in \mathrm{Pic}(C)$ is a fixed line bundle of even degree.

Let $\pi : X \to C$ be a blow-up of $C$.

Definition 2.8. The graph $A_X$ *associated to* $X$ is the subgraph of the dual graph $\Gamma_C$ of $C$ corresponding to the set of nodes of $C$ which are blown-up by $\pi$.

A necessary and sufficient condition for a subgraph $A$ of $\Gamma_C$ to be the graph associated to a blow-up of $C$ which is the support of some limit square root of $(C, N)$ is

(A) For every irreducible component $C_j$ of $C$, consider the vertex $v_j$ of $\Gamma_C$ corresponding to $C_j$. Then the number of edges of $A$ containing $v_j$ is congruent to $\deg_{C_j}(N)$ modulo 2.

We shall call *admissible* a subgraph of $\Gamma_C$ satisfying (A). One can see that there are $2^{b_1(\Gamma_C)}$ admissible subgraphs of $\Gamma_C$.

For example if $N = \omega_C$, then $\Gamma_C$ is always admissible.

Let $A_X$ be an admissible subgraph of $\Gamma_C$. Denote by $E_1, \ldots, E_m$ the exceptional components of $X$ and by $E_i \cap E_i^c = \{p_i, q_i\}$. Recall that $\tilde{X} := \overline{X - \cup_{1 \le i \le m} E_i}$. Consider the restriction $\tilde{\pi} : \tilde{X} \to C$. The dual graph of $\tilde{X}$ is $\overline{\Gamma_C - A_X}$. If $g^\nu$ is the genus of the normalization $C^\nu$ of $C$, then there are $2^{2g^\nu + b_1(\overline{\Gamma_C - A_X})}$ line bundles $\tilde{L} \in \mathrm{Pic}(\tilde{X})$ such that

$$\tilde{L}^{\otimes 2} = \tilde{\pi}^*(N)\left(-\sum_{1 \le i \le m}(p_i + q_i)\right).$$

In fact we have $2^{2g^\nu}$ choices for the pull-back of $\tilde{L}$ to $C^\nu$ and $2^{b_1(\overline{\Gamma_C - A_X})}$ gluings at nodes of $\tilde{C}$. If we glue $\tilde{L}$ to $\mathcal{O}_{E_i}(1)$ for $i = 1, \ldots, m$ (regardless of the gluing data, producing isomorphic limit square roots as explained in Lemma 2.5), we get a limit square root of $(C, N)$.

- **Fact:** the geometric multiplicity of the point $\xi = (X, G, \alpha)$ of the zero dimensional scheme $\overline{S}_C(N)$ is $2^{b_1(\Sigma_X)} = 2^{b_1(\Gamma_C) - b_1(\overline{\Gamma_C - A_X})}$.

In fact the order of ramification of $D_\xi/\mathrm{Aut}(\xi) \to D_C$ over the origin is obtained as follows. If $\tilde{X}$ has $\gamma$ connected components and $X$ has $m$ exceptional components, this order is $2^m/2^{\gamma-1} = 2^{b_1(\Sigma_X)}$ (notice that the image of the coboundary $\mathcal{C}^0(\Sigma_X, \mu_2) \to \mathcal{C}^1(\Sigma_X, \mu_2)$ has cardinality $2^{\gamma-1}$). Since the graph $\Sigma_X$ is obtained from $\Gamma_C$ by contracting the edges in $\overline{\Gamma_C - A_X}$, then $b_1(\Sigma_X) = b_1(\Gamma_C) - b_1(\overline{\Gamma_C - A_X})$.

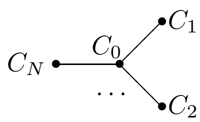
Notice that a limit square root supported on a quasistable curve $X$ has geometric multiplicity 1 if and only if $b_1(\Sigma_X) = 0$. In particular this is true if $X$ is either a stable curve or of compact type (i.e. its dual graph is a tree).

Notice that

$$\mathrm{length}\, \overline{S}_C(N) = 2^{b_1(\Gamma_C)}\ 2^{2g^\nu + b_1(\overline{\Gamma_C - A_X})}\ 2^{b_1(\Gamma_C) - b_1(\overline{\Gamma_C - A_X})} = 2^{2g^\nu + 2b_1(\Gamma_C)} = 2^{2g}.$$

We shall widely use the following examples in Chapter 3.

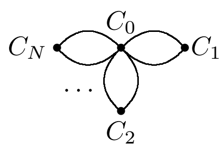
EXAMPLE 2.9. Consider a curve $C$ of genus $g$ whose dual graph $\Gamma_C$ is shown below.




Let us describe $\overline{S}_C(\omega_C \otimes T)$ where $T = \mathcal{O}_{\mathcal{C}}(D) \otimes \mathcal{O}_C \in \mathrm{Pic}(C)$ for a smoothing $\mathcal{C}$ of $C$ and a Cartier divisor $D$ of $\mathcal{C}$ supported on $C$.

Since $b_1(\Gamma_C) = 0$, there is only one blow-up $X \to C$ of $C$ such that $A_X$ is admissible. More precisely the edge of $\Gamma_C$ connecting $C_0$ to $C_j$ appears in $A_X$ if and only if $\deg_{C_j}(\omega_C \otimes T) \equiv 1\ (2)$. Let $C_1, \dots, C_d$ be the components satisfying the last condition. Set $n_j := C_0 \cap C_j$. Then a limit square root of $\omega_C \otimes T$ is given by gluing a square root of $\omega_C \otimes T \otimes \mathcal{O}_{C_0}(-\sum_{1 \le j \le d} n_j)$, a square root of $\omega_C \otimes T \otimes \mathcal{O}_{C_j}(-n_j)$ for $j = 1, \dots, d$, a square root of $\omega_C \otimes T \otimes \mathcal{O}_{C_j}$ for $j = d+1, \dots, N$ and $\mathcal{O}_E(1)$ for every exceptional component $E$ of $X$ (note that there is just one gluing datum because $X$ is of compact type). Since $X$ is of compact type, then $\overline{S}_C(N)$ is reduced.

EXAMPLE 2.10. Consider a curve $C$ of genus $g$ whose dual graph $\Gamma_C$ is as shown below.




Let us describe $\overline{S}_C(\omega_C \otimes T)$ where $T = \mathcal{O}_{\mathcal{C}}(D) \otimes \mathcal{O}_C \in \mathrm{Pic}(C)$ for a general smoothing $\mathcal{C}$ of $C$ and a Cartier divisor $D$ of $\mathcal{C}$ supported on $C$.

Let $X \to C$ be a blow-up of $C$. It is easy to see that $A_X$ is admissible if and only if for $j = 1, \dots, N$ either both the edges connecting $C_0$ to $C_j$ appear in $A_X$ or none of these edges appears. Notice that $b_1(\Gamma_C) = N$, hence there are $2^N$ admissible subgraphs of $\Gamma_C$.

Pick an admissible subgraph $A_X$ of $\Gamma_C$ and set $\{n_{j1}, n_{j2}\} := C_0 \cap C_j$. Assume that $n_{j1}, n_{j2}$ for $j = 1, \dots, d$ are the nodes which are blown-up to get $X$.

Glue a square root of $\omega_C \otimes T \otimes \mathcal{O}_{C_0}(-\sum_{1 \le j \le d}(n_{j1} + n_{j2}))$ and a square root of $\omega_C \otimes T \otimes \mathcal{O}_{C_j}$ for $j = d+1, \dots, N$. There are $2^{b_1(\overline{\Gamma_C - A_X})} = 2^{N-d}$ possible gluings. A limit square root of $\omega_C \otimes T$ is given by gluing a line bundle of $C_0 \cup C_{d+1} \cdots \cup C_N$ obtained as just explained, a square root of $\omega_C \otimes T \otimes \mathcal{O}_{C_j}(-n_{j1} - n_{j2})$ for $j = 1, \dots, d$ and $\mathcal{O}_E(1)$ for every exceptional component $E$ of $X$ (regardless of the gluing data, see Lemma 2.5 (iii)).

Since $b_1(\Sigma_X) = d$, then a limit square root of $\omega_C \otimes T$ supported on $X$ has multiplicity $2^d$.

### 2.2.1. The moduli space of stable spin curves.

Consider Cornalba's compactification $\overline{S_g}$ of the moduli space of theta characteristics on smooth curves. $\overline{S_g}$ parametrizes isomorphism classes of stable spin curves. The main difference with the previously recalled moduli spaces is the notion of isomorphisms, yielding a coarser equivalence relation (see Def. 2.11).

In Proposition 2.12 we shall see a typical unexpected phenomenon of one-dimensional subvarieties of $\overline{S_g}$ which will be behind the discussion of Chapter 3.

Definition 2.11. Let $\xi = (X, G, \alpha)$ and $\xi' = (X', G', \alpha')$ be stable spin curves respectively of the stable curves $C$ and $C'$. An isomorphism between $\xi$ and $\xi'$ is the datum of

- an isomorphism $\psi : C \to C'$
- an isomorphism of limit square roots of $(C, \omega_C)$ between $\xi$ and $\psi^*\xi$.

$\overline{S}_g$ is well-known as *moduli space of stable spin curves.*

If $C$ is a stable curve without non-trivial automorphisms, then the fiber of $\varphi$ over the point of $\overline{M_g}$ parametrizing the isomorphism class of $C$ is exactly $\overline{S}_C(\omega_C)$.

If $C$ is a stable curve without non-trivial automorphisms, then the local description of $\overline{S_g}$ at a stable spin curve of $C$ is given by (2.2) below Lemma 2.7 where we put $B = D_C$. Then if $C$ has two irreducible components and $\xi = (X, G, \alpha)$ is a stable spin curve of $C$ such that $\tilde{X}$ is connected, it is easy to see that $\mathrm{Aut}(\xi)$ acts trivially on $U_\xi$ and hence $\xi$ is a smooth point of $\overline{S_g}$ (see also the proof of the following Proposition).

It follows that the general curve of $\overline{S_g}$ passing through $\xi$ is smooth at $\xi$. We show that this is no longer true for subcurves of $\overline{S_g}$ obtained by pulling-back (via $\varphi : \overline{S_g} \to \overline{M_g}$) general curves of $\overline{M_g}$ through $\varphi(\xi)$.

Proposition 2.12. *Let $C$ be a stable curve without non-trivial automorphisms and with two smooth irreducible components. Let $x$ be the point of $\overline{M_g}$ parametrizing the isomorphism class of $C$. Consider the morphism $\varphi : \overline{S_g} \to \overline{M_g}$ and a general curve $B$ of $\overline{M_g}$ containing $x$.*

*Then the curve $\varphi^{-1}(B)$ of $\overline{S_g}$ is singular at a point $\xi = (X, G, \alpha)$ of $\varphi^{-1}(x)$ such that $X$ is the blow-up of $C$ at least at two nodes and $\tilde{X}$ is connected.*

Proof. The problem is local, hence we may assume that $B \subset D_C$ (recall that $D_C$ is the base of the universal deformation of $C$). Let $t_1, \dots, t_{3g-3}$ be the coordinates of $D_C$. If $n_1, \dots, n_\delta$ are the nodes of $C$, assume that $\{t_i = 0\}$ is the locus where the node $n_i$ persists for $i = 1, \dots, \delta$. Since $B$ is general, the implicit function theorem allows us to describe $B$ as

$$(t_1, t_1h_2(t_1), \dots, t_1h_{3g-3}(t_1))$$

where $h_j$ are analytic functions such that $h_j(0) \in \mathbb{C}^*$.

Let $\xi = (X, G, \alpha)$ be a stable spin curve of $C$ such that $X \to C$ is the blow-up of $C$ at the nodes $n_1, \dots, n_m$ of $C$ with $1 < m < \delta$. If we consider (see the discussion below Th. 2.6)

$$\rho : D_\xi := D_s \times D'_s \to D_C \qquad (s_1 \dots s_m, s_{m+1} \dots) \to (s_1^2 \dots s_m^2, s_{m+1} \dots)$$

the base of the universal deformation of $\xi$ is $U_\xi = \rho^{-1}(B)$ and is given by

$$U_\xi = \mathrm{v}(s_2^2 - s_1^2h_2(s_1^2), \dots s_m^2 - s_1^2h_m(s_1^2), s_{m+1} - s_1^2h_{m+1}(s_1^2), \dots s_{3g-3} - s_1^2h_{3g-3}(s_1^2)).$$

Let $\Gamma_C$ be the dual graph of $C$. We find how $\mathrm{Aut}(\xi)$ acts on $U_\xi$.

Since $1 < m < \delta$, then $A_X$ has exactly one vertex and the image of the coboundary operator $\mathcal{C}^0(\Sigma_X, \mu_2) \to \mathcal{C}^1(\Sigma_X, \mu_2)$ is trivial. Hence $\mathrm{Aut}(\xi)$ acts trivially on $U_\xi$ and the local picture of $\varphi^{-1}(B)$ at $\xi$ is given by $U_\xi$ (see (2.2) below Lemma 2.7) which is singular at the origin. $\square$

In the hypotesis of the previous Proposition, it is easy to see that if $X$ is the blow-up at most at one node or $C$, then $\varphi^{-1}(A)$ is smooth at $\xi$, while if $X$ is the blow-up at the whole set of nodes of $C$, then $\xi$ is a singular point.

## 2.3. The universal Picard variety

In [**C2**] L. Caporaso, using Geometric Invariant Theory, constructed a modular compactification $\overline{P_{d,g}}$ over $\overline{M_g}$ of the so-called universal Picard variety $Pic_{d,g}$ over $M_g$ whose set-theoretic description is given by

$$Pic_{d,g} = \{(C, L) : \ C \text{ smooth curve of genus } g,\ L \text{ line bundle of } C \text{ of degree } d\}/\text{iso}.$$

In this Section we recall some basic facts about the geometry of $\overline{P_{d,g}}$, stressing some properties which we shall use in the sequel.

The boundary points of $\overline{P_{d,g}}$ correspond to certain line bundles on quasistable curves having degree 1 on exceptional components and this is the main analogy between $\overline{P_{d,g}}$ and the notion of limit square roots of the previous Section.

Fix a large $d$ and consider the Hilbert scheme $\mathrm{Hilb}_{d,g}$ of connected curves of degree $d$ and genus $g$ in $\mathbb{P}^s$, where $s = d - g$. The group $SL(s+1)$ naturally acts on $\mathrm{Hilb}_{d,g}$. Fix a linearization for this action from now on. If we denote by $H_d \subset \mathrm{Hilb}_{d,g}$ the subset of GIT-semistable points representing connected curves, then $\overline{P_{d,g}}$ is the GIT-quotient (see [**C1**, Theorem 2.1])

$$q : H_d \longrightarrow \overline{P_{d,g}} = H_d/SL(s+1).$$

$\overline{P_{d,g}}$ is a modular compactification of the universal Picard variety $Pic_{d,g}$, that is its points have a geometrically meaningful description, which we shall briefly recall below.

Definition 2.13. Let $X$ be a quasistable curve and $L \in \mathrm{Pic}(X)$ be a line bundle of degree $d$. We say that the multidegree $\underline{\deg}\, L$ is *balanced* if

- $\deg_E L = 1$ for every exceptional component $E$ of $X$
- the multidegree $\underline{\deg} L$ satisfies the *Basic Inequality*, that is for every subcurve $Z$ of $X$

$$(BI) \qquad \left| \deg_Z L - \frac{d}{2g-2}(\deg_Z \omega_X) \right| \leq \frac{k_Z}{2}.$$

The notion of twisters of a nodal curve is introduced in order to control the non-separatedness of the Picard functor.

Definition 2.14. Let $X$ be a nodal curve and fix a smoothing $f : \mathcal{X} \to B$ of $X$. A line bundle $T \in \mathrm{Pic}(X)$ is said to be a $f-$*twister* of $X$ or simply a *twister* of $X$ if

$$T \simeq \mathcal{O}_{\mathcal{X}}(D) \otimes \mathcal{O}_X,$$

where $D$ is a Cartier divisor of $\mathcal{X}$ supported on irreducible components of $X$. We shall denote by $Tw_f(X)$ the set of all the $f-$twisters of $X$. When no confusion may arise we shall use also the suggestive notation $\mathcal{O}_f(D)$ for an $f-$twister of $X$.

Definition 2.15. Consider two balanced line bundles $L' \in \operatorname{Pic} X'$ and $L'' \in \operatorname{Pic} X''$, where $X'$ and $X''$ are quasistable curves. We say that $L'$ and $L''$ are *equivalent* if there exists a semistable curve $X$ obtained by a finite sequence of blow-ups both of $X'$ and of $X''$, and a twister $T$ of $X$ such that, denoting by $L'_X$ and $L''_X$ the pull-backs of $L'$ and $L''$ to $X$, we have

$$L'_X \simeq L''_X \otimes T.$$

The modular property of $\overline{P_{d,g}}$ follows from the following Theorem (see [**C2**, Proposition 3.1, Proposition 6.1 and Lemma 5.2] and [**CCC**, Theorem 5.1.6] for the proof).

Theorem 2.16. *Let $X \subset \mathbb{P}^s$ be a connected curve of genus $g$. Then*

(i) *The Hilbert point of $X$ is GIT-semistable if and only if $X$ is quasistable and $\mathcal{O}_X(1)$ is balanced.*

(ii) *Assume that the Hilbert point of $X$ and $X'$ are GIT-semistable. Then they are GIT-equivalent if and only if $\mathcal{O}_X(1)$ and $\mathcal{O}_{X'}(1)$ are equivalent.*

Therefore $\overline{P_{d,g}}$ parametrizes equivalence classes of balanced line bundles of degree $d$ on quasistable curves of genus $g$.

Moreover if $L'$ and $L''$ are two balanced and equivalent line bundles on quasistable curves of genus $g$, then their corresponding Hilbert points in $H_d$ are identified in $\overline{P_{d,g}}$.

There exists a natural injective morphism (see [**CCC**, Lemma-Definition 5.2.1.])

$$\chi : S_g \hookrightarrow \overline{P_{g-1,g}}. \tag{2.3}$$

Let us denote by $\hat{S}_g$ the closure of the image of $S_g$ in $\overline{P_{g-1,g}}$. We can view $\chi$ as a natural birational map between $\overline{S}_g$ and $\hat{S}_g$.

The modular property of $\hat{S}_g$ is explicit and is given by the following

Theorem 2.17. *The points of $\hat{S}_g$ are in bijection with equivalence classes of balanced line bundles $L \in Pic(X)$ where $X$ is a quasistable curve of genus $g$ such that there exists a twister $T$ of $X$ satisfying*

$$L^{\otimes 2} \simeq \omega_X \otimes T.$$

Proof. (See [**CCC**, Theorem 5.2.2]). ☐

Definition 2.18. Let $T \in \operatorname{Pic}(X)$ be a twister of a quasistable curve $X$. We say that $T$ is an *admissible twister* of $X$ if the multidegree $\frac{1}{2}\underline{\deg}\,(\omega_X \otimes T)$ is balanced. In this case if

$$T \simeq \mathcal{O}_f(D) \in \operatorname{Pic}(X)$$

for a smoothing $\mathcal{X}$ of $X$ and a Cartier divisor $D$ of $\mathcal{X}$, we say that $D$ is an *admissible divisor* of $\mathcal{X}$.

Definition 2.19. Let $L \in \operatorname{Pic}(X)$ be a line bundle such that $\alpha : L^{\otimes 2} \simeq \omega_X \otimes T$, where $T = \mathcal{O}_{\mathcal{X}}(D) \otimes \mathcal{O}_X$ is an admissible twister of $X$. We say that $(X, L)$ is a *D-twisted spin curve.*

Notice that $(X, L, \alpha)$ is a limit square root of $(X, \omega_X \otimes T)$. In the sequel, when we shall see a twisted spin curve as limit square root, we omit the given isomorphism $\alpha$ if no confusion may arise. A stable spin curve supported on a stable curve is a $0-$twisted spin curve.

## 2.4. The equivalence class of a line bundle

In [**F**] Fontanari showed that the morphism $\chi$ of (2.3) extends to a natural morphism

$$\chi : \overline{S_g} \longrightarrow \hat{S}_g.$$

Its set-theoretic description is as follows.

One can show that a stable spin curve $\xi$ is represented in $H_{g-1}$ by a GIT-semistable point whose orbit is closed (in $H_{g-1}$). $\chi(\xi)$ is the image via the quotient morphism $q : H_{g-1} \to \overline{P_{g-1,g}}$ of the closed orbit of the Hilbert point in $H_{g-1}$ representing $\xi$.

Moreover $\chi$ is a bijective morphism.

It is well-known that $(d-g+1, 2g-1) = 1$ if and only if $\overline{P_{d,g}}$ is a geometric quotient, that is $H_d$ has only GIT-stable points (see [**C1**, Proposition 6.2, Propostion 8.1]). In particular there are GIT-strictly semistable points in $H_{g-1}$.

Consider $q : H_{g-1} \to \overline{P_{g-1,g}}$. From the previous discussion, we argue that a twisted spin curve $(X, L)$ which is not a stable spin curve is represented in $H_{g-1}$ by a GIT-strictly semistable point such that $q(X, L) \in \hat{S}_g$ (Th. 2.17). This means that if $q(X, L) = \chi(\xi) \in \hat{S}_g$ for a stable spin curve $\xi$, the orbit of the Hilbert point representing $(X, L)$ is non-closed (in $H_{g-1}$) and its closure contains the (closed) orbits of the Hilbert point of $\xi$ (see the Fundamental Theorem of GIT, Theorem 1.1).

In this case $L$ and $G$ are equivalent line bundle according to Def. 2.15.

A natural question is

Question 2.20. *Let $(X, L)$ be a twisted spin curve. Describe a stable spin curve $\xi = (X', G, \alpha)$ such that $L$ and $G$ are equivalent.*

In the sequel we will answer to the posed question for twisted spin curves arising from general smoothings. Notice that when the stable model of $X$ in Question 2.20 has no nontrivial automorphisms, the stable spin curve $\xi$ containing a line bundle equivalent to $L$ is unique.

Let $X$ be a quasistable curve. From now on we will fix its decomposition $X = \cup_{1\leq i\leq\gamma} X_i$ into irreducible components. Set

$$X_i \cap X_i^c = \{p_{i1}, \ldots, p_{ih_i}\}.$$

Let $T \in Tw(X)$ be a twister of $X$. For every $X_i \subset X$ we have

$$T \otimes \mathcal{O}_{X_i} \simeq \mathcal{O}_{X_i}\left(\sum_{1\leq h\leq h_i} m_{ih} p_{ih}\right) \quad m_{ih} \in \mathbb{Z}$$

$$m_{ih} = -m_{jh'} \quad \text{if } p_{ih} = p_{jh'} \in X_i \cap X_j \tag{2.4}$$

$$m_{ih} > 0 \Rightarrow m_{ih'} > 0 \quad \text{if } p_{ih}, p_{ih'} \in X_i \cap X_j,\ i \neq j \tag{2.5}$$

$$m_{i_1h_1} < 0\,,\ m_{i_2h_2} \leq 0 \ldots m_{i_{N-1}h_{N-1}} \leq 0 \Rightarrow m_{i_Nh_N} > 0 \tag{2.6}$$

$$\text{if } p_{i_1h_1} \in X_{i_1} \cap X_{i_2}\,,\ p_{i_2h_2} \in X_{i_2} \cap X_{i_3} \ \ldots\ p_{i_Nh_N} \in X_N \cap X_{i_1}.$$

It follows from (2.4) that $T$ naturally defines a $1-$chain $\gamma_T \in \mathcal{C}^1(\Gamma_X, \mathbb{Z})$ whose coefficient on the half edges[1] of the dual graph $\Gamma_X$ are the $m_{ih}$.

DEFINITION 2.21. $\gamma_T$ is said to be the *1-chain of* $T$.

In the sequel we shall denote by

$$\mathrm{supp}T|_{X_i} := \{p_{ih} \text{ s.t. } m_{ih} \neq 0.\}$$

The geometry of the admissible twister of a quasistable curve is given by the following

LEMMA 2.22. *Let $X$ be a quasistable curve and let $T$ be a twister of $X$. The following properties are equivalent.*

(i) *$T$ is admissible.*

(ii) *The coefficients of the $1-$chain $\gamma_T \in \mathcal{C}^1(\Gamma_X, \mathbb{Z})$ of $T$ run over the set $\{-1, 0, 1\}$.*

*If $T$ is induced by a general smoothing we have also*

(iii) *There exists a partition of $X$ into subcurves $Z_1, \dots, Z_{d_T}$ such that*
  (a) *for every $h = 1, \dots, d_T$, we have $Z_h \neq \emptyset$.*
  (b) *for $h \neq h'$ we have $Z_h \cap Z_{h'} \neq \emptyset$ if and only if $|h - h'| \leq 1$*
  (c) *if we set $Z_0 \cap Z_1 := \emptyset$ and $Z_{d_T} \cap Z_{d_T+1} = \emptyset$, then for every $h = 1, \dots, d_T$ and $X_i \subset Z_h$*

$$T \otimes \mathcal{O}_{Z_h} \simeq \mathcal{O}_{Z_h}\left(\sum_{\substack{p \in Z_h \cap Z_{h+1} \\ q \in Z_h \cap Z_{h-1}}} (p - q)\right) \qquad \mathit{supp}T|_{X_i} \subset Z_h \cap Z_h^c.$$

PROOF. First of all it is easy to see that for any subcurve $Z \subset X$

$$|\deg_Z T| \leq k_Z \Leftrightarrow (BI) \quad \left|\frac{1}{2}\deg_Z(\omega_X \otimes T) - \frac{1}{2}\deg_Z \omega_X\right| \leq \frac{k_Z}{2} \Leftrightarrow T \text{ is admissible} \tag{2.7}$$

$(i) \Rightarrow (ii)$.

For each component $X_i$ of $X$, we denote by $v_{X_i}$ the corresponding vertex in $\Gamma_X$.

Assume by contradiction that there exists $p_{1h} \in X_1$ such that $m_{1h} \leq -2$. Consider the set of vertices $V$ of $\Gamma_X$ such that $v$ is in $V$ if and only if

- there exists a chain of edges of $\Gamma_X$ connecting $v$ and $v_{X_1}$
- the coefficients of $\gamma_T$ on each half edge of the chain run over $\{0, 1, -1\}$
- if we consider an edge $e$ of the chain, then the coefficient of $\gamma_T$ of the half edge of $e$ closer to $v$ is either 0 or $-1$.

$V$ is a proper subset of vertices of $\Gamma_X$, because combining (2.5) and (2.6) with the condition $m_{1h} \leq -2$, it is easy to see that the vertex of the component of $X$ intersecting $X_1$ and containing $p_{1h}$ is not in $V$.

Pick the proper subcurve $Z_V$ of $X$ corresponding to $V$. Since $T$ is admissible, it follows from (2.7) that $|\deg_{Z_V} T| \leq k_{Z_V}$. Hence by construction there exists a component $X_2 \neq X_1$ of $X$ such that $X_2 \nsubseteq Z_V$ and $Z_V \cap X_2 \neq \emptyset$ and such that there is $p_{2h} \in X_2$ with $m_{2h} \leq -2$.

Iterating this argument and applying (2.6), one finds an infinite number of distinct components of $X$, yielding a contradiction.

[1]Let $X^\nu$ be the normalization of $X$. The set of *half edges* of $\Gamma_X$ corresponds to the points of $C^\nu$ mapping to a node of $C$.

$(ii) \Rightarrow (i)$.
Pick any subcurve $Z$. From the given hypothesis on $\gamma_T$, each point $p_Z$ of $Z \cap Z^c$ contributes of an integer in $\{-1, 0, 1\}$ to $\deg_Z T$, then $|\deg_Z T| \leq k_Z$ and it follows from (2.7) that $T$ is admissible.

$(i), (ii) \Rightarrow (iii)$
Let $f : \mathcal{X} \to B$ be a general smoothing such that $T = \mathcal{O}_f(D)$ for a Cartier divisor $D$ of $\mathcal{X}$ supported on irreducible components of $X$. Write

$$D = \sum_{1 \leq i \leq \gamma} a_i X_i.$$

Modulo tensoring by the trivial twister $\mathcal{O}_f(nX)$ ($n >> 0$) of $X$ we may assume that the minimum of the $a_i$ is 1 and the maximum is a positive integer $d_T$. Set for $1 \leq h \leq d_T$

$$Z_h := \underset{a_i = h}{\cup} X_i \subset X.$$

In this way we have $Z_1 \neq \emptyset$ and for every $X_i \subset Z_1$

$$\mathrm{supp} T|_{X_i} \subset Z_1 \cap Z_1^c.$$

$$T \otimes \mathcal{O}_{Z_1} \simeq \mathcal{O}_{Z_1} \left( \sum_{p \in Z_1 \cap Z_1^c} m_p p \right) \quad 0 < m_p \in \mathbb{Z}$$

Since $T$ is admissible, using (2.7) we get $\left|\deg_{Z_1} T\right| \leq k_{Z_1}$ and hence

$$k_{Z_1} \leq \sum_{p \in Z_1 \cap Z_1^c} m_p = |\deg_{Z_1} T| \leq k_{Z_1} \tag{2.8}$$

which implies $m_p = 1$, $\forall\, p \in Z_1 \cap Z_1^c$.

If $Z_1 = X$ then the twister is trivial and there is nothing to prove. Otherwise $Z_2 \neq \emptyset$ because from $(ii)$ all the irreducible components of $X$ intersecting $Z_1$ are in $Z_2$. For every $X_i \subset Z_2$

$$\mathrm{supp} T|_{X_i} \subset Z_2 \cap Z_2^c.$$

$$T \otimes \mathcal{O}_{Z_2} \simeq \mathcal{O}_{Z_2} \left( \sum_{\substack{p \in Z_2 \cap Z_2^c \\ p \notin Z_2 \cap Z_1}} m_p p - \sum_{q \in Z_1 \cap Z_2} q \right) \quad 0 < m_p \in \mathbb{Z}$$

Arguing as for (2.8) for the subcurve $Z_1 \cup Z_2$ we get $m_p = 1$, $\forall\, p \in (Z_2 \cap Z_2^c) - Z_1$.

Iterating we get a partition satisfying $(a)$ and $(c)$ and it follows from $(ii)$ that it satisfies also $(b)$.

$(iii) \Rightarrow (ii)$ Obvious form (c). $\square$

DEFINITION 2.23. Let $T$ be an admissible twister of a quasistable curve $X$ and let $\gamma_T$ be its $1-$chain. A node of $X$ is said to be *T-twisted* if the half edges of $\Gamma_X$ corresponding to it appear with non trivial coefficient in $\gamma_T$ (and hence either 1 or $-1$, see Lemma 2.22 (ii)).

- The refined partition of a quasistable curve

Let $X$ be a quasistable curve and $T$ an admissible twister of $X$ induced by a general smoothing of $X$. Let $Z_1, \dots, Z_{d_T}$ be the partition of $X$ induced by $T$ (see Lemma 2.22(iii)). Let $\mathcal{E}_1, \dots, \mathcal{E}_{d_T}$ be respectively the union of the exceptional components of $X$ contained in $Z_1, \dots, Z_{d_T}$ and consider the partition of $X$ given by

$$\overline{Z_1 - \mathcal{E}_1}, \dots, \overline{Z_{d_T} - \mathcal{E}_{d_T}}\,, \mathcal{E}_1, \dots, \mathcal{E}_{d_T}.$$

Abusing notation denote by $Z_1, \dots, Z_{d_T}$ the first $d_T$ subcurves.

- We call $Z_1, \dots, Z_{d_T}, \mathcal{E}_1, \dots, \mathcal{E}_{d_T}$ *the refined partition of* $X$ induced by $T$.

By definition $\deg_E(\omega_X \otimes T) = 2$ for every exceptional component of $X$. Therefore, if $E \cap E^c = \{p, q\}$, we have $T \otimes E \simeq \mathcal{O}_E(p+q)$ and every exceptional node of $X$ is $T-$twisted.

In particular the subcurve $Z_h$ for $h \geq 2$ in a refined partition is non-empty, otherwise the properties of Lemma 2.22 (iii) cannot hold for the original partition. Obviously if $Z_1 = \emptyset$ then $\mathcal{E}_1 \neq \emptyset$ (see Lemma 2.22 (iii)(a)).

Now we can answer Question 2.20 for general smoothings.

THEOREM 2.24. *Let $X$ be a quasistable curve, $f : \mathcal{X} \to B$ be a general smoothing of $X$ and $T = \mathcal{O}_f(D)$ be an admissible twister of $X$. Let $Z_1, \dots, Z_{d_T}, \mathcal{E}_1, \dots, \mathcal{E}_{d_T}$ be the refined partition of $X$ induced by $T$ and $(X, L)$ be a $D-$twisted spin curve.*

*A stable spin curve $\xi = (X_L, G_L, \alpha)$ of $X$ which is equivalent to $(X, L)$ is given by the following data*

(i) *$X_L$ is obtained by blowing-up $X$ at each non-exceptional $T-$twisted node*

(ii) *if we set $Z_{d_T} \cap Z_{d_T+1} := \emptyset$, then the line bundle $G_L \in \mathrm{Pic}(X_L)$ is given by gluing*

$$L|_{Z_h} \otimes \mathcal{O}_{Z_h}\left(-\sum_{p \in Z_h \cap Z_{h+1}} p\right)$$

*for every $h = 1, \dots, d_T$ such that $Z_h \neq \emptyset$ and $\mathcal{O}_E(1)$ for every exceptional curve $E$ of $X_L$.*

PROOF. It follows from Lemma 2.22 (iii) (c) that for every $h = 1, \dots, d_T$ such that $Z_h \neq \emptyset$ there are line bundles $R_h \in \mathrm{Pic}(Z_h)$ with

$$R_h^{\otimes 2} \simeq \omega_{Z_h}$$

such that

$$L \otimes \mathcal{O}_{Z_h} \simeq R_h \otimes \mathcal{O}_{Z_h}\left(\sum_{p \in Z_h \cap Z_{h+1}} p\right).$$

Obviously for every exceptional component $E$ of $X$ we have $L \otimes \mathcal{O}_E = \mathcal{O}_E(1)$.

Let $\pi : X_L \to X$ be the blow-up of $X$ at each non-exceptional $T-$twisted node. Let $\mathcal{E}(\pi)$ be the set of exceptional components of $X_L$ contracted by $\pi$. Consider the following diagram

$$\begin{tikzcd}
\mathcal{X}_L \arrow[r] \arrow[d] & \tilde{\mathcal{X}} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\
B' \arrow[r, equal] & B' \arrow[r, "b"] & B
\end{tikzcd}$$

where $b$ is a base change of order two totally ramified over $0 \in B$ and $\mathcal{X}_L \to B'$ is the smoothing of $X_L$ obtained by suitably blowing-up the fiber product $\tilde{\mathcal{X}} := \mathcal{X} \times_b B'$. Notice that $\mathcal{X}_L$ is smooth at each exceptional node of an exceptional component of $\mathcal{E}(\pi)$ and has an $A_1-$singularity at the remaining nodes. We set $\mathcal{E}_{01} := 0$ (the zero divisor) and for $h = 2, \dots, d_T$

$$\mathcal{E}_{h-1,h} := \sum E \qquad \forall\, E \in \mathcal{E}(\pi) \text{ s.t. } E \cap Z_{h-1} \neq \emptyset\ ,\ E \cap Z_h \neq \emptyset.$$

Consider the Cartier divisor of $\mathcal{X}_L$

$$D_L := - \sum_{1 \leq h \leq d_T} (h\, Z_h + h\, \mathcal{E}_{h-1,h} + h\, \mathcal{E}_h)$$

and denote by $T_L$ the twister of $X_L$ given by $T_L := \mathcal{O}_{\mathcal{X}_L}(D_L) \otimes \mathcal{O}_{X_L}$. Consider the line bundle $G_L$ of $X_L$

$$G_L := \pi^* L \otimes T_L \in \text{Pic}\,(X_L). \tag{2.9}$$

By construction the pair $(X_L, G_L)$ satisfies $(i)$ and $(ii)$ of the Theorem. Since (2.9) says that $G_L$ and $L$ are equivalent, in order to conclude it suffices to show that $(X_L, G_L)$ yields a stable spin curve.

Let us check the last statement. By construction for every $h = 1, \dots, d_T$ such that $Z_h \neq \emptyset$ and for every exceptional curve $E$ of $X_L$ we have

$$G_L \otimes \mathcal{O}_{Z_h} = R_h \qquad G_L \otimes \mathcal{O}_E = \mathcal{O}_E(1).$$

We have to define an homomorphism $\alpha : (G_L)^{\otimes 2} \to \pi^*(\omega_C)$ satisfying the property of limit square root. Since $\tilde{X}$ is the disjoint union of the $Z_h$, for every $h$ we have a natural map

$$\alpha_h : (G_L \otimes \mathcal{O}_{Z_h})^{\otimes 2} \simeq R_h^{\otimes 2} \simeq \omega_{Z_h} \simeq \pi^*(\omega_C) \otimes \mathcal{O}_{Z_h} \left( - \sum_{p \in Z_h \cap Z_h^c} p \right) \hookrightarrow \pi^*(\omega_C) \otimes \mathcal{O}_{Z_h}$$

and the desired $\alpha$ is defined to agree with $\alpha_h$ on each $Z_h$ and to be zero on the exceptional components of $X$. □

Remark 2.25. The previous Theorem has also the following important interpretation.

Let $f : \mathcal{X} \to B$ be a general smoothing of $X$ and $T = \mathcal{O}_f(D)$ be an admissible twister of $X$. Let $(X, L)$ be a $D-$twisted spin curve and $\xi$ be the stable spin curve constructed in Th. 2.24 which is equivalent to $(X, L)$. It follows from the proof of Th. 2.24 that there exists a representative $(X_L, G_L, \alpha)$ of $\xi$ such that $L$ and $G_L$ are limits of the same family of line bundles on a base change of order two of the family $\mathcal{X} \to B$ totally ramified over 0 (see also below [**CCC**, Def. 5.1.4]).

Example 2.26. Let $C = C_1 \cup C_2$ be a stable curve of genus $g$ where $C_1, C_2$ are smooth curves such that $C_1 \cap C_2 = \{n_1, n_2\}$. If $\nu : C^\nu \to C$ is the normalization, denote by $\{p, q\} := \nu^{-1}\{n_1, n_2\} \cap C_1$. Notice that $C$ belongs to the set of curves of Example 2.10.

Pick a general smoothing $f : \mathcal{C} \to B$ of $C$. Consider the admissible divisor $D := C_2$ of $\mathcal{C}$. The partition of $X$ induced by $D$ is given by $Z_1 = C_1$ and $Z_2 = C_2$. The nodes $n_1, n_2$ are $D-$twisted. In fact notice that

$$\omega_C(D) \otimes \mathcal{O}_{C_1} = \omega_{C_1}(2p + 2q) \qquad \omega_C(D) \otimes \mathcal{O}_{C_2} = \omega_{C_2}.$$

Pick line bundles $R_1 \in \mathrm{Pic}(C_1)$ and $R_2 \in \mathrm{Pic}(C_2)$ such that $R_i^{\otimes 2} = \omega_{C_i}$ and let $(C, L), (C, L')$ be the two possible $D-$twisted spin curves (for $L, L' \in \mathrm{Pic}(C)$) obtained by gluing $R_1(p+q)$ and $R_2$ (in the two possible ways) so that

$$L|_{C_1} = L'|_{C_1} = R_1(p + q) \qquad L|_{C_2} = L'|_{C_2} = R_2.$$

Obviously $(C, L)$ and $(C, L')$ are not stable spin curves.

The stable spin curve which is equivalent both to $L$ and to $L'$ and described in Proposition 2.24 is obtained by taking the blow-up $X \to C$ of $C$ at the $D-$twisted nodes $n_1, n_2$ and gluing $R_1$ and $R_2$ to $\mathcal{O}_E(1)$ for every exceptional curve $E$ of $X$.

CHAPTER 3

# Spin curves over non stable curves

In this chapter we shall study spin curves on non-stable curves using degenerations of theta hyperplanes.

In Section 1 we will see how to get a well-defined configuration of theta hyperplanes on a singular curve. In Section 2 and 3 we shall give enumerative results of configurations on tacnodal, cuspidal and nodal curves. In particular we shall describe the zero dimensional scheme associated to these configurations. In Section 4 we shall give a modular interpretation of degenerations of odd theta characteristics for smoothing of tacnodal or cuspidal curves.

**Notation and Terminology 3.**

(1) We will denote by $H_g$ the irreducible component of the Hilbert scheme $\mathrm{Hilb}^{p(x)}[\mathbb{P}^{g-1}]$ of curves in $\mathbb{P}^{g-1}$ having Hilbert polynomial $p(x) = (2g-2)x - g + 1$ and containing smooth canonical curves. We denote by $u : \mathcal{U} \to H_g$ the universal family over $H_g$ and for a given $h \in H_g$ we write $W_h$ for the projective curve $u^{-1}(h)$ represented by $h$.

(2) We set $N_g := 2^{g-1}(2^g - 1)$ and $N_g^+ := 2^{g-1}(2^g + 1)$, respectively the numbers of odd and even theta characteristics of a smooth curve of genus $g$ (recall that odd and even refers to the parity of the number of sections of the line bundle).

(3) **The projective setup of theta hyperplanes**

Let $C$ be a canonical smooth curve of genus $g$. It is well-known that if $C$ is general, then a theta characteristic $L$ of $C$ has $h^0(L) \leq 1$ and $N_g$ of these are odd. Thus a general smooth canonical curve admits exactly $N_g$ hyperplanes cutting the double of a semicanonical divisor. In this case we say that $C$ is *theta generic* and we can collect these hyperplanes (called *theta hyperplanes*) in a configuration $\theta(C)$ which is a point of

$$\mathbb{P}_{N_g} := Sym^{N_g}(\mathbb{P}^{g-1})^\vee.$$

In [**CS1**][**C2**] and [**CS2**] one can find many interesting properties of these objects. In particular the authors focused on the problem of recovering a smooth canonical theta generic curve from the datum of its theta hyperplanes. The main ingredient employed was the degeneration to the so-called *split curves*, that is stable curves which are the union of two rational smooth curves. In [**C2**] one can find a definition and an explicit description of configurations of theta hyperplanes of split curves.

Let $W \subset \mathbb{P}^{g-1}$ be a projective Gorenstein curve of arithmetic genus $g$. We shall say that $W$ is *canonical* if $\mathcal{O}_W(1) \simeq \omega_W$. One can define configurations of theta hyperplanes for (possibly singular) canonical curves.

Let $g \geq 3$. Let $V \subset H_g$ be the open set parametrizing smooth theta generic canonical curves. Consider the morphism

$$\theta : V \longrightarrow \mathbb{P}_{N_g}$$

such that $\theta(h)$ is the configuration $\theta(W_h)$ of theta hyperplanes of $W_h$.

Now let $W$ be a canonical curve. Pick a projective smoothing $f : \mathcal{W} \to B$ of $W$ whose general fiber is theta generic. Consider the associated morphism

$$\gamma_f : B^* \longrightarrow H_g$$

of the restricted family $\mathcal{W}^* \to B^*$. The image of $\gamma_f$ lies in $V$. As $B$ is smooth and $\mathbb{P}_{N_g}$ projective, the composed morphism

$$\theta \circ \gamma_f : B^* \to \mathbb{P}_{N_g}$$

extends to all of $B$ and we get a configuration of hyperplanes $\theta_f(W)$. We can see it also as a (not necessarily reduced) hypersurface of degree $N_g$ whose irreducible components are hyperplanes. Moreover we can consider the $B-$curve

$$J_{\mathcal{W}} \longrightarrow B$$

which is the closure of the incidence correspondence

$$\{(t, H) : H \subset \theta(W_t) \ , \ t \neq 0\} \subset B \times (\mathbb{P}^{g-1})^\vee.$$

We shall denote by $J_f(W)$ its fiber over 0.

Definition 3.1. We call $\theta_f(W)$ the *configuration of theta hyperplanes of* $W$ and $J_f(W)$ the *zero dimensional scheme of theta hyperplanes* of $W$ whose elements are *theta hyperplanes of* $W$. We say that $W$ is *theta generic* if it has a finite number of theta hyperplanes.

(4) **The sections of a stable spin curve**
Let $C$ be a stable curve and let $\xi = (X, G, \alpha)$ be a stable spin curve of $C$ supported on a blow-up $\pi : X \to C$ of $C$. Let $\mathcal{E}(X)$ be the set of the exceptional components of $X$. Recall that the subcurve $\tilde{X}$ of $X$ is defined as

$$\tilde{X} := \overline{X - \cup_{E \in \mathcal{E}(X)} E}.$$

The line bundle $G$ is obtained by gluing theta characteristics on the connected components of $\tilde{X}$ to $\mathcal{O}_E(1)$ for every $E \in \mathcal{E}(X)$.

Let $Z_1, \cdots, Z_{d_G}$ be the connected components of $\tilde{X}$ to which $G$ restricts to an odd theta characteristic. We call them the *odd connected components of* $\tilde{X}$. The *even connected components of* $\tilde{X}$ are the ones to which $G$ restricts to an even theta characteristic.

Let $Z$ be *any* connected component of $\tilde{X}$. Since for every $E \in \mathcal{E}(X)$ we have $|E \cap E^c| = 2$ and $G|_E = \mathcal{O}_E(1)$, then a non-trivial section of $G|_Z$ uniquely extends to a section of $G$ vanishing on the other connected components of $\tilde{X}$. Among these, take the sections of $G$ restricting to independent sections of $G|_Z$. It is easy to see that all these sections (for $Z$ running over the connected components of $\tilde{X}$) form a basis for $H^0(X, G)$ and therefore

$$H^0(X, G) = \bigoplus_{\substack{Z \subset \tilde{X} \\ Z \text{ connected}}} H^0(Z, G|_Z).$$

Notice that $G$ is odd if and only if $d_G \equiv 1$ (2).

(5) **Smoothing line bundles and sections**

Let $W$ be a curve with nodes, cusps and tacnodes and let $f : \mathcal{W} \to B$ be a smoothing of $W$.

(a) Since the fibers of $f$ are local complete intersection, there exists a relative dualizing sheaf on $\mathcal{W}$, which we shall denote by $\omega_f$ (for details see [**DM**] and [**Ht2**]). If $\mathcal{W}$ is smooth, then one can always define $\omega_f = K_{\mathcal{W}} \otimes f^*(\omega_B^\vee)$ where $K_{\mathcal{W}}$ is the canonical line bundle of $\mathcal{W}$.

(b) Consider a Cartier divisor $D$ of $\mathcal{W}$ whose support is contained in $W$ and the line bundle $\omega_f(D) \in \mathrm{Pic}(\mathcal{W})$. The following fact is a topological property and its proof appeared in an early version of [**CCC**].
Let $L \in \mathrm{Pic}(W)$ be a line bundle with an isomorphism $\iota_0 : L^{\otimes 2} \to \omega_f(D) \otimes \mathcal{O}_W$. Then up to shrinking $B$ there exists a line bundle $\mathcal{L} \in \mathrm{Pic}\,\mathcal{W}$ extending $L$ and an isomorphism $\iota : \mathcal{L}^{\otimes 2} \to \omega_f(D)$ extending $\iota_0$. Moreover if $(\mathcal{L}', i')$ is another extension of $(L, \iota_0)$, then there exists an isomorphism $\chi : \mathcal{L} \to \mathcal{L}'$ restricting to the identity and with $\iota = \iota' \circ \chi^{\otimes 2}$.

(c) Consider a line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{W})$. Assume that for every $b \in B^*$ one has $h^0(W_b, \mathcal{L}|_{W_b}) = d \geq 1$. This is equivalent to the datum of the locally free sheaf $\mathcal{V}^* := g_*\mathcal{L}^*$ over $B^*$. Consider the subbundle $\mathcal{V} \subset g_*\mathcal{L}$ extending $\mathcal{V}^*$. The *space of the $f-$smoothable sections of $\mathcal{L}|_W$* is given by the $d-$dimensional subspace $V_0 \subset H^0(W, \mathcal{L}|_W)$, the fiber of $\mathcal{V}$ over $0 \in B$.

## 3.1. The case of a local complete intersection

We shall analyze configurations of theta hyperplanes of non-stable curves. We will find a sufficient condition for a curve to have a configuration of theta hyperplanes which does not depend on smoothing to theta generic curves. Then we write down explicit formulas for the reduced zero dimensional scheme of theta hyperplanes for nodal, cuspidal and tacnodal canonical curves.

LEMMA 3.2. *Let $W$ be a theta generic canonical curve parameterized by a smooth point of $H_g$. There exists a unique natural configuration of theta hyperplanes $\theta(W)$ such that when $W$ is smooth $\theta(W)$ is the image of the point of $H_g$ representing $W$ via the rational map $\theta : H_g --> \mathbb{P}_{N_g}$.*

PROOF. Let $U \subset H_g$ be the open set corresponding to theta generic curves on which $H_g$ is smooth and $U' \subset U$ the open set of $U$ corresponding to smooth curves. Let $h_0$ be the point of $U$ parametrizing $W$. Consider the incidence variety

$$\Gamma_{U'} := \{(h, \theta(W_h)) : h \in U'\} \subset U \times \mathbb{P}_{N_g}.$$

Let $\Gamma_U$ be the closure of $\Gamma_{U'}$ in $U \times \mathbb{P}_{N_g}$ and $\rho$ be the projection

$$\rho : \Gamma_U \longrightarrow U.$$

We observe that, since for every $h \in U$ the curve $W_h$ is theta generic, the morphism $\rho$ has always finite fibers. The morphism $\rho$ is bijective on $\Gamma_{U'}$, so it is a birational projective morphism. As $U$ is smooth and $\Gamma_U$ is irreducible (since $U'$ and hence $\Gamma_{U'}$ are irreducible) we can apply the Zariski Main Theorem obtaining that $\rho$ is bijective everywhere.

We can uniquely define $\theta(W) := \rho^{-1}(h_0)$. $\square$

We show that Lemma 3.2 works for theta generic l.c.i. canonical curves.

PROPOSITION 3.3. *Let $W$ be a canonical l.c.i. curve parametrized by $h \in H_g$. Then $H_g$ is smooth at $h$. In particular if $W$ is also theta generic, there exists a natural configuration of theta hyperplanes $\theta(W)$.*

PROOF. Let us show the first statement. Since $W$ is l.c.i., if $h^1(N_{W/\mathbb{P}^{g-1}}) = 0$ then $H_g$ is smooth at $h$ (see the smoothness criterion for the Hilbert scheme of Section 1.1.2).

Consider the exact sequence

$$0 \to \mathcal{I}_W/\mathcal{I}_W^2 \to \Omega^1_{\mathbb{P}^{g-1}|_W} \to \Omega^1_W \to 0,$$

with the exactness on the left because $W$ is a l.c.i. curve (see [**B**, 4.1.3.i]). By taking $\mathcal{H}om_{\mathcal{O}_W}(-, \mathcal{O}_W)$ we have

$$0 \to \mathcal{H}om_{\mathcal{O}_W}(\Omega^1_W, \mathcal{O}_W) \to \mathcal{T}_{\mathbb{P}^{g-1}}|_W \to N_{W/\mathbb{P}^{g-1}} \xrightarrow{\alpha} \mathcal{E}xt^1_{\mathcal{O}_W}(\Omega^1_W, \mathcal{O}_W) \to 0.$$

Let $\mathcal{N}'_W$ be the kernel of $\alpha$ and split the sequence into

$$0 \to \mathcal{H}om_{\mathcal{O}_W}(\Omega^1_W, \mathcal{O}_W) \to \mathcal{T}_{\mathbb{P}^{g-1}}|_W \to \mathcal{N}'_W \to 0$$

$$0 \to \mathcal{N}'_W \to N_{W/\mathbb{P}^{g-1}} \to \mathcal{E}xt^1_{\mathcal{O}_W}(\Omega^1_W, \mathcal{O}_W) \to 0.$$

By the long exact sequences in cohomology we get the two maps

$$H^1(W, \mathcal{T}_{\mathbb{P}^{g-1}|_W}) \to H^1(W, \mathcal{N}'_W) \to 0$$

$$H^1(W, \mathcal{N}'_W) \to H^1(W, N_{W/\mathbb{P}^{g-1}}) \to 0.$$

Hence if $h^1(W, \mathcal{T}_{\mathbb{P}^{g-1}|_W}) = 0$ it follows that $h^1(W, \mathcal{N}'_W) = 0$ and also $h^1(W, N_{W/\mathbb{P}^{g-1}}) = 0$.

From the Euler sequence of $\mathbb{P}^{g-1}$ restricted to $W$ we have

$$H^1(W, \mathcal{O}_W) \to H^1(W, \mathcal{O}_W(1)) \otimes H^0(W, \mathcal{O}_W(1))^\vee \to H^1(W, \mathcal{T}_{\mathbb{P}^{g-1}}|_W) \to 0$$

Since $\mathcal{O}_W(1) \simeq \omega_W$, dualizing we get

$$0 \to H^1(W, \mathcal{T}_{\mathbb{P}^{g-1}}|_W)^\vee \to H^0(W, \mathcal{O}_W) \otimes H^0(W, \omega_W) \xrightarrow{\beta} H^0(W, \omega_W).$$

Since $\beta$ is injective, then $h^1(W, T_{\mathbb{P}^{g-1}}|_W) = 0$.

The second part follows from Lemma 3.2. $\square$

We give a sufficient condition for a curve of $H_g$ to be canonical.

PROPOSITION 3.4. *Any irreducible curve parameterized by a point of $H_g$ is canonical.*

PROOF. Let $u : \mathcal{U} \to H_g$ be the universal family over $H_g$ and denote by $\varphi : \mathcal{U} \to \mathbb{P}^{g-1}$ the projection. Let $\mathcal{O}_{\mathcal{U}}(-1)$ be the dual of the pull-back of $\mathcal{O}_{\mathbb{P}^{g-1}}(1)$ via $\varphi$.

We have $h^0((\omega_u \otimes \mathcal{O}_{\mathcal{U}}(-1))|_{u^{-1}(h)}) = 1$ over the open set of smooth canonical curves hence by semicontinuity $h^0(\omega_W \otimes \mathcal{O}_W(-1)) \geq 1$. Thus if $W$ is integral, the degree-zero line bundle $\omega_W \otimes \mathcal{O}_W(-1)$ is trivial and hence $W$ is canonical. $\square$

## 3.2. Enumerative results

In this section we shall deal with enumerative problems on theta hyperplanes. In particular we shall write down formulas for the number of theta hyperplanes of curves with nodes, cusps and tacnodes. In [**C2**, Prop.1, Prop.2] one can find formulas for nodes and cusps. We generalize these results including also tacnodal curves.

As in [**C2**] we shall use the projection of a canonical integral curve from a singular point. Each theta hyperplane containing the singular point projects to a theta hyperplane of the projected curve. If one projects from a tacnode, the tacnode projects to a node. If $H$ is a theta hyperplane containing the tacnode, the projected theta hyperplane contains the node if and only if $H$ contains the tacnodal tangent.

Definition 3.5. We say that a curve is *semi-theta-generic* (s.t.g.) if it is obtained by identifying general point of its normalization and the connected components of its normalization are theta-generic curves.

Remark 3.6. We will see in Theorem 3.9 that an irreducible s.t.g. canonical curve with nodes, cusps and tacnodes is theta generic.

Notation 3.7. Let $g \geq 3$. In the sequel we shall denote by $W^g_{\tau\gamma\delta}$ an irreducible s.t.g. canonical curve with $\tau$ tacnodes, $\gamma$ cusps and $\delta$ nodes of genus $g$ and by $\tilde{g}$ the genus of its normalization. Observe that a theta hyperplane contains no nodal and no cuspidal tangents (recall that a s.t.g. curve is obtained by identifying general points of its normalization).

We denote by $t^j_{ikh}$ the number (when it is finite) of theta hyperplanes containing $i$ tacnodes, $j$ tacnodal tangents of these $i$ tacnodes, $k$ cusps and $h$ nodes. We call such a hyperplane *a theta hyperplanes of type* $(i,j,k,h)$. We call a theta hyperplane of type $(0,0,0,0)$ simply *a theta hyperplane of type 0*. We denote by $\theta_0(W^g_{\tau\gamma\delta})$ the set of the theta hyperplanes of type 0 and by $t_0$ their number (when it is finite).

Lemma 3.8. *Let $g \geq 3$ and $W^g_{\tau\gamma\delta}$ be as in Notation 3.7.*

(i) *If $R$ is an odd theta characteristic of $W^g_{\tau\gamma\delta}$, then $h^0(R) = 1$.*

(ii) *There exists a set bijection (recall the definition of $S^-(-)$ of Section 2.1)*

$$\theta_0(W^g_{\tau\gamma\delta}) \xrightarrow{\sim} S^-(W^g_{\tau\gamma\delta}).$$

*In particular $W^g_{\tau\gamma\delta}$ has a finite number of theta hyperplanes of type* 0.

Proof. (i) See Lemma 4.2 of Chapter 4.

(ii) Set $W := W^g_{\tau\gamma\delta}$. If $H$ is a theta hyperplane of type zero of $W$, consider the effective divisor $D_H$ given by the reduction modulo 2 of the divisor cut on $W$ by $H$.

Since $H$ is limit of theta hyperplanes of smooth curves and the parity of a semicanonical line bundle is stable under deformation, it follows that $\mathcal{O}_W(D_H)$ is an odd theta characteristic of $W$. From (i) it follows that any odd theta characteristic of $W$ has exactly one section. Hence we have a set injection

$$\theta_0(W) \hookrightarrow S^-(W).$$

If $R$ is an odd theta characteristic of $W$, let $D$ be the only effective divisor of $|R|$ and $H$ be the theta hyperplane cutting $2D$ on $W$.

Assume that $W$ has no tacnodes. Since a node or a cusp of $W$ are not Cartier divisor, it follows that $H$ is of type 0 and the injection is also a surjection.

Assume that $W$ has a tacnode. We show that $H$ contains no tacnodes of $W$. The only thing to check is that $H$ does not contain a tacnodal tangent (in fact if $H$ contains a tacnode without tangent, it cuts a divisor not divisible by 2 as Cartier divisor).

Assume that $H$ contains a tacnodal tangent. The equation of the tacnode in an analytic coordinate system $(x, y)$ of a smooth surface containing $W$ is $y^2 - x^4 = 0$. The local equation of the divisor cut by $H$ is given by $y$. If there exists $f$ such that $f^2 = y$ then $f^4 = x^4$ and hence $f = cx$ for a constant $c$. Thus $y = c^2x^2$ which cannot hold along the tacnodal singularity. □

THEOREM 3.9. *Let $g \geq 3$ and $W^g_{\tau\gamma\delta}$ be as in Notation 3.7.*
*If $j < i$ or $h \neq \delta$*

$$t^j_{ikh} = 2^{\tau-j+\delta-h-1}\binom{\tau}{i}\binom{i}{j}\binom{\delta}{h}\binom{\gamma}{k}(N^+_{\tilde{g}} + N_{\tilde{g}}).$$

*If $i = j$ and $h = \delta$*

$$t^i_{ik\delta} = \begin{cases} 2^{\tau-i}\binom{\tau}{i}\binom{\gamma}{k}N^+_{\tilde{g}} & \text{if } \tau - i + \gamma - k \equiv 1 \ (2) \\ \\ 2^{\tau-i}\binom{\tau}{i}\binom{\gamma}{k}N_{\tilde{g}} & \text{if } \tau - i + \gamma - k \equiv 0 \ (2) \end{cases}$$

*In particular $W^g_{\tau\gamma\delta}$ is theta generic.*

PROOF. The proof is by induction on $g$. The formulas hold in genus 3 (see [**CS1**, 3.2]).

First of all consider the case $(i, j, k, h) \neq (0, 0, 0, 0)$. We project the curve from a singular point (since $g \geq 4$ we can project at least one time). The number $t^j_{ikh}$ is obtained by multiplying the number of theta hyperplanes containing a fixed set of $i$ tacnodes, $j$ tacnodal tangents, $k$ cusps and $h$ nodes and the number $\alpha(i, j, k, h) := \binom{\tau}{i}\binom{i}{j}\binom{\gamma}{k}\binom{\delta}{h}$ of all possible fixed sets.

If $j < i$, we project from a tacnode contained in the theta hyperplane and whose tacnodal tangent is not contained in the hyperplane. The projected curve $W^{g-1}_{\tau-1,\gamma,\delta+1}$ has genus $g - 1$ and we can apply the induction. A theta hyperplane of type $(i, j, k, h)$ of $W^g_{\tau\gamma\delta}$ projects to a theta hyperplane of type $(i - 1, j, k, h)$ of $W^{g-1}_{\tau-1,\gamma,\delta+1}$, then

$$t^j_{ikh}(W^g_{\tau\gamma\delta}) = \alpha(i, j, k, h)\frac{t^j_{i-1,k,h}(W^{g-1}_{\tau-1,\gamma,\delta+1})}{\alpha(i - 1, j, k, h)}.$$

Since $\delta + 1 \neq h$ and $\tilde{g}$ is the genus of the normalization of both $W^{g-1}_{\tau-1,\gamma,\delta+1}$ and $W^g_{\tau\gamma\delta}$, by induction

$$\frac{t^j_{i-1,k,h}(W^{g-1}_{\tau-1,\gamma,\delta+1})}{\alpha(i - 1, j, k, h)} = 2^{\tau-1-j+\delta+1-h-1}(N^+_{\tilde{g}} + N_{\tilde{g}}) = 2^{\tau-j+\delta-h-1}(N^+_{\tilde{g}} + N_{\tilde{g}}).$$

The cases $i = j$ and $\delta \neq h$ are similar (projection from a tacnode if $i \neq 0$, from a node if $i = 0$ and $h \neq 0$ and from a cusp if $i = h = 0$).

If $i = j \neq 0$ and $\delta = h$, we project from a tacnode contained in the theta hyperplane, which contains its tacnodal tangent because $i = j$. We have

$$t^i_{ikh}(W^g_{\tau\gamma\delta}) = \alpha(i, i, k, \delta)\frac{t^{i-1}_{i-1,k,h+1}(W^{g-1}_{\tau-1,\gamma,\delta+1})}{\alpha(i - 1, i - 1, k, h + 1)}.$$

Being $\delta + 1 = h + 1$ and observing that the parity of $\tau - i + \gamma - k$ is preserved, by induction

$$\begin{cases} \dfrac{t^{i-1}_{i-1,k,h+1}(W^{g-1}_{\tau-1,\gamma,\delta+1})}{\alpha(i-1,i-1,k,h+1)} = 2^{\tau-1-i+1}N^+_{\tilde{g}} = 2^{\tau-i}N^+_{\tilde{g}} & \text{if } \tau-i+\gamma-k \equiv 1\ (2) \\ \dfrac{t^{i-1}_{i-1,k,h+1}(W^{g-1}_{\tau-1,\gamma,\delta+1})}{\alpha(i-1,i-1,k,h+1)} = 2^{\tau-1-i+1}N_{\tilde{g}} = 2^{\tau-i}N_{\tilde{g}} & \text{if } \tau-i+\gamma-k \equiv 0\ (2) \end{cases}.$$

The cases $i = j = 0$ and $\delta = h$ are similar (projection from a node if $\delta \neq 0$ and from a cusp if $\delta = 0$).

It follows from Lemma 3.8 that the number of theta hyperplane of type 0 is $|S^-(W)|$ and hence we are done by Proposition 2.1. □

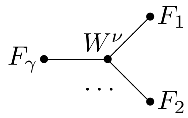
### 3.3. The multiplicity of a theta hyperplane

We complete the description of the zero dimensional scheme of theta hyperplanes of irreducible theta generic canonical curves with cusps and tacnodes computing the multiplicities of its points.

We solve the problem using twisted spin curves of the stable reduction of a general smoothing of these curves.

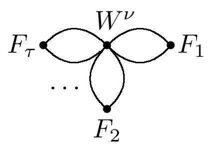
LEMMA 3.10. *Let $W$ be a curve and denote by $W^\nu$ its normalization. Let $\mathcal{W} \to B$ be a general smoothing of $W$ whose stable reduction $\mathcal{C}$ has central fiber $C$.*

(i) *Assume that $W$ is an irreducible curve whose singularities are exactly $\gamma$ cusps. Consider the base change $b : B' \to B$ of order 6 totally ramified over $0 \in B$. Then $\mathcal{C}$ is a smooth $B'$−surface and the dual graph of $C$ is*




*where $F_1, \dots, F_\gamma$ are elliptic curves.*

(ii) *Assume that $W$ is an irreducible curve whose singularities are exactly $\tau$ tacnodes. Consider the base change $b : B' \to B$ of order 4 totally ramified over $0 \in B$. Then $\mathcal{C}$ is a smooth $B'$−surface and the dual graph of $C$ is*




*where $F_1, \dots, F_\tau$ are elliptic curves.*

(iii) *Let $W$ and $\mathcal{W}$ be as in (i) or (ii) and let $f : \mathcal{C} \to B'$ be the stable reduction of $\mathcal{W}$. Let $F$ be the Cartier divisor of $\mathcal{C}$ which is the sum of the elliptic components $F_i$ with coefficients 1. Consider the fiber product $h : \mathcal{W}' = \mathcal{W} \times_B B' \to B'$. Then $\mathcal{C}$ is endowed with a $B'$−morphism $\varphi : \mathcal{C} \to \mathcal{W}'$ such that*

$$(PB) \quad \varphi^*(\omega_h) \simeq \omega_f(F).$$

PROOF. We follow [**BPV**, Theorem III-10.1] and [**HM**, Example pag.122]. Since $\mathcal{W}$ is general, it is a smooth surface.

(i) Let $\overline{\mathcal{W}}$ be the surface obtained by blowing-up $\mathcal{W}$ three times in correspondence of each cusp so that the reduced special fiber has normal crossings. Take a base change $b_1 : B_1 \to B$ of order 2 totally ramified over $0 \in B$ and the normalization $\mathcal{W}_1$ of the fiber product $\mathcal{W} \times_{b_1} B$. As explained in [**HM**], $\mathcal{W}_1$ is the double cover of $\mathcal{W}$ branched along the irreducible components of the special fiber of $\overline{\mathcal{W}}$ appearing with odd multiplicities and it is a smooth surface because this branch divisor is smooth. Take the base change $b_2 : B' \to B_1$ of order 3 totally ramified over $0 \in B_1$ and the normalization $\mathcal{C}'$ of the fiber product $\mathcal{W}_1 \times_{b_2} B'$. As before $\mathcal{C}'$ is the triple cover of $\mathcal{W}_1$ ramified along the irreducible components of the special fiber appearing with multiplicities not divisible by 3. Then $\mathcal{C}'$ is a smooth surface because the branch divisor is smooth. The irreducible components of the special fiber of $\mathcal{C}'$ are $\gamma$ elliptic curves, $W^\nu$ and some $(-1)$-curves. The surface $f : \mathcal{C} \to B'$ is obtained by contracting all the $(-1)$-curves contained in the special fiber.

(ii) The tacnodal case is similar combining two base changes $b_1 : B_1 \to B$ and $b_2 : B' \to B_1$ of order 2 totally ramified over 0.

(iii) Let $\mathcal{C}'$ be as in $(i)$. By the universal property of the fiber products we have a $B'-$morphism from $\mathcal{C}'$ to $\mathcal{W}'$ factorizing trough the $B'-$relative minimal model $\mathcal{C}$ of $\mathcal{C}'$.
We get the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\varphi"] \arrow[dr, "f"'] & \mathcal{W}' \arrow[r] \arrow[d, "h"] & \mathcal{W} \arrow[d] \\
& B' \arrow[r, "b"] & B
\end{tikzcd}$$

Since $\varphi$ is a birational morphism which is an isomorphism away from the special fibers, we have that $\omega_f$ and $\varphi^*(\omega_h)$ are isomorphic away from the special fiber $C$ of $\mathcal{C}$ and hence differ by a divisor of $\mathcal{C}$ supported on components of $C$. If $\nu : W^\nu \to W$ is the normalization, then

$$\varphi^*(\omega_h) \otimes \mathcal{O}_{W^\nu} \simeq \nu^*(\omega_W) \simeq \omega_{W^\nu}(2\sum_i (F_i \cap F_i^c))$$

and hence the divisor of $\mathcal{C}$ is exactly $F$ and the relation $(PB)$ follows.

□

Definition 3.11. The dual graphs of the previous Lemma are said to be respectively a *cuspidal* and *tacnodal graph centered at* $W^\nu$. The elliptic curves $F$ are said to be *elliptic tails.*

- Elliptic normal singularities

Fix the notation of the previous Lemma. We describe the singularities of $\mathcal{W}'$.
By the description of the first order deformation of a cusp and a tacnode (see [**HM**, 3-b (7)]), we can write $\mathcal{W}$ around a cusp (respectively a tacnode) as $\mathrm{v}(y^2 - x^3 + th_1(x,t)) \subset \mathbb{C}^3_{x,y,t}$ (respectively $\mathrm{v}(y^2 - x^4 + th_2(x,t) \subset \mathbb{C}^3_{x,y,t})$ where $h_1$ and $h_2$ are analytic functions in $x$ and $t$ such that $h_1(0,0), h_2(0,0) \neq 0$ (recall that $\mathcal{W}$ is a smooth surface) and the fibration is over $t$.

Since $\mathcal{W}'$ lives on a base change of order 6 (respectively 4), then locally it is given by $\mathrm{v}(y^2 - x^3 + t^6 h_1(x,t^6))$ (respectively $\mathrm{v}(y^2 - x^4 + t^4 h_2(x,t^4))$. We see that these singularities are analytically equivalent to the elliptic normal singularities described in the Examples of Chapter 1.1.

The surface $\mathcal{C}$ is obtained by contracting all the $(-1)$-curves of $(\mathcal{W}')^{can}$ contained in the special fiber. We saw that, in the tacnodal case, $(\mathcal{W}')^{can}$ is $B'-$minimal and hence the two surfaces coincide.

- Curves of twisted spin curves

Let $\mathcal{W} \to B$ be a general smoothing of a curve $W$ as in Lemma 3.10 and pick its stable reduction $f : \mathcal{C} \to B'$. For every admissible divisor $D$ of $\mathcal{C}$ (see Definition 2.17), consider the moduli space of Theorem 2.6

$$S_{\mathcal{N}_D} := \overline{S}_f(\mathcal{N}_D) \longrightarrow B' \qquad \mathcal{N}_D := \omega_f(D).$$

Consider the variety $\overline{S}_f^-(\omega_f^*) \subset S_{\mathcal{N}_D}$ parametrizing odd theta characteristics of the fibers of the family $\mathcal{C}^* \to (B')^*$. We shall denote by $S^-_{\mathcal{N}_D}$ the closure of $\overline{S}_f^-(\omega_f^*)$ in $S_{\mathcal{N}_D}$.

Notice that the curves $S^-_{\mathcal{N}_D}$ are all birational as $D$ varies, since they contain $\overline{S}_f^-(\omega_f^*)$ as open subscheme. Then they have the same normalization, which we shall denote by

$$\nu_D : S_f^\nu \longrightarrow S^-_{\mathcal{N}_D}.$$

For every admissible $D$ we have a rational $B'-$map

$$\mu_D : S^-_{\mathcal{N}_D} - - > J_{\mathcal{W}'}. \tag{3.10}$$

which is an isomorphism away from the central fiber. Obviously $\mu_D$ is defined at smooth points of the central fiber. Since $S_f^\nu$ is smooth we get a natural morphism

$$\psi : S_f^\nu \longrightarrow J_{\mathcal{W}'}. \tag{3.11}$$

With this setup, we are ready to compute the multiplicities of the theta hyperplanes. Let us start with some examples.

Example 3.12. (The "characteristic numbers" of theta hyperplanes)

We shall see that the multiplicities of a theta hyperplane containing exactly one cusp is 3, containing exactly one tacnode without the tacnodal tangent is 4 and containing a tacnodal tangent is 6. Below we give a motivation for these "characteristic numbers".

Let $W$ be a curve whose singularities are cusps (respectively tacnodes). Consider a general smoothing $\mathcal{W} \to B$ of $W$ to theta generic curves and its stable reduction $f : \mathcal{C} \to B'$ over a base change $B'$ of order 6 (respectively 4) totally ramified over $0 \in B$ (see Lemma 3.10). If $C$ is the central fiber of $\mathcal{C}$, we know that there exists a morphism $\varphi : C \to W$ contracting the elliptic tails of $C$ (see Lemma 3.10 (3)).

The multiplicities of the theta hyperplenes of $W$ will be determined by the description of the above morphism (3.11) $\psi : S_f^\nu \to J_{\mathcal{W}'}$ (recall that $\mathcal{W}' = \mathcal{W} \times_B B'$).

(a) $W$ has exactly 1 cusp

$C$ is a curve of compact type with two components, $F$ elliptic and $W^\nu$ of genus $g-1$. The stable spin curves of $C$ are supported on the blow-up of $C$ at its node (see Example 2.9).

- If we glue an even theta characteristic of $W^\nu$ and the odd theta characteristic of $F$ to $\mathcal{O}_E(1)$ ($E$ is the exceptional component), we will find a hyperplane (via $\psi$) of type zero of multiplicity is 1.
- If we glue an odd theta characteristic of $W^\nu$ and a fixed even theta characteristic of $F$ to $\mathcal{O}_E(1)$, we will find a hyperlane containing the cusp. The morphism $\varphi : C \to W$ contracts $F$ and the hyperplane does not change if we vary the 3 even theta characteristics of $F$ and 3 is its multiplicity.

(b) $W$ has exactly 1 tacnode

$C$ is a curve with two components, $F$ of genus 1 and $W^\nu$ of genus $g-2$ and $F \cap W^\nu$ are two nodes. We shall denote by $S^-_C$ the zero dimensional scheme which is the fiber of $S^-_{\omega_f} \to B'$ over 0. We distinguish three types of stable spin curves of $C$ (see Example 2.10).

- If the odd stable spin curve $\xi$ is supported on the blow-up of $C$ at the two nodes and is given by gluing any even theta characteristic of $F$ and an odd theta characteristic of $W^\nu$ to $\mathcal{O}_E(1)$ (for $E$ running over the set of exceptional components), we will find (via $\psi$) a hyperplane containing the tacnodal tangent. Again $\varphi : C \to W$ contracts $F$. Since $F$ has 3 even theta characteristics and $\xi$ has multiplicity 2 in $S^-_C$, the hyperplane has multiplicity 6.
- If the odd stable spin curve $\xi$ is supported on the blow-up of $C$ at the two nodes and is given by gluing an odd theta characteristic of $F$ and an even theta characteristic of $W^\nu$ to $\mathcal{O}_E(1)$, $\xi$ is a double point of $S^-_{\omega_f}$ (it has multiplicity 2 in $S^-_C$, see the below Lemma 3.14) and hence there are two points in $S^\nu_f$ over $\xi$. We will find (via $\psi$) two different hyperplanes of type zero having multiplicity 1.
- If the odd stable spin curve $\xi$ is supported on $C$, we will find a hyperlane containing the tacnode without the tacnodal tangent. Call $\{p, q\} := F \cap F^c$. The hyperplane does not change if we change 4 restrictions of $\xi$ to $F$. The multiplicity of the hyperplane is 4.

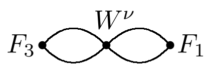
Example 3.13. (Idea of proof of Theorem 3.15)

Let $W$ be a irreducible theta generic curve of genus $g$ with exactly 3 tacnodes $t_1, t_2, t_3$ and $W^\nu$ be its normalization. Let $\mathcal{W} \to B$ be a general smoothing of $W$ to theta generic smooth curves and $\mathcal{C} \to B'$ be its stable reduction (see Lemma 3.10). We know that the special fiber $C$ of $\mathcal{C}$ has 3 elliptic tails $F_1, F_2, F_3$ and a tacnodal dual graph centered at $W^\nu$. Call $\{n_{h1}, n_{h2}\} = F_h \cap F_h^c$.

We find the multiplicity of a theta hyperplane of type $(2,1)$ containing $t_1, t_2$ and the tacnodal tangent of $t_1$.

FIRST STEP: from stable spin curves to twisted spin curves

Consider the rational maps (3.10) $\mu_D : S^-_{\mathcal{N}_D} - - > J_{\mathcal{W}'}$ extending to the morphism (3.11) $\psi : S^\nu_f \to J_{\mathcal{W}'}$. Let $\xi = (X, G, \alpha)$ be a stable spin curve in $S^-_{\omega_f}$ where $X$ is the blow-up of $C$ at all of its nodes except $F_2 \cap F_2^c$. Assume that $G \in \mathrm{Pic}(X)$ restricts to an even theta characteristic $R_1$ of $F_1$, to $\mathcal{O}_{F_3}$ and to the theta characteristic $R$ of $W^\nu \cup F_2$. The graph $\Sigma_X$ (obtained from the dual graph of $C$ by contracting the edges representing nodes which are not blown-up) is as shown below.

$\Sigma_X$ 


One proves (see Lemma 3.14) that $\xi$ is a singular point of $S^-_{\omega_f}$ with $2^{b_1(\Sigma_X)} = 2^2 = 4$ branches. Let $\nu_0 : S^\nu_f \to S^-_{\omega_f}$ be the normalization.

Consider the admissible divisor $D = F_1 + F_3$ of $\mathcal{C}$ and $S^-_{\mathcal{N}_D}$. Using Proposition 2.24 it follows that there is a set of 4 smooth points $(C, L_1), \ldots, (C, L_4)$ of $S^-_{\mathcal{N}_D}$ (and hence of $S^\nu_f$) which is exactly the set $\nu_0^{-1}(X, G, \alpha)$.

In order to find the images of the points of this set via $\psi$, it suffices to find the images of the 4 smooth points $(C, L)$ of $S_{\mathcal{N}_D}$ via $\mu_D$.

SECOND STEP: the behaviour of the smoothable sections of $L_1, L_2, L_3, L_4$

Set $\iota : L^{\otimes 2} \simeq \omega_f(D)$ and pick the unique $\mathcal{L}_i$ extending $(L_i, \iota)$ such that $\mathcal{L}_i^{\otimes 2} = \omega_f(D)$ and $\mathcal{L}_i \otimes \mathcal{O}_C = L_i$ (see Not.Ter. 3 (5)). Each one of $L_1, \ldots, L_4$ has exactly one $f-$smoothable section because, by the assumption on $\mathcal{W}$, the curves approaching $C$ are theta generic and hence $h^0(\mathcal{L}_i|_{C_b}) = 1$ for $0 \neq b \in B'$. We will see that these $f-$smoothable sections

- identically vanish on $F_1$
- vanish on a point of $F_2$

- are non-zero constants on $F_3$
- vanish on $g-4$ smooth points of $C$ on $W^\nu$ (the number depend by the chosen blow-up $X$ of $C$).

The theta hyperlanes associated to $L_1, L_2, L_3, L_4$ contain the tacnodal tangent of $t_1$, the tacnode $t_2$ without its tacnodal tangent and do not contain $t_3$.

THIRD STEP: the partition of $L_1, L_2, L_3, L_4$ induced by the smoothable sections

Using Theorem 2.24 we see that the line bundles $L \in \operatorname{Pic} C$ are obtained by gluing (with 4 suitable gluings)

$$R(n_{11}+n_{12}+n_{31}+n_{32}) \in \operatorname{Pic}(W^\nu \cup F_2)$$

$$R_1 \in \operatorname{Pic} F_1 \quad \mathcal{O}_{F_3} \in \operatorname{Pic} F_3$$

It is convenient to display the 4 line bundles $L$ in a table as follows.

Consider $M_1, M_2 \in \operatorname{Pic}(W^\nu \cup F_1 \cup F_2)$ obtained by gluing (with the same gluing data of the line bundles $L$ at the corresponding nodes)

$$R(n_{11}+n_{12}+n_{31}+n_{32}) \in \operatorname{Pic}(W^\nu \cup F_2) \quad R_1 \in \operatorname{Pic} F_1.$$

and similarly $K_1, K_2 \in \operatorname{Pic}(W^\nu \cup F_2 \cup F_3)$ by gluing

$$R(n_{11}+n_{12}+n_{31}+n_{32}) \in \operatorname{Pic}(W^\nu \cup F_2) \quad \mathcal{O}_{F_3} \in \operatorname{Pic} F_3.$$

Display all the line bundles in a table

| | $M_1$ | $M_2$ |
|---|---|---|
| $K_1$ | $L_{11}$ | $L_{12}$ |
| $K_2$ | $L_{21}$ | $L_{22}$ |

following the rule that each $L$ is obtained by gluing the $K$ of the corresponding row (resp. the $M$ of the corresponding column) at the nodes $F_1 \cap F_1^c$ (resp. at the nodes $F_3 \cap F_3^c$).

One proves that there are exactly 2 distinct sections each one of which is the smoothable section of the line bundles $L$ of a row of the table. Thus the images of $L_1, L_2, L_3, L_4$ via $\mu_D$ are exactly 2 distinct theta hyperplanes, one for each row of the table and we get a contribution of 2 to the multiplicity.

FOURTH STEP: the calculation of the multiplicity

If we change $\xi$ by changing the even theta characteristics of $F_1$ (among the 3 possible ones) and 4 restrictions of $\xi$ to $F_2$, we don't change the theta hyperplanes. We get a multiplicity $2 \cdot 3 \cdot 4 = 6 \cdot 4$ (see Theorem 3.15).

LEMMA 3.14. *Let $\mathcal{C} \to B$ be a general smoothing of a stable curve $C$ with a tacnodal dual graph. Consider the variety $S^-_{\omega_f}$ of odd stable spin curves of the fibers of $\mathcal{C}$. Let $\xi = (X, G, \alpha)$ be a stable spin curve of $C$ with $X \neq C$ viewed as point of $S^-_{\omega_f}$. Then $\xi$ is a singular point of $S^-_{\omega_f}$ with $2^{b_1(\Sigma_X)}$ branches.*

PROOF. Assume that $C$ has $2\tau$ nodes and let $F_1, \dots F_\tau$ be the elliptic curves of $C$ (see the notation of Lemma 3.10). Assume that the nodes $F_h \cap F_h^c$ for $1 \le h \le m$ are blown-up in $X \to C$. Notice that $m = b_1(\Sigma_X)$. Denote by $t_{2h}, t_{2h-1}$ the coordinates of $D_C$ (recall that $D_C$ is the base of the universal deformation of $C$) such that $\{t_{2h} = 0\}$ and $\{t_{2h-1} = 0\}$ are the loci preserving the nodes in $F_h \cap F_h^c$. Let $D_t$ be the space of the coordinates $t_{2h}, t_{2h-1}$ for $1 \leq h \leq m$ and write $D_C = D_t \times D'_t$.

Consider the arc $A$ in $D_C$ corresponding (up to restrict $B$) to the smoothing $\mathcal{C} \to B$. We proceed as in the proof of [**M**, Th. 2.6]. The implicit function theorem allows us to describe $A$, for some $1 \leq i \leq 3g-3$, as

$$(t_i h_1(t_i), \ldots, t_i, \ldots, t_i h_{3g-3}(t_i))$$

where, $h_j$ are analytic functions such that $h_j(0) \in \mathbb{C}^*$ for $j = 1, \ldots, 2m$ ($\mathcal{C}$ is smooth).

Consider as usual

$$D_\xi := D_s \times D'_s \xrightarrow{\quad\rho\quad} D_C = D_t \times D'_t$$

$$(s_1 \ldots s_{2m}, s_{2m+1}, \ldots, s_{3g-3}) \longrightarrow (s_1^2, \ldots, s_{2m}^2, s_{2m+1}, \ldots s_{3g-3})$$

The local picture of $S^-_{\omega_f}$ at $\xi$ is given by $U_\xi/\mathrm{Aut}(\xi)$ where $U_\xi = D_\xi \times_{D_C} A$ (see (2.2) below Th. 2.7). It suffices to show that $\rho^{-1}(A)/\mathrm{Aut}(\xi)$ has $2^m$ branches. $\rho^{-1}(A)$ is given by

$$\mathrm{v}(s_1^2 - s_i^2 h_1(s_i^2), \ldots, \hat{i}, \ldots, s_{2m}^2 - s_i^2 h_{2m}(s_i^2), \ldots, s_{3g-3} - s_i^2 h_{3g-3}(s_i^2)) \quad \text{if } i \leq 2m$$

$$\mathrm{v}(s_1^2 - s_i h_1(s_i), \ldots, s_{2m}^2 - s_i h_{2m}(s_i), \ldots, \hat{i}, \ldots, s_{3g-3} - s_i h_{3g-3}(s_i)) \quad \text{if } i > 2m.$$

We find how $\mathrm{Aut}(\xi)$ acts on $D_\xi$. It is easy to see that the image of the coboundary map

$$\mathrm{Aut}(\xi) \simeq \mathcal{C}^0(\Sigma_X, \mu_2) \simeq \mu_2^{m+1} \longrightarrow \mu_2^{2m} \simeq \mathcal{C}^1(\Sigma_X, \mu_2) \simeq \mathrm{Aut}_{D_C} D_\xi$$

is generated by the automorphisms $b_1, \ldots, b_m$ where $b_h$ (for $h = 1, \ldots, m$) acts on $D_\xi$ in the following way

$$b_h(s_1, \ldots, s_{2h-1}, s_{2h}, \ldots, s_{3g-3}) = (s_1, \ldots, -s_{2h-1}, -s_{2h}, \ldots, s_{3g-3}).$$

Set $w_{2h-1} := s_{2h-1}^2$, $w_{2h} := s_{2h}^2$, $z_h := s_{2h}s_{2h-1}$ for $h = 1, \ldots, m$ and $w_h = s_h$ for $h = 2m, \ldots, 3g-3$. Then $\rho^{-1}(A)/\mathrm{Aut}(\xi)$ is given by

$$\mathrm{v}(w_1 - w_i h_1(w_i), \ldots, w_{3g-3} - w_i h_{3g-3}(w_i), z_1^2 - w_1 w_2, \ldots, z_m^2 - w_{2m-1} w_{2m})$$

which is a singular point with $2^m$ branches. □

THEOREM 3.15. *Let $W$ be an irreducible theta generic canonical curve of genus $g$ whose singular points are only tacnodes. Then the multiplicity of a theta hyperplane of type $(i, j)$ is $4^{i-j}\, 6^j$.*

PROOF. Let $t_1, \ldots, t_\tau$ be the tacnodes of $W$. Let $\mathcal{W} \to B$ be a projective general smoothing of $W$ to theta generic smooth curves and let $f : \mathcal{C} \to B'$ be its stable reduction.

In the sequel we shall maintain the notations of Lemma 3.10. We know that the special fiber $C$ of $\mathcal{C}$ has a tacnodal dual graph centered at $W^\nu$. Denote by $\{n_{h1}, n_{h2}\} := F_h \cap F_h^c$.

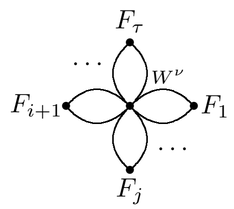
FIRST STEP: the reduction to twisted spin curves

For every admissible divisor $D$ of $\mathcal{C}$, consider the diagram (over $B'$)

$$\begin{tikzcd} S^{\nu}_f \arrow[r, "\nu_D"] \arrow[d, "\nu_0"'] \arrow[dr, "\psi"] & S^{-}_{\mathcal{N}_D} \arrow[d, dotted, "\mu_D"] \\ S^{-}_{\omega_f} \arrow[r, dotted, "\mu_0"] & J_{\mathcal{W}'} \end{tikzcd}$$

where $\nu_D$ is the normalization maps so that $\mu_D \circ \nu_D = \psi$ (where $\mu_D$ is defined). For every $D$ the base of the universal deformation of a $D-$twisted spin curve $(C, L)$ is $B'$ and $\mathrm{Aut}(C, L)$ acts trivially on $B'$. Hence $S^{-}_{\mathcal{N}_D}$ is smooth at the point $(C, L)$ (hence $\mu_D$ is defined there). Using this setup we will describe the map $\psi$ and the scheme structure of the fiber of $J_{\mathcal{W}'}$ over $0 \in B'$.

Let $\xi \in S^{-}_{\omega_f}$ be a stable odd spin curve supported on the blow-up $X$ of $C$ and pick a representative $(X, G, \alpha)$ of $\xi$. Assume that the nodes which are blown-up to get $X$ (for $i, j$ such that $1 \le j \le i \le \tau$) are $\{n_{h1}, n_{h2}\}$ for $h = 1, \dots, j$ and $h = i+1, \dots, \tau$ (see Example 2.10). Let $A_X$ be the graph associated to $X$ (obtained from $\Gamma_C$ by contracting the edges corresponding to the nodes which are not blown-up to get $X$). Then $A_X = \Sigma_X$ and is as shown below (there are loops from $F_1$ to $F_j$ and from $F_{i+1}$ to $F_\tau$).




In the first three Steps, $\xi$ will be fixed. Assume that $R_1, \dots, R_j$ are even theta characteristics respectively of $F_1, \dots, F_j$ and $R$ is a theta characteristic of $W^\nu \cup F_{j+1} \cdots \cup F_i$ so that $G$ has the following restrictions to the non-exceptional components of $X$

$$G|_{F_h} = R_h \quad (1 \le h \le j) \qquad G|_{F_h} = \mathcal{O}_{F_h} \quad (i < h \le \tau) \qquad G|_{W^\nu \cup F_{j+1} \cdots \cup F_i} = R$$

$$R_h^{\otimes 2} = \mathcal{O}_{F_h} \qquad R^{\otimes 2} = \omega_{W^\nu \cup F_{j+1} \cdots \cup F_i}.$$

In order to describe the map $\psi$ we choose another representative in the equivalence class of $\xi$ as follows. Define the Cartier divisor of $\mathcal{C}$ (which is a smooth surface, see Lemma 3.10)

$$D := \sum_{1 \le h \le j} F_h + \sum_{i < h \le \tau} F_h.$$

It is an admissible divisor of $\mathcal{C}$ (see Lemma 2.22). Then $G$ is equivalent to a line bundle $L \in \mathrm{Pic}(C)$ of a $D-$twisted spin curve $(C, L)$ if $L$ is obtained by gluing line bundles (with suitable gluings) in such a way that (see Theorem 2.24 and the notation of Definition 2.14)

$$L|_{F_h} = R_h \quad (1 \le h \le j) \qquad L|_{F_h} = \mathcal{O}_{F_h} \quad (i < h \le \tau)$$

$$L|_{W^\nu \cup F_{j+1} \cup \dots F_i} = R\left( \sum_{1 \le h \le j} (n_{h1} + n_{h2}) + \sum_{i < h \le \tau} (n_{h1} + n_{h2}) \right)$$

$$L^{\otimes 2} = \omega_C \otimes \mathcal{O}_f(D).$$

We have $b_1(\Sigma_X) = \tau - i + j$ and then $2^{\tau-i+j}$ gluings giving rise to $2^{\tau-i+j}$ different line bundles $L$. If $L$ is one of such line bundles, then it follows from Th. 2.24 and Remark 2.25 that there exists a representative $(X, G, \alpha)$ of $\xi$ such that $L$ and $G$ are limits of the same family of line bundles on a base change of order two of $f : \mathcal{C} \to B'$ totally ramified over 0. Hence $\xi$ and $(C, L)$ are the same point in $S^{\nu}_f$, that is $\nu_0(\nu_D^{-1}(C, L)) = \xi$.

It follows from Lemma 3.14 that the point $\xi$ of $S^{-}_{\omega_f}$ has $2^{\tau-i+j}$ branches and then $\nu_0(\nu_D^{-1}(C, L)) = \xi$ if and only if $L$ runs over the set of the above $2^{\tau-i+j}$ line bundles and

$$\forall\, X' \neq C \quad (X', L') \in S^{-}_{\mathcal{N}_D} \Longrightarrow \nu_D^{-1}(X', L') \cap \nu_0^{-1}(\xi) = \emptyset.$$

In order to find the image of the points of $S^{\nu}_f$ over $\xi$ (with representative $(X, G, \alpha)$), it suffices to find the images via the morphism $\mu_D$ of the above $D-$twisted spin curves $(C, L)$ (recall that $(C, L)$ is a smooth point of $S^{-}_{\mathcal{N}_D}$ because it is supported on $C$).

SECOND STEP: the behaviour of the smoothable sections of the line bundles $L$

Let $(C, L)$ be a $D-$twisted spin curve which is equivalent to $\xi = (X, G, \alpha)$. Set $\iota : L^{\otimes 2} \simeq \omega_f(D)$ and pick the line bundle $\mathcal{L}$ smoothing $(L, \iota)$ (see Not.Ter. 3 (5)). Since $f : \mathcal{C} \to B'$ is a smoothing to theta generic curves, there exists a unique $f-$smoothable section of $\mathcal{L}|_C = L$. We want to characterize its behavior on the irreducible components of $C$.

Recall that $\varphi : \mathcal{C} \to \mathcal{W}'$ is the canonical desingularization of $\mathcal{W}' = \mathcal{W} \times_B B'$. Consider the canonical desingularization $h_1 : \mathcal{W}_1 \to B'$ of $\mathcal{W}'$ at $t_1, \ldots, t_i$ so that there exists a birational morphism $\pi : \mathcal{C} \to \mathcal{W}_1$ which is an isomorphism away from the special fiber. Let $W_1 \subset \mathcal{W}_1$ be the central fiber. Thus $\pi : \mathcal{C} \to \mathcal{W}_1$ is the contraction of $F_{i+1}, \ldots, F_\tau$ to tacnodes of $W_1$ and $W_1$ has $F_1, \ldots, F_i$ as elliptic components.

We shall denote by $W_2 := \overline{W_1 - \cup_{1 \le h \le j} F_h}$ ($W_2$ has $F_{j+1}, \ldots, F_i$ as elliptic components).

- CLAIM: one can construct $2^{\tau-i+j}$ line bundles $P_1, P_2 \cdots$ in $\mathrm{Pic}(W_1)$ such that $1 = h^0(P_1) = h^0(P_2) = \cdots$ and such that $\{\pi^* P_1, \pi^* P_2 \cdots\}$ is exactly the set of line bundles $L$.

Let us prove the claim. Consider the theta characteristic $R$ of $W^{\nu} \cup F_{j+1} \cdots \cup F_i$ (see STEP I). Since the starting stable spin curve $\xi = (X, G, \alpha)$ is odd and the restrictions of $G$ are even on $F_1, \ldots, F_j$ and odd on $F_{i+1}, \ldots, F_\tau$, it follows that $R$ is odd (respectively even) if and only if $\tau - i$ is even (respectively odd) (see Not.Ter.3 (4)). It follows from Prop.2.2 that $R(\sum_{i<h\le\tau}(n_{h1}+n_{h2}))$ induces $2^{\tau-i}$ odd theta characteristics $P'_1, P'_2 \cdots$ on $W_2$ and by the theta genericity assumption

$$1 = h^0(P'_1) = h^0(P'_2) = \cdots .$$

Let $P'$ be one of these line bundles. Consider the Cartier divisor $D' := \sum_{1 \le h \le j} F_h$ of the total space $\mathcal{W}_1$ of the family $h_1 : \mathcal{W}_1 \to B'$. We construct the $2^{\tau+j-i}$ line bundles $P_1, P_2 \cdots$ by gluing

$$P'\left(\sum_{1 \le h \le j} (n_{h1} + n_{h2})\right) \in \mathrm{Pic}(W_2) \qquad R_h \in \mathrm{Pic}(F_h) \;\; 1 \le h \le j.$$

with suitable gluing data so that $\omega_{h_1}(D') \otimes \mathcal{O}_{W_1} = P_1^{\otimes 2} = P_2^{\otimes 2} = \cdots$ . Since $R_h$ is non effective, we have that if $P$ comes from $P'$, then $h^0(W_1, P) = h^0(W_2, P') = 1$. Pick one $P$ and the unique line bundle $\mathcal{P} \in \mathrm{Pic}(\mathcal{W}_1)$ such that $\mathcal{P}^{\otimes 2} = \omega_{h_1}(D')$ and $\mathcal{P}|_{W_1} = P$. Recall that $\pi : \mathcal{C} \to \mathcal{W}_1$ is birational. Arguing as for the relation $(PB)$ of Lemma 3.10

$$(\pi^*\mathcal{P})^{\otimes 2} = \pi^*(\omega_{h_1}(D')) \simeq \omega_f(D).$$

It follows that $\pi^*P$ is one of the line bundles $L$. Assume by contradiction that two distinct $P_1, P_2$ satisfy $\pi^*P_1 \simeq \pi^*P_2$. Then

$$(\pi^*\mathcal{P}_1)|_C \simeq (\pi^*\mathcal{P}_2)|_C \Rightarrow \pi^*\mathcal{P}_1 \simeq \pi^*\mathcal{P}_2.$$

Since $\pi$ is a birational morphism which is an isomorphism away from the special fiber and the degree of the restrictions of $P_1$ and $P_2$ to the irreducible components of $W_1$ are equal, we would have the contradiction

$$(\mathcal{P}_1)^* \simeq (\pi^*\mathcal{P}_1)^* \simeq (\pi^*\mathcal{P}_2)^* \simeq (\mathcal{P}_2)^* \Rightarrow P_1 \simeq P_2.$$

Thus $\{\pi^*P_1, \pi^*P_2, \cdots\}$ is exactly the set of line bundles $L$ and the claim is done.

For each $P$ we have $h^0(P) = 1$, then the unique section $s_P$ of $P$ is $h_1-$smoothable (recall that $h_1$ is the family $h_1 : \mathcal{W}_1 \to B'$). The $f-$smoothable section of $\pi^*P$ is given by $\pi^*s_P$.

The behavior of $\pi^*s_P$ is given by looking at $s_P$ and hence by construction

- $\pi^*s_P$ identically vanishes on $F_1, \ldots, F_j$
- $\pi^*s_P$ has a zero on each curve $F_{j+1}, \ldots, F_i$
- $\pi^*s_P$ is a non-zero constant on each curve $F_{i+1}, \ldots F_\tau$ (the section of each theta characteristic $P'$ of $W_2$ does not vanish on the tacnodes $t_{i+1}, \ldots, t_\tau$
- $\pi^*s_P$ has zeroes $\{l_1, \ldots, l_{g-i-j-1}\}_P$ on smooth points of $C$ on $W^\nu$ (which are zeroes of the section of the theta characteristic $P'$ of $W_2$ corresponding to $P$).

The theta hyperplane $\mu_D(C, \pi^*P)$ contains the tacnodal tangent of $t_1, \ldots, t_j$, the tacnodes $t_{j+1}, \ldots, t_i$ without tacnodal tangents and cut the smooth points $\{l_1, \ldots, l_{g-i-j-1}\}_P$ of $W$.

THIRD STEP: the partition induced by the smoothable sections

It is convenient to enumerate and display in a table the set of $2^{\tau+j-i}$ line bundles $L$ as follows. Consider the line bundles $M_1, \ldots, M_{2^j} \in \mathrm{Pic}(W^\nu \cup F_1 \cdots \cup F_i)$ obtained by gluing (with the same gluing data of the $L$ at the corresponding nodes) the following line bundles

$$R\left(\sum_{1\leq h\leq j}(n_{h1}+n_{h2}) + \sum_{i<h\leq\tau}(n_{h1}+n_{h2})\right) \in \mathrm{Pic}(W^\nu \cup F_{j+1}\ldots F_i)$$

$$R_h \in \mathrm{Pic}(F_h) \qquad 1 \leq h \leq j$$

Consider the line bundles $K_1, \ldots, K_{2^{\tau-i}} \in \mathrm{Pic}(W^\nu \cup F_{j+1}, \ldots, F_\tau)$ obtained by gluing (with the same gluing data of the $L$) the following line bundles

$$R\left(\sum_{1\leq h\leq j}(n_{h1}+n_{h2}) + \sum_{i<h\leq\tau}(n_{h1}+n_{h2})\right) \in \mathrm{Pic}(W^\nu \cup F_{j+1}\ldots F_i)$$

$$\mathcal{O}_{F_h} \in \mathrm{Pic}(F_h) \qquad i < h \leq \tau.$$

Display the $2^{\tau-i+j}$ line bundles $L$ in the table

| | $M_1$ | $M_2$ | $\cdot\ \cdot\ \cdot$ | $M_{2^j}$ |
|---|---|---|---|---|
| $K_1$ | $L_{11}$ | $L_{12}$ | $\cdot\ \cdot\ \cdot$ | $L_{1,2^j}$ |
| $K_2$ | $L_{21}$ | $L_{22}$ | $\cdot\ \cdot\ \cdot$ | $L_{2,2^j}$ |
| $\cdot$ | | | $\cdot\ \cdot\ \cdot$ | |
| $\cdot$ | | | $\cdot\ \cdot\ \cdot$ | |
| $K_{2^{\tau-i}}$ | $L_{2^{\tau-i},1}$ | $L_{2^{\tau-i},2}$ | $\cdot\ \cdot\ \cdot$ | $L_{2^{\tau-i},2^j}$ |

TABLE 1

following the rule that each line bundle $L$ is obtained by gluing the line bundle $K$ (respectively $M$) of the corresponding row (respectively column) at the nodes $n_{h1}, n_{h2}$ for $1 \leq h \leq j$ (respectively for $i < h \leq \tau$). Notice that the line bundles of each row (respectively column) have the same gluing data at $n_{h1}, n_{h2}$ for $i < h \leq \tau$ (respectively for $1 \leq h \leq j$).

Consider the set $\{s_{P_1}, s_{P_2} \dots\}$ of the $2^{\tau-i+j}$ smoothable sections of the line bundles $P$. Notice that if $P|_{W_2} = P'(\sum_{1\leq h\leq j}(n_{h1}+n_{h2}))$ then the section of $H^0(W_1, P)$ is the one restricting to the section of $H^0(W_2, P|_{W_2})$ vanishing on $n_{h1}, n_{h2}$ for $1 \leq h \leq j$ and vanishing on $F_h$ for $1 \leq h \leq j$. Therefore there are exactly $2^{\tau-i}$ distinct sections of type $s_{P_h}$ each one of which appears $2^j$ times. Call $s_1, \dots, s_{2^{\tau-i}}$ the distinct sections. Their pull-backs induce a partition of the set of the line bundles $L$ and hence a partition of the TABLE 1.

- CLAIM: the induced partition of TABLE 1 is by row.

Pick one of the sections $s_1, \dots, s_{2^{\tau-i}}$ : assume without loss of generality that it is $s_1$. Denote by $s_1'$ the restriction of $\pi^* s_1$ to $W^\nu \cup F_{j+1} \dots \cup F_i$ which is a section of

$$R\left(\sum_{1\leq h\leq j}(n_{h1}+n_{h2}) + \sum_{i<h\leq\tau}(n_{h1}+n_{h2})\right)$$

vanishing on the first $2j$ nodes. The line bundles $M$ (previously constructed and appearing in TABLE 1) are not effective on $F_1, \dots, F_j$, then $s_1'$ descends to a section of each $M$.

Recall that the line bundles $L$ on the same row of TABLE 1 have the same gluing data at $n_{h1}, n_{h2}$ for $i < h \leq \tau$. Thus in order to conclude this step it suffices to show that $s_1'$ respects one and only one gluing datum at these $\tau - i$ nodes (descending to a section of each line bundle of a row of Table 1).

Let $D(C)$ be the group of the spin gluing data of $C$. Since we are gluing fixed line bundles on the irreducible components of $C$, we can use the description of $D(C)$ given in Section 2.1. Consider generators $d_1, \dots, d_\tau$ of $D(C)$, where $d_h := d_{n_{h1}}$. Let $d, d' \in D(C)$ be two spin gluing data such that, for a fixed index $h$ with $i < h \leq \tau$, the generator $d_h$ appear in $d$ and not in $d'$. Assume that $s_1'$ respects $d$ (descending to a section $s_1$ of some $L$). We have seen that $s_1$ is a non zero constant on $F_h$ (recall that $L|_{F_h} = \mathcal{O}_{F_h}$) and hence it can' t respect $d'$ (which gives the opposite identification of $d$ at the node $n_{h1}$ and the same identification at $n_{h2}$). We conclude that $\pi^* s_1$ is a smoothable section for each line bundle of one row of TABLE 1.

FOURTH STEP: the calculation of the multiplicities

In the FIRST STEP we produced $2^{\tau-i+j}$ line bundles $L$ from a fixed stable spin $\xi$ and each $(C, L)$ is a $D-$twisted spin curve having multiplicity 1 in the fiber of $S_f^\nu \to B'$ over 0.

It follows from the conclusion of SECOND STEP that if $P$ is the unique line bundle of $W_1$ such that $\pi^* P = L$, then the theta hyperplane $\mu_D(C, L)$ is given by the span of the tacnodal tangents of $t_1, \dots, t_j$, the tacnodes $t_{j+1}, \dots, t_i$ and the smooth points $\{l_1, \dots, l_{g-i-j-1}\}_P$.

The THIRD STEP implies that $\mu_D(C,L) = \mu_D(C,L')$ if $L$ and $L'$ belong to the same row of TABLE 1 and we get $2^j$ of such twisted spin curves.

If we vary $(C,L)$ by gluing any one of the 3 even theta characteristics of $F_h$ for $1 \leq h \leq j$ we don't change the corresponding theta hyperplanes and we get $3^j$ of such twisted spin curves.

Moreover we shall see in Proposition 3.21 that any elliptic component $F_h$ of $C$ admits an automorphism fixing $F_h^c$ and exchanging any two of the four square roots of $\mathcal{O}_{F_h}(n_{h1} + n_{h2})$ for $j < h \leq i$. Using this it is easy too see that there is a partition of the $D-$twisted spin curves in sets of $4^{i-j}$ elements identifying via $\mu_D$.

We conclude that each theta hyperplane of type $(i,j)$ has multiplicity $2^j 3^j 4^{i-j} = 6^j 4^{i-j}$. □

Below we shall deal with the easier case of cuspidal singularities.

Theorem 3.16. *Let $W$ be an irreducible theta generic canonical curve of genus $g$ whose singular points are only cusps. Then the multiplicity of a theta hyperplane of type $k$ is $3^k$.*

Proof. Let $c_1, \dots, c_\gamma$ be the cusps of $W$. Let $\mathcal{W} \to B$ be a general projective smoothing of $W$ to theta generic smooth curves and $f : \mathcal{C} \to B'$ be its stable reduction with central fiber $C$. In the sequel we shall maintain the notations of Lemma 3.10. Denote by $n_h := F_h \cap F_h^c$

Fix the admissible divisor of $\mathcal{C}$

$$D = \sum_{1 \leq h \leq \gamma} F_h$$

and the smooth curve $S^-_{\mathcal{N}_D}$, all of whose points are supported on $C$. Consider the diagram

$$\begin{tikzcd} S_f^\nu \arrow[r, "\nu_D"] \arrow[dr, "\psi"'] & S^-_{\mathcal{N}_D} \arrow[d, "\mu_D"] \\ & J_{\mathcal{W}'} \end{tikzcd}$$

such that $\mu_D \circ \nu_D = \psi$. Recall that $\nu_0 : S_f^\nu \to S^-_{\omega_f}$ denotes the normalization

Fix a stable odd spin curve $(X, G, \alpha) \in S^-_{\omega_f}$ of $C$. It is supported on the blow-up $X$ of $C$ at all of its nodes (see Example 2.9). Assume that $R_1, \dots, R_k$ are even theta characteristics respectively of $F_1 \dots, F_k$ and $R$ is a theta characteristic of $W^\nu$ so that $G$ has the following restrictions to the irreducible non-exceptional components of $X$

$$G|_{F_h} = R_h \ \ (1 \leq h \leq k) \qquad G|_{F_h} = \mathcal{O}_{F_h} \ \ (k < h \leq \gamma) \qquad G|_{W^\nu} = R$$

It follows from Proposition 2.24 that $G$ is equivalent to $L \in \mathrm{Pic}(C)$ for a $D-$twisted spin curve $(C,L)$ if $L$ is obtained by gluing the following restrictions to the irreducible components of $C$ (notice that there is only one gluing since $C$ is of compact type)

$$L|_{W^\nu} = R\left(\sum_{1 \leq h \leq \gamma} n_h\right)$$

$$L|_{F_h} = R_h \ \ (1 \leq h \leq k) \qquad L|_{F_h} = \mathcal{O}_{F_h} \ \ (k < h \leq \gamma).$$

As in the tacnodal case this means that $\nu_0^{-1}(\xi) = \nu_D^{-1}(C,L)$ which is a point of $S_f^\nu$ (notice that, since $C$ is of compact type, then $S^-_{\omega_f} \to B'$ is étale everywhere and therefore the points of $S^-_{\omega_f}$ over $0 \in B'$ have one branch). In order to describe the morphism $\psi : S_f^\nu \to J_{\mathcal{W}'}$ it suffices to find the images of the $D-$twisted spin curves via the morphism $\mu_D : S^-_{\mathcal{N}_D} \to J_{\mathcal{W}'}$.

Arguing exactly as in Theorem 3.15 one can show that the $f$−smoothable section of a $D$−twisted spin curve

- identically vanish on $F_1, \dots, F_k$
- is a non-zero constant on each curve $F_{k+1}, \dots, F_\gamma$
- has $g-k-1$ zeroes on smooth points of $C$ on $W^\nu$ and two different sections have different sets of zeroes.

Since the morphism $\varphi : \mathcal{C} \to \mathcal{W}'$ contracts the elliptic curves $F_h$, the theta hyperplane $\mu_D(C, L)$ contains the cusps $c_1, \dots, c_k$ and does not contain the cusps $c_{k+1}, \dots, c_\gamma$.

Each $L$ has multiplicity 1 in the central fiber of $S^-_{\mathcal{N}_D}$ (see Example 2.9). If we change the $3^k$ even theta characteristics of $F_1, \dots, F_k$ to which $L$ restricts, then $\mu_D(C, L)$ does not change.

We conclude that each theta hyperplane of type $k$ has multiplicity $3^k$. □

Arguing exactly as for the previous two theorems we have

Theorem 3.17. *Let $W$ be an irreducible theta generic canonical curve of genus $g$ whose singular points are tacnodes and cusps. Then the multiplicity of a theta hyperplane of type $(i, j, k)$ is $4^{i-j}6^j3^k$.*

### 3.4. Twisted spin curves and the compactification

In this section we shall sum-up all the results of the previous sections in Theorem 3.22. In order to obtain a clear statement, we shall consider the case of curves whose singularities are only tacnodes or cusps, even if it will be evident how to proceed in the mixed case.

First of all we describe the elliptic tails arising from the stable reduction of a general smoothing of a cuspidal and tacnodal projective curve.

- The elliptic curve $F$ of the elliptic surface singularity $y^2 - x^3 + t^6 = 0$ (resp. $y^2 - x^4 + t^4 = 0$) has $j$−invariant $j(F) = 0$ (resp. $j(F) = 1728$).

In fact it is easy to check that $F$ is the double cover $\psi : F \to \mathbb{P}^1$ branched over $0, 1, \infty, -\omega$, where $\omega^3 = 1$ (resp. branched over $0, 1, \infty, -1$).

Let $W$ be a tacnodal curve as in Lemma 3.10 and let $F$ be an elliptic component of the stable reduction $C$ of a general smoothing of $W$. It is easy to see that $F$ admits a non-trivial involution $\psi$ fixing the ramifications points over $0, \infty$ and exchanging the ones over $1, -1$ and that the points $F \cap F^c$ lie over $0, \infty$. Thus if $\tau$ is the involution of $F$ exchanging the sheets we have

$$\mathrm{Aut}_W C \supseteq < \psi, \tau > \simeq \mu_2^2. \tag{3.12}$$

Definition 3.18. Let $W$ be an irreducible curve with cusps and tacnodes. Fix a general projective smoothing $\mathcal{W}$ of $W$. Let $s$ be a cusp (resp. a tacnode) of $W$. The *cuspidal (resp. tacnodal) blow-up of $W$ at $s$ with respect to $\mathcal{W}$* is the curve $C$ which is the central fiber of the stable reduction of $\mathcal{W}$ at $s$. The elliptic tails of $C$ are said to be *elliptic exceptional components*.

- The prototype of a tacnodal blow-up

One can find the explicit equation of a tacnodal blow-up by applying the canonical desingularization of the elliptic surface singularity $y^2 - x^4 + t^4 = 0$. A similar construction works also for the cuspidal blow-up.

Consider the blow-up $\mathcal{Z} \subset \mathbb{C}^2_{x,t} \times \mathbb{P}^1_{[s_0,s_1]}$ of $\mathbb{C}^2_{x,t}$ at the origin. Set $s := s_0/s_1$ and consider $\mathrm{v}(t - sx) \subset \mathcal{Z}$.
The canonical desingularization $\rho : \mathcal{W}^{can} \to \mathcal{W}$ of the surface singularity $\mathcal{W} = \mathrm{v}(y^2 - x^4 + t^4)$ is given by

$$\mathbb{C}^4_{x,t,s,y} \supset \mathrm{v}(t - sx\,,\, y^2 + s^4 - 1) = \mathcal{W}^{can} \longrightarrow \mathrm{v}(t - sx) \subset \mathcal{Z}$$

which is the double cover of $\mathcal{Z}$ ramified over $\mathrm{v}(s^4 - 1)$. The restriction of $\rho$ over the set of points with $t = 0$ gives a tacnodal blow-up of $y^2 - x^4 = 0$.

More explictly this blow-up is given by

$$\mathrm{v}(sx\,,\, y^2 + s^4 - 1) = \mathrm{v}(x\,,\, y^2 + s^4 - 1) \cup \mathrm{v}(s\,,\, y^2 + s^4 - 1) = F \cup C$$

which is the union of two smooth curves.

$F$ is an elliptic curve, because it is the double cover of $\mathrm{v}(x) \subset \tilde{\mathbb{C}}^2_{x,t}$ ramified over four points $(x, t, s_h) = (0, 0, i^h) \in \mathrm{v}(x)$ for $h = 1, \ldots, 4$. It is easy to check that $j(F) = 1728$.

Definition 3.19. Let $W$ be an irreducible projective curve whose singularities are only cusps. Fix a general projective smoothing $\mathcal{W}$ of $W$.

A *cuspidal spin curve on* $W$ *with respect to* $\mathcal{W}$ is a triple $(C, T, L)$, where

- $C$ is the cuspidal blow-up of $W$ at all of its cusps with respect to $\mathcal{W}$
- if $f$ is the stable reduction of $\mathcal{W}$, then $T$ is *the* twister $T \in Tw_f(C)$ induced by the (admissible) divisor which is the sum with coefficient 1 of *all* the exceptional elliptic components
- $L$ is a square root of $\omega_C \otimes T$.

Definition 3.20. Let $W$ be an irreducible projective curve whose singularities are only tacnodes. Fix a general projective smoothing $\mathcal{W}$ of $W$.

A *tacnodal spin curve on* $W$ *with respect to* $\mathcal{W}$ is a triple $(C, T, L)$, where

- $C$ is the tacnodal blow-up of $W$ at all of its tacnodes with respect to $\mathcal{W}$
- if $f$ is the stable reduction of $\mathcal{W}$, then $T$ is *a* twister $T \in Tw_f(C)$ induced by an (admissible) divisor which is the sum with coefficient 1 of *some* exceptional elliptic components
- $L$ is a square root of $\omega_C \otimes T$.

Notice that if $(C, T, L)$ is a tacnodal spin curve and $F$ is an elliptic exceptional component not contained in the divisor inducing $T$ with $\{p, q\} := F \cap F^c$, then $L|_F$ is a square root of $\mathcal{O}_F(p+q)$.

Proposition 3.21. *Let $W$ be an irreducible curve with a tacnode $t$. Let $C$ be a tacnodal blow-up of $W$ at $t$ with exceptional elliptic component $F$. Set $F \cap F^c = \{p, q\}$. If $G_1$ and $G_2$ are square roots of $\mathcal{O}_F(p+q)$, there exists $\sigma$ in $Aut_W C$ such that $\sigma^* G_1 \simeq G_2$.*

Proof. We know from (3.12) that $\mathrm{Aut}_W C \supseteq <\psi, \tau>$, where $\psi$ is an involution fixing only $F \cap F^c$ and $\tau$ is the involution of $F$ exchanging the sheets. Since the four square roots of $\mathcal{O}_F(p+q)$ are effective, we can pick the effective divisors $D_1, \ldots, D_4$ in their linear series such that $2D_i = p+q$. Since $2(\tau^*(D_i)) = \tau^*(2D_i) = \tau^*(p+q) = p+q$, we have (up to reorder the indices) $D_2 = \tau^*(D_1)$ and $D_3 = \tau^*(D_4)$. Moreover $2(\psi^*(D_i)) = \psi^*(2D_i) = \psi^*(p+q) = p+q$ and hence (up to reorded the indices) $D_1 = \psi^*(D_3)$ and $D_2 = \psi^*(D_4)$. □

We collect the main differences among spin curves in a table.

| SPIN CURVES | STABLE | CUSPIDAL | TACNODAL |
|---|---|---|---|
| typical base change | $t \to t^2$ | $t \to t^6$ | $t \to t^4$ |
| surface singularity | $y^2 - x^2 = t^2$<br>rational | $y^2 - x^3 = t^6$<br>elliptic | $y^2 - x^4 = t^4$<br>elliptic |
| exceptional curve | $E$ rational<br>$E^2 = -2$ | $F$ elliptic<br>$F^2 = -1$<br>$j(F) = 0$ | $F$ elliptic<br>$F^2 = -2$<br>$j(F) = 1728$ |

TABLE 2

We sum-up all the obtained results in the following Theorem. Recall the definition of $J_{\mathcal{W}}$ in Not.Ter. 3 (3).

THEOREM 3.22. *Let $W$ be an irreducible theta generic canonical curve of genus $g$ whose singular points are either only cusps or only tacnodes. Let $f : \mathcal{W} \to B$ be a projective smoothing of $W$ to theta generic curves.*

*Then $J_{\mathcal{W}}$ is a compactification of $J_{\mathcal{W}^*} \simeq S^-_{\omega^*_f}$ whose boundary points do not depend on the chosen family.*

*The boundary points of $J_{\mathcal{W}}$ correspond to spin curves of $W$ with respect to a fixed general projective smoothing of $W$. Denote by $(C, T, L)$ such a spin curve of $W$.*

*Let $W$ be cuspidal and $W^\nu, F_1, \ldots, F_\gamma$ be the irreducible components of $C$. Then $(C, T, L)$ and $(C, T, L')$ are identified in $J_{\mathcal{W}}$ if and only if*

- *$L|_{W^\nu} = L'|_{W^\nu}$;*
- *for $h = 1 \ldots, \gamma$, either $L|_{F_h} = L'|_{F_h} = \mathcal{O}_{F_h}$ or $L|_{F_h}, L|_{F_h}$ are even theta characteristics of $F_h$.*

*Let $W$ be tacnodal and $W^\nu, F_1, \ldots F_\tau$ be the irreducible components of $C$. Then $(C, T, L)$ and $(C, T', L')$ are identified in $J_{\mathcal{W}}$ if and only if*

- *$T = T'$;*
- *if $F_h$ is in the support of the divisor inducing $T$, then either $L|_{F_h} = L'|_{F_h} = \mathcal{O}_{F_h}$ or $L|_{F_h}, L|_{F_h}$ are even theta characteristics of $F_h$;*
- *if $F_1$ is the union of the elliptic exceptional components of $C$ to which $L, L'$ restrict to the trivial bundle and $F_2$ is the union of the elliptic exceptional components of $C$ not contained in the support of the divisor inducing $T$, then there is an automorphism $\sigma \in Aut_W C \cap Aut(F_2)$ such that $L'|_{W^\nu \cup F_1 \cup F_2} = (\sigma^* L)|_{W^\nu \cup F_1 \cup F_2}$.*

### 3.4.1. A quartic with an ordinary triple point.

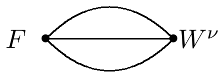
It is possible to generalize the techniques to more complicate type of singularities. In the sequel we consider the case of an irreducible curve of genus 3 with an ordinary triple point.

Let $W$ be a irreducible plane quartic with an ordinary triple point. There are 4 theta lines of type zero and 3 theta lines containing the singular point (the 3 lines of the tangent cone).

Let $\mathcal{W} \to B$ be the general projective smoothing of $W$ such that $\mathcal{W}$ locally is given by the equation $y^3 - x^3 + t = 0$. It is easy to see that its stable reduction $\mathcal{C} \to B'$ is given by

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\varphi"] \arrow[dr, "f"'] & \mathcal{W}' \arrow[r] \arrow[d] & \mathcal{W} \arrow[d] \\
& B' \arrow[r, "b"] & B
\end{tikzcd}$$

where $b$ is a base change of order 3 totally ramified over $0 \in B$. A local equation for $\mathcal{W}'$ is $y^3 - x^3 - t^3 = 0$. The central fiber of $\mathcal{C}$ is given by $C = W^\nu \cup F$ where $W^\nu$ is the normalization of $W$ and $F$ is an elliptic curve with $j(F) = 0$ and $F^3 = -3$. The dual graph of $C$ is as shown below.




Let us denote by $p, q, r$ the 3 nodes of $C$. Consider the birational curves $S^-_{\omega_f}$ and $S_{\mathcal{N}}$ where $\mathcal{N} := \omega_f(F)$ and their common normalization $S^\nu_f$ yielding a natural morphism $\psi : S^\nu_f \to J_{\mathcal{W}'}$.

Arguing as in the cuspidal and tacnodal case it is easy to see that the theta lines of type zero are given by the four smooth points $(C, L)$ of $S_{\mathcal{N}}$ (supported on $C$) with multiplicity 1 in the central fiber. Each one of these line bundles is equivalent to the unique odd spin curve supported on the blow-up of $C$ at the whole set of its nodes.

The theta line $l_p$ of $W$ containing the triple point and tangent to the branch corresponding to $p$ is given by taking the odd spin curves supported on the blow-up of $C$ at $p$. Since there are 4 such odd spin curves of multiplicity 2, then $l_p$ has multiplicity 8.

Arguing similarly, the multiplicity of the other two theta lines containing the triple point is 8.

CHAPTER 4

# Theta hyperplanes and stable reduction of curves

In this chapter we shall describe theta hyperplanes of canonical stable curves, giving an application to the stable reduction of curves.

In Section 1 we will see how to control degenerations of theta hyperplanes to canonical stable curves.

In Section 2 we will discuss ètale completions of curves of theta characteristics.

In Sections 3 and 4 we will recall some known results about the stable reduction of curves and we will use the results of Section 1 and the Geometric Invariant Theory to give a general computational approach to the stable reduction.

**Notation and Terminology 4.**

(1) Let $W$ be a canonical curve with nodes, cusps and tacnodes. A *theta hyperplane of type* 0 of $W$ is a hyperplane cutting a divisor $D$ divisible by 2 as Cartier divisor and not intersecting the singular locus of $W$.
In particular notice that the square root of $D$ yields a theta characteristic of $W$.

(2) The *valence* of a vertex of a graph is the number of the edges containing the vertex. A graph is said to be *Eulerian* if the valence of each vertex is even. Notice that a stable curve admits a semicanonical line bundle if and only if its dual graph is Eulerian. A graph is said to be *bipartite* if there exists a partition of the set of vertices into two classes such that every edge has its ends in different classes (see also [**D**, 1.6]).

## 4.1. Theta hyperplanes of stable curves

We describe configurations of theta hyperplanes of canonical stable curves. In [**CS2**, Lemma 2.4.1] one can find a description of theta hyperplanes of canonical stable curves all of whose components are non degenerate, that is either irreducible or with two rational components (i.e. the so-called split curves). In general, when the curve has degenerate components, the typical phenomenon is the existence of theta hyperplanes containing subcurves.

We start with the theta hyperplane of type 0 of a s.t.g. canonical stable curve. We shall restrict our analysis to curves with at most two irreducible components, which is what we will need in Section 4.4, even if one can get similar results also in the general case.

LEMMA 4.1. *Let $W$ be a s.t.g. curve with nodes, cusps and tacnodes with at most two irreducible components. Let $Y \subset W$ be an irreducible component of $W$ and $p_1, \dots, p_{2n}$ general points of $Y$ where $n \geq 1$. Assume that every semicanonical line bundle of $W$ has at most one section. If $R$ is a semicanonical line bundle of $W$ and $M$ is a line bundle of $W$ such that $M^{\otimes 2} = \mathcal{O}_W(p_1 + \cdots + p_{2n})$, then $h^0(R \otimes M^{-1}) = 0$.*

PROOF. Fix a semicanonical line bundle $R$ of $W$ and set $Y^s := Y^{sm} \cap W^{sm}$. Consider

$$U := \left\{ (M,(q_1,\dots,q_{2n}))\ :\ M^{\otimes 2} = \mathcal{O}_W\left(\sum_{1\le i\le 2n} q_i\right)\right\} \subset \mathrm{Pic}^n W \times \mathrm{Sym}^{2n} Y^s$$

and the projection onto its second factor

$$\varphi : U \longrightarrow \mathrm{Sym}^{2n} Y^s$$

whose finite fibers are isomorphic to the group $\mathrm{Pic}_W[2] \subset \mathrm{Pic}^0 W$ of $2-$torsion points. Moreover consider

$$V := \left\{(M,(q_1,\dots,q_{2n}))\ :\ h^0(R\otimes M^{-1}) \ge 1\right\} \subset \mathrm{Pic}^n W \times \mathrm{Sym}^{2n} Y^s.$$

It suffices to show that every irreducible component of $U$ has a point outside $V$. In particular it suffices to show the existence of a fiber of $\varphi$ with empty intersection with $V$.

Let $q$ be a general point of $Y$ and consider $2nq \in \mathrm{Sym}^{2n}Y^s$. It follows that

$$\varphi^{-1}(2nq) = \{(\mathcal{O}_W(nq)\otimes F\ ,\ 2nq)\ :\ F \in \mathrm{Pic}_W[2]\} \subset U.$$

Since $R' := R\otimes F^{-1} \in \mathrm{Pic}(W)$ is semicanonical for every $F \in \mathrm{Pic}_W[2]$ (hence with at most one section by hypothesis) and since $q$ is general and $n \ge 1$, we get

$$h^0(R\otimes(\mathcal{O}_W(-nq)\otimes F^{-1})) = h^0(R'\otimes \mathcal{O}_W(-nq)) = 0.$$

It follows that $\varphi^{-1}(2nq)\cap V = \emptyset$ and we are done. $\square$

LEMMA 4.2. *Let $W$ be as in Lemma 4.1 and satisfying one of the following conditions*

(i) *$W$ is irreducible*

(ii) *$W$ is general, stable and with two irreducible components.*

*If $R$ is an odd theta characteristic of $W$, then $h^0(R) = 1$. If $W$ is stable and $R$ is an even theta characteristic of $W$, then $h^0(R) = 0$. In particular if $W$ is stable and canonical, then it has a finite number of theta hyperplanes of type zero.*

PROOF. The proof is by induction on the (arithmetic) genus $g$ of $W$, since if $W$ has genus at most 1, any theta chacteristic has at most one section.

$(i)$ If $W$ has a node $n$, pick the normalization $\pi : W' \to W$ of $W$ at $n$. Let $p, q$ be the points of $W'$ over $n$. Consider the line bundle $R_1 := \pi^* R$ satisfying the relation $R_1^{\otimes 2} = \omega_{W'}(p+q)$. We get $R_1 = F\otimes B$, where $F^{\otimes 2} = \omega_{W'}$ and $B^{\otimes 2} = \mathcal{O}_{W'}(p+q)$. Notice that $\omega_{W'}\otimes R_1^{-1} = F\otimes B^{-1}$. By induction every semicanonical line bundle of $W'$ has at most one section and since $W$ is s.t.g., the points $p$ and $q$ of $W'$ are general. Hence we can apply Lemma 4.1 and then $R_1$ is non special. It follows that $h^0(R_1) = 1$ (Riemann-Roch). From

$$0 \to R \to \pi_*\pi^* R \to \mathcal{F}_n \to 0$$

where $\mathcal{F}_n$ is a torsion sheaf supported on $n$, we get $h^0(R) \le h^0(R_1) = 1$.

If $W$ has a cusp $c$, consider the normalization $\pi : W' \to W$ at $c$. Set $p = \pi^{-1}(c)$. Recall that the genus of $W$ is $g-1$.

We show by induction that if $R$ is a theta characteristic of $W$, then $h^0(R) \le 1$. In fact if $R$ is even, consider $\pi^*(R) \in \mathrm{Pic}(W')$. It follows from Proposition 2.2 that $\pi^*(R) = L(p)$ for an odd theta characteristic $L$ of $W'$ and by induction $h^0(L) = 1$. Since $(\pi^*R)^{\otimes 2} = \pi^*(\omega_W) = \omega_{W'}(2p)$ and $\deg R = g - 1$, we have

$$h^0(\pi^*R) = \deg R - g(W') + 1 + h^0(\omega_{W'} \otimes (\pi^*R)^{-1}) = 1 + h^0(L(-p)).$$

$W$ is s.t.g. then $p$ is general. Therefore $h^0(L(-p)) = 0$ and $h^0(R) \leq h^0(\pi^*(R)) = 1$. Hence $h^0(R) = 0$ ($R$ is even).

If $R$ is an odd theta characteristic of $W$, then it follows from Proposition 2.2 that $\pi^*R = L(p)$ for an even theta characteristic $L$ of $W'$ and hence $h^0(L) = 0$. Arguing as before we have $h^0(R) \leq h^0(\pi^*R) = 1 + h^0(L(-p)) = 1$ and hence $h^0(R) = 1$ ($R$ is odd).

If $W$ has a tacnode $t$, consider the normalization $\pi : W' \to W$ at $t$. Set $\{p, q\} = \pi^{-1}(t)$. Recall that the arithmetic genus of $W'$ is $g - 2$.

We show by induction that if $R$ is a theta characteristic of $W$, then $h^0(R) \leq 2$. In fact if $R$ is even, consider $\pi^*R \in \mathrm{Pic}(W')$. It follows from Proposition 2.2 that $\pi^*R = L(p + q)$ for an odd theta characteristic $L$ of $W'$. By induction $h^0(L) = 1$. Since $(\pi^*R)^{\otimes 2} = \pi^*(\omega_W) = \omega_{W'}(2p + 2q)$ we have

$$h^0(\pi^*R) = \deg R - g(W') + 1 + h^0(\omega_{W'} \otimes (\pi^*R)^{-1}) = 2 + h^0(L(-p - q)).$$

$W$ is s.t.g. then $p, q$ are general. Therefore $h^0(L(-p - q)) = 0$ and $h^0(R) \leq h^0(\pi^*R) = 2$.

Now if $R$ is an odd theta characteristic of $W$, then it follows from Proposition 2.2 that $\pi^*(R) = L(p + q)$ for an even theta characteristic $L$ of $W'$. By induction $h^0(L) \leq 2$. Arguing as before we have $h^0(\pi^*R) = 2 + h^0(L(-p - q))$ and $p, q$ are general points, hence $h^0(\pi^*R) = 2$. It follows that $h^0(R) \leq h^0(\pi^*R) = 2$ and then $h^0(R) = 1$ ($R$ is odd).

*(ii)* If an irreducible component of $W$ has a node $n$, we can argue as in *(i)* (applying Lemma 4.1 to a curve with two components).

If $W$ has two rational smooth components (i.e. $W$ is a split curve) the statement follows from [**C2**, Proposition 2].

If $W$ has a smooth component of genus at least 1, pick a degeneration of $W$ to a curve $W_0$ having a nodal component. Then $R$ specializes to a stable spin curve of $W_0$.

If the stable spin curve is supported on $W_0$, we are done by the first part of the proof and the semicontinuity. If the stable spin curve is supported on a proper blow-up $X_0$ of $W_0$, then we can apply the induction to $\tilde{X}_0$ and we are done by semicontinuity. ☐

In the sequel if $f : \mathcal{C} \to B$ is a smoothing of a canonical stable curve $C$, we denote by $S_C^-$ the fiber of $S_{\omega_f}^- \to B$ over $0 \in B$. Recall that in this hypothesis $J_f(C)$ is the central fiber of the family of theta hyperplanes $J_{\mathcal{C}}$ (see Not.Ter. 3 (3)).

THEOREM 4.3. *Let $C$ be a canonical general stable curve with at most two components. Let $f : \mathcal{C} \to B$ be a projective smoothing of $C$ to theta generic curves. There exists a morphism of zero dimensional schemes*

$$\mu : S_C^- \longrightarrow J_f(C)$$

*which does not depend on $f$. Moreover if $\xi = (X, G, \alpha)$ is an odd stable spin curve of $C$, then*

(a) *$\mu(\xi)$ contains all the nodes which are blown-up to get $X$.*
(b) *$\mu(\xi)$ contains the components of $\tilde{X}$ to which $G$ restricts to an even theta characteristic.*

PROOF. Consider the normalization $\nu : S_f^\nu \to S_{\omega_f}^-$. Since $S_{\omega_f}^-$ and $J_{\mathcal{C}}$ are isomorphic away from the central fiber and $S_f^\nu$ is smooth, we get a morphism $\mu'_f : S_f^\nu \to J_{\mathcal{C}}$ (which a priori depends on $f$.) We want to prove that $\mu'_f$ is constant along fibers of $\nu$.

Fix $\xi \in S_C^-$ supported on the blow-up $X$ of $C$. Since $S_{\omega_f}^-$ coarsely represents the functor of *odd* spin curves, a point in $\nu^{-1}(\xi)$ is in the isomorphism class of $\xi$. This means that any two point in $\nu^{-1}(\xi)$ are representatives $(X, G_1, \alpha_1), (X, G_2, \alpha_2)$ of $\xi$ and there is $\sigma \in \mathrm{Aut}_C(X)$ such that $G_2 \simeq \sigma^* G_1$.

We can describe $\mu'_f$ as follows. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\varphi'"] \arrow[dr, "f_X"'] & \mathcal{C}' \arrow[r, "\varphi"] \arrow[d] & \mathcal{C} \arrow[d, "f"] \\
& B' \arrow[r, "b"] & B
\end{tikzcd}$$

where $b$ is a base change of order 2 totally ramified over 0 and $\varphi'$ is a suitable blow-up of the fiber product $\mathcal{C}' := \mathcal{C} \times_b B'$ so that $f_X : \mathcal{X} \to B'$ is a smoothing of $X$. Pick a representative $(X, G, \alpha)$ of $\xi$. Let $\mathcal{E}$ be the effective Cartier divisor of $\mathcal{X}$ supported on exceptional components of $X$ so that we have an isomorphism $\iota : G^{\otimes 2} \xrightarrow{\sim} \omega_{f_X}(-\mathcal{E}) \otimes \mathcal{O}_X$. Let $\mathcal{G} \in \mathrm{Pic}\,\mathcal{X}$ be the unique line bundle smoothing $(G, \iota)$ (see Not.Ter. 2 (3) (b)). Since $\mathcal{C}$ is a smoothing to theta generic curves, then for $t \in (B')^*$ we have $h^0(X_t, \mathcal{G}|_{X_t}) = 1$. The image of $(X, G, \alpha)$ via $\mu'_f$ is given by the vanishing locus of the $f_X-$smoothable section of $G$.

Let $(X, G_1, \alpha_1)$ and $(X, G_2, \alpha_2)$ be in $\nu^{-1}(\xi)$ with $G_2 = \sigma^* G_1$ for $\sigma \in \mathrm{Aut}_C X$. Notice that since $C$ is general with at most two components, then $G_i$ has exactly one section (see Lemma 4.2). Thus if $s$ is the $f_X$ smoothable section of $G_1$, then $\sigma^* s$ is the $f_X$ smoothable section of $G_2$. The two smoothable sections have the same behavior away from the exceptional components of $X$ and vanish on a point of each exceptional component. Since the composition $\varphi \circ \varphi' : \mathcal{X} \to \mathcal{C}$ contracts the exceptional components, $\mu'_f$ is constant along the fibers of $\nu$. We get a map $\mu_f : S_C^- \to J_f(C)$.

Let $(X, G, \alpha)$ be a representative of $\xi$ and $E \subset X$ exceptional. The $f_X-$smoothable section of $G$ vanishes at one point of $E$ and this implies $(a)$. Let $Z$ be a component to which $G$ restricts to an even theta characteristic. Not.Ter. 3 and Lemma 4.2 imply that $G$ is non effective on $Z$ yielding $(b)$.

We prove that $\mu_f$ does not depend on $f$.

Let $C$ have one irreducible component. $\mu_f(\xi)$ is independent of $f$, because it is given by the linear span of the nodes of $C$ which are blown-up to get $X$ and of the (smooth) points of the effective divisor of the odd theta characteristic $G|_{\tilde{X}}$.

Let $C$ have two irreducible components.

Assume that $\tilde{X}$ has only one (odd) connected component. Notice that $h^0(G|_{\tilde{X}}) = 1$ (see Lemma 4.2). Using Lemma 4.1 it is easy to see that the section of $G|_{\tilde{X}}$ does not vanish on components of $X$. Hence $\mu_f(\xi)$ is given by the linear span of the nodes which are blown-up to get $X$ and of the (smooth) points of the effective divisor of $G|_{\tilde{X}}$. $\mu_f(\xi)$ is independent of $f$.

Assume that $\tilde{X}$ has two connected components, $X_1$ odd and $X_2$ even. Thus $\mu_f(\xi)$ contains $X_2$ by the above $(b)$. Since $X$ is non-degenerate, $\mu_f(\xi)$ doesn't contain $X_1$, hence $\mu_f(\xi)$ is given by the span of the linear space containing $X_2$ and the (smooth) points of the effective divisor of the odd theta characteristic $G|_{X_1}$. $\mu_f(\xi)$ is independent of $f$. □

#### 4.1.1. The multiplicities of $J(C)$ when $C$ has at most two components.

In the sequel we show how to use Theorem 4.3 to compute the multiplicities of the theta hyperplanes. If $C$ has at most two components, $J(C)$ will denote the zero dimensional scheme of theta hyperplane.

Let $C$ be a general stable projective curve of genus $g$ with $\delta$ nodes. Let $f$ be a projective smoothing of $C$ to theta generic curves and $\mu$ as in Th.4.3.

Assume that $C = C_1 \cup C_2$ with $g(C_i) = g_i$. Write $S^-_C = S_1 \cup S_2$, where a point is in $S_1$ if and only if it represents an odd stable spin curve $\xi = (X, G, \alpha)$ of $C$ with $\tilde{X}$ connected.

It follows from the proof of Th. 4.3 that $\mu(S_1) \cap \mu(S_2) = \emptyset$ and that $\mu$ is injective over $S_1$. Thus if $\xi = (X, G, \alpha) \in S_1$, then $\mu(\xi)$ has the same multiplicity of $\xi$ in $S^-_X$ that is $2^{b_1(\Sigma_X)}$.

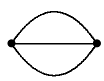
Let $X$ be the blow-up of $C$ at all of its nodes. Let $\xi_1 = (X, G_1, \alpha_1)$ and $\xi_2 = (X, G_2, \alpha_2)$ be in $S_2$. Then $\mu(\xi_1) = \mu(\xi_2)$ if and only if $\xi_1$ and $\xi_2$ have the same odd connected component $C_i$ and $G_1|_{C_i} \simeq G_2|_{C_i}$. Thus if $\xi = (X, G, \alpha) \in S_2$ and $C_i$ is the odd connected component of $\tilde{X}$, then the multiplicity of $\mu(\xi)$ is $2^{b_1(\Sigma_X)} N^+_{g_{3-i}}$, where $N^+_{g_{3-i}}$ is the number of even theta characteristics of the component $C_{3-i}$ of $X$.

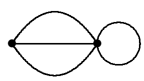
(SP) In particular if $C$ is a split curve of genus $g$, (i.e. a stable curve which is the union of two rational normal curves intersecting at $g+1$ points), then $\mu$ is always injective because there are no odd stable spin curves supported on the blow-up of $C$ at all of its nodes. In this case $\mu$ induces an isomorphism of zero dimensional schemes $S^-_C \simeq J(C)$.

If $C$ is irreducible, then $\mu$ is an isomorphism of zero dimensional scheme. Then a theta hyperplane containing $h$ nodes has multiplicity $2^h$.

Example 4.4. The genus 3 case is special because any odd spin curve of a stable plane quartic has exactly one section, i.e. it is theta generic. This yields a bijection between odd theta spin curves and theta lines of a plane quartic, which we shall describe below.

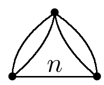
If $C$ has at most two components, the description is the above one. For example let $C$ be the union of a line and a (possibly nodal) cubic whose dual graph is one of the two graphs shown below.

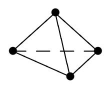




Consider an odd stable spin curve $(X, G, \alpha)$ with $\tilde{X}$ non connected. It is easy to see that there is just one such odd stable spin curve. Its corresponding theta line is the linear component of $C$.

Let $C$ be the union of two lines and a conic (possibly reducible) whose dual graph is one of the two graphs shown below.



In the left hand side case there are two types of odd stable spin curves supported on a blow-up $X$ of $C$ with $\tilde{X}$ connected, depending if the node $n$ on the two lines is blown-up or not. If $n$ is blown-up, there are 2 odd stable spin curves supported on the same curve, whose corresponding theta lines are the two tangents to the conic from $n$. If $n$ is not blown-up, there are 4 odd stable spin curves whose corresponding theta lines are the linear span of the 4 pairs of nodes where in each pair the first node is on the conic and on one line and the second node is on the conic and on the other line. Moreover there are exactly 2 odd stable spin curves supported on two blow-ups $X$ of $C$ with $\tilde{X}$ not connected (the blow-ups respectively of the 3 nodes of a linear component of $C$) and the corresponding theta lines are the 2 linear components of $C$.

In the right hand side case, there are 3 odd spin curves supported on blow-ups $X$ of $C$ with $\tilde{X}$ connected and the corresponding theta lines are the linear span of the two nodes which are blown-up. They are 3 distinct lines. Moreover there are 4 odd spin curves supported on a blow-up $X$ of $C$ with $\tilde{X}$ non connected (they are the blow-up of the three nodes on a component of $C$). The corresponding theta lines are the 4 linear components of $C$.

### 4.2. Étale completions of curves of theta characteristics

Let $f : \mathcal{C} \to B$ be a smoothing of a stable curve $C$. Consider the restricted family $\mathcal{C}^* \to B^*$. The modular curve $S_{\omega_f^*} \to B^*$ of theta characteristics of the fibers of $\mathcal{C}^*$ is étale over $B^*$. Obviously a flat completion of $S_{\omega_f^*}$ over $B$ is $S_{\omega_f}$.

We want to characterize the families $f : \mathcal{C} \to B$ such that $S_{\omega_f^*}$admits an *étale* completion over $B$. We will see that this property depends only on the dual graph of the special fiber $C$.

DEFINITION 4.5. Let $\Gamma$ be a graph. We say that $\Gamma$ is étale if any graph obtained by contracting the edges of an Eulerian subgraph of $\Gamma$ is bipartite.

Notice that an étale graph is bipartite.

PROPOSITION 4.6. *Let $C$ be a stable curve without non-trivial automorphisms. Let $f : \mathcal{C} \to B$ be a smoothing of $C$ with $B \subset \overline{M_g}$ and $\mathcal{C}$ smooth. The modular curve $S_{\omega_f^*} \to B^*$ admits an étale completion over $B$ if and only if the dual graph $\Gamma_C$ is étale.*

PROOF. Let $S_f^\nu$ be the normalization of $S_{\omega_f}$. If $S_{\omega_f^*} \to B^*$ admits an étale completion over $B$, then it is $S_f^\nu \to B$. We will show that the $S_f^\nu \to B$ is étale if and only if $\Gamma_C$ is étale.

Pick a stable spin curve $\xi = (X, L, \alpha)$ in the special fiber of $S_{\omega_f}$. Assume that $X \to C$ is the blow-up of $C$ at the nodes $n_1, \dots, n_m$ of $C$.

The problem is local, then we can assume $B \subset D_C$ (recall that $D_C$ is the base of the universal deformation of $C$). For $j = 1, \dots, m$ let $t_j$ be the coordinate of $D_C$ so that $\{t_j = 0\}$ is the locus where $n_j$ persists. Using the fact that $\mathcal{C}$ is smooth and the implicit function theorem, we can describe $B$ without loss of generality as

$$(t_1, t_1 h_2(t_1), \dots, t_1 h_m(t_1), \dots, t_1 h_{3g-3}(t_1))$$

where $h_j$ is an analytic function and $h_j(0) \in \mathbb{C}^*$ for $j = 1 \dots, m$.

If we consider the usual base change $D_\xi := D_s \times D'_s \stackrel{\rho}{\to} D_C = D_t \times D'_t$ given by

$$(s_1 \dots s_m, s_{m+1}, \dots, s_{3g-3}) \xrightarrow{\rho} (s_1^2, \dots, s_m^2, s_{m+1}, \dots s_{3g-3})$$

then

$$U_\xi = \mathrm{v}(s_2^2 - s_1^2 h_2(s_1^2), \dots, s_m^2 - s_1^2 h_m(s_1^2), s_{m+1} - s_1^2 h_{m+1}(s_1^2), \dots, s_{3g-3} - s_1^2 h_{3g-3}(s_1^2)).$$

The tangent cone of $U_\xi$ is given by

$$T(U_\xi) = \mathrm{v}(s_2^2 - s_1^2, \dots, s_m^2 - s_1^2, s_{m+1}, \dots) = \bigcup_{\epsilon_j \in \{1,-1\}} \mathrm{v}(\dots, s_j + \epsilon_j s_1, \dots, s_i, \dots)_{\substack{2 \le j \le m \\ m < i \le 3g-3}}$$

and hence $U_\xi$ has $2^{m-1}$ distinct branches. Notice that $\rho$ is a $2^m$–fold cover. The restriction of $\rho$ to each branch is a double cover of $B$ (hence ramified over the origin). In fact the involution $\theta$ of $D_\xi$ over $D_C$

$$(s_1, \dots, s_m, s_{m+1}, \dots, s_{3g-3}) \xrightarrow{\theta} (-s_1, \dots, -s_m, s_{m+1}, \dots, s_{3g-3})$$

acts on $U_\xi$ over $B$ preserving the components of $T(U_\xi)$ and hence the branches of $U_\xi$. Denote by $\nu : S_f^\nu \to S_{\omega_f}$ the normalization.

- CLAIM: $S^\nu_f \to B$ is unramified at the points of the fiber $\nu^{-1}(\xi)$ of $\nu : S^\nu_f \to S_{\omega_f}$ if and only if the above involution $\theta : D_\xi \to D_\xi$ is contained in the image of the coboundary operator $\delta : \mathrm{Aut}(\xi) \simeq \mathcal{C}^0(\Sigma_X, \mu_2) \to \mathcal{C}^1(\Sigma_X, \mu_2) \simeq \mathrm{Aut}_{D_C} D_\xi$.

In fact $S^\nu_f \to B$ is unramified at the points contained in $\nu^{-1}(\xi)$ if and only if the restriction of $\mu : U_\xi/\mathrm{Aut}(\xi) \to B$ to each branch is bijective. If $\theta$ is contained in $\delta(\mathcal{C}^0(\Sigma_X, \mu_2))$, then it acts on each branch of $U_\xi$ exchanging the sheets and hence the restriction of $\mu$ to the image of each branch of $U_\xi$ in $U_\xi/\mathrm{Aut}(\xi)$ is bijective. Conversely assume that the restriction of $\mu : U_\xi/\mathrm{Aut}(\xi) \to B$ to each branch is bijective. Then for every branch of $U_\xi$ there exists $\theta_{U_\xi} \in \delta(\mathcal{C}^0(\Sigma_X, \mu_2))$ restricting to the involution induced by $\theta$. If $\theta_{U_\xi} \neq \theta$, then either $\theta_{U_\xi}(s_1, \dots, s_h, \dots) = (-s_1, \dots, s_h, \dots)$ or $\theta_{U_\xi}(s_1, \dots, s_k, \dots) = (s_1, \dots, -s_k, \dots)$ for some $h, k \leq m$. In both cases all the components of $T(U_\xi)$ are not fixed, yielding a contradiction. Thus $\theta_{U_\xi} = \theta \in \delta(\mathcal{C}^0(\Sigma_X, \mu_2))$

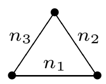
Notice that $\theta$ is represented by the chain of $\mathcal{C}^1(\Sigma_X, \mu_2)$ having $-1$ over all the edges. It is easy to see that it is contained in $\delta(\mathcal{C}^0(\Sigma_X, \mu_2))$ if and only if $\Sigma_X$ is a bipartite graph.

We are done, because if we take the subgraph $A_X$ of $\Gamma_C$ associated to $X$ (obtained by taking the edges corresponding to the nodes of $C$ which are blown-up to get $X$), then $\Sigma_X$ is obtained by contracting the edges of the Eulerian subgraph $\overline{\Gamma_C - A_X}$ of $\Gamma_C$.

$\square$

Example 4.7. The trees and the tacnodal graphs are étale, while the dual graph of an irreducible stable curve is not étale.

Example 4.8. Consider a stable curve $C$ whose dual graph is shown below. $\Gamma_C$ is not étale because it is not bipartite.




Let $D_t \subset D_C$ be the polydisc with coordinate $t_1, t_2, t_3$ so that $\{t_i = 0\}$ is the locus preserving $n_i$. Consider the arc $B = \mathrm{v}(t_2 - t_1 \,,\, t_3 - t_1) \subset D_t$. Let $\xi = (X, L, \alpha)$ be a spin curve supported on the blow-up of $C$ at $n_1, n_2, n_3$. If we consider $D_s$ with coordinate $s_1, s_2, s_3$, we have

$$U_\xi = \mathrm{v}(s_2 + s_1, s_3 + s_1) \cup \mathrm{v}(s_2 + s_1, s_3 - s_1) \cup \mathrm{v}(s_2 - s_1, s_3 + s_1) \cup \mathrm{v}(s_2 - s_1, s_3 - s_1) = \bigcup_{1 \leq i \leq 4} T_i.$$

where each $T_i$ has a double cover onto $B$. The image of $\delta : \mathcal{C}^0(\Sigma_X, \mu_2) \to \mathcal{C}^1(\Sigma_X, \mu_2)$ is given by $\{\theta_1, \theta_2, \theta_3, id\} \simeq \mu_2^2$, where

$$\theta_1(s_1, s_2, s_3) = (s_1, -s_2, -s_3) \quad \theta_2(s_1, s_2, s_3) = (-s_1, s_2, -s_3) \quad \theta_3(s_1, s_2, s_3) = (-s_1, -s_2, s_3)$$

$$T_1 \xleftrightarrow{\theta_1} T_4 \qquad T_1 \xleftrightarrow{\theta_2} T_3 \qquad T_1 \xleftrightarrow{\theta_3} T_2$$

$$T_2 \xleftrightarrow{\theta_1} T_3 \qquad T_2 \xleftrightarrow{\theta_2} T_4 \qquad T_3 \xleftrightarrow{\theta_3} T_4$$

Notice that $(s_1, s_2, s_3) \to (-s_1, -s_2, -s_3)$ is not contained in $\mathrm{Im}(\delta)$.

$T_1, \dots, T_4$ are identified in $U_\xi/\mathrm{Aut}(\xi)$ which is smooth and endowed with a double cover of $B$ having $\xi$ as a ramification point.

## 4.3. The stable reduction of curves

In [**Hs1**] and [**Hs2**], B. Hassett obtained results on the stable reduction of curves using the Minimal Model Program. We generalize some of these results using the canonical desingularization of surfaces. Moreover we shall see that the Geometric Invariant Theory yields a general approach to the stable reduction, which is purely computational.

By an *isolated plane curve singularity* we will mean an analytic neighborhood of one singular point of a reduced curve on a smooth surface.

Let $W$ be an isolated plane curve singularities. Let $\Delta$ be the complex $1-$dimensional disc. A *smoothing* of $W$ will be a surjective proper holomorphic map $f : \mathcal{W} \to \Delta$, where $\mathcal{W}$ is a complex surface and the fibers of $f$ are isolated plane curve singularities with $f^{-1}(0) = W$ and $f^{-1}(t)$ smooth for every $t \in \Delta - 0$. We say that a smoothing of $W$ is *general* if $\mathcal{W}$ is a smooth surface.

Let $W$ be an isolated plane curve singularity.

Let $\mathcal{W} \to \Delta$ be a smoothing of $W$ and $C$ be its stable reduction. Hence $C = W^\nu \cup W_T$, where $W^\nu$ is the normalization of $W$ and $W_T := \overline{C - W^\nu}$. Moreover $W^\nu \cap W_T = \{p_1, \dots, p_b\}$, where $b$ is the number of branches of $W$. The pointed curve $(W_T, p_1, \dots, p_b)$ is said to be *the tail of the stable reduction* (see [**Hs1**, Section 3] for details).

PROPOSITION 4.9. *Let $W$ be an isolated plane curve singularity with $b$ branches and genus $g$. Let $\tilde{g}$ be the genus of the normalization of $W$. Let $\mathcal{T}_W$ be the set of the tails obtained from all the smoothings of $W$. Then these tails are pointed stable curves and $\mathcal{T}_W$ is a connected closed subvariety of the moduli space $\overline{M_{\gamma,b}}$, where $\gamma = g - \tilde{g} - b + 1$.*

PROOF. See [**Hs1**, Proposition 3.2]. □

PROPOSITION 4.10. *Let $W$ be an isolated curve singularity of analytic type $y^q = x^p$ where $p, q$ are integers with $p \geq q > 1$ and set $b = gcd(p,q)$. There exist infinitely many curves which are stable reductions of smoothings of $W$ whose tails $(W_T, p_1, \dots, p_b)$ satisfy the following properties*

(1) *$p_1 + p_2 \cdots + p_b$ is a subcanonical divisor of $W_T$, that is*
$$\left(\frac{pq}{b} - \frac{p}{b} - \frac{q}{b} - 1\right)(p_1 + p_2 + \cdots + p_b) = K_{W_T}$$
(2) *$W_T$ is $q-$gonal, with $g^1_q = |\frac{q}{b}(p_1 + p_2 + \cdots + p_b)|$.*
(3) *If $p = q + 1 = 3$ (that is $W$ is a cuspidal singularity), then $\mathcal{T}_W = \overline{M_{1,1}}$.*

PROOF. See [**Hs1**, Theorem 6.2 and Theorem 6.3] and [**Hs2**, Proposition 4.1]. □

We show that Proposition 4.10 (1) holds for all the tails arising from a general smoothing of a singularity of analytic type $y^m = x^m$.

PROPOSITION 4.11. *Let $W$ be an isolated curve singularity curve of analytic type $y^m = x^m$, $m \geq 4$. Let $(W_T, p_1, \dots, p_m)$ be the pointed curve which is the tail of a general smoothing of $W$. Then the divisor $p_1 + \cdots + p_m$ is subcanonical for $W_T$.*

PROOF. It is easy to see that the tail $W_T$ is a $m-$fold cover $\psi : W_T \to \mathbb{P}^1$ totally ramified in $m$ points and $R(\psi) := (m-1)(\sum_{1 \leq i \leq m} p_i)$ is the ramification divisor of $\psi$.

Hence for every point $p$ of $\mathbb{P}^1$ we have
$$R(\psi) + \psi^*(-2p) = K_{W_T}.$$

It follows that

$$K_{W_T} = \frac{2}{m(m-1)}\left[\frac{m(m-1)}{2}\psi^*(-2p)\right] + R(\psi) = \frac{-2}{m(m-1)}\left[\sum_{1\le i<j\le m}(m(p_i+p_j))\right] + R(\psi) =$$

$$= \frac{-2}{m(m-1)}\left[m(m-1)\sum_{1\le i\le m} p_i\right] + R(\psi) = -2\sum_{1\le i\le m} p_i + R(\psi) = (m-3)\sum_{1\le i\le m} p_i.$$

□

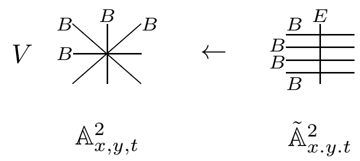
Now we consider elliptic tails arising from smoothings of a tacnodal singularity.

PROPOSITION 4.12. *Let $W$ be a singularity of analytic type $y^2 = x^4$. The set of the elliptic curves appearing as tail of smoothings of $W$ is the whole $\overline{M_1}$.*

PROOF. Consider the smoothings $\mathcal{W}_a$ of the tacnodal singularity $W$ given by the surfaces

$$\mathbb{A}^3_{x,y,t} \supset \mathcal{W}_a := \mathrm{v}(y^2 - x^4 + \frac{a}{2-a}x^3t + x^2t^2 - \frac{a}{2-a}xt^3) \longrightarrow \Delta_t \qquad a \in \mathbb{C}^*\backslash\{1,-1,2\}$$

where $t$ is the coordinate of the disc $\Delta_t$. Notice that $\mathcal{W}_a \to \mathbb{A}^2_{x,t}$ is a double cover ramified along the plane curve $V := \mathrm{v}(x^4 - \frac{a}{2-a}x^3t - x^2t^2 + \frac{a}{2-a}xt^3) \subset \mathbb{A}^2_{x,t}$. If $a$ is general, then $Z$ is an ordinary quadruple point and in this case the surface $\mathcal{W}_a$ has a normal elliptic singularity in the origin (see Section 1.2). The canonical desingularization of $\mathcal{W}_a$ is the double cover of the blow-up $\tilde{\mathbb{A}}^2_{x.y.t}$ of $\mathbb{A}^2_{x,y,t}$ as shown below, where $B$ denotes the curves of the branch locus and $E$ the exceptional curve (which is not contained in the branch locus).




The special fiber of the canonical desingularization is the stable reduction of $\mathcal{W}_a$ and contains an elliptic tail, which is the double cover of $E$ ramified over $E \cap E^c$. It is easy to see that, up to a projectivity, $E \cap E^c = \{[0,1],[1,1],[1,0],[a,1]\}$. Hence the general complex number is the $j-$invariants of an elliptic tail arising from a suitable smoothing of $W$. Since the set of the tails arising from smoothings of $W$ is a closed set (see Proposition 4.9), we are done. □

Using the same technique, one can find similar results for other types of singularities

## 4.4. A GIT-computational approach to the stable reduction

The group $SL(g)$ naturally acts over the space $\mathbb{P}_{N_g}$ of configurations of $N_g$ hyperplanes of $\mathbb{P}^{g-1}$. We shall describe a computational method to give negative results on the stable reduction of curves using the GIT-stability of configuration of theta hyperplanes. Since these configurations are completely known for many curves (see Th.4.3, Th.3.9, Th.3.16, Th.3.15), the following stability criterion from [**MFK**, Prop. 4.3] is explicit and computable.

**GIT stability criterion.** For any $\Omega \in Sym^N(\mathbb{P}^{g-1})^\vee$ and for $h = 1, \ldots, g-1$ let $\mu_{h-1}(\Omega)$ be the maximum multiplicity of an $(h-1)-$dimensional linear space of $\mathbb{P}^{g-1}$ as a subscheme of $\Omega$, viewed as a degree $N$ hypersurface of $\mathbb{P}^{g-1}$.

Then $\Omega$ is GIT semistable (respectively GIT stable) if and only if for every $h = 1, \ldots, g-1$ we have $\mu_{h-1}(\Omega) \le M_{h-1}$ (respectively $\mu_{h-1} < M_{h-1}$) where $M_{h-1} := N\frac{g-h}{g}$.

DEFINITION 4.13. A canonical curve is said to be *theta-stable* if

(i) it has a configuration of theta hyperplanes which does not depend on the smoothings to theta generic curves;
(ii) the configuration of (i) is GIT-stable.

LEMMA 4.14. *Let $W$ be a theta-stable canonical curve and $\mathcal{W} \to B$ be a smoothing of $W$ to theta generic curves. Let $C$ be a stable curve and $f : \mathcal{C} \to B$ be a smoothing of $C$ to theta generic curves. If $\theta_f(C)$ is GIT-stable and not conjugate to $\theta(W)$, then $\mathcal{C}$ is not the stable reduction of $\mathcal{W}$.*

PROOF. Assume the converse, that is (modulo a base change) $\mathcal{C} \to B$ is the stable reduction of $\mathcal{W} \to B$. Consider the canonical model of $\mathcal{C}^* \to B^*$ :

$$\varphi : \mathcal{C}^* \hookrightarrow \mathbb{P}(H^0(f_*\omega_f)^\vee).$$

The families $\mathcal{X} := Im(\varphi)$ and $\mathcal{W}^*$ are $B^*-$conjugate, i.e. there exists a morphism $\rho : B^* \to SL(g)$ such that for every $t \in B^*$ we have $X_t = W_t^{\rho(t)}$. Then $\theta(X_t) = \theta(W_t)^{\rho(t)}$. It follows that the configurations $\theta_f(C)$ and $\theta(W)$ are GIT-stable limits of conjugate families and thus they are conjugate yielding a contradiction. $\square$

REMARK 4.15. It may be useful to notice that the morphism $\varphi$ of the proof of the previous Theorem extends to all of $\mathcal{C}$ if and only if $C$ does not have a separating node (see [**Ct**]).

In the sequel we shall consider examples of theta-stable curves (cuspidal and tacnodal curves and some stable curves), computing the GIT-stability of configurations of theta hyperplanes.

THEOREM 4.16. *The following curves are theta-stable.*

(i) *A general irreducible theta generic canonical curve of genus $g$ whose singular points are only nodes or only cusps.*
(ii) *A general irreducible theta generic canonical curve of genus $g$ with $g \geq 26$ whose singular points are only tacnodes.*
(iii) *A general canonical stable curve of genus $g$ with two irreducible smooth components of the same genus.*

PROOF. Let $W$ be as in (i). It follows from Prop. 3.3, Th. 3.9 and Section 4.1.1 that $W$ has a configuration $\theta(W)$ of theta hyperplans independent of smoothings to theta generic curves.

Let us show the GIT-stability of $\theta(W)$. Since the locus of GIT-unstable points is closed, it suffices to show the GIT-stability in the case of a rational curve with $g$ cusps.

Let $W$ be rational. Since $W$ is general, if $\{c_1, \dots, c_h\}$ is a fixed subset of $h$ cusps of $W$, we have $\mu_{h-1} := \mu_{h-1}(\theta(W)) = \text{length}\, R_h$, where

$$R_h = \{H \in J(W) : \{c_1, \dots, c_h\} \subset H\}.$$

Then by Theorem 3.9 and Theorem 3.16 (notice that $N_{\tilde{g}} = N_0 = 0$ and $N_{\tilde{g}}^+ = N_0^+ = 1$)

$$\mu_{h-1} = \sum_{\substack{h \leq i \leq g-1 \\ g-i \equiv 1(2)}} \binom{g-h}{i-h} 3^i = \sum_{\substack{0 \leq i \leq g-1-h \\ g-i \equiv h+1(2)}} \binom{g-h}{i} 3^{i+h} =$$

$$= 3^h \left[ \sum_{\substack{0 \leq i \leq g-h \\ g-i-h \equiv 1(2)}} \binom{g-h}{i} 3^i \right] = 3^h 2^{g-h-1}(2^{g-h}-1).$$

The inequalities required by the GIT criterion are

$$\mu_{h-1} = 3^h\ 2^{g-h-1}\ (2^{g-h}-1) < M_{h-1} = 2^{g-1}\ (2^g-1)\ \frac{g-h}{g} \quad \forall\ 1 \leq h \leq g-1.$$

If we set $r := g-h$ with $1 \leq r \leq g-1$, these are

$$\frac{2^{r-1}(2^r-1)}{r\ 3^r} < \frac{2^{g-1}(2^g-1)}{g\ 3^g} \quad 1 \leq r \leq g-1. \tag{4.13}$$

Consider the function $F(x) := \frac{2^{x-1}(2^x-1)}{x\ 3^x}$. It is easy to check that it is strictly increasing for $x \geq 3$ and that $F(1) = F(2) < F(3)$. Thus $F(g) > F(r)$ for $g \geq 3$ and $1 \leq r \leq g-1$, which is (4.13).

Now we consider the tacnodal case (ii). First of all we need a combinatorial lemma. If $R$ is a zero dimensional scheme and $r \in R$ a point, we shall denote by $mult(r)$ its multiplicity.

LEMMA 4.17. *Let $R$ be a zero dimensional scheme of length $N$. Let $E_1, \ldots, E_m$ be subschemes of $R$. Let $M$ be a subset of indices $M \subset \{1, \ldots, m\}$. If we denote by $N(M) = length\ (\cap_{i \in M} E_i)$ and $N^s = \sum_{|M|=s} N(M)$ for $s = 1, \ldots, m$, then we have*

$$length\ (R - \cup_{i=1}^m\ E_i) = N - N^1 + N^2 - \cdots + (-1)^m\ N^m.$$

PROOF. Consider the right side term. Each $r \in R - \cup_{i=1}^m E_i$ contributes of $mult(r)$ to it while each $r$ which is in $h$ of the $E_i$ contributes of

$$mult(r) \left[ 1 - \binom{h}{1} + \binom{h}{2} \cdots + (-1)^h \binom{h}{h} \right] = mult(r)\ [(1-1)]^h = 0.$$

$\square$

Arguing as in the cuspidal case, it suffices to show the GIT-stability of the configuration of a tacnodal curve $W$ with a maximal set of tacnodes, that is $\frac{g}{2}$ tacnodes for $g$ even and $\frac{g-1}{2}$ tacnodes for $g$ odd. We denote by $\tau$ the number of the tacnodes of $W$.

Let $\{t_1, \ldots, t_a, t_{a+1}, \ldots, t_{a+b}\}$ be a fixed subset of tacnodes and $\{l_{a+1}, \ldots, l_{a+b}\}$ be the set of tacnodal tangents of $t_{a+1}, \ldots, t_{a+b}$. We denote by $\mu_{a,b} :=$ length $R_{a,b}$, where

$$R_{a,b} = \{H \in J(W) : \{t_1, \ldots, t_a\} \subset H, \{l_{a+1}, \ldots, l_{a+b}\} \subset H\}.$$

Since $W$ is general, we have that $\mu_{h-1} := \mu_{h-1}(\theta(W))$ is given by

$$\mu_{h-1} = \max_{a+2b=h} \mu_{a,b} \quad \forall\ 1 \leq h \leq g-1.$$

Consider

$$E_i = \{H \in J(W) : t_i \notin H\} \quad \forall\ 1 \leq i \leq a$$

and

$$E_i = \{H \in J(W) : l_i \notin H\} \quad \forall\ a+1 \leq i \leq a+b.$$

Then $R_{a,b} = J(W) - \bigcup_{1 \leq i \leq a+b} E_i$.

Let $M$ be a subset of indices $\{1,\ldots,a+b\} \supset M = \{i_1,\ldots,i_s\} = \{i_1,\ldots,i_r\} \cup \{i_{r+1},\ldots,i_{r+t}\}$, where $\{i_1,\ldots,i_r\} \subset \{1,\ldots,a\}$, $\{i_{r+1},\ldots,i_{r+t}\} \subset \{a+1,\ldots,a+b\}$ and $r+t=s$. Maintaining the notations of Lemma 4.17, we get $N(M) = \text{length}\,(\cap_{i\in M} E_i)$, where

$$\underset{i\in M}{\cap} E_i = \{H \in J(W) : t_{i_j} \notin H \quad \forall\, 1 \le j \le r \;,\; l_{i_j} \notin H \quad \forall\, r+1 \le j \le r+t\}.$$

The number $N(M)$ depends only on $r$ and $t$ and we denote it by $\tilde{\mu}_{r,t}$. It follows from Lemma 4.17 that

$$\mu_{a,b} = \text{length}\, R_{a,b} = N_g + \sum_{1\le s\le a+b} (-1)^s \,(\sum_{r+t=s} \tilde{\mu}_{r,t}) = N_g + \sum_{\substack{0\le r\le a\\ 0\le t\le b\\ (r,t)\neq(0,0)}} \binom{a}{r}\binom{b}{t}(-1)^{r+t}\tilde{\mu}_{r,t}.$$

Split $\tilde{\mu}_{r,t}$ into three terms (see Theorem 3.9 and Theorem 3.15)

$$x_1 = \sum_{\substack{j<i\le\tau-r\\ 0\le j\le\tau-s}} 6^j\, 4^{i-j}\, 2^{\tau-j-1} \binom{\tau-s}{j}\binom{\tau-r-j}{i-j}(N_{\tilde{g}} + N^+_{\tilde{g}});$$

$$x_2 = \sum_{\substack{0\le i\le\tau-s\\ \tau-i\equiv 1(2)}} 6^i\, 2^{\tau-i} \binom{\tau-s}{i} N^+_{\tilde{g}};$$

$$x_3 = \sum_{\substack{0\le i\le\tau-s\\ \tau-i\equiv 0(2)}} 6^i\, 2^{\tau-i} \binom{\tau-s}{i} N_{\tilde{g}}.$$

We have

$$\frac{x_1}{N_{\tilde{g}} + N^+_{\tilde{g}}} = 2^{\tau-1} \sum_{\substack{0<i\le\tau-r-j\\ 0\le j\le\tau-s}} \binom{\tau-s}{j}\binom{\tau-r-j}{i} 4^i\, 3^j =$$

$$= 2^{\tau-1} \sum_{0\le j\le\tau-s} 3^j \binom{\tau-s}{j} \,( \sum_{0\le i\le\tau-r-j} \binom{\tau-r-j}{i} 4^i - 1) =$$

$$= 2^{\tau-1} \sum_{0\le j\le\tau-s} 3^j \binom{\tau-s}{j} (5^{\tau-s-j}\, 5^{s-r} - 1) = 2^{\tau-1}\, (5^{s-r}\, 8^{\tau-s} - 4^{\tau-s}).$$

Moreover

$$\frac{x_2}{N^+_{\tilde{g}}} = 2^{\tau} \sum_{\substack{0\le i\le\tau-s\\ \tau-i\equiv 1(2)}} \binom{\tau-s}{i} 3^i.$$

Applying the formula

$$(\alpha \pm \beta)^{\tau-s} = \sum_{\tau-s-i\equiv 0(2)} \binom{\tau-s}{i} \alpha^i\, \beta^{\tau-s-i} \pm \sum_{\tau-s-i\equiv 1(2)} \binom{\tau-s}{i} \alpha^i\, \beta^{\tau-s-i},$$

we obtain

$$\sum_{\substack{0\le i\le\tau-s\\ \tau-i\equiv 1(2)}} \binom{\tau-s}{i} 3^i = \begin{cases} 2^{\tau-s-1}(2^{\tau-s}-1) & \text{if } \; s\equiv 0\;(2)\\[2ex] 2^{\tau-s-1}(2^{\tau-s}+1) & \text{if } \; s\equiv 1\;(2)\end{cases}.$$

It follows that

$$x_2 = 2^{2\tau-s-1}\, (2^{\tau-s} - (-1)^s)\, N^+_{\tilde{g}}.$$

Similarly we get

$$x_3 = \sum_{\substack{0\le i\le\tau-s\\ \tau-i\equiv 0(2)}} 3^i\, 2^{\tau} \binom{\tau-s}{i} N_{\tilde{g}} = 2^{2\tau-s-1}\, (2^{\tau-s} + (-1)^s)\, N_{\tilde{g}}.$$

We collect the three terms into

$$\tilde{\mu}_{r,t} = (2^{4\tau-3s-1}\,5^{s-r} - 2^{3\tau-2s-1} + 2^{3\tau-2s-1})\,(N_{\tilde{g}} + N_{\tilde{g}}^{+}) + (-1)^s\,2^{2\tau-s-1}\,(N_{\tilde{g}}^{+} - N_{\tilde{g}}) =$$
$$= 2^{2g-4\tau}\,(2^{4\tau-3s-1}\,5^{s-r}) - (-1)^s\,2^{g-\tau}\,2^{2\tau-s-1} = 2^{2g-3s-1}\,5^{s-r} - (-1)^s\,2^{g-s-1}.$$

So we can compute $\mu_{a,b}$

$$\mu_{a,b} = N_g + \sum_{\substack{t=0\\1\le r\le a}} (-1)^r \binom{a}{r}\tilde{\mu}_{r,0} + \sum_{\substack{0\le r\le a\\1\le t\le b}} \binom{a}{r}\binom{b}{t}(-1)^{r+t}\tilde{\mu}_{r,t}.$$

The first two terms are (in this case $s = r$)

$$N_g + \sum_{0\neq r\ even}\binom{a}{r}\tilde{\mu}_{r,0} - \sum_{r\ odd}\binom{a}{r}\tilde{\mu}_{r,0} =$$
$$= N_g + \sum_{0\neq r\ even}\binom{a}{r}(2^{2g-3r-1} - 2^{g-r-1}) - \sum_{r\ odd}\binom{a}{r}(2^{2g-3r-1} + 2^{g-r-1}) =$$
$$= N_g + 2^{2g-3a-1}\sum_{1\le r\le a}\binom{a}{r}(-1)^r\,2^{3a-3r} - 2^{g-a-1}\sum_{1\le r\le a}2^{a-r}\binom{a}{r} =$$
$$= N_g + 2^{2g-3a-1}\Big(\sum_{0\le r\le a}\binom{a}{r}(-1)^r\,8^{a-r} - 8^a\Big) - 2^{g-r-1}\Big(\sum_{0\le r\le a}2^{a-r}\binom{a}{r} - 2^a\Big) =$$
$$= 2^{g-1}\,2^g - 1) + 2^{2g-3a-1}\,(7^a - 8^a) - 2^{g-a-1}\,(3^a - 2^a) = 7^a\,2^{2g-3a-1} - 3^a\,2^{g-a-1}.$$

The last term in $\mu_{a,b}$ (recall that $t = s - r$) is

$$\sum_{0\le r\le a}(-1)^r\binom{a}{r}\sum_{1\le t\le b}(-1)^t\binom{b}{t}(5^t\,2^{2g-3s-1} - (-1)^s\,2^{g-s-1}) =$$
$$= \sum_{0\le r\le a}(-1)^r\binom{a}{r}2^{2g-3r-3b-1}\sum_{1\le t\le b}(-1)^t\binom{b}{t}5^t\,8^{b-t} - \sum_{0\le r\le a}(-1)^{2s}\binom{a}{r}2^{g-r-b-1}\sum_{1\le t\le b}\binom{b}{t}2^{b-t} =$$
$$= \sum_{0\le r\le a}(-1)^r\binom{a}{r}2^{2g-3r-3b-1}\Big(\sum_{0\le t\le b}(-1)^t\binom{b}{t}5^t\,8^{b-t} - 8^b\Big) - \sum_{0\le r\le a}\binom{a}{r}2^{g-r-b-1}\Big(\sum_{0\le t\le b}\binom{b}{t}2^{b-t} - 2^b\Big) =$$
$$= \sum_{0\le r\le a}(-1)^r\binom{a}{r}2^{2g-3r-3b-1}(3^b - 8^b) - \sum_{0\le r\le a}\binom{a}{r}2^{g-r-b-1}\,(3^b - 2^b) =$$
$$= 2^{2g-3b-3a-1}(3^b - 8^b)\sum_{0\le r\le a}(-1)^r\binom{a}{r}8^{a-r} - 2^{g-b-a-1}\,(3^b - 2^b)\sum_{0\le r\le a}\binom{a}{r}2^{a-r} =$$
$$= 7^a\,2^{2g-3a-3b-1}\,(3^b - 8^b) - 3^a\,2^{g-b-a-1}\,(3^b - 2^b).$$

Thus we obtain

$$\mu_{a,b} = 7^a\,3^b\,2^{2g-3a-3b-1} - 3^a\,3^b\,2^{g-b-a-1}.$$

The stability condition is (for all $a, b$ such that $a + 2b = h$ and $1 \le h \le g - 1$)

$$\mu_{a,b} = 7^a\,3^b\,2^{2g-3a-3b-1} - 3^a\,3^b\,2^{g-b-a-1} < (2^{2g-1} - 2^{g-1})\frac{g-a-2b}{g}.$$

Since

$$\frac{3^{a+b}}{2^{a+b}} > \frac{g-a-2b}{g}$$

it suffices to show that

$$\frac{7^a}{8^a}\frac{3^b}{8^b} < \frac{g-a-2b}{g}$$

or also (recall that $a + 2b = h$)

$$g\frac{7^{a+2b}}{8^{a+2b}} + a + 2b = g\frac{7^h}{8^h} + h < g.$$

If we set $r = g - h$ for $1 \leq r \leq g-1$, the last inequality is

$$\frac{8^r}{r\ 7^r} < \frac{8^g}{g\ 7^g} \quad \forall\, 1 \leq r \leq g-1.$$

By the given hypothesis $g \geq 26$. Consider the function $F := \frac{8^x}{x\ 7^x}$. It is strictly increasing for $x \geq 8$, hence $F(g) > F(r)$ for $8 \leq r \leq g-1$. Being $F(26) > F(1)$ and $F$ decreasing for $x \leq 7$, we get $F(g) \geq F(26) > F(1) \geq F(r)$ for $1 \leq r \leq 7$.

We check by inspection the result for $g \leq 25$. Below we sum-up the maximal number of tacnodes that a curve with a GIT-stable configuration may have.

| (g,$\tau$) | h | (a,b) | | (g,$\tau$) | h | (a,b) | |
|---|---|---|---|---|---|---|---|
| (3,1) | 1 | (1,0) | unstable | (15,7) | 13 | (13,0) | unstable |
| (4,1) | 1 | (1,0) | unstable | (16,8) | 14 | (14,0) | unstable |
| (5,1) | 1 | (1,0) | unstable | (17,8) | 15 | (15,0) | unstable |
| (6,1) | 1 | (1,0) | unstable | (18,9) | 16 | (16,0) | unstable |
| (7,1) | 1 | (1,0) | unstable | (19,9) | - | - | stable |
| (8,2) | 2 | (2,0) | unstable | (20,10) | 19 | (19,0) | unstable |
| (9,3) | 4 | (4,0) | unstable | (21,10) | - | - | stable |
| (10,3) | 5 | (5,0) | unstable | (22,11) | 21 | (10,0) | unstable |
| (11,4) | 7 | (7,0) | unstable | (23,11) | - | - | stable |
| (12,5) | 8 | (8,0) | unstable | (24,12) | 23 | (23,0) | unstable |
| (13,6) | 10 | (10,0) | unstable | (25,12) | - | - | stable |
| (14,6) | 11 | (11,0) | unstable | ($g \geq 26, \tau$) | - | - | stable |

Let $C$ be a curve as in (iii). As a consequence of Theorem 4.3 we have seen that a general stable projective $C$ curve with two components (of the same genus) has a configuration of theta hypeperplanes $\theta(C)$ which does not depend on smoothings to theta generic curves. Below we shall check its GIT-stability. Using the same techniques we can see that the GIT-stability holds also when the components are nodal with the same genus.

Let $g_1$ be the genus of the irreducible components of $C$ and $\delta$ be the number of its nodes so that $g = 2g_1 + \delta - 1$. If $\Lambda$ is a linear space, we shall denote by $\mu_\Lambda$ the length of the scheme of theta hyperplanes of $C$ containing $\Lambda$.

First of all assume $C$ not to be split (that is $g_1 \neq 0$). In this case a maximal sets of nodes of $C$ in general position is given by any set of $\delta - 1$ nodes.

Since $C$ is general it is not restrictive to check the GIT-stability criterion only for the linear spaces spanned either by the irreducible components of $C$ or by sets of nodes of $C$.

Let $\Lambda_1$ be the linear space spanned by an irreducible component $C_1$ of $C$ (recall that we are in the case $g_1 \neq 0$). The theta hyperplanes containing $\Lambda_1$ correspond to the odd stable spin curves supported on the blow-up of $C$ at all of its nodes and obtained by gluing an even theta characteristic on $C_1$. Thus it is easy to compute $\mu_{\Lambda_1}$ by looking at this subscheme of $S_C^-$ and we have to check

$$\mu_{\Lambda_1} = 2^{\delta-1} N_{g_1} N_{g_1}^{+} = 2^{g-2}(2^{2g_1} - 1) < M_{g_1+\delta-2} = 2^{g-1}(2^g - 1)g_1/g,$$

which is true since the function $F := (2^x - 1)/x$ is strictly increasing for $x \geq 1$.

Let $\Lambda_{h-1}$ be spanned by $h$ nodes for $1 \leq h \leq \delta - 2$. Then $\mu_{h-1} = \mu_{\Lambda_{h-1}}$ which is given by $\mu_{h-1} = 2\mu_{\Lambda_1} + \mu$, where $\mu$ is the length of the subscheme of theta hyperplanes containing $\Lambda_h$ and missing the linear spans of the components of $C$.

Let us compute $\mu$. Let $L$ be the union of the edges of the dual graph $\Gamma_C$ of $C$ corresponding to the nodes spanning $\Lambda_h$. Then $\mu$ is the length of the subscheme of $S_C^-$ of the odd stable spin curves $(X, G)$ such that $\Sigma_X = L \cup \Sigma'_X$, where $\Sigma'_X$ is a subgraph of $\overline{\Gamma_C - L}$ and $\Sigma_X \neq \Gamma_C$. Notice that there are $2^{b_1(\overline{\Gamma_C - L})} - 1$ subgraphs $\Sigma'_X$ of $\overline{\Gamma_C - L}$ such that $\Sigma'_X \cup L$ is equal to $\Sigma_X$ for an odd stable spin curve $(X, G)$ of $C$ and for each one of these there are $2^{b_1(\Gamma_C)-1}$ odd stable spin curves (with multiplicity). Since $b_1(\overline{\Gamma_C - L}) = \delta - h - 1$ and $b_1(\Gamma_C) = \delta - 1$, we have $\mu = 2^{\delta-2}(2^{\delta-h-1} - 1)$ and hence we have to check

$$\mu_{h-1} = 2\mu_{\Lambda_1} + \mu = 2^\delta(2^{2g_1} - 1) + 2^{\delta-2}(2^{\delta-h-1} - 1) < M_{h-1} = 2^{g-1}(2^g - 1)\frac{g-h}{g}.$$

If $2g_1 \geq \delta - h - 1$, it suffices to check that

$$5g(2^{2g_1} - 1) < 2^{2g_1}(2^g - 1)(g - h)$$

which is true since $5g < (g - h)(2^g - 1)$.

If $2g_1 < \delta - h - 1$, it suffices to check that

$$5g(2^{\delta-h-1} - 1) < 2^{2g_1}(2^g - 1)(g - h).$$

Since $5(2^{\delta-h-1} - 1) < 4(2^{g-h} - 1)$ and $2g_1 \geq 2$, it suffices $g(2^{g-h} - 1) < (g - h)(2^g - 1)$ which is given using the function $F$.

Let $\Lambda_3$ be spanned by the whole set of nodes of $C$. We have to check the inequality

$$\mu_{\Lambda_3} = 2\mu_{\Lambda_1} = 2^{g-1}(2^{2g_1} - 1) < M_{\delta-2} = 2^{g-1}(2^g - 1)2g_1/g,$$

which is given again using the function $F$.

Now assume that $C$ is a split curve. It follows from [**C2**, Proposition 2, Lemma 3] that for $h = 1, \ldots, g-1$

$$\mu_{h-1} = \sum_{\substack{h \leq i \leq g-1 \\ g-i \equiv 1\ (2)}} \binom{g+1-h}{i-h} 2^{g-i-1}\, 2^i = 2^{g-1}\left[\sum_{\substack{0 \leq i \leq g+1-h \\ g-i-h \equiv 1\ (2)}} \binom{g+1-h}{i} - 1\right] = 2^{g-1}(2^{g-h} - 1)$$

and we have to check

$$\mu_{h-1} = 2^{g-1}(2^{g-h} - 1) \leq M_{h-1} = 2^{g-1}(2^g - 1)\frac{g-h}{g}$$

which is given using the function $F$ □

Combining Lemma 4.14 and Theorem 4.16 we show the typical result one can obtain. Similar results can be found analyzing the GIT-stability of configurations of theta hyperplanes of curves with other types of singularities.

Corollary 4.18. *Let $W$ be a general irreducible theta generic canonical curve of genus $g$ whose singular point are either only tacnodes (for $g \geq 26$) or only cusps.*

*Then a general canonical stable curve $C$ either irreducible or with two irreducible smooth components of the same genus does not appear as central fiber of the stable reduction of any smoothing of $W$ to theta generic curves.*

Proof. We have only to check that the configurations of theta hyperplanes of $W$ and of the stable curve are not conjugate.

If $W$ is cuspidal, then $3^k$ appears as multiplicity of a theta hyperplane of type $k$ for some $k \geq 1$. If $W$ is tacnodal, then for $1 \leq i \leq 2$, there exists a theta hyperplane of type $(i, i)$ appearing with multiplicity $6^i$ (see Th. 3.9 and Th. 3.15).

If $C$ is irreducible the multiplicity of a theta hyperplane of $C$ is $2^b$ for a non negative integer $b$. This multiplicity is never equal to $3^k, 6, 36$.

Let $C$ be reducible. Notice that $C$ has at least 3 nodes because such curve, being projective, has a very ample dualizing sheaf (see [**Ct**]). Therefore Section 4.1.1 implies that the multiplicity of a theta hyperplane of $C$ is either $2^b$ for a non negative integer $b$ or is $2^b N_{g_1}^+$ for $b \geq 2$, where $N_{g_1}^+$ is the number of the even theta characteristics of a component of $C$ of genus $g_1$. It is easy to see that this multiplicity is never equal to $3^k, 6, 36$. $\square$

# Bibliography


[AK] A. Altman and S. Kleiman, *Compactifying the Picard scheme*, Adv. Math. **35** (1980), 50–112.

[ACGH1] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris, J. *Geometry of algebraic curves.* Vol. I. Fundamental Principles of Mathematical Sciences. vol 267, Springer-Verlag, New York, 1985. 4

[ACGH2] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris. *Geometry of algebraic curves* Volume II, in preparation.

[B] F. Bardelli. *Lectures on stable curves.* Lectures on Riemann surfaces (Trieste, 1987). World Sci. Publishing, 1989, 648-704.

[BPV] W. Barth, C. Peters and A. Van de Ven. *Compact complex surfaces* Results in Mathematics and Related Areas, vol. 4, Springer-Verlag, Berlin 1984.

[C1] L. Caporaso. *A compactification of the universal Picard variety over the moduli space of stable curves* J. Amer. Math. Soc. vol 7 1994 num.3, 589-660.

[C2] L. Caporaso *On modular properties of odd theta-characteristics.* Advances in algebraic geometry motivated by physics (Lowell, MA, 2000), Contemp. Math. vol. 276, 101-114, Amer. Math. Soc. 2001.

[CC] L. Caporaso and C. Casagrande. *Combinatorial properties of stable spin curves* Communications in Algebra 2003 vol. 31 n. 8, 3653–3672.

[CCC] L. Caporaso, C. Casagrande and M. Cornalba *Moduli of roots of line bundles on curves* Transaction of the Amer. Math. Soc., to appear.

[CS2] L. Caporaso and E. Sernesi, *Characterizing curves by their odd theta-characteristics*, J. Reine Angew. Math. vol. 562, 2003, 101–135.

[CS1] L. Caporaso and E. Sernesi *Recovering plane curves from their bitangents.* J. Algebraic Geom. vol. 12, 2003, n. 2, 225–244.

[DM] P. Deligne and D. Mumford *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. 1969, n. 36,75-109.

[Ct] F. Catanese. *Pluricanonical-Gorenstein-curves.* Enumerative geometry and classical algebraic geometry (Nice, 1981). Progr. Math. vol. 24, 51-95, Birkhäuser Boston, MA, 1982.

[Co] M. Cornalba. *Moduli of curves and theta-characteristics.* Lectures on Riemann surfaces (Trieste, 1987), 560-589 World Sci. Publishing Teaneck, NJ 1989.

[D] R. Diestel. *Graph theory.* Graduate Text in Mathematics vol. 173, Springer-Verlag, New York, 1997.

[E] D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry* Graduate Texts in Mathematics, 150. Springer-Verlag New-Jork,1995.

[Es] E. Esteves, *Compactifying the relative Jacobian over families of reduced curves*, Trans. Amer. Math. Soc. **353** (2001), 3045–3095.

[F] C. Fontanari *On the geometry of the compactification of the universal Picard variety.* To appear in: Rendiconti dei Lincei.

[G1] D. Gieseker. *Geometric invariant theory and applications to moduli problems.* Invariant theory (Montecatini, 1982) Lecture Notes in Math. Vol. 996, 45–73 Springer, Berlin 1983.

[G2] D. Gieseker. *Lectures on moduli of curves*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics. Vol. 69. Published for the Tata Institute of Fundamental Research, Bombay 1982.

[GH] P. Griffiths and J. Harris. *Principles of algebraic geometry* Wiley Classics Library. John Wiley & Sons Inc. New York, 1994.

[Ha1] J. Harris. *Algebraic geometry: a first course*, Graduate Texts in Mathematics, vol 133. Springer-Verlag, New York 1995.

[Ha2] J. Harris. *Theta-characteristics on algebraic curves* Trans. Amer. Math. Soc. vol. 271, 1982, n. 2, 611–638.

[HM] J. Harris I. Morrison. *Moduli of Curves* Graduate Texts in Matehematics. vol. 187, Springer-Verlag, New-Jork.

[Ht1] R. Hartshorne. *Algebraic geometry*, Graduate Text in Mathematics, No. 52, Springer-Verlag, New Jork.

[Ht2] R. Hartshorne. *Residues and duality.* Lecture Notes in Mathematics, Springer-Verlag, 1966, n. 20.

[Hs1] B. Hassett. *Local stable reduction of plane curve singularities.* Journal für die Reine und Angewandte Mathematik. vol. 520, 2000, 169–194.

[Hs2] B. Hassett. *Stable log surfaces and limits of quartic plane curves* Manuscripta Mathematica. vol. 100, 1999, n. 4, 469–487.

[I] J. Igusa, *Fiber systems of Jacobian varieties*, Amer. J. Math. **78** (1956), 171–199.

[L] H. B. Laufer, *On minimally elliptic singularities*, Amer. J. Math. **99** n. 6 (1977), 1257–1295.

[M] L. Mainò. *Moduli of enriched stable curves* Harvard Ph.D. Thesis.

[Mu] D. Mumford. *Stability of projective varieties* Enseignement Math. (2) 1977, vol.23 n. 1-2, 39–110.

[MFK] D. Mumford, D., J. Fogarty and F. Kirwan. *Geometric invariant theory* E.M.G. 34. Third edition, Springer, New York, 1994.

[S] E. Sernesi. *Deformation of schemes*, Preliminary version april 2004, in preparation.

[N] P. Newstead. *Introduction to moduli problems and orbit spaces*, Tata Institute of Fundamental Research Lectures on Mathematics and Phisics, vol. 51, Tata Institute of Fundamental Research, Bombay, 1978.

[OS] T. Oda and C. S. Seshadri, *Compactifications of the generalized Jacobian variety*, Trans. Amer. Math. Soc. **253** (1979), 1–90.

[P] R. Pandharipande, *A compactification over* $\overline{M}_g$ *of the universal moduli space of slope-semi-stable vector bundles*, J. Amer. Math. Soc. **9** (1996), 425–471.

[W] P. Wagreich *Elliptic singularities of surfaces.* American Journal of Mathematics, 1970 vol. 92, 419-454