Tesi di Dottorato

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Nonlinear parabolic equations with general measure data

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NONLINEAR PARABOLIC EQUATIONS WITH GENERAL MEASURE DATA

Tesi di Dottorato

di

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Introduction

This thesis is devoted to the study of a class of nonlinear parabolic initial boundary value problems with measure data, in bounded domains. If $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, is a bounded open set, let $A(u) = -\mathrm{div}(a(t,x,\nabla u))$ be an operator acting from the space $L^p(0,T;W_0^{1,p}(\Omega))$ into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$, $p > 1$, and satisfying the Leray-Lions assumptions (see (1.3.5)–(1.3.7) below) which imply appropriate coercivity and monotonicity properties. We study, under suitable hypotheses, the existence and the asymptotic behavior of solutions of initial boundary problems of the type

$$
\begin{cases} u_t + A(u) = \mu & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{1}
$$

where μ is a general bounded Radon measure on $Q = (0,T)\times\Omega$, and $u_0 \in L^1(\Omega)$, $T > 0$. To fix the ideas, one can consider, as a special example of (1), the p-Laplace initial boundary value problem:

$$
\begin{cases} u_t - \mathrm{div}(|\nabla u|^{p-2}\nabla u) = \mu & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{2}
$$

If both A and μ do not depend on time, then A reduces to an elliptic pseudomonotone operator satisfying the classical Leray-Lions assumptions acting from $W_0^{1,p}(\Omega)$ into its dual space $W^{-1,p'}(\Omega)$; in this case we will investigate the asymptotic behavior of the solutions of problem (1) as t goes to infinity, proving that it converges, in a suitable way, to the stationary solution of the same problem, that is to the solution v of the elliptic boundary value problem

$$
\begin{cases} -\mathrm{div}(a(x,\nabla v)) = \mu & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}
$$

The difficulties in the study of such problems concern the possibly very singular right hand side that forces the choice of a suitable formulation that ensures both existence and uniqueness of the solution.

If $\mu \in W^{-1,p'}(\Omega)$, a unique solution of problem (3) exists in a variational sense (see [**LL**]); on the other hand, if μ is a bounded Radon measure on Ω the question of existence and uniqueness of solution for problem (3) was extensively studied, from the work of G. Stampacchia in [**S**], for linear operators, and from [**BG**] in the nonlinear case; the literature in this topic is wide and a full list of references can be found in [**B6**], [**BGO**], and [**DMOP**]; indeed, the introduction of the notions of *duality*, *entropy*, and *renormalized* solution allowed the authors to prove existence of solution and, in most case, uniqueness.

On the other hand, in the parabolic case a similar approach was followed; if $\mu \in L^{p'}(Q)$ and $u_0 \in L^2(\Omega)$, (1) has a unique solution in a suitable energy space and in $C(0,T;L^2(\Omega))$ (see [**L**]). Moreover, if μ is a bounded Radon measure on Q that does not charge the sets of zero parabolic p-capacity (see Definition 1.36 below), the so-called *soft measures* or *absolutely continuous measures* with respect to the parabolic p-capacity, the notion of both entropy and renormalized solution for problem (1) can be given to ensure existence and uniqueness of solution (see [**Pr2**], [**BM**], [**Po1**] and [**DPP**] for the general case); in [**DP**] the authors proved that these two notions of solution actually turn out to coincide. Note that these solutions, as well as in the elliptic case, do not belong to the energy space while, as a key property in this framework, their truncations $T_k(u)$ (where $T_k(s) = \max(-k, \min(k,s))$) do; that is $T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega))$.

If μ is a general, possibly singular, Radon measure on Q, then a distributional solution of (1) was proved to exist in [**BDGO**]; unfortunately, as in the elliptic case, this notion is too weak to ensure uniqueness as simple examples show; we will deal with this question in Chapter 4.

In Chapter 1 we first recall some basic tools and preliminary results concerning PDE theory and, in particular, we will state a generalized integration by parts formula, a Simon type compactness result and a useful trace result contained in [**Po1**]; moreover, we introduce the notation we will use throughout the thesis. Then, we give a brief review of what has been done (up to now) in the theory of both elliptic and parabolic differential problems with measure data.

Chapter 2 is devoted to the study of the asymptotic behavior, as T tends to infinity, of the entropy solution of problem

$$
\begin{cases} u_t + A(u) = \mu & \text{in } (0,T)\times\Omega \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(0,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{4}
$$

where A is a pseudomonotone operator of Leray-Lions type, $u_0 \in L^1(\Omega)$ is a nonnegative function, and μ is a nonnegative bounded Radon measure on Q not charging the sets of parabolic zero p-capacity (we will denote by $M_0(Q)$ the space of such measures); here both a and μ are supposed to be independent on time. Note that the uniqueness of such a solution allow us to deal with a unique function well defined for any $T > 0$. We first characterize the measures we consider; indeed, it is easy to see that, if $\mu \in M_0(Q)$ does not depend on time, the $\mu = f - \operatorname{div}(G)$ with $f \in L^1(\Omega)$ and $G \in (L^{p'}(\Omega))^N$ that is, thank to a result of [**BGO**], $\mu \in M_0(\Omega)$, the space of all bounded Radon measures on Ω that do not charge the sets of zero elliptic p-capacity. The main result of this chapter is the proof that, under suitable hypotheses, the entropy solution $u(t,x)$ of problem (4) converges to the stationary solution v of problem (3), at least in $L^1(\Omega)$. The proof of this fact is achieved in several steps: we first prove it in the easier case when $u_0 = 0$ with the use of a comparison lemma which shows, in particular, that $u(t,x)$ is monotone nondecreasing in time. To deal with the general case, we prove an improved comparison result, generalizing a result of [**Pa**] which dealt with elliptic problems, between entropy sub and super solutions; this fact, together with standard compactness arguments, allows us to conclude. Note that, to treat the general case, we also prove a technical lemma that involves a homogenization argument and, in particular, a nonlinear G-compactness theorem contained in [**CDD**]. In the last section we show how, in the linear case, the same result can be obtained for a general, possibly singular, datum μ using the framework of duality solutions that apply to the parabolic setting as well as to the elliptic one. All these results are contained in [**Pe1**].

In Chapter 3, whose main issues are contained in a joint work with Tommaso Leonori[1] (see [**LP**]), we give the same type of result for a rather different class of operators; in fact, we study a *quasilinear problem* whose model is

$$\begin{cases} u_t - \Delta u + g(u)|\nabla u|^2 = f & \text{in } (0,T)\times\Omega,\\ u(0,x) = u_0(x) & \text{in } \Omega,\\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{5}$$

where $u_0 \in L^1(\Omega)$ is nonnegative, $g : \mathbb{R} \to \mathbb{R}$ is a real function in $C^1(\mathbb{R})$ such that

$$g(s)s \geq 0,\ \forall s \in \mathbb{R}, \tag{6}$$

$$g'(s) > 0, \forall s \in \mathbb{R}, \tag{7}$$

and $f(x) \in L^1(Q)$ is a nonnegative function independent on time; in the literature, the absorption term $|\nabla u|^2$ is said to have a natural growth since it forces, in some sense, the solution to belong to the energy space $L^2(0,T;H^1_0(\Omega))$. These kind of equations, that naturally arise from a class of variational problems, have been largely studied recently; the assumptions on the nonlinearity g, namely (6) and (7), are rather standard since they ensure, for instance, the uniqueness of the solution.

[1]Department of Mathematics, University of Tor Vergata, Rome.

Actually, the asymptotic result is obtained via a suitable use of a comparison result contained in [**BaM**], and then applying arguments similar to those of Chapter 2. We first prove the result in the homogeneous case (proving that the solution is monotone nondecreasing in t), then for *special initial data* (that is, for instance, taking u_0 as the solution of the stationary problem), and finally, in the general case, by a suitable approximation argument.

As we said before, to apply arguments of Chapter 2 and 3 we need to impose a restriction on the regularity of the datum μ which, essentially, has not to charge the sets of zero parabolic p-capacity; if μ is a general, possibly singular, bounded Radon measure (we say that $\mu \in M(Q)$) we need to prove that a solution exists in a sense which should ensure uniqueness; this machinery was developed, in the elliptic case, with the use of the notion of renormalized solution extending the one of entropy solution.

Chapter 4 of this thesis is devoted to the proof of the existence of a renormalized solution for problem (1) and to the study of its main properties.

We first introduce our main assumptions on the operator a and on the data, recalling a fundamental decomposition theorem for general measures in $M(Q)$ proved in [**DPP**]; that is, if $\mu \in M(Q)$ then we have

$$\mu = f - \operatorname{div}(G) + g_t + \mu_s, \tag{8}$$

in the sense of distributions, for some $f \in L^1(Q)$, $G \in (L^{p'}(Q))^N$, $g \in L^p(0,T;W_0^{1,p}(\Omega))$, and $\mu_s \perp p$-capacity ; note that the decomposition of the absolutely continuous part of μ is not uniquely determined. We denote by μ_0 the absolutely continuous part of the measure μ with respect of the p-capacity, that is, $\mu_0 = f - \operatorname{div}(G) + g_t$ using the notation of (8).

In Section 4.2 we give the definition of a renormalized solution for problem (1) and we prove a basic estimate enjoyed by any of these solutions (see Proposition 4.3). If g is as in (8), for the sake of simplicity, we will often refer to the renormalized solution u as well as its regular translation $v = u - g$.

In our setting, let $\mu \in M(Q)$ and $u_0 \in L^1(\Omega)$. A measurable function u is a *renormalized solution* of problem (1) if, there exists a decomposition (f, G, g) of μ_0 such that $v \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$ for every $q < p - \frac{N}{N+1}$, $T_k(v) \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$ for every $k > 0$, and for every $S \in W^{2,\infty}(\mathbb{R})$ ($S(0) = 0$) such that S' has compact support on $\mathbb{R}$, we have

$$\begin{aligned} &-\int_\Omega S(u_0)\varphi(0)\,dx - \int_0^T \langle \varphi_t, S(v)\rangle\,dt + \int_Q S'(v)a(t,x,\nabla u)\cdot\nabla\varphi\,dxdt \\ &\quad + \int_Q S''(v)a(t,x,\nabla u)\cdot\nabla v\,\varphi\,dxdt = \int_Q S'(v)\varphi\,d\hat{\mu}_0, \end{aligned} \tag{9}$$

for every $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, $\varphi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, with $\varphi(T,x) = 0$, such that $S'(v)\varphi \in L^p(0,T;W_0^{1,p}(\Omega))$. Moreover, for every $\psi \in C(\overline{Q})$ we have

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{n\leq v<2n\}} a(t,x,\nabla u)\cdot\nabla v\ \psi\ dxdt = \int_Q \psi\ d\mu_s^+, \tag{10}$$

and

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{-2n<v\leq -n\}} a(t,x,\nabla u)\cdot\nabla v\ \psi\ dxdt = \int_Q \psi\ d\mu_s^-, \tag{11}$$

where μ_s^+ and μ_s^- are respectively the positive and the negative part of the singular part μ_s of μ.

The feature of the definition of renormalized solution relies on the reconstruction properties (10) and (11); they show that, in some sense, the energy of a renormalized solution u, where it is *large*, goes to reconstruct the singular part of the measure μ.

The basic estimate enjoyed by any renormalized solution is

$$\int_Q |\nabla T_k(v)|^p\ dxdt \leq C(k+1), \tag{12}$$

for any $k > 0$, where C is a positive constant.

Section 4.3 is devoted to the proof that any renormalized solution, actually its regular translation v, admits a cap_p-quasi continuous representative (i.e. continuous everywhere but on a set of arbitrary small p-capacity) defined cap_p-almost everywhere (Theorem 4.11); to this aim, we use an estimate which allows us to conclude that any function in the space

$$\{u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q); u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)\},$$

admits a cap_p-quasi continuous representative defined cap_p-almost everywhere; this fact, by proving that v is finite cap_p-almost everywhere, will prove Theorem 4.11. Notice that this interesting property justifies, in some sense, the fact that we use $S(v)$ against μ_0 (the absolutely continuous part of μ) in the renormalized formulation (see Corollary 4.9), where S is a real bounded Lipschitz continuous function on $\mathbb{R}$. In Section 4.4 we introduce the setting of the approximation argument which we shall use to prove existence of a renormalized solution; in particular, we first discuss how the definition of renormalized solution does not depend on the decomposition of μ_0 we approximate, and then we state a standard compactness result (Proposition 4.15) that will be central in the rest of the chapter.

A key role to prove existence is the proof of Theorem 4.20 to which Section 4.5 is devoted; that is, the strong convergence of the truncations $T_k(v^\varepsilon)$ of the approximating sequence. Here, following an idea of [**DMOP**] we construct suitable cut-off functions

ψ_δ that allow us to work , separately, *far from* and *near to* the set E where the singular part of the measure μ is concentrated.

In Section 4.6 we prove that a renormalized solution actually exists using the result of Theorem 4.20; to do that we make use again of the cut-off functions ψ_δ to split the proof into its easier part (far from E) and its harder part (near to E).

Finally, in Section 4.7 we try to emphasize the fact that, as in the elliptic case, the notion of renormalized solution should be the right one to ensure uniqueness since we prove that, in the linear case, this fact turns out to be true; indeed, if A is linear, we show that the renormalized solution u turns out to be a solution in a duality sense as so it is unique. In the last part of this section we also mention, as an interesting application of properties (10) and (11), that an *Inverse Maximum Principle* (we use the terminology introduced in [**DuP**] in the case of the Laplace operator) apply to general parabolic pseudomonotone operators with singular measure data; that is, for instance: *nonnegative renormalized solutions arise from nonnegative singular measure data.* All the results of Chapter 4 are contained in [**Pe2**].

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CHAPTER 1

A review on some preliminary tools and basic results

1.1. Notations, functional spaces and basic tools

We set by $\mathbb{R}^N$ the N-Euclidian space (simply $\mathbb{R}$ if $N = 1$) on which the standard Lebesgue measure is considered, as defined on the σ-algebra of Lebesgue measurable sets. The scalar product between two vectors a, b in $\mathbb{R}^N$ will be denoted by $a \cdot b$ or simply ab in most cases. Given a bounded open set Ω of $\mathbb{R}^N$, whose boundary will be denoted by $\partial\Omega$, and given a positive T, we shall consider the cylinder $Q_T = (0, T) \times \Omega$ (or simply Q where there is no possibility of confusion), setting by $C_0(Q)$ and $C_0^\infty(Q)$, the space of continuous, respectively C^∞, functions with compact support in Ω, while $C(\overline{\Omega})$ will denote functions that are continuous in the whole closed set $\overline{\Omega}$; moreover we will indicate by $C_0^\infty([0, T] \times \Omega)$ (resp. $C_0^\infty([0, T) \times \Omega)$) the set of all C^∞ functions with compact support on the set $[0, T] \times \Omega$) (resp. on $[0, T) \times \Omega$).

For the sake of simplicity here we will denote by D any bounded open subset of $\mathbb{R}^N$ (that in the rest of this thesis would indicate Ω, $(0, T) \times \Omega$, Q, etc...). We will deal with the space $M(D)$ of Radon measures μ on D that, by means of *Riesz's representation theorem*, turns out to coincide with the dual space of $C_0(D)$ with the topology of locally uniform convergence; we shall identify the element μ in $M(D)$ with the real valued additive set function associated, which is defined on the σ-algebra of Borel subsets of D and is finite on compact subsets. Thus with μ^+ and μ^- we mean, respectively, the positive and the negative variation of the Hahn decomposition of μ, that is $\mu = \mu^+ - \mu^-$, while the total variation of μ will be denoted by $|\mu| = \mu^+ + \mu^-$. Since we will always deal with the subset of $M(D)$ of the measures with bounded total variation on D, to simplify the notation we will denote also by $M(D)$ this subset. The restiction of a measure μ on a subset E is denoted by $\mu \llcorner E$ and is defined as follows:

$$\mu \llcorner E(B) = \mu(E \cap B), \quad \text{for every Borel subset } B \subseteq D.$$

If $\mu = \mu \llcorner E$ we wll say that μ is concentrated on E.

For $1 \leq p \leq \infty$, we denote by $L^p(D)$ the space of Lebesgue measurable functions (in fact, equivalence classes, since almost everywhere equal functions are identified)

$u : D \to \mathbb{R}$ such that, if $p < \infty$

$$\|u\|_{L^p(D)} = \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{p}} < \infty,$$

or which are essentially bounded (w.r.t Lebesgue measure) if $p = \infty$. For the definition, the main properties and results on Lebesgue spaces we refer to [**B**]. For a function u in a Lebesgue space we set by $\dfrac{\partial u}{\partial x_i}$ (or simply u_{x_i}) its partial derivative in the direction x_i defined in the sense of distributions, that is

$$\langle u_{x_i}, \varphi \rangle = - \int_D u \varphi_{x_i} \, dx,$$

and we denote by $\nabla u = (u_{x_1}, ..., u_{x_N})$ the gradient of u defined this way.

The Sobolev space $W^{1,p}(D)$ with $1 \leq p \leq \infty$, is the space of functions u in $L^p(D)$ such that $\nabla u \in (L^p(D))^N$, endowed with its natural norm $\|u\|_{W^{1,p}(D)} = \|u\|_{L^p(D)} + \|\nabla u\|_{L^p(D)}$, while $W^{1,p}_0(D)$ will indicate the closure of $C^\infty_0(D)$ with respect to this norm. We still follow [**B**] for basic results on Sobolev spaces. Let us just recall that, for $1 < p < \infty$, the dual space of $L^p(D)$ can be identified with $L^{p'}(D)$, where $p' = \frac{p}{p-1}$ is the Hölder conjugate exponent of p, and that the dual space of $W^{1,p}_0(D)$ is denoted by $W^{-1,p'}(D)$. By a well known result, any element of $T \in W^{-1,p'}(D)$ can be written in the form $T = -\mathrm{div}(G)$ where $G \in (L^{p'}(D))^N$.

For every $0 < p < \infty$, we introduce the Marcinkiewicz space $M^p(D)$ of measurable functions f such that there exists $c > 0$, with

$$\mathrm{meas}\{x : |f(x)| \geq k\} \leq \frac{c}{k^p},$$

for every positive k; it turns out to be a Banach space endowed with the norm

$$\|f\|_{M^p(D)} = \inf \left\{ c > 0 : \mathrm{meas}\{x : |f(x)| \geq k\} \leq \left(\frac{c}{k} \right)^p \right\} .$$

Let us recall that, since D is bounded, then for $p > 1$ we have the following continuous embeddings

$$L^p(D) \hookrightarrow M^p(D) \hookrightarrow L^{p-\varepsilon}(D),$$

for every $\varepsilon \in (0, p-1]$.

Finally, let us spend a few words on how positive constant will be denoted hereafter. If no otherwise specified, we will write C to denote any positive constant (possibly different) which only depends on the data, that is on quantities that are fixed in the assumptions (N, Ω, Q, p, and so on...); in any case such constants never depend on the different indexes having a limit we often introduce, for instance $\varepsilon, \delta, \eta$ that vanish or n, k that go to infinity; here and in the rest of the thesis $\omega(\nu, \eta, \varepsilon, n, h, k)$ will indicate any quantity that vanishes as the parameters go to their (obvious, if not explicitly stressed) limit point with the same order in which they appear, that is, for example

$$\lim_{\nu\to 0}\limsup_{\delta\to 0}\limsup_{n\to+\infty}\limsup_{\varepsilon\to 0}|\omega(\varepsilon,n,\delta,\nu)|=0.$$

Moreover, for the sake of simplicity, in what follows, the convergences, even if not explicitly stressed, may be understood to be taken possibly up to a suitable subsequence extraction.

We already said that we refer to [**B**] for most basic tools in Lebesgue theory and Sobolev spaces; however, among them, let us recall explicitly some that will play a crucial role in the methods we use.

(1) *Generalized Young's inequality:* for $1<p<\infty$, $p'=\frac{p}{p-1}$ and any positive ε we have:

$$ab\le\varepsilon^p\frac{a^p}{p}+\frac{1}{\varepsilon^{p'}}\frac{b^{p'}}{p'},\qquad\forall a,b>0.$$

(2) *Hölder's inequality:* for $1<p<\infty$, $p'=\frac{p}{p-1}$, we have, for every $f\in L^p(D)$ and every $g\in L^{p'}(D)$:

$$\int_D|fg|\,dx\le\left(\int_D|f|^p\right)^{\frac{1}{p}}\left(\int_D|g|^{p'}\right)^{\frac{1}{p'}}.$$

(3) Let $1<p<\infty$, $p'=\frac{p}{p-1}$, $\{f_n\}\subset L^p(D)$, $\{g_n\}\subset L^{p'}(D)$ be such that f_n strongly converges to f in $L^p(D)$ and g_n weakly converges to g in $L^{p'}(D)$. Then

$$\lim_{n\to\infty}\int_D f_n\,g_n\,dx=\int_D fg\,dx\,.$$

The same conclusion holds true if $p=1$, $p'=\infty$ and the weak convergence of g_n is replaced by the $*$-weak convergence in $L^\infty(D)$. Moreover if f_n strongly converges to zero in $L^p(D)$, and g_n is bounded in $L^{p'}(D)$, we also have

$$\lim_{n\to\infty}\int_D f_n\,g_n\,dx=0\,.$$

(4) Let f_n converge to f in measure and suppose that:

$$\exists\, C>0,\quad q>1:\quad\|f_n\|_{L^q(D)}\le C,\quad\forall\, n.$$

Then

$$f_n\longrightarrow f\quad\text{strongly in } L^s(D),\ \text{for every } 1\le s<q.$$

(5) *Fatou's lemma:* Let $1\le p<\infty$, and let $\{f_n\}\subset L^p(D)$ be a sequence such that $f_n\to f$ a.e. in D and $f_n\ge h(x)$ with $h(x)\in L^1(D)$, then

$$\int_D f\,dx\le\liminf_{n\to\infty}\int_D f_n\,dx.$$

(6) *Generalized Lebesgue theorem:* Let $1 \le p < \infty$, and let $\{f_n\} \subset L^p(D)$ be a sequence such that $f_n \to f$ a.e. in D and $|f_n| \le g_n$ with g_n strongly convergent in $L^p(D)$, then $f \in L^p(D)$ and f_n strongly converges to f in $L^p(D)$.

(7) Let $\{f_n\} \subset L^1(D)$ and $f \in L^1(D)$ be such that, $f_n \ge 0$, $f_n \to f$ a.e. in D, and

$$\lim_{n\to\infty} \int_D f_n \, dx = \int_D f \, dx,$$

then f_n strongly converges to f in $L^1(D)$.

(8) *Vitali's theorem:* Let $1 \le p < \infty$, and let $\{f_n\} \subset L^p(D)$ be a sequence such that $f_n \to f$ a.e. in D and

$$\lim_{\mathrm{meas}(E)\to 0} \sup_n \int_E |f_n|^p \, dx = 0. \tag{1.1.1}$$

Then $f \in L^p(D)$ and f_n strongly converges to f in $L^p(D)$.

(9) Let $\{f_n\} \subset L^1(D)$ and $\{g_n\} \subset L^\infty(D)$ be two sequences such that

$$f_n \longrightarrow f \quad \text{weakly in } L^1(D),$$

$$g_n \longrightarrow g \quad \text{a.e. in } D \text{ and } *\text{-weakly in } L^\infty(D).$$

Then

$$\lim_{n\to\infty} \int_D f_n \, g_n \, dx = \int_D f g \, dx\,.$$

REMARK 1.1. Property (1.1.1) is the so called equi-integrability property of the sequence $\{|f_n|^p\}$. We recall that *Dunford-Pettis theorem* ensures that a sequence in $L^1(D)$ is weakly convergent in $L^1(D)$ if and only if it is equi-integrable. Moreover, results (4), (6) and (7) can be proven as an easy consequences of Vitali's theorem and so we will refer to them as Vitali's theorem as well. For the same reason we will refer to result (9) as *Egorov theorem*.

For functions in the Sobolev space $W_0^{1,p}(D)$ we will often use *Sobolev's theorem* stating that, if $p < N$, $W_0^{1,p}(D)$ continuously injects into $L^{p^*}(D)$ with $p^* = \frac{Np}{N-p}$; if $p = N$, $W_0^{1,p}(D)$ continuously injects into $L^q(D)$ for every $q < \infty$, while, if $p > N$, $W_0^{1,p}(D)$ continuously injects into $C(\overline{D})$. Let us also recall *Rellich's theorem* stating that, if $p < N$, the injection of $W_0^{1,p}(D)$ into $L^q(D)$ is compact for every $1 \le q < p^*$, and *Poincaré's inequality*, that is, there exists $C > 0$ such that

$$\|u\|_{L^p(D)} \le C \|\nabla u\|_{(L^p(D))^N}\,,$$

for every $u \in W_0^{1,p}(D)$, so that $\|\nabla u\|_{(L^p(D))^N}$ can be used as equivalent norm on $W_0^{1,p}(D)$.

We will often use the following result due to G. Stampacchia.

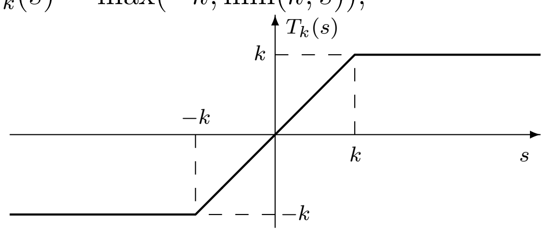
THEOREM 1.2. *Let $G : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function such that $G(0) = 0$. Then for every $u \in W_0^{1,p}(D)$ we have $G(u) \in W_0^{1,p}(D)$ and $\nabla G(u) = G'(u)\nabla u$ almost everywhere in D.*

PROOF. See [**S**]. □

Theorem 1.2 has an important consequence, that is

$$\nabla u = 0 \quad \text{a.e. in } F_c = \{x : u(x) = c\},$$

for every $c > 0$. Hence, we are able to consider the composition of function in $W_0^{1,p}(D)$ with some useful auxiliary function. One of the most used will be the truncation function at level $k > 0$, that is $T_k(s) = \max(-k, \min(k, s))$;



thus, if $u \in W_0^{1,p}(D)$, we have that $T_k(u) \in W_0^{1,p}(D)$ and $\nabla T_k(u) = \nabla u \chi_{\{u<k\}}$ a.e. on D, for every $k > 0$.

If u is such that its truncation belongs to $W_0^{1,p}(D)$, then we can define an *approximated gradient* of u defined as the a.e. unique measurable function $v : D \to \mathbb{R}^N$ such that

$$v = \nabla T_k(u) \tag{1.1.2}$$

almost everywhere on the set $\{|u| \leq k\}$, for every $k > 0$ (see for instance [**B6**])

1.1.1. Spaces of functions with values in a Banach space. Now we want to recall some feature about spaces of functions with values in a Banach space, that is one of the most important tool to deal with evolution problems.

Given a real Banach space V, we will denote by $C^\infty(\mathbb{R}; V)$ the space of functions $u : \mathbb{R} \to V$ which are infinitely many times differentiable (according to the definition of Frechet differentiability in Banach space) and by $C_0^\infty(\mathbb{R}; V)$ the space of functions in $C^\infty(\mathbb{R}; V)$ having compact support. As we mentioned above, for $a, b \in \mathbb{R}$, $C_0^\infty([a, b]; V)$ will be the space of restrictions to $[a, b]$ of functions of $C_0^\infty(\mathbb{R}; V)$, and $C([a, b]; V)$ the space of all continuous functions from $[a, b]$ into V.

We recall that a function $u : [a, b] \to V$ is said to be Lebesgue measurable if there exists a sequence $\{u_n\}$ of step functions (i.e. $u_n = \sum_{j=1}^{k_n} a_j^n \chi_{A_j^n}$ for a finite number k_n

of Borel subsets $A_j^n \subset [a,b]$ and with $a_j^n \in V$) converging to u almost everywhere with respect to the Lebesgue measure in $[a,b]$.

Then for $1 \leq p < \infty$, $L^p(a,b;V)$ is the space of measurable functions $u : [a,b] \to V$ such that

$$\|u\|_{L^p(a,b;V)} = \left(\int_a^b \|u\|_V^p \; dt \right)^{\frac{1}{p}} < \infty,$$

while $L^\infty(a,b;V)$ is the space of measurable functions such that:

$$\|u\|_{L^\infty(a,b;V)} = \sup_{[a,b]} \|u\|_V < \infty.$$

Of course both spaces are meant to be quotiented, as usual, with respect to the almost everywhere equivalence. The reader can find a presentation of these topics in [**DL**].

Let us recall that, for $1 \leq p \leq \infty$, $L^p(a,b;V)$ is a Banach space, moreover if for $1 \leq p < \infty$ and V', the dual space of V, is separable, then the dual space of $L^p(a,b;V)$ can be identified with $L^{p'}(a,b;V')$.

For our purpose V will mainly be either the Lebesgue space $L^p(\Omega)$ or the Sobolev space $W_0^{1,p}(\Omega)$, with $1 \leq p < \infty$ and Ω is a bounded open set of $\mathbb{R}^N$. Since in this case V is separable we have that $L^p(a,b;L^p(\Omega)) = L^p((a,b)\times\Omega)$, the ordinary Lebesgue space defined in $(a,b)\times\Omega$ and $L^p(a,b;W_0^{1,p}(\Omega))$ consists of all functions $u : [a,b]\times\Omega \to \mathbb{R}$ which belong to $L^p((a,b)\times\Omega)$ and such that $\nabla u = (u_{x_1},...,u_{x_N})$ belongs to $(L^p((a,b)\times\Omega))^N$ (often, for simplicity, we will indicate this space only by $L^p((a,b)\times\Omega)$); moreover,

$$\left(\int_a^b \int_\Omega |\nabla u|^p \; dxdt \right)^{\frac{1}{p}}$$

defines an equivalent norm by Poincaré's inequality.

Given a function in $L^p(a,b;V)$ it is possible to define a time derivative of u in the space of vector valued distributions $\mathcal{D}'(a,b;V)$ which is the space of linear continuous functions from $C_0^\infty(a,b)$ into V (see [**Sc**]). In fact, the definition is the following:

$$\langle u_t, \psi \rangle = -\int_a^b u\,\psi_t \; dt\,, \qquad \forall \;\; \psi \in C_0^\infty(a,b),$$

where the equality is meant in V. If $u \in C^1(a,b;V)$ this definition clearly coincides with the Frechet derivative of u. In the following, when u_t is said to belong to a space $L^q(a,b;\tilde{V})$ ($\tilde{V}$ being a Banach space) this means that there exists a function $z \in L^q(a,b;\tilde{V}) \cap \mathcal{D}'(a,b;V)$ such that:

$$\langle u_t, \psi \rangle = -\int_a^b u\,\psi_t \; dt = \langle z, \psi \rangle\,, \qquad \forall \;\; \psi \in C_0^\infty(a,b).$$

In the following, we will also use sometimes the notation $\dfrac{\partial u}{\partial t}$ instead of u_t. We racall the following classical embedding result

THEOREM 1.3. *Let H be an Hilbert space such that:*

$$V \underset{dense}{\hookrightarrow} H \hookrightarrow V' .$$

Let $u \in L^p(a,b;V)$ be such that u_t, defined as above in the distributional sense, belongs to $L^{p'}(a,b;V')$. Then u belongs to $C([a,b];H)$.

PROOF. [**DL**], Chapter XVIII, Section 2, Theorem 1. □

This result also allows us to deduce, for functions u and v enjoying these properties, the integration by parts formula:

$$\int_a^b \langle v, u_t\rangle \, dt + \int_a^b \langle u, v_t\rangle \, dt = (u(b), v(b)) - (u(a), v(a)) \,, \tag{1.1.3}$$

where $\langle \cdot,\cdot\rangle$ is the duality between V and V' and $(\cdot,\cdot)$ the scalar product in H. Notice that (1.1.3) makes sense thanks to Theorem 1.3. Its proof relies on the fact that $C_0^\infty(a,b;V)$ is dense in the space of functions $u \in L^p(a,b;V)$ such that $u_t \in L^{p'}(a,b;V')$ endowed with the norm $\|u\| = \|u\|_{L^p(a,b;V)} + \|u_t\|_{L^{p'}(a,b;V')}$, together with the fact that (1.1.3) is true for u, $v \in C_0^\infty(a,b;V)$ by the theory of integration and derivation in Banach spaces. Note however that in this context (1.1.3) is subject to the verification of the hypotheses of Theorem 1.3; if, for instance, $V = W_0^{1,p}(\Omega)$, then

$$W_0^{1,p}(\Omega) \underset{dense}{\hookrightarrow} L^2(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$$

only if $p \geq \frac{2N}{N+2}$; for the sake of simplicity we will often work under this bound, that actually turns out to be only technical.

1.1.2. Further useful results. Here we give some further results that will be very useful in what follows; the first one contains a generalization of the integration by parts formula (1.1.3) where the time derivative of a function is less regular than there, an its proof can be found in [**DP**] (see also [**CW**]).

LEMMA 1.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous piecewise C^1 function such that $f(0) = 0$ and f' is zero away from a compact set of $\mathbb{R}$; let us denote $F(s) = \int_0^s f(r)dr$. If $u \in L^p(0,T;W_0^{1,p}(\Omega))$ is such that $u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$ and if $\psi \in C^\infty(\overline{Q})$, then we have*

$$\int_0^T \langle u_t, f(u)\psi\rangle \, dt = \int_\Omega F(u(T))\psi(T)\, dx - \int_\Omega F(u(0))\psi(0)\, dx - \int_Q \psi_t \, F(u)\, dxdt. \tag{1.1.4}$$

Now we state three embedding theorems that will play a central role in our work; the first one is an Aubin-Simon type result that we state in a form general enough to our purpose, while the second one is a generalization of Theorem 1.3; the third one is the well-known *Gagliardo-Nirenberg embedding theorem* followed by an important consequence of it for the evolution case.

THEOREM 1.5. *Let u^n be a sequence bounded in $L^q(0,T;W_0^{1,q}(\Omega))$ such that u_t^n is bounded in $L^1(Q)+L^{s'}(0,T;W^{-1,s'}(\Omega))$ with $q,s>1$, then u^n is relatively strongly compact in $L^1(Q)$, that is, up to subsequences, u^n strongly converges in $L^1(Q)$ to some function $u\in L^1(Q)$.*

PROOF. See [**Si**], Corollary 4. □

Let us define, for every $p>1$, the space S^p as

$$S^p=\{u\in L^p(0,T;W_0^{1,p}(\Omega));u_t\in L^1(Q)+L^{p'}(0,T;W^{-1,p'}(\Omega))\},\tag{1.1.5}$$

endowed with its natural norm $\|u\|_{S^p}=\|u\|_{L^p(0,T;W_0^{1,p}(\Omega))}+\|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}$. We have the following trace result:

THEOREM 1.6. *Let $p>1$, then we have the following continuous injection*

$$S^p\hookrightarrow C(0,T;L^1(\Omega)).$$

PROOF. See [**Po1**], Theorem 1.1. □

THEOREM 1.7 (Gagliardo-Nirenberg). *Let v be a function in $W_0^{1,p}(\Omega)\cap L^\rho(\Omega)$ with $q\geq 1$, $\rho\geq 1$. Then there exists a positive constant C, depending on N, q and ρ, such that*

$$\|v\|_{L^\gamma(\Omega)}\leq C\|\nabla v\|^{\theta}_{(L^q(\Omega))^N}\|v\|^{1-\theta}_{L^\rho(\Omega)}\,,$$

for every θ and γ satisfying

$$0\leq\theta\leq 1,\quad 1\leq\gamma\leq+\infty,\quad \frac{1}{\gamma}=\theta\left(\frac{1}{q}-\frac{1}{N}\right)+\frac{1-\theta}{\rho}\,.$$

PROOF. See [**N**], Lecture II. □

An immediate consequence of the previous result is the following embedding result:

COROLLARY 1.8. *Let $v\in L^q(0,T;W_0^{1,q}(\Omega))\cap L^\infty(0,T;L^\rho(\Omega))$, with $q\geq 1$, $\rho\geq 1$. Then $v\in L^\sigma(Q)$ with $\sigma=q\frac{N+\rho}{N}$ and*

$$\int_Q|v|^\sigma\,dxdt\leq C\|v\|^{\frac{\rho q}{N}}_{L^\infty(0,T;L^\rho(\Omega))}\int_Q|\nabla v|^q\,dxdt\,.$$

PROOF. See [**DiB**], Proposition 3.1. □

Finally we want to recall some useful density results. Let us call $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$ endowed with its natural norm $\|\cdot\|_{W_0^{1,p}(\Omega)} + \|\cdot\|_{L^2(\Omega)}$ and

$$W = \left\{ u \in L^p(0,T;V),\ u_t \in L^{p'}(0,T;V') \right\}, \tag{1.1.6}$$

endowed with its natural norm $\|u\|_W = \|u\|_{L^p(0,T;V)} + \|u_t\|_{L^{p'}(0,T;V')}$. We have the following

THEOREM 1.9. *Let $1 < p < \infty$, then $C_0^\infty([0,T] \times \Omega)$ is dense in W.*

PROOF. See [**DPP**], Theorem 2.11. □

Let us emphasize that, if $u \in W \cap L^\infty(Q)$, then the approximating sequence of functions in $C_0^\infty([0,T] \times \Omega)$ that exists thanks to Theorem 1.9, can be chosen to be bounded.

To conclude let us state a straightforward consequence of Corollary 2.3.2 in [**Dr**], (where we suppose $p \geq \frac{2N}{N+2}$ just for simplicity).

PROPOSITION 1.10. *If $u \in S^p \cap L^\infty(Q)$ then there exists a sequence of uniformly bounded functions $u^n \in C^\infty([0,T], W_0^{1,p}(\Omega))$ that converges to u in S^p; that is, if $u_t = v^{(1)} + v^{(2)} \in L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))$, then u^n converges to u in $L^p(0,T;W_0^{1,p}(\Omega))$, $u_t^n = v^{n,(1)} + v^{n,(2)}$ with $v^{n,(1)}$ that converges to $v^{(1)}$ in $L^1(Q)$ and $v^{n,(2)}$ converges to $v^{(2)}$ in $L^{p'}(0,T;W^{-1,p'}(\Omega))$.*

1.2. Elliptic case

Here we want to give a brief review on basic results concerning elliptic differential problems with measure data. We shall begin by recalling the variational case and the linear case with the definition of *duality solution*, then we will discuss the case of general Leray-Lions type pseudomonotone operators with measures, and finally we will mention the case of lower order terms with natural growth. Next section will be devoted to discuss the parabolic case associated with these problems.

Let $1 < p < +\infty$ and $\Omega \subseteq \mathbb{R}^N$ a bounded open set and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ a *Carathéodory function* (i.e., $a(\cdot,\xi)$ is measurable for every $\xi \in \mathbb{R}^N$, and $a(x,\cdot)$ is continuous for a.e. $x \in \Omega$) such that

$$a(x,\xi) \cdot \xi \geq \alpha |\xi|^p, \tag{1.2.1}$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$, with α a positive constant;

$$|a(x,\xi)| \leq \beta(b(x) + |\xi|^{p-1}), \tag{1.2.2}$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$, with β a positive constant and b a nonnegative function in $L^{p'}(\Omega)$;

$$(a(x,\xi) - a(x,\eta)) \cdot (\xi - \eta) > 0, \tag{1.2.3}$$

for a.e. $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N, \xi \neq \eta$.

Under these assumptions

$$A : u \mapsto -\mathrm{div}(a(x, \nabla u)),$$

turns out to be a continuous, coercive, pseudomonotone operator from $W_0^{1,p}(\Omega)$ into its dual space $W^{-1,p'}(\Omega)$.

Remark 1.11. First of all observe that $a(x,0) = 0$, for a.e. $x \in \Omega$. In fact, from (1.2.1), for $t > 0$ fixed, one has

$$a(x, t\xi) \cdot \xi \geq \alpha t^{p-1} |\xi|^p,$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$; so, using the fact that

$$a(x, t\xi) \xrightarrow{t \to 0^+} a(x, 0),$$

for any $\xi \in \mathbb{R}^N$ and for a.e. $x \in \Omega$ (thanks to the continuity of a with respect to its second argument), we obtain

$$a(x,0) \cdot \xi \geq 0 \ \ \forall \xi \in \mathbb{R}^N;$$

therefore, taking $-\xi$ in the place of ξ, in the above inequality, we conclude $a(x,0) \cdot \xi = 0$ for a.e. $x \in \Omega$ and $\forall\, \xi \in \mathbb{R}^N$ and so $a(x,0) = 0$ for a.e. $x \in \Omega$.

We are interested in the study of the following problem:

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) = \mu & \text{in}\, \Omega, \\ u = 0 & \text{on}\ \partial\Omega, \end{cases} \tag{1.2.4}$$

where $\mu \in M(\Omega)$; let us first focus on the case where $\mu \in W^{-1,p'}(\Omega)$.

1.2.1. Variational case. Observe that, if $p > N$, then, by Sobolev embedding theorem, we have that $M(\Omega) \subset W^{-1,p'}(\Omega)$ and so we come back to this case.

Definition 1.12. Let $1 < p < +\infty$, then if $f \in W^{-1,p'}(\Omega)$, u is a *variational solution* of problem (1.2.4) if:

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega a(x,\nabla u) \cdot \nabla \varphi \; dx = \langle f, \varphi \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)}, \quad \forall \varphi \in W_0^{1,p}(\Omega). \end{cases} \tag{1.2.5}$$

Remark 1.13. In the linear case, i.e. when $a(x,\cdot) = A(x)$ with $A(x)$ is an $N \times N$ matrix with coefficients $L^\infty(\Omega)$ and $p = 2$, the existence of a variational solution $u \in H_0^1(\Omega)$ can be easily done via the use of *Lax-Milgram theorem.*

Let us state the existence theorem for variational solutions of problem (1.2.4), its proof relies on an application of *Schauder fixed point theorem.*

Theorem 1.14. *The operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined as $A(u) = -\mathrm{div}(a(\cdot,\nabla u))$ is surjective, and so, if $f \in W^{-1,p'}(\Omega)$ then there exists $u \in W_0^{1,p}(\Omega)$ such that $A(u) = f$ in $W^{-1,p'}(\Omega)$, that is,*

$$\int_\Omega a(x,\nabla u)\cdot\nabla\varphi\; dx = \langle f,\varphi\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}, \qquad \forall\varphi \in W_0^{1,p}(\Omega).$$

Proof. See [**LL**]. □

The variational solution of problem (1.2.4) is also unique, in fact we have the following

Theorem 1.15. *The variational solution of problem (1.2.4) is unique, that is, if $f \in W^{-1,p'}(\Omega)$ then there exists a unique $u \in W_0^{1,p}(\Omega)$ such that $A(u) = f$ in $W^{-1,p'}(\Omega)$.*

Proof. Let $u, v \in W_0^{1,p}(\Omega)$ such that $A(u) = f$ and $A(v) = f$ in $W^{-1,p'}(\Omega)$. Using (1.2.5) for both solutions and subtracting the one from he other we obtain

$$\int_\Omega (a(x,\nabla u) - a(x,\nabla v))\cdot\nabla\varphi\; dx = 0,$$

for all $\varphi \in W_0^{1,p}(\Omega)$. So taking $\varphi = u - v$ as test function, and using assumption (1.2.1), we have easily conclude that $u = v$. □

Remark 1.16. The variational problem turns out to admit a solution even if a is more general; for instance a could depend explicitly (and continuously) from u with suitable change of assumption (1.2.2); however uniqueness is not guaranteed in general unless in the case $1 < p \leq 2$ with a stronger assumption on a (see [**BG2**]).

1.2.2. Linear case. Let $A(x)$ be a $N \times N$ matrix with entries $a_{i,j}(x) \in L^\infty(\Omega)$ satisfying assumption (1.2.1) ($p = 2$), and consider the linear problem

$$\begin{cases} L(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2.6}$$

where $L(\xi) = -\mathrm{div}(A(x)\xi)$ and $\mu \in M(\Omega)$. Let us consider L^* as the adjoint operator of L defined by $L^*(u) = -\mathrm{div}(A^*(x)\nabla u)$ for all $u \in H_0^1(\Omega)$, where $A^*(x)$ is the transpose matrix of $A(x)$. If $f \in W^{-1,p'}(\Omega)$, with $p' > N$ we can consider

$$\begin{cases} L^*(v) = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2.7}$$

Let v be the variational solution of problem (1.2.7); thanks to standard elliptic regularity results we have that $v \in C(\overline{\Omega})$ and

$$\|v\|_{C(\overline{\Omega})} \leq \lambda \|f\|_{W^{-1,p'}(\Omega)}. \tag{1.2.8}$$

So, for every $p' > N$, we can define

$$G^*_{p'} : \; W^{-1,p'}(\Omega) \longrightarrow C(\overline{\Omega}),$$

as

$$G^*_{p'}(f) = v.$$

$G^*_{p'}$ turns out to be linear and continuous; thus we can define the *Green operator* as

$$G^* : \; \bigcup_{p'>N} W^{-1,p'}(\Omega) \longrightarrow C_0(\Omega),$$

with

$$G^* \mid_{W^{-1,p'}(\Omega)} = G^*_{p'}.$$

This argument justifies the definition of *duality solution* given by G. Stampacchia in [**S**], for the problem

$$\begin{cases} L(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2.9}$$

DEFINITION 1.17. Let $\mu \in M(\Omega)$ we will say that $u \in L^1(\Omega)$ is a *duality solution* of problem (1.2.9) if

$$\int_\Omega ug \; dx = \int_\Omega G^*(g) \; d\mu, \tag{1.2.10}$$

for all $g \in L^\infty(\Omega)$.

A duality solution, easily, turns out to be a distributional solution of problem (1.2.9) and, if it exists, is obviously unique as an easy consequence of its definition.

THEOREM 1.18. *Let $\mu \in M(\Omega)$, then there exists a unique duality solution of problem (1.2.9). Moreover, $u \in W_0^{1,q}(\Omega)$ with $q < \frac{N}{N-1}$.*

PROOF. See [**S**]. □

REMARK 1.19. Notice that the regularity of the duality solution, that is $u \in W_0^{1,q}(\Omega)$ with $q < \frac{N}{N-1}$, is sharp and cannot be, in general, improved; in fact one can think about the fundamental solution of the Laplace operator in a ball. So, in general, we deal with solutions that do not belong to the usual energy space; however notice that, as we will see below, these *infinity energy solutions* turn out to have finite energy truncations at any level.

1.2.3. Weak solutions for monotone operators with measure data. A whole theory was recently developed about the Dirichlet problem

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) = \mu & \text{in}\,\Omega, \\ u = 0 & \text{on}\ \partial\Omega, \end{cases} \tag{1.2.11}$$

where $\mu \in M(\Omega)$ and a satisfies assumption (1.2.1)–(1.2.3). As we said before, the interest in study problem (1.2.11) arises if $p \leq N$, since, if $p > N$ then $M(\Omega) \subset W^{-1,p'}(\Omega)$ and one can apply classical variational results (see Theorem 1.14 and Theorem 1.15). On the other hand, if $p \leq N$ the solution of problem (1.2.11) cannot be expected to belong to $W_0^{1,p}(\Omega)$, nor is clear in which sense the solution should be considered. As we said before, in the linear case, the notion of duality solution provided the right tool to get existence and uniqueness for such a problem with general measure data; unfortunately this method does not apply in the case of general nonlinear operators. In this case, the key point is to look for solutions of problem (1.2.11) as limit of an approximating sequence of regular solutions. Henceforward, we will say that a sequence $\{\mu_n\} \subset M(\Omega)$ converges *tightly* (or, equivalently, in the *narrow topology of measures*) to a measure μ if

$$\lim_{n\to\infty} \int_\Omega \varphi \, d\mu_n = \int_\Omega \varphi \, d\mu, \quad \forall\, \varphi \in C(\overline{\Omega}).$$

Let us remark that μ_n converges tightly to μ if and only if μ_n converges to μ weak-$*$ in $M(\Omega)$ and $\mu_n(\Omega)$ converges to $\mu(\Omega)$. Via a standard convolution argument one can easily prove the following

THEOREM 1.20. *Let $\mu \in M(\Omega)$. Then there exists a sequence $\{f_n\} \subset C^\infty(\Omega)$ such that*

$$\|f_n\|_{L^1(\Omega)} \leq |\mu|_{M(\Omega)}\,,$$

and

$$f_n \longrightarrow \mu \quad \textit{tightly in} \quad M(\Omega).$$

Thanks to Theorem 1.20 a method for solving problem (1.2.11) is to find *a priori* estimates which only depend on the norm of the datum μ in $M(\Omega)$ and then look for compactness results which allow to pass to the limit in the approximating problems.

This method has been proved to work in [**BG**] and yields a function u that is a distributional solution of (1.2.11). However, u only belongs to the space $W_0^{1,q}(\Omega)$ for every $q < \frac{N(p-1)}{N-1}$ and its regularity is optimal as shown by simple examples like, for instance, the fundamental solution of the p-laplacian on a ball of $\mathbb{R}^N$. Since $\frac{N(p-1)}{N-1} > 1$ if and only if $p > 2 - \frac{1}{N}$, for smaller values of p we cannot even use the framework of Sobolev spaces to deal with (1.2.11), so that, this lower bound on p is required in [**BG**]. Even if this bound can be overcome by a suitable use of the estimate of the truncations of the approximating solution (see [**B6**]), we will often use, in the following, similar bounds to avoid technicalities.

Anyway, for $1 < p < \infty$, we can provide the definition of weak solution of problem (1.2.11), where the gradient of u is understood to be the approximated gradient of u defined in (1.1.2). For simplicity let us define

$$\mathcal{T}_0^{1,p}(\Omega) = \left\{ u \ \text{ measurable} : T_k(u) \in W_0^{1,p}(\Omega), \forall\ k > 0 \right\};$$

notice that $\mathcal{T}_0^{1,p}(\Omega)$ is not a linear space as simple examples show; however, if u is in $\mathcal{T}_0^{1,p}(\Omega)$ and φ is in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ then $u + \varphi$ belongs to $\mathcal{T}_0^{1,p}(\Omega)$ (see [**DMOP**]).

Definition 1.21. A measurable function $u \in \mathcal{T}_0^{1,p}(\Omega)$, for every $k > 0$, is a *weak solution* of problem (1.2.11) if $a(x, \nabla u)$ belongs to $(L^1(\Omega))^N$ and the equation is satisfied in the sense of distributions, that is

$$\int_\Omega a(x, \nabla u) \cdot \nabla \varphi \, dx = \int_\Omega \varphi \, d\mu, \quad \forall\ \varphi \in C_0^\infty(\Omega).$$

In [**BG**], for $p > 2 - \frac{1}{N}$, and in [**B6**] (see also [**BGO**]) in the general case, the problem of existence of weak solutions of (1.2.11) is solved by using the following tools, which we here recall being key results for the whole theory.

Lemma 1.22. *Let $C > 0$ and let $\{u^n\} \subset \mathcal{T}_0^{1,p}(\Omega)$ be such that:*

$$\int_\Omega |\nabla T_k(u^n)|^p \, dx \le C(k+1) \quad \forall\ k > 0.$$

Then, if $p < N$, u^n is bounded in $M^{\frac{N(p-1)}{N-p}}(\Omega)$ and $|\nabla u^n|$ is bounded in $M^{\frac{N(p-1)}{N-1}}(\Omega)$; if $p = N$, u^n is bounded in $M^q(\Omega)$ for every $q < \infty$ and $|\nabla u^n|$ is bounded in $M^r(\Omega)$ for every $r < N$. Moreover, there exists a measurable function $u \in \mathcal{T}_0^{1,p}(\Omega)$ and a subsequence, not relabeled, such that:

$$u^n \longrightarrow u \quad \textit{a.e. in } \Omega,$$

$$T_k(u^n) \longrightarrow T_k(u) \quad \textit{weakly in } W_0^{1,p}(\Omega) \textit{ and a.e. in } \Omega \textit{ for every } k > 0.$$

PROOF. As far as the estimates are concerned, see [**B6**], Lemma 4.1 and Lemma 4.2 if $p < N$, while for the case $p = N$ see [**BPV**], Lemma 2.5. The convergence results are contained in Theorem 6.1 of [**B6**]. $\square$

LEMMA 1.23. *Let* $u^n \in W_0^{1,p}(\Omega)$ *a sequence of solutions of*

$$\begin{cases} -\operatorname{div}(a(x,\nabla u^n)) = f^n - \operatorname{div}(F^n) & in\,\Omega, \\ u^n = 0 & on\ \partial\Omega, \end{cases}$$

where $f^n \in L^\infty(\Omega)$ *is such that* $\|f^n\|_{L^1(\Omega)} \leq C$, *and* F^n *strongly converges in* $(L^{p'}(\Omega))^N$. *Then there exists* $u \in \mathcal{T}_0^{1,p}(\Omega)$ *and a subsequence, not relabeled, such that:*

$$u^n \longrightarrow u \quad a.e.\ in\ \ \Omega,$$

$$\nabla u^n \longrightarrow \nabla u \quad a.e.\ in\ \ \Omega,$$

and

$$a(x,\nabla u^n) \longrightarrow a(x,\nabla u) \quad strongly\ in\ (L^1(\Omega))^N.$$

PROOF. See [**B6**],[**DV**], [**BGO**] and [**BPV**]. $\square$

Thanks to Lemma 1.22 and Lemma 1.23 one can easily prove the existence result for problem (1.2.11).

THEOREM 1.24. *Assume (1.2.1)–(1.2.3), and let* $\mu \in M(\Omega)$. *Then there exists a weak solution* u *of problem (1.2.11) in* $\mathcal{T}_0^{1,p}(\Omega)$. *Moreover, if* $p < N$, u *belongs to* $M^{\frac{N(p-1)}{N-p}}(\Omega)$ *and* $|\nabla u|$ *belongs to* $M^{\frac{N(p-1)}{N-1}}(\Omega)$, *while, if* $p = N$, u *belongs to* $M^q(\Omega)$ *for every* $q < \infty$ *and* $|\nabla u|$ *belongs to* $M^r(\Omega)$ *for every* $r < N$

PROOF. See [**B6**], Theorem 6.1, for $p < N$, and [**BPV**], Theorem 2.6 for $p = N$. $\square$

1.2.4. Serrin counterexample: lack of uniqueness. Nothing has been said about uniqueness of weak solutions of (1.2.11), which is still open, even for linear operator with smooth data. In fact, in [**Se**], J. Serrin shown that, if $N = 2$, and $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$ and for every fixed $0 < \varepsilon < 1$, then there exists a matrix A^ε, such that

- $a^\varepsilon_{i,j}$ are measurabe functions defined on Ω, $\forall\ i,j = 1,2$,
- $a^\varepsilon_{i,j} \in L^\infty(\Omega)$, $\forall\ i,j = 1,2$,
- $A^\varepsilon(x)\ \xi \cdot \xi \geq \alpha_\varepsilon |\xi|^2$, for a.e. $x \in \Omega$, and for any $\xi \in \mathbb{R}^2$, with $\alpha > 0$,

and

$$
\begin{cases} u \in W_0^{1,q}(\Omega), & \forall\, 1 \le q < \frac{2}{1+\varepsilon}, \\ \displaystyle\int_\Omega A^\varepsilon(x)\nabla u \cdot \nabla\varphi \; dx = 0, & \forall \varphi \in C_0^\infty(\Omega), \end{cases} \tag{1.2.12}
$$

admits at least two solutions.

The Serrin's coefficients are

$$
a_{i,j} = \left(\frac{1}{\varepsilon^2} - 1\right)\frac{x_i x_j}{r^2} + \delta_{i,j}, \tag{1.2.13}
$$

for $i,j = 1,2.$, where $r = \sqrt{x_1^2 + x_2^2}$ and $\delta_{i,j}$ stands for the Kronecker symbol; if $v(x)$ is the unique variational solution (see for instance [**E**]) of problem

$$
\begin{cases} -\operatorname{div}(A^\varepsilon(x)\nabla v) = 0, & \text{in } \Omega, \\ v = x_1 & \text{on } \partial\Omega, \end{cases}
$$

then $u = x_1 r^{-N+1-\varepsilon} - v(x)$ is a nontrivial (the trivial solution is obviously $u = 0$) solution of problem (1.2.12). Let us notice that, this pathological solution found by Serrin belongs to $W_0^{1,q}(\Omega)$ for every $q \in [1, \frac{2}{1+\varepsilon})$, this is coherent with the uniqueness result of Theorem 1.15.

In [**Pr1**], the author extended such a counterexample to the case $N \ge 3$. For instance, if $N = 3$ the matrix is:

$$
A^\varepsilon = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

where $a_{i,j}$ are the same coefficients defined in (1.2.13).

However, since it is not been proved that such a solution belongs to $\mathcal{T}_0^{1,p}(\Omega)$, the uniqueness of weak solution is still an open problem.

1.2.5. Elliptic p-capacity. The attempt to find a different formulation for (1.2.11) which could allow to have both existence and uniqueness has been developed in [**B6**] and in [**DMOP**] where the notions of *entropy solution* and *renormalized solution* have been respectively introduced. Both these definitions, which have been proved to be equivalent (see [**DMOP**]), ask for solutions in $\mathcal{T}_0^{1,p}(\Omega)$ and use a weak formulation of the equation where nonlinear test functions depending on u are used to restrict the equation on the subsets where u is bounded. Both these approaches are able to get uniqueness provided μ belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$. In terms of measures, this restriction has a straight relationship with the notion of elliptic p-capacity, as it was proved in [**BGO**]. In order to recall this result, we need first to introduce the notion of p-capacity (see also Section 2 in [**DMOP**] and the references quoted therein, in particular [**HKM**] where fine properties and estimates are presented).

For $p > 1$, the elliptic p-capacity of a compact set K of Ω can be defined as follows:

$$\operatorname{cap}_p^e(K,\Omega) = \inf_{\varphi\in\mathcal{C}(K,\Omega)} \left\{ \int_\Omega |\nabla\varphi|^p \, dx \right\}, \tag{1.2.14}$$

where $\mathcal{C}(K,\Omega) = \{\varphi \in C_0^\infty(\Omega) : \varphi \geq \chi_K\}$, and, as usual, we use the convention that $\inf\emptyset = +\infty$; then one can extend this definition by regularity to any Borel subset of Ω.

Let us also recall that a function u is said to be cap_p^e quasi continuous if for every $\varepsilon > 0$ there exists a set $E \subseteq \Omega$ such that $\operatorname{cap}_p^e(E) \leq \varepsilon$ and u is continuous in $\Omega\backslash E$. It is well known that every function $u \in W_0^{1,p}(\Omega)$ admits a unique cap_p^e quasi continuous representative $\tilde{u} \in W_0^{1,p}(\Omega)$, that is a function $\tilde{u}$ which is equal to u almost everywhere in Ω and is cap_p^e quasi continuous. Moreover the values of $\tilde{u}$ are defined cap_p^e quasi everywhere. Thanks to this fact it is also possible to prove the following: for any Borel set $B \subseteq \Omega$, we have

$$\operatorname{cap}_p^e(B,\Omega) = \inf \left\{ \int_\Omega |\nabla v|^p \, dx : 0 \leq v \in W_0^{1,p}(\Omega),\ v = 1 \ \operatorname{cap}_p^e\text{-a.e. on } B \right\}. \tag{1.2.15}$$

Moreover, if u belongs to $W_0^{1,p}(\Omega)$, and μ is a bounded Radon measure such that $\mu(E) = 0$ for every $E \subset \Omega$ such that $\operatorname{cap}_p^e(E) = 0$, we have that u is measurable with respect to μ and, if u is also bounded, then u belongs to $L^\infty(\Omega,\mu)$ (see Proposition 2.7 of [**DMOP**]).

Let us now recall some fundamental results on the link between p-capacity and Radon measures.

THEOREM 1.25. *Let μ belong to $M(\Omega)$. Then $\mu(E) = 0$ for every subset $E \subseteq \Omega$ such that $\operatorname{cap}_p^e(E) = 0$ if and only if μ belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$.*

PROOF. See [**BGO**], Theorem 2.1. □

A further decomposition result for measures is the following.

THEOREM 1.26. *Let μ belong to $M(\Omega)$. Then there exists a unique couple of measures (μ_0,λ) such that $\mu_0,\lambda \in M(\Omega)$, $\mu_0(B) = 0$ for every subset B such that $\operatorname{cap}_p^e(B) = 0$ while λ is concentrated on a subset E of zero $p-$capacity, and $\mu = \mu_0 + \lambda$. By Theorem 1.25 we then have that there exist $f \in L^1(\Omega)$, $F \in (L^{p'}(\Omega))^N$, such that:*

$$\mu = f - \operatorname{div}(F) + \lambda.$$

Moreover, if $\mu \geq 0$, we have $\mu_0 \geq 0$, $\lambda \geq 0$ and also f can be chosen nonnegative.

PROOF. For the decomposition result, see [**FST**], Lemma 2.1, and again [**BGO**] for the last part. □

Let us remark that, since $L^1(\Omega) \cap W^{-1,p'}(\Omega) \neq \{0\}$, there is not a unique way (not even a *better* way), in Theorem 1.25, to write $\mu_0 = f - \mathrm{div}(F)$, with $f \in L^1(\Omega)$ and $F \in (L^{p'}(\Omega))^N$.

1.2.6. Entropy solutions. Thanks to Theorem 1.25 one can provide the definition of *entropy solution* for problem (1.2.11) whose introduction is motivated by Theorem 1.29 below.

Definition 1.27. Let μ be a measure in $L^1(\Omega) + W^{-1,p'}(\Omega)$. Then $u \in \mathcal{T}_0^{1,p}(\Omega)$ is an *entropy solution* of problem (1.2.11), if, for any $k > 0$, it satisfies

$$\int_\Omega a(x, \nabla u) \cdot \nabla T_k(u - \varphi)\, dx \leq \int_\Omega T_k(u - \varphi)\, d\mu, \tag{1.2.16}$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Remark 1.28. Let us observe that both terms in (1.2.16) are well defined ; in fact, the left hand side of (1.2.16) can be rewritten as:

$$\int_{\{|u| \leq \mathrm{M}\}} a(x, \nabla T_{\mathrm{M}}(u)) \cdot \nabla T_k(u - \varphi)\, dx,$$

where $\mathrm{M} = k + \|\varphi\|_{L^\infty(\Omega)}$, since $\nabla T_k(u - \varphi) = 0$ a.e. on $\{|u| > M\}$; now, thanks to (1.2.2), we have that $a(x, \nabla T_{\mathrm{M}}(u)) \in (L^{p'}(\Omega))^N$, while $\nabla T_{\mathrm{k}}(u - \varphi) \in (L^p(\Omega))^N$, since $(u - \varphi) \in \mathcal{T}_0^{1,p}(\Omega)$; Theorem 1.25 gives sense to the right hand side of (1.2.16) and it turns out to be independent on the different decompositions of μ.

Notice that this definition can not be extended directly to the general case of $\mu \in M(\Omega)$ because of the possible lack of $\mu-$measurability of the integrand on the right hand side of (1.2.16).

Finally notice that such a solution turns out to be a distributional solution of problem (1.2.11) (see [**B6**], [**BGO**]).

Now we can state the main result about entropy solution:

Theorem 1.29. *Let μ be a measure in $L^1(\Omega) + W^{-1,p'}(\Omega)$. Then there exists a unique entropy solution of problem (1.2.11).*

Proof. See [**BGO**], Theorem 3.2 and Theorem 3.3. $\square$

Let us emphasize that in the proof of uniqueness of Theorem 1.29 in [**BGO**] the authors used the following result on the behavior of the *energy* of the solution u on the set where it is large, this kind of results will have a central role in our work.

LEMMA 1.30. *Let $u \in \mathcal{T}_0^{1,p}(\Omega)$ be an entropy solution of problem (1.2.11), with μ a measure in $L^1(\Omega) + W^{-1,p'}(\Omega)$ and let us define $B_{h,k} = \{x \in \Omega : h \le |u| \le h + k\}$, for every $h, k > 0$; then*

$$\lim_{h\to+\infty} \int_{B_{h,k}} |\nabla u|^p \, dx = 0.$$

1.2.7. Renormalized solutions. As we said before, the notion of entropy solution can not be generalized directly to the case of a general, possibly singular, measure in $M(\Omega)$. In [**DMOP**], the authors, by mean of the notion of *renormalized solution* (introduced first in [**DPL**] for first order hyperbolic equations, and then developed in many papers; see [**BDGM**], [**M**], [**BM**]), extended this concept to general measure data. In this paper they give four definitions of renormalized solution that turn out to be equivalent. If μ is a measure in $M(\Omega)$ we will denote with μ_0 its absolutely continuous part with respect to the p-capacity, and with μ_s^+ and μ_s^-, respectively, the positive and the negative variation of the singular part of μ: moreover, we will say that a function $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfies condition P_u if there exists $k > 0$ and two functions $w^{+\infty}$, $w^{-\infty} \in C_b^1(\Omega)$, such that

$$\begin{cases} w = w^{+\infty} & \text{a.e. in } \{u > k\}, \\ w = w^{-\infty} & \text{a.e. in } \{u < -k\}. \end{cases} \tag{1.2.17}$$

DEFINITION 1.31. Let $\mu \in M(\Omega)$. A function $u \in \mathcal{T}_0^{1,p}(\Omega)$ is a *renormalized solution* of problem (1.2.11), if the following conditions hold:

(a) $|\nabla u|^{p-1} \in L^q(\Omega) \qquad \forall\, q < \frac{N}{N-1}$;

(b) for any $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ that satisfies condition P_u, then

$$\int_\Omega a(x, \nabla u) \cdot \nabla w \, dx = \int_\Omega w \, d\mu_0 + \int_\Omega w^{+\infty} \, d\mu_s^+ - \int_\Omega w^{-\infty} \, d\mu_s^- . \tag{1.2.18}$$

REMARK 1.32. Notice that all terms in (1.2.18) are well defined; in fact, as far as the first term is concerned, it can be rewritten as

$$\int_{\{|u|\le k\}} a(x, \nabla u) \cdot \nabla w \, dx + \int_{\{|u|>k\}} a(x, \nabla u) \cdot \nabla w \, dx; \tag{1.2.19}$$

for $k > 0$ and $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfying condition P_u; so,

$$\int_{\{|u|\le k\}} a(x, \nabla u) \cdot \nabla w \, dx = \int_{\{|u|\le k\}} a(x, \nabla T_k(u)) \cdot \nabla w \, dx, \tag{1.2.20}$$

is well defined since, thanks to assumption (1.2.2), $a(x, \nabla T_k(u)) \in (L^{p'}(\Omega))^N$ and $\nabla w \in (L^p(\Omega))^N$. On the other hand, the second term of (1.2.20) makes sense since, w satisfy assumption P_u, and so $\nabla w \in L^\infty(\{|u| > k\})$, while $a(x, \nabla u) \in (L^q(\Omega))^N$ for any $q < \frac{N}{N-1}$. The right hand side of (1.2.18) makes sense as well, since, using Theorem 1.25

$$\int_\Omega w \, d\mu_0,$$

is well defined because of the fact that $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, while there are no problem to give sense at the last two terms of (1.2.18) , since $w^{+\infty}$ e $w^{-\infty}$ are two bounded and continuous functions on Ω. Let us also observe that we can choose in (1.2.18) $w \in C_0^\infty(\Omega)$ (with $w^{+\infty} = w^{-\infty} = w$), and so a renormalized solution turns out to be a distributional solution of problem (1.2.11).

As we mentioned above, a renormalized solution turns out to coincide with an entropy solution if $\mu \in M_0(\Omega)$; actually we can easily prove the following

PROPOSITION 1.33. *Let $\mu \in M_0(\Omega)$. Then, problem 1.2.11 has at most one renormalized solution.*

PROOF. Thanks to Theorem 1.29, it will be enough to prove that, if u is a renormalized solution of problem (1.2.11), then u is an entropy solution of the same problem. For any $h > 0$, we can choose in (1.2.18), $w = T_h(u - \varphi)$, with $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$; in fact, we have

$$w = T_h(T_{h+M}(u) - \varphi),$$

where $M = \|\varphi\|_{L^\infty(\Omega)}$, and so $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$; moreover, w satisfy condition P_u since we can choose $w^{+\infty} = h$, $w^{-\infty} = -h$ and $k = h + M$. Hence, using $w = T_h(u - \varphi)$ in (1.2.18) one can readily check that u is an entropy solution (with equality sing) being $\mu_s^+ = \mu_s^- = 0$. ☐

The main result in [**DMOP**] is the following:

THEOREM 1.34. *Let $\mu \in M(\Omega)$. Then there exists a renormalized solution of problem (1.2.11).*

The proof of the above result basically relies on the proof of the strong convergence in $W_0^{1,p}(\Omega)$ of the truncates of approximating sequence of solutions.

Let us give another definition of renormalized solution (equivalent to the one given in Definition 1.31, Theorem 2.33 in [**DMOP**]); this definition emphasizes a *reconstruction property* of renormalized solutions that we will try to adapt to the parabolic case in Chapter 4.

DEFINITION 1.35. Let $\mu \in M(\Omega)$. A function $u \in \mathcal{T}_0^{1,p}(\Omega)$ is a *renormalized solution* of problem (1.2.11), if condition (a) of Definition 1.31 is satisfied and if the following conditions hold:

(b) for any $\varphi \in C(\overline{\Omega})$ we have

$$\lim_{n\to\infty} \frac{1}{n} \int_{\{n\le u<2n\}} a(x,\nabla u)\cdot\nabla u\,\varphi\; dx = \int_\Omega \varphi\; d\mu_s^+, \tag{1.2.21}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \int_{\{-2n<u<\le -n\}} a(x,\nabla u)\cdot\nabla u\,\varphi\; dx = \int_\Omega \varphi\; d\mu_s^-; \tag{1.2.22}$$

(c) for every S in $W^{1,\infty}(\mathbb{R})$ with compact support in $\mathbb{R}$ we have

$$\int_\Omega a(x,\nabla u)\cdot\nabla u\, S'(u)\; dx + \int_\Omega a(x,\nabla u)\cdot\nabla\varphi\, S(u)\; dx = \int_\Omega S(u)\,\varphi\; d\mu_0\,, \tag{1.2.23}$$

for every $\varphi \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ such that $S(u)\varphi \in W_0^{1,p}(\Omega)$.

In [**DMOP**] the authors also proved some interesting partial uniqueness results; for instance, under slightly stronger assumptions on a, if the difference between two renormalized solutions is bounded then these turn out to coincide. This type of results, as well as Definition 1.35 itself, suggest that a key role in the notion of renormalized solution is played by the behavior of the energy of the solution on the set where it turns out to be large. Finally notice that, in the linear case, the renormalized solution is unique for any measure in $M(\Omega)$, since it turns out to coincide with a duality solution of the same problem (see [**DMOP**], Theorem 10.1).

1.2.8. Lower order terms with natural growth. A large number of papers was devoted to the study of both elliptic and parabolic problems with nonlinear absorption terms with natural growth in the gradient whose model is

$$\begin{cases} -\Delta u + g(u)|\nabla u|^2 = f & \text{in } (0,T)\times\Omega,\\ u(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2.24}$$

These type of problems has, even in the more general case of superlinear growth with respect to the gradient, an interesting variational meaning; actually, if we consider a functional on $W_0^{1,p}(\Omega)$ as

$$J(v) = \frac{1}{p}\int_\Omega a(x,v)|\nabla v|^p - \int_\Omega f v\,,$$

with $f \in L^{p'}(\Omega)$ e $0<\alpha\le a(x,s)\le\beta$ $\alpha,\beta>0$. Then $J(v)$ is convex with respect to ∇v and weakly lower semicontinuous, and so there exists $u\in W_0^{1,p}(\Omega)$ such that $J(u)\le J(v)$, for any $v\in W_0^{1,p}(\Omega)$. Let us formally write down the *Euler equation* for u; we readily obtain

$$\int_\Omega a(x,u)|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi + \frac{1}{p}\int_\Omega a_s(x,u)|\nabla u|^p\varphi = \int_\Omega f\varphi\; dx\,, \tag{1.2.25}$$

for any $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$; the weak form of problem (1.2.25), that is

$$\begin{cases} -\mathrm{div}(a(x,u)|\nabla u|^{p-2}\nabla u) + \dfrac{1}{p} a_s(x,u)|\nabla u|^p = f \quad \text{in } \Omega, \\ \\ u \in W_0^{1,p}(\Omega)\,, \quad f \in L^{p'}(\Omega). \end{cases} \tag{1.2.26}$$

For an exhaustive review on this topic we refer to [**Po2**], and references therein; actually, we will focus our attention on the quasilinear case (1.2.24) with natural growth (that is $p = 2$ in (1.2.25)); here $g : \mathbb{R} \to \mathbb{R}$ is a real function in $C^1(\mathbb{R})$ such that

$$g(s)s \geq 0, \ \forall s \in \mathbb{R},$$
$$g'(s) > 0, \forall s \in \mathbb{R}$$

and $f(x) \in L^1(\Omega)$. In a more general context, in [**BGO2**], it was proved that (1.2.24) admits a weak solution, that is a function $u \in H_0^1(\Omega)$ such that $g(u)|\nabla u|^2 \in L^1(\Omega)$ satisfies

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega g(u)|\nabla u|^2 \varphi = \int_\Omega f \varphi\,,$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

1.3. Parabolic case

Following the line of the previous section for elliptic problems, here we want to give some basic knowledge on what has been done, up to now, about partial differential equations of parabolic type with measures as data. We will first introduce the notion of parabolic p-capacity and then we shall deal with initial boundary value problems related to parabolic operators of Leray-Lions type.

1.3.1. Parabolic p-capacity. We recall the notion of *parabolic p-capacity* associated to our problem (for further details see [**P**], [**DPP**]).

Definition 1.36. Let $Q = Q_T = (0,T) \times \Omega$ for any fixed $T > 0$, and let us recall that $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$, endowed with its natural norm $\|\cdot\|_{W_0^{1,p}(\Omega)} + \|\cdot\|_{L^2(\Omega)}$ and

$$W = \left\{ u \in L^p(0,T;V), \ u_t \in L^{p'}(0,T;V') \right\}, \tag{1.3.1}$$

endowed with its natural norm $\|u\|_W = \|u\|_{L^p(0,T;V)} + \|u_t\|_{L^{p'}(0,T;V')}$. So, if $U \subseteq Q$ is an open set, we define the *parabolic p-capacity* of U as

$$\mathrm{cap}_p(U) = \inf\{\|u\|_W : u \in W, u \geq \chi_U \text{ a.e. in } Q\},$$

where as usual we set $\inf \emptyset = +\infty$, then for any Borel set $B \subseteq Q$ we define

$$\mathrm{cap}_p(B) = \inf\{\mathrm{cap}_p(U), U \text{ open set of } Q, B \subseteq U\}.$$

As we mentioned above, we will denote with $M(Q)$ the set of all Radon measures with bounded variation on Q, while $M_0(Q)$ will denote the set of all measures with bounded variation over Q that do not charge the sets of zero p-capacity, that is if $\mu \in M_0(Q)$, then $\mu(E) = 0$, for all $E \subseteq Q$ such that $\mathrm{cap}_p(E) = 0$.

In [**DPP**] the authors give another notion of parabolic capacity, equivalent to the one given here as far as sets of zero capacity are concerned; this definition of capacity can be alternatively given starting from the compact sets in Q, as follows. As we said, we denote $C_0^\infty([0,T] \times \Omega)$ the space of restrictions to Q of smooth functions in $\mathbb{R} \times \mathbb{R}^N$ with compact support in $\mathbb{R} \times \Omega$.

Definition 1.37. Let K be a compact subset of Q. The capacity of K is defined as:

$$\mathrm{CAP}(K) = \inf\left\{\|u\|_W : u \in C_c^\infty([0,T] \times \Omega),\ u \geq \chi_K\right\}.$$

The capacity of any open subset U of Q is then defined by:

$$\mathrm{CAP}(U) = \sup\left\{\mathrm{CAP}(K),\ K \text{ compact, } K \subset U\right\},$$

and the capacity of any Borel set $B \subset Q$ by

$$\mathrm{CAP}(B) = \inf\left\{\mathrm{CAP}(U),\ U \text{ open subset of } Q,\ B \subset U\right\}.$$

This second definition of capacity, that enjoys the subadditivity property as well as the first we gave, given for compact subsets is used in [**DPP**] to prove Theorem 1.9 above, and it will turn out to be very useful to our aim since it allows to extend the notion of parabolic capacity to sets with respect to any open set contained in Q.

Proposition 1.38. *Let B be a Borel subset of Q. Then one has* $\mathrm{CAP}(B) = 0$ *if and only if* $\mathrm{cap}_p(B) = 0$.

Proof. See [**DPP**], Proposition 2.14. □

In [**DPP**] the authors also proved the following decomposition theorem:

Theorem 1.39. *Let μ be a bounded measure on Q. If $\mu \in M_0(Q)$ then there exist $h \in L^{p'}(0,T; W^{-1,p'}(\Omega))$, $g \in L^p(0,T; W_0^{1,p}(\Omega) \cap L^2(\Omega))$ and $f \in L^1(Q)$, such that*

$$\int_Q \varphi \, d\mu = \int_0^T \langle h, \varphi \rangle \, dt - \int_0^T \langle \varphi_t, g \rangle \, dt + \int_Q f\varphi \, dxdt, \tag{1.3.2}$$

for any $\varphi \in C_c^\infty([0,T] \times \Omega)$, where $\langle \cdot, \cdot \rangle$ denotes the duality between $(W_0^{1,p}(\Omega) \cap L^2(\Omega))'$ and $W_0^{1,p}(\Omega) \cap L^2(\Omega)$.

Proof. See [**DPP**], Theorem 1.1. □

So, if μ is in $M(Q)$, thanks to a well known decomposition result (see for instance [**FST**]), we can split it into a sum (uniquely determined) of its *absolutely continuous* part μ_0 with respect to p-capacity and its *singular* part μ_s (that is μ_s is concentrated on a set of zero p-capacity). Hence, if $\mu \in M(Q)$, by Theorem 1.39, we have

$$\mu = f - \operatorname{div}(G) + g_t + \mu_s, \tag{1.3.3}$$

in the sense of distributions, for some $f \in L^1(Q)$, $G \in (L^{p'}(Q))^N$, $g \in L^p(0,T;W_0^{1,p}(\Omega))$, and $\mu_s \perp p$-capacity; note that the decomposition of the absolutely continuous part of μ in Theorem 1.39 is not uniquely determined.

Let us state some further results about parabolic p-capacity; the first two are characterizations of the relationship between sets of zero parabolic capacity and sections of the parabolic cylinder with both zero $\mathcal{L}^N$ measure sets and zero elliptic p-capacity sets, while the third one shows that any function in W admits a cap_p-quasi continuous representative. Let us recall that a function u is called cap–quasi continuous if for every $\varepsilon > 0$ there exists an open set F_ε, with $\mathrm{cap}_p(F_\varepsilon) \le \varepsilon$, and such that $u_{|Q\setminus F_\varepsilon}$ (the restriction of u to $Q \setminus F_\varepsilon$) is continuous in $Q \setminus F_\varepsilon$. As usual, a property will be said to hold cap–quasi everywhere if it holds everywhere except on a set of zero capacity.

Theorem 1.40. *Let B be a Borel set in Ω. Let $t_0 \in (0,T)$. One has*

$$\mathrm{cap}_p(\{t_0\} \times B) = 0 \quad \textit{if and only if} \quad \mathrm{meas}_{\,\Omega}(B) = 0.$$

Proof. See [**DPP**], Theorem 2.15. □

Notice that, by virtue of Theorem 1.40, if a measure is concentrated on a section $\{t_0\} \times \Omega$, it does not charge sets of zero parabolic capacity if and only if it belongs to $L^1(\Omega)$.

Theorem 1.41. *Let $B \subset \Omega$ be a Borel set, and $0 \le t_0 < t_1 \le T$. Then we have*

$$\mathrm{cap}_p((t_0,t_1) \times B) = 0 \quad \textit{if and only if} \quad \mathrm{cap}_p^e(B) = 0.$$

Proof. See [**DPP**], Theorem 2.16. □

Theorem 1.42. *Any element v of W has a cap–quasi continuous representative $\tilde{v}$ which is cap–quasi everywhere unique, in the sense that two cap–quasi continuous representatives of v are equal except on a set of zero capacity.*

Proof. See [**DPP**], Lemma 2.20. □

1.3.2. Parabolic problems with measure data. As before, let Ω be a bounded, open subset of $\mathbb{R}^N$, T a positive number and $Q = (0,T) \times \Omega$. Here, as well as for the elliptic case, we want to give a brief review on results concerning parabolic initial boundary value problem

$$\begin{cases} u_t + A(u) = \mu & \text{in } (0,T) \times \Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{1.3.4}$$

where A is a nonlinear pseudomonotone and coercive operator in divergence form which acts from the space $L^p(0,T;W_0^{1,p}(\Omega))$ into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$.

Let $a : (0,T)\times\Omega\times\mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function (i.e., $a(\cdot,\cdot,\xi)$ is measurable on Q for every ξ in $\mathbb{R}^N$, and $a(t,x,\cdot)$ is continuous on $\mathbb{R}^N$ for almost every (t,x) in Q), such that the following holds:

$$a(t,x,\xi)\cdot\xi \geq \alpha\,|\xi|^p\,, \quad p > 1\,, \tag{1.3.5}$$

$$|a(t,x,\xi)| \leq \beta\,[b(t,x) + |\xi|^{p-1}]\,, \tag{1.3.6}$$

$$[a(t,x,\xi) - a(t,x,\eta)](\xi - \eta) > 0\,, \tag{1.3.7}$$

for almost every (t,x) in Q, for every ξ, η in $\mathbb{R}^N$, with $\xi \neq \eta$, where α and β are two positive constants, and b is a nonnegative function in $L^{p'}(Q)$.

We define the differential operator

$$A(u) = -\mathrm{div}(a(t,x,\nabla u))\,, \qquad u \in L^p(0,T;W_0^{1,p}(\Omega))\,.$$

Under assumptions (1.3.5), (1.3.6) and (1.3.7), A is a coercive and pseudomonotone operator acting from the space $L^p(0,T;W_0^{1,p}(\Omega))$ into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$.

As a model example, problem (1.3.4) includes the p-Laplace evolution problem:

$$\begin{cases} u_t - \mathrm{div}(|\nabla u|^{p-2}\nabla u) = \mu & \text{in } (0,T) \times \Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{1.3.8}$$

We are interested in the study of problem (1.3.4) in presence of measure data μ and u_0. If $\mu \in L^{p'}(Q)$ and $u_0 \in L^2(\Omega)$, (1.3.4) has a unique solution in $W \cap C(0,T;L^2(\Omega))$ (where W was defined in (1.1.6)) in the weak sense, that is

$$-\int_\Omega u_0\,\varphi(0)\,dx - \int_0^T \langle \varphi_t, u\rangle\,dt + \int_Q a(t,x,\nabla u)\cdot\nabla\varphi\,dxdt = \int_0^T \langle f,\varphi\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\,dt,$$

for all $\varphi \in W$ such that $\varphi(T) = 0$ (see [**L**]).

Under the general assumption that μ and u_0 are bounded measures, the existence of a distributional solution was proved in [**BDGO**], by approximating (1.3.4) with problems having regular data and using compactness arguments.

Unfortunately, as in the elliptic case, due to the lack of regularity of the solutions, the distributional formulation is not strong enough to provide uniqueness, as it can be proved by adapting to the parabolic case the counterexample of J. Serrin cited above for the stationary problem (see (1.2.13)).

In case of linear operators the difficulty can be overcome again by defining the solution in a duality sense, by adapting the techniques of the stationary case; in fact, for simplicty, let us consider the linear parabolic problem

$$\begin{cases} u_t + L(u) = f & \text{in } (0,T)\times\Omega, \\ u(0) = 0, & \text{in } \Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{1.3.9}$$

with $f \in L^1(Q)$, where $L(u) = -\mathrm{div}(M(t,x)\nabla u)$, and M is a matrix with bounded, measurable entries, and satisfying the ellipticity assumption (1.3.5) ($p = 2$).

One can say that $u \in L^1(Q)$ is a *duality solution* of problem (1.3.9) if

$$\int_Q u\, g\; dxdt = \int_Q f\, w\; dxdt\,,$$

for every $g \in L^\infty(Q)$, and w is the solution of the *retrograde problem*

$$\begin{cases} -w_t - \mathrm{div}(M^*(t,x)\nabla w) = g & \text{in } (0,T)\times\Omega, \\ w(T,x) = 0 & \text{in } \Omega, \\ w(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{1.3.10}$$

where $M^*(t,x)$ is the transposed matrix of $M(t,x)$. In fact, let us fix $r, q \in \mathbb{R}$ such that

$$r,\, q > 1, \qquad \frac{N}{q} + \frac{2}{r} < 2\,,$$

and let us consider $g \in L^r(0,T;L^q(\Omega)) \cap L^\infty(Q)$. Let w be the solution of problem (1.3.10); standard parabolic regularity results say that

$$\|w\|_{L^\infty(Q)} \le C\|g\|_{L^r(0,T;L^q(\Omega))},$$

and so the linear and continuous functional

$$\Lambda : L^r(0,T;L^q(\Omega)) \mapsto \mathbb{R},$$

defined by

$$\Lambda(g) = \int_Q f\, w\; dxdt,$$

is well defined, since

$$|\Lambda(g)| \leq \|f\|_{L^1(Q)}\|w\|_{L^\infty(Q)} \leq C\|f\|_{L^1(Q)}\|g\|_{L^r(0,T;L^q(\Omega))}.$$

So, by *Riesz's representation theorem* there exists a unique $u \in L^{r'}(0,T;L^{q'}(\Omega))$ such that

$$\Lambda(g) = \int_Q u\, g\; dxdt,$$

for any $g \in L^r(0,T;L^q(\Omega))$.

This easily implies

THEOREM 1.43. *If $f \in L^1(Q)$, then there exists a unique duality solution of problem (1.3.9).*

With slightly modification on the proof one can prove that a unique solution in a duality sense also exists for linear problems with a smooth initial data.

A standard approximation argument shows that a unique solution even exists for problem

$$\begin{cases} u_t + L(u) = \mu & \text{in } (0,T)\times\Omega, \\ u(0) = u_0 & \text{in } \Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{1.3.11}$$

for any $\mu \in M(Q)$ and $u_0 \in L^1(\Omega)$. In this case one can prove that there exists a unique $u \in L^1(Q)$ such that

$$-\int_\Omega u_0 w(0)\; dx + \int_Q u\, g\; dxdt = \int_Q w\; d\mu, \tag{1.3.12}$$

for every $g \in C_0^\infty(Q)$; notice that all terms in the above formulation are well defined thanks to standard parabolic regularity results (see [**LSU**], [**E**]).

However, for nonlinear operators a new concept of solution needs to be defined to get a well–posed problem. In case of problem (1.3.4) with L^1 data, this was done independently in [**BM**] and in [**Pr2**] (see also [**AMST**]), where the notions of *renormalized* solution, and of *entropy* solution, respectively, were introduced. Both these approaches allow to obtain existence and uniqueness of solutions if $\mu \in L^1(Q)$ and $u_0 \in L^1(\Omega)$.

Let us give the notion of entropy solution for parabolic problem (1.3.4) with a general $\mu \in M_0(Q)$, recalling that

$$S^p = \{u \in L^p(0,T;W_0^{1,p}(\Omega)); u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)\},$$

and denoting, for $k > 0$, by

$$\Theta_k(z) = \int_0^z T_k(s)\; ds,$$

the primitive function of the truncation function.

DEFINITION 1.44. Let $\mu \in M_0(Q)$, $(f, g, -\mathrm{div}(G))$ a decomposition of μ and $u_0 \in L^1(\Omega)$. A measurable function u is an *entropy solution* of (1.3.4) if

$$T_k(u-g) \in L^p(0,T;W_0^{1,p}(\Omega)) \text{ for every } k>0, \tag{1.3.13}$$

$$t \in [0,T] \mapsto \int_\Omega \Theta_k(u-g-\varphi)(t,x)\, dx \tag{1.3.14}$$

is a continuous function for all $k \geq 0$ and all $\varphi \in S^p \cap L^\infty(Q)$, and moreover

$$\begin{aligned} &\int_\Omega \Theta_k(u-g-\varphi)(T,x)\, dx - \int_\Omega \Theta_k(u-g-\varphi)(0,x)\, dx \\ &+ \int_0^T \langle \varphi_t, u-g-\varphi\rangle\, dt + \int_Q a(t,x,\nabla u)\cdot \nabla T_k(u-g-\varphi)\, dxdt \\ &\leq \int_Q f T_k(u-g-\varphi)\, dxdt + \int_Q G\cdot \nabla T_k(u-g-\varphi)\, dxdt, \end{aligned} \tag{1.3.15}$$

for all $k \geq 0$ and all $\varphi \in S^p \cap L^\infty(Q)$.

In [**DPP**] the authors extend the result of existence and uniqueness to a larger class of measures which includes the L^1 case. Precisely, they prove (in the framework of renormalized solutions) that problem (1.3.4) has a unique solution for every u_0 in $L^1(\Omega)$ and for every measure μ which does not charge the sets of zero capacity, that is $\mu \in M_0(Q)$.

As we have seen before, the importance of the measures not charging sets of zero capacity was first observed in the stationary case in [**BGO**].

In order to use a similar approach in the evolution case, in [**DPP**] is developed the theory of capacity related to the parabolic operator $u_t + A(u)$ and then investigated the relationships between time–space dependent measures and capacity (see Theorems 1.39, 1.40 and 1.41 above).

As far as the initial datum is concerned, considering measure data which do not charge sets of zero parabolic capacity leads to take u_0 in $L^1(\Omega)$, so that no improvement can be obtained with respect to previous results. In fact, in virtue of Theorem 1.40, if a measure is concentrated on a section $\{t_0\} \times \Omega$, it does not charge sets of zero parabolic capacity if and only if it belongs to $L^1(\Omega)$. Here we introduced the capacity on subsets of the open set Q, but a different choice could also be done to compute the capacity of subsets of $[0,T] \times \Omega$. In this latter context one could take $t_0 = 0$ in Theorem 1.40 and regard u_0 as a measure concentrated at $t = 0$, which explains why we take $u_0 \in L^1(\Omega)$. However, this argument also suggest that there is no real need to define the capacity up to $t = 0$.

Thanks to the decomposition result of Theorem 1.39, if μ is *absolutely continuous* with respect to capacity (these are called *soft measures*) we can still set our problem (1.3.4) in the framework of renormalized solutions. The idea is that, since μ can be splitted as in (1.3.3), problem (1.3.4) can be formally rewritten as $(u-g)_t + A(u) = f - \mathrm{div}(G)$, and the renormalization argument can be applied to the difference $u-g$. Let us introduce the definition of renormalized solution of (1.3.4) given in [**DPP**].

DEFINITION 1.45. Let $\mu \in M_0(Q)$, $(f, g, -\mathrm{div}(G))$ a decomposition of μ and $u_0 \in L^1(\Omega)$. A measurable function u is a *renormalized solution* of (1.3.4) if

$$u - g \in L^\infty(0,T;L^1(\Omega))\,,\ T_k(u-g) \in L^p(0,T;W_0^{1,p}(\Omega)) \text{ for every } k>0, \tag{1.3.16}$$

$$\lim_{n\to\infty} \int\limits_{\{n\le|u-g|\le n+1\}} |\nabla u|^p\,dxdt = 0\,; \tag{1.3.17}$$

moreover, for every $S \in W^{2,\infty}(\mathbf{R})$ such that S' has compact support,

$$\begin{aligned}&(S(u-g))_t - \mathrm{div}(a(t,x,\nabla u)S'(u-g)) + S''(u-g)a(t,x,\nabla u)\nabla(u-g)\\ &= S'(u-g)f + S''(u-g)G\cdot\nabla(u-g) - \mathrm{div}(G\,S'(u-g))\end{aligned} \tag{1.3.18}$$

in the sense of distributions, and

$$S(u-g)(0) = S(u_0) \text{ in } L^1(\Omega). \tag{1.3.19}$$

THEOREM 1.46. *Let μ be a bounded measure on Q which does not charge the subsets of Q of zero capacity, and let $u_0 \in L^1(\Omega)$. Then there exists a unique renormalized solution u of (1.3.4). Moreover u satisfies the additional regularity: $u \in L^\infty(0,T;L^1(\Omega))$ and $T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega))$, for every $k>0$.*

PROOF. See [**DPP**], Theorem 1.3. □

Notice that the notion of renormalized solution and entropy solution for parabolic problem (1.3.4) turn out to be equivalent as proved in [**DP**]; in Chapter 4 we extend the notion of renormalized solution for general measure data $\mu \in M(Q)$ and so, thanks to this result, this notion will turn out to be coherent with all definitions of solution given before for problem (1.3.4).

To conclude this section, let us just mention the case of quasilinear parabolic equation with absorption term in relation with the stationary case (1.2.24); as in this case, to our purpose is enough to deal with the model case

$$\begin{cases} u_t - \Delta u + g(u)|\nabla u|^2 = f & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{1.3.20}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $N \geq 2$, $u_0 \in L^1(\Omega)$, and $g : \mathbb{R} \to \mathbb{R}$ is a real function in $C^1(\mathbb{R})$ such that

$$g(s)s \geq 0, \ \forall s \in \mathbb{R}, \tag{1.3.21}$$

$$g'(s) > 0, \forall s \in \mathbb{R}, \tag{1.3.22}$$

and $f(x) \in L^1(Q)$.

For a solution of problem (1.3.20) we mean a function $u \in L^2(0,T;H_0^1(\Omega))$ which satisfies

$$\int_0^T \langle u_t, \varphi\rangle + \int_Q \nabla u\cdot\nabla\varphi + \int_Q g(u)|\nabla u|^2\varphi = \int_Q f\varphi, \tag{1.3.23}$$

for any $\varphi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$ and such that $g(u)|\nabla u|^2$ belongs to $L^1(Q)$. Here the symbol $\langle\cdot,\cdot\rangle$ denote the duality between functions of $L^2(0,T;H^{-1}(\Omega)) + L^1(Q)$ and functions in $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$; in fact, such a solutions turns out to have time derivative in $L^2(0,T;H^{-1}(\Omega)) + L^1(Q)$, and in particular, thanks to Theorem 1.6, they belong to $C(0,T;L^1(\Omega))$, and so we mean that the initial datum is achieved in the sense of $L^1(\Omega)$.

Problem (1.3.20) has a solution in the sense of (1.3.23) as proved in [**DO2**] (see also [**Po1**], and [**BP**], and references therein for improvements to more general cases).

CHAPTER 2

Asymptotic behavior of solutions for parabolic operators of Leray-Lions type and measure data

A large number of papers was devoted to the study of asymptotic behavior for solution of parabolic problems under various assumptions and in different contexts: for a review on classical results see [**F**], [**A**], [**Sp**], and references therein. More recently in [**G**] the same problem was studied for bounded data and a class of operators rather different to the one we will discuss.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set, $N \geq 2$, and let $p > 1$; we are interested in the asymptotic behavior with respect to the time variable t of the entropy solution of parabolic problems whose model is

$$\begin{cases} u_t(t,x) - \Delta_p u(t,x) = \mu & \text{in } (0,T) \times \Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{2.0.1}$$

where $T > 0$ is any positive constant, $u_0 \in L^1(\Omega)$ is a nonnegative function, and $\mu \in M_0(Q)$ is a nonnegative measure with bounded variation over $Q = (0,T) \times \Omega$ which does not charge the sets of zero p-capacity in accordance with Definition 1.36; moreover we suppose that μ does not depend on the time variable t (i.e. there exists a bounded Radon measure ν on Ω such that, for any Borel set $B \subseteq \Omega$, and $0 < t_0,\ t_1 < T$, we have $\mu(B \times (t_0, t_1)) = (t_1 - t_0)\nu(B)$). Actually we shall investigate the limit as T tends to infinity of the solution $u(T,x)$ of the problem (2.0.1). Observe that by virtue of uniqueness results concerning entropy solutions of (2.0.1) we have $u_T(t,x) = u_{T'}(t,x)$ a.e. in Ω, for all $T > T'$ and $t \in (0,T')$, where the index T and T' indicate that we deal with the solution of problem (2.0.1) respectively on Q_T and $Q_{T'}$; so the function $u(t,x) = u_T(t,x),\ T > t$ is well defined for all $t > 0$; we are interested in the asymptotic behavior of $u(t,x)$ as t tends to infinity. Actually, for a larger class of problem than (2.0.1), we shall prove that, as t tends to infinity, $u(t,x)$ converges in $L^1(\Omega)$ to $v(x)$, the entropy solution of the corresponding elliptic problem

$$\begin{cases} -\Delta_p v(x) = \mu & \text{in } \Omega, \\ v(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.0.2}$$

2.1. General assumptions and main result

For the sake of exposition we recall our assumption on the operator; let $a : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function (i.e. $a(\cdot,\xi)$ is measurable on Ω, $\forall\xi \in \mathbb{R}^N$, and $a(x,\cdot)$ is continuous on $\mathbb{R}^N$ for a.e. $x \in \Omega$) such that the following holds:

$$a(x,\xi)\cdot\xi \geq \alpha|\xi|^p, \tag{2.1.1}$$

$$|a(x,\xi)| \leq \beta[b(x) + |\xi|^{p-1}], \tag{2.1.2}$$

$$(a(x,\xi) - a(x,\eta))\cdot(\xi-\eta) > 0, \tag{2.1.3}$$

for almost every $x \in \Omega$, for all $\xi,\ \eta \in \mathbb{R}^N$ with $\xi \neq \eta$, where $p > 1$ and $\alpha,\ \beta$ are positive constants and b is a nonnegative function in $L^{p'}(\Omega)$. We shall deal with the solutions of the initial boundary value problem

$$\begin{cases} u_t + A(u) = \mu & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{2.1.4}$$

where $A(u) = -\mathrm{div}(a(x,\nabla u))$, μ is a nonnegative measure with bounded variation over Q that does not depend on time, and $u_0 \in L^1(\Omega)$.

Recall that $M_0(\Omega)$ is the set of all measures with bounded variation over Ω that do not charge the sets of zero elliptic p-capacity, that is if $\mu \in M_0(\Omega)$, then $\mu(E) = 0$, for all $E \in \Omega$ such that $\mathrm{cap}_p^e(E) = 0$; analogously we denote $M_0(Q)$ the set of all measures with bounded variation over Q that does not charge the sets of zero parabolic p-capacity, that is if $\mu \in M_0(Q)$ then $\mu(E) = 0$, for all $E \in Q$ such that $\mathrm{cap}_p(E) = 0$.

As we said before, in [**B6**] (for more details see also [**BGO**]) the concept of entropy solution of the elliptic boundary value problem associated to (2.1.4) was introduced: let $\mu \in M_0(\Omega)$ be a measure with bounded variation over Ω which does not charge the sets of zero elliptic p-capacity; we know that v is an entropy solution for the boundary value problem

$$\begin{cases} A(v) = \mu & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1.5}$$

if v is finite a.e., its truncated function $T_k(v) \in W_0^{1,p}(\Omega)$, for all $k > 0$, and it holds

$$\int_\Omega a(x,\nabla v)\cdot\nabla T_k(v-\varphi)\ dx \leq \int_\Omega T_k(v-\varphi)\ d\mu, \tag{2.1.6}$$

for all $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, for all $k > 0$; observe that the gradient of such a solution v is not in general well defined in the sense of distributions, anyway it is possible to give a sense to (2.1.6), using the notion of *approximated gradient* of v, defined as the a.e.

unique measurable function that coincides a.e. with $\nabla T_k(v)$, over the set where $|v| \le k$, for every $k > 0$ (see Section 1.1). Such a solution exists and is unique for all measures in $M_0(\Omega)$ (see Theorem 1.29), and turns out to be a distributional solution of problem (2.1.5); moreover such a solution satisfies (2.1.6) with the equality sign (see [**DMOP**]). We finally remind the analogous definition given in the parabolic case in [**Pr2**] (see also [**DP**]). Let $k > 0$ and define

$$\Theta_k(z) = \int_0^z T_k(s)\, ds,$$

as the primitive function of the truncation function; let $\mu \in M_0(Q)$ and $u_0 \in L^1(\Omega)$, then we say that $u(t,x) \in C(0,T;L^1(\Omega))$ is an entropy solution of the problem

$$\begin{cases} u_t + A(u) = \mu & \text{in } (0,T) \times \Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(t,x) = 0, & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{2.1.7}$$

if, for all $k > 0$, we have that $T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega))$, and it holds

$$\begin{aligned} &\int_\Omega \Theta_k(u-\varphi)(T)\, dx - \int_\Omega \Theta_k(u_0 - \varphi(0))\, dx \\ &+ \int_0^T \langle \varphi_t, T_k(u-\varphi)\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)}\, dt \\ &+ \int_Q a(x,\nabla u)\cdot \nabla T_k(u-\varphi)\, dxdt \le \int_Q T_k(u-\varphi)\, d\mu, \end{aligned} \tag{2.1.8}$$

for any $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q) \cap C([0,T];L^1(\Omega))$ with $\varphi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$.

Remark 2.1. The entropy solution u of the problem (2.1.7) exists and is unique (see Theorem 1.46) and is such that $|a(x,\nabla u)| \in L^q(Q)$ for all $q < 1 + \dfrac{1}{(N+1)(p-1)}$, even if its approximated gradient may not belong to any Lebesgue space.

Finally observe that, if $p > \frac{2N+1}{N+1}$, the solution is regular enough to be continuous with values $L^1(\Omega)$; in fact such a solution, as we will explain below, turns out to belong to $L^s(0,T;W_0^{1,s}(\Omega))$, with $u_t \in L^1(Q) + L^{s'}(0,T;W^{-1,s'}(\Omega))$, for suitable $s > 1$, and so $u \in C(0,T;L^1(\Omega))$ thanks to the trace result of Theorem 1.6.

Using Theorem 1.41 we derive that measures of $M_0(Q)$ which do not depend on time can actually be identified with a measure in $M_0(\Omega)$; recall that, in [**BGO**] is proved that, if $\mu \in M_0(\Omega)$, then it may be decomposed as $\mu = f - \operatorname{div}(g)$, where $f \in L^1(\Omega)$ and $g \in (L^{p'}(\Omega))^N$. So, if μ is a measure of $M_0(Q)$ which does not depend on time and B is a Borel set in Ω of zero elliptic p-capacity, then thanks to Theorem 1.41 we deduce that

$\mathrm{cap}_p(B \times (0,T)) = 0$ and so $\mu(B \times (0,T)) = 0$; since μ is supposed to be independent on time, we have

$$0 = \mu(B \times (0,T)) = T\nu(B),$$

with $\nu \in M(\Omega)$, and so $\nu(B) = 0$, thus $\nu \in M_0(\Omega)$. Hence, from now on, we shall always identify μ and ν.

Moreover, notice that, if $\mu \geq 0$ is a measure in $M_0(\Omega)$, then f can be chosen to be nonnegative its decomposition as proved in Theorem 1.26.

Before passing to the statement and the proof of our main result let us state some interesting results about the entropy solution v of the elliptic problem (2.1.5); first of all, let us suppose $p > \frac{2N}{N+1}$, and observe that, in this case, such a solution actually turns out to be an entropy solution of the initial boundary value problem (2.1.7) with initial datum $u_0(x) = v(x)$, for all $T > 0$, since we have

$$\int_\Omega \Theta_k(v-\varphi)(T)\, dx - \int_\Omega \Theta_k(v-\varphi)(0)\, dx$$

$$= \int_Q \frac{d}{dt}\Theta_k(v-\varphi)\, dxdt = \int_0^T \int_\Omega T_k(v-\varphi)(v-\varphi)_t\, dxdt$$

$$= -\int_0^T \langle \varphi_t, T_k(v-\varphi)\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\, dt$$

that can be cancelled out with the analogous term in (2.1.8) getting the right formulation (2.1.6) for v.

As we said before, for technical reasons we shall use the stronger assumption that $p > \frac{2N+1}{N+1}$ throughout this chapter; notice that, in this case, the gradient of the entropy solution v (that coincides with the distributional one) actually belong to some Lebesgue space.

Moreover, observe that, if $\mu \in M_0(\Omega)$, and $\mu \geq 0$ then the entropy solution v of the elliptic problem is nonnegative; indeed, choosing in (2.1.6) as test function $\varphi = T_h(v^+)$, we get

$$\int_\Omega a(x,\nabla v)\cdot \nabla T_k(v - T_h(v^+))\, dx \leq \int_\Omega T_k(v - T_h(v^+))\, d\mu; \tag{2.1.9}$$

now, as we can write $\mu = f - \mathrm{div}(g) \in L^1(\Omega) + W^{-1,p'}(\Omega)$, and $T_k(v - T_h(v^+))$ converges, as h tends to infinity, to $T_k(-v^-)$ almost everywhere, $*$-weakly in $L^\infty(\Omega)$, and weakly in $W_0^{1,p}(\Omega)$ (see for instance [**BGO**]), we have

$$\lim_{h\to+\infty} \int_\Omega T_k(v - T_h(v^+))\, d\mu = \int_\Omega T_k(-v^-)\, d\mu \leq 0;$$

On the other hand, observing that, for $0 \leq v \leq h$ we have $T_k(v - T_h(v^+)) = T_k(-v^-) = 0$, we can split the left hand side of (2.1.9) into three terms:

$$\int_\Omega a(x, \nabla v) \cdot \nabla T_k(v - T_h(v^+))\ dx = \int_{\{v \leq 0\}} a(x, \nabla v) \cdot \nabla T_k(v)\ dx$$

$$+ \int_{\{h < v \leq h+k\}} a(x, \nabla v) \cdot \nabla (v - h)\ dx + \int_{\{v > h+k\}} a(x, \nabla v) \cdot \nabla h\ dx,$$

and the last term is obviously zero, while, using hypothesis (2.1.3), the second term is positive and we can drop it; therefore, passing to the limit on h in the right hand side of (2.1.9), and using hypothesis (2.1.1), we obtain

$$\alpha \int_\Omega |\nabla T_k(v^-)|^p\ dx = \alpha \int_{\{v \leq 0\}} |\nabla T_k(v^-)|^p\ dx$$

$$\leq \int_\Omega a(x, \nabla v) \cdot \nabla T_k(v - T_h(v^+))\ dx \leq 0,$$

and so $v \geq 0$ a.e. in Ω. Arguing analogously, with the use of a standard Landes regularization argument, one can prove that the entropy solution $u(t, x)$ of problem (2.1.4) turns out to be nonnegative if $\mu \geq 0$.

Now, we can state our main result for the homogeneous case, that is for problem (2.1.4); Section 2.2 will be devoted to the proof of this result, while in Section 2.3 we shall prove the same result for the nonhomogeneus problem with nonnegative initial data in $L^1(\Omega)$. Finally in Section 2.4, we will prove that the same result hold true also for general, possibly singular, measure data in the linear case.

THEOREM 2.2. *Let $\mu \in M_0(Q)$ be independent on the variable t, $p > \frac{2N+1}{N+1}$, and let $\mu \geq 0$; let $u(t, x)$ be the entropy solution of problem (2.1.4) with $u_0 = 0$, and $v(x)$ the entropy solution of the corresponding elliptic problem (2.1.5). Then*

$$\lim_{t \to +\infty} u(t, x) = v(x),$$

in $L^1(\Omega)$.

2.2. Homogeneous case

First of all, let us state and prove a comparison result that plays a key role in the proof of our main result. In the proof of this result and in what follows we will use several facts proved in [**Pr2**] for L^1 data and whose generalization to data in $L^1(Q) + L^p(0, T; W_0^{1,p}(\Omega))$ is quite simple; in particular, we will use the fact that the approximating sequences of variational solutions are strongly compacts in $C(0, T; L^1(\Omega))$.

LEMMA 2.3. *Let $u_0, v_0 \in L^1(\Omega)$, such that $0 \leq u_0 \leq v_0$, and let $\mu \in M_0(\Omega)$; if u and v are, respectively, the entropy solutions of the problems*

$$\begin{cases} u_t + A(u) = \mu & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0 & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{2.2.1}$$

and

$$\begin{cases} v_t + A(v) = \mu & \text{in } (0,T)\times\Omega, \\ v(0,x) = v_0 & \text{in } \Omega, \\ v(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{2.2.2}$$

then $u \leq v$ a.e. in Ω, for all $t \in (0,T)$.

PROOF. First of all, let us suppose u_0, $v_0 \in L^2(\Omega)$; we will use an approximation argument. Consider the entropy solutions of the same problems with datum $F \in W^{-1,p'}(\Omega)$ instead of μ; let us call also these solutions u and v. These solutions coincide with the variational ones with this kind of data. Therefore we can use the variational formulation of problems (2.2.1) and (2.2.2), integrating between 0 and t, for any $t \leq T$. Using $\varphi = (u-v)^+$ as test function, and then subtracting, we get

$$\begin{aligned} 0 &= \int_{Q_t} (u-v)_t (u-v)^+ \, dxdt \\ &\quad + \int_{Q_t} (a(x,\nabla u) - a(x,\nabla v)) \cdot \nabla (u-v)^+ \, dxdt \\ &= \frac{1}{2}\int_\Omega \int_0^t \frac{d}{dt}[(u-v)^+]^2 \, dxdt \\ &\quad + \int_{Q_t} (a(x,\nabla u) - a(x,\nabla v)) \cdot \nabla (u-v)^+ \, dxdt \\ &= \frac{1}{2}\int_\Omega [(u-v)^+]^2(t) \, dx - \frac{1}{2}\int_\Omega [(u-v)^+]^2(0) \, dx \\ &\quad + \int_{Q_t} (a(x,\nabla u) - a(x,\nabla v)) \cdot \nabla (u-v)^+ \, dxdt. \end{aligned}$$

Since the last term is positive we can drop it, while the second one is zero as $u_0 \leq v_0$. Therefore we have, for all $t \in (0,T)$

$$(u-v)^+ = 0 \text{ a.e. in } \Omega,$$

so $u \leq v$ a.e. in Ω, for all $t \in (0,T)$.

Now, let us consider u and v as the entropy solutions of problems (2.2.1) and (2.2.2) with nonnegative data in $M_0(\Omega)$, and u_0, v_0 in $L^1(\Omega)$ as initial data; as we can write $\mu = f - \operatorname{div}(g) \in L^1(\Omega) + W^{-1,p'}(\Omega)$, we can approximate the L^1 term f with a sequence f_n of nonnegative regular functions that converges to f in $L^1(\Omega)$ and such that $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$; moreover, let us consider two sequences of smooth functions $u_{0,n}$ and $v_{0,n}$ such that $u_{0,n}$ converges to u_0, and $v_{0,n}$ converges to v_0 in $L^1(\Omega)$, with $0 \leq u_{0,n} \leq v_{0,n}$.

So, we can apply the result proved above finding two sequence of solutions u_n and v_n of problems (2.2.1) and (2.2.2) with data $f_n - \operatorname{div}(g)$ and $u_{0,n}$, $v_{0,n}$ as initial data, for which the comparison result holds true; so $u_n \leq v_n$ for all $t \in (0,T)$, a.e. in Ω. Now, as we said before, the solutions of (2.2.1) and (2.2.2) obtained as limits of u_n and v_n are unique and coincide with the unique entropy solution of the limit problem with datum μ, and initial datum, respectively, u_0 and v_0; then we have that the sequences u_n and v_n converge, respectively, to u and v a.e. in Ω, for all fixed $t \in (0,T)$. So $u \leq v$ a.e. in Ω, for all fixed $t \in (0,T)$. □

Now, we are able to prove Theorem 2.2.

Proof of Theorem 2.2. For the sake of simplicity here we will denote by Q the parabolic cylinder $(0,1)\times\Omega$; let $n \in \mathbb{N}\cup\{0\}$, and define $u^n(t,x)$ as the entropy solution of the initial boundary value problem

$$
\begin{cases}
u^n_t + A(u^n) = \mu & \text{in } (0,1)\times\Omega,\\
u^n(0,x) = u(n,x) & \text{in } \Omega,\\
u^n(t,x) = 0 & \text{on } (0,1)\times\partial\Omega,
\end{cases}
\tag{2.2.3}
$$

Recall that, since $u \in C(0,T;L^1(\Omega))$, then $u(n,x) \in L^1(\Omega)$ is well defined; moreover, observe that, by virtue of uniqueness of entropy solution and the definition of u^n, recalling that $u(0,x) = 0$, we have that $u(n,x) = u^{n-1}(1,x)$ a.e. in Ω, for $n \geq 1$. Now, applying Lemma 2.3, and recalling that v, the entropy solution of problem (2.1.5), is also an entropy solution of problem (2.1.7) with $v \geq 0$ itself as initial datum, we get immediately that

$$
u(t,x) \leq v(x), \quad \text{for all } \ t \in (0,T), \text{a.e. in } \ \Omega,
\tag{2.2.4}
$$

being $u(t,x)$ solution of the same problem with $u(0,x) = 0$ as initial datum. Moreover, applying again Lemma 2.3, we get that, for every $n \geq 0$

$$
u^n(t,x) \leq v(x), \quad \text{for all } \ t \in (0,1), \text{a.e. in } \ \Omega,
\tag{2.2.5}
$$

Finally, if we consider a parameter $s > 0$ we have that both $u(t,x)$ and $u_s(t,x) \equiv u(t+s,x)$ are solutions of problem (2.1.7) with, respectively, 0 and $u(s,x) \geq 0$ as initial datum; so, again from Lemma 2.3 we deduce that $u(t+s,x) \geq u(t,x)$ for $t,s > 0$, and so u is a monotone nondecreasing function in t. In particular $u(n,x) \le u(m,x)$ for all $n, m \in \mathbb{N}$ with $n < m$, and so we have

$$u^n(t,x) \le u^{n+1}(t,x),$$

for all $n \ge 0$ and $t > 0$.

Therefore, from the monotonicity of u^n, it follows that exists a function $\tilde{u}$ such that, $u^n(t,x)$ converges to $\tilde{u}(t,x)$ a.e. on Q as n tends to infinity.

Now, let us look for some *a priori estimates* concerning the sequence u^n.

Let us fix n and take $\varphi = 0$ in the entropy formulation for u^n; we get

$$\begin{aligned} &\int_\Omega \Theta_k(u^n)(1) + \alpha \int_Q |\nabla T_k(u^n)|^p \, dxdt \\ &\le k(|\mu|_{M_0(Q)} + \|u(n,x)\|_{L^1(\Omega)}) \\ &\le k(|\mu|_{M_0(Q)} + \|v\|_{L^1(\Omega)}) = Ck. \end{aligned} \tag{2.2.6}$$

Therefore, for every fixed $k > 0$, from the first term on the left hand side of (2.2.6), recalling that $u^n(t,x)$ is monotone nondecreasing in t, we get, arguing as in [**BDGO**], that u^n is uniformly bounded in $L^\infty(0,1;L^1(\Omega))$, while from the second one we have that $T_k(u^n)$ is uniformly bounded in $L^p(0,1;W_0^{1,p}(\Omega))$.

We can improve this kind of estimate by using the *Gagliardo-Nirenberg* Corollary 1.8. Indeed, this way, we get

$$\int_Q |T_k(u^n)|^{p+\frac{p}{N}} \, dxdt \le Ck \tag{2.2.7}$$

and so, we can write

$$k^{p+\frac{p}{N}} \text{meas}\{|u^n| \ge k\} \le \int_{\{|u^n| \ge k\}} |T_k(u^n)|^{p+\frac{p}{N}} \, dxdt \le \int_Q |T_k(u^n)|^{p+\frac{p}{N}} \, dxdt \le Ck;$$

then,

$$\text{meas}\{|u^n| \ge k\} \le \frac{C}{k^{p-1+\frac{p}{N}}} \ . \tag{2.2.8}$$

Therefore, the sequence u^n is uniformly bounded in the *Marcinkiewicz* space $M^{p-1+\frac{p}{N}}(Q)$; that implies, since in particular $p > \frac{2N}{N+1}$, that u^n is uniformly bounded in $L^m(Q)$ for all $1 \le m < p - 1 + \frac{p}{N}$.

We are interested about a similar estimate on the gradients of functions u^n; let us emphasize that these estimate hold true for all functions satisfying (2.2.6), so we will not write the index n. First of all, observe that

$$\text{meas}\{|\nabla u| \ge \lambda\} \le \text{meas}\{|\nabla u| \ge \lambda; |u| \le k\} + \text{meas}\{|\nabla u| \ge \lambda; |u| > k\}. \tag{2.2.9}$$

With regard to the first term to the right hand side of (2.2.9) we have

$$\begin{aligned} \operatorname{meas}\{|\nabla u| \geq \lambda; |u| \leq k\} &\leq \frac{1}{\lambda^p} \int_{\{|\nabla u| \geq \lambda; |u| \leq k\}} |\nabla u|^p \, dx \\ &= \frac{1}{\lambda^p} \int_{\{|u| \leq k\}} |\nabla u|^p \, dx = \frac{1}{\lambda^p} \int_Q |\nabla T_k(u)|^p \, dx \leq \frac{Ck}{\lambda^p}; \end{aligned} \tag{2.2.10}$$

while for the last term in (2.2.9), thanks to (2.2.8), we can write

$$\operatorname{meas}\{|\nabla u| \geq \lambda; |u| > k\} \leq \operatorname{meas}\{|u| \geq k\} \leq \frac{\overline{C}}{k^\sigma},$$

with $\sigma = p - 1 + \frac{p}{N}$. So, finally, we get

$$\operatorname{meas}\{|\nabla u| \geq \lambda\} \leq \frac{\overline{C}}{k^\sigma} + \frac{Ck}{\lambda^p},$$

and we can have a better estimate by taking the minimum over k of the right hand side; the minimum is achieved for the value

$$k_0 = \left(\frac{\sigma C}{\overline{C}}\right)^{\frac{1}{\sigma+1}} \lambda^{\frac{p}{\sigma+1}},$$

and so we get the desired estimate

$$\operatorname{meas}\{|\nabla u| \geq \lambda\} \leq C\lambda^{-\gamma} \tag{2.2.11}$$

with $\gamma = p(\frac{\sigma}{\sigma+1}) = \frac{Np+p-N}{N+1} = p - \frac{N}{N+1}$; this estimate is the same obtained in [**BG**].

Then, coming back to our case, we have found that, for every $n \geq 0$, $|\nabla u^n|$ is equi-bounded in $M^\gamma(Q)$, with $\gamma = p - \frac{N}{N+1}$, and so, since $p > \frac{2N+1}{N+1}$, $|\nabla u^n|$ is uniformly bounded in $L^s(Q)$ with $1 \leq s < p - \frac{N}{N+1}$.

Now, we shall use the above estimates to prove some *compactness* results that will be useful to pass to the limit in the entropy formulation for u^n. Indeed, thanks to these estimates, we can say that there exists a function $\overline{u} \in L^q(0,1; W_0^{1,q}(\Omega))$, for all $q < p - \frac{N}{N+1}$, such that u^n converges to $\overline{u}$ weakly in $L^q(0,1; W_0^{1,q}(\Omega))$. Observe that, obviously, we have $\overline{u} = \tilde{u}$ a.e. in Q. On the other hand from the equation we deduce that $u_t^n \in L^1(Q) + L^{s'}(0,1; W^{-1,s'}(\Omega))$ uniformly with respect to n, where $s' = \frac{q}{p-1}$, for all $q < p - \frac{N}{N+1}$, and so, thanks to the *Aubin-Simon* type result of Theorem 1.5, we have that u^n actually converges to $\overline{u}$ in $L^1(Q)$. Moreover, using the estimate (2.2.6) on the truncations of u^n, we deduce, from the boundedness and continuity of $T_k(s)$, that, for every $k > 0$

$$T_k(u^n) \rightharpoonup T_k(\overline{u}), \quad \text{weakly in} \quad L^p(0,1; W_0^{1,p}(\Omega)),$$

$$T_k(u^n) \to T_k(\overline{u}), \quad \text{strongly in} \quad L^p(Q).$$

Finally, the sequence u^n satisfies the hypotheses of Theorem 3.3 in [**BDGO**], and so we get

$$\nabla u^n \to \nabla \overline{u} \ \text{ a.e. in } \Omega.$$

All these results allow us to pass to the limit in the entropy formulation of u^n; indeed, for all $k > 0$, u^n satisfies

$$\int_\Omega \Theta_k(u^n - \varphi)(1)\ dx \tag{2.2.12}$$

$$-\int_\Omega \Theta_k(u^n(0,x) - \varphi(0))\ dx \tag{2.2.13}$$

$$+\int_0^1 \langle \varphi_t, T_k(u^n - \varphi)\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\ dt \tag{2.2.14}$$

$$+\int_Q a(x,\nabla u^n)\cdot \nabla T_k(u^n - \varphi)\ dxdt \tag{2.2.15}$$

$$\leq \int_Q T_k(u^n - \varphi)\ d\mu, \tag{2.2.16}$$

for all $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q) \cap C([0,T];L^1(\Omega))$ with $\varphi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$. Let us analyze this inequality term by term: since $T_k(u^n - \varphi)$ converges to $T_k(\overline{u} - \varphi)$ $*$-weakly in $L^\infty(Q)$, and $T_k(u^n - \varphi)$ converges to $T_k(\overline{u} - \varphi)$ also weakly in $L^p(0,1;W_0^{1,p}(\Omega))$, we have

$$\int_Q T_k(u^n - \varphi)\ d\mu \xrightarrow{n} \int_Q T_k(\overline{u} - \varphi)\ d\mu;$$

moreover, we can write

$$\begin{aligned}(2.2.15) = &\int_Q (a(x,\nabla u^n) - a(x,\nabla\varphi))\cdot \nabla T_k(u^n - \varphi)\ dxdt \\ &+ \int_Q a(x,\nabla\varphi)\cdot\nabla T_k(u^n - \varphi)\ dxdt,\end{aligned} \tag{2.2.17}$$

and the second term to the right hand side of (2.2.17) converges, as n tends to infinity, to

$$\int_Q a(x,\nabla\varphi)\cdot \nabla T_k(\overline{u} - \varphi)\ dxdt,$$

while to deal with the nonnegative first term of the right hand side of (2.2.17), we must use the a.e. convergence of the gradients; then, applying *Fatou's lemma*, we get

$$\int_Q (a(x,\nabla\overline{u}) - a(x,\nabla\varphi))\cdot\nabla T_k(\overline{u}-\varphi)\ dxdt$$

$$\leq \liminf_n \int_Q (a(x,\nabla u^n) - a(x,\nabla\varphi))\cdot\nabla T_k(u^n-\varphi)\ dxdt.$$

Our goal is to prove that $\overline{u} = v$ almost everywhere in Ω; to do that, it is enough to prove that $\overline{u}$ does not depend on time, and that (2.2.12)+(2.2.13)+(2.2.14) converges to zero as n tends to infinity; indeed, if that holds true, we obtain that $\overline{u}$ satisfies the entropy formulation for the elliptic problem (2.1.5), and so, since the entropy solution is unique, we get that $\overline{u} = v$ a.e. in Ω.

Let us prove first that $\overline{u}$ does not depend on time; let us denote by $w(x)$ the almost everywhere limit of the monotone nondecreasing sequence $u^n(0,x) = u(n,x)$, hence, using the comparison Lemma 2.3, we have that, for fixed $t\in(0,1)$

$$u^n(0,x) \leq u^n(t,x) = u(n+t,x) \leq u(n+1,x) = u^{n+1}(0,x),$$

and, since both u^n and u^{n+1} converge to $w(x)$ that does not depend on time, such happens also for the a.e. limit of $u^n(t,x)$ that is $\overline{u} = w$.

Now, using the *monotone convergence theorem*, we get

$$\lim_n[(2.2.12)+(2.2.13)] = \int_\Omega \Theta_k(\overline{u}-\varphi)(1)\ dx - \int_\Omega \Theta_k(w(x)-\varphi(0))\ dx$$

$$= \int_\Omega\int_0^1 \frac{d}{dt}\Theta(\overline{u}-\varphi)\ dtdx = \int_0^1 \langle(\overline{u}-\varphi)_t, T_k(\overline{u}-\varphi)\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\ dt,$$

while, since $T_k(u^n-\varphi)$ converges to $T_k(\overline{u}-\varphi)$ weakly in $L^p(0,1;W_0^{1,p}(\Omega))$, we have

$$(2.2.14) \xrightarrow{n} \int_0^1 \langle\varphi_t, T_k(\overline{u}-\varphi)\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\ dt.$$

Finally we can sum all these terms and, since $\overline{u}$ does not depend on time, we find

$$\lim_n[(2.2.12)+(2.2.13)+(2.2.14)] = \int_0^1 \langle\overline{u}_t, T_k(\overline{u}-\varphi)\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\ dt = 0;$$

and, as we mentioned above, this is enough to prove that $\overline{u}(t,x) = v(x)$. $\square$

2.3. Nonhomogeneous case

Now we deal with the general case of problem (2.1.4) with a nonhomogeneous initial datum u_0, that is

$$\begin{cases} u_t + A(u) = \mu & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{2.3.1}$$

with $\mu \in M_0(Q)$ satisfying the usual hypotheses used throughout this chapter, and u_0 a nonnegative function in $L^1(\Omega)$. We shall prove the following result:

THEOREM 2.4. *Let $\mu \in M_0(Q)$ be independent of the variable t, $p > \frac{2N+1}{N+1}$, $u_0 \in L^1(\Omega)$, and let $\mu, u_0 \geq 0$; moreover, let $u(t,x)$ be the entropy solution of problem (2.3.1), and v the entropy solution of the corresponding elliptic problem (2.1.5). Then*

$$\lim_{t\to+\infty} u(t,x) = v(x),$$

in $L^1(\Omega)$.

Most part of our work will be concerned with comparison between suitable entropy subsolutions and supersolutions of problem (2.1.4). The notion of entropy subsolution and supersolution for the parabolic problem will be given as a natural extension of the one for the elliptic case (see [**Pa**]).

DEFINITION 2.5. A function $\underline{u}(t,x) \in C([0,T];L^1(\Omega))$ is an *entropy subsolution* of problem (2.3.1) if, for all $k > 0$, we have that $T_k(\underline{u}) \in L^p(0,T;W_0^{1,p}(\Omega))$, and holds

$$\begin{aligned} &\int_\Omega \Theta_k(\underline{u}-\varphi)^+(T)\,dx - \int_\Omega \Theta_k(\underline{u}_0-\varphi(0))^+\,dx \\ &+\int_0^T \langle \varphi_t, T_k(\underline{u}-\varphi)^+\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\,dt \\ &+\int_Q a(x,\nabla \underline{u})\cdot\nabla T_k(\underline{u}-\varphi)^+\,dxdt \leq \int_Q T_k(\underline{u}-\varphi)^+\,d\mu, \end{aligned} \tag{2.3.2}$$

for all $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q) \cap C([0,T];L^1(\Omega))$ with $\varphi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $\underline{u}(0,x) \equiv \underline{u}_0(x) \leq u_0$ almost everywhere on Ω with $\underline{u}_0 \in L^1(\Omega)$.

On the other hand, $\overline{u}(t,x) \in C([0,T];L^1(\Omega))$ is an *entropy supersolution* of problem (2.3.1) if, for all $k>0$, we have that $T_k(\overline{u}) \in L^p(0,T;W_0^{1,p}(\Omega))$, and holds

$$
\begin{aligned}
&\int_\Omega \Theta_k(\overline{u}-\varphi)^-(T)\ dx - \int_\Omega \Theta_k(\overline{u}_0-\varphi(0))^-\ dx \\
&+\int_0^T \langle \varphi_t, T_k(\overline{u}-\varphi)^-\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\ dt \\
&+\int_Q a(x,\nabla\overline{u})\cdot\nabla T_k(\overline{u}-\varphi)^-\ dxdt \ge \int_Q T_k(\overline{u}-\varphi)^-\ d\mu,
\end{aligned}
\tag{2.3.3}
$$

for all $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q) \cap C([0,T];L^1(\Omega))$ with $\varphi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $\overline{u}(0,x) \equiv \overline{u}_0(x) \le u_0$ almost everywhere on Ω with $\overline{u}_0 \in L^1(\Omega)$.

Observe that, an entropy solution of problem (2.3.1) turns out to be both an entropy subsolution and an entropy supersolution of the same problem as an easy approximation argument shows. Thanks to Definition 2.5 it is possible to improve straightforwardly the comparison Lemma 2.3, by comparing both subsolution and supersolution with the unique entropy solution of problem (2.3.1) using a similar approximation argument; actually, we shall state this result in a simpler case that will be enough to prove Theorem 2.4.

LEMMA 2.6. *Let $\mu \in M_0(\Omega)$, and let $\underline{u}$ and $\overline{u}$ be, respectively, an entropy subsolution and an entropy supersolution of problem (2.3.1), and let u be the unique entropy solution of the same problem. Then $\underline{u} \le u \le \overline{u}$.*

PROOF. Let us prove only that $\underline{u} \le u$ being the other case analogous. We know that the entropy solution u is found as limit of regular functions u_n solutions of approximating problems with smooth data. So, if $\mu = f - \mathrm{div}(g)$ with $f \in L^1(\Omega)$ and $g \in (L^{p'}(\Omega))^N$ we can choose a sequence of smooth functions f_n that converges to f in $L^1(\Omega)$; let us call $\mu_n = f_n - \mathrm{div}(g)$. Moreover, let $u_{0,n}$ be a sequence of regular functions that converges to u_0 in $L^1(\Omega)$; so, we call u_n the variational solution of problem

$$
\begin{cases}
(u_n)_t + A(u^n) = \mu_n & \text{in } (0,T)\times\Omega,\\
u_n(0,x) = \tilde{u}_{0,n} & \text{in } \Omega,\\
u_n(t,x) = 0 & \text{on } (0,T)\times\partial\Omega,
\end{cases}
\tag{2.3.4}
$$

where $\tilde{u}_{0,n} = \min(u_{0,n}, \underline{u}(0))$; notice that also $\tilde{u}_{0,n}$ converges to $\underline{u}(0)$ in $L^1(\Omega)$.

Now, we can choose $T_k(\underline{u}-u_n)^+$ as test function obtaining

$$\int_0^T \langle (u_n)_t, T_k(\underline{u}-u_n)^+\rangle \; dt + \int_Q a(x,\nabla u_n)\cdot \nabla T_k(\underline{u}-u_n)^+ \; dxdt$$

$$= \int_Q \mu_n T_k(\underline{u}-u_n)^+ \; dxdt;$$

On the other hand being $\underline{u}$ a subsolution we have

$$\int_\Omega \Theta_k(\underline{u}-u_n)^+(T)\; dx - \int_\Omega \Theta_k(\underline{u}-u_n)^+(0)\; dx$$

$$+ \int_0^T \langle (u_n)_t, T_k(\underline{u}-u_n)^+\rangle \; dt$$

$$+ \int_Q a(x,\nabla \underline{u})\cdot \nabla T_k(\underline{u}-u_n)^+ \; dxdt \leq \int_Q T_k(\underline{u}-u_n)^+ \; d\mu dt.$$

So, we can subtract this relation from the previous one, recalling that $(\underline{u}-u_n)^+(0) = 0$, to obtain

$$\int_\Omega \Theta_k(\underline{u}-u_n)^+(T)\; dx + \int_Q (a(x,\nabla \underline{u}) - a(x,\nabla u_n))\cdot \nabla T_k(\underline{u}-u_n)^+ \; dxdt$$

$$\leq \int_Q T_k(\underline{u}-u_n)^+ \; d\mu dt - \int_Q \mu_n T_k(\underline{u}-u_n)^+ \; dxdt + \int_\Omega \Theta_k(\underline{u}(0)-\tilde{u}_{n,0})^+(0)\; dx.$$

Finally, the monotonicity assumption on a, and *Fatou's lemma*, yield that $\underline{u}\leq u$, since, using again the stability result cited above we deduce that the left hand side of the above inequality tends to zero as n goes to infinity. Observe that, we actually proved that, for every fixed $T>0$, a.e. on Ω, we have $\underline{u}(T,x)\leq u(T,x)$. □

Proof of Theorem 2.4. If v is the entropy solution of problem (2.1.5), we proved that v is also an entropy solution of the initial boundary value problem (2.1.4) with v itself as initial datum. Therefore, by comparison Lemma 2.6, if $0\leq u_0\leq v$, we have that the solution $u(t,x)$ of (2.1.4) converges to v in $L^1(\Omega)$ as t tends to infinity; in fact, we proved it for the entropy solution with homogeneous inital datum in Theorem 2.2 while v is a stationary entropy solution.

Now, let us take $\hat{u}(t,x)$ the solution of problem (2.3.1) with $u_0 = v^\tau$ as initial datum for some $\tau>1$, where v^τ is the entropy solution of the elliptic problem (2.1.5) with μ_τ

as datum instead of $\mu = f - \operatorname{div}(g)$ ($f \geq 0$ in $L^1(\Omega)$, $g \in (L^{p'}(\Omega))^N$), where

$$\mu_\tau = \begin{cases} \tau\mu & \text{if } f = 0, \\ \tau f - \operatorname{div}(g) & \text{if } f \neq 0. \end{cases}$$

Observe that, since $\tau > 1$, by virtue of comparison results we have immediately $v \leq v^\tau$.

Hence, since v^τ does not depend on time, we have that it is a supersolution of the parabolic problem (2.3.1), and, recalling that v is a subsolution of the same problem (being $v \leq v^\tau$ as initial data), we can apply the comparison lemma again finding that $v(x) \leq \hat{u}(t,x) \leq v^\tau(x)$ a.e. in Ω, for all positive t.

Now, thanks to the fact that the datum μ does not depend on time, we can apply the comparison result also between $\hat{u}(x, t+s)$ solution with $u_0 = \hat{u}(x,s)$, with s a positive parameter, and $\hat{u}(t,x)$, the solution with $u_0 = v^\tau$ as initial datum; so we obtain $\hat{u}(x,t+s) \leq \hat{u}(t,x)$ for all $t, s > 0$, a.e. in Ω. So, by virtue of this monotonicity result we have that there exists a function $\overline{v} \geq v$ such that $\hat{u}(t,x)$ converges to $\overline{v}$ a.e. in Ω as t tends to infinity. Clearly $\overline{v}$ does not depend on t and we can develop the same argument used for the homogeneous case, starting from the analogous estimate

$$\int_\Omega \Theta_k(\hat{u}^n)(t) + \alpha \int_Q |\nabla T_k(\hat{u}^n)|^p \, dxdt \leq k(|\mu|_{M_0(Q)} + \|u_0\|_{L^1(\Omega)}) = Ck, \tag{2.3.5}$$

to prove that we can pass to the limit in the entropy formulation, and so, by uniqueness result, we can obtain that $\overline{v} = v$. So, we have proved that the result holds for the solution starting from $u_0 = v^\tau$ as initial datum, with $\tau > 1$. Since we proved before that the result holds true also for the solution starting from $u_0 = 0$, then, again applying a comparison argument, we can conclude in the same way that this result holds true for solution starting from u_0 such that $0 \leq u_0 \leq v^\tau$ as initial datum, for fixed $\tau > 1$.

Finally let us consider the general case of a solution $u(t,x)$ with initial datum $u_0 \in L^1(\Omega)$ and let suppose first that $\mu \neq 0$; let us define the monotone nondecreasing (in τ) family of functions

$$u_{0,\tau} = \min(u_0, v^\tau).$$

As we have shown above, for every fixed $\tau > 1$, $u_\tau(t,x)$, the entropy solution of problem (2.3.1) with $u_{0,\tau}$ as initial datum, converges to v a.e. in Ω, as t tends to infinity. Moreover, we also have that $T_k(u_\tau(t,x))$ converges to $T_k(v)$ weakly in $W_0^{1,p}(\Omega)$ as t diverges, for every fixed $k > 0$.

Now, let us state the following lemma, that will be useful in the sequel: we shall prove it below, when the proof of Theorem 2.4 will be completed.

LEMMA 2.7. *Let $\mu \neq 0$ be a nonnegative measure in $M_0(\Omega)$, and $\tau > 0$. Moreover, let v_τ be the entropy solution of problem*

$$\begin{cases} A(v_\tau) = \mu_\tau & \text{in } \Omega, \\ v_\tau = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3.6}$$

then, we have

$$\lim_{\tau \to \infty} v_\tau(x) = +\infty$$

almost everywhere on Ω.

So, thanks to *Lebesgue theorem*, and Lemma 2.7, we can easily check that $u_{0,\tau}$ converges to u_0 in $L^1(\Omega)$ as τ tends to infinity. Therefore, using a stability result of entropy solution (see for instance [**Po1**]) we obtain that $T_k(u_\tau(t,x))$ converges to $T_k(u(t,x))$ strongly in $L^p(0,T;W_0^{1,p}(\Omega))$ as τ tends to infinity.

Now, making the same calculations used in [**Pr2**] to prove the uniqueness of entropy solutions applied to u and u_τ, where u_τ is considered as the solution obtained as limit of approximating solutions with smooth data, we can easly find, for any fixed $\tau > 1$, the following estimate

$$\int_\Omega \Theta_k(u - u_\tau)(t)\, dx \leq \int_\Omega \Theta_k(u_0 - u_{0,\tau})\, dx,$$

for every $k,\ t > 0$. Then, let us divide the above inequality by k, and let us pass to the limit as k tends to 0; we obtain

$$\|u(t,x) - u_\tau(t,x)\|_{L^1(\Omega)} \leq \|u_0(x) - u_{0,\tau}(x)\|_{L^1(\Omega)}, \tag{2.3.7}$$

for every $t > 0$. Hence, we have

$$\|u(t,x) - v(x)\|_{L^1(\Omega)} \leq \|u(t,x) - u_\tau(t,x)\|_{L^1(\Omega)} + \|u_\tau(t,x) - v(x)\|_{L^1(\Omega)};$$

then, thanks to the fact that the estimate in (2.3.7) is uniform in t, for every fixed ε, we can choose $\bar\tau$ large enough such that

$$\|u(t,x) - u_{\bar\tau}(t,x)\|_{L^1(\Omega)} \leq \frac{\varepsilon}{2},$$

for every $t > 0$; on the other hand, thanks to the result proved above, there exists $\bar t$ such that

$$\|u_{\bar\tau}(t,x) - v(x)\|_{L^1(\Omega)} \leq \frac{\varepsilon}{2},$$

for every $t > \bar t$, and this proves our result if $\mu \neq 0$.

If $\mu = 0$ we can consider, for every $\varepsilon > 0$, the solution u^ε of problem

$$\begin{cases} u^\varepsilon_t + A(u^\varepsilon) = \varepsilon & \text{in } (0,T) \times \Omega, \\ u^\varepsilon(0,x) = u_0 & \text{in } \Omega, \\ u^\varepsilon(t,x) = 0 & \text{on } (0,T) \times \partial\Omega; \end{cases} \tag{2.3.8}$$

thanks to Lemma 2.6 we get that

$$u(t, x) \leq u^{\varepsilon}(t, x), \tag{2.3.9}$$

for every fixed $t \in (0, T)$, and almost every $x \in \Omega$, by the previous result we obtain

$$\lim_{T \to \infty} u^{\varepsilon}(T, x) = v^{\varepsilon}(x),$$

where v^{ε} is the entropy solution of the elliptic problem associated to (2.3.8), and, moreover, by virtue of the stability result for this problem, we know that $v^{\varepsilon}(x)$ converges to 0 as ε tends to 0; so, almost everywhere on Ω, using (2.3.9), we have

$$0 \leq \limsup_{T \to \infty} u(T, x) \leq v^{\varepsilon}(x),$$

and, since ε is arbitrary, we can conclude using Vitali's Theorem and (2.3.9). $\square$

PROOF OF LEMMA 2.7. Let us first suppose that $\mu \in W^{-1,p'}(\Omega)$, so that we have $\mu = -\mathrm{div}(g)$ with $g \in (L^{p'}(\Omega))^N$; v_τ then solves

$$\int_{\Omega} a(x, \nabla v_\tau) \cdot \nabla \varphi \, dx = \tau \int_{\Omega} g \cdot \nabla \varphi \, dx, \tag{2.3.10}$$

for every $\varphi \in W_0^{1,p}(\Omega)$; let us take $\varphi = v_\tau$ as test function in (2.3.10); so, using assumption (2.1.1), we get

$$\alpha \int_{\Omega} |\nabla v_\tau|^p \, dx \leq \tau \|g\|_{(L^{p'}(\Omega))^N} \|v_\tau\|_{W_0^{1,p}(\Omega)};$$

now, since $v_\tau \neq 0$, we can divide the above expression by $\tau \|v_\tau\|_{W_0^{1,p}(\Omega)}$, getting

$$\frac{1}{\tau} \left(\int_{\Omega} |\nabla v_\tau|^p \, dx \right)^{\frac{p-1}{p}} = \left(\int_{\Omega} \left| \nabla \left(\frac{v_\tau}{\tau^{\frac{1}{p-1}}} \right) \right|^p \, dx \right)^{\frac{p-1}{p}} \leq \|g\|_{(L^{p'}(\Omega))^N}.$$

Therefore, we have that $\dfrac{v_\tau}{\tau^{\frac{1}{p-1}}}$ is bounded in $W_0^{1,p}(\Omega)$, and so there exists a function $v \in W_0^{1,p}(\Omega)$ and a subsequence, such that $\dfrac{v_\tau}{\tau^{\frac{1}{p-1}}}$ weakly converges in $W_0^{1,p}(\Omega)$ (and then a.e.) to v as τ tends to infinity. So, it is enough to prove that $v > 0$ almost everywhere on Ω to conclude our proof.

To this aim, for every $\tau > 0$, let us define

$$a_\tau(x, \xi) = \frac{1}{\tau} a(x, \tau^{\frac{1}{p-1}} \xi);$$

we can easily check that such an operator satisfies assumptions (2.1.1), (2.1.2) and (2.1.3), with the same constants α and β. Notice that in the model case of the p-laplacian, thanks to its homogeneity property, we have $a_\tau \equiv a$. Now, $\dfrac{v_\tau}{\tau^{\frac{1}{p-1}}}$ satisfies the elliptic problem

$$
\begin{cases} -\mathrm{div}(a_\tau(x, \nabla \dfrac{v_\tau}{\tau^{\frac{1}{p-1}}})) = \mu & \text{in } \Omega, \\ \\ \dfrac{v_\tau}{\tau^{\frac{1}{p-1}}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3.11}
$$

in a variational sense; indeed, for every $\varphi \in W_0^{1,p}(\Omega)$, we have

$$
\int_\Omega a_\tau(x, \nabla \left(\frac{v_\tau}{\tau^{\frac{1}{p-1}}}\right)) \cdot \nabla\varphi \, dx = \frac{1}{\tau} \int_\Omega a(x, \nabla v_\tau) \cdot \nabla\varphi \, dx = \int_\Omega g \cdot \nabla\varphi \, dx. \tag{2.3.12}
$$

Moreover, thanks to Theorem 4.1 in [**CDD**], we have that the family of operators $\{a_\tau\}$ has a G-limit in the class of Leray-Lions type operators; that is, there exists a Carathéodory function $\overline{a}$ satisfying assumptions (2.1.1), (2.1.2) and (2.1.3), and a sequence of indices $\tau(k)$ (called τ again), such that

$$
a_\tau \xrightarrow{G} \overline{a};
$$

so, because of that, being v the weak limit of $\dfrac{v_\tau}{\tau^{\frac{1}{p-1}}}$ in $W_0^{1,p}(\Omega)$, we get that

$$
a(x, \nabla \left(\frac{v_\tau}{\tau^{\frac{1}{p-1}}}\right)) \overset{\tau\to\infty}{\longrightarrow} \overline{a}(x, \nabla v),
$$

weakly in $(L^{p'}(\Omega))^N$. Therefore, using this result in (2.3.12) we have

$$
\int_\Omega \overline{a}(x, \nabla v) \cdot \varphi \, dx = \int_\Omega g \cdot \nabla\varphi \, dx,
$$

for every $\varphi \in W_0^{1,p}(\Omega)$; and so, v is a variational solution of problem

$$
\begin{cases} -\mathrm{div}(\overline{a}(x, \nabla v)) = \mu & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}
$$

Then, recalling that $\mu \neq 0$ and using a suitable Harnack type inequality (see for instance [**T**]), we deduce that $v > 0$ almost everywhere on Ω.

Now, if $\mu \in M_0(\Omega)$, we have $\mu_\tau = \tau f - \mathrm{div}(g)$, with $f \neq 0$ a nonnegative function in $L^1(\Omega)$; we can suppose, without loss of generality, that $\mu_\tau = \tau\chi_E - \mathrm{div}(g)$ for a suitable set $E \subseteq \Omega$ of positive measure; indeed, f, being nonidentically zero, it turns out to be strictly bounded away from zero on a suitable $E \subseteq \Omega$, and so there exists a constant c such that $f \geq c\chi_E$, and then, once we proved our result for such a μ_τ, we can easily prove the statement by applying again a comparison argument. Now, reasoning analogously as above we deduce that $\dfrac{v_\tau}{\tau^{\frac{1}{p-1}}}$ solves the elliptic problem

$$(2.3.13)\qquad \begin{cases} a_\tau(x,(\dfrac{v_\tau}{\tau^{\frac{1}{p-1}}})) = \chi_B - \dfrac{1}{\tau}\mathrm{div}(g) & \text{in } \Omega, \\ \dfrac{v_\tau}{\tau^{\frac{1}{p-1}}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover,

$$\chi_B - \frac{1}{\tau}\mathrm{div}(g) \longrightarrow \chi_B,$$

strongly in $W^{-1,p'}(\Omega)$ as τ tends to infinity. Therefore, since G-convergence is stable under such type a of convergence of data, we have, that the weak limit v of $\dfrac{v_\tau}{\tau^{\frac{1}{p-1}}}$ in $W_0^{1,p}(\Omega)$, solves

$$\begin{cases} -\mathrm{div}(\overline{a}(x,\nabla v)) = \chi_B & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

and so we may conclude, as above, that $v > 0$ almost everywhere on Ω, that implies that v_τ goes to infinity as τ tends to infinity. $\square$

Remark 2.8. Let us observe that the results of Theorem 2.2 and Theorem 2.4 can be improved depending on the regularity of the data, and so of the solution of the elliptic problem (see for instance (2.2.4)), for such regularity results we refer to standard elliptic regularity results and to [**BGO**].

2.4. General measure data: linear case

If μ is a general measure in $M(Q)$ that does not depend on time we can use the concept of duality solution as defined in Section 1.3 to prove the asymptotic result for linear problems. Let us consider the linear problem

$$(2.4.1)\qquad \begin{cases} u_t - \mathrm{div}(M(x)\nabla u) = \mu & \text{in } (0,T)\times\Omega, \\ u(0) = u_0 & \text{in } \Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases}$$

with M a bounded matrix satisfying assumption (2.1.1) ($p = 2$), μ a nonnegative measure not depending on time, and $u_0 \in L^1(\Omega)$ nonnegative.

Let us prove the following preliminary result:

PROPOSITION 2.9. *Let $\mu \in M(Q)$ that does not depend on time and let v be the duality solution of elliptic problem*

$$\begin{cases} -\operatorname{div}(M(x)\nabla v) = \mu & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.4.2}$$

Then v is the unique solution of the parabolic problem

$$\begin{cases} v_t - \operatorname{div}(M(x)\nabla v) = \mu & \text{in } (0,T)\times\Omega, \\ v(0) = v(x), & \text{in } \Omega, \end{cases} \tag{2.4.3}$$

in the duality sense introduced in (1.3.12), for any fixed $T > 0$.

PROOF. We have to check that v is a solution of problem (2.4.3), to do that let us multiply (1.3.10) by $T_k(v)$ and integrate on Q; we obtain

$$-\int_0^T \langle w_t, T_k(v)\rangle\, dt + \int_Q M^*(x)\nabla w \cdot \nabla T_k(v)\, dxdt$$

$$= \int_Q T_k(v)\, g\, dxdt.$$

Now, integrating by parts we have

$$-\int_0^T \langle w_t, T_k(v)\rangle\, dt = \int_\Omega w(0)v(x) + \omega(k),$$

while

$$\int_Q T_k(v)\, g\, dxdt = \int_Q v\, g\, dxdt + \omega(k).$$

Finally, thanks to a result of [**DMOP**] we have

$$\int_Q M^*(x)\nabla w \cdot \nabla T_k(v)\, dxdt = \int_Q M(x)\nabla T_k(v)\cdot\nabla w\, dxdt = \int_0^T \int_\Omega w\, d\lambda_k(x)\, dt,$$

where λ_k are measures in $M_0(\Omega)$ that converge to μ tightly; thus, recalling that w is bounded, and using dominated convergence theorem, we have

$$\int_Q M^*(x)\nabla w\cdot\nabla T_k(v)\, dxdt = \int_Q w\, d\mu + \omega(k);$$

gathering together all these facts we have that v is a duality solution of (2.4.1) with itself as initial datum. □

Proposition 2.9 allows us to deduce that the duality solution of problem (2.4.1) u belongs to $C(0,T;L^1(\Omega))$ for any fixed $T>0$; indeed $z=v-u$ uniquely solves problem

$$\begin{cases} z_t - \operatorname{div}(M(x)\nabla z) = 0 & \text{in } (0,T)\times\Omega, \\ z(0) = v - u_0 & \text{in } \Omega, \\ z = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{2.4.4}$$

in the duality sense, and so $z \in C(0,T;L^1(\Omega))$ thanks to Theorem 1.6, since z turns out to be an entropy solution in the sense of Definition 1.44.

So, we have that that u satisfies

$$\int_Q u\,g\;dxdt = \int_Q w\;d\mu + \int_\Omega u_0\,w(0)\;dx, \tag{2.4.5}$$

for any $g\in L^\infty(Q)$ and w is the unique solution of the retrograde problem

$$\begin{cases} -w_t - \operatorname{div}(M^*(x)\nabla w) = g & \text{in } (0,T)\times\Omega, \\ w(T,x) = 0 & \text{in } \Omega, \\ w(t,x) = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{2.4.6}$$

This way, for fixed μ and $g\in L^\infty(Q)$ one can uniquely determine u and w, solution of the above problems, defined for any time $T>0$. Moreover, recalling that through the change of variable $s=T-t$, w turns out to solve a linear parabolic problem, if $g\ge 0$, by classical comparison result, one has that $w(t,x)$ is decreasing in time, for any $T>0$.

Moreover, let us give the following definition:

Definition 2.10. A function $u\in L^1(Q)$ is a *duality supersolution* of problem (2.4.1) if

$$\int_Q u\,g\;dxdt \ge \int_Q w\;d\mu + \int_\Omega u_0 w(0)\;dx,$$

for any bounded $g\ge 0$, and w solution of (2.4.6), while u is a *duality subsolution* if $-u$ is a duality supersolution.

Lemma 2.11. *Let $\overline{u}$ and $\underline{u}$ be respectively a duality supersolution and a duality subsolution for problem (2.4.1). Then $\underline{u}\le\overline{u}$.*

Proof. Actually, simply subtract the two formulations one from the other to obtain

$$\int_Q (\underline{u}-\overline{u})g\;dxdt \le 0,$$

for any $g\ge 0$, and so $\underline{u}\le\overline{u}$. □

Now we can state our asymptotic result where as usual, thanks to the uniqueness result, we can think to the solution of problem (2.4.1) as defined for any $t>0$.

THEOREM 2.12. *Let u be a duality solution of problem (2.4.1), and v be the duality solution of elliptic problem*

$$\begin{cases} -\mathrm{div}(M(x)\nabla v) = \mu & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.4.7}$$

Then $u(t,x)$ converges to $v(x)$ in $L^1(\Omega)$ as t diverges.

PROOF. Let us first suppose $u_0 = 0$. Thanks to the result of Lemma 2.11, arguing as in the proof of Theorem 2.2, we have that $u(t,x)$ is increasing in time and it converges to $\tilde{v}(x)$ almost everywhere and in $L^1(\Omega)$ since $u(t,x) \leq v(x)$.

Now, recalling that u is obtained as limit of regular solutions with smooth data μ_τ, we can define $u_\tau^n(t,x)$ as the solution of

$$\begin{cases} (u_\tau^n)_t - \mathrm{div}(M(x)\nabla u_\tau^n) = \mu_\tau & \text{in } (0,1)\times\Omega, \\ u_\tau^n(0) = u_\tau(n,x) & \text{in } \Omega \\ u_\tau^n = 0 & \text{on } (0,1)\times\partial\Omega. \end{cases} \tag{2.4.8}$$

On the other hand, if $g \geq 0$, we define $w^n(t,x)$ as

$$\begin{cases} -w_t^n - \mathrm{div}(M^*(x)\nabla w^n) = g & \text{in } (0,1)\times\Omega, \\ w^n(1,x) = w(n+1,x) & \text{in } \Omega, \\ w^n = 0 & \text{on } (0,1)\times\partial\Omega. \end{cases} \tag{2.4.9}$$

Therefore, by comparison principle, we have that w^n is increasing with respect to n and, arguing as in the proof of Theorem 2.2, its limit $\tilde{w}$ does not depend on time and is the solution of

$$\begin{cases} -\mathrm{div}(M^*(x)\nabla \tilde{w}) = g & \text{in } \Omega, \\ \tilde{w}(x) = 0 & \text{on } \partial\Omega; \end{cases} \tag{2.4.10}$$

so that, using u_τ^n (2.4.9) and w^n in (2.4.8), integrating by parts and subtracting we obtain

$$\int_0^1\int_\Omega u^n\, g - \int_0^1\int_\Omega w^n\, d\mu + \int_\Omega u^n(0)w^n(0)\, dx - \int_\Omega u^n(1)w^n(1)\, dx + \omega(\tau) = 0.$$

Hence, we can pass to the limit on n using monotone convergence theorem obtaining

$$\int_\Omega \tilde{v}\, g - \int_\Omega \tilde{w}\, d\mu\, dx = 0, \tag{2.4.11}$$

and so $v = \tilde{v}$.

If g has no sign we can reason separately with g^+ and g^- obtaining (2.4.11) and then using the linearity of (2.4.5) to conclude.

Finally, if $u_0 \in L^1(\Omega)$ we can reason, with many simplifications, as in the proof of Theorem 2.4, considering $v^\tau \equiv \tau v$ and recalling that, if $\mu \neq 0$, then $v > 0$. $\square$

CHAPTER 3

Asymptotic behavior of solutions for parabolic equations with natural growth terms and irregular data

We are interested at the asymptotic behavior as t tends to $+\infty$ of solutions $u \in L^2(0,T;H_0^1(\Omega))$ of the problem

$$\begin{cases} u_t - \Delta u + g(u)|\nabla u|^2 = f & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{3.0.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $N \geq 3$, $u_0 \in L^1(\Omega)$ nonnegative, while $g : \mathbb{R} \to \mathbb{R}$ is a real function in $C^1(\mathbb{R})$ such that

$$g(s)s \geq 0, \ \forall s \in \mathbb{R}, \tag{3.0.2}$$

$$g'(s) > 0, \forall s \in \mathbb{R} \tag{3.0.3}$$

and $f(x) \in L^1((0,T)\times\Omega)$ is a nonnegative function independent on time; in what follows we will often use the notation $Q = Q_T = \Omega \times (0,T)$.

As we mentioned above, this kind of problem has been largely studied in different context: in particular, for $g \equiv 1$ and with any power-like nonlinearity with respect to $|\nabla u|$ (the so called Viscous Hamilton-Jacobi equation) a certain number of papers has been devoted to the study of large time behavior of solution under suitable assumptions (see for instance [**BKL**] and references therein). Let us emphasize that for a purely PDE approach to this problem the main tool in order to obtain such kind of results is strictly related to the proof of some comparison principle. In our problem the main difficulty relies on the dependence on both u and its gradient of the nonlinear term.

Let us remind that, for a solution we mean a function $u \in L^2(0,T;H_0^1(\Omega))$ which satisfies

$$\int_0^T \langle u_t, \varphi\rangle + \int_Q \nabla u \cdot \nabla\varphi + \int_Q g(u)|\nabla u|^2\varphi = \int_Q f\varphi, \tag{3.0.4}$$

for any $\varphi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$ and such that $g(u)|\nabla u|^2$ belongs to $L^1(Q)$. As we said before, here the symbol $\int_0^T \langle\cdot,\cdot\rangle$ denote the duality between elements of

$L^2(0,T;H^{-1}(\Omega))+L^1(Q)$ and functions in $L^2(0,T;H_0^1(\Omega))\cap L^\infty(Q)$; in fact, such a solutions turns out to have time derivative in $L^2(0,T;H^{-1}(\Omega))+L^1(Q)$, and in particular, thanks to Theorem 1.6, they belong to $C(0,T;L^1(\Omega))$, and so we recall that the initial datum is achieved in the sense of $L^1(\Omega)$. For such a duality there exists a generalized integration by parts formula (see Lemma 1.4) that we will use.

As we will see (Lemma 3.4 and Lemma 3.5 below) the solution of problem (3.0.1) is unique and so if we consider $T'>T$, we have $u_T(x,t)\equiv u_{T'}(t,x)$, $\forall t<T$ and for almost every $x\in\Omega$; thus we can look at $u(t,x)$ as the unique solution of (3.0.1) defined $\forall t>0$.

We want to prove that $u(t,x)$ converges, for t that tends to $+\infty$, to $v(x)$ which is the unique solution of the elliptic problem

$$\begin{cases} -\Delta v+g(v)|\nabla v|^2=f & \text{in } \Omega\,,\\ v(x)=0 & \text{on } \partial\Omega\,, \end{cases} \tag{3.0.5}$$

such that $g(v)|\nabla v|^2\in L^1(\Omega)$; in other words $v(x)$ satisfies

$$\int_\Omega \nabla v\cdot\nabla\varphi+\int_\Omega g(v)|\nabla v|^2\varphi=\int_\Omega f\varphi\,, \tag{3.0.6}$$

for every $\varphi\in H_0^1(\Omega)\cap L^\infty(\Omega)$.

Our main result concerning the asymptotic behavior of solutions is the following:

THEOREM 3.1. *‘ Let f, $u_0\in L^1(\Omega)$ be nonnegative functions and let $u(t,x)$ be the weak solution of the problem (3.0.1). Then*

$$\lim_{t\to+\infty}u(t,x)=v(x)\qquad \textit{in } L^1(\Omega).$$

In order to prove Theorem 3.1 we will use several comparison results, and Section 3.1 is devoted to the statement and the proof of them, while in Section 3.2 we will prove Theorem 3.1. Actually we will see that the convergence of solution $u(t,x)$ to $v(x)$ could be stronger that the one obtained under the general assumptions of Theorem 3.1 provided that the data are more regular.

REMARK 3.2. Observe that $v(x)$ is solution of the problem (3.0.1) with itself as initial datum, being $v(x)$ independent from t.

Finally, thanks to the sign condition on f and g we have that both $u(t,x)$ and $v(x)$ are nonnegative. Indeed, for any $k>0$, consider $T_k(u^-)$ (where $T_k(s)=\max(-k,\min(k,s))$ is the usual truncation function) as test function in (3.0.1); we have

$$\int_0^T\langle u_t,T_k(u^-)\rangle_{H^{-1},H_0^1}+\int_Q\nabla u\cdot\nabla T_k(u^-)+\int_Q g(u)|\nabla u|^2T_k(u^-)=\int_Q fT_k(u^-)\,.$$

Then, from the sign condition on $g(s)$ we deduce that $g(u)|\nabla u|^2 T_k(u^-) \leq 0$, so

$$\int_Q \Theta_k^-(u)_t \, dx - \int_Q |\nabla T_k(u^-)|^2 \, dxdt \geq 0 \,,$$

where $\Theta_k^-(s) = \int_0^s T_k(r^-)dr$; thus, recalling that $u_0 \geq 0$, we obtain $u^-(t,x) \equiv 0$ a.e. in Q; the same holds for v using (3.0.6).

3.1. Comparison results

As we said before, the most important tool in order to prove the above result relies in some comparison result between subsolutions and supersolutions of problem (3.0.1); let us introduce the following definition:

DEFINITION 3.3. We say that $z \in L^2(0,T;H^1(\Omega))$ is a *subsolution* of problem (3.0.1) if $g(z)|\nabla z|^2 \in L^1(Q)$ and

$$\begin{cases} z_t(t,x) - \Delta z(t,x) + g(z)|\nabla z|^2 \leq f & \text{in } (0,T) \times \Omega, \\ z(0,x) \leq u_0(x) & \text{in } \Omega, \\ z(t,x) \leq 0 & \text{on } (0,T) \times \partial\Omega \,. \end{cases} \tag{3.1.1}$$

On the other hand $w \in L^2(0,T;H^1(\Omega))$ is a *supersolution* of problem (3.0.1) if $g(z)|\nabla z|^2 \in L^1(Q)$ and

$$\begin{cases} w_t - \Delta w(t,x) + g(w)|\nabla w|^2 \geq f & \text{in } (0,T) \times \Omega, \\ w(0,x) \geq u_0(x) & \text{in } \Omega, \\ w(t,x) \geq 0 & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{3.1.2}$$

where first equation both in (3.1.1) and (3.1.2) are understood in their weak sense; i.e. $z(t,x)$ satisfies

$$\int_0^T \langle z_t, \varphi \rangle + \int_Q \nabla z \cdot \nabla \varphi + \int_Q g(z)|\nabla z|^2 \varphi \leq \int_Q f\varphi \,, \tag{3.1.3}$$

$$\forall \varphi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q) \,, \quad \varphi \geq 0 \text{ a.e. in } Q,$$

and $w(t,x)$ satisfies

$$\int_0^T \langle w_t, \varphi \rangle + \int_Q \nabla w \cdot \nabla \varphi + \int_Q g(w)|\nabla w|^2 \varphi \geq \int_Q f\varphi \,, \tag{3.1.4}$$

$$\forall \varphi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q) \,, \quad \varphi \geq 0 \text{ a.e. in } Q.$$

Now we are able to state and prove two comparison lemmas that will play the key role in the proof of our main result. The first one concerns the elliptic case for unbounded sub and supersolutions with no restrictions on the sign of the datum, while in Lemma 3.5 we prove the same result in the parabolic case with general nonnegative data; these calculations are inspired by [**BaM**].

Moreover, observe that the following comparison results between sub and supersolutions easily imply the uniqueness of solution for the corresponding problem; this result simply follows by observing that any solution turns out to be both a subsolution and a supersolution for the problem.

LEMMA 3.4. *Let $v_1(x)$ and $v_2(x)$ be respectively a subsolution and a supersolution of the elliptic problem (3.0.5) with $f \in L^1(\Omega)$ and g satisfying (3.0.2) and (3.0.3). Then $v_1(x) \leq v_2(x)$ a.e. in Ω.*

PROOF. Let us define the following change of variable:

$$\Psi(t) = \int_0^t e^{-2G(s)} ds, \quad \text{where} \quad G(s) = \int_0^s g(\sigma) d\sigma \,. \tag{3.1.5}$$

Consider now the function

$$\varphi(s) = \Psi^{-1}(s)$$

which is a function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, $\varphi \in C^2(\mathbb{R})$, such that $\varphi'(s) > 0$, $\forall s \in \mathbb{R}$, and let us consider

$$v = \varphi(u), \qquad v_1 = \varphi(u_1) \text{ and } \quad v_2 = \varphi(u_2) \,.$$

Since v_1 and v_2 are respectively a subsolution and a supersolution of problem (3.0.5), then u_1 and u_2 satisfy the following inequalities:

$$-\mathrm{div}\,(\varphi'(u_1)\nabla u_1) + g(\varphi(u_1))\varphi'(u_1)^2 |\nabla u_1|^2 \leq f$$

and

$$-\mathrm{div}\,(\varphi'(u_2)\nabla u_2) + g(\varphi(u_2))\varphi'(u_2)^2 |\nabla u_2|^2 \geq f \,.$$

Hence, by subtracting the first from the second in their weak sense (i.e. with nonnegative test functions η that belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$), we obtain

$$\int_\Omega [\varphi'(u_1)\nabla u_1 - \varphi'(u_2)\nabla u_2] \cdot \nabla \eta \; dx$$

$$+ \int_\Omega \left[g(\varphi(u_1))\varphi'(u_1)^2 |\nabla u_1|^2 - g(\varphi(u_2))\varphi'(u_2)^2 |\nabla u_2|^2 \right] \eta \; dx \leq 0 \,.$$

Now, applying the fundamental theorem of calculus both to first and second term and defining

$$w = u_1 - u_2 \,,$$

we have, recalling that thanks to the assumptions (3.0.2), (3.0.3) and the definition of Ψ, w belongs to $L^\infty(\Omega)$,

$$\begin{aligned}&\int_\Omega\int_0^1 [\varphi''(u)w\nabla u+\varphi'(u)\nabla w]\cdot\nabla\eta\,d\tau dx\\&+\int_\Omega\int_0^1\big[g'(\varphi(u))\varphi'(u)^3w|\nabla u|^2+2g(\varphi(u))\varphi'(u)\varphi''(u)w|\nabla u|^2\\&\qquad+2g(\varphi(u))\varphi'(u)^2\nabla u\cdot\nabla w\big]\,\eta\,d\tau dx\le 0\,,\end{aligned}$$

where $u=\tau u_1+(1-\tau)u_2$. Let us choose $\eta=w^+$ as test function in the previous inequality, so we have:

$$\begin{aligned}&\int_\Omega\int_0^1 \varphi''(u)w\nabla u\cdot\nabla w^+ +\varphi'(u)|\nabla w|^2\,d\tau dx\\&+\int_\Omega\int_0^1[g'(\varphi(u))\varphi'(u)^3(w^+)^2|\nabla u|^2+2g(\varphi(u))\varphi'(u)\varphi''(u)(w^+)^2|\nabla u|^2\\&\qquad+2g(\varphi(u))\varphi'(u)^2w\nabla u\cdot\nabla w^+]d\tau dx\le 0\,.\end{aligned}$$

Thanks to Young's inequality, we have:

$$\left|\varphi''(u)w\nabla u\cdot\nabla w^+\right|\le\frac12\varphi'(u)|\nabla w|^2+\frac12\frac{\varphi''(u)^2}{\varphi'(u)}(w^+)^2|\nabla u|^2\,,$$

$$\left|2g(\varphi(u))\varphi'(u)^2w\nabla u\cdot\nabla w^+\right|\le\frac12\varphi'(u)|\nabla w|^2+2g(\varphi(u))^2\varphi'(u)^3(w^+)^2|\nabla u|^2\,,$$

and so

$$\begin{aligned}&\int_\Omega\int_0^1 g'(\varphi(u))\varphi'(u)^3(w^+)^2|\nabla u|^2+2g(\varphi(u))\varphi'(u)\varphi''(u)(w^+)^2|\nabla u|^2\\&-2g(\varphi(u))^2\varphi'(u)^3(w^+)^2|\nabla u|^2-\frac12\frac{\varphi''(u)^2}{\varphi'(u)}(w^+)^2|\nabla u|^2\;dxd\tau\le 0\,,\end{aligned}$$

that implies

$$\begin{aligned}&\int_\Omega|w^+|^2\int_0^1|\nabla u|^2\varphi'(u)^3\left[g'(u)+2g(\varphi(u))\frac{\varphi''(u)}{\varphi'(u)^2}\right.\\&\qquad\left.-2g(\varphi(u))^2-\frac12\frac{\varphi''(u)^2}{\varphi'(u)^4}\right]\;dxd\tau\le 0\,.\end{aligned}\tag{3.1.6}$$

Now observe that

$$\varphi(\Psi(t))=t,\qquad \varphi'(\Psi(t))\Psi'(t)=1$$

and

$$\varphi''(\Psi(t))\Psi'(t)^2+\varphi'(\Psi(t))\Psi''(t)=0\,.$$

Recalling the definition of Ψ in (3.1.5) we have:

$$\Psi'(t) = e^{-2G(t)}, \qquad \Psi''(t) = -2g(t)\Psi'(t)\,.$$

Then

$$\varphi''(\Psi(t))\Psi'(t)^2 = 2g(t)\varphi'(\Psi(t))\Psi'(t),$$

and so

$$\frac{\varphi''}{\varphi'^2} = 2g\,.$$

By replacing the last expression in (3.1.6), we obtain

$$\int_\Omega |w^+|^2 \int_0^1 |\nabla u|^2 \varphi'(u)^3 [g'(u) + 4(g(\varphi(u)))^2$$

$$-2g(\varphi(u))^2 - \tfrac{1}{2}4g(\varphi(u))^2]\; dx d\tau \le 0\,,$$

that implies $w^+(x) \equiv 0$; thus $u_1(x) \le u_2(x)$ a.e. in Ω; this concludes the proof since $\varphi'(s) > 0$. $\square$

LEMMA 3.5. *Let $f \ge 0$ in $L^1(Q)$, g satisfying (3.0.2) and (3.0.3) and let u_1, u_2 be, respectively, a subsolution and a supersolution of problem (3.0.1). Then $u_1 \le u_2$ a.e. in Q.*

PROOF. Consider, as in the previous proof,

$$w = \Psi(u) = \int_0^u e^{-G(s)} ds \quad \text{where} \quad G(s) = \int_0^s g(t)dt\,.$$

Thanks to the structure of the change of variable we can easily check that w satisfies the following differential equation:

$$w_t - \Delta w = f e^{-G(\phi(w))} \quad \text{where} \quad \phi = \Psi^{-1}\,.$$

We remind that, since g is increasing, then $g \notin L^1(\mathbb{R})$, and so
$e^{-G(s)} \in L^1(\mathbb{R})$; hence

$$\Psi : L^2(0,T;H_0^1(\Omega)) \to L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)\,.$$

Moreover $\Psi(s)$ is an increasing function. Therefore let us consider

$$w_1 = \Psi(u_1) \quad \text{and} \quad w_2 = \Psi(u_2)\,.$$

Now let us take $\varphi = (w_1 - w_2)^+$ as test function in the weak formulation of sub and supersolution for u_1 and u_2; then taking the difference we obtain

$$\int_0^T \langle (w_1 - w_2)_t, (w_1 - w_2)^+ \rangle + \int_Q \nabla(w_1 - w_2) \cdot \nabla(w_1 - w_2)^+$$

$$\leq \int_Q f(e^{-G(\phi(w_1))} - e^{-G(\phi(w_2))})(w_1 - w_2)^+ .$$

Let us observe that the function $\eta(\tau) = e^{G(\phi(\tau))}$ is monotone nonincreasing, so we deduce that

$$\int_\Omega |(w_1 - w_2)^+|^2(T) + \int_Q |\nabla(w_1 - w_2)^+|^2 \leq \int_\Omega |(w_1 - w_2)^+|^2(0) , \tag{3.1.7}$$

and so, since $(w_1 - w_2)^+(0) \equiv 0$, it follows that

$$w_1(t,x) \leq w_2(t,x) \quad \text{a.e. in } \; Q ,$$

that easily implies that $u_1(t,x) \leq u_2(t,x)$ a.e. in Q. Notice that, by (3.1.7), the result hold true also for every fixed $0 \leq t \leq T$, a.e. on Ω. □

Remark 3.6. We would like to emphasize a general fact that we will often use in what follows: the solution of problem (3.0.1) actually belongs to $L^\infty(\Omega \times (t^*, T))$, $\forall t^* \in (0,T)$, if the initial datum $u_0 \in L^1(\Omega)$ and f is regular. Indeed, consider the heat potential $\overline{u}(t,x)$ in $\mathbb{R}^N \times (0,T)$ associated to our problem, that is the solution of

$$\begin{cases} \overline{u}_t(t,x) - \Delta\overline{u}(t,x) = f(x) & \text{in } \; \mathbb{R}^N \times (0,T), \\ \overline{u}(0,x) = \overline{u}_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{3.1.8}$$

provided that $\overline{u}(t,x)$ vanishes for $|x|$ that diverges, and $\overline{u}_0$ is the trivial extension of u_0 at 0 outside Ω; due to the sign assumption on g, $\overline{u}(t,x)$, restricted to Ω, turns out to be a supersolution of problem (3.0.1), and so

$$0 \leq u(t,x) \leq \overline{u}(t,x), \quad \text{a.e. on } \; (0,T) \times \Omega .$$

Now, thanks to classical results in potential theory we have, under suitable hypotheses on the data (see for instance [**DiB2**]) that $\overline{u} \in L^\infty(\mathbb{R}^N \times (0,T))$ and so also u turns out to be bounded. For our aim it is enough to observe that the previous result holds true if $u_0 \in L^1(\Omega)$ and $f \in L^\infty(Q)$ (for further details and sharp hypotheses on the data see again [**DiB2**]).

3.2. Asymptotic behavior

In this section we will first state and prove our asymptotic preliminary results and then we will deal with the proof of Theorem 3.1; let us recall that 2^* denotes the Sobolev conjugate exponent of 2, that is $2^* = \frac{2N}{N-2}$. From now on we will denote by $v(x)$ the unique solution of problem (3.0.5).

THEOREM 3.7. *Let $f \in L^1(\Omega)$ be a nonnegative function, g satisfying (3.0.2) and (3.0.3) and let $u(t,x)$ be the solution of the problem*

$$\begin{cases} u_t - \Delta u + g(u)|\nabla u|^2 = f & \text{in } \Omega \times (0,T)\,, \\ u(0,x) = 0 & \text{in } \Omega\,, \\ u(t,x) = 0 & \text{on } \partial\Omega \times (0,T)\,, \end{cases} \tag{3.2.1}$$

Then

$$\lim_{t\to+\infty} u(t,x) = v(x) \qquad \text{in } L^{2^*}(\Omega).$$

PROOF. Let us introduce for any $\tau > 1$ the sequence of functions in $Q_1 = \Omega \times (0,1)$ defined by

$$u^\tau(t,x) = u(\tau + t, x)\,.$$

Since $u(t,x)$ is the solution of problem (3.2.1), and recalling that f does not depend on time, we deduce that u^τ solves

$$\begin{cases} u^\tau_t - \Delta u^\tau + g(u^\tau)|\nabla u^\tau|^2 = f & \text{in } (0,1)\times\Omega, \\ u^\tau(0,x) = u(\tau,x) & \text{in } \Omega, \\ u^\tau(t,x) = 0 & \text{on } (0,1)\times\partial\Omega\,. \end{cases} \tag{3.2.2}$$

Moreover from Lemma 3.5, since $v(x)$ is a supersolution of both (3.2.1) and (3.2.2), we obtain the following estimates:

$$u(t,x) \le v(x), \qquad \forall t \in \mathbb{R}\,, \text{ a.e. in } \Omega$$

and

$$u^\tau(t,x) \le v(x), \qquad \forall t \in (0,1)\,, \forall \tau \in \mathbb{N} \text{ a.e. in } \Omega\,. \tag{3.2.3}$$

Now we can compare $u(t,x)$ with $u_s(t,x) = u(s+t,x)$, $s > 0$. Since $f(x)$ does not depend on t, then both $u(t,x)$ and $u_s(t,x)$ solve (3.2.1) with respectively $u_{01} \equiv 0$ and $u_{02} = u(s,x) \ge 0$ as initial datum. Using again Lemma 3.5 we deduce that $u(t,x) \le u(t+s,x)\,, \forall s \in \mathbb{R}$. Thus $u(t,x)$ is a monotone nondecreasing function with respect to the variable t: this fact implies that also $u^\tau(t,x)$, by its definition, is a monotone nondecreasing sequence with respect to the parameter τ, that is:

$$u^\tau(t,x) \le u^{\tau+1}(t,x) \qquad \forall t \in (0,1)\,, \text{ a.e. in } \Omega\,.$$

Therefore there exists a function $\widetilde{u}(t,x)$ such that $u^\tau(t,x)$ converges a.e. in Q_1 to $\widetilde{u}(t,x)$ as τ tends to $+\infty$. Moreover, thanks to (3.2.3) we have, using the *dominated convergence theorem*, that

$$u^\tau(t,x) \longrightarrow \widetilde{u}(t,x) \qquad \text{in } L^2(Q_1)\,.$$

On the other hand, using the monotonicity of $u(t,x)$ with respect to t and (3.2.3), there exists a function $\overline{u}(x) \in L^2(\Omega)$ such that $u(t,x)$ converges to $\overline{u}(x)$ in $L^2(\Omega)$. So, in particular, $u^\tau(0,x)$ converges to the same function as τ diverges. Thus, stability results for such type of equation (see for instance [**DO**], [**Po1**]) imply that u^τ converges to $\widetilde{u}$ strongly in $L^2(0,1;H_0^1(\Omega))$, and $g(u^\tau)|\nabla u^\tau|^2$ converges to $g(\widetilde{u})|\nabla\widetilde{u}|^2$ strongly in $L^1(Q_1)$. Observe that the function $\widetilde{u}(t,x)$ does not depend on time, in fact

$$u^\tau(0,x) \le \tilde{u} = u(x,t+\tau) \le u(\tau+1,x) = u^{\tau+1}(0,x)\ ;$$

so, being the limit of $u^\tau(0,x)$ and $u^{\tau+1}(0,x)$ as τ diverges the same, we deduce that $\widetilde{u}(t,x) \equiv \widetilde{u}(x)$. Moreover, the above calculation easily implies that $\widetilde{u}(x) \equiv \overline{u}(x)$ a.e. in Ω.

To conclude let us prove that $\overline{u}(x)$ solves the elliptic problem (3.0.5). Consider the weak formulation of (3.2.2) and choose $\Psi(x) \in H_0^1(\Omega)\cap L^\infty(\Omega)$ as test function; we obtain

$$\int_0^1 \langle u_t^\tau, \Psi(x)\rangle + \int_{\Omega\times(0,1)} \nabla u^\tau\cdot\nabla\Psi(x)$$
$$+\int_{\Omega\times(0,1)} g(u^\tau)|\nabla u^\tau|^2\Psi(x) = \int_{\Omega\times(0,1)} f\Psi(x)\,.$$

Thanks to the stability result cited above, integrating by parts, the last expression tends to

$$\int_\Omega \nabla\overline{u}(x)\cdot\nabla\Psi(x) + \int_\Omega g(\overline{u}(x))|\nabla\overline{u}(x)|^2\Psi(x) = \int_\Omega f\Psi(x)$$

and then

$$\overline{u}(x) = \widetilde{u}(x) = v(x)\,,$$

where $v(x)$ is the unique solution of problem (3.0.5). Finally, let us observe that, thanks to (3.2.3) and Sobolev embedding theorem, $u(t,x)$ actually converges to $v(x)$ in $L^{2^*}(\Omega)$. $\square$

THEOREM 3.8. *Let $f \in L^\infty(\Omega)$, $u_0(x) \in L^1(\Omega)$ two nonnegative functions and let $u(t,x)$ be the solution of the problem*

$$\begin{cases} u_t - \Delta u + g(u)|\nabla u|^2 = f & \text{in } \Omega\times(0,T) \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u(t,x) = 0 & \text{on } \partial\Omega\times(0,T)\,. \end{cases} \tag{3.2.4}$$

Then

$$\lim_{t\to+\infty} u(t,x) = v(x) \qquad \textit{in } L^p(\Omega)\,,$$

for every $p \geq 1$.

PROOF. We divide the proof in few steps.
Step 1. Suppose that u is the solution of the following problem:

$$\begin{cases} u_t - \Delta u + g(u)|\nabla u|^2 = f & \text{in } \Omega\times(0,T) \\ u(0,x) = \tau v(x) + h & \text{in } \Omega \\ u(t,x) = 0 & \text{on } \partial\Omega\times(0,T)\,, \end{cases} \tag{3.2.5}$$

where $\tau > 1$ and $h \geq 0$.

Observe that $\tau v(x) + h$ is a supersolution for (3.2.4): in fact, using the monotonicity of $g(s)$,

$$\begin{aligned} &\frac{d}{dt}(\tau v(x)+h) - \Delta(\tau v(x)+h) + g(\tau v(x)+h)|\nabla \tau v(x) + h|^2 \\ &\qquad\qquad \geq -\Delta v + g(v)|\nabla v|^2 = f\,, \end{aligned}$$

$$\tau v(x) + h \geq v(x) \text{ a.e. in } \Omega\,,$$

and

$$\tau v(x) + h \geq 0 \text{ on } \partial\Omega\times(0,T)\,.$$

So, by the comparison lemma we proved above,

$$v(x) \leq u(t,x) \leq \tau v(x) + h \quad \text{ a.e. in } \quad Q \tag{3.2.6}$$

Moreover we can compare $u(t,x)$ with $u(t+s,x)$, $s\in\mathbb{R}^+$: in fact we have that $u(t,x)$ is the solution of the problem (3.0.1) that for $t=0$ achieves the value $\tau v(x)+h$ while $u(t+s,x)|_{t=0} = u(s,x)$. Since $\tau v(x)+h$ is a supersolution and again by the comparison lemma we deduce, being $\tau v(x)+h \geq u(s,x)$, that $u(t,x) \geq u(t+s,x)$ a.e. on Q. Let us consider now the sequence of problems

$$\begin{cases} u^\tau_t - \Delta u^\tau + g(u^\tau)|\nabla u^\tau|^2 = f & \text{in } \Omega\times(0,1), \\ u^\tau(0,x) = u(\tau,x) & \text{in } \Omega' \\ u^\tau(t,x) = 0 & \text{on } \partial\Omega\times(0,1)\,, \end{cases} \tag{3.2.7}$$

for any $\tau\in\mathbb{N}$. Thanks to stability result (see again [**DO**], [**Po1**]), we have that $u^\tau(t,x)$ converges in $L^2(0,1;H^1_0(\Omega))$ to a function $\overline{u}(t,x)$ and thanks to monotonicity with respect to t variable we have that $\overline{u}$ does not depend on t. So we can pass to the limit in the weak formulation of problem (3.0.1) and conclude that $\overline{u}(x)$ is a stationary solution of (3.0.1), and by virtue of uniqueness result for the elliptic problem (3.2.7), $\overline{u}(x) \equiv v(x)$.

Observe that from (3.2.6), the a.e. convergence of $u(t,x)$ to $v(x)$ and the boundedness of $v(x)$ actually $u(t,x)$ converges to $v(x)$ in $L^p(\Omega)$, $\forall p \geq 1$.

Step 2. Now let us consider an initial datum $u_0(x)$ such that $0 \leq u_0(x) \leq \tau v(x) + h$ for some $\tau \geq 1$ and $h > 0$. Let $u_1(t,x)$ be the solution of

$$\begin{cases} u_{1t} - \Delta u_1 + g(u_1)|\nabla u_1|^2 = f & \text{in } \Omega \times (0,T) \\ u_1(0,x) = 0 & \text{in } \Omega \\ u_1(t,x) = 0 & \text{on } \partial\Omega \times (0,T)\,, \end{cases}$$

and $u_2(t,x)$ the solution of

$$\begin{cases} u_{2t} - \Delta u_2 + g(u_2)|\nabla u_2|^2 = f & \text{in } \Omega \times (0,T) \\ u_2(0,x) = \tau v(x) + h & \text{in } \Omega \\ u_2(t,x) = 0 & \text{on } \partial\Omega \times (0,T)\,; \end{cases}$$

finally let us consider the solution $u(t,x)$ of

$$\begin{cases} u_t - \Delta u + g(u)|\nabla u|^2 = f & \text{in } \Omega \times (0,T) \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u(t,x) = 0 & \text{on } \partial\Omega \times (0,T)\,. \end{cases}$$

Thanks to the comparison lemma we have that

$$u_1(t,x) \leq u(t,x) \leq u_2(t,x) \quad \text{a.e. in } Q\,.$$

So, passing to the limit with respect to the t variable, we have that

$$v(x) = \lim_{t\to+\infty} u_1(t,x) \leq \liminf_{t\to\infty} u(t,x) \leq \limsup_{t\to\infty} u(t,x) \leq \lim_{t\to+\infty} u_2(t,x) = v(x)\,.$$

Then the limit with respect to the t variable of $u(t,x)$ exists, coincides with $v(x)$ and the convergence is in $L^p(\Omega)$.

Observe that in the last case if we choose as initial datum any $u_0(x) \in L^\infty(\Omega)$, we can always find τ and h such that we go back to the previous case (in particular the choice can be $h = \|u_0\|_{L^\infty(\Omega)}$ and any $\tau > 1$).

Step 3. Finally, let us consider a general nonnegative initial datum $u_0 \in L^1(\Omega)$: thanks to Remark 3.6 we have that any solution $u(t,x)$ of (3.0.1) with $f \in L^\infty(\Omega)$ is such that $u(t^*,x) \in L^\infty(\Omega)$, for some $0 < t^* < T$. So, thanks to uniqueness of solution we can consider the problem

$$\begin{cases} u_t - \Delta u + g(u)|\nabla u|^2 = f & \text{in } \Omega \times (t^*, T) \\ u(t^*, x) = u(t^*, x) & \text{in } \Omega \\ u(t, x) = 0 & \text{on } \partial\Omega \times (t^*, T) . \end{cases}$$

Now $u(t^*, x) \in L^\infty(\Omega)$ and so again we come back to the previous case. □

The difficulty to generalize our result to the case where both f and u_0 are in $L^1(\Omega)$ (Theorem 3.1) relies on the lack of some continuous dependence of the solutions from their initial data uniformly with respect to the variable t. To avoid this fact in the proof of our main result we will use a change of variable, shifting the problem to a more regular one (as in the proof of Lemma 3.5). However, if u_0 is more regular we can prove the following partial result:

THEOREM 3.9. *Let $f \in L^1(\Omega)$, $u_0(x) \in L^\infty(\Omega)$ be two nonnegative functions and let $u(t, x)$ be the solution of the problem*

$$\begin{cases} u_t - \Delta u + g(u)|\nabla u|^2 = f & \textit{in } \Omega \times (0, T) \\ u(0, x) = u_0(x) & \textit{in } \Omega \\ u(t, x) = 0 & \textit{on } \partial\Omega \times (0, T) . \end{cases}$$

Then

$$\lim_{t \to +\infty} u(t, x) = v(x) \qquad \textit{in } L^{2^*}(\Omega).$$

PROOF. Let us consider u_n the solution of the problem

$$\begin{cases} (u_n)_t(t, x) - \Delta u_n(t, x) + g(u_n)|\nabla u_n|^2 = f_n & \text{in } (0, T) \times \Omega, \\ u_n(0, x) = u_0(x) & \text{in } \Omega, \\ u_n(t, x) = 0 & \text{on } (0, T) \times \partial\Omega, \end{cases}$$

where f_n is a monotone nondecreasing sequence of regular approximating functions such that

$$f_n \to f \text{ strongly in } L^1(\Omega) .$$

Moreover, let us define $\tilde{u}$ the solution of problem

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} + g(\tilde{u})|\nabla \tilde{u}|^2 = f & \text{in } \Omega \times (0, T) \\ \tilde{u}(0, x) = v(x) + h & \text{in } \Omega \\ \tilde{u}(t, x) = 0 & \text{on } \partial\Omega \times (0, T) , \end{cases} \tag{3.2.8}$$

where as before we can choose $h = \|u_0\|_{L^\infty(\Omega)}$, and let us call $\tilde{u}_n$ the solution of problem (3.2.8) with f_n instead of f as datum. So, thanks to the monotonicity assumptions on f_n and the comparison result proved we easily obtain

$$u_n(t,x) \le u(t,x),$$

and

$$u_n(t,x) \le \tilde{u}_n(t,x);$$

so, passing to the limit over n in the last inequality and using again the stability result cited above, we actually have

$$u_n(t,x) \le u(t,x) \le \tilde{u}(t,x)$$

a.e. on Q_T, for all $T > 0$. So we can pass to the limit over t obtaining

$$v_n(x) \le \liminf_{t\to\infty}\ u(t,x) \le \limsup_{t\to\infty} u(t,x) \le v(x) \text{ a.e. in } \Omega\,.$$

once we proved that the convergence result holds true for $\tilde{u}$. Notice that the above estimate allows us to conclude using again stability results for elliptic problems and the dominated convergence theorem. So, let us prove that $\tilde{u}(t,x)$ converges to $v(x)$; applying the comparison lemma to $\tilde{u}_n$ and reasoning as in the proof of Theorem 3.8 we easily see that

$$\tilde{u}_n(t,x) \le \tilde{u}_n(t+s,x)$$

a.e. in Ω, $t,s > 0$. So, by stability, this monotonicity property with respect to t still holds true for the limit $\tilde{u}(t,x)$ that decreases, as t diverges, to some function $\overline{u}(x)$, that turns out to be $v(x)$ reasoning exactly as in the proof of Theorem 3.8 with the use of functions u^τ. □

Remark 3.10. Notice that, in view of Lemma 3.5 we can make the same calculation as in the proof of Theorem 3.8 to see that the result of Theorem 3.9 holds true even if the initial datum $u_0 \le nv(x)$, with $n \in \mathbb{N}$. We will use this fact in the sequel.

Now we are able to prove our main result.

Proof of Theorem 3.1. Consider our parabolic problem (3.0.1) and its associated problem

$$\begin{cases} w_t - \Delta w = f e^{-G(\phi(w))} & \text{in } \Omega \times (0,T)\,, \\ w(0,x) = \Psi(u_0) & \text{in } \Omega\,, \\ w(t,x) = 0 & \text{on } \partial\Omega \times (0,T), \end{cases}$$

through the change of variable

$$w = \Psi(u) = \int_0^u e^{-G(s)} ds \quad \text{where} \quad G(s) = \int_0^s g(t)dt \quad \text{and } \varphi(s) = \Psi^{-1}(s)\,.$$

Let us define the following sequences $\{u_n\}$ and $\{w_n\}$ of solutions of problems

$$\begin{cases} (u_n)_t(t,x) - \Delta u_n(t,x) + g(u_n)|\nabla u_n|^2 = f & \text{in } (0,T)\times\Omega\,, \\ u_n(0,x) = u_{n,0}(x) = \min(nv(x), u_0) & \text{in } \Omega\,, \\ u_n(t,x) = 0 & \text{on } (0,T)\times\partial\Omega\,, \end{cases} \tag{3.2.9}$$

and

$$\begin{cases} (w_n)_t - \Delta w_n = f e^{-G(\phi(w_n))} & \text{in } \Omega\times(0,T)\,, \\ w_n(0,x) = \Psi(u_{n,0}) & \text{in } \Omega\,, \\ w(t,x) = 0 & \text{on } \partial\Omega\times(0,T)\,. \end{cases} \tag{3.2.10}$$

From previous results we know that $u_n(t,x)$ converges for t that tends to $+\infty$ to $v(x)$ in $L^{2^*}(\Omega)$ and, thanks to the fact that Ψ is Lipschitz continuous, we also deduce that

$$\lim_{t\to\infty} \Psi(u_n(t,x)) = \Psi(v(x)) \tag{3.2.11}$$

in $L^{2^*}(\Omega)$. So, up to subsequences the almost everywhere limit of $w_n(t,x)$ with respect to the time variable is $\Psi(v(x))$.

Now, since $u_n(t,x) \le u(t,x)$ and so $w_n(t,x) \le w(t,x)$, arguing as in the proof of Lemma 3.5 with $\varphi = (w - w_n)$ as test function we can obtain an estimate as (3.1.7), that is $\forall t > 0$

$$\int_\Omega |(w-w_n)|^2(t) \le \int_\Omega |(w_0 - w_{n,0})|^2(0)\,.$$

Let us choose a positive ε: thanks to the above estimate we can choose n large enough, such that

$$\|w - w_n\|_{L^2(\Omega)} \le \frac{\varepsilon}{2}\,.$$

We can deduce from the above inequality and (3.2.11):

$$\|w - \Psi(v(x))\|_{L^2(\Omega)} \le \frac{\varepsilon}{2} + \|\Psi(u_n) - \Psi(v(x))\|_{L^2(\Omega)} \le \varepsilon\,,$$

provided t is large enough. Hence, once again up to subsequences, $\Psi(u_n)$ converges almost everywhere to $\Psi(v(x))$ and thanks to continuity of φ, u_n converges a.e. to $v(x)$ too. So the a.e. limit of $u(t,x)$ with respect to the time variable actually coincides with $v(x)$. Thus to conclude it is enough to prove that this convergence is in $L^1(\Omega)$. In order to do it let us recall (see Remark 3.6) that our solution is always dominated by the solution $\tilde u$ of the heat equation with the same data. We know (see Theorem 2.4) that this solution converges in $L^1(\Omega)$ to the entropy solution of

$$\begin{cases} -\Delta \tilde v = f & \text{in } \Omega\,, \\ \tilde v = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Since

$$u(t,x) \leq \tilde{u}(t,x) \xrightarrow{L^1(\Omega)} \tilde{v}(x)$$

we can conclude, thanks to Vitali's lemma, that

$$u(t,x) \longrightarrow v(x) \qquad \text{in } L^1(\Omega)\,.$$

$\square$

CHAPTER 4

Renormalized solutions of nonlinear parabolic equations with general measure data

Let $\Omega \subseteq \mathbb{R}^N$ a bounded open set, $N \geq 2$, and let $p > 1$; in this chapter we prove existence of a renormalized solution for parabolic problems whose model is

$$
\begin{cases} u_t - \Delta_p u = \mu & \text{in } (0,T) \times \Omega, \\ u(0,x) = u_0 & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{4.0.12}
$$

where $T > 0$ is any positive constant, $\mu \in M(Q)$ is a any measure with bounded variation over $Q = (0,T) \times \Omega$, and $u_0 \in L^1(\Omega)$, and $-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the usual $p-$Laplace operator.

We will deal with *parabolic p-capacity* associated to our problem as introduced in Definition 1.36. As before, we denote with $M(Q)$ the set of all Radon measures with bounded variation on Q, while $M_0(Q)$ will denote the set of all measures with bounded variation over Q which do not charge the sets of zero p-capacity, that is if $\mu \in M_0(Q)$, then $\mu(E) = 0$, for all $E \in Q$ such that $\mathrm{cap}_p(E) = 0$.

As we said before, in [**DPP**] the authors give another notion of parabolic capacity (see Definition 1.37), equivalent to the one of Definition 1.36 as far as sets of zero capacity are concerned; this new notion is defined on compact set by minimizing the same energy over all functions $\varphi \in C_0^\infty(Q)$ greater than the characteristic function of the set. Therefore, thanks to this approach, we can also define this notion of parabolic capacity of a set with respect to any open $U \subseteq Q$ and this will turn out to be very useful in what follows (see for instance Lemma 4.17 below).

We remind that, if $\mu \in M(Q)$ is a general measure with bounded total variation on Q, we can split it into a sum (uniquely determined) of its *absolutely continuous* part μ_0 with respect to p-capacity and its *singular* part μ_s (that is μ_s is concentrated on a set of zero p-capacity). Hence, if $\mu \in M(Q)$, by Theorem 1.39, we have

$$
\mu = f - \mathrm{div}(G) + g_t + \mu_s, \tag{4.0.13}
$$

in the sense of distributions, for some $f \in L^1(Q)$, $G \in (L^{p'}(Q))^N$, $g \in L^p(0,T; W_0^{1,p}(\Omega))$, and $\mu_s \perp p$-capacity ; recall that the decomposition of the absolutely continuous part of μ in Theorem 1.39 is not uniquely determined.

Moreover, let

$$W_1 = \{u \in L^p(0,T; W_0^{1,p}(\Omega)),\ u_t \in L^{p'}(0,T; W^{-1,p'}(\Omega))\},$$

then, in our setting, any function of W_1 will admits a cap_p-quasi continuous representative (that is, it coincides cap_p-quasi everywhere with a function that is continuous everywhere but on a set of arbitrary small capacity, see Section 1.3).

4.1. General assumptions

Let $a : (0,T)\times\Omega\times\mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function (i.e. $a(\cdot,\cdot,\xi)$ is measurable on Ω, $\forall\xi \in \mathbb{R}^N$, and $a(t,x,\cdot)$ is continuous on $\mathbb{R}^N$ for a.e. $(t,x) \in Q$) such that the following holds:

$$a(t,x,\xi)\cdot\xi \geq \alpha|\xi|^p, \tag{4.1.1}$$

$$|a(t,x,\xi)| \leq \beta[b(t,x) + |\xi|^{p-1}], \tag{4.1.2}$$

$$(a(t,x,\xi) - a(t,x,\eta))\cdot(\xi-\eta) > 0, \tag{4.1.3}$$

for almost every $(t,x) \in Q$, for all $\xi,\ \eta \in \mathbb{R}^N$ with $\xi \neq \eta$, where $p > 1$ and $\alpha,\ \beta$ are positive constants and b is a nonnegative function in $L^{p'}(Q)$. As usual, for every $u \in L^p(0,T; W_0^{1,p}(\Omega))$, let us define the differential operator

$$A(u) = -\text{div}(a(t,x,\nabla u)),$$

that, thanks to its properties, turns out to be a coercive pseudomonotone operator acting from the space $L^p(0,T; W_0^{1,p}(\Omega))$ into its dual $L^{p'}(0,T; W^{-1,p'}(\Omega))$. We shall deal with the solutions of the initial boundary value problem

$$\begin{cases} u_t + A(u) = \mu & \text{in } (0,T)\times\Omega \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(0,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{4.1.4}$$

where μ is a measure with bounded variation over Q, and $u_0 \in L^1(\Omega)$.

As we know, if $\mu \in W'(where \mathrm{W} is defined in (1.3.1))$, and $u_0 \in L^2(\Omega)$, problem (4.1.4) has a unique solution $u \in W \cap C([0,T]; L^2(\Omega))$ in the variational sense, that is

$$-\int_\Omega u_0\ \varphi(0)\ dx - \int_0^T \langle\varphi_t, u\rangle\ dt + \int_Q a(t,x,\nabla u)\cdot\nabla\varphi\ dxdt = \int_0^T \langle f,\varphi\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\ dt,$$

for all $\varphi \in W$ such that $\varphi(T) = 0$ (see [**L**], and [**DPP**]).

Also recall that, in [**B6**] (for more details see also [**BGO**]) the concept of entropy solution of the elliptic boundary value problem associated to (4.1.4) was introduced for a measure $\mu \in M_0(\Omega)$ while the entropy solution u of the problem (4.1.4) exists and is unique as shown in [**DPP**] if $\mu \in M_0(Q)$ (see also [**DP**]). Moreover, the solution is such that $|a(t,x,\nabla u)| \in L^q(Q)$ for all $q < 1 + \dfrac{1}{(N+1)(p-1)}$, even if its gradient may not belong to any Lebesgue space.

Our purpose is to extend all these definitions to general measure data.

For the sake of simplicity we will make a further assumption on the range of p; as in Chapter 2, we assume $p > \frac{2N+1}{N+1}$ that is a standard assumption giving good compactness results and we will assume it throughout the rest of this thesis. Let us observe that, in this setting, the spaces W and W_1 turn out to coincide.

4.2. Definition of renormalized solution and main result

Here we give the definition of renormalized solution following the idea of [**BP**] (se also [**DMOP**] and [**DPP**]). To simplify the notation, let us define $v = u - g$ in what follows, where u is the solution and g is the *time-derivative* part of μ_0, and $\hat{\mu}_0 = \mu - g_t - \mu_s = f - \operatorname{div}(G)$; moreover we understood that

$$\int_Q w \; d\hat{\mu}_0 = \int_Q fw \; dxdt + \int_Q G \cdot \nabla w \; dxdt,$$

for every $w \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$.

Definition 4.1. Assume (4.1.1)–(4.1.3), let $\mu \in M(Q)$, and $u_0 \in L^1(\Omega)$. A measurable function u is a *renormalized solution* of problem (4.1.4) if, there exists a decomposition (f,G,g) of μ_0 such that $v \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$ for every $q < p - \frac{N}{N+1}$, $T_k(v) \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$ for every $k > 0$, and for every $S \in W^{2,\infty}(\mathbb{R})$ $(S(0) = 0)$ such that S' has compact support on $\mathbb{R}$, we have

$$\begin{aligned} &-\int_\Omega S(u_0)\varphi(0)\;dx - \int_0^T \langle \varphi_t, S(v)\rangle\;dt + \int_Q S'(v)a(t,x,\nabla u)\cdot\nabla\varphi\;dxdt \\ &\quad+ \int_Q S''(v)a(t,x,\nabla u)\cdot\nabla v\;\varphi\;dxdt = \int_Q S'(v)\varphi\;d\hat{\mu}_0, \end{aligned} \tag{4.2.1}$$

for every $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, $\varphi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, with $\varphi(T,x) = 0$, such that $S'(v)\varphi \in L^p(0,T;W_0^{1,p}(\Omega))$. Moreover, for every $\psi \in C(\overline{Q})$ we have

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{n \le v < 2n\}} a(t,x,\nabla u)\cdot\nabla v\;\psi\;dxdt = \int_Q \psi\;d\mu_s^+, \tag{4.2.2}$$

and

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{-2n < v \le -n\}} a(t,x,\nabla u)\cdot\nabla v\;\psi\;dxdt = \int_Q \psi\;d\mu_s^-, \tag{4.2.3}$$

where μ_s^+ and μ_s^- are respectively the positive and the negative part of the singular part μ_s of μ.

REMARK 4.2. First of all, notice that, thanks to our regularity assumptions and the choice of S, all terms in (4.2.1), (4.2.2), and (4.2.3) are well defined; in what follows we will often make a little abuse of notation referring to v as a *renormalized solution* of problem (4.1.4). Observe that condition $\varphi(T,x)=0$ is well defined in the sense of $L^2(\Omega)$ thanks to Theorem 1.3.

Also, observe that (4.2.1) implies that equation

$$
\begin{aligned}
&(S(u-g))_t - \operatorname{div}(a(t,x,\nabla u)S'(u-g)) + S''(u-g)a(t,x,\nabla u)\cdot\nabla(u-g)\\
&= S'(u-g)f + G\cdot S''(u-g)\nabla(u-g) - \operatorname{div}(GS'(u-g))
\end{aligned}
\tag{4.2.4}
$$

is satisfied in the sense of distributions, and, since $(S(u-g))_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$, we can use in (4.2.4) not only functions in $C_0^\infty(Q)$ but also in $L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$. Let us also observe that, since for such S we have $S(v) = S(T_M(v)) \in L^p(0,T;W_0^{1,p}(\Omega))$ (if $\operatorname{supp}(S')\subset[-M,M]$) and $S(v)_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$ then $S(v)\in C([0,T];L^1(\Omega))$ (see Theorem 1.6) and one can say that the initial datum is achieved in a weak sense, that is $S(v)(0) = S(u_0)$ in $L^1(\Omega)$ for every S (see [**DPP**] for more details).

Finally, we want to stress that Definition 4.1 actually extends all the notions of solutions studied up to now and, in particular, if $\mu \in M_0(Q)$ it turns out to coincide with the notion of *entropy solution* as shown in [**DP**]; notice that, in this case, entropy and renormalized solutions turn out to be unique.

Let us first show the following interesting property of renormalized solutions:

PROPOSITION 4.3. *Let $v = u-g$ be a renormalized solution of problem (4.1.4). Then, for every, $k>0$, we have*

$$
\int_Q |\nabla T_k(v)|^p \, dxdt \le \tilde{C}(k+1),
\tag{4.2.5}
$$

where $\tilde{C}$ is a positive constant not depending on k.

PROOF. Obviously we can prove it without loss of generality for k large enough. First of all observe that, thanks to (4.1.1), (4.2.2) and (4.2.3), using Young's inequality one can easily show that there exists a positive constant M such that

$$
\frac{1}{n}\int_{\{n\le|v|<2n\}} |\nabla u|^p dxdt \le M.
\tag{4.2.6}
$$

On the other hand, using the definition of v, we have

$$
\int_Q |\nabla T_k(v)|^p \, dxdt \le C\int_{\{|v|<k\}} |\nabla u|^p dxdt + C.
\tag{4.2.7}
$$

Hence, we have to control the first term on the right hand side of (4.2.7); using (4.2.6), we have

$$\int_{\{|v|<k\}} |\nabla u|^p dxdt \leq \sum_{n=0}^{[\log_2 k]+1} \int_{\{2^n \leq |v| < 2^{n+1}\}} |\nabla u|^p dxdt + \int_{\{0 \leq |v| < 1\}} |\nabla u|^p dxdt$$

$$\leq M \sum_{n=0}^{[\log_2 k]+1} 2^n \ + \ C = M(2^{[\log_2 k]+2} - 1) + C \leq C(k+1),$$

that, together with (4.2.7) yields (4.2.5).

$\square$

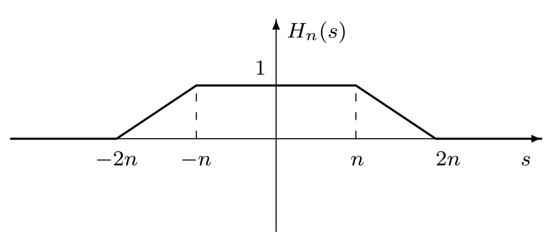
The main result of this chapter is the following one:

THEOREM 4.4. *Assume (4.1.1)–(4.1.3), let* $\mu \in M(Q)$ *and* $u_0 \in L^1(\Omega)$*. Then there exists a renormalized solution of problem (4.1.4).*

4.3. cap_p-quasi continuous representative of a renormalized solution

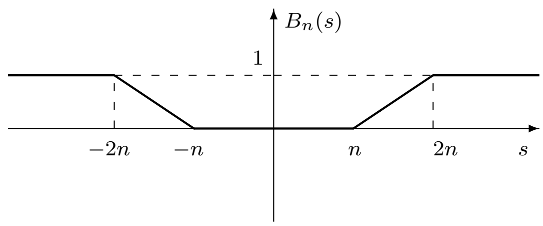
Now we prove some essential property of renormalized solutions that will be useful throughout the rest of the chapter; in particular we shall prove that a renormalized solution (actually the *regular translation* of it) v is finite cap_p-quasi everywhere and it admits a cap_p-quasi continuous representative (and we will always refer to it). To this aim we introduce the following function:

$$H_n(s) = \begin{cases} 1 & \text{if } |s| \leq n, \\[2ex] \dfrac{2n-s}{n} & \text{if } n < s \leq 2n, \\[2ex] \dfrac{2n+s}{n} & \text{if } -2n < s \leq -n, \\[2ex] 0 & \text{if } |s| > 2n. \end{cases} \tag{4.3.1}$$



Let us also introduce another auxiliary function that we will often use in the following; this function can be introduced in terms of $H_n(s)$ defined in (4.3.1):

$$B_n(s) = 1 - H_n(s). \tag{4.3.2}$$



Let us introduce some new notation: if F is a function of one real variable, then $\overline{F}$ will denote its primitive function, that is $\overline{F}(s) = \displaystyle\int_0^s F(r)\ dr$; where there will be no possibility of misunderstanding, we will indicate simply with S the space S^p introduced in Chapter 1, that is

$$S = \{u \in L^p(0,T;W_0^{1,p}(\Omega)); u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)\}, \tag{4.3.3}$$

endowed with its natural norm $\|u\|_S = \|u\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}$, and its subspace W_2 as

$$W_2 = \{u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q); u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)\}, \tag{4.3.4}$$

endowed with its natural norm $\|u\|_{W_2} = \|u\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u\|_{L^\infty(Q)}$ $+ \|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}$; for any $p > 1$, following the outlines of [**DPP**] let us also define $\tilde{W} \equiv W_1 \cap L^\infty(0,T;L^2(\Omega))$ and for all $z \in \tilde{W}$, let us denote

$$[z]_W = \|z\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|z_t\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))} + \|z\|^2_{L^\infty(0,T;L^2(\Omega))} \ .$$

In the proof of Lemma 2.17 of [**DPP**] the authors show that

Lemma 4.5. *Let $u \in W_2$, then there exists $z \in \tilde{W}$ such that $|u| \le z$ and*

$$\begin{aligned}[z]_W \le C\Big(&\|u\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t^1\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \\ &+\|u\|_{L^\infty(Q)}\|u_t^2\|_{L^1(Q)} + \|u\|^2_{L^\infty(0,T;L^2(\Omega))}\Big),\end{aligned} \tag{4.3.5}$$

where $u_t^1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, $u_t^2 \in L^1(Q)$ is a decomposition of u_t, that is $u_t = u_t^1 + u_t^2$.

Remark 4.6. Observe that u_t^1 and u_t^2 can be chosen such that

$$\|u_t^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} + \|u_t^2\|_{L^1(Q)} \le 2\|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)},$$

and so (4.3.5) easily implies

$$
\begin{aligned}
[z]_W \le C \Big(&\|u\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)} \\
&+\|u\|_{L^\infty(Q)}\|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)} + \|u\|^2_{L^\infty(0,T;L^2(\Omega))}\Big),
\end{aligned}
\tag{4.3.6}
$$

that was a result of Lemma 2.17 in [**DPP**]. For the sake of simplicity let us define

$$
\begin{aligned}
[u]_* = {} &\|u\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t^1\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \\
&+\|u\|_{L^\infty(Q)}\|u_t^2\|_{L^1(Q)} + \|u\|^2_{L^\infty(0,T;L^2(\Omega))},
\end{aligned}
$$

and

$$
\begin{aligned}
[u]_{**} = {} &\|u\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)} \\
&+\|u\|_{L^\infty(Q)}\|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)} + \|u\|^2_{L^\infty(0,T;L^2(\Omega))}.
\end{aligned}
$$

Now our aim is to prove the following result:

THEOREM 4.7. *Let $u \in W_2$; then u admits a unique cap_p-quasi continuous representative defined cap_p-quasi everywhere.*

To prove Theorem 4.7 we need first a *capacitary estimate*, this is the goal of next result:

LEMMA 4.8. *Let $u \in W_2$ be a cap_p-quasi continuous function, then, for every $k > 0$,*

$$
\mathrm{cap}_p(\{|u| > k\}) \le \frac{C}{k} \max\left([u]_*^{\frac{1}{p}}, [u]_*^{\frac{1}{p'}}\right). \tag{4.3.7}
$$

PROOF. We divide the proof in two steps.

Step 1. Let $u \in C_0^\infty([0,T]\times\Omega)$, so the set $\{|u| > k\}$ is open and we can estimate in terms of the norm of W, by Lemma 4.5 there exists $z \in \tilde{W}$ such that $|u| \le z$ and (4.3.5) holds true, so, recalling that $\tilde{W} \subseteq W$ continuously, $\dfrac{z}{k}$ is a good function to test capacity of the set $\{|u| > k\}$ and we can write

$$
\begin{aligned}
\mathrm{cap}_p(\{|u| > k\}) &\le \frac{\|z\|_W}{k} \le \frac{C}{k}\|z\|_{\tilde{W}} = \frac{C}{k}\Big(\|z\|_{L^p(0,T;W_0^{1,p}(\Omega))} \\
&+\|z\|_{L^\infty(0,T;L^2(\Omega))} + \|z_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}\Big) \le \frac{C}{k}\left([u]_*^{\frac{1}{p}} + [u]_*^{\frac{1}{2}} + [u]_*^{\frac{1}{p'}}\right) \\
&\le \frac{C}{k}\max\left([u]_*^{\frac{1}{p}}, [u]_*^{\frac{1}{p'}}\right).
\end{aligned}
$$

Step 2. Let $u \in W_2$ and cap_p-quasi continuous, then, for every fixed $\varepsilon > 0$, there exists an open set A_ε such that $\mathrm{cap}_p(A_\varepsilon) \leq \varepsilon$ and $u_{|_{Q\setminus A_\varepsilon}}$ is continuous. Hence, the set $\{|u_{|_{Q\setminus A_\varepsilon}}| > k\} \cap (Q\setminus A_\varepsilon)$ is open in $Q\setminus A_\varepsilon$, that is there exists an open set $U \in \mathbb{R}^{N+1}$ such that

$$\{|u_{|_{Q\setminus A_\varepsilon}}| > k\} \cap (Q\setminus A_\varepsilon) = U \cap (Q\setminus A_\varepsilon)\,.$$

Therefore, the set

$$\{|u| > k\} \cup A_\varepsilon = \{|u_{|_{Q\setminus A_\varepsilon}}| > k\} \cap (Q\setminus A_\varepsilon) \cup A_\varepsilon = (U \cup A_\varepsilon) \cap Q$$

turns out to be open; let z the function given in Lemma 4.5 and let $w \in W$ such that $w \geq \chi_{A_\varepsilon}$ and

$$\|w\|_W \leq \mathrm{cap}_p(A_\varepsilon) + \varepsilon \leq 2\varepsilon;$$

we have that $w + \dfrac{z}{k} \geq 1$ almost everywhere on $\{|u| > k\} \cup A_\varepsilon$, so

$$\begin{aligned}\mathrm{cap}_p(\{|u| > k\} \cup A_\varepsilon) &\leq \|w\|_W + \frac{\|z\|_W}{k}\\ &\leq \frac{\|z\|_W}{k} + 2\varepsilon.\end{aligned}$$

Finally, recalling the monotonicity of the capacity and thanks the arbitrary choice of ε we can conclude as in Step 1. $\square$

An interesting consequence of these results, whose proof can be checked arguing as in [**DMOP**] and [**HKM**] for the elliptic case, is the following

COROLLARY 4.9. *Let $u \in W_2$ a and $\mu_0 \in M_0(Q)$. Then, (the* cap_p *quasi continuous representative of) u is measurable with respect to μ_0. Moreover (the* cap_p*-quasi continuous representative of) u belongs to $L^\infty(Q, \mu_0)$, hence to $L^1(Q, \mu_0)$.*

REMARK 4.10. Obviously we also have that

$$\mathrm{cap}_p(\{|u| > k\}) \leq \frac{C}{k} \max\left([u]_{**}^{\frac{1}{p}}, [u]_{**}^{\frac{1}{p'}}\right); \tag{4.3.8}$$

therefore, thanks to Young's inequality and to the fact that W_2 is continuously embedded in $C([0,T]; L^1(\Omega))$ (see [**Po1**]) we deduce that

$$\mathrm{cap}_p(\{|u| > k\}) \leq \frac{C}{k} \max\left(\|u\|_{W_2}^p, \|u\|_{W_2}^{p'}\right). \tag{4.3.9}$$

Proof of Theorem 4.7. Let us first observe that there are no difficulties in approximating, for instance via convolution, a function $u \in W_2$ with smooth functions $u^m \in C_0^\infty([0,T] \times \Omega)$ in the norm

$$\|u^m\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t^m\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}$$

with $\|u^m\|_{L^\infty(Q)} \leq C$ (see [**Dr**] and [**DPP**]); so let u^m be a sequence like this, actually we can construct u^m such that

$$\sum_{m=1}^{\infty} 2^m \max\left([u^{m+1} - u^m]_{**}^{\frac{1}{p}}, [u^{m+1} - u^m]_{**}^{\frac{1}{p'}}\right)$$

is finite. Now, for every m and r, let us define

$$\omega^m = \{|u^{m+1} - u^m| > \frac{1}{2^m}\} \quad \text{and} \quad \Omega^r = \bigcup_{m \geq r} \omega^m.$$

Now we can apply Lemma 4.8 and recalling (4.3.8) we obtain

$$\mathrm{cap}_p(\omega^m) \leq C 2^m \max\left([u^{m+1} - u^m]_{**}^{\frac{1}{p}}, [u^{m+1} - u^m]_{**}^{\frac{1}{p'}}\right),$$

and so, by subadditivity

$$\mathrm{cap}_p(\Omega^r) \leq C \sum_{m \geq r} 2^m \max\left([u^{m+1} - u^m]_{**}^{\frac{1}{p}}, [u^{m+1} - u^m]_{**}^{\frac{1}{p'}}\right),$$

that implies

$$\lim_{r \to \infty} \mathrm{cap}_p(\Omega^r) = 0. \tag{4.3.10}$$

Moreover, for every $y \notin \Omega^r$ we have that

$$|u^{m+1} - u^m|(y) \leq \frac{1}{2^m}$$

for any $m \geq r$, and so u^m converges uniformly on the complement of Ω^r and pointwise on the complement of $\bigcap_{r=1}^{\infty} \Omega^r$. But, for any $l \in \mathbb{N}$, we have

$$\mathrm{cap}_p\left(\bigcap_{r=1}^{\infty} \Omega^r\right) \leq \mathrm{cap}_p(\Omega^l),$$

and so, by (4.3.10), we conclude that $\mathrm{cap}_p\left(\bigcap_{r=1}^{\infty} \Omega^r\right) = 0$; therefore the limit of u^m is cap_p-quasi continuous and is defined cap_p-quasi everywhere.

Let us call $\tilde{u}$ this cap_p-quasi continuous representative of u, and let z be another cap_p-quasi continuous representative of u; thanks to Lemma 4.8 (and in particular using its consequence (4.3.9)), for any $\varepsilon > 0$, we have

$$\mathrm{cap}_p(\{|\tilde{u} - z| > \varepsilon\}) \leq \frac{C}{\varepsilon} \max\left(\|\tilde{u} - z\|_{W_2}^p, \|\tilde{u} - z\|_{W_2}^{p'} \right) = 0$$

since $\tilde{u} = z$ in W_2 and this concludes the proof. □

Now we want to prove that, if $v = u - g$ is a renormalized solution then, it is finite cap_p-quasi everywhere and it admits a cap_p-quasi continuous representative.

THEOREM 4.11. *Let $v = u - g$ a renormalized solution of problem (4.1.4). Then v admits a cap_p-quasi continuous representative finite cap_p-quasi everywhere.*

PROOF. Let us indicate with $\overline{H}_n(s)$ the primitive function of $H_n(s)$. Observe that from (4.2.5) we readily have a similar estimate for $\overline{H}_n(v)$, that is

$$\int_Q |\nabla \overline{H}_n(v)|^p \, dxdt \leq C(n+1) \tag{4.3.11}$$

and so, choosing $\overline{H}_n(v)$ and φ in the renormalized formulation of v (4.2.1) we have

$$\begin{aligned} &- \int_Q \varphi_t \, \overline{H}_n(v) \, dxdt \\ &\quad + \int_Q H_n(v) a(t,x,\nabla u) \cdot \nabla \varphi \, dxdt \\ &= \int_Q H_n(v) \varphi \, d\hat{\mu}_0 \\ &\quad + \frac{1}{n} \int_{\{n < v \leq 2n\}} a(t,x,\nabla u) \cdot \nabla v \, \varphi \, dxdt \\ &- \frac{1}{n} \int_{\{-2n < v \leq -n\}} a(t,x,\nabla u) \cdot \nabla v \, \varphi \, dxdt, \end{aligned} \tag{4.3.12}$$

and so we deduce that, in the sense of distribution

$$\frac{d}{dt}\left(\overline{H}_n(v)\right) \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q);$$

therefore, since $\overline{H}_n(v) \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, thanks to Theorem 4.7, $\overline{H}_n(v)$ turns out to admit a cap_p-quasi continuous representative finite cap_p-quasi everywhere and so to conclude the prove is enough to prove that v is finite cap_p-quasi everywhere.

Actually, from (4.3.12) we have

$$\frac{d}{dt}\left(\overline{H}_n(v)\right) = \operatorname{div}(H_n(v)a(t,x,\nabla u)) + \frac{1}{n}a(t,x,\nabla u)\cdot\nabla v\chi_{\{n<v\leq 2n\}}$$

$$-\frac{1}{n}a(t,x,\nabla u)\cdot\nabla v\chi_{\{-2n<v\leq -n\}} \ + H_n(v)\hat{\mu}_0,$$

and so, from the estimate (4.3.11), we easily deduce that there exists a decomposition of $(\overline{H}_n(v))_t$ such that

$$\|(\overline{H}_n(v))^1_t\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq Cn,$$

and

$$\|(\overline{H}_n(v))^2_t\|_{L^1(Q)} \leq C.$$

Now, thanks to Theorem 4.7, we have that $\overline{H}_n(v)$ admits a cap_p-quasi continuous representative and moreover $\{|v|>n\}\equiv\{|\overline{H}_n(v)|>n\}$. We can apply Lemma 4.8 to obtain

$$\mathrm{cap}_p(\{|v|>n\}) = \mathrm{cap}_p(\{|\overline{H}_n(v)|>n\}) \leq \frac{C}{n}\max\left([\overline{H}_n(v)]_*^{\frac{1}{p}}, [\overline{H}_n(v)]_*^{\frac{1}{p'}}\right); \tag{4.3.13}$$

hence, using the obvious estimate

$$\|\overline{H}_n(v)\|^2_{L^\infty(0,T;L^2(\Omega))} \leq \|\overline{H}_n(v)\|_{L^\infty(Q)}\|\overline{H}_n(v)\|_{L^\infty(0,T;L^1(\Omega))},$$

the fact that $v\in L^\infty(0,T;L^1(\Omega))$, and (4.3.11), we can conclude from (4.3.13) that

$$\mathrm{cap}_p(\{|v|>n\}) \leq \frac{C}{n}\max(n^{\frac{1}{p}}, n^{\frac{1}{p'}}),$$

that allow us to conclude that v is finite cap_p-quasi everywhere and so the proof of Theorem 4.11. □

4.4. Approximating measures; basic estimates and compactness

First of all, we want to discuss how the renormalized solution does not depend on the decomposition of the regular part of the measure μ_0; to to that we made use of the following result proved in [**DPP**]:

LEMMA 4.12. *Let $\mu_0\in M_0(Q)$, and let $(f,-\operatorname{div}(G_1),g_2)$ and $(\tilde{f},-\operatorname{div}(\tilde{G}_1),\tilde{g}_2)$ be two different decompositions of μ according to Theorem 1.39. Then we have $(g_2-\tilde{g}_2)_t = \tilde{f}-f-\operatorname{div}(\tilde{G}_1)+\operatorname{div}(G_1)$ in distributional sense, $g_2-\tilde{g}_2\in C([0,T];L^1(\Omega))$ and $(g_2-\tilde{g}_2)(0)=0$.*

PROOF. Lemma 2.29 in [**DPP**], pag. 22. □

If $\mu \in M_0(Q)$, the definition of renormalized solution does not depend on the decomposition of the absolutely continuous part of μ as shown in Proposition 3.10 in [**DPP**]. Next result try to stress the fact that even for general measure data this fact should be true; actually for technical reason we prove that the the definition of renormalized solution is stable under *bounded perturbations* of the decomposition of μ_0 (see also Remark 4.14).

PROPOSITION 4.13. *Let u be a renormalized solution of (4.1.4). Then u satisfies Definition 4.1 for every decomposition $(\tilde{f}, -\mathrm{div}(\tilde{G}), \tilde{g})$ such that $g-\tilde{g} \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$.*

SKETCH OF THE PROOF. Assume that u satisfies Definition 4.1 for $(f, -\mathrm{div}(G), g)$ and let $(\tilde{f}, -\mathrm{div}(\tilde{G}), \tilde{g})$ be a different decomposition of μ_0 such that $g - \tilde{g}$ is bounded. Thanks to Lemma 4.12 we readily have that $\tilde{v} = u - \tilde{g} \in L^\infty(0,T;L^1(\Omega))$; to prove that $T_k(u-\tilde{g}) \in L^p(0,T;W_0^{1,p}(\Omega))$ for every $k > 0$ we can reason as in the proof of Proposition 3.10 in [**DPP**] with $S(v) = \overline{H}_n(v)$ and using the fact that thanks to (4.2.2) and (4.2.3) we have

$$\lim_{n\to\infty} \frac{1}{n} \int_{\{n \le |u-g| < 2n\}} |\nabla u|^p \, dx \le C.$$

To prove that the reconstruction properties (4.2.2) and (4.2.3) are satisfied for $\tilde{v}$ we have to be more careful.

To prove (4.2.2) we choose $\beta_h(\overline{h}_n(v)+g-\tilde{g})\psi$ (where $\beta_h(s) = B_h(s^+)$ and $\psi \in C^1(\overline{Q})$) and $S'(s) = h_n(s) = H_n(s^+)$ in (4.2.4), calling $\gamma_n = \overline{h}_n(v) + g - \tilde{g}$ and using Lemma 4.12, to obtain

$$\int_0^T \langle (\gamma_n)_t, \beta_h(\gamma_n)\psi \rangle \, dt \tag{A}$$

$$+ \int_Q h_n(v) a(t,x,\nabla u) \cdot \nabla \beta_h(\gamma_n) \; \psi \, dxdt \tag{B}$$

$$- \frac{1}{n} \int_{\{n \le v < 2n\}} a(t,x,\nabla u) \cdot \nabla v \quad \beta_h(\gamma_n) \; \psi \, dxdt \tag{C}$$

$$+ \int_Q h_n(v) a(t,x,\nabla u) \cdot \nabla \psi \; \beta_h(\gamma_n) \, dxdt \tag{D}$$

$$= \int_Q [(h_n(v) - 1)f + \tilde{f}] \beta_h(\gamma_n) \psi \, dxdt \tag{E}$$

$$+ \int_Q [(h_n(v) - 1)G + \tilde{G}] \cdot \nabla(\beta_h(\gamma_n)\psi) \, dxdt \tag{F}$$

$$+ \int_Q G \cdot \nabla h_n(v) \quad \beta_h(\gamma_n) \; \psi \, dxdt \,. \tag{G}$$

Now, integrating by parts in (A) we have

$$(\mathrm{A}) = -\int_Q \beta_h(\gamma_n)\psi_t \, dxdt + \int_\Omega \beta_h(\gamma_n)(T)\psi(T) \, dx - \int_\Omega \beta_h(\overline{h}_n(u_0))\psi(0) \, dx = \omega(n,h)\,;$$

while, thanks to the properties of β_h and to the fact $h_n(v)$ strongly converges to 1 in $L^p((0,T);W^{1,p}(\Omega))$ (this fact essentially relies on the estimate (4.2.5)) we have that

$$(\mathrm{E}), (\mathrm{F}) = \omega(n,h)\,.$$

On the other hand, using the Hölder inequality and the fact that $0 \leq \beta_h(\gamma_n) \leq 1$, we have

$$|(\mathrm{G})| \leq C\left(\int_Q |G|^{p'} \, dxdt\right)^{\frac{1}{p'}} \left(\int_Q |\nabla h_n(v)|^p \, dxdt\right)^{\frac{1}{p}} = \omega(n)\,,$$

again using the fact that $h_n(v)$ strongly converges to 1 in $L^p((0,T);W^{1,p}(\Omega))$.

Moreover, since $g - \tilde{g}$ is bounded, we can truncate v on the set $\{|\gamma_n| \leq 2h\}$, that is $u = T_M(v) + g$ on $\{|v| \leq M\} \subseteq \{|\gamma_n| \leq 2h\}$, for suitable $M > 0$ not dependent on n. Hence, since $T_{2h}(\gamma_n)$ is weakly compact in $L^p(0,T;W_0^{1,p}(\Omega))$ (actually arguing as in [**DPP**], that is taking $T_{2h}(\gamma_n)$ as test function in (4.2.4), one can show that it is bounded, uniformly with respect to n, in $L^p(0,T;W_0^{1,p}(\Omega))$), we have

$$(\mathrm{B}) = \frac{1}{h}\int_{\{h \leq \gamma_n < 2h\}} h_n(v) a(t,x,\nabla u)\nabla\gamma_n \ \psi \, dxdt$$

$$= \frac{1}{h}\int_{\{h \leq \tilde{v} < 2h\}} a(t,x,\nabla u)\nabla\tilde{v} \ \psi \, dxdt + \omega(n),$$

and, on the other hand, since $h_n(v)$ strongly converges to 1 in $L^p((0,T);W^{1,p}(\Omega))$, we have

$$-(\mathrm{C}) = \frac{1}{n}\int_{\{n\le v<2n\}\cap\{|\gamma_n|\ge 2h\}} a(t,x,\nabla u)\cdot\nabla v\ \psi\ dxdt$$

$$+\frac{1}{n}\int_{\{n\le v<2n\}\cap\{|\gamma_n|<2h\}} a(t,x,\nabla u)\cdot\nabla v\quad \beta_h(\gamma_n)\ \psi\ dxdt$$

$$= \frac{1}{n}\int_{\{n\le v<2n\}\cap\{|\gamma_n|\ge 2h\}} a(t,x,\nabla u)\cdot\nabla v\ \psi\ dxdt + \omega(n)$$

$$= \frac{1}{n}\int_{\{n\le v<2n\}} a(t,x,\nabla u)\cdot\nabla v\ \psi\ dxdt$$

$$-\frac{1}{n}\int_{\{n\le v<2n\}\cap\{|\gamma_n|<2h\}} a(t,x,\nabla u)\cdot\nabla v\ \psi\ dxdt + \omega(n)$$

$$= \int_Q \psi\ d\mu_s^+ + \omega(n);$$

Collecting together all these facts we derive that

$$\lim_{h\to\infty}\frac{1}{h}\int_{\{h\le\tilde v<2h\}} a(t,x,\nabla u)\nabla\tilde v\ \psi\ dxdt = \int_Q \psi\ d\mu_s^+.$$

Then we conclude by density for every $\psi\in C(\overline{Q})$; the proof of (4.2.3) can be treated analogously.

Finally the fact that $\tilde v$ satisfies equation (4.2.4) can be proved as in the proof of Proposition 3.10 in [**DPP**]. □

REMARK 4.14. Let us stress the fact that in Proposition 4.13 we deal with small perturbations of the *time derivative* part of μ_0 because of technical reasons; actually the proof of Proposition 3.10 of [**DPP**] is given by suitable estimates and with the use of Fatou's lemma. Unfortunately, as far as the reconstruction properties (4.2.2) and (4.2.3) are concerned, we have to make use of a more subtle analysis on each term. The requirement on $g-\tilde g$ arise from this fact.

However, in the linear case, we can drop this stronger assumption proving the result in its general form. Indeed as we will see later (see the proof of Theorem 4.24), in this case a renormalized solution turns out to be a duality solution and so, using the duality formulation for u and Lemma 4.12, we can easily conclude.

Now, let us come back to the existence of a renormalized solution for problem (4.1); as we said before, if $\mu \in M(Q)$ we can split it this way:

$$\mu = f - \operatorname{div}(G) + g_t + \mu_s, \tag{4.4.1}$$

for some $f \in L^1(Q)$, $G \in (L^{p'}(Q))^N$, $g \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, and $\mu_s \perp p$-capacity, that is, μ_s is concentrated on a set $E \subset Q$ with $\text{cap}_p(E) = 0$. There are many ways to approximate this measure looking for existence of solution of problem (4.1.4); we make the following choice, let

$$\mu^\varepsilon = f^\varepsilon - \operatorname{div}(G^\varepsilon) + g^\varepsilon_t + \lambda^\varepsilon_\oplus - \lambda^\varepsilon_\ominus, \tag{4.4.2}$$

where $f^\varepsilon \in C^\infty_0(Q)$ is a sequence of functions that converges to f in $L^1(Q)$, $G^\varepsilon \in C^\infty_0(Q)$ is a sequence of functions that converges to G in $(L^{p'}(Q))^N$, $g^\varepsilon \in C^\infty_0(Q)$ is a sequence of functions that converges to g in $L^p(0,T;W^{1,p}_0(\Omega))$, and $\lambda^\varepsilon_\oplus \in C^\infty_0(Q)$ (respectively $\lambda^\varepsilon_\ominus$) is a sequence of nonnegative functions that converges to μ^+_s (respectively μ^-_s) in the narrow topology of measures. Moreover let $u^\varepsilon_0 \in C^\infty_0(\Omega)$ that approaches u_0 in $L^1(\Omega)$. Notice that this approximation can be easily obtained via a standard convolution argument, and we can also assume

$$\|\mu^\varepsilon\|_{L^1(Q)} \leq C|\mu|, \qquad \|u^\varepsilon_0\|_{L^1(\Omega)} \leq C\|u_0\|_{L^1(\Omega)} .$$

Let us call u^ε the solution of problem

$$\begin{cases} u^\varepsilon_t + A(u^\varepsilon) = \mu^\varepsilon & \text{in } (0,T)\times\Omega \\ u^\varepsilon(0,x) = u^\varepsilon_0 & \text{in } \Omega, \\ u^\varepsilon(0,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{4.4.3}$$

that exists and is unique, and let $v^\varepsilon = u^\varepsilon - g^\varepsilon$. Approximation (4.4.2) yields standard compactness results (see [**BDGO**],[**DO**], and [**DPP**]) that we collect in the following

PROPOSITION 4.15. *Let u^ε and v^ε as defined before. Then*

$$\|u^\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \tag{4.4.4}$$

$$\int_Q |\nabla T_k(u^\varepsilon)|^p \, dxdt \leq Ck, \tag{4.4.5}$$

$$\|v^\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \tag{4.4.6}$$

$$\int_Q |\nabla T_k(v^\varepsilon)|^p \, dxdt \leq C(k+1). \tag{4.4.7}$$

Moreover, there exists a measurable function u such that $T_k(u)$ and $T_k(v)$ belong to $L^p(0,T;W^{1,p}_0(\Omega))$, u and v belong to $L^\infty(0,T;L^1(\Omega))$, and, up to a subsequence, for any

$k > 0$, and for every $q < p - \frac{N}{N+1}$, we have

$$u^\varepsilon \longrightarrow u \quad \textit{a.e. on} \quad Q \quad \textit{weakly in } L^q(0,T;W_0^{1,q}(\Omega)) \quad \textit{and strongly in } L^1(Q),$$

$$v^\varepsilon \longrightarrow v \quad \textit{a.e. on} \quad Q \quad \textit{weakly in } L^q(0,T;W_0^{1,q}(\Omega)) \quad \textit{and strongly in } L^1(Q),$$

$$T_k(u^\varepsilon) \rightharpoonup T_k(u) \quad \textit{weakly in} \quad L^p(0,T;W_0^{1,p}(\Omega)) \quad \textit{and a.e. on } Q,$$

$$T_k(u^\varepsilon) \rightharpoonup T_k(u) \quad \textit{weakly in} \quad L^p(0,T;W_0^{1,p}(\Omega)) \quad \textit{and a.e. on } Q,$$

$$\nabla u^\varepsilon \longrightarrow \nabla u \quad \textit{a.e. on} \quad Q,$$

$$\nabla v^\varepsilon \longrightarrow \nabla v \quad \textit{a.e. on} \quad Q.$$

Sketch of the proof. Here we give just an idea on how (4.4.4)–(4.4.7) can be obtained following the outlines of [**DPP**]. First of all, we choose $T_k(u^\varepsilon)$ as test function in (4.4.3) and we integrate in $]0,t[$ to get:

$$\int_\Omega \Theta_k(u^\varepsilon)(t)\,dx + \int_0^t\int_\Omega a(s,x,\nabla u^\varepsilon)\cdot\nabla T_k(u^\varepsilon)\,dxds = \int_0^t\int_\Omega \mu^\varepsilon\,T_k(u^\varepsilon)\,dxds + \int_\Omega \Theta_k(u_0^\varepsilon)\,dx\,,$$

which yields, from (4.1.1) and the fact that $\|u_0^\varepsilon\|_{L^1(\Omega)}$ and $\|\mu^\varepsilon\|_{L^1(Q)}$ are bounded:

$$\int_\Omega \Theta_k(u^\varepsilon)(t)\,dx + \int_0^t\int_\Omega |\nabla T_k(u^\varepsilon)|^p\,dxds \le Ck\,.$$

Since $\Theta_k(s) \ge 0$ and $|\Theta_1(s)| \ge |s| - 1$, we get

$$\int_\Omega |u^\varepsilon(t)|\,dx + \int_0^t\int_\Omega |\nabla(T_k(u^\varepsilon))|^p\,dxdt \le C(k+1) \qquad \forall k > 0\,,\ \forall t \in [0,T]\,. \tag{4.4.8}$$

Taking the supremum on $(0,T)$ we obtain the estimate of u^ε in $L^\infty(0,T;L^1(\Omega))$. Similarly we can get the estimates on $v^\varepsilon = u^\varepsilon - g^\varepsilon$: let us choose $T_k(v^\varepsilon)$ as test function in (4.4.3). Integrating by parts (recall that g^ε has compact support in Q, so that

$v^\varepsilon(0) = u^\varepsilon(0) = u_0^\varepsilon$) and using (4.1.1) this gives:

$$\int_\Omega \Theta_k(v^\varepsilon)(t)\,dx + \alpha \int_0^t\!\!\int_\Omega |\nabla u^\varepsilon|^p\,\chi_{\{|v^\varepsilon|\le k\}} dxds$$

$$\le \int_\Omega \Theta_k(u_0^\varepsilon)\,dx + \int_Q f^\varepsilon T_k(v^\varepsilon)\,dxdt + \int_0^t\!\!\int_\Omega G^\varepsilon\cdot\nabla u^\varepsilon\,\chi_{\{|v^\varepsilon|\le k\}} dxds$$

$$- \int_0^t\!\!\int_\Omega G^\varepsilon\cdot\nabla g^\varepsilon\,\chi_{\{|v^\varepsilon|\le k\}} dxds + \int_0^t\!\!\int_\Omega a(s,x,\nabla u^\varepsilon)\cdot\nabla g^\varepsilon\,\chi_{\{|v^\varepsilon|\le k\}} dxds$$

$$+ \int_Q T_k(v^\varepsilon)\,d\lambda_\oplus^\varepsilon - \int_Q T_k(v^\varepsilon)\,d\lambda_\ominus^\varepsilon.$$

Using assumption (4.1.2) and by means of Young's inequality we obtain:

$$\int_\Omega \Theta_k(v^\varepsilon)(t)\,dx + \frac{\alpha}{2}\int_0^t\!\!\int_\Omega |\nabla u^\varepsilon|^p\,\chi_{\{|v^\varepsilon|\le k\}} dxdt \le k\int_Q |f^\varepsilon|\,dxdt$$

$$+C\int_Q |G^\varepsilon|^{p'}\,dxdt + C\int_Q |\nabla g^\varepsilon|^p\,dxdt + C\int_Q |b(t,x)|^{p'} dxdt + k\int_\Omega |u_0^\varepsilon|\,dx\,.$$

$$+k\int_Q d\lambda_\oplus^\varepsilon + k\int_Q d\lambda_\ominus^\varepsilon\,.$$

Since G^ε is bounded in $L^{p'}(Q)$, g^ε is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, f^ε, $\lambda_\oplus^\varepsilon$ and $\lambda_\ominus^\varepsilon$ are bounded in $L^1(Q)$ and u_0^ε is bounded in $L^1(\Omega)$, we obtain

$$\int_\Omega \Theta_1(v^\varepsilon)(t)\,dx \le C \qquad \forall t\in(0,T)\,,$$

which implies the estimate of v^ε in $L^\infty(0,T;L^1(\Omega))$, and also

$$\int_Q |\nabla u^\varepsilon|^p\,\chi_{\{|v^\varepsilon|\le k\}} dxdt \le C\,(k+1)\,,$$

which yields that $T_k(v^\varepsilon)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ for any $k>0$ (recall that g^ε itself is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$). □

Remark 4.16. Let us observe that from Proposition 4.15, thanks to assumption (4.1.2) on a and Vitali's theorem, we easily deduce that $a(t,x,\nabla u^\varepsilon)$ is strongly compact in $L^1(Q)$.

4.5. Strong convergence of truncates

In this section we shall prove the strong convergence of truncates of renormalized solutions of problem (4.1.4); to do that we will crossover the approach used in [**DMOP**] for the elliptic case with the one in [**BP**].

With the symbol $T_k(v)_\nu$ we indicate the Landes time-regularization of the truncate function $T_k(v)$; this notion, introduced in [**La**], was fruitfully used in several papers afterwards (see in particular [**DO**], [**BDGO**], and [**BP**]). Let z_ν be a sequence of functions such that

$$z_\nu \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\,, \qquad \|z_\nu\|_{L^\infty(\Omega)} \le k\,,$$

$$z_\nu \longrightarrow T_k(u_0) \quad \text{a.e. in } \ \Omega \ \text{ as } \nu \text{ tends to infinity,}$$

$$\frac{1}{\nu}\|z_\nu\|^p_{W_0^{1,p}(\Omega)} \longrightarrow 0 \ \text{ as } \nu \text{ tends to infinity.}$$

Then, for fixed $k > 0$, and $\nu > 0$, we denote by $T_k(v)_\nu$ the unique solution of the problem

$$\begin{cases} \dfrac{d\,T_k(v)_\nu}{dt} = \nu(T_k(v) - T_k(v)_\nu) & \text{in the sense of distributions,} \\[2ex] T_k(v)_\nu(0) = z_\nu & \text{in } \Omega. \end{cases}$$

Therefore, $T_k(v)_\nu \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ and $\dfrac{dT_k(v)}{dt} \in L^p(0,T;W_0^{1,p}(\Omega))$, and it can be proved (see also [**La**]) that, up to subsequences, as ν diverges

$$T_k(v)_\nu \longrightarrow T_k(v) \quad \text{strongly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and a.e. in } Q,$$

$$\|T_k(v)_\nu\|_{L^\infty(Q)} \le k \quad \forall \nu > 0.$$

First of all, let us state a preliminary result about the capacity of compact sets, and then our basic result about approximate capacitary potential.

Lemma 4.17. *Let K be a compact subset of $Q = (0,T)\times\Omega$ such that $\mathrm{cap}_p(K,Q) = 0$, then for every open set U such that $K \subseteq U \subseteq Q$, we have*

$$\mathrm{cap}_p(K,U) = 0.$$

Proof. Let $\psi_\delta \in C_0^\infty(Q)$ be a sequence approximating the capacity of K in Q (as in Definition 1.37), and φ a cut-off function for K in U, that is a function in $C_0^\infty(U)$ such that $\varphi \equiv 1$ on K and extended to zero on $Q\backslash U$; therefore we have

$$\mathrm{CAP}(K,U) \le \|\psi_\delta\varphi\|_W = \|\psi_\delta\varphi\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|(\psi_\delta\varphi)_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))};$$

easily we have that

$$\|\psi_\delta\varphi\|_{L^p(0,T;W_0^{1,p}(\Omega))} \leq C\|\psi_\delta\|_W.$$

On the other hand, if $(\psi_\delta)_t = -\mathrm{div}(F_\delta)$ in the sense of $L^{p'}(0,T;W^{-1,p'}(\Omega))$, we have, reasoning by a density argument, that, for every $v \in L^p(0,T;W_0^{1,p}(\Omega))$

$$\langle(\psi_\delta\varphi)_t, v\rangle_{L^{p'}(0,T;W^{-1,p'}(\Omega)),L^p(0,T;W_0^{1,p}(\Omega))}$$

$$= \int_Q F_\delta \cdot \nabla(v\varphi)\ dxdt + \int_Q \psi_\delta\varphi_t v\ dxdt,$$

and so, using Theorem 1.3 (recall that $p > \frac{2N+1}{N+1}$), Hölder's inequality and Sobolev embeddings, one can check that

$$\|(\psi_\delta\varphi)_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq C\|\psi_\delta\|_W,$$

that implies the result thanks to the choice of ψ_δ and Proposition 1.38. □

LEMMA 4.18. *Let $\mu_s = \mu_s^+ - \mu_s^- \in M(Q)$ where μ_s^+ and μ_s^- are concentrated, respectively, on two disjoint sets E^+ and E^- of zero p-capacity. Then, for every $\delta > 0$, there exist two compact sets $K_\delta^+ \subseteq E^+$ and $K_\delta^- \subseteq E^-$ such that*

$$\mu_s^+(E^+ \backslash K_\delta^+) \leq \delta, \qquad \mu_s^-(E^- \backslash K_\delta^-) \leq \delta, \tag{4.5.1}$$

and there exist $\psi_\delta^+,\ \psi_\delta^- \in C_0^1(Q)$, such that

$$\psi_\delta^+,\ \psi_\delta^- \equiv 1 \text{ respectively on } K_\delta^+,\ K_\delta^-, \tag{4.5.2}$$

$$0 \leq \psi_\delta^+,\ \psi_\delta^- \leq 1, \tag{4.5.3}$$

$$\mathrm{supp}(\psi_\delta^+) \cap \mathrm{supp}(\psi_\delta^-) \equiv \emptyset. \tag{4.5.4}$$

Moreover

$$\|\psi_\delta^+\|_S \leq \delta, \quad \|\psi_\delta^-\|_S \leq \delta, \tag{4.5.5}$$

and, in particular, there exists a decomposition of $(\psi_\delta^+)_t$ and a decomposition of $(\psi_\delta^-)_t$ such that

$$\|(\psi_\delta^+)_t^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq \frac{\delta}{3}, \quad \|(\psi_\delta^+)_t^2\|_{L^1(Q)} \leq \frac{\delta}{3}, \tag{4.5.6}$$

$$\|(\psi_\delta^-)_t^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq \frac{\delta}{3}, \quad \|(\psi_\delta^-)_t^2\|_{L^1(Q)} \leq \frac{\delta}{3}, \tag{4.5.7}$$

and both ψ_δ^+ and ψ_δ^- converge to zero $*$-weakly in $L^\infty(Q)$, in $L^1(Q)$, and, up to subsequences, almost everywhere as δ vanishes.

Moreover, if $\lambda_\oplus^\varepsilon$ and $\lambda_\ominus^\varepsilon$ are as in (4.4.2) we have

$$\int_Q \psi_\delta^- \, d\lambda_\oplus^\varepsilon = \omega(\varepsilon,\delta), \qquad \int_Q \psi_\delta^- \, d\mu_s^+ \leq \delta, \tag{4.5.8}$$

$$\int_Q \psi_\delta^+ \, d\lambda_\ominus^\varepsilon = \omega(\varepsilon,\delta), \qquad \int_Q \psi_\delta^+ \, d\mu_s^- \leq \delta, \tag{4.5.9}$$

$$\int_Q (1-\psi_\delta^+\psi_\eta^+) \, d\lambda_\oplus^\varepsilon = \omega(\varepsilon,\delta,\eta), \qquad \int_Q (1-\psi_\delta^+\psi_\eta^+) \, d\mu_s^+ \leq \delta+\eta, \tag{4.5.10}$$

$$\int_Q (1-\psi_\delta^-\psi_\eta^-) \, d\lambda_\ominus^\varepsilon = \omega(\varepsilon,\delta,\eta), \qquad \int_Q (1-\psi_\delta^-\psi_\eta^-) \, d\mu_s^- \leq \delta+\eta. \tag{4.5.11}$$

Proof. Let us fix $\delta > 0$, so thanks to the regularity of the measure μ_s, there exist two disjoint compact sets $K_\delta^+ \subseteq E^+$ and $K_\delta^- \subseteq E^-$, such that (4.5.1) are satisfied and there exist two open sets U_δ^+ and U_δ^-, disjoint, containing respectively K_δ^+ and K_δ^-. Now, thanks to Lemma 4.17, since $\mathrm{cap}_p(K_\delta^+, Q) = 0$ (resp. $\mathrm{cap}_p(K_\delta^-, Q) = 0$), we have that $\mathrm{cap}_p(K_\delta^+, U_\delta^+) = 0$ (resp. $\mathrm{cap}_p(K_\delta^-, U_\delta^-) = 0$). Hence, by definition of parabolic p-capacity there exists two functions $\varphi_\delta^+ \in C_0^\infty(U_\delta^+)$ (resp. $\varphi_\delta^- \in C_0^\infty(U_\delta^-)$) such that, for any $\delta' > 0$, we have

$$\|\varphi_\delta^+\|_W \leq \delta', \qquad (\text{resp. } \|\varphi_\delta^-\|_W \leq \delta') \tag{4.5.12}$$

and

$$\varphi_\delta^+ \geq \chi_{K_\delta^+} \qquad (\text{resp. } \varphi_\delta^- \geq \chi_{K_\delta^-}),$$

where we have extended these functions to zero, respectively, in $Q\backslash U_\delta^+$ and $Q\backslash U_\delta^-$; we will choose the value of δ' in a suitable way later.

Now, let us define

$$\psi_\delta^+ = \overline{H}(\varphi_\delta^+), \qquad \psi_\delta^- = \overline{H}(\varphi_\delta^-), \tag{4.5.13}$$

where $\overline{H}(s)$ is the primitive of the continuous function

$$H(s) = \begin{cases} \dfrac{4}{3} & \text{if } |s| \leq \frac{1}{2}, \\ \text{affine} & \text{if } \frac{1}{2} < |s| \leq 1, \\ 0 & \text{if } |s| > 1. \end{cases} \tag{4.5.14}$$

Readily we can observe that (4.5.2), (4.5.3) and (4.5.4) are satisfied. Moreover we have

$$\|\psi_\delta^+\|_S = \|\psi_\delta^+\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|(\psi_\delta^+)\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}$$

$$\leq \|\psi_\delta^+\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|(\psi_\delta^+)^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} + \|(\psi_\delta^+)^2\|_{L^1(Q)},$$

for every decomposition of $(\psi_\delta^+)_t$. From now on we deal only with ψ_δ^+ since the same argument holds for ψ_δ^-. Let us observe that, in the sense of distribution, we have $(\psi_\delta^+)_t = H(\varphi_\delta^+)(\varphi_\delta^+)_t$, and so, if $(\varphi_\delta^+)_t = -\mathrm{div}(F_\delta^+)$ in $L^{p'}(0,T;W^{-1,p'}(\Omega))$, we have that, for any $v \in L^p(0,T;W_0^{1,p}(\Omega))$

$$\langle(\psi_\delta^+)_t, v\rangle_{L^{p'}(0,T;W^{-1,p'}(\Omega)),L^p(0,T;W_0^{1,p}(\Omega))}$$

$$= \int_Q H(\varphi_\delta^+)\ F_\delta^+ \cdot \nabla v\ dxdt - \frac{8}{3}\int_{\{\frac{1}{2}\leq\varphi_\delta^+\leq 1\}} F_\delta^+ \cdot \nabla\varphi_\delta^+ v\ dxdt,$$

Therefore from (4.5.12) we have

$$\|\psi_\delta^+\|_{L^p(0,T;W_0^{1,p}(\Omega))} \leq C\delta',$$

$$\|(\psi_\delta^+)_t^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq C\delta'$$

and, using the Young's inequality,

$$\|(\psi_\delta^+)_t^2\|_{L^1(Q)} \leq C(\delta'^p + \delta'^{p'}).$$

So, we can actually choose δ' small enough such that (4.5.5), (4.5.6) and (4.5.7) are satisfied; moreover, thanks to Theorem 1.5 we also have that these functions tends to zero in $L^1(Q)$ as δ goes to zero and so, up to subsequences, almost everywhere; due to this fact the convergence to zero $*$-weakly in $L^\infty(Q)$ as δ vanishes is obvious.

Now, if $\lambda_\oplus^\varepsilon$ is as in the statement we have, for every $\delta > 0$,

$$0 \leq \int_Q \psi_\delta^-\ d\lambda_\oplus^\varepsilon = \int_Q \psi_\delta^-\ d\mu_s^+ + \omega(\varepsilon),$$

while recalling (4.5.1) we have

$$0 \leq \int_Q \psi_\delta^-\ d\mu_s^+ = \int_{U_\delta^-} \psi_\delta^-\ d\mu_s^+ \leq \mu_s^+(U_\delta^-) \leq \mu_s^+(Q\backslash U_\delta^+)$$

$$\leq \mu_s^+(Q\backslash K_\delta^+) = \mu_s^+(E^+\backslash K_\delta^+) \leq \delta.$$

Therefore (4.5.8) is proved, and (4.5.9) can be obtained analogously. Now, let δ and η two nonnegative fixed values: we have

$$0 \leq \int_Q (1 - \psi_\delta^+\psi_\eta^+)\ d\lambda_\oplus^\varepsilon = \int_Q (1 - \psi_\delta^+\psi_\eta^+)\ d\mu_s^+ + \omega(\varepsilon);$$

on the other hand, since $1-\psi_\delta^+\psi_\eta^+$ is in $C(\overline{Q})$, and is identically zero on $K_\delta^+\cap K_\eta^+$, using again (4.5.1) we can obtain

$$0\le\int_Q(1-\psi_\delta^+\psi_\eta^+)\,d\mu_s^+=\int_{Q\setminus(K_\delta^+\cap K_\eta^+)}(1-\psi_\delta^+\psi_\eta^+)\,d\mu_s^+$$

$$\le\mu_s^+(Q\setminus(K_\delta^+\cap K_\eta^+))\le\mu_s^+(Q\setminus K_\delta^+)+\mu_s^+(Q\setminus K_\eta^+)\le\delta+\eta.$$

This proves (4.5.10) while the proof of (4.5.11) is analogous. □

Remark 4.19. This result is actually enough to our aim; however, one would like to have a stronger and reasonable result that, up to now, is still an open problem. Can one choose ψ_δ^+ and ψ_δ^- as *uniformly bounded* functions in $C_0^\infty(Q)$ vanishing in the W_1 norm? This fact, for instance, should allow us to prove the reverse implication of the decomposition Theorem 1.39; that is, if $\mu\in M(Q)$ admits a decomposition as in (1.3.2), then $\mu\in M_0(Q)$.

In what follows we will ever refer to subsequences of ψ_δ^+ and ψ_δ^- that satisfy all the convergence results stated in Lemma 4.18.

The essential key in the proof of Theorem 4.4 is the following

Theorem 4.20. *Let v^ε and v as before. Then, for every $k>0$*

$$T_k(v^\varepsilon)\longrightarrow T_k(v)\quad\textit{strongly in}\ \ L^p(0,T;W_0^{1,p}(\Omega)).$$

Proof. Our aim is to prove the following asymptotic estimate:

$$\limsup_{\varepsilon\to0}\int_Q a(t,x,\nabla u^\varepsilon)\cdot\nabla T_k(v^\varepsilon)\,dxdt\le\int_Q a(t,x,\nabla u)\cdot\nabla T_k(v)\,dxdt. \tag{4.5.15}$$

The result will readily follow from (4.5.15) by a quite standard argument. We shall prove it in several steps.

Step 0. Near E and far from E.

For every $\delta,\eta>0$, let ψ_δ^+, ψ_η^+, ψ_δ^-, and ψ_η^- as in Lemma 4.18 and let E^+ and E^- be the sets where, respectively, the positive variation and the negative variation of the singular part of μ are concentrated; setting $\Phi_{\delta,\eta}=\psi_\delta^+\psi_\eta^++\psi_\delta^-\psi_\eta^-$, we can write

$$\begin{aligned}&\int_Q a(t,x,\nabla u^\varepsilon)\cdot\nabla(T_k(v^\varepsilon)-T_k(v)_\nu)H_n(v^\varepsilon)\,dxdt\\ &=\int_Q a(t,x,\nabla u^\varepsilon)\cdot\nabla(T_k(v^\varepsilon)-T_k(v)_\nu)H_n(v^\varepsilon)\Phi_{\delta,\eta}\,dxdt\\ &+\int_Q a(t,x,\nabla u^\varepsilon)\cdot\nabla(T_k(v^\varepsilon)-T_k(v)_\nu)H_n(v^\varepsilon)(1-\Phi_{\delta,\eta})\,dxdt\,.\end{aligned}\tag{4.5.16}$$

Now, if $n > k$, since $a(t,x,\nabla T_{2n}(u^\varepsilon)) \cdot \nabla T_k(v)_\nu$ is weakly compact in $L^1(Q)$ as ε goes to zero, $H_n(v^\varepsilon)$ converges to $H_n(v)$ $*$-weakly in $L^\infty(Q)$, and almost everywhere on Q, thanks to Egorov theorem, we have

$$\limsup_{\varepsilon\to 0} \int_Q a(t,x,\nabla u^\varepsilon) \cdot \nabla(T_k(v^\varepsilon) - T_k(v)_\nu) H_n(v^\varepsilon) \Phi_{\delta,\eta} \, dxdt$$

$$= \limsup_{\varepsilon\to 0} \left[\int_Q a(t,x,\nabla u^\varepsilon) \nabla T_k(v^\varepsilon) \Phi_{\delta,\eta} \, dxdt \right] - \int_Q a(t,x,\nabla u) \cdot \nabla T_k(v)_\nu H_n(v) \Phi_{\delta,\eta} \, dxdt.$$

So we have

$$\limsup_{\varepsilon,\nu} \int_Q a(t,x,\nabla u^\varepsilon) \cdot \nabla(T_k(v^\varepsilon) - T_k(v)_\nu) H_n(v^\varepsilon) \Phi_{\delta,\eta} \, dxdt$$

$$= \limsup_{\varepsilon\to 0} \left[\int_Q a(t,x,\nabla u^\varepsilon) \nabla T_k(v^\varepsilon) \Phi_{\delta,\eta} \, dxdt \right] - \int_Q a(t,x,\nabla u) \cdot \nabla T_k(v) H_n(v) \Phi_{\delta,\eta} \, dxdt.$$

Since $0 \le H_n(v) \le 1$ and $\Phi_{\delta,\eta}$ tends to zero $*$-weakly in $L^\infty(Q)$ as δ goes to zero,

$$\int_Q a(t,x,\nabla u) \cdot \nabla T_k(v) H_n(v) \Phi_{\delta,\eta} \, dxdt = \omega(\delta).$$

Therefore, if we prove that

$$\limsup_{\varepsilon,\delta,\eta} \int_Q a(t,x,\nabla u^\varepsilon) \nabla T_k(v^\varepsilon) \Phi_{\delta,\eta} \, dxdt \le 0, \tag{4.5.17}$$

then we can conclude

$$\limsup_{\varepsilon,\nu,\delta,n,\eta} \int_Q a(t,x,\nabla u^\varepsilon) \cdot \nabla(T_k(v^\varepsilon) - T_k(v)_\nu) H_n(v^\varepsilon) \Phi_{\delta,\eta} \, dxdt \le 0. \tag{4.5.18}$$

Step 1. Near to E.
Let us check (4.5.17). If $\mu^\varepsilon = \hat{\mu}_0^\varepsilon + \lambda_\oplus^\varepsilon - \lambda_\ominus^\varepsilon$, then, choosing $(k - T_k(v^\varepsilon)) H_n(v^\varepsilon) \psi_\delta^+ \psi_\eta^+$ as test function in the weak formulation of u^ε, defining $\Gamma_{n,k}(s) = \displaystyle\int_0^s (k - T_k(r)) H_n(r) \, dr$,

and integrating by parts, we obtain

$$\begin{aligned}
&-\int_Q \Gamma_{n,k}(v^\varepsilon)\ \frac{d}{dt}(\psi_\delta^+\psi_\eta^+)\ dxdt \\
&\qquad+\int_Q (k-T_k(v^\varepsilon))H_n(v^\varepsilon)a(t,x,\nabla u^\varepsilon)\cdot\nabla(\psi_\delta^+\psi_\eta^+)\ dxdt \\
&\qquad+\int_Q a(t,x,\nabla u^\varepsilon)\cdot\nabla H_n(v^\varepsilon)\ (k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\ dxdt \\
&-\int_Q a(t,x,\nabla u^\varepsilon)\cdot\nabla T_k(v^\varepsilon)\ H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+\ dxdt \\
&=\quad \int_Q (k-T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+\ d\hat\mu_0^\varepsilon \\
&\qquad+\int_Q (k-T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+\ d\lambda_\oplus^\varepsilon \\
&-\int_Q (k-T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+\ d\lambda_\ominus^\varepsilon;
\end{aligned}$$

so, for $n>k$, we have

$$\begin{aligned}
&\int_Q a(t,x,\nabla(T_k(v^\varepsilon)+g^\varepsilon))\cdot\nabla(T_k(v^\varepsilon)+g^\varepsilon)\psi_\delta^+\psi_\eta^+\ dxdt && \text{(A)}\\
&\qquad+\int_Q (k-T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+\ d\lambda_\oplus^\varepsilon && \text{(B)}\\
&=-\int_Q \Gamma_{n,k}(v^\varepsilon)\ \frac{d}{dt}(\psi_\delta^+\psi_\eta^+)\ dxdt && \text{(C)}\\
&\qquad+\frac{2k}{n}\int_{\{-2n<v^\varepsilon\le -n\}} a(t,x,\nabla u^\varepsilon)\cdot\nabla v^\varepsilon\ \psi_\delta^+\psi_\eta^+\ dxdt && \text{(D)}
\end{aligned}$$

$$+\int_Q (k-T_k(v^\varepsilon))H_n(v^\varepsilon)a(t,x,\nabla u^\varepsilon)\cdot\nabla(\psi_\delta^+\psi_\eta^+)\ dxdt \tag{E}$$

$$-\int_Q (k-T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+\ d\hat{\mu}_0^\varepsilon \tag{F}$$

$$+\int_Q (k-T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+\ d\lambda_\ominus^\varepsilon \tag{G}$$

$$+\int_Q a(t,x,\nabla(T_k(v^\varepsilon)+g^\varepsilon))\cdot\nabla g^\varepsilon\ \psi_\delta^+\psi_\eta^+\ dxdt; \tag{H}$$

here, we have used the fact that

$$\begin{aligned}
&\int_Q a(t,x,\nabla u^\varepsilon)\cdot\nabla T_k(v^\varepsilon)\ \psi_\delta^+\psi_\eta^+\ dxdt\\
&=\int_Q a(t,x,\nabla(T_k(v^\varepsilon)+g^\varepsilon))\cdot\nabla T_k(v^\varepsilon)\psi_\delta^+\psi_\eta^+\ dxdt\\
&=\int_Q a(t,x,\nabla(T_k(v^\varepsilon)+g^\varepsilon))\cdot\nabla(T_k(v^\varepsilon)+g^\varepsilon)\psi_\delta^+\psi_\eta^+\ dxdt\\
&\quad-\int_Q a(t,x,\nabla(T_k(v^\varepsilon)+g^\varepsilon))\cdot\nabla g^\varepsilon\ \psi_\delta^+\psi_\eta^+\ dxdt.
\end{aligned}\tag{4.5.19}$$

Let us analyze term by term using in particular Proposition 4.15 and Lemma 4.18; due to the fact that $\Gamma_{n,k}(v^\varepsilon)$ converges to $\Gamma_{n,k}(v)$ weakly in $L^p(0,T;W_0^{1,p}(\Omega))$, we obtain, observing that $\Gamma_{n,k}(v)\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$

$$-(\mathrm{C})=\int_Q \Gamma_{n,k}(v)\ \frac{d\psi_\delta^+}{dt}\ \psi_\eta^+\ dxdt$$

$$+\int_Q \Gamma_{n,k}(v)\ \frac{d\psi_\eta^+}{dt}\ \psi_\delta^+\ dxdt+\omega(\varepsilon)=\omega(\varepsilon,\delta);$$

now, since $(k-T_k(v^\varepsilon))H_n(v^\varepsilon)$ converges to $(k-T_k(v))H_n(v)$ $*$-weakly in $L^\infty(Q)$, we have

$$(\mathrm{E})=\int_Q (k-T_k(v))H_n(v)a(t,x,\nabla T_{2n}(v)+g)\cdot\nabla(\psi_\delta^+\psi_\eta^+)\ dxdt+\omega(\varepsilon)=\omega(\varepsilon,\delta);$$

moreover, since $H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+$ weakly converges to $H_n(v)\psi_\delta^+\psi_\eta^+$ in $L^p(0,T;W_0^{1,p}(\Omega))$ and using again Lemma 4.18, we easily have

$$-(\mathrm{F})=\int_Q (k-T_k(v))H_n(v)\psi_\delta^+\psi_\eta^+\ d\hat{\mu}_0+\omega(\varepsilon)=\omega(\varepsilon,\delta);$$

while, using (4.5.9) we have

$$0 \leq (\mathrm{G}) \leq 2k \int_Q \psi_\delta^+ \psi_\eta^+ \, d\lambda_\ominus^\varepsilon = 2k \int_Q \psi_\delta^+ \psi_\eta^+ \, d\mu_s^- + \omega(\varepsilon) = \omega(\varepsilon, \delta),$$

and we readily have that $(\mathrm{H}) = \omega(\varepsilon, \delta)$.

It remains to control term (D); we want to stress the fact that the use of the double cut-off function $\psi_\delta^+ \psi_\eta^+$ was introduced essentially to control this term. Suppose we proved that $(\mathrm{D}) = \omega(\varepsilon, \delta, n, \eta)$ and let us conclude the proof of (4.5.17); actually, collecting all we shown above, we have

$$(\mathrm{A}) + (\mathrm{B}) = \omega(\varepsilon, \delta, n, \eta),$$

and, observing that both (A) and (B) are nonnegative, we can conclude that

$$\int_Q a(t, x, \nabla(T_k(v^\varepsilon) + g^\varepsilon)) \nabla(T_k(v^\varepsilon) + g^\varepsilon) \psi_\delta^+ \psi_\eta^+ \, dxdt = \omega(\varepsilon, \delta, \eta), \tag{4.5.20}$$

and

$$\int_Q (k - T_k(v^\varepsilon)) H_n(v^\varepsilon) \psi_\delta^+ \psi_\eta^+ \, d\lambda_\oplus^\varepsilon = \omega(\varepsilon, \delta, n, \eta). \tag{4.5.21}$$

On the other hand, reasoning as before with $(k + T_k(v^\varepsilon)) H_n(v^\varepsilon) \psi_\delta^- \psi_\eta^-$ as test function we can obtain

$$\int_Q a(t, x, \nabla(T_k(v^\varepsilon) + g^\varepsilon)) \nabla(T_k(v^\varepsilon) + g^\varepsilon) \psi_\delta^- \psi_\eta^- \, dxdt = \omega(\varepsilon, \delta, \eta), \tag{4.5.22}$$

and

$$\int_Q (k + T_k(v^\varepsilon)) H_n(v^\varepsilon) \psi_\delta^- \psi_\eta^- \, d\lambda_\ominus^\varepsilon = \omega(\varepsilon, \delta, n, \eta). \tag{4.5.23}$$

But, (4.5.20) and (4.5.22) together with (4.5.19) (that obviously holds true even with $\psi_\delta^- \psi_\eta^-$ in place of $\psi_\delta^+ \psi_\eta^+$) yield (4.5.17), while both (4.5.21) and (4.5.23) show an interesting property of approximating renormalized solutions; they suggest that, in some sense, v^ε (and so the solution u^ε) tends to be, respectively, large (larger than any $k > 0$) on the set where the singular measure μ_s^+ is concentrated, and small (smaller than any $k < 0$) on the set where the singular measure μ_s^- is concentrated.

So, to conclude let us check that (D) $= \omega(\varepsilon, \delta, n, \eta)$ (and the analogous property in the case with $\psi_\delta^- \psi_\eta^-$). First of all, since $0 \le \psi_\delta^+ \le 1$, we have that

$$
\begin{aligned}
\text{(D)} = \frac{2k}{n} \int_{\{-2n<v^\varepsilon\le -n\}} & a(t,x,\nabla(T_{2n}(v^\varepsilon)+g^\varepsilon)) \cdot \nabla(T_{2n}(v^\varepsilon)+g^\varepsilon)\ \psi_\delta^+ \psi_\eta^+ \ dxdt \\
-\frac{2k}{n} \int_{\{-2n<v^\varepsilon\le -n\}} & a(t,x,\nabla(T_{2n}(v^\varepsilon)+g^\varepsilon)) \cdot \nabla g^\varepsilon\ \psi_\delta^+ \psi_\eta^+ \ dxdt \\
\le \frac{2k}{n} \int_{\{-2n<v^\varepsilon\le -n\}} & a(t,x,\nabla(T_{2n}(v^\varepsilon)+g^\varepsilon)) \cdot \nabla(T_{2n}(v^\varepsilon)+g^\varepsilon)\ \psi_\eta^+ \ dxdt + \omega(\varepsilon,\delta) \\
= \frac{2k}{n} \int_{\{-2n<v^\varepsilon\le -n\}} & a(t,x,\nabla u^\varepsilon) \cdot \nabla v^\varepsilon\ \psi_\eta^+ \ dxdt + \omega(\varepsilon,\delta,n),
\end{aligned}
$$

where to get last equality we used (4.1.2), Hölder's inequality and the estimate on the truncates of Proposition 4.15; therefore, we have just to prove that

$$
\frac{1}{n} \int_{\{-2n<v^\varepsilon\le -n\}} a(t,x,\nabla u^\varepsilon) \cdot \nabla v^\varepsilon\ \psi_\eta^+ = \omega(\varepsilon,n,\eta).
$$

To emphasize this interesting property that, at first glance, could appear in contrast with the reconstruction property (4.2.3), we will prove it in the following

LEMMA 4.21. *Let u^ε be a solution of problem (4.4.3) and ψ_η^+, ψ_η^- as in Lemma 4.18. Then*

$$
\frac{1}{n} \int_{\{-2n<v^\varepsilon\le -n\}} a(t,x,\nabla u^\varepsilon) \cdot \nabla v^\varepsilon\ \psi_\eta^+ \ dxdt = \omega(\varepsilon,n,\eta), \tag{4.5.24}
$$

and

$$
\frac{1}{n} \int_{\{n\le v^\varepsilon<2n\}} a(t,x,\nabla u^\varepsilon) \cdot \nabla v^\varepsilon\ \psi_\eta^- \ dxdt = \omega(\varepsilon,n,\eta). \tag{4.5.25}
$$

PROOF. Let us prove (4.5.25); if $\beta_n(s) = B_n(s^+)$, we can choose $\beta_n(v^\varepsilon)\psi_\eta^-$ as test function for problem (4.4.3), and rearranging conveniently all terms, we have

$$\frac{1}{n}\int_{\{n\le v^\varepsilon<2n\}} a(t,x,\nabla(T_{2n}(v^\varepsilon)+g^\varepsilon))\cdot\nabla(T_{2n}(v^\varepsilon)+g^\varepsilon)\psi_\eta^-\ dxdt \tag{A}$$

$$+\int_Q \beta_n(v^\varepsilon)\psi_\eta^-\ d\lambda_\ominus^\varepsilon \tag{B}$$

$$=\int_Q \overline{\beta}_n(v^\varepsilon)\frac{d\psi_\eta^-}{dt}\ dxdt \tag{C}$$

$$-\int_Q a(t,x,\nabla u^\varepsilon)\cdot\nabla\psi_\eta^-\beta_n(v^\varepsilon)\ dxdt \tag{D}$$

$$+\int_Q \beta_n(v^\varepsilon)\psi_\eta^-\ d\hat{\mu}_0^\varepsilon \tag{E}$$

$$+\int_Q \beta_n(v^\varepsilon)\psi_\eta^-\ d\lambda_\oplus^\varepsilon \tag{F}$$

$$+\frac{1}{n}\int_{\{n\le v^\varepsilon<2n\}} a(t,x,\nabla(T_{2n}(v^\varepsilon)+g^\varepsilon))\cdot\nabla g^\varepsilon\ \psi_\eta^-\ dxdt. \tag{G}$$

Observing that both terms on the left hand side of the above equality are nonnegative, let us analyze the right hand side term by term; thanks to Proposition 4.15 and to the fact that $\beta_n(v)$ converges to 0 a.e. on Q and $*$-weakly in $L^\infty(Q)$, we have

$$(\mathrm{D}) = \omega(\varepsilon,n),$$

while, since $\overline{\beta}_n(v^\varepsilon)$ converges to $\overline{\beta}_n(v)$ as ε goes to zero, and $\overline{\beta}_n(v)$ tends to 0 in $L^1(Q)$ as n diverges, again thanks to Proposition 4.15 we easily obtain

$$(\mathrm{C}) = \omega(\varepsilon,n);$$

moreover, again thanks to Proposition 4.15 and to the definition of β_n, in particular using the fact that $\beta_n(v)$ strongly converges to 0 in $L^p(0,T;W_0^{1,p}(\Omega))$ (again this fact is an easy consequence of the estimate on truncates of Proposition 4.15) and $*$-weakly in $L^\infty(Q)$, we have that (E) $= \omega(\varepsilon,n)$ and, using (4.1.2) and Hölder's inequality as before, we have

$$(G) = \omega(\varepsilon,n);$$

finally, thanks to (4.5.8),

$$(\mathrm{F}) \le \int_Q \psi_\eta^-\ d\lambda_\oplus^\varepsilon = \omega(\varepsilon,\eta).$$

Putting together all these facts we obtain (4.5.25), while (4.5.24) can be proved in an analogous way choosing $B_n(s^-)$ and ψ_η^+ as test functions in (4.4.3). $\square$

REMARK 4.22. Notice that the result of Lemma 4.21 turns out to hold true even for more general functions ψ_η^+ and ψ_η^- in $W^{1,\infty}(Q)$ which satisfy

$$0 \le \psi_\eta^+ \le 1 \qquad\qquad 0 \le \psi_\eta^- \le 1,$$

and

$$0 \le \int_Q \psi_\eta^+ \, d\mu_s^- \le \eta \qquad\qquad 0 \le \int_Q \psi_\eta^- \, d\mu_s^+ \le \eta,$$

since the reminder term of the integration by parts easily vanishes as first ε goes to zero and then n diverges. We will use this fact later.

Step 2. Far from E.
We first prove a result that will be essential to deal with the second term in the right hand side of (4.5.16):

LEMMA 4.23. *Let $h, k > 0$, and u^ε and $\Phi_{\delta,\eta}$ as before, then*

$$\int_{\{h \le |v^\varepsilon| < h+k\}} |\nabla u^\varepsilon|^p (1 - \Phi_{\delta,\eta}) = \omega(\varepsilon, h, \delta, \eta) \tag{4.5.26}$$

PROOF. Let $\psi(s) = T_k(s - T_h(s))$ and let us multiply the formulation of u^ε by the test function $\psi(v^\varepsilon)(1 - \Phi_{\delta,\eta})$; integrating, if $\Theta_{k,h}(s) = \displaystyle\int_0^s \psi(\sigma)\, d\sigma$, we have

$$\begin{aligned}
&\int_Q \Theta_{k,h}(v^\varepsilon)_t (1 - \Phi_{\delta,\eta}) \, dxdt \\
&\qquad + \int_Q a(t, x, \nabla u^\varepsilon) \cdot \nabla T_k(v^\varepsilon - T_h(v^\varepsilon)) \, (1 - \Phi_{\delta,\eta}) \, dxdt \\
&\quad - \int_Q a(t, x, \nabla u^\varepsilon) \cdot \nabla \Phi_{\delta,\eta} \, T_k(v^\varepsilon - T_h(v^\varepsilon)) \, dxdt \\
&= \int_Q f^\varepsilon \, T_k(v^\varepsilon - T_h(v^\varepsilon))(1 - \Phi_{\delta,\eta}) \, dxdt \\
&\qquad + \int_Q G^\varepsilon \cdot \nabla (T_k(v^\varepsilon - T_h(v^\varepsilon))(1 - \Phi_{\delta,\eta})) \, dxdt \\
&\qquad + \int_Q T_k(v^\varepsilon - T_h(v^\varepsilon))(1 - \Phi_{\delta,\eta}) d\lambda_\oplus^\varepsilon \\
&\quad - \int_Q T_k(v^\varepsilon - T_h(v^\varepsilon))(1 - \Phi_{\delta,\eta}) d\lambda_\ominus^\varepsilon.
\end{aligned}$$

Now using the fact that

$$
\begin{aligned}
&\int_Q a(t,x,\nabla u^\varepsilon)\cdot\nabla T_k(v^\varepsilon - T_h(v^\varepsilon))\ (1-\Phi_{\delta,\eta})\ dxdt \\
&= \int_{\{h\le|v^\varepsilon|<h+k\}} a(t,x,\nabla u^\varepsilon)\cdot\nabla u^\varepsilon\ (1-\Phi_{\delta,\eta})\ dxdt \\
&\quad - \int_{\{h\le|v^\varepsilon|<h+k\}} a(t,x,\nabla u^\varepsilon)\cdot\nabla g^\varepsilon\ (1-\Phi_{\delta,\eta})\ dxdt,
\end{aligned}
\tag{4.5.27}
$$

and, using Young's inequality we obtain

$$
\begin{aligned}
&\left|\int_Q G^\varepsilon\cdot\nabla T_k(v^\varepsilon - T_h(v^\varepsilon))\ (1-\Phi_{\delta,\eta})\ dxdt\right| \le C_1\int_{\{h\le|v^\varepsilon|<h+k\}} |G^\varepsilon|^{p'}(1-\Phi_{\delta,\eta})\ dxdt \\
&+C_2\int_{\{h\le|v^\varepsilon|<h+k\}} |\nabla u^\varepsilon|^p\ (1-\Phi_{\delta,\eta})\ dxdt + \left|\int_{\{h\le|v^\varepsilon|<h+k\}} G^\varepsilon\cdot\nabla g^\varepsilon(1-\Phi_{\delta,\eta})\ dxdt\right|;
\end{aligned}
$$

where, when we use Young's inequality, we can choose C_2 small as we want (for instance $C_2<\dfrac{\alpha}{3}$); in the same way we can deal with the second term on the right hand side of (4.5.27) after we used assumption (4.1.2) on a; therefore, using assumption (4.1.1) on a in the first term of the right hand side of (4.5.27), noticing that $\Theta_{k,h}(s)$ is nonnegative for any $s \in \mathbb{R}$ and integrating by parts, we obtain

$$\int_Q \Theta_{k,h}(v^\varepsilon)\frac{d\Phi_{\delta,\eta}}{dt}\ dxdt \tag{A}$$

$$+\int_{\{h\le|v^\varepsilon|<h+k\}} |\nabla u^\varepsilon|^p\ (1-\Phi_{\delta,\eta})\ dxdt \tag{B}$$

$$\le C\int_{\{h\le|v^\varepsilon|<h+k\}} |G^\varepsilon|^{p'}\ (1-\Phi_{\delta,\eta})\ dxdt \tag{C}$$

$$+\,C\int_{\{h\le|v^\varepsilon|<h+k\}} |\nabla g^\varepsilon|^p\ (1-\Phi_{\delta,\eta})\ dxdt \tag{D}$$

$$+\int_{\{|v^\varepsilon|\ge h\}} |f^\varepsilon|(1-\Phi_{\delta,\eta})\ dxdt \tag{E}$$

$$+\,C\int_{\{h\le|v^\varepsilon|<h+k\}} |b(t,x)|^{p'}\ (1-\Phi_{\delta,\eta})\ dxdt \tag{F}$$

$$+\int_Q T_k(v^\varepsilon - T_h(v^\varepsilon))(1-\Phi_{\delta,\eta})d\lambda^\varepsilon_\oplus \tag{G}$$

$$-\int_Q T_k(v^\varepsilon - T_h(v^\varepsilon))(1-\Phi_{\delta,\eta})d\lambda^\varepsilon_\ominus \tag{H}$$

$$+\int_\Omega \Theta_{k,h}(u_0^\varepsilon)\ dx. \tag{I}$$

First of all, thanks to the definition of $\Theta_{k,h}(s)$, the strong compactness in $L^1(Q)$ of both v^ε and u_0^ε, using Vitali's theorem we readily have

$$\text{(A)}, \text{(I)} = \omega(\varepsilon, h),$$

while thanks to the equi-integrability property

$$\text{(C)} + \text{(D)} + \text{(E)} + \text{(F)} = \omega(\varepsilon, h);$$

finally, thanks to (4.5.8) and (4.5.10) we have

$$|\text{(G)}| \le k\left|\int_Q (1-\psi_\delta^+\psi_\eta^+)\ d\lambda^\varepsilon_\oplus - \int_Q \psi_\delta^-\psi_\eta^-\ d\lambda^\varepsilon_\oplus\right| = \omega(\varepsilon,\delta,\eta);$$

analogously using (4.5.9) and (4.5.11) one has $|\text{(H)}| = \omega(\varepsilon,\delta,\eta)$; collecting together all these facts we obtain (4.5.26). $\square$

Now, let us analyze the second term in the right hand side of (4.5.16), that is, in some sense, far from the set where the singular measure is concentrated.

We can write, for $n > k$

$$\begin{aligned}
&\int_Q a(t,x,\nabla u^\varepsilon)\cdot\nabla(T_k(v^\varepsilon)-T_k(v)_\nu)H_n(v^\varepsilon)(1-\Phi_{\delta,\eta})\ dxdt \\
&= \int_{\{|v^\varepsilon|\le k\}} a(t,x,\nabla u^\varepsilon)\cdot\nabla(v^\varepsilon-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\ dxdt \\
&\quad -\int_{\{|v^\varepsilon|> k\}} a(t,x,\nabla u^\varepsilon)\cdot\nabla T_k(v)_\nu H_n(v^\varepsilon)(1-\Phi_{\delta,\eta})\ dxdt;
\end{aligned} \tag{4.5.28}$$

First of all, thanks to Egorov theorem and to Proposition 4.15, and since $|T_k(v)_\nu| \le k$ a.e. on Q recalling definition of $T_k(v)_\nu$, we have

$$\begin{aligned}
&\int_{\{|v^\varepsilon|> k\}} a(t,x,\nabla u^\varepsilon)\cdot\nabla T_k(v)_\nu H_n(v^\varepsilon)(1-\Phi_{\delta,\eta})\ dxdt \\
&= \int_{\{|v^\varepsilon|> k\}} a(t,x,\nabla(T_{2n}(v^\varepsilon)+g^\varepsilon))\cdot\nabla T_k(v)_\nu H_n(v^\varepsilon)(1-\Phi_{\delta,\eta})\ dxdt \\
&= \int_{\{|v^\varepsilon|> k\}} a(t,x,\nabla(T_{2n}(v)+g))\cdot\nabla T_k(v)_\nu H_n(v)(1-\Phi_{\delta,\eta})\ dxdt + \omega(\varepsilon) \\
&= \omega(\varepsilon,\nu).
\end{aligned} \tag{4.5.29}$$

To deal with the first term on the right hand side of (4.5.28) we adapt a method introduced, for the parabolic case, in [**Po1**]; for $h > 2k$ let us define

$$w^\varepsilon = T_{2k}(v^\varepsilon - T_h(v^\varepsilon) + T_k(v^\varepsilon) - T_k(v)_\nu);$$

notice that $\nabla w^\varepsilon = 0$ if $|v^\varepsilon| > h + 4k$, thus the estimate on $T_k(v^\varepsilon)$ of Proposition 4.15 implies that w^ε is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$; therefore we easily have that

$$w^\varepsilon \longrightarrow T_{2k}(v - T_h(v) + T_k(v) - T_k(v)_\nu) \quad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and a.e. on } Q.$$

Hence, let us multiply by $w^\varepsilon(1-\Phi_{\delta,\eta})$ the equation solved by u^ε and integrate to obtain

$$\int_0^T \langle v_t^\varepsilon, w^\varepsilon(1-\Phi_{\delta,\eta})\rangle \, dt \tag{A}$$

$$+\int_Q a(t,x,\nabla u^\varepsilon)\cdot\nabla w^\varepsilon(1-\Phi_{\delta,\eta})\, dxdt \tag{B}$$

$$-\int_Q a(t,x,\nabla u^\varepsilon)\cdot\nabla\Phi_{\delta,\eta}\; w^\varepsilon\, dxdt \tag{C}$$

$$=\int_Q f^\varepsilon \; w^\varepsilon(1-\Phi_{\delta,\eta})\, dxdt \tag{D}$$

$$+\int_Q G^\varepsilon\cdot\nabla(w^\varepsilon(1-\Phi_{\delta,\eta}))\, dxdt \tag{E}$$

$$+\int_Q w^\varepsilon(1-\Phi_{\delta,\eta})d\lambda_\oplus^\varepsilon \tag{F}$$

$$-\int_Q w^\varepsilon(1-\Phi_{\delta,\eta})d\lambda_\ominus^\varepsilon. \tag{G}$$

Let us analyze term by term the above identity; first of all, thanks to the properties of w^ε and to *Lebesgue's theorem* we have that (D) $=\omega(\varepsilon,\nu,h)$; while, on the other hand, we have

$$\text{(E)}=\int_{\{h\le v<h+2k\}} G\cdot\nabla v\;(1-\Phi_{\delta,\eta})\, dxdt+\omega(\varepsilon,\nu,h),$$

and using Young's inequality and Lemma 4.23, we have that

$$\int_{\{h\le v<h+2k\}} G\cdot\nabla v\;(1-\Phi_{\delta,\eta})\, dxdt=\omega(h,\delta,\eta).$$

Now, reasoning as in the proof of Lemma 4.23, thanks to (4.5.8)–(4.5.11), using the fact that $|w^\varepsilon|\le 2k$, we have that both (F) $=\omega(\varepsilon,\delta,\eta)$ and (G) $=\omega(\varepsilon,\delta,\eta)$, while thanks to Proposition 4.15 and to the definition of w^ε we have

$$\text{(C)}=\omega(\varepsilon,\nu,h).$$

Let us now analyze term (B); if we define $M=h+4k$ we have

$$\int_Q a(t,x,\nabla u^\varepsilon)\cdot\nabla w^\varepsilon(1-\Phi_{\delta,\eta})\, dxdt$$

$$=\int_Q a(t,x,\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le M\}})\cdot\nabla w^\varepsilon(1-\Phi_{\delta,\eta})\, dxdt.$$

Now, if $E_\varepsilon = \{|v^\varepsilon - T_h(v^\varepsilon) + T_k(v^\varepsilon) - T_k(v)_\nu| \le 2k\}$ and $h \ge 2k$ we can split it as

$$
\begin{aligned}
&\int_Q a(t,x,\nabla u^\varepsilon)\cdot\nabla w^\varepsilon(1-\Phi_{\delta,\eta})\,dxdt\\
&= \int_Q a(t,x,\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le k\}})\cdot\nabla(v^\varepsilon - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\,dxdt\\
&\quad + \int_{\{|v^\varepsilon|>k\}} a(t,x,\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le M\}})\cdot\nabla(v^\varepsilon - T_h(v^\varepsilon))\,(1-\Phi_{\delta,\eta})\chi_{E_\varepsilon}\,dxdt\\
&\quad - \int_{\{|v^\varepsilon|>k\}} a(t,x,\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le M\}})\cdot\nabla T_k(v)_\nu\,(1-\Phi_{\delta,\eta})\chi_{E_\varepsilon}\,dxdt.
\end{aligned}
\tag{4.5.30}
$$

Let us analyze the second term in the right hand side of (4.5.30); since $v^\varepsilon - T_h(v^\varepsilon) = 0$ if $|v^\varepsilon| \le h$, we have

$$
\begin{aligned}
&\left|\int_{\{|v^\varepsilon|>k\}} a(t,x,\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le M\}})\cdot\nabla(v^\varepsilon - T_h(v^\varepsilon))\,(1-\Phi_{\delta,\eta})\chi_{E_\varepsilon}\,dxdt\right|\\
&\le \int_{\{h\le|v^\varepsilon|<h+4k\}} |a(t,x,\nabla u^\varepsilon)||\nabla v^\varepsilon|\,dxdt,
\end{aligned}
$$

and using assumption (4.1.2) on a and Young's inequality we get:

$$
\begin{aligned}
&\int_{\{h\le|v^\varepsilon|<h+4k\}} |a(t,x,\nabla u^\varepsilon)||\nabla v^\varepsilon|\,(1-\Phi_{\delta,\eta})\,dxdt\\
&\le C\int_{\{h\le|v^\varepsilon|<h+4k\}} |\nabla u^\varepsilon|^p\,(1-\Phi_{\delta,\eta})\,dxdt\\
&\quad + C\int_{\{h\le|v^\varepsilon|<h+4k\}} |\nabla g^\varepsilon|^p\,(1-\Phi_{\delta,\eta})\,dxdt\\
&\quad + C\int_{\{h\le|v^\varepsilon|<h+4k\}} |b(t,x)|^{p'}\,(1-\Phi_{\delta,\eta})\,dxdt.
\end{aligned}
$$

Thus, using equi-integrability and Lemma 4.23 we obtain

$$
\int_{\{|v^\varepsilon|>k\}} a(t,x,\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le M\}})\cdot\nabla(v^\varepsilon - T_h(v^\varepsilon))\,(1-\Phi_{\delta,\eta})\chi_{E_\varepsilon}\,dxdt = \omega(\varepsilon,h,\delta,\eta).
\tag{4.5.31}
$$

Let us now analyze the third term in the right hand side of (4.5.30); since, thanks to Proposition 4.15, we have

$$\int_{\{|v^\varepsilon|>k\}} a(t,x,\nabla u^\varepsilon \chi_{\{|v^\varepsilon|\le M\}}) \cdot \nabla T_k(v)\ (1-\Phi_{\delta,\eta})\chi_{E_\varepsilon}\ dxdt = \omega(\varepsilon),$$

then

(4.5.32)

$$\int_{\{|v^\varepsilon|>k\}} a(t,x,\nabla u^\varepsilon \chi_{\{|v^\varepsilon|\le M\}}) \cdot \nabla T_k(v)_\nu\ (1-\Phi_{\delta,\eta})\chi_{E_\varepsilon} dxdt$$

$$= \int_{\{|v^\varepsilon|>k\}} a(t,x,\nabla u^\varepsilon \chi_{\{|v^\varepsilon|\le M\}}) \cdot \nabla(T_k(v)_\nu - T_k(v))\ (1-\Phi_{\delta,\eta})\chi_{E_\varepsilon} dxdt + \omega(\varepsilon),$$

so, thank to the fact that $T_k(v)_\nu$ strongly converges to $T_k(v)$ in $L^p(0,T;W_0^{1,p}(\Omega))$ and using again Proposition 4.15 we have

$$\int_{\{|v^\varepsilon|>k\}} a(t,x,\nabla u^\varepsilon \chi_{\{|v^\varepsilon|\le M\}}) \cdot \nabla(T_k(v)_\nu - T_k(v))\ (1-\Phi_{\delta,\eta})\chi_{E_\varepsilon}\ dxdt = \omega(\varepsilon,\nu),$$

that together with (4.5.32) yields

$$\int_{\{|v^\varepsilon|>k\}} a(t,x,\nabla u^\varepsilon \chi_{\{|v^\varepsilon|\le M\}}) \cdot \nabla T_k(v)_\nu\ (1-\Phi_{\delta,\eta})\chi_{E_\varepsilon}\ dxdt = \omega(\varepsilon,\nu). \tag{4.5.33}$$

So, putting together (4.5.30), (4.5.31) and (4.5.33) we get

$$(\mathrm{B}) = \int_Q a(t,x,\nabla u^\varepsilon \chi_{\{|v^\varepsilon|\le k\}}) \cdot \nabla(v^\varepsilon - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\ dxdt + \omega(\varepsilon,\nu,h,\delta,\eta),$$

and then, gathering together all the above results

$$\begin{aligned} &\int_0^T \langle v_t^\varepsilon, w^\varepsilon(1-\Phi_{\delta,\eta})\rangle\ dt \\ &\qquad + \int_Q a(t,x,\nabla u^\varepsilon \chi_{\{|v^\varepsilon|\le k\}}) \cdot \nabla(v^\varepsilon - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\ dxdt \\ &= \omega(\varepsilon,\nu,h,\delta,\eta). \end{aligned} \tag{4.5.34}$$

If we prove that

$$\int_0^T \langle v_t^\varepsilon, w^\varepsilon(1-\Phi_{\delta,\eta})\rangle\ dt \ge \omega(\varepsilon,\nu,h), \tag{4.5.35}$$

then we obtain our estimate *far from E*:

$$\limsup_{\varepsilon,\nu,\delta,n,\eta} \int_Q a(t,x,\nabla u^\varepsilon) \cdot \nabla(T_k(v^\varepsilon) - T_k(v)_\nu) H_n(v^\varepsilon)(1-\Phi_{\delta,\eta})\ dxdt \le 0. \tag{4.5.36}$$

So, let us prove (4.5.35). Observing that, thanks to the fact that $|T_k(v)_\nu| \leq k$, we can write (recalling that $h > k > 0$)

$$w^\varepsilon = T_{h+k}(v^\varepsilon - T_k(v)_\nu) - T_{h-k}(v^\varepsilon - T_k(v^\varepsilon));$$

we have,

$$\begin{aligned}
&\int_0^T \langle v_t^\varepsilon, w^\varepsilon(1-\Phi_{\delta,\eta})\rangle \, dt \\
&= \int_0^T \langle (T_k(v)_\nu)_t, T_{h+k}(v^\varepsilon - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle \, dt \\
&\quad + \int_Q S_{h+k}(v^\varepsilon - T_k(v)_\nu)_t \, (1-\Phi_{\delta,\eta}) \, dxdt \\
&\quad - \int_Q G_{h-k}(v^\varepsilon)_t \, (1-\Phi_{\delta,\eta}) \, dxdt,
\end{aligned}$$

where

$$S_{h+k}(s) = \int_0^s T_{h+k}(\sigma) \, d\sigma,$$

and

$$G_{h-k}(s) = \int_0^s T_{h-k}(\sigma - T_k(\sigma)) \, d\sigma.$$

First of all, thanks to the definition of $T_k(v)_\nu$ we have

$$\begin{aligned}
&\int_0^T \langle (T_k(v)_\nu)_t, T_{h+k}(v^\varepsilon - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle \, dt \\
&= \nu \int_Q (T_k(v) - T_k(v)_\nu) T_{h+k}(v - T_k(v)_\nu)(1-\Phi_{\delta,\eta}) \, dxdt + \omega(\varepsilon) \\
&= \nu \int_{\{|v|\leq k\}} (v - T_k(v)_\nu) T_{h+k}(v - T_k(v)_\nu)(1-\Phi_{\delta,\eta}) \, dxdt \\
&\quad + \nu \int_{\{v>k\}} (k - T_k(v)_\nu) T_{h+k}(v - T_k(v)_\nu)(1-\Phi_{\delta,\eta}) \, dxdt \\
&\quad + \nu \int_{\{v<-k\}} (-k - T_k(v)_\nu) T_{h+k}(v - T_k(v)_\nu)(1-\Phi_{\delta,\eta}) \, dxdt + \omega(\varepsilon);
\end{aligned}$$

and the three terms in the right hand side are all nonnegative, so we can drop it to obtain

$$\int_0^T \langle (T_k(v)_\nu)_t, T_{h+k}(v^\varepsilon - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle \, dt \geq \omega(\varepsilon), \tag{4.5.37}$$

while integrating by parts we have

$$\begin{aligned}
&\int_Q S_{h+k}(v^\varepsilon - T_k(v)_\nu)_t \ (1-\Phi_{\delta,\eta}) \ dxdt \\
&\quad - \int_Q G_{h-k}(v^\varepsilon)_t \ (1-\Phi_{\delta,\eta}) \ dxdt \\
&= \int_Q S_{h+k}(v^\varepsilon - T_k(v)_\nu) \ \frac{d\Phi_{\delta,\eta}}{dt} \ dxdt \\
&\quad - \int_Q G_{h-k}(v^\varepsilon) \ \frac{d\Phi_{\delta,\eta}}{dt} \ dxdt \\
&\quad + \int_\Omega S_{h+k}(v^\varepsilon - T_k(v)_\nu)(T) \ dx \\
&\quad - \int_\Omega G_{h-k}(v^\varepsilon)(T) \ dx \\
&\quad + \int_\Omega G_{h-k}(u_0^\varepsilon) \ dx \\
&\quad - \int_\Omega S_{h+k}(u_0^\varepsilon - z_\nu) \ dx
\end{aligned}$$

Reasoning as in the proof of Lemma 2.1 in [**Po1**] we can easily show that both

$$\int_\Omega S_{h+k}(v^\varepsilon - T_k(v)_\nu)(T) \ dx - \int_\Omega G_{h-k}(v^\varepsilon)(T) \ dx \geq 0,$$

and

$$\int_\Omega G_{h-k}(u_0^\varepsilon) \ dx - \int_\Omega S_{h+k}(u_0^\varepsilon - z_\nu) \ dx = \omega(\varepsilon, \nu, h).$$

Therefore we have proved that

$$\begin{aligned}&\int_0^T \langle v_t^\varepsilon, w^\varepsilon(1-\Phi_{\delta,\eta})\rangle\ dt\\ &\geq \int_Q S_{h+k}(v^\varepsilon - T_k(v)_\nu)\ \frac{d\Phi_{\delta,\eta}}{dt}\ dxdt\\ &\quad - \int_Q G_{h-k}(v^\varepsilon)\ \frac{d\Phi_{\delta,\eta}}{dt}\ dxdt + \omega(\varepsilon,\nu,h),\end{aligned}$$

so, to conclude we have to check that

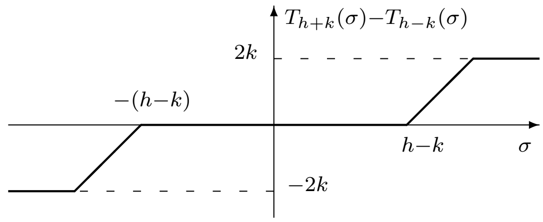
$$\begin{aligned}&\int_Q S_{h+k}(v^\varepsilon - T_k(v)_\nu)\ \frac{d\Phi_{\delta,\eta}}{dt}\ dxdt\\ &\quad - \int_Q G_{h-k}(v^\varepsilon)\ \frac{d\Phi_{\delta,\eta}}{dt}\ dxdt \geq \omega(\varepsilon,\nu,h);\end{aligned} \tag{4.5.38}$$

actually, thanks to Proposition 4.15 and to the properties of $T_k(v)_\nu$ we have

$$\begin{aligned}&\int_Q S_{h+k}(v^\varepsilon - T_k(v)_\nu)\ \frac{d\Phi_{\delta,\eta}}{dt}\ dxdt\\ &\quad - \int_Q G_{h-k}(v^\varepsilon)\ \frac{d\Phi_{\delta,\eta}}{dt}\ dxdt\\ &\geq \int_Q S_{h+k}(v - T_k(v))\ \frac{d\Phi_{\delta,\eta}}{dt}\ dxdt\\ &\quad - \int_Q G_{h-k}(v)\ \frac{d\Phi_{\delta,\eta}}{dt}\ dxdt + \omega(\varepsilon,\nu)\\ &= \int_Q F_h(v)\ \frac{d\Phi_{\delta,\eta}}{dt} + \omega(\varepsilon,\nu),\end{aligned}$$

where $F_h(s) = S_{h+k}(s - T_k(s)) - G_{h-k}(s)$; that is, if $h > 2k$,

$$F_h(s) = \begin{cases} \displaystyle\int_0^{s-k} (T_{h+k}(\sigma) - T_{h-k}(\sigma))\ d\sigma & \text{if } \ s > h-k \\ 0 & \text{if } \ |s| \leq h-k \\ \displaystyle\int_0^{s+k} (T_{h+k}(\sigma) - T_{h-k}(\sigma))\ d\sigma & \text{if } \ s < -(h-k), \end{cases}$$



So, $F_h(v)$ converges almost everywhere to 0 on Q and, since $v \in L^1(Q)$, we can apply *dominated convergence theorem* to conclude that (4.5.38) holds true.

Step 3. Strong convergence of truncates.
Collecting together (4.5.16),(4.5.18), and (4.5.36) we have, taking again $n > k$,

$$\limsup_{\varepsilon,\nu,n} \left(\int_Q a(t,x,\nabla u^\varepsilon) \cdot \nabla T_k(v^\varepsilon) \, dxdt \right.$$
$$\left. - \int_Q a(t,x,\nabla u^\varepsilon) \cdot \nabla T_k(v)_\nu H_n(v^\varepsilon) \, dxdt \right) \leq 0, \tag{4.5.39}$$

therefore, since using Egorov theorem and Proposition 4.15 we have

$$\int_Q a(t,x,\nabla u^\varepsilon) \cdot \nabla T_k(v)_\nu H_n(v^\varepsilon) \, dxdt$$
$$= \int_Q a(t,x,\nabla u) \cdot \nabla T_k(v) \, dxdt + \omega(\varepsilon,\nu,n),$$

then (4.5.39) implies (4.5.15).

Now, recalling that

$$\int_Q a(t,x,\nabla(T_k(v^\varepsilon)+g^\varepsilon)) \cdot \nabla T_k(v^\varepsilon) \, dxdt$$
$$= \int_Q a(t,x,\nabla(T_k(v^\varepsilon)+g^\varepsilon)) \cdot \nabla(T_k(v^\varepsilon)+g^\varepsilon) \, dxdt \tag{4.5.40}$$
$$- \int_Q a(t,x,\nabla(T_k(v^\varepsilon)+g^\varepsilon)) \cdot \nabla g^\varepsilon \, dxdt,$$

using Fatou's lemma, and Proposition 4.15 we can easily conclude that

$$\int_Q a(t,x,\nabla(T_k(v^\varepsilon)+g^\varepsilon)) \cdot \nabla(T_k(v^\varepsilon)+g^\varepsilon) \, dxdt$$
$$= \int_Q a(t,x,\nabla(T_k(v)+g)) \cdot \nabla(T_k(v)+g) \, dxdt + \omega(\varepsilon); \tag{4.5.41}$$

Thus, being nonnegative, $a(t,x,\nabla(T_k(v^\varepsilon)+g^\varepsilon))\nabla(T_k(v^\varepsilon)+g^\varepsilon)$ actually converges to $a(t,x,\nabla(T_k(v)+g))\nabla(T_k(v)+g)$ in $L^1(Q)$; hence, using assumption (4.1.1)

$$\alpha|\nabla(T_k(v^\varepsilon)+g^\varepsilon)|^p \le a(t,x,\nabla(T_k(v^\varepsilon)+g^\varepsilon))\cdot\nabla(T_k(v^\varepsilon)+g^\varepsilon),$$

and so, by Vitali's theorem, recalling that g^ε strongly converges to g in $L^p(0,T;W_0^{1,p}(\Omega))$, we get

$$T_k(v^\varepsilon) \longrightarrow T_k(v) \quad \text{strongly in} \quad L^p(0,T;W_0^{1,p}(\Omega)).$$

$\square$

4.6. Existence of a renormalized solution

Now we are able to prove that problem (4.1.4) has a renormalized solution.

Proof of Theorem 4.4. Let $S\in W^{2,\infty}(\mathbb{R})$ such that S' has a compact support as in Definition 4.1, and let $\varphi\in C_0^1([0,T)\times\Omega)$; then the approximating solutions u^ε (and v^ε) satisfy

$$\begin{aligned}
-\int_\Omega S(u_0^\varepsilon)\varphi(0)\,dx &- \int_0^T \langle \varphi_t, S(v^\varepsilon)\rangle \\
&+\int_Q S'(v^\varepsilon)a(t,x,\nabla u^\varepsilon)\cdot\nabla\varphi\,dxdt \\
&+\int_Q S''(v^\varepsilon)a(t,x,\nabla u^\varepsilon)\cdot\nabla v^\varepsilon\,\varphi\,dxdt \\
&= \int_Q S'(v^\varepsilon)\varphi\,d\hat{\mu}^\varepsilon + \int_Q S'(v^\varepsilon)\varphi\,d\lambda_\oplus^\varepsilon - \int_Q S'(v^\varepsilon)\varphi\,d\lambda_\ominus^\varepsilon.
\end{aligned} \tag{4.6.1}$$

Thanks to Theorem 4.20 all but the last term easily pass to the limit on ε; actually the only terms that give some problems are the last two. We can write

$$\int_Q S'(v^\varepsilon)\varphi\,d\lambda_\oplus^\varepsilon = \int_Q S'(v^\varepsilon)\varphi\,\psi_\delta^+ d\lambda_\oplus^\varepsilon + \int_Q S'(v^\varepsilon)\varphi\,(1-\psi_\delta^+)d\lambda_\oplus^\varepsilon, \tag{4.6.2}$$

where ψ_δ^+ is defined as in Lemma 4.18; thus

$$\left|\int_Q S'(v^\varepsilon)\varphi\,(1-\psi_\delta^+)d\lambda_\oplus^\varepsilon\right| \le C\int_Q (1-\psi_\delta^+)\,d\lambda_\oplus^\varepsilon = \omega(\varepsilon,\delta),$$

while choosing $S'(v^\varepsilon)\varphi\psi_\delta^+$ in the formulation of u^ε one gets,

$$
\begin{aligned}
\int_Q S'(v^\varepsilon)\varphi\psi_\delta^+ \, d\lambda_\oplus^\varepsilon = & -\int_Q S'(v^\varepsilon)\varphi\psi_\delta^+ \, d\hat{\mu}^\varepsilon + \int_Q S'(v^\varepsilon)\varphi\psi_\delta^+ \, d\lambda_\ominus^\varepsilon \\
& - \int_Q (\varphi\psi_\delta^+)_t \, S(v^\varepsilon) \, dxdt \\
& + \int_Q S'(v^\varepsilon) a(t,x,\nabla u^\varepsilon)\cdot\nabla(\varphi\psi_\delta^+) \, dxdt \\
& + \int_Q S''(v^\varepsilon) a(t,x,\nabla u^\varepsilon)\cdot\nabla v^\varepsilon \, \varphi\psi_\delta^+ \, dxdt;
\end{aligned}
\tag{4.6.3}
$$

now, thanks to Proposition 4.15 and the properties of ψ_δ^+, we readily have

$$\int_Q S'(v^\varepsilon)\varphi\psi_\delta^+ \, d\hat{\mu}^\varepsilon = \omega(\varepsilon,\delta),$$

and, thanks to (4.5.9),

$$\left|\int_Q S'(v^\varepsilon)\varphi\psi_\delta^+ \, d\lambda_\ominus^\varepsilon\right| \le C\int_Q \psi_\delta^+ \, d\lambda_\ominus^\varepsilon = \omega(\varepsilon,\delta)$$

while, since $S(v)\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$ and using (4.5.6),

$$\int_Q (\varphi\psi_\delta^+)_t \, S(v^\varepsilon) \, dxdt = \omega(\varepsilon,\delta);$$

moreover, since $a(t,x,\nabla u^\varepsilon)$ is strongly compact in $L^1(Q)$, $S'(v^\varepsilon)$ is bounded, and ψ_δ^+ converges to zero in $L^p(0,T;W_0^{1,p}(\Omega))$ as δ goes to zero, we have

$$\int_Q S'(v^\varepsilon) a(t,x,\nabla u^\varepsilon)\cdot\nabla(\varphi\psi_\delta^+) \, dxdt = \omega(\varepsilon,\delta),$$

and, finally, using Theorem 4.20 and the fact that $\nabla u^\varepsilon = T_M(v^\varepsilon) + g^\varepsilon$ on the set $\{v^\varepsilon \le M\}$,

$$\int_Q S''(v^\varepsilon) a(t,x,\nabla u^\varepsilon)\cdot\nabla v^\varepsilon \, \varphi\psi_\delta^+ \, dxdt = \omega(\varepsilon,\delta).$$

Therefore, from (4.6.2) we deduce

$$\int_Q S'(v^\varepsilon)\varphi \, d\lambda_\oplus^\varepsilon = \omega(\varepsilon). \tag{4.6.4}$$

Analogously we can prove that

$$\int_Q S'(v^\varepsilon)\varphi \, d\lambda_\ominus^\varepsilon = \omega(\varepsilon). \tag{4.6.5}$$

Then u satisfies equation (4.2.1) with $\varphi \in C_0^1([0,T)\times\Omega)$; now, an easy density argument shows that u satisfies the same formulation with $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ such that $\varphi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, and $\varphi(T,x)=0$.

To prove existence it remains, then, to prove properties (4.2.2) and (4.2.3); so let us take and $H_n(v)(1-\psi_\delta^-)\varphi$ as test function in the formulation of u, where $\varphi \in C_0^\infty(Q)$. We obtain

$$
\begin{aligned}
&-\int_Q ((1-\psi_\delta^-)\varphi)_t\ \overline{H}_n(v)\ dxdt \\
&\quad+\int_Q H_n(v)a(t,x,\nabla u)\cdot\nabla((1-\psi_\delta^-)\varphi)\ dxdt \\
&=\int_Q H_n(v)(1-\psi_\delta^-)\varphi\ d\hat{\mu}_0 \\
&\quad+\frac{1}{n}\int_{\{n<v\le 2n\}} a(t,x,\nabla u)\cdot\nabla v\ (1-\psi_\delta^-)\varphi\ dxdt \\
&\quad-\frac{1}{n}\int_{\{-2n\le v<-n\}} a(t,x,\nabla u)\cdot\nabla v\ (1-\psi_\delta^-)\varphi\ dxdt.
\end{aligned}
\tag{4.6.6}
$$

Now, recalling that u^ε is also a distributional solution with datum μ^ε we have

$$
\begin{aligned}
&-\int_Q ((1-\psi_\delta^-)\varphi)_t v^\varepsilon\ dxdt+\int_Q a(t,x,\nabla u^\varepsilon)\cdot\nabla((1-\psi_\delta^-)\varphi)\ dxdt \\
&=\int_Q (1-\psi_\delta^-)\varphi\ d\hat{\mu}_0^\varepsilon+\int_Q (1-\psi_\delta^-)\varphi\ d\lambda_\oplus^\varepsilon-\int_Q (1-\psi_\delta^-)\varphi\ d\lambda_\ominus^\varepsilon,
\end{aligned}
\tag{4.6.7}
$$

for every $\varphi \in C_0^\infty(Q)$.

Therefore, let us take the difference between (4.6.6) and (4.6.7); we obtain

$$-\int_Q ((1-\psi_\delta^-)\varphi)_t\,\overline{H}_n(v)\,dxdt + \int_Q ((1-\psi_\delta^-)\varphi)_t v^\varepsilon\,dxdt \tag{A}$$

$$+\int_Q H_n(v)a(t,x,\nabla u)\cdot\nabla((1-\psi_\delta^-)\varphi)\,dxdt \tag{B}$$

$$-\int_Q a(t,x,\nabla u^\varepsilon)\cdot\nabla((1-\psi_\delta^-)\varphi)\,dxdt \tag{C}$$

$$-\int_Q H_n(v)(1-\psi_\delta^-)\varphi\,d\hat{\mu}_0 + \int_Q (1-\psi_\delta^-)\varphi\,d\hat{\mu}_0^\varepsilon \tag{D}$$

$$+\frac{1}{n}\int_{\{-2n\le v<-n\}} a(t,x,\nabla u)\cdot\nabla v\,(1-\psi_\delta^-)\varphi\,dxdt \tag{E}$$

$$+\int_Q (1-\psi_\delta^-)\varphi\,d\lambda_\oplus^\varepsilon \tag{F}$$

$$-\int_Q (1-\psi_\delta^-)\varphi\,d\lambda_\ominus^\varepsilon \tag{G}$$

$$=\frac{1}{n}\int_{\{n<v\le 2n\}} a(t,x,\nabla u)\cdot\nabla v\,(1-\psi_\delta^-)\varphi\,dxdt. \tag{H}$$

First of all, we easily have

$$\text{(A)} = \omega(\varepsilon, n),$$

and, thanks to Proposition 4.15,

$$\text{(B)} + \text{(C)} = \omega(\varepsilon, n).$$

Now, since $H_n(v)$ strongly converges to 1 in $L^p(0,T;W^{1,p}(\Omega))$ (thanks to the estimate on the truncates of Proposition 4.3, as we said before) and then $H_n(v)(1-\psi_\delta^-)\varphi$ to $(1-\psi_\delta^-)\varphi$ in $L^p(0,T;W_0^{1,p}(\Omega))$, we have

$$\text{(D)} = \omega(\varepsilon, n).$$

Moreover, thanks to (4.5.11),

$$\text{(G)} = \omega(\varepsilon, \delta)$$

and thanks to Lemma 4.21 (see also Remark 4.22), and to Theorem 4.20, we have

$$\text{(E)} = \omega(n, \delta).$$

Finally, using again Theorem 4.20, and Lemma 4.21, we have

$$\text{(H)} = \frac{1}{n}\int_{\{n<v\le 2n\}} a(t,x,\nabla u)\cdot\nabla v\,\varphi\,dxdt + \omega(n,\delta),$$

while, by construction of $\lambda_{\oplus}^{\varepsilon}$, we have

$$(\mathrm{F}) = \int_Q \varphi \ d\mu_s^+ + \omega(\varepsilon, \delta).$$

Putting together all the above results we obtain (4.2.2) for every $\varphi \in C_0^\infty(Q)$. Now, if $\varphi \in C^\infty(\overline{Q})$ we can split

$$\begin{aligned}
&\frac{1}{n}\int_{\{n<v\leq 2n\}} a(t,x,\nabla u)\cdot\nabla v \ \varphi \ dxdt \\
&= \frac{1}{n}\int_{\{n<v\leq 2n\}} a(t,x,\nabla u)\cdot\nabla v \ \varphi\psi_\delta^+ \ dxdt \\
&\quad + \frac{1}{n}\int_{\{n<v\leq 2n\}} a(t,x,\nabla u)\cdot\nabla v \ \varphi(1-\psi_\delta^+) \ dxdt,
\end{aligned} \tag{4.6.8}$$

and, thanks to what we proved before

$$\lim_{n\to\infty}\frac{1}{n}\int_{\{n<v\leq 2n\}} a(t,x,\nabla u)\cdot\nabla v \ \varphi\psi_\delta^+ \ dxdt = \int_Q \varphi \ d\mu_s^+ + \omega(\delta). \tag{4.6.9}$$

On the other hand, reasoning as before, we are under the assumption of Lemma 4.21 (see Remark 4.22), so we have

$$\frac{1}{n}\int_{\{n<v^\varepsilon\leq 2n\}} a(t,x,\nabla u^\varepsilon)\cdot\nabla v^\varepsilon \ \varphi(1-\psi_\delta^+) \ dxdt = \omega(\varepsilon, n, \delta),$$

that, gathered together with the strong convergence of truncates proved in Theorem 4.20, yields

$$\frac{1}{n}\int_{\{n<v\leq 2n\}} a(t,x,\nabla u)\cdot\nabla v \ \varphi(1-\psi_\delta^+) \ dxdt = \omega(n, \delta). \tag{4.6.10}$$

Finally, putting together (4.6.8), (4.6.9) and (4.6.10) we get (4.2.2) for every $\varphi \in C^\infty(\overline{Q})$, and, reasoning by density, for every $\varphi \in C(\overline{Q})$, which concludes the proof of (4.2.2). To obtain (4.2.3) we can reason as before using ψ_δ^+ in the place of ψ_δ^- and viceversa, and this concludes the proof of Theorem 4.4. □

4.7. Uniqueness in the linear case and inverse maximum principle

4.7.1. Uniqueness in the linear case. In this section we try to stress the fact that the notion of renormalized solution, as in the elliptic case, should be the right one to get uniqueness. As we said before if the datum μ belongs to $M_0(Q)$ the renormalized solution turns out to be unique (see [**DPP**]); the same happens for a general measure datum and $u_0 \in L^1(\Omega)$ as initial condition, if the operator is linear, that is if

$$a(t,x,\xi) = M(t,x)\cdot\xi, \tag{4.7.1}$$

where M is a matrix with bounded, measurable entries, and satisfying the ellipticity assumption (4.1.1) (obviously with $p = 2$). In fact we have

THEOREM 4.24. *Let M as in (4.7.1). Then the renormalized solution of problem*

$$\begin{cases} u_t - \operatorname{div}(M(t,x)\nabla u) = \mu & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0 & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{4.7.2}$$

is unique.

PROOF. We will proof this result by showing that a renormalized solution of problem (4.7.2) is a solution in a *duality sense*; uniqueness will follow immediately as in the elliptic case where the notion of duality solution was introduced and studied in [**S**] (See Section 1.2). So, let $w \in L^2(0,T;H_0^1(\Omega)) \cap C(\overline{Q})$ such that $w_t \in L^2(0,T;H^{-1}(\Omega))$ with $w(T) = 0$ and choose $\overline{H}_n(v)$ and w as test functions in (4.2.1), we have

$$\begin{aligned}
&- \int_\Omega \overline{H}_n(u_0)w(0)\,dx && \text{(A)}\\
&- \int_0^T \left\langle w_t, \overline{H}_n(v)\right\rangle\,dt && \text{(B)}\\
&+ \int_Q H_n(v)M(t,x)\nabla u\cdot\nabla w\,dxdt && \text{(C)}\\
&- \frac{1}{n}\int_{\{n\le v<2n\}} M(t,x)\nabla u\cdot\nabla v\;w\,dxdt && \text{(D)}\\
&+ \frac{1}{n}\int_{\{-2n<v\le -n\}} M(t,x)\nabla u\cdot\nabla v\;w\,dxdt && \text{(E)}\\
= &\int_Q H_n(v)\;w\;d\hat{\mu}_0, && \text{(F)}
\end{aligned}$$

and by properties (4.2.2) and (4.2.3) we readily obtain

$$\text{(A)} + \text{(B)} + \text{(C)} - \text{(F)} = \int_Q w\;d\mu_s + \omega(n), \tag{4.7.3}$$

where $\mu_s = \mu_s^+ - \mu_s^-$.

On the other hand if $\psi \in C_0^\infty(Q)$ we can choose w as the solution of the *parabolic retrograde problem*

$$\begin{cases} -w_t - \operatorname{div}(M^*(t,x)\nabla w) = \psi & \text{in } (0,T)\times\Omega, \\ w(T,x) = 0 & \text{in } \Omega, \\ w(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{4.7.4}$$

where $M^*(t,x)$ is the transposed matrix of $M(t,x)$; now, since both $H_n(v)$ and g are good test functions for this problem, and recalling that both $\overline{H}_n(v)$ converges to v in $L^1(Q)$ and $\overline{H}_n(u_0)$ converges to u_0, we have

$$\begin{aligned}
(\mathrm{A}) + (\mathrm{B}) + (\mathrm{C}) - (\mathrm{F}) &= -\int_\Omega \overline{H}_n(u_0)w(0)\;dx \\
&\quad - \int_0^T \left\langle w_t, \overline{H}_n(v)\right\rangle\;dt + \int_Q \nabla\overline{H}_n(v)\cdot M^*(t,x)\nabla w\;dxdt \\
&\quad - \int_Q H_n(v)\;w\;d\hat{\mu}_0 + \int_Q H_n(v)\nabla g\cdot M^*(t,x)\nabla w\;dxdt \\
&= -\int_\Omega \overline{H}_n(u_0)w(0)\;dx + \int_Q \overline{H}_n(v)\psi\;dxdt \\
&\quad - \int_Q H_n(v)\;w\;d\hat{\mu}_0 + \int_Q H_n(v)\nabla g\cdot M^*(t,x)\nabla w\;dxdt \\
&= -\int_\Omega u_0 w(0)\;dx + \int_Q v\psi\;dxdt \\
&\quad - \int_Q w\;d\hat{\mu}_0 + \int_Q \nabla g\cdot M^*(t,x)\nabla w\;dxdt + \omega(n) \\
&= -\int_\Omega u_0 w(0)\;dx + \int_Q v\psi\;dxdt - \int_Q w\;d\hat{\mu}_0 + \int_Q g\;\psi dxdt \\
&\quad + \int_0^T \langle w_t, g\rangle\;dt + \omega(n) \\
&= -\int_\Omega u_0 w(0)\;dx + \int_Q u\psi\;dxdt - \int_Q w\;d\mu_0 + \omega(n);
\end{aligned} \tag{4.7.5}$$

therefore, comparing (4.7.3) and (4.7.5), we obtain

$$-\int_\Omega u_0 w(0)\ dx + \int_Q u\psi\ dxdt = \int_Q w\ d\mu,$$

for every $\psi \in C_0^\infty(Q)$, and so u is the unique solution of problem (4.7.2), since two different solutions u_1 and u_2 of the same problem must satisfy

$$\int_Q (u_1 - u_2)\psi\ dxdt = 0$$

for every $\psi \in C_0^\infty(Q)$. □

4.7.2. Inverse maximum principle for general parabolic operators. In the elliptic case, an easy consequence of the definition and existence of a renormalized solution (see [**DMOP**]) is the so called *Inverse maximum principle* for general monotone operators proved independently in [**DuP**] in the model case of the Laplace operator. This result has a large number of interesting applications; for instance it allows to prove a generalized *Kato's inequality* when Δu is a measure (see [**BrPo**]). In the same way for parabolic equations, a straightforward consequence of Definition 4.1 and Theorem 4.4, using again the notation $v = u - g$, is the following result where, for technical reasons we must assume a stronger hypothesis on g (see also Remark 4.26 below).

THEOREM 4.25 (Parabolic "inverse" maximum principle). *Let $\mu \in M(Q)$, and suppose that there exists $g \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ such that μ can be decomposed as in (4.4.1); let u be the renormalized solution of problem (4.1.4). Then, if $u \geq 0$, we have $\mu_s \geq 0$.*

REMARK 4.26. Notice that obviously the result has an easy *nonpositive* counterpart and that Theorem 4.25 apply, in particular, for purely singular data. Also observe that the stronger assumption on g is rather technical and relies on the fact that we are not able to prove that, in the *decomposition* Theorem 1.39, g could be chosen to be bounded, this question being still an open problem. Finally notice that the sign assumption on u in Theorem 4.25 can be relaxed; actually, because of the reconstruction property (4.2.3), the same result holds true even if u is only supposed to be bounded from below.

APPENDIX A

Further remarks and open problems

1. Asymptotic behavior

Both in Chapter 2 and Chapter 3 the sign assumptions on the data are rather technical since they allow us to work with the trivial subsolution $u \equiv 0$. Obviously the same results of both Theorem 2.4 and Theorem 3.1 can be obtained for nonpositive data and one would like to prove them for general sign data; however notice that, for instance, comparison Lemma 3.5 can not be improved directly to this case because of technical reasons even if Theorem 3.7 and Theorem 3.8 can be proved in this general setting with slight modifications of their proofs (splitting both f and u_0 in their positive and negative part with a suitable choice of sub and supersolutions).

As we said before, in many cases, the convergences in norm to the stationary solution can be improved depending on the regularity of the limit solution (or equivalently to the regularity of the datum); indeed, consider (2.2.4) in the proof of Theorem 2.2, that is

$$u(t,x) \leq v(x), \quad \text{for all } \ t \in (0,T), \text{a.e. in } \ \Omega;$$

so, if, for instance, $\mu \in L^q(\Omega)$ with $q > \frac{N}{p}$, then Stampacchia's type estimates ensure that the solution v of the stationary problem

$$\begin{cases} A(v) = \mu & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

is in $L^\infty(\Omega)$ and so the convergence of $u(t,x)$ to v of Theorem 2.2 is at least $*$-weak in $L^\infty(\Omega)$ and almost everywhere. Reasoning similarly one can refine, depending on the data, the asymptotic results of both Chapter 2 and Chapter 3.

In Chapter 2 we use the assumption $p > \frac{2N+1}{N+1}$. This bound is essentially used to apply Theorem 1.6 that ensures that the solution of the evolution problem actually belongs to $C(0,\infty;L^1(\Omega))$. However, if $1 < p \leq \frac{2N+1}{N+1}$ something can be said; let us look, for simplicity, at the proof of Theorem 2.2. A priori we only know that $u(t,x)$ is bounded in $L^\infty(0,T;L^1(\Omega))$, for every $T > 0$; using this fact one can prove a similar result for the solution $u(t,x)$ of problem

$$\begin{cases} u_t + A(u) = \mu & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0 & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{A.1.1}$$

Indeed, the main difficulty relies on finding a suitable sequence $t_n \in \mathbb{R}^+$ of values such that $t_n \to \infty$ and $t_{n+1} - t_n = 1$, and $u(t_n, x)$ is a well defined function of $L^1(\Omega)$; once we proved it, one can reason analogously as in the proof of Theorem 2.2 to prove, for instance, the following

Theorem A.1. *Let $\mu \in M_0(Q)$ be independent on the variable t, and let $\mu \geq 0$; let $u(t,x)$ be the entropy solution of problem (A.1.1) with $u_0 = 0$, and $v(x)$ the entropy solution of the corresponding elliptic problem. Then there exists a sequence of values $t_n \in \mathbb{R}^+$ such that*

$$\lim_{n\to+\infty} u(t_n, x) = v(x),$$

in $L^1(\Omega)$.

Such a sequence obviously exists depending on the representative of $u(t,x)$ we choose; indeed, if we call $E_n = [n, n+1]$, and we denote by B_n the set of values t of E_n where $u(t,x)$ is well defined, we can easily observe that the set $\displaystyle\bigcap_{n\geq 0}(B_n - n)$ has Lebesgue measure 1 and so there exist (many) values $0 \leq \delta \leq 1$ such that $t_n \equiv \delta + n$ satisfies our requirement.

If we consider the problem

$$\begin{cases} u_t + A(u) = 0 & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0 & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{A.1.2}$$

with $u_0 \in L^1(\Omega)$, an interesting question is on what can be said on the rate of decay of the norm of the solution as t diverges. A huge number of papers was devoted to such a question in the past concerning different problems and in different contexts. The first result one would obtain is an estimate of the type

$$\|u(t,x)\|_{L^q(\Omega)} \leq \frac{C}{t^\gamma}\|u_0\|_{L^1(\Omega)}, \tag{A.1.3}$$

for any $q < \infty$ for a suitable power γ depending on q; from (A.1.3) one should also prove, with the use of a Moser type iteration method, an estimate on the L^∞ norm of u, that is

$$\|u(t,x)\|_{L^\infty(\Omega)} \leq \frac{C}{t^\sigma}\|u_0\|_{L^1(\Omega)}, \tag{A.1.4}$$

for a power $\sigma > 1$; in a slightly different context, these type of computations are contained, for instance, in [**Po3**].

Moreover, in [**BKL**], the authors classify, the behavior of solution of problem like

$$\begin{cases} u_t - \Delta u + |\nabla u|^q = 0 & \text{in } (0,T) \times \mathbb{R}^N, \\ u(0,x) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{A.1.5}$$

depending on $q > 1$, with a nonnegative initial datum in $L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$. This problem is quite different from the one we analyzed in Chapter 3 due to the lack of the nonlinear term g in the absorbing term, to the presence of the power q, and to the fact that they consider problems defined in the whole space. However, it should be interesting to deal with the study of asymptotic behavior for *mixed* problems of the type

$$\begin{cases} u_t - \Delta u + g(u)|\nabla u|^q = 0 & \text{in } (0,T) \times \Omega, \\ u(0,x) = u_0 & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T) \times \partial\Omega. \end{cases} \tag{A.1.6}$$

with the same assumptions of Chapter 3 on g and for any q. One should expect, possibly depending on the growth of g, that, as in [**BKL**], there exists a critical value q_c such that, if $q < q_c$ the behavior of the solution u is as in Chapter 3, while, if $q \geq q_c$, the absorbing term $g(u)|\nabla u|^q$ becomes dominant yielding a concentration phenomenon. Notice that, comparison results for problem of such a type are an hard task to achieve, so, to overcome many technical difficulties, one may consider a regularizing zero order term in problem (A.1.6).

Let us finally spend a few words on the fact that the operator a does not depend on t in Chapter 2; even if this is a standard assumption in the study of this type of problems, something has been done, in the linear case and with smooth data, considering suitable dependence on t in the principal part of the operator. In particular, in [**A**], the author, introduce the notion of *G-convergence* for linear parabolic operators (the so called *PG*-convergence) and prove several results related to the asymptotic behavior of solution of parabolic problem with dependence on t of the matrix $M(t,x)$ defining the differential operator. One of the results was the following

THEOREM A.2 (Arosio). *Let u be the solution of the linear problem*

$$\begin{cases} u_t - \operatorname{div}(M(t,x)\nabla u) = -\operatorname{div}(G) & \textit{in } (0,T) \times \Omega, \\ u(0,x) = u_0 & \textit{in } \Omega, \\ u(t,x) = 0 & \textit{on } (0,T) \times \partial\Omega, \end{cases} \tag{A.1.7}$$

with $G \in (L^{p'}(\Omega))^N$ and $u_0 \in H_0^1(\Omega)$; moreover, suppose that the linear operators defined by the matrices $M_k(t,x) \equiv M(t+k,x)$ PG-converge to the linear operator associated to *a matrix $M_\infty(x)$, then $u(t,x)$ converges, as t tends to infinity, to u_∞ in $L^2(\Omega)$, where u_∞ is the solution of the linear Dirichlet problem associated to $M_\infty(x)$.*

An interesting question would be whether or not a similar result can be achieved in the nonlinear framework and with measure data, that is in the setting of Chapter 2.

2. Renormalized solutions and soft measures

As we mentioned above, the most hard task in the framework of renormalized solutions with general measure data, as well as in the elliptic case, is to prove the uniqueness of the solution of problem

$$\begin{cases} u_t + A(u) = \mu & \text{in } (0,T)\times\Omega \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(0,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{A.2.1}$$

with possibly singular data, being $A(u) = -\mathrm{div}(a(t,x,\nabla u))$ a nonlinear pseudomonotone operator.

In [**DMOP**], in the elliptic case, the authors proved some partial uniqueness results using stronger assumptions on A, namely the strong monotonicity and the Lipschitz continuity, or the Hölder continuity with respect to the gradient (these hypotheses are satisfied, for instance, by the function $a(t,x,\xi) = |\xi|^{p-2}\xi$); essentially they prove that, if u and $\hat{u}$ are two renormalized solutions for the elliptic problem and if they satisfy a further compatibility property (e.g. if its difference is bounded), then they turn out to coincide. Note that, in the proof of such results, they prove that $T_k(u-\hat{u})$ is in $W_0^{1,p}(\Omega)$ in order to use it as test function.

One would like to attain the same type of results; actually, a difficulty relies on the proof that $T_k(u-\hat{u})$ can be chosen as test function in the renormalized formulation since in [**DMOP**], to prove this fact, they use other equivalent definitions of such a solution. In fact, as we said before, in [**DMOP**] are introduced four definitions of renormalized solution that turn out to coincide ([**DMOP**], Theorem 2.33); thanks to this result one can prove more interesting properties of renormalized solution and even the proof of the existence can be obtained in an easier way (see [**DMOP**] and [**Ma**] for the elliptic case). However, in the parabolic case, one can easily extend these definitions, but there are several technical difficulties in the proof of equivalences.

Another possible extension of our result could be the proof of existence of a renormalized solution for problem

$$\begin{cases} u_t + A(u) = \mu & \text{in } (0,T)\times\Omega \\ u(0,x) = \nu & \text{in } \Omega, \\ u(0,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{A.2.2}$$

where both μ and ν are, possibly singular, general measure data; this should be done using the method of Chapter 4 with the one of [**BP**] where the authors prove the existence of a renormalized solution for problem (A.2.2) when ν is a general measure in $M(\Omega)$ and $\mu \in L^1(Q)$.

Let us stress a fact concerning the decomposition Theorem 1.39. In Remark 4.19 we suggested the following natural question

OPEN PROBLEM A.3. Consider Lemma 4.18. Can one choose ψ_δ^+ and ψ_δ^- as *uniformly bounded* functions in $C_0^\infty(Q)$ vanishing in the W_1 norm?

If this were true, for instance, it should allow us to prove the reverse implication of the decomposition Theorem 1.39; if $\mu \in M(Q)$ admits a decomposition as in (1.3.2), then $\mu \in M_0(Q)$; that is we could prove the following *Representation Theorem*

CONJECTURE A.4. Let μ be a bounded measure on Q. Then $\mu \in M_0(Q)$ if and only if there exist $h \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, $g \in L^p(0,T;W_0^{1,p}(\Omega)\cap L^2(\Omega))$ and $f \in L^1(Q)$, such that

$$\int_Q \varphi \; d\mu = \int_0^T \langle h, \varphi\rangle \; dt - \int_0^T \langle \varphi_t, g\rangle \; dt + \int_Q f\varphi \; dxdt, \tag{A.2.3}$$

for any $\varphi \in C_c^\infty([0,T]\times\Omega)$, where $\langle\cdot,\cdot\rangle$ denotes the duality between $(W_0^{1,p}(\Omega)\cap L^2(\Omega))'$ and $W_0^{1,p}(\Omega)\cap L^2(\Omega)$.

Let us finally spend a few words on Remark 4.26 that suggests another way to prove Conjecture A.4. As we said, the stronger assumption on g made in Theorem 4.25 is rather technical and relies on the fact that we are not able to prove the following

OPEN PROBLEM A.5. Is it true that, in the Theorem 1.39, g can be chosen to be bounded?

Note that, in Theorem 1.39, the presence of the term g should be due to take into account those measures that, in some sense, are concentrated on time-dependent jumps. If g could be chosen to be bounded, then the proof of Conjecture A.4 should easily follow be taking inspiration from the proof of a result in [**GR**].

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