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Traveling waves of director fields

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Introduction

The present PhD thesis mainly concerns the construction and the properties of axially symmetric traveling wave solutions for the heat flow of harmonic maps from an infinitely long vertical cylinder of radius R, $\Omega = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 < R^2\} \subset \mathbb{R}^3$, to the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$:

$$u_t = \Delta u + |\nabla u|^2 u \quad \text{in } \Omega \times \mathbb{R}. \tag{1}$$

Here $u = u(x,t)$ is a map from $\Omega \times \mathbb{R}$ in $\mathbb{S}^2$, Δu is the vector field given by $(\Delta u_1, \Delta u_2, \Delta u_3)$ and

$$|\nabla u|^2 = \sum_{i,\,j=1}^{3} \left| \frac{\partial u_i}{\partial x_j} \right|^2 ,$$

where u_1, u_2, u_3 are the three scalar components of the director field u.

Equation (1) can be viewed as the simplest possible one within a class of evolution equations for director fields which naturally arise in applications (see for example [30] and [5] for a list of references). Equations similar to (1) naturally appear in the study of the orientation of nematic liquid crystals and of microscopic magnetic dipoles composing ferromagnetic materials, where the director field u represents the orientation of the particles or microscopic dipoles. But even the simplest mathematical models used in these applications do not reduce to (1): in each of them (1) is part of a system of PDEs and the equation itself contains additional terms. These additional terms, so as the other equations forming the model, are important for the global dynamics of the system which the model refers to. Equation (1) can be viewed as the "heat equation for director fields" in the sense that its nonlinearity merely reflects the constraint $|u| = 1$. Surprisingly enough, the fundamental mathematical issues which make this class of evolution equations so interesting, in particular the formation of defects in the vector field u and nonuniqueness phenomena for the associated initial-boundary value problems, can be already observed for this "simple" equation. A detailed study of the properties of (1) should lead to better insight in the possible local behavior of a solution around its defects and its relation with the nonuniqueness phenomena of the flow.

We observe that equation (1) can also be used to solve an interesting problem in Differential Geometry. Given two riemannian manifolds M and N, with N compact and $\dim(M) = \dim(N)$, and a continuous map $u_0 : M \longrightarrow N$, one can ask whether there exists or not a harmonic map from M to N which is homotopic to u_0. For instance, if $N = \mathbb{S}^2 \subset \mathbb{R}^3$ and $M = \mathbb{B}^2$, where $\mathbb{B}^2$ is the unit ball of $\mathbb{R}^2$, one could associate to (1) the initial and boundary conditions

$$\begin{cases} u(x,0) = u_0(x) & \text{in } \mathbb{B}^2 \\ u(x,t) = u_0(x) & \text{in } \partial\mathbb{B}^2 \times \mathbb{R}^+ . \end{cases}$$

If there exists a solution u of this differential problem which is continuous up to $t = \infty$ and $u_t \to 0$ for $t \to \infty$, then the limit map $u(x, \infty)$ is both harmonic, since it solves the equation

$$\Delta w + |\nabla w|^2 w = 0 , \tag{2}$$

and homotopic to u_0 (see [28] for more details).

In cylindrical coordinates (r, θ, x_3), axially symmetric solutions of (1) can be represented in the form

$$u(r, \theta, x_3, t) = (\cos\theta \sin h, \sin\theta \sin h, \cos h), \tag{3}$$

where $h = h(r, x_3, t)$, the so-called *angle function*, satisfies the scalar equation (see [12],[13])

$$h_t = h_{rr} + h_{x_3x_3} + \frac{h_r}{r} - \frac{\sin(2h)}{2r^2} \quad \text{for } 0 < r < R, x_3 \in \mathbb{R}, t \in \mathbb{R}. \tag{4}$$

It is well known (see [28] for instance) that the initial and boundary value problem for the harmonic map flow:

$$\begin{cases} u_t - \Delta u = |\nabla u|^2 u & \text{in } \Omega \times \mathbb{R} \\ u(x, 0) = u_0(x) & \text{in } \Omega \\ u(x, t) = u_0(x) & \text{in } \partial\Omega \times \mathbb{R}^+ \end{cases} \tag{5}$$

(here Ω must be understood as a smooth bounded domain in $\mathbb{R}^3$, for instance the unit ball $\mathbb{B}^3$) may not have a global classical solution even if the initial and boundary data u_0 is a smooth function. For $\Omega = \mathbb{B}^3$ this is true even in the class of axially symmetric solutions of (5) when the initial and boundary data u_0 is itself axially symmetric, as shown in [12]. Actually, the smoothness of u_0 only ensures local existence and uniqueness (with respect to time) of the classical solution of (5): it may happen that there exists a finite time $T > 0$ such that a classical solution $u = u(x, t)$ is defined in all the time interval $[0, T)$ but

$$\limsup_{t \to T^-} \left(\sup_{x \in \Omega} |\nabla u(x, t)| \right) = \infty .$$

When this blow-up phenomenon actually happens the time T is called first time of blow-up . For $t > T$ it does not make sense to look for classical solutions of (5), but only for weak ones. However, for these last ones there is no uniqueness (see [28]).

For axially symmetric solutions of harmonic map flow the singularities can only occur along the x_3-axis, as follows from equation (4) and standard regularity theory. Moreover, due to the axial symmetry, in the points of continuity along the x_3-axis only two values can be attained: the north pole $\mathbf{N} = (0, 0, 1)$ and the south pole $\mathbf{S} = (0, 0, -1)$, corresponding to values of the angle function which are even integer multiples and odd integer multiples of π respectively. Then, until the first blow-up time the vector solution, unique and smooth, must be identically equal to $\mathbf{N}$ or to $\mathbf{S}$ on the x_3-axis and its angle function must be equal to a fixed integer multiple of π. At the first blow-up time there is the formation of point singularities along this axis. In each of them the angle function of the vector solution u suddenly switches from an integer multiple of π to another one. Then around such a point the vector u rapidly changes its orientation performing one or more half revolutions on the unit sphere $\mathbb{S}^2$. Recently ([2], [27]) it has been shown that nonuniqueness of axially symmetric solutions of harmonic map flow is directly related to the occurrence of point singularities in the solutions: in the special case of the unit ball in $\mathbb{R}^3$ as spatial domain and the function $x/|x|$ as initial and boundary condition, the evolution of the point singularity on the vertical axis of the ball can be prescribed, i.e. given any function $\zeta_0(t) : [0, \infty) \longrightarrow (-1, 1)$ there exists an axially symmetric solution of the heat flow (with the same initial and boundary condition!) which is regular in its domain except of the set $\{(x_1, x_2, x_3, t) = (0, 0, \zeta_0(t), t), t \geq 0\}$. The proof of this nonuniqueness phenomenon is based on the construction of quite complicated comparison functions for equation (4). This construction strongly uses the fact that the angle function associated to $x/|x|$, i.e.

$$h_0(r, x_3) \;=\; \arccos\left(\frac{x_3}{\sqrt{r^2 + x_3^2}}\right), \tag{6}$$

takes values only in the interval $[0, \pi]$, and it cannot be used to study nonuniqueness phenomena when the angle function h_0 of the data u_0 does not satisfy the condition $k\pi \leq h_0 \leq (k+1)\pi$ for some $k \in \mathbb{Z}$. Unfortunately the latter condition is usually not satisfied in the cases in which the initial function is smooth and the first time of blow-up is finite.

For more general axially symmetric initial functions nonuniqueness results can still be obtained, but it is much harder to find appropriate comparison functions. In this context it turns out to be useful to construct axially symmetric traveling wave solutions of (1) with a point singularity on the x_3-axis: at least in some cases, these traveling waves are the appropriate comparison functions, as we shall see in Chapter 3.

In the case of axially symmetric traveling wave solutions the angle function h takes the form:

$$h(r, x_3, t) = \psi(r, x_3 - ct)\,,$$

where $c \in \mathbb{R}$ is known as wave speed and $\psi = \psi(r, z)$, the so-called *shape function*, is a solution to the scalar equation:

$$\psi_{rr} + \psi_{zz} + \frac{\psi_r}{r} + c\psi_z - \frac{\sin(2\psi)}{2r^2} = 0 \qquad \text{for } 0 < r < R, z \in \mathbb{R}\,. \tag{7}$$

These traveling waves are interesting mathematical objects themselves. They offer an example of solutions of the harmonic map flow having a point singularity moving along an axis with constant speed. Moreover, it is possible to construct these traveling wave solutions so that the point singularity has either topological degree 1 or 0. In the first case, at any time the vector solution u rapidly switches around the point singularity from the direction $\mathbf{S}$ to $\mathbf{N}$ (or viceversa) by performing an half revolution on $\mathbb{S}^2$. In the second case, the vector solution u rapidly performs a complete revolution on $\mathbb{S}^2$, starting from the initial orientation $\mathbf{N}$ to come back to it (see figure on the top of the next page).

We used several different constructions to obtain traveling wave solutions of (1). A first construction of variational type is exposed in the first chapter and is similar to that one introduced by Lucia, Muratov and Novaga in the context of Ginzburg-Landau problems in cylinders [25, 23, 24]. Just as in [23], the constructed traveling wave attains a value independent of z on the lateral surface of the cylinder Ω and connects two locally stable and axially symmetric steady states at $x_3 = \pm\infty$. Assuming that the associated

(a) (b)

Figure 1: Behavior near a point singularity of degree 1 (a) and 0 (b)

"energy" at $x_3 = +\infty$ is greater than the one at $x_3 = -\infty$ (the energy is the Dirichlet integral on the disk of radius R, $\int_{\mathbb{B}^2_R} |\nabla u|^2 \, \mathrm{d}\, x_1 \, \mathrm{d}\, x_2$, evaluated at $x_3 = \pm\infty$), the speed c of the wave is positive and determined by the radius of the cylinder Ω and the boundary value. The shape function of the wave has a unique singular point on the x_3-axis of topological degree 1. By the translation invariance along the x_3-axis, we may assume that the singular point is the origin $\mathbf{0} = (0, 0, 0)$. We shall prove that the limit behavior around $\mathbf{0}$ is given by the field $x/|x|$, a result similar to the one proved in [1] for harmonic maps from $\mathbb{B}^3 \subset \mathbb{R}^3$ in $\mathbb{S}^2 \subset \mathbb{R}^3$ minimizing the Dirichlet energy under a suitable boundary condition.

Since the Ginzburg-Landau heat flow:

$$u_t = \Delta u + \frac{(1 - |u|^2)}{\epsilon^2} u$$

can be seen as a "penalty approximation" of the harmonic map flow (see [28]), it may not be surprising that a variational technique used in the Ginzburg-Landau context can be adjusted to work also in the harmonic maps context. In view of the bistable character of the Dirichlet integral, what is really surprising is the fact that for our problem it is possible to construct a traveling wave for *every* value of the wave speed, a result which will be proved in Chapter 4. Naturally, waves with different speeds also have different shape functions, even if the boundary data, the limit states at $x_3 = \pm\infty$ and the position of the point singularity are always the same.

This remarkable phenomenon is intrinsically connected to the nonuniqueness of the general solution of the harmonic heat flow, as highlighted by the proof of the nonuniqueness of the traveling waves. If $u_0 = u_0(x_1, x_2, x_3)$ denotes the wave which we obtain by the variational construction and $c_0 > 0$ its speed, the function $u_0(x_1, x_2, x_3 - c_0 t)$ is a solution to (5) which at any time t has a singularity of degree 1 at the point $(0, 0, c_0t)$. If now c is a given value, positive or negative, with $c \neq c_0$, we shall prove that (5) admits a solution u_c which, at any time t, has a (unique) singularity of degree 1 at the point $(0, 0, ct)$. In other words, for our special choice of u_0, it is possible to construct for every prescribed value $c \in \mathbb{R}$ a solution of (5) which moves the point singularity of u_0 along the x_3-axis with constant speed c.

For $t \to \infty$, the time evolution of u_c resembles more and more to a traveling wave and actually it is possible to prove that

$$v_c(x_1, x_2, x_3) := \lim_{t\to\infty} u_c(x_1, x_2, x_3 + ct, t)$$

is the shape function of a traveling wave with speed c. We remark that

$$v_c\Big|_{\partial\Omega} = u_0\Big|_{\partial\Omega}$$

and that v_c has a unique point singularity of degree 1 in $\mathbf{0}$.

Of course the traveling waves with speed $c \neq c_0$ can be distinguished from u_0 by their construction, which is not variational, and by their wave speeds. We conjecture though that there is a different way to distinguish them: we believe that for $c \neq c_0$ the tangent map of v_c at the origin is not $x/|x|$. If so, it is natural to ask whether the structure of the nonuniqueness phenomena of problem (5) is the following: nonuniqueness is caused by a certain freedom to prescribe the speed of a point singularity, as suggested by the results in [27] which we have discussed before, but the choice of the speed is intrinsically related to the local behavior of the solution near the singularity. In particular, one may wonder whether there is a unique solution of (5) whose local behavior near a singularity $x_0(t)$ satisfies the symmetry properties obtained for minimizers in [1] (for example, behavior of the type $\frac{x-x_0(t)}{|x-x_0(t)|}$ etc.).

Also in Chapter 2 we shall use a variational technique to obtain axially symmetric traveling wave solutions to (1) whose angle function on the lateral surface of the cylinder Ω is a given decreasing function of the variable x_3. Apparently, the construction is very similar to that one of the first chapter. But, while in the first chapter the existence of a traveling wave relies on the bistable character of the Dirichlet integral and on the different nature of the two steady states at $x_3 = \pm\infty$, the method used in the second chapter actually works thanks to the boundary condition, which forces solutions of (7) to move in the x_3-direction with a given speed $c > 0$. At first sight, this construction may seem a bit artificial but it allows us to construct traveling wave solutions to (1) whose point singularity, located on the x_3-axis, can also have topological degree 0, other than 1. For this purpose, it is necessary to add a *relaxation* term to the target functional.

Below we briefly resume the contents and the organization of the thesis.

In Chapter 1 we construct axially symmetric traveling wave solutions of (1) with a given constant angle function at the boundary and a unique point singularity of degree 1 on the vertical axis of the cylinder Ω. In addition we describe the limit behavior of the waves as the radius of the cylinder Ω tends to ∞, and determine the limit behavior near the point singularity.

The second chapter concerns the construction of axially symmetric traveling wave solutions to (1) whose angle function on the lateral surface of the cylinder Ω is a prescribed decreasing function of the variable x_3. Each of these traveling waves has a unique point singularity on the x_3-axis whose topological degree can be either 0 or 1.

The third chapter describes a simple application of the traveling waves constructed in the first two chapters. They are used as comparison functions in the study of the nonuniqueness properties of (4) for a suitable **smooth** initial and boundary data.

In the last chapter we shall show that the traveling wave problem considered in Chapter 1 has a solution for *every prescribed* wave speed $c \in \mathbb{R}$.

Finally, the appendix collects some technical propositions and results often used before.

Part of Chapter 1 is contained in the preprint "Traveling wave solutions of harmonic heat flow" coauthored by M. Bertsch and C. Muratov, which will appear in "Calculus of Variations and Partial Differential Equations" The results of Chapters 2 and 4 are contained in two preprints with M. Bertsch, "Traveling wave solutions of the heat flow of director fields" and "Nonuniqueness of the traveling wave speed for harmonic heat flow". In addition the results of Chapter 3 will be contained in a preprint which I am preparing.

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Chapter 1

Traveling wave solutions of the harmonic heat flow

Let u be a unit vector in $\mathbb{R}^3$ defined on the disk $D_R \subset \mathbb{R}^2$ of radius R. Considering the Dirichlet integral $\int_{D_R} |\nabla u|^2 \,\mathrm{d}\,x$ for $u \in H^1(D_R;\mathbb{S}^2)$, the corresponding Euler-Lagrange equation is (see [28])

$$\Delta u + |\nabla u|^2 u = 0 \quad \text{in } D_R := \{(x_1,x_2) : x_1^2 + x_2^2 < R^2\}. \tag{1.1}$$

Given a constant $b > \frac{1}{R}$, we associate to equation (1.1) the boundary condition

$$u(x_1,x_2) = u_b(x_1,x_2) := \left(\frac{2bx_1}{1+b^2R^2}, \frac{2bx_2}{1+b^2R^2}, \frac{1-b^2R^2}{1+b^2R^2}\right) \text{ for } (x_1,x_2) \in \partial D_R. \tag{1.2}$$

Setting $r := \sqrt{x_1^2 + x_2^2}$, the following two functions are solutions of (1.1)-(1.2):

$$\begin{aligned} u_+(x_1,x_2) &:= \left(\frac{2bx_1}{1+b^2r^2}, \frac{2bx_2}{1+b^2r^2}, \frac{1-b^2r^2}{1+b^2r^2}\right), \\ u_-(x_1,x_2) &:= \left(\frac{2bR^2x_1}{r^2+b^2R^4}, \frac{2bR^2x_2}{r^2+b^2R^4}, \frac{r^2-b^2R^4}{r^2+b^2R^4}\right). \end{aligned}$$

Observe that $|u_+| = |u_-| = 1$, and that, since $bR > 1$,

$$\int_{D_R} |\nabla u_-|^2 \,\mathrm{d}\,x = \frac{8\pi}{1+b^2R^2} < \int_{D_R} |\nabla u_+|^2 \,\mathrm{d}\,x = \frac{8b^2R^2\pi}{1+b^2R^2}. \tag{1.3}$$

More precisely, u_- is a global minimizer of the Dirichlet integral in $H^1(D_R;\mathbb{S}^2)$ subject to the boundary condition (1.2), while u_+ is a local minimizer.

In the present chapter we consider traveling wave solutions of the equation

$$u_t \;=\; \Delta u + |\nabla u|^2 u \tag{1.4}$$

in $\Omega_R \times \mathbb{R}$, where $\Omega_R = D_R \times \mathbb{R} \equiv \{(x_1,x_2,x_3) : x_1^2 + x_2^2 < R^2\}$, which connect u_- at $x_3 = -\infty$ to u_+ at $x_3 = \infty$:

$$u(x_1,x_2,x_3,t) = v(x_1,x_2,x_3 - ct) \in \mathbb{S}^2,$$

where $c \in \mathbb{R}$ and the function $v = v(x_1, x_2, z)$ is a solution of the problem

$$\begin{cases} \Delta v + c v_z + |\nabla v|^2 v = 0 \text{ and } |v| = 1 & \text{in } D_R \times \mathbb{R} \\ v(x_1, x_2, \pm\infty) = u_\pm(x_1, x_2) & \text{for } (x_1, x_2) \in D_R \\ v = u_b & \text{on } \partial D_R \times \mathbb{R}. \end{cases} \tag{1.5}$$

In other words, the traveling wave is a connecting orbit between the two harmonic maps u_- and u_+.

In view of the energy inequality (1.3) and the bistable character of the Dirichlet integral, we expect that there exists a solution v for a certain *positive* wave speed, c_R. Actually, a similar result holds for bistable Ginzburg-Landau systems ([23]), which can be considered as approximations of our problem (see [28]). In the present chapter we shall construct a traveling wave with speed $c_R > 0$. In the fourth chapter we shall use this construction to prove the existence of a traveling wave for *all* wave speeds $c \in \mathbb{R}$, a most surprising result which is undoubtfully counterintuitive, in particular if $c < 0$. As we already explained in the introduction, this result is intimately related to a nonuniqueness property of initial-boundary value problems for equation (1.4) (see also [3], [2], [27]).

Before stating the main results of this chapter we observe that the asymptotic states $u_\pm$ are axially symmetric and can be written as

$$u_\pm(x_1, x_2) = \left(\frac{x_1}{r} \sin\theta_\pm(r), \frac{x_2}{r} \sin\theta_\pm(r), \cos\theta_\pm(r) \right),$$

where

$$\begin{aligned} \theta_+(r) &:= 2\arctan(br) && \text{for } 0 < r \le R, \\ \theta_-(r) &:= 2\arctan\left(\frac{bR^2}{r}\right) = \pi - 2\arctan\left(\frac{r}{bR^2}\right) && \text{for } 0 < r \le R. \end{aligned}$$

Therefore it is natural to consider axially symmetric traveling waves:

$$v(x_1, x_2, z) = \left(\frac{x_1}{r} \sin\theta(r,z), \frac{x_2}{r} \sin\theta(r,z), \cos\theta(r,z) \right), \tag{1.6}$$

where the *angle function* θ is a solution of the problem

$$(\mathrm{I}_{c,R}) \qquad \begin{cases} \theta_{rr} + \dfrac{1}{r}\theta_r + \theta_{zz} + c\theta_z - \dfrac{\sin(2\theta)}{2r^2} = 0 & \text{in } (0,R) \times \mathbb{R} \\ \theta(R, z) = 2\arctan(bR) & \text{for } z \in \mathbb{R} \\ \theta(r, \pm\infty) = \theta_\pm(r) & \text{for } 0 < r < R. \end{cases}$$

The axial symmetry implies that $v(0,0,z) = (0,0,\pm1)$. Since $v(0,0,\pm\infty) = u_\pm(0,0) = (0,0,\pm1)$, we expect that any solution of problem $\mathrm{I}_{c,R}$ has at least one singular point at the cylinder axis (although, in principle, the singularity could also occur at $z = \pm\infty$). This is confirmed by the following result:

Theorem 1.1. *Let $b, R > 0$ be such that $bR > 1$. Then there exists $c_R > 0$ such that Problem $I_{c_R,R}$ has a solution, θ_R, which satisfies:*
(i) θ_R is real analytic in $[0,R] \times \mathbb{R} \setminus \{(0,0)\}$;
(ii) $\theta_R(0,z) = \pi$ if $z < 0$, $\theta_R(0,z) = 0$ if $z > 0$;
(iii) θ_R is strictly decreasing with respect to z in $(0,R) \times \mathbb{R}$;
(iv) the limits of θ_R to $\theta_\pm$ as $z \to \pm\infty$ are uniform with respect to r.

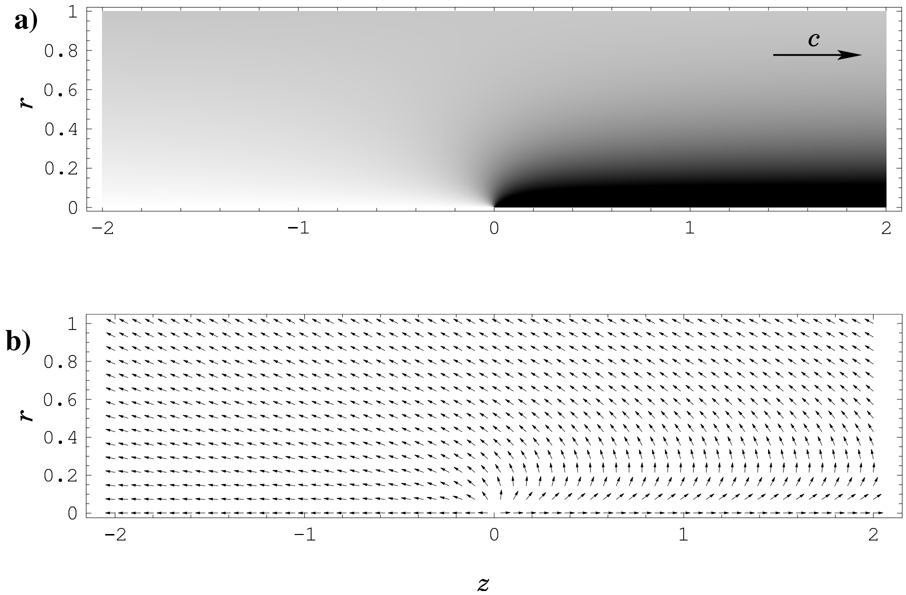

Figure 1.1: Qualitative form of the traveling wave solution from Theorem 1.1. In (a), the angle variable θ as a function of z and r is shown as a density plot, with black corresponding to $\theta = 0$ and white to $\theta = \pi$. In (b), the corresponding vector field is plotted. The wave is moving from left to right.

The translation invariance of Problem $\mathrm{I}_{c,R}$ with respect to z implies that θ_R belongs to a one-parameter family of solutions of Problem $\mathrm{I}_{c,R}$. If $bR < 1$, the energy inequality (1.3) is reversed and, due to the symmetry of the problem, Theorem 1.1 continues to hold with $c_R < 0$.

The second main result of this chapter concerns the limit problem as $R \to \infty$:

$$(\mathrm{I}_{c,\infty}) \quad \begin{cases} \theta_{rr} + \dfrac{1}{r}\theta_r + \theta_{zz} + c\theta_z - \dfrac{\sin(2\theta)}{2r^2} = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R} \\ \theta(r,\infty) = 2\arctan(br) & \text{for } r > 0 \\ \theta(r,-\infty) = \pi & \text{for } r > 0. \end{cases}$$

Observe that in this case the equilibrium solution $2\arctan(br)$ is no longer isolated and belongs to the continuum $\{2\arctan(ar) : a \geq 0\}$.

Theorem 1.2. *Let $b > 0$ and let c_R be defined by Theorem 1.1 for all $R > \frac{1}{b}$. Then $c_R \to c_\infty$ as $R \to \infty$ for some $c_\infty \in \mathbb{R}^+$ and Problem $I_{c_\infty,\infty}$ has a solution, θ_∞, which satisfies:*
(i) θ_∞ is real analytic in $\mathbb{R}^+ \times \mathbb{R} \setminus \{(0,0)\}$;
(ii) $\theta_\infty(0,z) = \pi$ if $z < 0$, $\theta_\infty(0,z) = 0$ if $z > 0$;
(iii) θ_∞ is strictly decreasing with respect to z in $\mathbb{R}^+ \times \mathbb{R}$;
(iv) the limits of θ_∞ as $z \to \pm\infty$ are uniform with respect to r.

The last main result of the chapter is related to the behavior of the traveling wave θ_R in a neighborhood of the origin. In terms of the vector function v_R defined by (1.6) for $\theta = \theta_R$, the statement of the next theorem can be formulated by saying that, in the neighborhood of its point singularity, $v_R(x) \approx x/|x|$.

Theorem 1.3. *Given $b, R > 0$ such that $bR > 1$, let c_R and θ_R be defined as in Theorem 1.1. Then as $\rho \to 0^+$*

$$\theta_R(\rho\cos\phi, \rho\sin\phi) \to \frac{\pi}{2} - \phi$$

loc. uniformly in $[-\pi/2, \pi/2]$.

Let us note that the problem of existence of traveling wave solutions for scalar reaction-diffusion equations has been studied in great detail (see, e.g. [4, 32]). In particular, problems in infinite cylinders with Dirichlet boundary data were treated in [11, 17, 31]. In our case, the situation is complicated by the fact that the nonlinearity in Problem $\mathrm{I}_{c_R,R}$ becomes singular as $r \to 0$. This is why a variational approach to this problem can be particularly useful. It is also worth mentioning that a result similar to Theorem 1.3 has been proved in [1] for minimizing harmonic maps.

The proof of Theorem 1.1 is based on the solution of a constrained minimization problem which is similar to one introduced by Lucia, Muratov and Novaga in the context of Ginzburg-Landau problems in cylinders [25, 23, 24] (see also the work [17] of Heinze for a related approach). Methods illustrated in the appendix will be used to handle some specific technicalities related to director fields and axial symmetry. In section 1.1 we introduce and solve the constrained minimization problem, and in section 1.2 we prove Theorem 1.1. In section 1.3 we consider the limit $R \to \infty$ and prove Theorem 1.2. Finally, in section 1.4 we determine the behavior of the traveling wave θ_R in a neighborhood of the origin.

1.1 The constrained minimization problem

In the following, we follow closely the arguments of [24]. Formally the equation for $\theta(r, z)$,

$$\theta_{rr} + \frac{1}{r}\theta_r + \theta_{zz} + c\theta_z - \frac{\sin(2\theta)}{2r^2} = 0 \quad \text{in } (0, R) \times \mathbb{R}, \tag{1.7}$$

is the Euler-Lagrange equation of the functional

$$\int_{\mathbb{R}} \mathrm{d}\, z \int_0^R \frac{1}{2} r\, \mathrm{e}^{cz} \left(\theta_r^2 + \theta_z^2 + \frac{\sin^2\theta}{r^2} - (\theta_+')^2 - \frac{\sin^2\theta_+}{r^2} \right) \mathrm{d}\, r.$$

The terms of the integrand containing $\theta_+(r) = 2\arctan(br)$ have been added to make the functional finite for certain functions θ behaving like θ_+ as $z \to \infty$. More precisely, setting $f = \theta - \theta_+$ and denoting by $L^2_{c,r}((0, R) \times (-\infty, M))$, with $-\infty < M \le \infty$, the set of Lebesgue measurable functions f on $(0, R) \times (-\infty, M)$ such that

$$\int_{-\infty}^M \mathrm{d}\, z \int_0^R r\, \mathrm{e}^{cz} f^2\, \mathrm{d}\, r < \infty,$$

we define the sets

$$\begin{aligned} Y^M_{c,R} = \Big\{ f \in L^2_{c,r}((0,R)\times(-\infty,M));\ f_r, f_z, \frac{\sin f}{r} \in L^2_{c,r}((0,R)\times(-\infty,M)),\\ f(R,z) = 0 \text{ for a.e. } z \in (-\infty, M) \Big\}, \\ Y_{c,R} = \Big\{ f \in L^2_{c,r}((0,R)\times\mathbb{R});\ f \in Y^M_{c,R} \text{ for all } M < \infty \Big\}. \end{aligned}$$

For all $f \in Y_{c,R}$ we define the functional

$$\Phi_{c,R}(f) := \lim_{M\to\infty} \Phi_{c,R}^M(f) := \lim_{M\to\infty} \int_{-\infty}^{M} \mathrm{d}\, z \int_0^R r\, \mathrm{e}^{cz} \left(\frac{1}{2} f_r^2 + \frac{1}{2} f_z^2 + V(r,f)\right) \mathrm{d}\, r,$$

where

$$V(r,f) := \frac{1}{2r^2} \left(\sin^2(\theta_+ + f) - f \sin(2\theta_+) - \sin^2\theta_+\right).$$

It follows from the following proposition that $\Phi_{c,R} : Y_{c,R} \to (-\infty, \infty]$ is well-defined. Before stating it we recall an auxiliary result which we shall use several times (see Appendix, Lemma A.1 and Corollary A.3).

Lemma 1.4. *For all $w \in H^1_{\mathrm{loc}}(0,\infty) \subset C((0,\infty))$ and $0 < \rho_1 < \rho_2$*

$$\int_{\rho_1}^{\rho_2} \frac{r}{2} \left(w_r^2 + \frac{\sin^2 w}{r^2}\right) \mathrm{d}\, r \geq |\cos(w(\rho_2)) - \cos(w(\rho_1))|\,.$$

If $k \in \mathbb{Z}$, $a \in \mathbb{R}$ and $w(r) = k\pi + 2\arctan(ar)$, then, for all $0 \leq \rho_1 < \rho_2$,

$$\int_{\rho_1}^{\rho_2} \frac{r}{2} \left(w_r^2 + \frac{\sin^2 w}{r^2}\right) \mathrm{d}\, r = |\cos(w(\rho_2)) - \cos(w(\rho_1))| = \left|\frac{1-a^2\rho_2^2}{1+a^2\rho_2^2} - \frac{1-a^2\rho_1^2}{1+a^2\rho_1^2}\right|.$$

Proposition 1.5. *Let $f \in Y_{c,R}$ and $\theta = f + \theta_+$. Then*
(i) for a.e. $z \in \mathbb{R}$

$$\int_0^R r\left(\frac{f_r^2}{2} + V(r,f)\right) \mathrm{d}\, r = \int_0^R \frac{r}{2}\left(\theta_r^2 + \frac{\sin^2\theta}{r^2} - (\theta_+')^2 - \frac{\sin^2\theta_+}{r^2}\right) \mathrm{d}\, r\,;$$

(ii) there exists $C > 0$ such that for all $0 < M < \infty$ and $b > \frac{1}{R}$

$$\Sigma_{c,R}^M(f) := \int_{-\infty}^{M} \mathrm{e}^{cz} \mathrm{d}\, z \int_0^R \frac{r}{2}\left(\theta_r^2 + \frac{\sin^2\theta}{r^2} - (\theta_+')^2 - \frac{\sin^2\theta_+}{r^2}\right) \mathrm{d}\, r \geq -Cb^2\|f\|^2\,, \tag{1.8}$$

where $\|\cdot\|$ is the norm in the weighted space $L^2_{c,r}((0,R)\times\mathbb{R})$;
(iii) $\Sigma_{c,R}(f) := \lim_{M\to\infty} \Sigma_{c,R}^M(f)$ exists, and $-Cb^2\|f\|^2 \leq \Sigma_{c,R}(f) \leq \infty$.

Proof: (i) For a.e. $z \in \mathbb{R}$, $f(\cdot\,,z) \in C([0,R]) \cap H^1_r(0,R)$ and $\frac{\sin f(r,z)}{r} \in L^2_r(0,R)$ (see [30]). Here the subscript r in L^2_r and H^1_r indicates that the usual L^p or Sobolev spaces are to be considered with the weight function r. Since $\theta(\cdot\,,z) \in C([0,R]) \cap H^1_r(0,R)$ and $\frac{\sin\theta(r,z)}{r} \in L^2_r(0,R)$ for a.e. $z \in \mathbb{R}$, integration by parts yields that

$$\int_0^R r f_r(r,z)\theta_+' \,\mathrm{d}\, r = -\int_0^R \frac{\sin(2\theta_+)f}{2r} \,\mathrm{d}\, r \quad \text{for a.e. } z \in \mathbb{R}.$$

Hence (i) follows from the definition of θ and V.
(ii) Since $bR > 1$, $\theta(R,z) = 2\arctan(bR) > \frac{\pi}{2}$. Let $\rho_b \in (0,R)$ be such that $\theta_+(\rho_b) = \frac{\pi}{3}$.

For every $z \in \mathbb{R}$ we define $A_z = \{r \in [3^{-1/2}\rho_b, \rho_b]\ ;\ |f(r,z)| > \frac{\pi}{6}\}$ and, denoting by μ the 1-dimensional Lebesgue measure,

$$B = \left\{ z \in \mathbb{R}\ :\ \mu(A_z) \geq \rho_b \left(1 - \frac{1}{\sqrt{3}}\right)\right\} = \left\{ z \in \mathbb{R}\ :\ \mu(A_z) = \rho_b \left(1 - \frac{1}{\sqrt{3}}\right)\right\}.$$

A simple computation shows that $\|f\|^2 \geq \frac{\pi^2\rho_b^2}{36\sqrt{3}}(1 - 3^{-1/2}) \int_B \mathrm{e}^{cz}\,\mathrm{d}\,z$, where $\|\cdot\|$ is the norm in $L^2_{c,r}((0,R)\times\mathbb{R})$. Hence $\int_B \mathrm{e}^{cz}\,\mathrm{d}\,z \leq Kb^2\,\|f\|^2$ for some $K > 0$. On the other hand, if $z \notin B$ there exists $\rho(z) \in [\rho_b 3^{-1/2}, \rho_b]$ such that $|f(\rho(z), z)| \leq \frac{\pi}{6}$, and therefore $|\theta(\rho(z), z)| \leq \frac{\pi}{2}$. Since $\theta(R, z) > \frac{\pi}{2}$, there exists $r(z) \in [0, R]$ such that $\theta(r(z), z) = \frac{\pi}{2}$. For a.e. $z \in \mathbb{R}$, $f(0,z)$ is a multiple of π ([30]), and, using Lemma 1.4 and computing the following integral separately over $(0, r(z))$ and $(r(z), R)$, we obtain that

$$\int_0^R \frac{r}{2}\left(\theta_r^2(r,z) + \frac{\sin^2(\theta(r,z))}{r^2}\right)\mathrm{d}\,r \geq \frac{2b^2R^2}{1+b^2R^2} \quad \text{for a.e. } z \in \mathbb{R}\setminus B.$$

By Lemma 1.4

$$\Sigma(z) := \int_0^R \frac{r}{2}\left(\theta_r^2(r,z) + \frac{\sin^2(\theta(r,z))}{r^2} - (\theta_+')^2 - \frac{\sin^2\theta_+}{r^2}\right)\mathrm{d}\,r =$$

$$\int_0^R \frac{r}{2}\left(\theta_r^2(r,z) + \frac{\sin^2(\theta(r,z))}{r^2}\right)\mathrm{d}\,r - \frac{2b^2R^2}{1+b^2R^2} \quad \text{for a.e. } z \in \mathbb{R}.$$

Hence $\Sigma(z) \geq 0$ for a.e. $z \notin B$, and, for every $M > 0$,

$$\Sigma^M_{c,R}(f) \geq \int_{(-\infty,M]\cap B} \Sigma(z)\,\mathrm{e}^{cz}\,\mathrm{d}\,z \geq \frac{-2b^2R^2}{1+b^2R^2}\int_{(-\infty,M]\cap B} \mathrm{e}^{cz}\,\mathrm{d}\,z \geq -2Kb^2\|f\|^2.$$

(iii) For every $M > 0$ we have that

$$\Sigma^M_{c,R}(f) = \int_{(-\infty,M]\setminus B} \Sigma(z)\,\mathrm{e}^{cz}\,\mathrm{d}\,z + \int_{(-\infty,M]\cap B}\left(G(z) - \frac{2b^2R^2}{1+b^2R^2}\right)\mathrm{e}^{cz}\,\mathrm{d}\,z,$$

where $G(z) = \int_0^R \frac{r}{2}\left(\theta_r^2(r,z) + \sin^2(\theta(r,z))r^{-2}\right)\mathrm{d}\,r$. Since G is nonnegative, Σ is nonnegative in $\mathbb{R}\setminus B$, and $\int_B \mathrm{e}^{cz}\,\mathrm{d}\,z \leq Kb^2\,\|f\|^2 < \infty$, the result follows at once. □

Observe that, reasoning as in the proof of (ii), we obtain that

$$\begin{aligned}\Phi_{c,R}(f) - \Sigma^M_{c,R}(f) \geq\ & \Sigma_{c,R}(f) - \Sigma^M_{c,R}(f) = \\ & \lim_{N\to\infty}\left(\Sigma^N_{c,R}(f) - \Sigma^M_{c,R}(f)\right) \geq -Cb^2\|f\|^2.\end{aligned} \tag{1.9}$$

Corollary 1.6. *The functional $\Phi_{c,R} : Y_{c,R} \to (-\infty, \infty]$ is well defined and $\Phi_{c,R}(f) \geq -Cb^2\|f\|^2$ for all $f \in Y_{c,R}$, where $\|\cdot\|$ is the norm in $L^2_{c,r}((0,R)\times\mathbb{R})$. If $\theta = f + \theta_+$, then*

$$\Phi_{c,R}(f) = \int_{\mathbb{R}} \mathrm{e}^{cz}\,\mathrm{d}\,z \int_0^R \frac{r}{2}\left(\theta_r^2 + \theta_z^2 + \frac{\sin^2\theta}{r^2} - (\theta_+')^2 - \frac{\sin^2\theta_+}{r^2}\right)\mathrm{d}\,r\,. \tag{1.10}$$

Setting

$$\Gamma_{c,R}(f) := \int_{\mathbb{R}} \mathrm{d}\, z \int_0^R \frac{r}{2} \,\mathrm{e}^{cz}\, f_z^2 \,\mathrm{d}\, r,$$

Corollary 1.6 and the following result imply that $\Phi_{c,R}$ is bounded from below on the set

$$\mathcal{X}_{c,R} := \{ f \in Y_{c,R}; \Gamma_{c,R}(f) = 1 \}.$$

Lemma 1.7. *For all $f \in Y_{c,R}$ such that $f_z \in L^2_{c,r}((0,R) \times \mathbb{R})$*

$$\Gamma_{c,R}(f) \geq \frac{c^2}{8} \|f\|^2 ,$$

where $\|\cdot\|$ is the norm in $L^2_{c,r}((0,R) \times \mathbb{R})$. Moreover, for all $z \in \mathbb{R}$

$$\int_0^R r f^2(r,z) \,\mathrm{d}\, r \;\leq\; \frac{2\,\mathrm{e}^{-cz}}{c} \Gamma_{c,R}(f). \tag{1.11}$$

Proof: For a.e. $r \in (0,R)$ the function $f(r,z)\,\mathrm{e}^{\frac{cz}{2}}$ belongs to $H^1(\mathbb{R})$ and hence vanishes as $z \to \pm\infty$. Therefore, integrating by parts and using Hölder's inequality, we obtain that $\frac{c}{2}\|f\|^2 \leq \|f\| \cdot \|f_z\| = \|f\| \sqrt{2\Gamma_{c,R}(f)}$. The proof of (1.11) is equally simple: given any $z \in \mathbb{R}$, from the inequality

$$\int_z^\infty \mathrm{e}^{cy} \,\mathrm{d}\, y \int_0^R r \left(\sqrt{c} f + \frac{f_z}{\sqrt{c}} \right)^2 \mathrm{d}\, r \geq 0$$

we derive that

$$c \int_z^\infty \mathrm{d}\, y \int_0^R r\, \mathrm{e}^{cy}\, f^2 \,\mathrm{d}\, r + \frac{1}{c} \int_z^\infty \mathrm{d}\, y \int_0^R r\, \mathrm{e}^{cy}\, f_z^2 \,\mathrm{d}\, r \geq$$

$$- \int_z^\infty \mathrm{d}\, y \int_0^R r\, \mathrm{e}^{cy} \frac{\partial}{\partial y} \left(f^2 \right) \mathrm{d}\, r \;=\; \int_0^R r\, \mathrm{e}^{cz}\, f^2(r,z) \;\mathrm{d}\, r + c \int_z^\infty \mathrm{d}\, y \int_0^R r\, \mathrm{e}^{cy}\, f^2 \,\mathrm{d}\, r$$

and this last inequality implies (1.11). $\square$

In the remainder of this section we shall solve the following constrained minimization problem for all $c > 0$:

(MP) Find $h \in \mathcal{X}_{c,R}$ such that $\Phi_{c,R}(h) = \mathcal{I}_{c,R} := \inf_{f \in \mathcal{X}_{c,R}} \Phi_{c,R}(f)$.

Lemma 1.8. *There exists $M_0 = M_0(b,c)$ such that for any $M \geq M_0$ and $f \in \mathcal{X}_{c,R}$*

$$\Phi_{c,R}(f) \geq \Phi^M_{c,R}(f) .$$

Proof: Given any $f \in \mathcal{X}_{c,R}$, we define ρ_b, $A_z, \Sigma(z)$ ($z \in \mathbb{R}$) and B as in the proof of Proposition 1.5. It follows from (1.11) that

$$\frac{\pi^2 \rho_b^2}{108} \leq \int_{\frac{\rho_b}{\sqrt{3}}}^{\rho_b} r f^2(r,z) \,\mathrm{d}\, r \leq \frac{2\,\mathrm{e}^{-cz}}{c}$$

for $z \in B$. Since $\rho_b = \frac{1}{\sqrt{3b}}$, this implies that there exists $M_0 = M_0(b,c)$ such that $B \subseteq (-\infty, M_0]$. At the same time, for any $z \in \mathbb{R} \setminus B$ we have that $\Sigma(z) \geq 0$ (see proof of Proposition 1.5). So, if $M \geq M_0$, then

$$\Phi_{c,R}(f) - \Phi_{c,R}^M(f) \geq \Sigma_{c,R}(f) - \Sigma_{c,R}^M(f) \;=\; \int_M^\infty \mathrm{e}^{cz}\, \Sigma(z)\,\mathrm{d}\,z \geq 0\,.$$

$\square$

Proposition 1.9. *Let $\{h_n\}$ be a minimizing sequence for $\Phi_{c,R}$ in $\mathcal{X}_{c,R}$. Then there exist $h \in Y_{c,R}$ and a subsequence, which we denote again by $\{h_n\}$, such that $h_n \to h$ a.e. in $(0,R) \times \mathbb{R}$ and, for every $M > 0$,*

$$h_{nr} \rightharpoonup h_r, \quad h_{nz} \rightharpoonup h_z, \quad \frac{\sin(h_n + \theta_+)}{r} \rightharpoonup \frac{\sin(h + \theta_+)}{r} \tag{1.12}$$

in $L^2_{c,r}((0,R) \times (-\infty, M))$ as $n \to \infty$, and $\Phi_{c,R}(h) \leq \mathcal{I}_{c,R}$.

Proof: Since $\Gamma_{c,R}(h_n) = 1$, the functions h_{nz} and, by Lemma 1.7, h_n are uniformly bounded in $L^2_{c,r}((0,R) \times \mathbb{R})$. Fixing $M > 0$, we claim that

$$h_{nr} \text{ and } \frac{\sin(h_n + \theta_+)}{r} \text{ are uniformly bounded in } L^2_{c,r}((0,R) \times (-\infty, M)). \tag{1.13}$$

By (1.9) and Lemma 1.7,

$$\int_{-\infty}^M \mathrm{d}\,z \int_0^R \frac{r\,\mathrm{e}^{cz}}{2} h_{nr}^2 \,\mathrm{d}\,r \;=\; \Sigma_{c,R}^M(h_n) - \int_{-\infty}^M \mathrm{d}\,z \int_0^R r\,\mathrm{e}^{cz}\, V(r, h_n)\,\mathrm{d}\,r \;\leq$$

$$\Phi_{c,R}(h_n) + K_1 + \int_{-\infty}^M \mathrm{d}\,z \int_0^R \mathrm{e}^{cz} \left(\frac{h_n \sin(2\theta_+)}{2r} + \frac{\sin^2\theta_+}{2r} \right) \mathrm{d}\,r\,,$$

where K_1 is a constant depending only on b and c. Observe that $\frac{h_n \sin(2\theta_+)}{2r} = h_n \left(r\theta_+' \right)'$ and the $L^2_r(0,R)$-norm of $\frac{\sin\theta_+}{r}$ and θ_+' are bounded by 2. Hence, integrating by parts and applying Hölder's inequality, we obtain that

$$\int_{-\infty}^M \mathrm{d}\,z \int_0^R r\,\mathrm{e}^{cz}\, h_{nr}^2 \,\mathrm{d}\,r \;\leq\; C_1 \left(1 + \sqrt{\int_{-\infty}^M \mathrm{d}\,z \int_0^R r\,\mathrm{e}^{cz}\, h_{nr}^2\,\mathrm{d}\,r} \right),$$

where C_1 depends on b, c, M and $\mathcal{I}_{c,R}$.

Similarly, it follows from the equality

$$\int_{-\infty}^M \mathrm{d}\,z \int_0^R r\,\mathrm{e}^{cz} \frac{\sin^2(h_n + \theta_+)}{2r^2}\,\mathrm{d}\,r \;=\; \Sigma_c(h_n, M) - \frac{1}{2} \int_{-\infty}^M \mathrm{d}\,z \int_0^R r\,\mathrm{e}^{cz}\, h_{nr}^2 \,\mathrm{d}\,r +$$

$$\int_{-\infty}^M \mathrm{e}^{cz}\,\mathrm{d}\,z \int_0^R \frac{\sin(2\theta_+)h_n}{2r}\,\mathrm{d}\,r + \int_{-\infty}^M \mathrm{e}^{cz}\,\mathrm{d}\,z \int_0^R \frac{\sin^2(\theta_+)}{2r}\,\mathrm{d}\,r\,,$$

that also $\frac{\sin(h_n+\theta_+)}{r}$ is uniformly bonded in $L^2_{c,r}((0,R) \times (-\infty, M))$, and we have proved (1.13).

In view of the uniform bounds on h_n, it follows from a standard diagonal procedure that, up to a subsequence, there exists a limit function $h \in Y_{c,R}$ (observe that, by the compactness of the usual trace operator, h vanishes at $r = R$ for a.e. $z \in \mathbb{R}$). If for every $M > 0$ we define the functional

$$\mathscr{E}^M_{c,R}(\cdot) = \int_{-\infty}^{M} \mathrm{e}^{cz}\,\mathrm{d}\,z \int_0^R \frac{r}{2}\left((\cdot)_r^2 + (\cdot)_z^2 + \frac{\sin^2(\cdot)}{r^2} - (\theta_+')^2 - \frac{\sin^2\theta_+}{r^2}\right)\mathrm{d}\,r =$$

$$= \int_{-\infty}^{M} \mathrm{e}^{cz}\,\mathrm{d}\,z \int_0^R \frac{r}{2}\left((\cdot)_r^2 + (\cdot)_z^2 + \frac{\sin^2(\cdot)}{r^2}\right)\mathrm{d}\,r - \frac{2b^2R^2\,\mathrm{e}^{cM}}{c(1+b^2R^2)}\,,$$

then, by Proposition 1.5, $\Phi^M_{c,R}(h) = \mathscr{E}^M_{c,R}(h+\theta_+)$ and $\Phi^M_{c,R}(h_n) = \mathscr{E}^M_{c,R}(h_n+\theta_+)$ for every $n \in \mathbb{N}$. By Fatou's Lemma

$$\Phi^M_{c,R}(h) \le \liminf_{n\to\infty} \Phi^M_{c,R}(h_n) \qquad \text{for all } M > 0,$$

and, by Lemma 1.8, for all $M > M_0$

$$\Phi^M_{c,R}(h) \le \liminf_{n\to\infty} \Phi_{c,R}(h_n) = \mathcal{I}_{c,R}.$$

The thesis follows then from the definition of $\Phi_{c,R}$. □

Since $\Gamma_{c,R}(h) \le 1$, we don't know whether h is a solution of Problem (MP). The following result provides a criterium for the existence of a minimizer.

Lemma 1.10. *If $\Phi_{c,R}(w) \le 0$ (resp. < 0) and $\Gamma_{c,R}(w) > 0$ for some $w \in Y_{c,R}$, then $\mathcal{I}_{c,R} \le 0$ ($\mathcal{I}_{c,R} < 0$) and Problem (MP) has a solution.*

Proof: Reasoning along the lines of [24], we set $a := -c^{-1}\log(\Gamma_{c,R}(w))$ and $w_a(r,z) := w(r, z-a)$. Then $\Gamma_{c,R}(w_a) = \mathrm{e}^{ca}\,\Gamma_{c,R}(w) = 1$ and $\Phi_{c,R}(w_a) = \mathrm{e}^{ca}\,\Phi_{c,R}(w) \le 0$ (resp. < 0). Hence $w_a \in \mathcal{X}_{c,R}$ and $\mathcal{I}_{c,R} \le 0$ (resp. $\mathcal{I}_{c,R} < 0$).

If $\mathcal{I}_{c,R} = 0$, w_a itself is a minimizer.

If $\mathcal{I}_{c,R} < 0$, we use the function h defined by Proposition 1.9 to construct a minimizer: since $0 < \Gamma_{c,R}(h) \le 1$, $d := -c^{-1}\log(\Gamma_{c,R}(h)) \ge 0$; setting $h_d(r,z) := h(r, z-d)$ we have that $\Gamma_{c,R}(h_d) = 1$ and $\Phi_{c,R}(h_d) = \mathrm{e}^{cd}\,\Phi_{c,R}(h) \le \Phi_{c,R}(h) \le \mathcal{I}_{c,R}$. Hence h_d is a solution of Problem (MP). □

Proposition 1.11. *Let $b, R > 0$ be such that $bR > 1$. Then there exists $c^*_R = c^*_R(b)$ such that for every $c \in (0, c^*_R)$ Problem (MP) has a solution and $\mathcal{I}_{c,R} < 0$.*

Proof: In view of Lemma 1.10, it is enough to prove that there exists $c^*_R > 0$ such that for all $0 < c < c^*_R$ there exists $f \in Y_{c,R}$ such that $\Phi_{c,R}(f) < 0$ and $\Gamma_{c,R}(f) > 0$.

We define the function

$$\vartheta(r,z) := \max\left(2\arctan(br), 2\arctan\left(\frac{A(z)}{r}\right)\right) \quad \text{for } (r,z) \in [0,R]\times\mathbb{R}\,,$$

where

$$A(z) = \begin{cases} 0 & \text{if } z \ge 1 \\ bR^2(1-z)^2 & \text{if } 0 < z < 1 \\ bR^2 & \text{if } z \le 0\,. \end{cases}$$

Observe that $\vartheta(r,z) = 2\arctan(br)$ if $z \geq 1$, $\vartheta(r,z) = 2\arctan(bR^2r^{-1})$ if $z \leq 0$, and, for $z \in (0,1)$,

$$\vartheta(r,z) = \begin{cases} 2\arctan(\frac{A(z)}{r}) & r < \sqrt{\frac{A(z)}{b}} \\ 2\arctan(br) & r \geq \sqrt{\frac{A(z)}{b}}. \end{cases}$$

Since $(A')^2A^{-1} = 4bR^2$ in $(0,1)$, one easily checks that the function $f := \vartheta - 2\arctan(br)$ belongs to $Y_{c,R}$. It follows from (1.10) that

$$\Phi_{c,R}(f) = \int_{-\infty}^{0} 2\,\mathrm{e}^{cz}\,\frac{1-b^2R^2}{1+b^2R^2}\,\mathrm{d}\,z + \int_0^1 2\,\mathrm{e}^{cz}\,\frac{1-bA(z)}{1+bA(z)}\,\mathrm{d}\,z +$$

$$\int_0^1 \mathrm{e}^{cz}(A'(z))^2\left(\log\left(1+\frac{1}{bA(z)}\right) - \frac{1}{1+bA(z)}\right)\mathrm{d}\,z \leq \frac{2}{c}\left(\frac{1-b^2R^2}{1+b^2R^2} + \mathrm{e}^c - 1\right)$$

$$+ \int_0^1 \mathrm{e}^{cz}\,\frac{(A'(z))^2}{bA(z)}\,\mathrm{d}\,z = \frac{2}{c}\left(\frac{1-b^2R^2}{1+b^2R^2} + (\mathrm{e}^c - 1)(1+2R^2)\right).$$

Hence there exists $c_R^* = c_R^*(b) > 0$ such that $\Phi_{c,R}(f) < 0$ for all $c \in (0, c_R^*)$. □

Now we are ready to prove the main result of this section:

Theorem 1.12. *Let $b, R > 0$ be such that $bR > 1$.*
(i) For all $c > 0$ the constrained minimization problem (MP) has a solution, $h_{c,R}$.
(ii) There exists $c_R = c_R(b) > 0$ such that $\mathcal{I}_{c_R,R} = 0$, and

$$\mathcal{I}_{c,R} = 1 - \left(\frac{c_R}{c}\right)^2 \quad \textit{for all } c > 0. \tag{1.14}$$

Proof: Let $c, \tilde{c} > 0$ and let $T : \mathcal{X}_{\tilde{c},R} \longrightarrow \mathcal{X}_{c,R}$ be the map defined by $T(f)(r,z) \equiv f(r, \frac{c}{\tilde{c}}z + \beta)$, where $\beta = \frac{1}{\tilde{c}}\log\left(\frac{c}{\tilde{c}}\right)$. One easily verifies that T is well-defined and bijective, and that

$$\Phi_c(T(f)) = 1 + \left(\frac{\tilde{c}}{c}\right)^2(\Phi_{\tilde{c}}(f) - 1) \quad \text{for all } f \in \mathcal{X}_{\tilde{c},R}. \tag{1.15}$$

Let $c_R^*(b)$ be defined by Proposition 1.11, let $\tilde{c} \in (0, c_R^*)$ and let $h_{\tilde{c},R}$ be a minimizer of $\Phi_{\tilde{c},R}$ on $\mathcal{X}_{\tilde{c},R}$. Since T is bijective, relation (1.15) implies that $T(h_{\tilde{c},R})$ is a minimizer of $\Phi_{c,R}$ on $\mathcal{X}_{c,R}$, and that

$$h_{c,R}(r,z) = h_{\tilde{c},R}\left(r, \frac{c}{\tilde{c}}z + \frac{1}{\tilde{c}}\log\left(\frac{c}{\tilde{c}}\right)\right) \quad \text{if } c > 0.$$

In particular

$$\mathcal{I}_{c,R} = 1 + \left(\frac{\tilde{c}}{c}\right)^2(\mathcal{I}_{\tilde{c},R} - 1). \tag{1.16}$$

Since $\mathcal{I}_{\tilde{c},R} < 0$, it follows from (1.16) that there exists $c_R > \tilde{c}$ such that $\mathcal{I}_{c_R,R} = 0$. Replacing $\tilde{c}$ by c_R in (1.16), we obtain (1.14). □

Corollary 1.13. *Let c_R be defined by Theorem 1.12. Then $\Phi_{c_R,R}(w) \geq 0$ for all $w \in Y_{c_R,R}$.*

The proof is immediate: if $\Phi_{c_R,R}(w) < 0$ for some $w \in Y_{c_R,R}$, then $\Gamma_{c_R,R}(w) > 0$ and, by Lemma 1.10, $\mathcal{I}_{c_R,R} < 0$. On the other hand, by definition, $\mathcal{I}_{c_R,R} = 0$ and we have found a contradiction.

1.2 Proof of Theorem 1.1

In this section we shall prove our first main result:

Theorem 1.14. *Let $bR > 1$ and let c_R and $h_{c,R}$ be defined by Theorem 1.12. Then there exists $z_R \in \mathbb{R}$ such that the function*

$$\theta_R(r,z) := \theta_+(r) + h_{c_R,R}(r, z+z_R)$$

satisfies all properties listed in Theorem 1.1.

We shall often omit the subscripts of c_R, $h_{c_R,R}$, $\Gamma_{c_R,R}$ and $\Phi_{c_R,R}$.

The proof of Theorem 1.14 consists of several steps. First we introduce some function spaces. Let V be the Hilbert space

$$V := \left\{\eta : \frac{\eta}{r} \in L^2_{c,r}((0,R)\times\mathbb{R}); \eta_r, \eta_z \in L^2_{c,r}((0,R)\times\mathbb{R}); \eta(R,z) = 0 \text{ for a.e. } z\right\}$$

with scalar product

$$\langle u, v\rangle_V = \int_{\mathbb{R}} \mathrm{d}\, z \int_0^R r\, \mathrm{e}^{cz} \left(\frac{uv}{r^2} + u_r v_r + u_z v_z\right) \mathrm{d}\, r\,.$$

We remark that if $\eta \in V$, then $\eta(0,z) = 0$ for a.e. $z \in \mathbb{R}$ (see [30]). For each $M > 0$ let S_M be the subspace of V containing the functions $\eta \in C^1([0,R] \times \mathbb{R})$ such that $\operatorname{supp}(\eta) \subseteq [0,R] \times (-\infty, M]$, $\eta(R,z) = 0$ for $z \in \mathbb{R}$, and $\|\eta\|_V < \infty$. Let V_M be the closure of S_M in V. Then V_M is a Hilbert space with scalar product $\langle \cdot\,,\,\cdot\rangle_V$.

Lemma 1.15. *Let $c = c_R$ and $h = h_{c_R,R}$. For all $\varepsilon \in (0,1)$ there exist $M > 0$ and $\eta \in V_M$ such that $\langle h_z, \eta_z\rangle > 2(1-\varepsilon)\Gamma(h)$ and $\|\eta_z\|^2 < (1+\varepsilon)^2\|h_z\|^2$, where $\langle \cdot\,,\,\cdot\rangle$ and $\|\cdot\|$ are the scalar product and norm in $L^2_{c,r}((0,R)\times\mathbb{R})$.*

Proof: Since $\|h_z\|^2 = 2\Gamma(h)$, we have that $2(1-\varepsilon)\Gamma(h) - \langle h_z, \eta_z\rangle = \langle h_z, h_z - \eta_z\rangle - \varepsilon\|h_z\|^2 \le \|h_z\|\,(\|h_z - \eta_z\| - \varepsilon\|h_z\|)$. Hence it is sufficient to show that for all $\varepsilon \in (0,1)$ there exist $M > 0$ and $\eta \in V_M$ such that $\|h_z - \eta_z\| < \varepsilon\|h_z\|$. Let $\{g_n\} \subset C^1_0((0,R)\times\mathbb{R})$ be a sequence such that $g_n \to h_z$ in $L^2_{c,r}((0,R)\times\mathbb{R})$. For every $n \in \mathbb{N}$ we define

$$\eta_n(r,z) := -\int_z^\infty g_n(r,t)\,\mathrm{d}\,t \qquad (r,z) \in [0,R]\times\mathbb{R}\,.$$

For all $n \in \mathbb{N}$ there exists $M_n > 0$ such that $\eta_n = 0$ in $[0,R]\times(M_n,\infty)$. Moreover, $\eta_n(R,z) = 0$ for $z \in \mathbb{R}$ and $\|\eta_n\|_V < \infty$. Therefore $\eta_n \in V_{M_n}$ for all n, and, since $(\eta_n)_z = g_n$, the proof is complete. □

Proposition 1.16. *Let $c = c_R$ and $h = h_{c_R,R}$. Then h is a distributional solution of the equation*

$$h_{zz} + ch_z + h_{rr} + \frac{h_r}{r} - \frac{\sin(2h+2\theta_+) - \sin(2\theta_+)}{2r^2} = 0 \quad \text{in } (0,R)\times\mathbb{R}. \tag{1.17}$$

Proof: For $M > 0$ we define the following functionals on V_M:

$$F_M(\eta) \;=\; \Phi(h+\eta) \quad \text{and} \quad G_M(\eta) \;=\; \Gamma(h+\eta)\,.$$

G_M is locally Lipschitz continuous on V_M, Frechet differentiable in zero and its differential in 0 is $\nabla G_M(\eta) \;=\; \langle h_z, \eta_z\rangle$. By Lemma 1.15, $\nabla G_M \not\equiv 0$ on V_M if M is large enough.

Also F_M is differentiable in 0 and its differential in 0 is

$$\nabla F_M(\eta) \;=\; \int_{\mathbb{R}} \mathrm{d}\,z \int_0^R r\,\mathrm{e}^{cz}\left(\nabla h \nabla \eta + \frac{\sin(2h+2\theta_+) - \sin(2\theta_+)}{2r^2}\eta\right) \mathrm{d}\,r\,.$$

Let $\mathcal{G}_M := \{\eta \in V_M : G_M(\eta) = G_M(0)\} = \{\eta \in V_M : \Gamma(h+\eta) = 1\}$. Since $\eta + h \in \mathcal{X}_{c_R,R}$ for all $\eta \in \mathcal{G}_M$, we have that $\Phi(h) \le \Phi(h+\eta)$ if $\eta \in \mathcal{G}_M$. By the Lagrange's multiplier theorem and the inclusion $V_M \subseteq V_{M'}$ for $M' > M$, there exists $\lambda \in \mathbb{R}$ such that $\nabla F_M \;=\; \lambda \nabla G_M$ on V_M for all $M > 0$. In particular, for all $\eta \in C_0^1((0,R)\times\mathbb{R})$ we have that

$$\int_{\mathbb{R}} \mathrm{d}\,z \int_0^R r\,\mathrm{e}^{cz}\left(h_r\eta_r + (1-\lambda)h_z\eta_z + \frac{\sin(2h+2\theta_+) - \sin(2\theta_+)}{2r^2}\eta\right) \mathrm{d}\,r \;=\; 0, \qquad (1.18)$$

i.e. h is a distributional solution of the equation

$$(1-\lambda)(h_{zz} + ch_z) + h_{rr} + \frac{h_r}{r} - \frac{\sin(2h+2\theta_+) - \sin(2\theta_+)}{2r^2} \;=\; 0 \quad \text{in } (0,R)\times\mathbb{R}.$$

It remains to prove that $\lambda = 0$. By Lemma 1.15, applied with $\varepsilon = \frac{1}{2}$, there exist $M > 0$ and $\eta \in V_M$ such that $\langle h_z, \eta_z\rangle > \Gamma(h) \;=\; 1$ and $\|\eta_z\| < \frac{3}{2}\|h_z\| \;=\; \frac{3}{\sqrt{2}}$, where the scalar product and the norm are taken in $L^2_{c,r}((0,R)\times\mathbb{R})$.

First we suppose that $\lambda > 0$. Let $a < 0$ and $\eta_a := a\eta$. Then

$$\nabla F_M(\eta_a) = \lambda \nabla G_M(\eta_a) = \lambda a \langle h_z, \eta_z\rangle < \lambda a < 0,$$

whence

$$\Phi(h+\eta_a) < \Phi(h) + \lambda a + \|\eta\|_V\, o(a) \quad \text{as } a \to 0^-.$$

Since $\Phi(h) = 0$, we can choose $a < 0$ so small that $\Phi(h+\eta_a) < 0$. On the other hand, since $h + \eta_a \in Y_{c_R,R}$ it follows from Corollary 1.13 that $\Phi(h+\eta_a) \ge 0$ and we have found a contradiction.

Hence $\lambda \le 0$. Arguing by contradiction we suppose that $\lambda < 0$. Reasoning as before, with $a > 0$ instead of $a < 0$, the result follows at once. □

Standard regularity theory (see [22]) implies

Corollary 1.17. *Let $c = c_R$ and $h = h_{c_R,R}$. Then h is real analytic in $(0,R]\times\mathbb{R}$, h is a classical solution of (1.17) in $(0,R]\times\mathbb{R}$, and $h(R,z) \;=\; 0$ for every $z \in \mathbb{R}$.*

Proposition 1.18. *Let $h = h_{c_R,R}$. Then*

$$0 < h(r,z) < \varphi(r) := \pi - 2\arctan\left(\frac{r}{bR^2}\right) - 2\arctan(br) \qquad (1.19)$$

for $0 < r \le R$ and $z \in \mathbb{R}$.

Proof: Let $f_1(r,z) = \max(0, h(r,z))$, $f_2(r,z) = \min(f_1(r,z), \varphi(r))$, $\theta_1(r,z) = f_1(r,z) + \theta_+(r)$ and $\theta_2(r,z) = f_2(r,z) + \theta_+(r)$. Then we have trivially $f_i \in Y_{c_R,R}$ $(i = 1,2)$ and $|(f_2)_z| \le |(f_1)_z| \le |h_z| \Rightarrow \Gamma(f_2) \le \Gamma(f_1) \le \Gamma(h)$. Thanks to Proposition 1.5, we can prove the inequality $\Sigma(f_i) \le \Sigma(h)$ by showing that for every $z \in \mathbb{R}$

$$\int_0^R H(r,z;\theta_i)\,\mathrm{d}\,r - \int_0^R H(r,z;\theta) \le 0 \tag{1.20}$$

where $\theta(r,z) = h(r,z) + \theta_+(r)$ and

$$H(r,z;u) = \frac{r}{2}\left(u_r^2(r,z) + \frac{\sin^2(u(r,z))}{r^2}\right).$$

We fix $z \in \mathbb{R}$ arbitrarily. Since $h(r,z)$ is real analytic in $(0,R)$, we may write

$$E_-(z) \equiv \{r \in (0,R)\,;\, \theta(r,z) < \theta_+(r)\} = \bigcup_{n \in \mathcal{T} \subseteq \mathbb{Z}} (\alpha_n, \beta_n), \tag{1.21}$$

where $0 \le \alpha_n < \beta_n \le \alpha_{n+1} < \beta_{n+1} \le R$ for $n, n+1 \in \mathcal{T}$. We observe that, for all $n \in \mathcal{T}$, $\theta(\beta_n, z) = \theta_+(\beta_n)$ and, if $\alpha_n > 0$, $\theta(\alpha_n, z) = \theta_+(\alpha_n)$. Then $\int_0^R (H(r,z;\theta_1) - H(r,z;\theta))\,\mathrm{d}\,r = \int_{E_-(z)} (H(r,z;\theta_+) - H(r,z;\theta))\,\mathrm{d}\,r = \sum_{n \in \mathcal{T}} \int_{\alpha_n}^{\beta_n} (H(r,z;\theta_+) - H(r,z;\theta))\,\mathrm{d}\,r$. By Corollary A.3

$$\int_{\alpha_n}^{\beta_n} (H(r,z;\theta_+) - H(r,z;\theta))\,\mathrm{d}\,r \le 0 \quad \text{if } \alpha_n > 0. \tag{1.22}$$

We observe that $\alpha_n = 0$ may happen for at most one value of n, and if so we may assume without loss of generality that $\alpha_0 = 0$. In this case $\theta(0,z) = k\pi$ with $k \in \mathbb{Z}, k \le 0$ and by Corollary A.3 we obtain

$$\int_0^{\beta_0} (H(r,z;\theta_+) - H(r,z;\theta))\,\mathrm{d}\,r \le 0 \quad \text{if } \alpha_0 = 0. \tag{1.23}$$

Since (1.22) and (1.23) imply (1.20) for $i = 1$, we get $\Sigma(f_1) \le \Sigma(h) \Rightarrow \Phi(f_1) \le \Phi(h) = 0$. At the same time $\Gamma(f_1) > 0$, since $\Gamma(f_1) = 0$ would imply $f_1 \equiv 0$ and then we would have $0 = \Sigma(f_1) \le \Sigma(h) = \Phi(h) - \Gamma(h) = -1$, which is clearly absurd. Arguing as in the first part of the proof of Lemma 1.10, there exists a constant k such that the function $f_1(r, z-k)$ belongs to $\mathcal{X}_{c_R,R}$ and is a minimizer of Problem (MP). By standard regularity theory $f_1(r,z-k)$ is smooth in $(0,R) \times \mathbb{R}$ and, by the strong maximum principle, $f_1(r,z-k) > 0$ for all $(r,z) \in (0,R) \times \mathbb{R}$. Hence $f_1 = h$ in $(0,R) \times \mathbb{R}$ and we have proved the first inequality in (1.19).

Now we can say that $f_2 = \min(h(r,z), \varphi(r)) \Rightarrow \theta_2 = \min(\theta(r,z), \theta_-(r))$. Arguing as before, with $E_-(z)$ replaced by $E_+(z) = \{r \in (0,R)\,;\, \theta(r,z) > \theta_-(r)\}$, only the proof of inequality (1.23) needs to be slightly modified. So we suppose that there exist $z \in \mathbb{R}$ and $\beta_0 \in (0,R]$ such that

$$\theta(r,z) > \theta_-(r) \text{ for } 0 < r < \beta_0 \quad \text{and} \quad \theta(\beta_0, z) = \theta_-(\beta_0). \tag{1.24}$$

Since $bR > 1$, by applying Theorem (A.6) to the function $\theta_-(r) = \pi - 2\arctan(r/(bR^2))$ we find that (1.23) is still true. Since (1.22) and (1.23) imply (1.20) for $i = 2$, we get $\Sigma(f_2) \le \Sigma(h) \Rightarrow \Phi(f_2) \le \Phi(h) = 0$. Arguing in the same way we did for f_1 we deduce the existence of $k \in \mathbb{R}$ such that $f_2(r, z-k) < \varphi(r)$ for all $(r,z) \in (0,R) \times \mathbb{R}$. Hence $f_2 = h$ in $(0,R) \times \mathbb{R}$ and (1.19) is completely proved. ☐

Lemma 1.19. *Let $h = h_{c_R,R}$. Then $h(\cdot, z) \to 0$ in $C^2_{\mathrm{loc}}((0, R])$ as $z \to \infty$.*

Proof: Let $\rho \in (0, R)$ be fixed and let $W_\rho(z) = \int_\rho^R h^2(r, z)\,\mathrm{d}\,r$. It follows from Lemma 1.7 that $\int_{\mathbb{R}} \mathrm{e}^{cz}\, W_\rho(z)\,\mathrm{d}\,z \le \frac{8}{c^2\rho}$, whence $\int_0^\infty W_\rho(z)\,\mathrm{d}\,z < \infty$. Standard Schauder estimates (see [15]) imply that there exists $K = K(c, b, \lambda, \rho) > 0$ such that

$$\|h\|_{C^4([\rho,R]\times\mathbb{R})} \le K. \tag{1.25}$$

Hence W_ρ is uniformly Lipschitz continuous in $\mathbb{R}$ and $h(\cdot, z) \to 0$ in $L^2(\rho, R)$ as $z \to \infty$. The convergence in $C^2([\rho, R]\times\mathbb{R})$ follows from (1.25) and the arbitrariness of ρ completes the proof. □

Our next step will be showing that we can choose the minimizing sequence $\{h_n\}$ such that its limit h is strictly decreasing with respect to z in $(0, R) \times \mathbb{R}$. To do it, we have to apply a one-dimensional rearrangement technique (with respect to z) to $\theta(r, z) = h(r, z) + \theta_+(r)$, or, equivalently, to h. Actually, since it is not possible to work directly with z, we shall apply the rearrangement to the variable $x = \mathrm{e}^{c_R z}$. To this end we consider the transformation

$$x = \mathrm{e}^{c_R z} > 0 \leftrightarrow z = c_R^{-1} \log x$$

and the associated bijective map

$$\begin{array}{rccc} T: & \mathcal{D}_{c_R,R} & \mapsto & \mathcal{S}_{c_R,R} \\ & f(r,z) & \mapsto & f(r, c_R^{-1}\log x) \end{array}$$

whose domain is given by the set

$$\mathcal{D}_{c_R,R} = \{g \in Y_{c_R,R} \mid \Gamma(g) < \infty\}$$

and the image by

$$\mathcal{S}_{c_R,R} = \left\{ g \in L^2_r((0,R)\times\mathbb{R}^+) \;\middle|\; g_r, \frac{\sin(g)}{r} \in L^2_r((0,R)\times(0,a)) \;\; \forall a > 0, \right.$$
$$\left. \int_0^\infty \mathrm{d}\,x \int_0^R r x^2 g_x^2\,\mathrm{d}\,r < \infty \text{ and } g(R, x) \equiv 0 \text{ (a.e.)} \right\}. \tag{1.26}$$

For every $f \in \mathcal{D}_{c_R,\,R}$ the equalities

$$\Phi(f) = \Psi(T(f)) = Q(T(f) + \theta_+(r)) \tag{1.27}$$

are trivially true if we define

$$\Psi(g) = \frac{1}{c_R} \lim_{a\to\infty} \int_0^a \mathrm{d}\,x \int_0^R r \left(\frac{c_R^2 x^2 g_x^2}{2} + \frac{g_r^2}{2} + V(r, g) \right) \mathrm{d}\,r$$

and

$$Q(\vartheta) = \frac{1}{2c_R} \lim_{a\to\infty} \int_0^a \mathrm{d}\,x \int_0^R r \left(c_R^2 x^2 \vartheta_x^2 + \vartheta_r^2 + \frac{\sin^2(\vartheta)}{r^2} - G_b(r) \right) \mathrm{d}\,r$$

where V is the same function as in section 1.1, and

$$G_b(r) \;=\; \frac{\sin^2(\theta_+(r))}{r^2} + \left| \frac{d}{dr}(\theta_+(r)) \right|^2. \tag{1.28}$$

Lemma 1.20. *Let $h = h_{c_R,R}$. Then $h_z < 0$ in $(0,R) \times \mathbb{R}$.*

Proof: Let
$$\bar{h} = T(h), \quad \vartheta = \bar{h} + \theta_+(r). \tag{1.29}$$
Then, thanks to Corollary 1.17, to Proposition 1.18 and to Lemma 1.19 we can say that $\bar{h}$ and ϑ satisfies properties (P1)-(P4) of Appendix B, with $\ell(r) = 0$ and $\ell(r) = 2\arctan(br)$ respectively. If we denote by $\bar{h}^*, \vartheta^*$ the onedimensional decreasing rearrangements of $\bar{h}$ and ϑ with respect to the variable x (see Appendix B for the exact definition), then it follows at once from Propositions B.9, B.10, and B.13 that the norms of $\bar{h}^*$ and $x\bar{h}^*_x$ in $L^2_r((0,R) \times \mathbb{R}^+)$ are both finite and $Q(\vartheta^*) \leq Q(\vartheta)$. Hence, by (1.27), $\Phi(T^{-1}(\bar{h}^*)) \leq \Phi(h)$. If we define $\tilde{h} = T^{-1}(\bar{h}^*)$, then $\tilde{h} \in Y_{c_R,R}$, is nondecreasing with respect to z and satisfies $0 < \Gamma(\tilde{h}) \leq 1$ and $\Phi(\tilde{h}) \leq \Phi(h) = 0$. The inequality $\Gamma(\tilde{h}) \leq 1$ follows from Proposition B.10 and definition of T, and $\Gamma(\tilde{h}) > 0$ follows from the observation that $\forall r \in (0,R)$ $\sup_{z\in\mathbb{R}} \tilde{h}(r,z) = \sup_{z\in\mathbb{R}} h(r,z) > 0$, since $\Gamma(\tilde{h}) = 0$ would imply $\tilde{h} \equiv 0$. Arguing as in the proof of Lemma 1.10, a suitable translation of $\tilde{h}$ with respect to z yields a minimizer of Problem (MP) which is decreasing in z. The strict monotonicity follows from the strong maximum principle. $\square$

Remark 1.21. The validity of Proposition B.8 relies on Proposition B.3, for which it is crucial that the function F does not depend on x. This explains why we cannot apply the rearrangement technique directly to the functional Φ in the original z variable. On the other hand, the form of the functional Q and the key inequality (B.9) applied to the function $P(x) = x^2$ make the method work in the x variable.

Proposition 1.22. *Let $h = h_{c_R,R}$. Then there exists $z_R \in \mathbb{R}$ such that $h(0,z) = \pi$ if $z < z_R$ and $h(0,z) = 0$ if $z > z_R$.*

Proof: Since $h(0,z) = \lim_{r\to 0^+} h(r,z) = k(z)\pi$ for some $k(z) \in \mathbb{Z}$ for a.e. $z \in \mathbb{R}$ ([30]), Theorem 1.18 implies that $k(z)$ is either 0 or 1. Hence, by Lemma 1.20, there are three possibilities for the behavior of $h(0,z)$:

(A) $h \in C([0,R] \times \mathbb{R})$ and $h(0,z) = 0$ for all $z \in \mathbb{R}$;

(B) there exists $z_R \in \mathbb{R}$ such that $h(0,z) = \pi$ if $z < z_R$ and $h(0,z) = 0$ if $z > z_R$;

(C) $h \in C([0,R] \times \mathbb{R})$ and $h(0,z) = \pi$ for all $z \in \mathbb{R}$.

We have to prove that cases (A) and (C) do not occur.

Arguing by contradiction, we first suppose that case (C) occurs. Let $\theta = h + \theta_+$, $a > b$ and $0 < \rho < 1$. By Lemma 1.19, there exists z_ρ such that $0 < \theta(\frac{\rho}{a}, z) < 2\arctan\rho$ for all $z \geq z_\rho$. Since $\theta(0,z) = \pi$ for all $z \in \mathbb{R}$, it follows from Lemma 1.4 that

$$\int_{z_\rho}^{\infty} \mathrm{e}^{cz}\,\mathrm{d}\,z \int_0^R \frac{r}{2}\left(\theta_r^2 + \frac{\sin^2\theta}{r^2} - \left(\theta_+'\right)^2 - \frac{\sin^2\theta_+}{r^2}\right)\mathrm{d}\,r$$
$$\geq \int_{z_\rho}^{\infty} 2\left(\frac{1-\rho^2}{1+\rho^2}\right)\mathrm{e}^{cz}\,\mathrm{d}\,z = \infty.$$

Hence $\Phi(h) = \infty$ and we have found a contradiction.

It remains to exclude case A: suppose that $h \in C([0,R]\times\mathbb{R})$ and $h(0,z)=\theta(0,z)=0$ for all $z\in\mathbb{R}$. Then, by Lemma 1.4,

$$\int_0^R \left(\frac{1}{2}|h_r|^2 + V(r,h)\right) \mathrm{d}\, r = \int_0^R \left(\theta_r^2 + \frac{\sin^2(\theta)}{r^2} - (\theta_+')^2 - \frac{\sin^2(\theta_+)}{r^2}\right) \mathrm{d}\, r \geq 0$$

for a.e. $z\in\mathbb{R}$. Hence $\Phi(h)\geq\Gamma(h)=1$. But $\Phi(h)=0$ and we have found a contradiction. □

Proposition 1.23. *Let $h = h_{c_R,R}$. Then*

$$h(\cdot,z) \to \begin{cases} 0 & \text{as } z\to\infty \\ \varphi & \text{as } z\to-\infty \end{cases}$$

in $C^2_{\mathrm{loc}}((0,R])$ and uniformly in $[0,R]$, where $\varphi(r)$ is defined by (1.19).

Proof: The convergence to 0 as $z\to\infty$ is an immediate consequence of Proposition 1.22 and Lemma's 1.19 and 1.20.

Since $h_z\leq 0$, the limit $H(r) := \lim_{z\to-\infty} h(r,z)$ is well-defined for all $r\in[0,R]$ and satisfies $0\leq H\leq\varphi$ and $H(R)=0$. By (1.25), $h(\cdot,z)\to H$ in $C^2_{\mathrm{loc}}((0,R])$ as $z\to-\infty$, and, for all $r\in(0,R]$, $h_z(r,z)$ and $h_{zz}(r,z)$ vanish as $z\to-\infty$. Hence $H\in C^2((0,R])$ and satisfies

$$H_{rr} + \frac{H_r}{r} - \frac{\sin(2H+2\theta_+)-\sin(2\theta_+)}{2r^2} = 0 \quad \text{in } (0,R).$$

It follows from Proposition 1.22 and Lemma 1.20 that H is continuous down to $r=0$ and $H(0)=\pi$. Setting $\theta_- = H+\theta_+$, we have that θ_- is a classical solution of

$$\begin{cases} \psi_{rr} + \frac{\psi_r}{r} - \frac{\sin(2\psi)}{2r^2} = 0 & \text{in } (0,R) \\ \psi(0)=\pi, \quad \psi(R) = 2\arctan(bR) & \\ \theta_+(r)\leq\psi(r)\leq\varphi(r)+\theta_+(r) & \text{in } [0,R]\,. \end{cases} \tag{1.30}$$

This problem has a unique solution, $\pi - 2\arctan(b^{-1}R^{-2}r)$, and hence $H=\varphi$ in $(0,R)$.

As before, the uniform convergence to φ in $[0,R]$ follows from Proposition 1.22 and Lemma 1.20. □

Proposition 1.24. *Let θ be the function given by*

$$\theta(r,z) = \theta_+(r) + h_{c_R,R}(r,z+z_R)\,,$$

let I be an open nonempty interval and $k\in\mathbb{Z}$ a constant such that $\theta(0,z) = k\pi$ for $z\in I$. Then θ is real analytic in $[0,1)\times I$.

Proof: It is enough to prove that θ is real analytic in a neighborhood of $(0,z)$ for all $z\in I$. The monotonicity with respect to z implies that θ is continuous in $[0,1)\times I$. Then the function

$$u(x_1,x_2,z) := \left(\frac{x_1}{r}\sin\theta(r,z), \frac{x_2}{r}\sin\theta(r,z), \cos\theta(r,z)\right), \quad r = \sqrt{x_1^2+x_2^2},$$

is a continuous weak solution of $\Delta u + |\nabla u|^2 u + cu_z = 0$ in $D_R \times I$. It is well known (see [19] and [16]) that weak solutions are real analytic in open sets in which they are continuous, and hence u is analytic in $D \times I$. Since the first component of u, u_1, vanishes in $\{(0,0)\} \times I$ and $u_1(r,0,z) = \sin(\theta(r,z))$, the analyticity of the function arcsin in a neighborhood of the origin implies that, given $z \in I$, θ is real analytic in a neighborhood of $(0,z)$. □

Theorem 1.14 follows almost at once from Propositions 1.22 and 1.23, Corollary 1.17, Lemma 1.20 and Proposition 1.24.

1.3 Proof of Theorem 1.2

In this section we consider the limit of $h_{c_R,R}$ as $R \to \infty$ to construct a solution of Problem $\mathrm{I}_{c_\infty,\infty}$, where c_∞ is the limit of c_R as $R \to \infty$. Here c_R and $h_{c_R,R}$ are defined by Theorem 1.12 (throughout this section we shall assume that $b > 0$ is fixed and $bR > 1$).

We first prove the existence of the limit speed c_∞.

Lemma 1.25. *The wave speed c_R is nondecreasing with respect to R and*

$$c_\infty := \lim_{R\to\infty} c_R < \infty.$$

Proof: Let $0 < \rho < R$ and

$$w(r,z) = \begin{cases} h_{c_\rho,\rho}(r,z) & \text{if } 0 \le r \le \rho, z \in \mathbb{R} \\ 0 & \text{if } \rho < r \le R, z \in \mathbb{R}. \end{cases}$$

Since $w \in Y_{c_\rho,R}$ and $\Phi_{c_\rho,R}(w) = \Phi_{c_\rho,\rho}(h_{c_\rho,\rho}) = 0$, it follows from Lemma 1.10 that $\mathcal{I}_{c_\rho,R}(w) \le 0$. Hence, by (1.14), $c_\rho \le c_R$.

It remains to show that $c_R \le C$ for a constant C which does not depend on R. By Proposition 1.5, there exists a constant K such that

$$0 = \Phi_{c_R,R}(h_{c_R,R}) = \Sigma_{c_R,R}(h_{c_R,R}) + 1 \ge -Kb^2\|h_{c_R,R}\|^2 + 1$$

for all b and R, and by Lemma 1.7,

$$0 \ge \frac{-8Kb^2}{c_R^2} + 1 \Rightarrow c_R \le \sqrt{8K}b\,.$$

□

The following result can be viewed as a stronger version of Proposition 1.22.

Lemma 1.26. *There exist $z_-^*, z_+^* \in \mathbb{R}$ and $0 < r^* < \frac{1}{b}$ such that for all $R > \frac{1}{b}$*

$$h_{c_R,R} + \theta_+ > \frac{\pi}{2} \quad \textit{in } [0,R] \times (-\infty, z_-^*) \tag{1.31}$$

and

$$h_{c_R,R} + \theta_+ < \frac{\pi}{2} \quad \textit{in } [0,r^*] \times (z_+^*, \infty) \tag{1.32}$$

Proof: To prove (1.31) we argue by contradiction and suppose that for all $n \in \mathbb{N}$ there exist $R_n > \frac{1}{b}$, $r_n \in [0, R_n]$ and $z_n \to -\infty$ as $n \to \infty$ such that

$$\theta_n(r_n, z_n) \leq \frac{\pi}{2},$$

where $\theta_n = h_{c_{R_n}, R_n} + \theta_+$. The monotonicity with respect to z implies that

$$\theta_n(r_n, z) \leq \frac{\pi}{2} \quad \text{for } z \geq z_n.$$

Setting

$$A_n = \int_{-\infty}^{z_n} \mathrm{d}\, z \int_0^{R_n} \frac{r\, \mathrm{e}^{c_{R_n} z}}{2} \left((\theta_n)_r^2 + \frac{\sin^2 \theta_n}{r^2} - (\theta_+')^2 - \frac{\sin^2 \theta_+}{r^2} \right) \mathrm{d}\, r,$$

it follows from Lemma 1.4, applied in the intervals $(0, r_n)$ and (r_n, R_n) for $z > z_n$, that

$$0 = \Phi_{c_{R_n}, R_n}(h_n) \geq \Gamma_{c_{R_n}, R_n}(h_n) + A_n = 1 + A_n.$$

Hence $A_n \leq -1$. On the other hand, using again Lemma 1.4,

$$A_n \geq 2 \int_{-\infty}^{z_n} \mathrm{e}^{c_{R_n} z} \frac{1 - b^2 R_n^2}{1 + b^2 R_n^2} \mathrm{d}\, z \geq -2 \int_{-\infty}^{z_n} \mathrm{e}^{c_{R_n} z} \mathrm{d}\, z \to 0 \quad \text{as } n \to \infty,$$

where we have used that $z_n \to -\infty$ and c_{R_n} is uniformly bounded (by Lemma 1.25). Hence we have found a contradiction.

It remains to prove (1.32). Let $a > b$ and $\rho > 0$ be such that $a\rho < 1$, i.e. $2 \arctan(a\rho) < \frac{\pi}{2}$. It follows from the proof of Lemma 1.19 that the convergence of $h_{c_R, R}$ to 0 in $C([\rho, R])$ as $z \to \infty$ is uniform with respect to R. Hence, setting $\theta_{c_R, R} = h_{c_R, R} + \theta_+$ there exists $z_\rho \in \mathbb{R}$ such that for all $R > \frac{1}{b}$

$$\theta_{c_R, R} < 2 \arctan(ar) \quad \text{in } [\rho, R] \times (z_\rho, \infty).$$

We claim that there exists $z_+^* > z_\rho$ such that (1.32) holds with $r^* = \rho$. Arguing by contradiction we suppose that there exists $z_n \to \infty$, $r_n \in (0, \rho)$ and $R_n > \frac{1}{b}$ such that, setting $\theta_n = \theta_{c_{R_n}, R_n}$,

$$\theta_n(r_n, z_n) = \frac{\pi}{2}.$$

Hence

$$\theta_n(r_n, z) \geq \frac{\pi}{2} \quad \text{if } z \leq z_n.$$

Applying, for $z_\rho < z < z_n$, Lemma 1.4 to the intervals $(0, r_n)$, (r_n, ρ) and (ρ, R_n), we find that

$$\begin{aligned} &\int_{z_\rho}^{z_n} \mathrm{e}^{c_{R_n} z}\, \mathrm{d}\, z \int_0^{R_n} \frac{r}{2} \left((\theta_n)_r^2 + \frac{\sin^2 \theta_n}{r^2} - \left(\theta_+'\right)^2 - \frac{\sin^2 \theta_+}{r^2} \right) \mathrm{d}\, r \\ &\qquad \geq \int_{z_\rho}^{z_n} 2 \left(\frac{1 - (a\rho)^2}{1 + (a\rho)^2} \right) \mathrm{e}^{c_{R_n} z}\, \mathrm{d}\, z \to \infty \quad \text{as } z_n \to \infty, \end{aligned}$$

since c_{R_n} is uniformly bounded. Hence $0 = \Phi_{c_{R_n}, R_n}(h_{c_{R_n}, R_n}) \to \infty$ as $n \to \infty$ and we have found a contradiction. $\square$

For any $M > 0$ we define L^+_M as the Hilbert space formed by all the functions f which are measurable on $\mathbb{R}^+ \times (-\infty, M)$ and for which

$$\|f\|_{M,+} := \int_{-\infty}^{M} \mathrm{d}\, z \int_0^\infty r\, \mathrm{e}^{c_R z}\, |f|^2\, \mathrm{d}\, r < \infty\,,$$

with the natural scalar product. Similarly we define the Hilbert space L^+_∞ with the norm

$$\|f\|_{\infty,+} := \int_{\mathbb{R}} \mathrm{d}\, z \int_0^\infty r\, \mathrm{e}^{c_R z}\, |f|^2\, \mathrm{d}\, r$$

In what follows we shall denote by h_R the function

$$h_R(r,z) \;=\; \begin{cases} h_{c_R,R}(r,z) & \text{if } r \le R \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1.27. *For any $R > \frac{1}{b}$ and $M > 0$ we have*

1. $\left\|\frac{\partial h_R}{\partial z}\right\|_{\infty,+} \le \sqrt{2}$
2. $\|h_R\|_{\infty,+} \le \frac{\sqrt{8}}{c_R}$
3. $\left\|\frac{\partial h_R}{\partial r}\right\|_{M,+} \le \mathcal{Q}$, $\left\|\frac{\sin(h_R)}{r}\right\|_{M,+} \le \mathcal{Q}'$

where $\mathcal{Q}$ and $\mathcal{Q}'$ are constants depending only on b, c_R and M.

Proof: It is sufficient to prove the estimates for the functions $h_{c_R,R}$. Given any $R > 1/b$, we can repeat for $h_{c_R,R}$ the same arguments used in the proof of Proposition 1.9 to obtain the estimates for a generic element h_n of a minimizing sequence. Since $\Phi_{c_R,R}(h_{c_R,R}) = \mathcal{I}_{c_R,R} = 0$, the constants $\mathcal{Q}$ and $\mathcal{Q}'$ only depend on b, c_R and M. $\square$

Theorem 1.28. *There exist $z_\infty \in \mathbb{R}$ and a function $h_\infty \in C^2((0,\infty)\times\mathbb{R}) \cap C^0([0,\infty)\times \mathbb{R}\setminus\{(0,z_\infty)\})$ such that:*
(i) h_∞ solves the differential equation

$$h_{zz} + c_\infty h_z + h_{rr} + \frac{h_r}{r} - \frac{\sin(2h+2\theta_+) - \sin(2\theta_+)}{2r^2} \;=\; 0 \quad \text{in } (0,\infty)\times\mathbb{R} \tag{1.33}$$

(ii) $h_\infty(r,z) \to 0$ as $z \to +\infty$ and $h_\infty(r,z) \to \pi - \theta_+$ as $z \to -\infty$ uniformly with respect to r;
(iii) $h_\infty(0,z) = \pi$ if $z < z_\infty$, $h_\infty(0,z) = 0$ if $z > z_\infty$;
(iv) h_∞ is strictly decreasing with respect to z in $\mathbb{R}^+\times\mathbb{R}$;
(v) h_∞ is real analytic in $[0,\infty)\times\mathbb{R}\setminus\{(0,z_\infty)\}$.

Proof: Thanks to the Lemma 1.25 and to the uniform bounds $0 \le h_R \le \pi$, through Schauder estimates we can get to say that for any $\rho > 0$ there exists a constant $K = K(\rho, b)$ such that for all $R > 1/b \quad \|h_R\|_{C^4([\rho,R]\times\mathbb{R})} \le K$. By using the previous estimate and Proposition 1.27 together with Lemma 1.25, we deduce the existence of a sequence $R_n \to \infty$ and of a function $h \in C^2(\mathbb{R}^+\times\mathbb{R})$ such that

A1) $h_{R_n} \to h$ in $C^2([\rho,\rho'] \times \mathbb{R})$ for any $0 < \rho < \rho'$;

A2) $h, h_z \in L_{\infty,+}$ and $h_r, \sin(h)r^{-1} \in L_{M,+}$ for any $M > 0$;

A3) $0 \leq h \leq \pi - \theta_+$;

A4) $h_z \leq 0$.

From (A1) follows that h is a solution of (1.33). From (A2) follows (see [30]) that for a.e. $z \in \mathbb{R}$ there exists $h(0,z) = \lim_{r\to 0} h(r,z) = k(z)\pi$ with $k(z) \in \mathbb{Z}$. In view of (A3) and (A4) only one of the following cases can occur:
(A) $h(0,z) = 0$ for all $z \in \mathbb{R}$;
(B) $h(0,z) = \pi$ for all $z \in \mathbb{R}$;
(C) there exists $\zeta \in \mathbb{R}$ such that $h(0,z) = 0$ for $z > \zeta$, $h(0,z) = \pi$ for $z < \zeta$. Since both (A) and (B) are excluded by Lemma 1.26, we conclude that (C) must occur.

The monotonicity of h with respect to z implies that $h \in C^0([0,\infty) \times \mathbb{R} \setminus \{(0,\zeta)\})$. By reasoning as in the proof of Proposition 1.24 it easily follows that $\theta := h + \theta_+$ is real analytic on the set $[0,\infty) \times \mathbb{R} \setminus \{(0,\zeta)\}$. So, the same is true for h.

By the strong maximum principle, $h_z < 0$ and $0 < h < \pi - \theta_+$ in the set $\mathbb{R}^+ \times \mathbb{R}$.

It follows from (A1) and Proposition 1.23 that $h(r,z) \to 0$ as $z \to +\infty$ and $h(r,z) \to \pi - \theta_+$ as $z \to -\infty$ uniformly in $[\rho,\rho']$, for any $0 < \rho < \rho'$. Then, (A3) and (A4) imply that in both cases the convergence is actually uniform with respect to $r \in [0,\infty)$. Setting $z_\infty = M_0$ and $h_\infty = h$ the proof is complete. □

One easily checks that c_∞ and $\theta_\infty(r,z) := h_\infty(r, z + z_\infty) + \theta_+(r)$ satisfy Theorem 1.2.

1.4 Behavior near the point of singularity

Let $b, R > 0$ be such that $bR > 1$. To simplify the notations, in what follows we shall denote by c the value $c_R > 0$ and by θ the function θ_R of Theorem 1.1. By h we shall denote the function $h(r,z) = \theta(r,z) - \theta_+(r)$ where $\theta_+(r) = 2\arctan(br)$. Let $D_1 = \{(r,z) \in (0,+\infty) \times \mathbb{R} \mid r^2 + z^2 < 1\}$. For any $\varepsilon \in (0,R)$ we define the function

$$\begin{array}{rccc} \theta_\varepsilon : & D_1 & \longrightarrow & \mathbb{R} \\ & (r,z) & \longrightarrow & \theta(\varepsilon r, \varepsilon z) \end{array}$$

We shall determine the limit behavior of θ in the neighborhood of the origin by studying the convergence properties of the sequence $\{\theta_\varepsilon\}_{\varepsilon\in(0,R)}$ as $\varepsilon \to 0$. To this aim some preliminary results are required.

For every $\rho \in (0,R]$ we define

$$D_\rho = \{(r,z) \in (0,+\infty) \times \mathbb{R} \mid r^2 + z^2 < \rho\}, \quad \partial^+ D_\rho = \partial D_\rho \cap \{r > 0\},$$

$$H_\rho \equiv \left\{ w \in L^2_r(D_\rho) \mid \exists w_r, w_z \in L^2_r(D_\rho), \frac{\sin w}{r} \in L^2_r(D_\rho) \right\}$$

and for $v \in H_\rho$:

$$E_\rho(v) = \iint_{D_\rho} \frac{r\, \mathrm{e}^{cz}}{2} \left(v_r^2 + v_z^2 + \frac{\sin^2 v}{r^2} \right) \mathrm{d}\, r\, \mathrm{d}\, z\,.$$

Lemma 1.29. *For every* $\rho \in (0, R]$

$$E_\rho(\theta) = \inf_{v \in H_\rho,\, v|_{\partial^+ D_\rho} = \theta|_{\partial^+ D_\rho}} E_\rho(v)$$

Proof: By contradiction, let $v \in H_\rho$ be such that $v|_{\partial^+ D_\rho} = \theta|_{\partial^+ D_\rho}$ and $E_\rho(v) < E_\rho(\theta)$. If we define, for $(r, z) \in [0, R] \times \mathbb{R}$,

$$\tilde{\theta}(r, z) = \begin{cases} v(r, z) & \text{if } r^2 + z^2 < \rho^2 \\ \theta(r, z) & \text{if } r^2 + z^2 \geq \rho^2 \end{cases}$$

and $\tilde{h}(r, z) = \tilde{\theta}(r, z) - \theta_+(r)$, then we have that $\tilde{h} \in Y_{c,R}$ (see section 1.1) and $\Gamma_{c,R}(\tilde{h}) < \infty$. Moreover, by using Proposition (1.5) we find that

$$\Phi_{c,R}(\tilde{h}) = \int_{\mathbb{R}} \mathrm{d}\, z \int_0^R \frac{r\, \mathrm{e}^{cz}}{2} \left(\tilde{\theta}_r^2 + \tilde{\theta}_z^2 + \frac{\sin^2 \tilde{\theta}}{r^2} - (\theta_+')^2 - \frac{\sin^2 \theta_+}{r^2} \right) \mathrm{d}\, r =$$

$$\iint_{\{r^2+z^2<\rho^2\}} \frac{r\, \mathrm{e}^{cz}}{2} \left(v_r^2 + v_z^2 + \frac{\sin^2 v}{r^2} - (\theta_+')^2 - \frac{\sin^2 \theta_+}{r^2} \right) \mathrm{d}r\, \mathrm{d}z$$

$$+ \iint_{\{r^2+z^2\geq\rho^2\}\cap[0,R]\times\mathbb{R}} \frac{r\, \mathrm{e}^{cz}}{2} \left(\theta_r^2 + \theta_z^2 + \frac{\sin^2 \theta}{r^2} - (\theta_+')^2 - \frac{\sin^2 \theta_+}{r^2} \right) \mathrm{d}\, r\, \mathrm{d}\, z \Longrightarrow$$

$$\Phi_{c,R}(\tilde{h}) - \Phi_{c,R}(h) = E_\rho(v) - E_\rho(\theta) < 0\,.$$

Thanks to Proposition 1.14 we have that, for a suitable $z_R \in \mathbb{R}$, $h(r, z - z_R)$ is a solution to problem (MP) with $c = c_R$ and then $\Phi_{c,R}(h) = 0$. From the previous inequality we derive $\Phi_{c,R}(\tilde{h}) < 0$, while from Corollary 1.13 we know that $\Phi_{c,R}(\tilde{h}) \geq 0$. Hence we have found a contradiction. $\square$

In the following we shall denote by ν and τ the following vector fields defined in $\mathbb{R}^2 \setminus \{0\}$:

$$\nu(r, z) = \left(\frac{r}{\sqrt{r^2 + z^2}}, \frac{z}{\sqrt{r^2 + z^2}} \right), \quad \tau(r, z) = \left(\frac{-z}{\sqrt{r^2 + z^2}}, \frac{r}{\sqrt{r^2 + z^2}} \right).$$

From lemma 1.29 we derive that

Lemma 1.30. *For every* $\rho \in (0, R]$

$$\iint_{D_\rho} (1 + cz) \frac{r\, \mathrm{e}^{cz}}{2} \left(\theta_r^2 + \theta_z^2 + \frac{\sin^2 \theta}{r^2} \right) \mathrm{d}\, r\, \mathrm{d}\, z =$$

$$\rho \left(\int_{\partial^+ D_\rho} \frac{r\, \mathrm{e}^{cz}}{2} \left(\theta_r^2 + \theta_z^2 + \frac{\sin^2 \theta}{r^2} \right) - \int_{\partial^+ D_\rho} r\, \mathrm{e}^{cz} \left| \nabla \theta \cdot \nu \right|^2 \right) \tag{1.34}$$

Proof: The idea of this proof comes from [10], pag. 102 (even if the author works with vector fields instead of angle functions). For every $\rho \in (0, R)$ and $k > 0$ such that $[\rho, \rho + k] \subset (0, R)$ we define

$$\phi_{\rho,k}(s) = \begin{cases} 1 & \text{if } s \leq \rho \\ 1 - \frac{s-\rho}{k} & \text{if } s \in (\rho, \rho + k) \\ 0 & \text{if } s \geq \rho + k\,. \end{cases}$$

We now freeze the values of ρ and k. Then we may simplify the notations by writing ϕ in place of $\phi_{\rho,k}$. Let $\delta = \delta(\rho,k) \in (0,1)$ be a positive number such that $\delta < \frac{R}{\rho+k} - 1$ and let I be the open interval $(-\delta,\delta)$. For every $t \in I$ and $(r,z) \in D_{\rho+k}$ we set

$$\lambda(r,z,t) \;=\; 1 + t\phi(\sqrt{r^2+z^2}), \quad \theta_t(r,z) \;=\; \theta(\lambda(r,z,t)r, \lambda(r,z,t)z)\,.$$

We remark that λ is bounded together with its first derivatives. A simple computation shows that $\forall t \in I$:

$$E_\rho(\theta_t) \;=\; \iint_{D_{(1+t)\rho}} \frac{r\,\mathrm{e}^{\frac{cz}{1+t}}}{2(1+t)}\left(\theta_r^2 + \theta_z^2 + \frac{\sin^2\theta}{r^2}\right)\mathrm{d}\,r\,\mathrm{d}\,z < \infty$$

while $E_{\rho+k}(\theta_t) - E_\rho(\theta_t) \le C < \infty$, where C is a constant depending on ρ and k. The last inequality comes from the regularity of θ outside the origin. Then, for every $t \in I$ $\theta_t \in H_{\rho+k}$ and $\theta_t|_{\partial^+ D_{\rho+k}} = \theta|_{\partial^+ D_{\rho+k}}$. Thanks to Lemma 1.29 we have that $\forall t \in I$ $E_{\rho+k}(\theta) \le E_{\rho+k}(\theta_t)$.

We want to show now that $F(t) \equiv E_{\rho+k}(\theta_t)$ is differentiable in $t=0$. Since $F(t) = E_\rho(\theta_t) + G_1(t) + G_2(t) + G_3(t)$ with

$$G_1(t) \;\equiv\; \iint_{D_{\rho+k}\setminus D_\rho} \frac{r\,\mathrm{e}^{cz}}{2}\left|\frac{\partial\theta_t}{\partial r}\right|^2 \mathrm{d}\,r\,\mathrm{d}\,z,\; G_2(t) \;\equiv\; \iint_{D_{\rho+k}\setminus D_\rho} \frac{r\,\mathrm{e}^{cz}}{2}\left|\frac{\partial\theta_t}{\partial z}\right|^2 \mathrm{d}\,r\,\mathrm{d}\,z$$

$$\text{and} \qquad G_3(t) \;\equiv\; \iint_{D_{\rho+k}\setminus D_\rho} \frac{r\,\mathrm{e}^{cz}}{2}\frac{\sin^2\theta_t}{r^2}\,\mathrm{d}\,r\,\mathrm{d}\,z$$

we are led to prove that F, G_1, G_2 and G_3 are differentiable in 0. Since λ is bounded and Lipschitz continuous with $\lambda \in [1-\delta, 1+\delta] \subset (0,2)$ and θ is smooth outside the origin, the differentiability of G_i ($i=1,2,3$) in 0 is obvious and a straightforward computation shows that

$$\begin{aligned} G_1'(0) \;=\; \iint_{D_{\rho+k}\setminus D_\rho} \frac{r\,\mathrm{e}^{cz}}{2}\Bigg\{2\theta_r\Bigg(&\phi(\sqrt{r^2+z^2})\,(\theta_r + r\theta_{rr} + z\theta_{rz}) \\ &+\phi'(\sqrt{r^2+z^2})\frac{r(r\theta_r + z\theta_z)}{\sqrt{r^2+z^2}}\Bigg)\Bigg\}\,\mathrm{d}\,r\,\mathrm{d}\,z\,, \\ G_2'(0) \;=\; \iint_{D_{\rho+k}\setminus D_\rho} \frac{r\,\mathrm{e}^{cz}}{2}\Bigg\{2\theta_z\Bigg(&\phi(\sqrt{r^2+z^2})\,(\theta_z + r\theta_{zr} + z\theta_{zz}) \\ &+\phi'(\sqrt{r^2+z^2})\frac{z(r\theta_r + z\theta_z)}{\sqrt{r^2+z^2}}\Bigg)\Bigg\}\,\mathrm{d}\,r\,\mathrm{d}\,z \end{aligned}$$

and

$$G_3'(0) \;=\; \iint_{D_{\rho+k}\setminus D_\rho} \frac{r\,\mathrm{e}^{cz}}{2}\phi(\sqrt{r^2+z^2})\frac{\sin(2\theta)}{r}\,(r\theta_r + z\theta_z)\,\mathrm{d}\,r\,\mathrm{d}\,z\,.$$

On the other hand, if we define

$$f(r,z,t) = \frac{r\,\mathrm{e}^{\frac{cz}{1+t}}}{2(1+t)}\left(\theta_r^2 + \theta_z^2 + \frac{\sin^2\theta}{r^2}\right)\,,$$

then for every $t \in I$

$$E_\rho(\theta_t) = \iint_{D_{(1+t)\rho} \setminus D_\rho} f(r,z,t)\,\mathrm{d}\,r\,\mathrm{d}\,z + \iint_{D_\rho} f(r,z,t)\,\mathrm{d}\,r\,\mathrm{d}\,z$$

and therefore, if $t \in I \setminus \{0\}$:

$$\frac{E_\rho(\theta_t) - E_\rho(\theta_0)}{t} = \frac{1}{t}\int_\rho^{(1+t)\rho} \mathrm{d}\,\sigma \int_{-\pi/2}^{\pi/2} \sigma f(\sigma\cos\varphi, \sigma\sin\varphi, t)\,\mathrm{d}\,\varphi +$$

$$+\frac{1}{t}\left(\iint_{D_\rho} f(r,z,t)\,\mathrm{d}\,r\,\mathrm{d}\,z - \iint_{D_\rho} f(r,z,0)\,\mathrm{d}\,r\,\mathrm{d}\,z\right).$$

Since $I = (-\delta,\delta)$ with $\delta \in (0,1)$, $f(\sigma\cos\varphi, \sigma\sin\varphi, t)$ is continuous in $t \in I$ uniformly with respect to $\sigma \in [(1-\delta)\rho, (1+\delta)\rho]$ and $\varphi \in [-\pi/2, \pi/2]$. Then

$$\lim_{t\to 0}\left(\frac{1}{t}\int_\rho^{(1+t)\rho} \mathrm{d}\,\sigma \int_{-\pi/2}^{\pi/2} \sigma f(\sigma\cos\varphi, \sigma\sin\varphi, t)\,\mathrm{d}\,\varphi\right) =$$

$$\lim_{t\to 0}\left(\frac{1}{t}\int_\rho^{(1+t)\rho}\left(\int_{\partial^+ D_\sigma} f(r,z,0)\right)\mathrm{d}\,\sigma\right) = \rho\int_{\partial^+ D_\rho} f(r,z,0)\,.$$

At the same time, the function

$$t \longrightarrow \iint_{D_\rho} f(r,z,t)\,\mathrm{d}\,r\,\mathrm{d}\,z$$

can be derived under the integral thanks to the Lebesgue's theorem. Therefore

$$\lim_{t\to 0}\left(\frac{1}{t}\left(\iint_{D_\rho} f(r,z,t)\,\mathrm{d}\,r\,\mathrm{d}\,z - \iint_{D_\rho} f(r,z,0)\,\mathrm{d}\,r\,\mathrm{d}\,z\right)\right) =$$

$$\iint_{D_\rho} \frac{\partial f}{\partial t}(r,z,0)\,\mathrm{d}\,r\,\mathrm{d}\,z = -\iint_{D_\rho}(1+cz)\frac{r\,\mathrm{e}^{cz}}{2}\left(\theta_r^2 + \theta_z^2 + \frac{\sin^2\theta}{r^2}\right)\mathrm{d}\,r\,\mathrm{d}\,z\,.$$

The differentiability of F in 0 together with the inequality $F(t) = E_{\rho+k}(\theta_t) \geq E_{\rho+k}(\theta) = F(0)$ for any $t \in I$ implies that

$$0 = \frac{dF}{dt}(0) = G_1^{'}(0;k) + G_2^{'}(0;k) + G_3^{'}(0;k) +$$

$$-\iint_{D_\rho}(1+cz)\frac{r\,\mathrm{e}^{cz}}{2}\left(\theta_r^2 + \theta_z^2 + \frac{\sin^2\theta}{r^2}\right)\mathrm{d}\,r\,\mathrm{d}\,z + \rho\int_{\partial^+ D_\rho} f(r,z,0)\,. \tag{1.35}$$

Here we have written $G_i^{'}(0;k)$ in place of $G_i^{'}(0)$ to highlight the parametric dependence of these quantities on k. If we leave ρ fixed and we consider k as an independent variable varying in an interval $(0, K(\rho))$, we can pass to the limit in (1.35) as $k \to 0^+$. Since $\lim_{k\to 0^+} G_3^{'}(0;k) = 0$ and

$$\lim_{k\to 0^+} G_1^{'}(0;k) = -\int_{\partial^+ D_\rho} r^2\,\mathrm{e}^{cz}\,\theta_r \nabla\theta\cdot\nu\,,$$

$$\lim_{k\to 0^+} G_2'(0;k) = -\int_{\partial^+ D_\rho} rz\,\mathrm{e}^{cz}\,\theta_z \nabla\theta\cdot\nu$$

we obtain the identity (1.34) for $\rho \in (0,R)$. But, thanks to the continuity of θ, (1.34) holds true for $\rho = R$. □

Thanks to the previous lemma we can now prove an adaptation to our traveling waves of the classical monotonicity formula for stationary harmonic maps. From this formula we shall derive that the quantity $E_1(\theta_\varepsilon)$ is a bounded function of ε for $\varepsilon \to 0^+$. To simplify the statement of the next result we define the following functions of $\rho \in (0,R]$:

$$\mathcal{G}(\rho) = \int_{\partial^+ D_\rho} \frac{r\,\mathrm{e}^{cz}}{2}\left(\theta_r^2 + \theta_z^2 + \frac{\sin^2\theta}{r^2}\right), \quad \mathcal{N}(\rho) = \int_{\partial^+ D_\rho} \frac{r\,\mathrm{e}^{cz}}{2}\left|\nabla\theta\cdot\nu\right|^2,$$

$$\mathcal{F}(\rho) = \int_0^\rho \mathcal{G}(\sigma)\,\mathrm{d}\,\sigma, \quad \mathcal{M}(\rho) = \int_0^\rho \mathcal{N}(\sigma)\,\mathrm{d}\,\sigma\,.$$

Remark 1.31. It is easy to check that for every $\rho \in (0,R]$:

$$\mathcal{G}(\rho) = \int_{\partial^+ D_1} \frac{r\,\mathrm{e}^{c\rho z}}{2}\left(|\nabla\theta_\rho|^2 + \frac{\sin^2\theta_\rho}{r^2}\right),$$

$$\mathcal{N}(\rho) = \int_{\partial^+ D_1} \frac{r\,\mathrm{e}^{c\rho z}}{2}\left|\nabla\theta_\rho\cdot\nu\right|^2,$$

$$\mathcal{F}(\rho) = E_\rho(\theta) = \rho \iint_{D_1} \frac{r\,\mathrm{e}^{c\rho z}}{2}\left(|\nabla\theta_\rho|^2 + \frac{\sin^2\theta_\rho}{r^2}\right)\mathrm{d}\,r\,\mathrm{d}\,z\,,$$

$$\mathcal{M}(\rho) = \iint_{D_\rho} \frac{r\,\mathrm{e}^{cz}}{2}\left|\nabla\theta\cdot\nu\right|^2\mathrm{d}\,r\,\mathrm{d}\,z = \rho\iint_{D_1} \frac{r\,\mathrm{e}^{c\rho z}}{2}\left|\nabla\theta_\rho\cdot\nu\right|^2\mathrm{d}\,r\,\mathrm{d}\,z\,.$$

Lemma 1.32. *One has that:*
(i) $\frac{d}{d\rho}\left(\frac{\mathrm{e}^{c\rho}\mathcal{F}(\rho)}{\rho}\right) \geq 0$,
(ii) $\frac{F(\rho)}{\rho}$ *is a bounded function,*
(iii) $\exists\ \lim_{\rho\to 0^+}\frac{\mathcal{F}(\rho)}{\rho} \in [0,\infty)$,
(iv) $\lim_{\rho\to 0^+}\frac{\mathcal{M}(\rho)}{\rho} = 0$.

Proof: Since (ii) and (iii) easily follow from (i), we only have to prove (i) and (iv). From (1.34) we derive that for every $\rho \in (0,R]$:

$$(1+c\rho)\frac{\mathcal{F}(\rho)}{\rho} \geq \mathcal{G}(\rho) - 2\mathcal{N}(\rho) \geq (1-c\rho)\frac{\mathcal{F}(\rho)}{\rho}\,.$$

Since $\mathcal{G} \equiv \mathcal{F}'$ we get

$$\mathcal{F}'(\rho) \geq (1-c\rho)\frac{\mathcal{F}(\rho)}{\rho} \Rightarrow \frac{d}{d\rho}\left(\frac{\mathcal{F}(\rho)}{\rho}\right) + c\frac{\mathcal{F}(\rho)}{\rho} \geq 0\,,$$

and (i) follows. Let $K > 0$ such that $\frac{\mathcal{F}(\rho)}{\rho} \leq K$. We have that

$$\mathcal{G}(\rho) - 2\mathcal{N}(\rho) \geq (1-c\rho)\frac{\mathcal{F}(\rho)}{\rho} \quad \forall\rho\in(0,R] \Rightarrow$$

$$\mathcal{F}(\rho) - \int_0^\rho \frac{\mathcal{F}(\sigma)}{\sigma}\,\mathrm{d}\,\sigma + c\int_0^\rho \mathcal{F}(\sigma)\,\mathrm{d}\,\sigma \geq 2\mathcal{M}(\rho) \Rightarrow$$

$$\frac{\mathcal{F}(\rho)}{\rho} - \frac{1}{\rho}\int_0^\rho \frac{\mathcal{F}(\sigma)}{\sigma}\,\mathrm{d}\,\sigma + \frac{cK\rho}{2} \geq 2\frac{\mathcal{M}(\rho)}{\rho} \geq 0\,. \tag{1.36}$$

Because of (i) the function $\frac{\mathcal{F}(\rho)}{\rho}$ can be extended to a bounded continuous function over $[0, R]$. Then

$$\lim_{\rho\to 0}\left(\frac{\mathcal{F}(\rho)}{\rho} - \frac{1}{\rho}\int_0^\rho \frac{\mathcal{F}(\sigma)}{\sigma}\,\mathrm{d}\,\sigma\right) = 0$$

and, thanks to (1.36), we obtain (iv).

In the remainder of this section we shall denote by $\mathcal{E}$ the following functional defined for $v \in H_1$:

$$\mathcal{E}(v) = \iint_{D_1} \frac{r}{2}\left(v_r^2 + v_z^2 + \frac{\sin^2 v}{r^2}\right)\mathrm{d}\,r\,\mathrm{d}\,z \tag{1.37}$$

Lemma (1.32) and Remark (1.31) imply that

Proposition 1.33. *(i) There exists $K \in \mathbb{R}^+$ such that $\mathcal{E}(\theta_\varepsilon) \leq K$ $\forall \varepsilon \in (0, R)$.*
(ii)

$$\lim_{\varepsilon\to 0^+} \iint_{D_1} \frac{r}{2}\,|\nabla\theta_\varepsilon \cdot \nu|^2\,\mathrm{d}\,r\,\mathrm{d}\,z = 0\,.$$

Moreover, we can easily prove that

Proposition 1.34. *For every $\rho \in (0, R)$ there exists $C = C(\rho)$ such that $\|\theta_\varepsilon\|_{C^3([\rho,R]\times\mathbb{R})} \leq C$ for all $\varepsilon \in (0, 1)$.*

Proof: For all $\varepsilon \in (0, 1)$ we have $0 \leq \theta_\varepsilon \leq \pi$ in $[0, R/\varepsilon] \times \mathbb{R}$. Moreover, θ_ε solves the equation

$$\psi_{rr} + \frac{\psi_r}{r} - \frac{\sin(2\psi)}{2r^2} + \psi_{zz} + c\varepsilon\psi_z = 0$$

in $(0, R/\varepsilon) \times \mathbb{R}$. At last, $\theta_\varepsilon(R/\varepsilon, z) = 2\arctan(bR)$. By using classical Schauder type estimates and the invariance of the previous equation with respect to z-translations we obtain the thesis. □

In order to state the first important theorem of this section, concerning the behavior of the sequence $\{\theta_\varepsilon\}$ for ε approaching to zero, we need a last lemma:

Lemma 1.35. *Let H_1^0 be the closed subspace of H_1 given by*

$$\{v \in H_1 \mid v|_{\partial^+ D_1} = 0\}\,.$$

For every $f \in H_1^0$ and $\varepsilon \in (0, R)$

$$|\mathcal{E}_\varepsilon(f + \theta_\varepsilon) - \mathcal{E}(f + \theta_\varepsilon)| \leq c\varepsilon\,\mathrm{e}^{c\varepsilon}\,Q(f)\,,$$

where

$$\mathcal{E}_\varepsilon(v) \equiv \iint_{D_1} \frac{r\,\mathrm{e}^{c\varepsilon z}}{2}\left(v_r^2 + v_z^2 + \frac{\sin^2 v}{r^2}\right)\mathrm{d}\,r\,\mathrm{d}\,z$$

for $v \in H_1$ and $Q(f) > 0$ is a constant depending only on f.

Proof: By using the standard inequality $|\,\mathrm{e}^{c\varepsilon z}-1| \le \mathrm{e}^{c\varepsilon}\,|c\varepsilon z|$ we get $|\mathcal{E}_\varepsilon(f+\theta_\varepsilon)-\mathcal{E}(f+\theta_\varepsilon)| \le \mathrm{e}^{c\varepsilon}\, c\varepsilon\, \mathcal{E}(f+\theta_\varepsilon) \le 2\,\mathrm{e}^{c\varepsilon}\, c\varepsilon(\mathcal{E}(f)+\mathcal{E}(\theta_\varepsilon))$ and the thesis follows then from Proposition (1.33). $\square$

Theorem 1.36. *There exist $\psi \in H_1$ and a decreasing sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0, \min\{1, R\})$ with $\varepsilon_n \to 0$ as $n \to \infty$ such that:*
(i)

$$\nabla\theta_{\varepsilon_n} \rightharpoonup \nabla\psi\,, \qquad \frac{\sin\theta_{\varepsilon_n}}{r} \rightharpoonup \frac{\sin\psi}{r}$$

in $L^2_r(D_1)$;
(ii) for every $\rho \in (0,1)$

$$\theta_{\varepsilon_n} \to \psi$$

in $C^2(\overline{D}_1 \cap \{r \ge \rho\})$.
Moreover, if $\{\varepsilon_n\}$ is any sequence converging to 0 for which (i) and (ii) are true, then the limit function $\psi \in H_1$ and satisfies:
(iii) $\psi(\overline{D}_1 \cap \{r > 0\}) \subseteq [0,\pi]$ and $\psi_z \le 0$;
(iv)

$$\iint_{D_1} \frac{r}{2}\,|\nabla\psi\cdot\nu|^2\,\mathrm{d}\,r\,\mathrm{d}\,z = 0\,;$$

(v)

$$\mathcal{E}(\psi) = \inf_{\{v\in H_1\,|\,v|_{\partial^+D_1}=\psi|_{\partial^+D_1}\}} \mathcal{E}(v)\,.$$

Proof: (i) and (ii) easily follow from Propositions (1.33) and (1.34).
(iii) follows from the inequalities:

$$2\arctan(b\varepsilon r) \le \theta_\varepsilon(r,z) = \theta(\varepsilon r, \varepsilon z) \le \pi - 2\arctan\left(\frac{\varepsilon r}{bR^2}\right)$$

and $\frac{\partial\theta_\varepsilon}{\partial z}(r,z) < 0$ for $(r,z) \in (0, \frac{R}{\varepsilon}) \times \mathbb{R}$. (iv) is a consequence of Proposition (1.33). To conclude we only need to prove that $\mathcal{E}(\psi) \le \mathcal{E}(v)$ for every $v \in H_1$ with $v|_{\partial^+D_1} = \psi|_{\partial^+D_1}$. If we write $v = f + \psi$, then $f \in H_1^0$ and

$$\mathcal{E}(v) - \mathcal{E}(\psi) = \frac{1}{2}\|\nabla f\|^2 + \langle\nabla f, \nabla\psi\rangle + \iint_{D_1} \frac{\sin^2(f+\psi) - \sin^2\psi}{2r}\,\mathrm{d}\,r\,\mathrm{d}\,z \tag{1.38}$$

where the norm $\|\cdot\|$ and the scalar product $\langle\cdot\,,\cdot\rangle$ are those ones of $L^2_r(D_1)$. Similarly, for every $n \in \mathbb{N}$

$$\mathcal{E}(f+\theta_{\varepsilon_n}) - \mathcal{E}(\theta_{\varepsilon_n}) = \frac{1}{2}\|\nabla f\|^2 + \langle\nabla f, \nabla\theta_{\varepsilon_n}\rangle + \iint_{D_1} \frac{\sin^2(f+\theta_{\varepsilon_n}) - \sin^2\theta_{\varepsilon_n}}{2r}\,\mathrm{d}\,r\,\mathrm{d}\,z \tag{1.39}$$

Thanks to simple trigonometric identities, we can write

$$\iint_{D_1} \frac{\sin^2(f+\psi) - \sin^2\psi}{2r}\,\mathrm{d}\,r\,\mathrm{d}\,z = I_1 + I_2$$

and, for each $n \in \mathbb{N}$,

$$\iint_{D_1} \frac{\sin^2(f+\theta_{\varepsilon_n}) - \sin^2\theta_{\varepsilon_n}}{2r}\,\mathrm{d}\,r\,\mathrm{d}\,z = I_{1,n} + I_{2,n}$$

where

$$I_1 = \iint_{D_1} \frac{\sin^2 f \cos(2\psi)}{2r} \,\mathrm{d}\, r \,\mathrm{d}\, z,$$

$$I_2 = \iint_{D_1} \frac{\sin f \sin\psi \cos f \cos\psi}{r} \,\mathrm{d}\, r \,\mathrm{d}\, z,$$

$$I_{1,n} = \iint_{D_1} \frac{\sin^2 f \cos(2\theta_{\varepsilon_n})}{2r} \,\mathrm{d}\, r \,\mathrm{d}\, z \quad \text{and}$$

$$I_{2,n} = \iint_{D_1} \frac{\sin f \sin\theta_{\varepsilon_n} \cos f \cos\theta_{\varepsilon_n}}{r} \,\mathrm{d}\, r \,\mathrm{d}\, z.$$

Thanks to (ii) we have $I_{1,n} \to I_1$ for $n \to \infty$. (ii) also implies that

$$\frac{\sin f \cos f \cos\theta_{\varepsilon_n}}{\sqrt{r}} \to \frac{\sin f \cos f \cos\psi}{\sqrt{r}}$$

in $L^2(D_1)$ as $n \to \infty$. On the other hand, (i) implies that

$$\frac{\sin\theta_{\varepsilon_n}}{\sqrt{r}} \rightharpoonup \frac{\sin\psi}{\sqrt{r}}$$

in $L^2(D_1)$. Therefore, $I_{2,n} \to I_2$ for $n \to \infty$. But then from (i), (1.38) and (1.39) we get that $\mathcal{E}(f + \theta_{\varepsilon_n}) - \mathcal{E}(\theta_{\varepsilon_n}) \to \mathcal{E}(v) - \mathcal{E}(\psi)$ when $n \to \infty$. On the other hand, for every $n \in \mathbb{N}$

$$\begin{aligned} &\mathcal{E}(f + \theta_{\varepsilon_n}) - \mathcal{E}(\theta_{\varepsilon_n}) = \\ &\mathcal{E}(f + \theta_{\varepsilon_n}) - \mathcal{E}_{\varepsilon_n}(f + \theta_{\varepsilon_n}) + \mathcal{E}_{\varepsilon_n}(f + \theta_{\varepsilon_n}) - \mathcal{E}_{\varepsilon_n}(\theta_{\varepsilon_n}) + \mathcal{E}_{\varepsilon_n}(\theta_{\varepsilon_n}) - \mathcal{E}(\theta_{\varepsilon_n}) \geq \\ &\mathcal{E}(f + \theta_{\varepsilon_n}) - \mathcal{E}_{\varepsilon_n}(f + \theta_{\varepsilon_n}) + \mathcal{E}_{\varepsilon_n}(\theta_{\varepsilon_n}) - \mathcal{E}(\theta_{\varepsilon_n}) \end{aligned} \tag{1.40}$$

since Lemma (1.29) implies $\mathcal{E}_{\varepsilon_n}(f + \theta_{\varepsilon_n}) \geq \mathcal{E}_{\varepsilon_n}(\theta_{\varepsilon_n})$. Then $\mathcal{E}(v) \geq \mathcal{E}(\psi)$ immediately follows from Lemma (1.35).

In the following we shall denote by $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0, R)$ a sequence converging to 0 for which the corresponding sequence $\{\theta_{\varepsilon_n}\}$ satisfies the statements (i) and (ii) of Theorem 1.36. Our purpose now is to show that

Claim 1.37. *The limit function ψ is given by the formula*

$$\psi(r, z) = \frac{\pi}{2} - \arctan\left(\frac{z}{r}\right).$$

The first step in this direction is given by

Proposition 1.38. *The following three cases can occur:*

A) $\psi \equiv 0$,

B) $\psi \equiv \pi$ *or*

C) $\psi \in C^0(\overline{D}_1 \setminus \{(0,0)\})$ *and*

$$\psi(0, z) = \begin{cases} 0 & \text{if } z > 0 \\ \pi & \text{if } z < 0\,. \end{cases}$$

In addition ψ is a function attaining values in $[0,\pi]$, which is smooth in $\overline{D}_1 \cap \{r>0\}$, non increasing with respect to z and constant along each radius coming out of the origin.

Proof: The last part of the statement is a direct consequence of Theorem 1.36. Since $\psi \in H_1$ and $0 \le \psi \le \pi$ we have that for a.e. $z \in (-1,1)$ exists $\psi(0,z) \equiv \lim_{r\to 0^+} \psi(r,z) \in \{0,\pi\}$. Since $\psi_z \le 0$, we can have $\psi(0,z)=0$ for all $z \in (-1,1)$, or $\psi(0,z)=\pi$ for all $z \in (-1,1)$, or

$$\psi(0,z) = \begin{cases} 0 & \text{if } z > \bar{z} \\ \pi & \text{if } z < \bar{z} \end{cases}$$

for a suitable $\bar{z} \in (0,\pi)$. In the first case, being ψ smooth in $\overline{D}_1 \cap \{r>0\}$ with $\psi_z \le 0$, one has $\psi \in C^0(\overline{D}_1)$. Then, since ψ must be constant along each radius coming out of the origin, we get $\psi \equiv 0$.

The same argument allows to say that in the second case $\psi \equiv \pi$.

In the last case the smoothness of ψ in $\overline{D}_1 \cap \{r>0\}$ and its monotonicity with respect to z permit to deduce that $\psi \in C^0(\overline{D}_1 \setminus \{(0,\bar{z})\})$. Then, since ψ must be constant along each radius coming out of the origin, $\bar{z}>0$ would imply $\psi \equiv \pi$ while $\bar{z}<0$ would imply $\psi \equiv 0$. Therefore $\bar{z}=0$. $\square$

To prove Claim 1.37 we first need to exclude the cases $\psi \equiv 0$ and $\psi \equiv \pi$. Since the arguments needed to prove that $\psi \not\equiv 0$ are similar to the ones used to show that $\psi \not\equiv \pi$, we shall only prove the latter assertion.

Proposition 1.39. *$\psi \not\equiv \pi$.*

Proof: By contradiction we assume $\psi \equiv \pi$. Let $0 < \bar{\rho} \ll 1$ be a fixed value and let σ, ζ be two positive numbers such that $\sigma = \frac{\zeta}{8}$, $\zeta \le \frac{\log 2}{2c}$ and $\sigma + \zeta \le \sqrt{1-\bar{\rho}^2}$. For every $z \in (-\zeta-\sigma, \zeta+\sigma)$ we define:

$$\alpha(z) \;=\; \begin{cases} \sigma^2(\sigma+\zeta-z)^{-2} & \text{if } z \in [\zeta, \zeta+\sigma) \\ 1 & \text{if } z \in [-\zeta,\zeta] \\ \sigma^2(\sigma+\zeta+z)^{-2} & \text{if } z \in (-\zeta-\sigma, -\zeta] \end{cases}$$

and for $(r,z) \in [0,\bar{\rho}] \times [-1,1]$:

$$\omega(r,z) = \begin{cases} 0 & \text{if } z \ge \zeta+\sigma \\ \pi - 2\arctan(\alpha(z) r) & \text{if } z \in (-\zeta-\sigma, \zeta+\sigma) \\ 0 & \text{if } z \le -\zeta-\sigma\,. \end{cases}$$

Thanks to Theorem 1.36 we know that, for a suitable decreasing sequence $\{\varepsilon_n\}$ which converges to zero as $n \to \infty$, we have $\theta_{\varepsilon_n} \to \pi$ in $C^2(\overline{D}_1 \cap \{r \ge \rho\})$ for every $\rho > 0$. Therefore, if we take $\rho \in (0,\bar{\rho}) \subset (0,1)$, we can say that there exists $\nu = \nu(\rho) \in \mathbb{N}$ such that $\forall n \ge \nu(\rho)$

$$\pi \ge \theta_{\varepsilon_n}(r,z) \ge \pi - 2\arctan\rho$$

for all $(r,z) \in \overline{D}_1 \cap \{r \ge \rho\}$. Then for every $n \ge \nu(\rho)$

$$\pi \ge \theta_{\varepsilon_n}(\rho,z) \ge \pi - 2\arctan\rho \tag{1.41}$$

$\forall z \in [-\sqrt{1-\rho^2}, \sqrt{1-\rho^2}]$. For every $\rho \in (0,\bar\rho)$ and $n \geq \nu(\rho)$ we define a function $v_{\rho,n}$ in the following way:

$$v_{\rho,n}(r,z) = \begin{cases} \max\{\omega(r,z)\,,\,\theta_{\varepsilon_n}(r,z)\} & \text{if } r<\rho \\ \theta_{\varepsilon_n}(r,z) & \text{if } r \geq \rho \end{cases}$$

at each point $(r,z) \in \overline{D}_1$. Thanks to (1.41) we can say that $v_{\rho,n}$ is a continuous function out of $\{r=0\}$. Moreover, $\forall r \in (0,\rho)$ we have $\omega(r,\sqrt{1-r^2}) = 0$, since $\sqrt{1-r^2} > \sqrt{1-\rho^2} > \sqrt{1-\bar\rho^2} \geq \sigma+\zeta$, and

$$v_{\rho,n}|_{\partial^+ D_1} = \theta_{\varepsilon_n}|_{\partial^+ D_1}\,.$$

At last, it is easy to check that $\omega, \omega_r, \omega_z, \frac{\sin\omega}{r} \in L^2_r([0,\bar\rho]\times[-1,1])$:

1. $0 \leq \omega \leq \pi$,
2. $\forall \rho \in (0,\bar\rho]$

$$\int_0^\rho \frac{r}{2}\left(\omega_r^2 + \frac{\sin^2\omega}{r^2}\right)\bigg|_z \,\mathrm{d}\,r = \begin{cases} 0 & \text{if } |z| > \sigma+\zeta \\ \frac{2\rho^2}{1+\rho^2} & \text{if } z \in (-\zeta,\zeta) \\ \frac{2\alpha^2(z)\rho^2}{1+\alpha^2(z)\rho^2} & \text{if } z \in (-\zeta-\sigma,-\zeta)\cup(\zeta,\zeta+\sigma)\,, \end{cases}$$

3. $\forall \rho \in (0,\bar\rho]$

$$\int_0^\rho \frac{r}{2}\omega_z^2(r,z)dr = \begin{cases} 0 & \text{if } |z| > \zeta+\sigma \text{ or } |z|<\zeta \\ \frac{|\alpha'(z)|^2}{\alpha^4(z)}\left(\log(1+\alpha^2\rho^2) - \frac{\alpha^2\rho^2}{1+\alpha^2\rho^2}\right) & \text{if } z \in (-\zeta-\sigma,-\zeta)\cup(\zeta,\zeta+\sigma)\,. \end{cases}$$

We remark that, thanks to the inequality $\log(1+x) \leq \sqrt{x}$:

$$\frac{|\alpha'(z)|^2}{\alpha^4(z)}\left(\log(1+\alpha^2\rho^2) - \frac{\alpha^2\rho^2}{1+\alpha^2\rho^2}\right) \leq \frac{|\alpha'(z)|^2}{\alpha^3(z)}\rho = \frac{4\rho}{\sigma^2}\,.$$

Thanks to the properties of ω we can say that $\forall \rho \in (0,\bar\rho)$ and $n \geq \nu(\rho)$ the function $v_{\rho,n} \in H_1$ and

$$\mathcal{E}_{\varepsilon_n}(v_{\rho,n}) - \mathcal{E}_{\varepsilon_n}(\theta_{\varepsilon_n}) = \iint_{(0,\rho)\times(-\zeta-\sigma,\zeta+\sigma)} \frac{r\,\mathrm{e}^{c\varepsilon_n z}}{2}\left(|\nabla v_{\rho,n}|^2 + \frac{\sin^2 v_{\rho,n}}{r^2}\right)\mathrm{d}\,r\,\mathrm{d}\,z -$$

$$-\iint_{(0,\rho)\times(-\zeta-\sigma,\zeta+\sigma)} \frac{r\,\mathrm{e}^{c\varepsilon_n z}}{2}\left(|\nabla \theta_{\varepsilon_n}|^2 + \frac{\sin^2\theta_{\varepsilon_n}}{r^2}\right)\mathrm{d}\,r\,\mathrm{d}\,z \leq J_1+J_2+J_3,$$

where

$$J_1 = \int_{-\zeta}^{\zeta}\mathrm{d}\,z\int_0^\rho \frac{r\,\mathrm{e}^{c\varepsilon_n z}}{2}\left(\left|\frac{\partial v_{\rho,n}}{\partial r}\right|^2 + \frac{\sin^2 v_{\rho,n}}{r^2}\right)\mathrm{d}\,r -$$

$$-\int_{-\zeta}^{\zeta}\mathrm{d}\,z\int_0^\rho \frac{r\,\mathrm{e}^{c\varepsilon_n z}}{2}\left(\left|\frac{\partial \theta_{\varepsilon_n}}{\partial r}\right|^2 + \frac{\sin^2\theta_{\varepsilon_n}}{r^2}\right)\mathrm{d}\,r\,,$$

$$J_2 = \int_{\zeta}^{\zeta+\sigma} \mathrm{d}\,z \int_0^{\rho} \frac{r\,\mathrm{e}^{c\varepsilon_n z}}{2}\left(\omega_r^2 + \frac{\sin^2\omega}{r^2}\right)\mathrm{d}\,r + \int_{-\zeta-\sigma}^{-\zeta} \mathrm{d}\,z \int_0^{\rho} \frac{r\,\mathrm{e}^{c\varepsilon_n z}}{2}\left(\omega_r^2 + \frac{\sin^2\omega}{r^2}\right)\mathrm{d}\,r$$

and

$$J_3 = \int_{-\zeta-\sigma}^{\zeta+\sigma} \mathrm{d}\,z \int_0^{\rho} \frac{r\,\mathrm{e}^{c\varepsilon_n z}}{2}\omega_z^2\,\mathrm{d}\,r\,.$$

It is easy to check that

$$J_3 \le 2\int_{\zeta}^{\zeta+\sigma} \mathrm{e}^{c\varepsilon_n z}\,\mathrm{d}\,z \int_0^{\rho} \frac{r}{2}\omega_z^2\,\mathrm{d}\,r \le \frac{8\rho}{\sigma^2}\,\mathrm{e}^{c\varepsilon_n\zeta}\,\frac{\mathrm{e}^{c\varepsilon_n\sigma}-1}{c\varepsilon_n}$$

and

$$J_2 \le 4\int_{\zeta}^{\zeta+\sigma} \mathrm{e}^{c\varepsilon_n z}\,\mathrm{d}\,z = 4\,\mathrm{e}^{c\varepsilon_n\zeta}\,\frac{\mathrm{e}^{c\varepsilon_n\sigma}-1}{c\varepsilon_n}\,.$$

To estimate J_1 we need to put together several arguments. First of all, since

$$\theta_{\varepsilon_n}(\rho,z) \ge \omega(\rho,z) \quad \forall z \in [-\sqrt{1-\rho^2}, \sqrt{1-\rho^2}]\,,$$

and

$$\lim_{r\to 0^+} \theta_{\varepsilon_n}(r,z) = \pi = \lim_{r\to 0^+} \omega(r,z)$$

for every $z \in (-\zeta, 0)$, it is possible to apply Corollary A.3 and deduce that $\forall z \in (-\zeta, 0)$

$$\int_0^{\rho} \frac{r}{2}\left(\left|\frac{\partial v_{\rho,n}}{\partial r}\right|^2 + \frac{\sin^2 v_{\rho,n}}{r^2}\right)\Bigg|_z \mathrm{d}\,r \le \int_0^{\rho} \frac{r}{2}\left(\left|\frac{\partial \theta_{\varepsilon_n}}{\partial r}\right|^2 + \frac{\sin^2 \theta_{\varepsilon_n}}{r^2}\right)\Bigg|_z \mathrm{d}\,r\,. \tag{1.42}$$

On the other hand, if $z \in (0, \zeta)$, then it is possible to define

$$\tilde\rho(z) = \inf\{r \in [0,\rho] \mid \theta_{\varepsilon_n}(r,z) \ge \omega(r,z)\}$$

and say that $\tilde\rho(z) \in (0,\rho]$ (due to the properties of θ and ω), $\theta_{\varepsilon_n}(r,z) < \omega(r,z)$ for $r \in [0, \tilde\rho(z))$, $\theta_{\varepsilon_n}(\tilde\rho(z), z) = \omega(\tilde\rho(z), z)$ and, since $\theta_{\varepsilon_n}(\rho,z) \ge \omega(\rho,z)$,

$$\int_0^{\rho} \frac{r}{2}\left(\left|\frac{\partial v_{\rho,n}}{\partial r}\right|^2 + \frac{\sin^2 v_{\rho,n}}{r^2}\right)\Bigg|_z \mathrm{d}\,r - \int_0^{\rho} \frac{r}{2}\left(\left|\frac{\partial \theta_{\varepsilon_n}}{\partial r}\right|^2 + \frac{\sin^2 \theta_{\varepsilon_n}}{r^2}\right)\Bigg|_z \mathrm{d}\,r \le$$

$$\int_0^{\tilde\rho(z)} \frac{r}{2}\left(\omega_r^2 + \frac{\sin^2\omega}{r^2}\right)\Bigg|_z \mathrm{d}\,r - \int_0^{\tilde\rho(z)} \frac{r}{2}\left(\left|\frac{\partial \theta_{\varepsilon_n}}{\partial r}\right|^2 + \frac{\sin^2 \theta_{\varepsilon_n}}{r^2}\right)\Bigg|_z \mathrm{d}\,r \tag{1.43}$$

as follows if we apply Corollary A.3 to the interval $[\tilde\rho(z), \rho]$. By using this same Corollary together with Lemma A.1 we derive that the right hand side of (1.43) is less or equal to

$$|\cos\omega(\tilde\rho(z),z) - \cos\omega(0,z)| - |\cos\theta_{\varepsilon_n}(\tilde\rho(z),z) - \cos\theta_{\varepsilon_n}(0,z)| =$$

$$1 + \cos\omega(\tilde\rho(z),z) - (1 - \cos\theta_{\varepsilon_n}(\tilde\rho(z),z)) = 2\cos\omega(\tilde\rho(z),z) =$$

$$-2\cos(2\arctan(\tilde\rho(z))) \le -2\cos(2\arctan\rho) = -2 + \frac{4\rho^2}{1+\rho^2} \le -2 + 4\rho^2\,.$$

Putting together (1.42) and this last estimate we get that

$$J_1 \leq \int_0^\zeta (-2+4\rho^2)\,\mathrm{e}^{c\varepsilon_n z}\,\mathrm{d}\,z = (-2+4\rho^2)\left(\frac{\mathrm{e}^{c\varepsilon_n\zeta}-1}{c\varepsilon_n}\right).$$

From the estimates just obtained for J_1, J_2 and J_3 we finally derive that $\forall\rho \in (0,\bar{\rho})$ and $n \geq \nu(\rho)$

$$\mathcal{E}_{\varepsilon_n}(v_{\rho,n})-\mathcal{E}_{\varepsilon_n}(\theta_{\varepsilon_n}) \leq (-2+4\rho^2)\left(\frac{\mathrm{e}^{c\varepsilon_n\zeta}-1}{c\varepsilon_n}\right)+4\,\mathrm{e}^{c\varepsilon_n\zeta}\ \frac{\mathrm{e}^{c\varepsilon_n\sigma}-1}{c\varepsilon_n}+\frac{8\rho}{\sigma^2}\,\mathrm{e}^{c\varepsilon_n\zeta}\ \frac{\mathrm{e}^{c\varepsilon_n\sigma}-1}{c\varepsilon_n}. \tag{1.44}$$

Since $\sigma, \zeta \leq \frac{\log 2}{2c}$ the following inequalities are true:

$$\sigma \leq \frac{\mathrm{e}^{c\varepsilon_n\sigma}-1}{c\varepsilon_n} \leq 2\sigma, \quad \zeta \leq \frac{\mathrm{e}^{c\varepsilon_n\zeta}-1}{c\varepsilon_n} \leq 2\zeta\,.$$

At the same time $\sigma = \zeta/8$ and we obtain from (1.44)

$$\mathcal{E}_{\varepsilon_n}(v_{\rho,n})-\mathcal{E}_{\varepsilon_n}(\theta_{\varepsilon_n}) \leq -2\zeta+8\rho^2\zeta+(1+2c\varepsilon_n\zeta)\zeta+\frac{128\rho}{\zeta}(1+2c\varepsilon_n\zeta) \leq$$

$$(-1+\log 2)\zeta+8\rho^2\zeta+\frac{128\rho}{\zeta}(1+\log 2).$$

with $\rho \in (0,\bar{\rho})$ and $n \geq \nu(\rho)$. But $\log 2 < 1$ and taking ρ sufficiently small and $n \geq \nu(\rho)$, we deduce that $\mathcal{E}_{\varepsilon_n}(v_{\rho,n})-\mathcal{E}_{\varepsilon_n}(\theta_{\varepsilon_n}) < 0$. Since from Lemma 1.29 we know that for every $n \in \mathbb{N}$

$$\mathcal{E}_{\varepsilon_n}(\theta_{\varepsilon_n}) = \inf_{\{v\in H_1 \,|\, v|_{\partial^+ D_1}=\theta_{\varepsilon_n}|_{\partial^+ D_1}\}} \mathcal{E}_{\varepsilon_n}(v) \tag{1.45}$$

we have just obtained the desired contradiction. □

After excluding the cases (A) and (B) of Proposition 1.38 we obtain:

Proposition 1.40. *There exists a positive constant A such that the limit function ψ is given by*

$$\psi(r,z) = 2\arctan\left(A\tan\left(\frac{\pi}{4}-\frac{\arctan(z/r)}{2}\right)\right)$$

for $(r,z) \in \overline{D}_1 \cap \{r>0\}$ and

$$\psi(0,z) = \begin{cases} 0 & \text{if } z>0 \\ \pi & \text{if } z<0\,. \end{cases}$$

Proof: From Proposition 1.38 we know that ψ is constant along each radius coming out of the origin. Then for every $\rho \in (0,1]$ and $\varphi \in [-\pi/2,\pi/2]$

$$\psi(\rho\cos\varphi,\rho\sin\varphi) = \psi(\cos\varphi,\sin\varphi)$$

and to prove the result it is enough to show that the function

$$g(\varphi) := \psi(\cos\varphi,\sin\varphi)$$

is given by

$$g(\varphi) = 2\arctan\left(A\tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)\right) \tag{1.46}$$

for a suitable $A > 0$. We remark that, due to Proposition 1.38, $g \in C^0([-\pi/2, \pi/2])$. By Theorem 1.36, statement (v), we know that ψ is a smooth solution to the Euler-Lagrange equation of the functional (1.37), i.e.

$$\psi_{zz} + \frac{1}{r}\frac{\partial}{\partial r}\Big(r\psi_r\Big) = \frac{\sin(2\psi)}{2r^2},$$

in $\overline{D}_1 \cap \{r > 0\}$. Since ψ is constant along each radius coming out of the origin, $r\psi_r + z\psi_z = 0$, and therefore ψ solves

$$\psi_{zz} - \frac{z}{r}\psi_{zr} = \frac{\sin(2\psi)}{2r^2}.$$

This implies that $\forall \varphi \in (-\pi/2, \pi/2)$

$$g''(\varphi) = \frac{\sin(2g)}{2\cos^2\varphi} + \tan(\varphi)g'(\varphi)$$

and therefore g solves the following differential problem

$$\begin{cases} \frac{d}{d\varphi}\Big(\cos(\varphi)g'(\varphi)\Big) = \frac{\sin(2g)}{2\cos\varphi} & \forall \varphi \in (-\pi/2, \pi/2) \\ g'(\varphi) = \frac{\psi_z(\cos\varphi, \sin\varphi)}{\cos\varphi} \leq 0 & \forall \varphi \in (-\pi/2, \pi/2) \\ g(\pi/2) = 0, & g(-\pi/2) = \pi\,. \end{cases} \tag{1.47}$$

If we multiply the differential equation of g by $\cos(\varphi)g'(\varphi)$ we obtain that

$$\frac{d}{d\varphi}\Big(\cos^2\varphi|g'(\varphi)|^2\Big) = \frac{d}{d\varphi}\Big(\sin^2 g\Big),$$

and there exists a constant C such that $\cos^2\varphi|g'(\varphi)|^2 - \sin^2 g = C$ for every $\varphi \in (-\pi/2, \pi/2)$.

If $C < 0$, then $\sin^2 g \geq -C > 0$, which cannot be true since g is continuous in $[-\pi/2, \pi/2]$ and $g(\pi/2) = 0, g(-\pi/2) = \pi$. On the other hand, if $C > 0$ then $|g'(\varphi)| \geq \frac{C}{\cos\varphi} \Rightarrow g'(\varphi) \leq -\frac{C}{\cos\varphi}$, which cannot be true since g is bounded. Hence $C = 0$ and

$$\cos\varphi|g'(\varphi)| = |\sin g| \Rightarrow -\cos(\varphi)g'(\varphi) = \sin g$$

since $0 \leq g \leq \pi$ and $g' \leq 0$. By integrating the latter differential equation and taking into account that $0 \leq g \leq \pi$, $g \not\equiv 0$ and $g \not\equiv \pi$, we obtain (1.46) with $A > 0$. □

Remark 1.41. From Proposition 1.40 we derive that

1. ψ is smooth on $\overline{D}_1 \setminus \{(0,0)\}$,
2. ψ is a strictly decreasing function of the angle $\varphi := \arctan(z/r)$,
3. $\mathcal{E}(\psi) = 2$.

To prove Claim 1.37 we only need to show that:

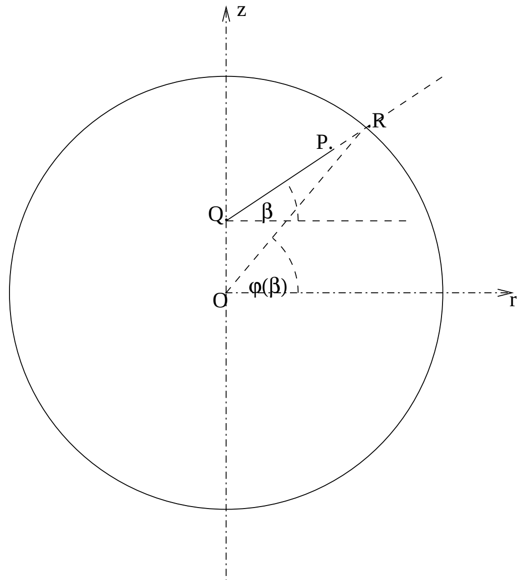

Figure 1.2: The value of v_a in P is equal to the value of ψ in R

Proposition 1.42. *The constant A of Proposition 1.40 is actually* 1.

Proof: Given any $a \in (-1,1)$, let (l,β) be the polar coordinates of the plan r-z centered at the point $(0,a)$:

$$\begin{cases} r = l\cos\beta \\ z = a + l\sin\beta\,. \end{cases}$$

Given any point $P=(r,z)$, $l = l(P)$ is its distance from the point $Q=(0,a)$:

$$l = \sqrt{r^2 + (z-a)^2}$$

and $\beta = \beta(P)$ is the angle formed by the vector $\overrightarrow{QP}$ with the direction $\overrightarrow{r} = (1,0)$. It is simple to verify that the point (r,z) belongs to $\overline{D}_1$ if and only if its polar coordinates (l,β) satisfy the constraints $\beta \in [-\pi/2,\pi/2]$, $0 \le l \le L(\beta)$, where

$$L(\beta) = \sqrt{1 - a^2\cos^2\beta} - a\sin\beta\,.$$

In particular, the points of $\partial^+ D_1$ are those ones having polar coordinates $(L(\beta),\beta)$ for $\beta \in (-\pi/2,\pi/2)$. We remark that for $a=0$ l and β are the usual polar coordinates.

Let $v_a = v_a(r,z)$ be the function defined on $\overline{D}_1$ by

$$v_a(l\cos\beta, a + l\sin\beta) = \psi(L(\beta)\cos\beta, a + L(\beta)\sin\beta)$$

for $\beta \in [-\pi/2,\pi/2]$, $0 < l \le L(\beta)$. We remark that the value of v_a in a point $P \in \overline{D}_1$ is given by the value of ψ in the intersection of the line passing through P and $Q=(0,a)$ with $\overline{\partial^+ D_1}$ (see figure). Since ψ is constant along each radius coming out of $(0,0)$, for $a=0$ we have $v_a = \psi$. If we denote by $q = q(\beta)$ the function defined by

$$q(\beta) = \psi(L(\beta)\cos\beta, a + L(\beta)\sin\beta)\,,$$

we have

$$|\nabla v_a|^2(l\cos\beta, a+l\sin\beta) = \frac{|q'(\beta)|^2}{l^2}$$

and

$$\mathcal{E}(v_a) = \int_{-\pi/2}^{\pi/2} \mathrm{d}\,\beta \int_0^{L(\beta)} \frac{\cos\beta}{2}\left(|q'(\beta)|^2 + \frac{\sin^2 q}{\cos^2\beta}\right)\mathrm{d}\,l =$$

$$\frac{1}{2}\int_{-\pi/2}^{\pi/2} L(\beta)\cos\beta\left(|q'(\beta)|^2 + \frac{\sin^2 q}{\cos^2\beta}\right)\mathrm{d}\,\beta\,.$$

Since for $\beta \in [-\pi/2, \pi/2]$

$$L^2(\beta)\cos^2\beta + (a + L(\beta)\sin\beta)^2 = 1\,,$$

for every $\beta \in [-\pi/2, \pi/2]$ there exists $\varphi(\beta) \in [-\pi/2, \pi/2]$ such that

$$\begin{cases} \cos\varphi(\beta) = L(\beta)\cos\beta \\ \sin\varphi(\beta) = a + L(\beta)\sin\beta\,. \end{cases}$$

$\varphi(\beta)$ is a smooth function of β and

$$\frac{\cos^2\varphi(\beta)}{\cos^2\beta} = 1 + a^2 - 2a\sin\varphi(\beta)\,, \quad \varphi'(\beta) = \frac{1 - 2a\sin\varphi(\beta) + a^2}{1 - a\sin\varphi(\beta)} > 0\,.$$

By the definition of $\varphi(\beta)$, $q(\beta) = g(\varphi(\beta))$, where $g = g(\varphi)$ is the same function as in the proof of Proposition 1.40, and Then

$$\mathcal{E}(v_a) = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos\varphi(\beta)\Bigg(|g'(\varphi(\beta))|^2|\varphi'(\beta)|^2 +$$

$$+\frac{\sin^2 g(\varphi(\beta))}{\cos^2\varphi(\beta)}(1 + a^2 - 2a\sin\varphi(\beta))\Bigg)\mathrm{d}\,\beta =$$

$$\frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos\varphi\left(|g'(\varphi)|^2\frac{1 - 2a\sin\varphi + a^2}{1 - a\sin\varphi} + \frac{\sin^2 g}{\cos^2\varphi}(1 - a\sin\varphi)\right)\mathrm{d}\,\varphi < \infty$$

If we think a as a variable in $(-1, 1)$, the formula just obtained tells us that $\mathcal{E}(v_a)$ is a smooth function of a and

$$\frac{d}{da}\Big(\mathcal{E}(v_a)\Big)\Big|_{\{a=0\}} = -\frac{1}{2}\int_{-\pi/2}^{\pi/2} \sin\varphi\cos\varphi\left(|g'(\varphi)|^2 + \frac{\sin^2 g(\varphi)}{\cos^2\varphi}\right)\mathrm{d}\,\varphi\,. \tag{1.48}$$

Since $v_a|_{\partial^+ D_1} = \psi|_{\partial^+ D_1}$ for each $a \in (-1, 1)$, it follows from Theorem 1.36 that $\mathcal{E}(v_a) \geq \mathcal{E}(\psi) = \mathcal{E}(v_0)$ for every $a \in (-1, 1)$. Therefore formula (1.48) implies that

$$-\frac{1}{2}\int_{-\pi/2}^{\pi/2} \sin\varphi\cos\varphi\left(|g'(\varphi)|^2 + \frac{\sin^2 g(\varphi)}{\cos^2\varphi}\right)\mathrm{d}\,\varphi = 0\,. \tag{1.49}$$

On the other hand, if we use the formula for ψ given by Proposition 1.40, we obtain that

$$-\frac{1}{2}\int_{-\pi/2}^{\pi/2}\sin\varphi\cos\varphi\left(|g'(\varphi)|^2+\frac{\sin^2 g(\varphi)}{\cos^2\varphi}\right)\mathrm{d}\,\varphi=$$

$$-\int_{-\pi/2}^{\pi/2}\tan\varphi\,\frac{4A^2\cos^2\varphi}{(1+A^2+(1-A^2)\sin\varphi)^2}\,\mathrm{d}\,\varphi=-\int_{-1}^{1}\frac{4A^2x}{(1+A^2+(1-A^2)x)^2}\,\mathrm{d}\,x\,. \quad (1.50)$$

If $A=1$, the last term in the previous identity is equal to

$$-\int_{-1}^{1}x\,\mathrm{d}\,x=0\,.$$

In the case $A\neq 1$ we can perform the substitution $u=1+A^2+(1-A^2)x$ so finding that the last term of (1.50) is equal to

$$-\frac{4A^2}{(1-A^2)^2}\int_{2A^2}^{2}\frac{u-(1+A^2)}{u^2}\,\mathrm{d}\,u=-\frac{2f(A)}{(1-A^2)^2}$$

with $f(A)=A^4-1-2A^2\log A$. Since $f(A)\neq 0$ for $A\neq 1$, we deduce that (1.49) cannot be satisfied for $A\neq 1$. Therefore $A=1$. □

Remark 1.43. The argument used in the proof of the previous Proposition is an adaptation to the axially symmetric case of a similar argument found in [1].

As a direct consequence of Propositions 1.33, 1.34, Theorem 1.36 and Claim 1.37 we obtain that:

Proposition 1.44. *If ψ is the function:*

$$\psi(r,z)=\frac{\pi}{2}-\arctan\left(\frac{z}{r}\right),$$

then, as $\varepsilon\to 0$,
(i)

$$\nabla\theta_\varepsilon\rightharpoonup\nabla\psi\,,\qquad\frac{\sin\theta_\varepsilon}{r}\rightharpoonup\frac{\sin\psi}{r}\qquad\text{in } L^2_r(D_1)\,;$$

(ii) for every $\rho\in(0,1)$

$$\theta_\varepsilon\to\psi\qquad\text{in } C^2(\overline{D}_1\cap\{r\geq\rho\})\,.$$

If we denote by $G=G(\rho,\varphi)$ the function defined by

$$G(\rho,\varphi)=\theta(\rho\cos\varphi,\rho\sin\varphi)$$

for $\rho\in(0,R]$ and $\varphi\in[-\pi/2,\pi/2]$, then it follows from the previous Proposition that for every $\alpha\in(0,\pi/2)$

$$G(\rho,\varphi)\to\frac{\pi}{2}-\varphi \quad (1.51)$$

in $C^2([-\pi/2+\alpha,\pi/2-\alpha])$ as $\rho\to 0^+$. In particular, the convergence is locally uniform and Theorem 1.3 is proved.

Chapter 2

Traveling wave solutions of the heat flow of director fields having a zero degree singularity

In this chapter we shall construct axially symmetric traveling wave solutions of (1) having a point singularity of degree zero on the axis of the cylinder $\Omega \equiv \{(x_1, x_2, x_3) | x_1^2 + x_2^2 < 1\}$. If $h(r, x_3, t) = \psi(r, x_3 - ct)$ is the angle function of a such traveling wave, then, as told in the introduction, ψ satisfies the singular elliptic equation (7) ($z = x_3 - ct$). To this equation we add a boundary condition at $r = 1$:

$$\psi(1, z) = g(z), \tag{2.1}$$

where g is a given function which satisfies, for some $z_0 < z_1$ and $0 < B < A$,

$$g \in C^4(\mathbb{R}), \; g' \leq 0 \text{ in } \mathbb{R}, \; g = A \text{ in } (-\infty, z_0), \; g = B \text{ in } (z_1, \infty). \tag{2.2}$$

To ensure that the traveling waves have a point singularity, we shall always choose $A > \pi$ and $0 < B < \pi/2$.

At first glance condition (2.1) may seem artificial. In a way it forces solutions to move in the x_3-direction with prescribed speed $c > 0$, and one could argue that this trivially imposes the existence of traveling wave solutions with the same velocity. On the other hand, condition (2.1) enables us to construct traveling waves with a point singularity of topological degree 0, which turn out to be useful as comparison functions for solutions of initial-boundary value problems, as we shall in Chapter 3 (actually we shall also construct waves with a degree-1 singularity).

As in chapter 1, we shall construct axially symmetric traveling waves which are non-increasing with respect to z, this means that point singularities, which necessarily belong to the z-axis due to the axial symmetry, occur at points $(r, z) = (0, \bar{z})$ at which ψ is discontinuous. Moreover, $\psi(0, z)$ is necessarily a multiple of π whenever $(0, z)$ is a point of continuity.

In what follows we shall denote (with abuse of notation) the function $\psi(r, z)$ by $h(r, z)$. The main theorems of the chapter show that it is possible to have both singular points at which h jumps from π to 0 (Theorem 2.1) and ones at which h jumps from 2π to 0 (Theorem 2.2). In the first case the topological degree of the point singularity is 1, in the latter case it is 0.

Theorem 2.1. *Let $c > 0$ and let $g(z)$ be a given function satisfying (2.2) with*

$$\pi < A < 3\pi/2 \quad \text{and} \quad 0 < B < \pi/2. \tag{2.3}$$

Then there exists a function $h_1 : [0,1] \times \mathbb{R} \to \mathbb{R}$ which is smooth in $(0,1] \times \mathbb{R}$ and satisfies equations (7) and (2.1). In addition the following properties are satisfied:
(i) there exists $\bar{z}_1$ such that h_1 is continuous in $\{(0,z) : z \neq \bar{z}_1\}$, $h_1(0,z) = 0$ if $z > \bar{z}_1$ and $h_1(0,z) = \pi$ if $z < \bar{z}_1$;
(ii) $h_1(r,z)$ is nonincreasing with respect to z;
(iii) $h_1(r,z) \to 2\arctan(br)$ uniformly with respect to $r \in [0,1]$ as $z \to \infty$, where b is defined by $2\arctan b = B$;
(iv) $h_1(r,z) \to \pi + 2\arctan(a_1 r)$ uniformly with respect to $r \in [0,1]$ as $z \to -\infty$, where a_1 is defined by $\pi + 2\arctan a_1 = A$;
(v) h_1 is real analytic in $[0,1) \times \mathbb{R} \setminus \{(0,\bar{z}_1)\}$.

Theorem 2.2. *Let $c > 0$ and let $g(z)$ be a given function satisfying (2.2) with*

$$\pi < A < 3\pi \quad \text{and} \quad 0 < B < \pi/2. \tag{2.4}$$

Then there exists a function $h_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ which satisfies Theorem 2.1 with properties (i), (iv) and (v) replaced by:
(i) there exists $\bar{z}_2$ such that h_2 is continuous in $\{(0,z) : z \neq \bar{z}_2\}$, $h_2(0,z) = 0$ if $z > \bar{z}_2$ and $h_2(0,z) = 2\pi$ if $z < \bar{z}_2$;
(iv) $h_2(r,z) \to 2\pi + 2\arctan(a_2 r)$ uniformly with respect to $r \in [0,1]$ as $z \to -\infty$, where a_2 is defined by $2\pi + 2\arctan a_2 = A$;
(v) h_2 is real analytic in $[0,1) \times \mathbb{R} \setminus \{(0,\bar{z}_2)\}$.

As in the previous chapter, our approach will be variational, but in the case of Theorem 2.2 the minimization problem involves a variant of the relaxed energy introduced by Bethuel, Brezis and Coron in [6] and used by Hardt, Poon and Lin in [18] to construct axially symmetric harmonic maps with zero-degree singularities. In addition, due to the boundary condition (2.1) which prescribes the wave speed c, we do not introduce a constraint in the minimization problem. The proof of the monotonicity of the solutions with respect to z relies again on a rearrangement technique.

The chapter is organized as follows. In section 2.1 we introduce the two minimization problems. In section 2.2 we collect some preliminary results. In section 2.3 we prove the existence of minimizers and in section 2.4 we show their monotonicity with respect to z. In section 2.5 we prove that the minimizers have a singularity. In section 2.6 we discuss the behavior of the singularities as $c \to \infty$.

2.1 Variational formulation

Let $c > 0$. Equation (7) is the Euler-Lagrange equation of the functional

$$\Phi_c(f) = \int_{\mathbb{R}} \mathrm{d}\, z \int_0^1 \mathrm{d}\, r \left\{ \frac{r}{2} \mathrm{e}^{cz} \left(f_z^2 + f_r^2 + \frac{\sin^2 f}{r^2} - G_b(r) \right) \right\}. \tag{2.5}$$

The function $G_b(r)$ is chosen in such a way that $\Phi_c(f)$ is convergent as $z \to \infty$ for all functions f belonging to a suitable class which contains the function $2\arctan(br)$, describing the desired behavior of the traveling waves as $z \to \infty$ (see point (iii) of Theorems 2.1 and 2.2):

$$G_b(r) = \frac{\sin^2(2\arctan(br))}{r^2} + \left|\frac{d}{dr}(2\arctan(br))\right|^2 . \tag{2.6}$$

A straightforward calculation shows that

$$\int_0^1 \frac{r}{2} G_b(r)\,\mathrm{d}\,r = \frac{2b^2}{1+b^2} . \tag{2.7}$$

On the other hand, it is well-known (see also Theorem A.6 in the Appendix) that, if $0 < b < 1$,

$$\int_0^1 \frac{r}{2}\left(f_r^2 + \frac{\sin^2 f}{r^2}\right)\mathrm{d}\,r \geq \frac{2b^2}{1+b^2} \quad \text{if } f \in H^1_{\mathrm{loc}}((0,1]) \text{ and } f(1) = 2\arctan b. \tag{2.8}$$

We define the class of functions

$$\mathcal{W} = \left\{ v \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}; L^2_r(0,1)) \cap L^2_{\mathrm{loc}}(\mathbb{R}; H^1_r(0,1)) \,;\, \frac{\sin v}{r} \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2_r(0,1)) \right\} ,$$

where the subscript r (in L^2_r, H^1_r etc.) indicates that the usual L^p or Sobolev spaces are to be considered with the weight function r. If $f \in \mathcal{W}$, then for a.e. $z \in \mathbb{R}$ the function $f(\cdot, z)$ is defined almost everywhere in $(0,1)$, $f(\cdot,z) \in H^1_r(0,1)$, and $\frac{\sin f(\cdot,z)}{r} \in L^2_r(0,1)$. This implies (see [30]) that, for almost every $z \in \mathbb{R}$, $f(\cdot,z) \in C^0([0,\,1])$ and

$$f(0,z) \;=\; k(z)\pi \quad \text{for some } k(z) \in \mathbb{Z}. \tag{2.9}$$

If $f \in \mathcal{W}$, the trace of f at $r = 1$ is well-defined. If $f(1,z) \equiv g(z)$ for a.e. $z \in \mathbb{R}$, it follows from (2.2), (2.7), (2.8) and the monotone convergence theorem that

$$\Phi_c(f) \;=\; \lim_{\substack{\alpha \to -\infty \\ \beta \to \infty}} \int_\alpha^\beta \mathrm{d}\,z \int_0^1 \mathrm{d}\,r \left\{ \frac{r}{2}\,\mathrm{e}^{cz}\left(f_z^2 + f_r^2 + \frac{\sin^2 f}{r^2} - G_b(r) \right) \right\}$$

is well-defined and attains values in $(-\infty, \infty]$. More precisely, for such functions f we have that

$$\Phi_c(f) \;\geq\; -\int_{-\infty}^{z_1} \mathrm{d}\,z \int_0^1 \frac{r}{2}\,\mathrm{e}^{cz}\, G_b(r)\,\mathrm{d}\,r = -\frac{2b^2\,\mathrm{e}^{cz_1}}{c(1+b^2)}. \tag{2.10}$$

We define, for each $c > 0$,

$$\mathcal{W}^c = \{ f \in \mathcal{W} \,;\; f(1,z) \;\equiv\; g(z),\, \Phi_c(f) < \infty \}$$

(observe that $\mathcal{W}^c \neq \emptyset$; it contains the function $2\arctan(br) + (g(z) - 2\arctan b)r$). Since (2.10) holds in $\mathcal{W}^c$ we can formulate our first minimization problem:

First variational problem: find $h_1 \in \mathcal{W}^c$ which minimizes Φ_c in $\mathcal{W}^c$.

Its solution will be the traveling wave of Theorem 2.1.

In order to prove Theorem 2.2 we need a suitable variant of the concept of relaxed energy, introduced in [6]. Let

$$\mathfrak{C} = \left\{\xi \in C^1([0,1] \times \mathbb{R})\,;\, \operatorname{supp}(\xi) \subseteq [0,1] \times [-M, M] \quad \text{for some } M > 0\right\}$$

and

$$\mathfrak{C}^c = \{\xi \in \mathfrak{C}\,;\, |\nabla \xi(r,z)| \le \mathrm{e}^{cz} \text{ in } [0,1] \times \mathbb{R}\}\,.$$

We define for every $f \in \mathcal{W}$ and $\xi \in \mathfrak{C}$,

$$L(f,\xi) := \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}\,z \int_0^1 \sin f\,(f_z \xi_r - f_r \xi_z)\,\mathrm{d}\,r \;-\frac{1}{2}\int_{\mathbb{R}} \cos(f(1,z))\xi_z(1,z)\,\mathrm{d}\,z. \tag{2.11}$$

We observe that $L(f,\xi)$ is well-defined and $L(f,-\xi) = -L(f,\xi)$. Hence

$$L_c(f) \;:=\; \sup_{\xi \in \mathfrak{C}^c} L(f,\xi) \in [0,\infty] \quad \text{for } f \in \mathcal{W}.$$

It turns out that $L_c < \infty$ in $\mathcal{W}^c$:

Theorem 2.3. *Let $f \in \mathcal{W}^c$ and let $P_f = \{z \in \mathbb{R}\,;\, \cos(f(0,z)) \;=\; -1\}$. Then*

$$L_c(f) \;=\; \int_{P_f} \mathrm{e}^{cz}\,\mathrm{d}\,z \quad < \infty\,.$$

We observe that, by (2.9), P_f is well-defined and Lebesgue-measurable. We shall prove Theorem 2.3 in section 2.2.

Theorem 2.2 corresponds to the following minimization problem:

Second variational problem: find $h_2 \in \mathcal{W}^c$ which minimizes $\Phi_c + 2L_c$ in $\mathcal{W}^c$.

2.2 Preliminaries, proof of Theorem 2.3

We introduce the following coordinate transformation:

$$x = \mathrm{e}^{cz} > 0 \leftrightarrow z = c^{-1}\log x. \tag{2.12}$$

It transforms equation (7) into

$$(rh_r)_r + c^2 r(x^2 h_x)_x - \frac{\sin(2h)}{2r} = 0 \quad \text{in } (0,1) \times \mathbb{R}^+,$$

which is the Euler-Lagrange equation of the functional

$$\Psi_c(f) \;=\; \frac{1}{2c}\int_0^\infty \mathrm{d}\,x \int_0^1 r\left(c^2 x^2 f_x^2 + f_r^2 + \frac{\sin^2 f}{r^2} - G_b(r)\right)\mathrm{d}\,r.$$

Transformation (2.12) induces naturally a bijective map $T : \mathcal{W} \to T(\mathcal{W})$, $f(r,z) \mapsto f(r, c^{-1}\log x)$, and

$$T(\mathcal{W}) = \{f \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^+; L^2_r(0,1)) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+; H^1_r(0,1)); \frac{\sin v}{r} \in L^2_{\mathrm{loc}}(\mathbb{R}^+; L^2_r(0,1))\}.$$

In particular

$$T(\mathcal{W}^c) = \{f \in T(\mathcal{W})\,;\; f(1,x) \,\equiv\, g(c^{-1}\log(x)),\; \Psi_c(f) < \infty\},$$

$$\Psi_c(f) \;=\; \lim_{\substack{\alpha\to 0^+\\ \beta\to\infty}} \frac{1}{2c}\int_\alpha^\beta \mathrm{d}\,x\int_0^1 r\left(c^2x^2f_x^2 + f_r^2 + \frac{\sin^2 f}{r^2} - G_b(r)\right)\mathrm{d}\,r. \tag{2.13}$$

We observe that

$$\Phi_c(f) \;=\; \Psi_c(T(f)) \quad \text{for } f\in\mathcal{W}^c\,. \tag{2.14}$$

We set

$$\mathcal{L}(f,\xi) = \frac{1}{2}\int_{\mathbb{R}^+}\mathrm{d}\,x\int_0^1 \sin(f)(f_x\xi_r - f_r\xi_x)\mathrm{d}\,r \;-\; \frac{1}{2}\int_{\mathbb{R}^+}\cos(f(1,x))\xi_x(1,x)\,\mathrm{d}\,x$$

for every $f\in T(\mathcal{W})$ and $\xi\in T(\mathfrak{C})$. It follows easily that

$$T(\mathfrak{C}) = \left\{\xi\in C^1([0,1]\times\mathbb{R}^+) : \mathrm{supp}(\xi)\subseteq[0,1]\times[M^{-1},M] \text{ for some } M>1\right\},$$

$$T(\mathfrak{C}^c) = \left\{\xi\in T(\mathfrak{C}) \;:\; \frac{1}{x^2}\xi_r^2 + c^2\xi_x^2 \le 1 \text{ in } [0,1]\times\mathbb{R}^+\right\},$$

and $\mathcal{L}(T(f),T(\xi)) \;=\; L(f,\xi)$ for each $\xi\in\mathfrak{C}$ and $f\in\mathcal{W}$. Hence, defining

$$\mathcal{L}_c(f) \;=\; \sup_{\xi\in T(\mathfrak{C}^c)} \mathcal{L}(f,\xi) \ge 0 \quad \text{for } f\in T(\mathcal{W}),$$

we obtain that

$$L_c(f) \;=\; \mathcal{L}_c(T(f)) \quad \text{for all } f\in\mathcal{W}\,. \tag{2.15}$$

In order to prove Theorem 2.3 we need the following result.

Proposition 2.4. *For all* $f\in T(\mathcal{W})$

$$\mathcal{L}(f,\xi) \;=\; -\frac{1}{2}\int_{\mathbb{R}^+}\cos(f(0,x))\xi_x(0,\,x)\;\mathrm{d}\,x \quad \textit{for } \xi\in T(\mathfrak{C})\,,$$

and

$$\mathcal{L}_c(f) \;=\; \sup_{\{\lambda\in C_0^1(\mathbb{R}^+)\,;\,|\lambda'|\le 1/c\}} \left(-\frac{1}{2}\int_{\mathbb{R}^+}\cos(f(0,x))\lambda'(x)\;\mathrm{d}\,x\right).$$

Proof: The first statement implies at once the second one. If ξ is sufficiently smooth, the first statement follows from an integration by parts in (2.11) (observe that for all $f\in\mathcal{W}$ we have, in addition to (2.9), that $\cos f(\cdot,z)$ is absolutely continuous in [0, 1] for a.e. $z\in\mathbb{R}$, and $\cos f(r,\cdot)$ is locally absolutely continuous in $\mathbb{R}$ for a.e. $r\in(0,1)$). A standard approximation argument completes the proof of the first statement. □

Proposition 2.5. *Let* $w\in T(\mathcal{W}^c)$*, let* $E_w = \{x\in\mathbb{R}^+\,;\,\cos(w(0,x)) \;=\; -1\}$ *and let* μ *denote the 1-dimensional Lebesgue measure. Then*

$$\mathcal{L}_c(w) \;=\; \frac{1}{c}\mu(E_w) < \infty\,.$$

Proof: First we prove that $\mu(E_w) < \infty$. Arguing by contradiction we suppose that $\mu(E_w) = \infty$. Let z_1 be defined by (2.2) and set $x_1 = \mathrm{e}^{cz_1}$. Then $\mu(E_w \cap (x_1, \infty)) = \infty$. For all $x \in E_w \cap (x_1, \infty)$ we have that $w(0, x) = k(x)\pi$, with $k(x)$ odd, and $w(1, x) = 2 \arctan b$. Hence it follows from (2.7) and Lemma A.1 that for any $x \in E_w \cap (x_1, \infty)$

$$\int_0^1 \frac{r}{2}\left(c^2x^2w_x^2 + w_r^2 + \frac{\sin^2 w}{r^2} - G_b(r)\right) \mathrm{d}\, r \geq \frac{2}{1+b^2} - \frac{2b^2}{1+b^2} = 2\frac{1-b^2}{1+b^2}.$$

On the other hand, by (2.8), the same integral is nonnegative if $x \geq x_1$ and uniformly bounded from below if $0 < x \leq x_1$. Since $\mu(E_w \cap (x_1, \infty)) = \infty$ and $1 - b^2 > 0$, this implies that $\Psi_c(w) = \infty$. Hence $w \notin T(\mathcal{W}^c)$ and we have found a contradiction.

Let $\lambda \in C_0^1(\mathbb{R}^+)$ such that $|\lambda'| \leq c^{-1}$. Then $-\frac{1}{2}\int_{\mathbb{R}^+} \cos(w(0, x))\lambda'(x)\ \mathrm{d}\, x = -\frac{1}{2}\int_{\mathbb{R}^+ \setminus E_w} \lambda'(x)\ \mathrm{d}\, x + \frac{1}{2}\int_{E_w} \lambda'(x)\ \mathrm{d}\, x = -\frac{1}{2}\int_{\mathbb{R}^+} \lambda'(x)\ \mathrm{d}\, x + \int_{E_w} \lambda'(x)\ \mathrm{d}\, x = \int_{E_w} \lambda'(x)\ \mathrm{d}\, x \leq \mu(E_w)/c$ and hence, by Proposition 2.4, $\mathcal{L}_c(w) \leq \mu(E_w)/c$.

It remains to prove that $\mathcal{L}_c(w) \geq \mu(E_w)/c$. Let $\varepsilon > 0$. Then there exists $x_\varepsilon > 0$ such that $\ell_\varepsilon \equiv \mu(E_w \cap (0, x_\varepsilon)) > \mu(E_w) - \varepsilon$. Let λ_ε be the function

$$\lambda_\varepsilon(x) \ = \begin{cases} \frac{x}{c} & \text{if } x \in (0, x_\varepsilon] \\ \frac{2x_\varepsilon - x}{c} & \text{if } x \in (x_\varepsilon, 2x_\varepsilon] \\ 0 & \text{if } x > 2x_\varepsilon\ . \end{cases}$$

It follows from Proposition 2.4 and a straightforward approximation argument that $\mathcal{L}_c(w) \geq -\frac{1}{2}\int_{\mathbb{R}^+} \cos(w(0, x))\lambda_\varepsilon'(x)\ \mathrm{d}\, x$. Hence

$$\begin{aligned} \mathcal{L}_c(w) &\geq -\frac{1}{2}\int_{\mathbb{R}^+} \cos(w(0, x))\lambda_\varepsilon'(x)\ \mathrm{d}\, x \ = \ -\frac{1}{2c}\int_0^{x_\varepsilon} \cos(w(0, x))\ \mathrm{d}\, x + \\ &+\frac{1}{2c}\int_{x_\varepsilon}^{2x_\varepsilon} \cos(w(0, x))\ \mathrm{d}\, x \ = \ -\frac{1}{2c}\Big(\mu((0, x_\varepsilon) \setminus E_w) - \mu((0, x_\varepsilon) \cap E_w)\Big) + \\ &+\frac{1}{2c}\Big(\mu((x_\varepsilon, 2x_\varepsilon) \setminus E_w) - \mu((x_\varepsilon, 2x_\varepsilon) \cap E_w)\Big) > -\frac{1}{2c}(x_\varepsilon - 2\ell_\varepsilon) + \frac{1}{2c}(x_\varepsilon - 2\varepsilon)\,, \end{aligned}$$

since $\mu((x_\varepsilon, 2x_\varepsilon) \cap E_w) \leq \mu(E_w \setminus (0, x_\varepsilon)) = \mu(E_w \setminus (E_w \cap (0, x_\varepsilon))) = \mu(E_w) - \ell_\varepsilon < \varepsilon$ and $\mu((x_\varepsilon, 2x_\varepsilon) \setminus E_w) = x_\varepsilon - \mu((x_\varepsilon, 2x_\varepsilon) \cap E_w) > x_\varepsilon - \varepsilon$. Hence $\mathcal{L}_c(w) > (\mu(E_w) - 2\varepsilon)/c$ and since $\varepsilon > 0$ can be chosen arbitrarily small the proof is complete. $\square$

Theorem 2.3 follows at once from (2.15), Proposition 2.5, and the relation

$$\int_{P_f} \mathrm{e}^{cz}\, \mathrm{d}\, z = \frac{1}{c}\int_{E_{T(f)}} \mathrm{d}\, x = \frac{1}{c}\mu\left(E_{T(f)}\right).$$

We conclude this section with a technical result which we shall use in section 2.5 .

Proposition 2.6. *Let $0 < b < 1$, $w \in T(\mathcal{W}^c)$, $k \in \mathbb{Z} \setminus \{0\}$ and $0 < \sigma < \sigma_b$, where*

$$\sigma_b = \arccos\left(\frac{3b^2 - 1}{1 + b^2}\right).$$

Then

$$\mu(\{x > 0; w(0, x) = k\pi\}) = \lim_{r \to 0^+} \mu(\{x > 0; k\pi - \sigma \leq w(r, x) < k\pi + \sigma\}) < \infty.$$

Proof: Let $n \in \mathbb{N}$ and $0 < r < 1$, and set

$$\begin{aligned} S_n &= \{0 < x < n\,;\, w(0,x) = k\pi\} \\ S_{r,n} &= \{0 < x < n\,;\, k\pi - \sigma \le w(r,x) < k\pi + \sigma\}) \\ F_{r,n} &= \{x > n\,;\, k\pi - \sigma \le w(r,x) < k\pi + \sigma\}). \end{aligned}$$

Since, for a.e. $x > 0$, $w(\cdot,x) \in C^0([0,1])$ and $w(0,x) = j(x)\pi$ for some $j(x) \in \mathbb{Z}$, the characteristic function of the set $\{x > 0\,;\, k\pi - \sigma \le w(r,x) < k\pi + \sigma\})\}$ converges a.e. to the characteristic function of $\{x > 0\,;\, w(0,x) = k\pi\}$ (here we have used that $\sigma < \pi$). Hence, by Lebesgue's theorem $\mu(S_{r,n}) \to \mu(S_n)$ as $r \to 0$ for all $n \in \mathbb{N}$.

It is easy to complete the proof if we show that for all $\varepsilon > 0$ there exists $\nu \in \mathbb{N}$ such that $\mu(F_{r,n}) < \varepsilon$ for all $n \ge \nu$ and $0 \le r \le 1$.

Arguing by contradiction we suppose that there exists $\varepsilon > 0$ such that for every $\nu \in \mathbb{N}$ there exist $n = n(\nu) \ge \nu$ and $0 \le r_n \le 1$ such that $\mu(F_{r_n,n}) \ge \varepsilon$. Choosing $\nu \ge x_1 \equiv \mathrm{e}^{cz_1}$, $w(1,x) = 2\arctan b$ for every $x \in F_{r_n,n}$. On the other hand, since $k \ne 0$, $w(r_n,x) \ge \pi - \sigma$ or $w(r_n,x) < -\pi + \sigma$ if $x \in F_{r_n,n}$. Hence, by Lemma A.1, for all $n = n(\nu)$ and $x \in F_{r_n,n}$

$$\int_{r_n}^1 \frac{r}{2}\left(w_r^2(r,x) + \frac{\sin^2 w(r,x)}{r^2}\right) \mathrm{d}\, r \ge |\cos(2\arctan b) + \cos(\sigma)|.$$

In view of (2.7) it is natural to require that the right hand side is larger than $\frac{2b^2}{1+b^2}$, which leads at once to the condition $\sigma < \sigma_b$. Hence there exists $C = C(b,\sigma) > 0$ such that for $n = n(\nu)$ and $\nu \ge x_1$

$$\int_n^\infty \mathrm{d}\, x \int_0^1 \frac{r}{2}\left(w_r^2(r,x) + \frac{\sin^2 w(r,x)}{r^2} - G_b(r)\right) \mathrm{d}\, r \ge C\mu(F_{r_n,n}) \ge C\varepsilon.$$

On the other hand, since $w \in T(\mathcal{W}^c)$, the latter integral vanishes as $n \to \infty$, and we have found a contradiction. $\square$

2.3 Existence of minimizers

In this section we prove the following result:

Theorem 2.7. *Let g satisfy (2.2), with $0 < B < \frac{\pi}{2}$ and $A > B$, and let $b \in (0,1)$ be defined by $2\arctan b = B$. Then the first and the second variational problem have a solution, h_1 and h_2 respectively, which satisfy the following properties:*

(i) h_1 and h_2 are real analytic in $(0,1) \times \mathbb{R}$ and continuous up to $r = 1$, and satisfy equations (7) and (2.1).

(ii) If $\pi < A < \frac{3\pi}{2}$, then $2\arctan(br) < h_1(r,z) < \pi + 2\arctan(a_1 r)$ for $(r,z) \in (0,1) \times \mathbb{R}$, where $a_1 \in (0,1)$ is defined by $\pi + 2\arctan a_1 = A$.

(iii) If $\pi < A < 3\pi$, then $2\arctan(br) < h_2(r,z) < 2\pi + 2\arctan(a_2 r)$ for $(r,z) \in (0,1) \times \mathbb{R}$, where $a_2 \in \mathbb{R}$ is defined by $2\pi + 2\arctan a_2 = A$.

(iv) $h_i(r,z) \to 2\arctan(br)$, $(i = 1,2)$, uniformly with respect to $r \in [0,1]$ as $z \to \infty$.

Proof: We only sketch the proof in case of the second variational problem. Since great parts of it are standard, we omit all details except of the less standard ones. We set

$$\mathcal{I} = \inf\{\Phi_c(h) + 2L_c(h); h \in \mathcal{W}^c\}.$$

By (2.10), $\mathcal{I} \geq -\frac{2b^2\,\mathrm{e}^{cz_1}}{c(1+b^2)}$. Let $\{h_n\}$ be a minimizing sequence and let $\sigma > 0$. We set, for all $f \in W^{1,2}_r((0,1)\times(-\sigma,\sigma))$,

$$E_{c,\sigma}(f) \;=\; \int_{-\sigma}^{\sigma} \mathrm{d}\,z \int_0^1 \mathrm{d}\,r \left\{ \frac{r}{2}\,\mathrm{e}^{cz} \left(f_z^2 + f_r^2 + \frac{\sin^2 f}{r^2} \right) \right\}$$

$$\Phi_{c,\sigma}(f) \;=\; E_{c,\sigma}(f) - \int_{-\sigma}^{\sigma} \mathrm{d}\,z \int_0^1 \mathrm{d}\,r \left(\frac{r}{2}\,\mathrm{e}^{cz}\, G_b(r) \right) .$$

Then $\Phi_{c,\sigma}(h_n)$ is uniformly bounded with respect to both σ and n. In addition, $\{h_n\}$ is bounded in $W^{1,2}_r((0,1)\times(-\sigma,\sigma))$ for all σ, and, by a standard diagonal procedure, there exist h, belonging to $W^{1,2}_r((0,1)\times(-\sigma,\sigma))$ for all $\sigma > 0$, and a subsequence of $\{h_n\}$, which we shall denote again by $\{h_n\}$, such that

$$h(1,z) \;=\; g(z) \quad \text{for a.e. } z \in \mathbb{R},$$

$$h_n \rightharpoonup h \quad \text{in } W^{1,2}_r((0,1)\times(-\sigma,\sigma)) \quad \text{and} \quad h_n \to h \quad \text{a.e. in } (0,1)\times\mathbb{R},$$

and

$$\frac{\sin h_n}{r} \rightharpoonup \frac{\sin h}{r} \quad \text{in } L^2((-\sigma,\sigma); L^2_r(0,1))$$

(indeed, $\frac{\sin h_n}{r}$ is uniformly bounded in $L^2((-\sigma,\sigma); L^2_r(0,1))$ and the weak convergence follows from Dominated Convergence Theorem applied to the sequence $\{f \sin h_n\} = \{\frac{\sin h_n}{\sqrt{r}}\sqrt{r} f\}$, with $f \in L^2((-\sigma,\sigma); L^2_r(0,1))$).

Setting $f_n = h_n - h$, the identity $E_{c,\sigma}(h_n) \;=\; E_{c,\sigma}(f_n) + E_{c,\sigma}(h) + R$, with

$$R = \int_{-\sigma}^{\sigma} \mathrm{d}\,z \int_0^1 \left\{ r\,\mathrm{e}^{cz} \left(f_{nr}h_r + f_{nz}h_z + \frac{\sin f_n \sin h \cos h_n}{r^2} \right) \right\} \mathrm{d}\,r\,,$$

implies that

$$E_{c,\sigma}(h_n) \;=\; E_{c,\sigma}(h) + E_{c,\sigma}(h_n - h) + o(1) \quad \text{as } n \to \infty. \tag{2.16}$$

We fix $\sigma > 0$ and $\xi \in \mathfrak{C}^c$ such that $\operatorname{supp}(\xi) \subseteq [0,1]\times[-\sigma,\sigma]$. We claim that

$$2L(h_n,\xi) - 2L(h,\xi) \geq -E_{c,\sigma}(h_n - h) + o(1) \quad \text{as } n \to \infty. \tag{2.17}$$

This inequality follows easily from the decomposition $2L(h_n,\xi) - 2L(h,\xi) \;=\; I_{1,n} + I_{2,n} + I_{3,n} + I_{4,n}$, where

$I_{1,n} \;=\; \int_{-\sigma}^{\sigma} \mathrm{d}\,z \int_0^1 \mathrm{d}\,r \{\sin f_n \cos h (f_{nz}\xi_r - f_{nr}\xi_z)\}$,

$I_{2,n} \;=\; \int_{-\sigma}^{\sigma} \mathrm{d}\,z \int_0^1 \mathrm{d}\,r \{\sin h (\cos f_n - 1)(f_{nz}\xi_r - f_{nr}\xi_z)\}$,

$I_{3,n} \;=\; \int_{-\sigma}^{\sigma} \mathrm{d}\,z \int_0^1 \mathrm{d}\,r \{\sin h (f_{nz}\xi_r - f_{nr}\xi_z)\}$,

$I_{4,n} \;=\; -\int_{-\sigma}^{\sigma} \mathrm{d}\,z \int_0^1 \mathrm{d}\,r \{(\sin h - \sin h_n)(h_z\xi_r - h_r\xi_z)\}$;

$|I_{1,n}| \leq \int_{-\sigma}^{\sigma} \mathrm{d}\,z \int_0^1 r \{\frac{|\sin f_n|}{r} |\nabla f_n|\,|\nabla \xi|\}\,\mathrm{d}\,r \leq \int_{-\sigma}^{\sigma} \mathrm{d}\,z \int_0^1 r\,\mathrm{e}^{cz} \{\frac{\sin^2 f_n}{2r^2} + |\nabla f_n|^2\}\,\mathrm{d}\,r = E_{c,\sigma}(f_n)$, and $I_{i,n} \to 0$ as $n \to \infty$ for $i = 2,3,4$.

Combining (2.16) and (2.17) and taking σ and ξ as before, we have that

$$\Phi_{c,\sigma}(h) + 2L(h,\xi) \leq \Phi_{c,\sigma}(h_n) + 2L(h_n,\xi) + o(1) \leq \Phi_{c,\sigma}(h_n) + 2L_c(h_n) + o(1). \quad (2.18)$$

Arguing as in the proof of (2.10), we obtain that $\Phi_c(h_n) \geq \Phi_{c,\sigma}(h_n) - \frac{2b^2\,\mathrm{e}^{-c\sigma}}{c(1+b^2)}$ for all $\sigma > z_1$. Since $\Phi_c(h_n) + 2L_c(h_n) \to \mathcal{I}$ as $n \to \infty$, it follows from (2.18) that $\Phi_{c,\sigma}(h) + 2L(h,\xi) \leq \mathcal{I} + \frac{2b^2\,\mathrm{e}^{-c\sigma}}{c(1+b^2)}$ for all $\xi \in \mathfrak{C}^c$ and $\sigma > z_1$ such that $\mathrm{supp}(\xi) \subseteq [0,1] \times [-\sigma,\sigma]$. Letting $\sigma \to \infty$ we find that $\Phi_c(h) + 2L(h,\xi) \leq \mathcal{I}$ for all $\xi \in \mathfrak{C}^c$, and hence h solves the second variational problem.

It remains to prove points (i)-(iv) of Theorem 2.7. The proof of (i) is standard. The proofs of (ii) and (iii) are similar and we omit the one of (ii).

Proof of (iii). Let $f_1(r,z) = \max\{2\arctan(br), h_2(r,z)\}$. Then $f_1 \in \mathcal{W}$, $f_1(1,z) = g(z)$ for a.e. $z \in \mathbb{R}$, $|f_{1r}| \leq \max(|h_{2r}|, \frac{2b}{1+b^2r^2})$, and $|f_{1z}| \leq |h_{2z}|$. We fix $z \in \mathbb{R}$ arbitrarily. Since $h_2(r,z) - 2\arctan(br)$ is real analytic in $(0,1)$, we may write

$$E_-(z) \equiv \{r \in (0,1)\,;\, h_2(r,z) < 2\arctan(br)\} = \bigcup_{n \in \mathcal{T} \subseteq \mathbb{Z}} (\alpha_n, \beta_n), \quad (2.19)$$

where $0 \leq \alpha_n < \beta_n \leq \alpha_{n+1} < \beta_{n+1} \leq 1$ for $n, n+1 \in \mathcal{T}$. We observe that, for all $n \in \mathcal{T}$, $h_2(\beta_n, z) = 2\arctan(b\beta_n)$ and, if $\alpha_n > 0$, $h_2(\alpha_n, z) = 2\arctan(b\alpha_n)$. We set

$$H(r,z;u) = \frac{r}{2}\left(u_r^2(r,z) + u_z^2(r,z) + \frac{\sin^2 u(r,z)}{r^2}\right).$$

Then

$$\int_0^1 (H(r,z;f_1) - H(r,z;h_2))\,\mathrm{d}\,r = \int_{E_-(z)} (H(r,z;2\arctan(br)) - H(r,z;h_2))\,\mathrm{d}\,r$$

$$= \sum_{n \in \mathcal{T}} \int_{\alpha_n}^{\beta_n} (H(r,z;2\arctan(br)) - H(r,z;h_2))\,\mathrm{d}\,r.$$

By Corollary A.3

$$\int_{\alpha_n}^{\beta_n} (H(r,z;2\arctan(br)) - H(r,z;h_2))\,\mathrm{d}\,r \leq 0 \quad \text{if } \alpha_n > 0. \quad (2.20)$$

We observe that $\alpha_n = 0$ may happen for at most one value of n, and if so we may assume without loss of generality that $\alpha_0 = 0$. Since $0 < b < 1$, it follows in this case from Theorem A.6 that also

$$\int_0^{\beta_0} (H(r,z;2\arctan(br)) - H(r,z;h_2))\,\mathrm{d}\,r \leq 0 \quad \text{if } \alpha_0 = 0. \quad (2.21)$$

Hence, by (2.20) and (2.21),

$$\int_0^1 H(r,z;f_1)\,\mathrm{d}\,r - \int_0^1 H(r,z;h_2)\,\mathrm{d}\,r \leq 0. \quad (2.22)$$

Since (2.22) holds for a.e. $z \in \mathbb{R}$ we conclude that $\Phi_c(f_1) \leq \Phi_c(h_2)$. In particular $f_1 \in \mathcal{W}^c$. In addition it follows from Theorem 2.3 that $L_c(f_1) \leq L_c(h_2)$. This implies that f_1 is a solution of the second variational problem. By standard regularity theory f_1 is smooth in $(0,1)\times\mathbb{R}$ and, by the strong maximum principle, $f_1(r,z) > 2\arctan(br)$ for all $(r,z) \in (0,1)\times\mathbb{R}$. Hence $f_1 = h_2$ in $(0,1)\times\mathbb{R}$ and we have proved the first inequality in (iii).

Similarly we define $f_2 = \min\{2\pi + 2\arctan(a_2 r), h_2(r,z)\}$. Arguing as before, with $E_-(z)$ replaced by $E_+(z) = \{r \in (0,1)\,;\, h_2(r,z) > 2\pi + 2\arctan(a_2 r)\}$, only the inequality (2.21) needs to be slightly modified. So we suppose that there exist $z \in \mathbb{R}$ and $\beta_0 \in (0,1]$ such that

$$h_2(r,z) > 2\pi + 2\arctan(a_2 r) \text{ for } 0 < r < \beta_0 \quad\text{and}\quad h_2(\beta_0, z) = 2\pi + 2\arctan(a_2\beta_0). \tag{2.23}$$

In view of (2.9) we may assume without loss of generality that $h_2(0,z) = k_0(z)\pi$ for some $k_0(z) \in \mathbb{Z}$. By (2.23) we have that $k_0(z) \geq 2$. If $k_0(z) = 2$ or if $k_0(z) \geq 4$, we obtain from Lemma A.1 that (2.21) still holds, with $2\arctan(br)$ replaced by $2\pi + 2\arctan(a_2 r)$. In the remaining case, $k_0(z) = 3$, (2.21) is replaced by the inequality

$$\int_0^{\beta_0} \left(H(r,z;2\pi + 2\arctan(a_2 r)) \;-\; H(r,z;h_2)\right) \mathrm{d}\,r \leq 2 \quad \text{if } h_2(0,z) = 3\pi,$$

which follows easily from Lemma A.1. This means that the inequality $\Phi_c(f_2) \leq \Phi_c(h_2)$ is not necessarily valid, but since $\cos(h_2(0,z)) = -1$ if $k_0(z) = 3$, it follows easily from Theorem 2.3 that the inequality $\Phi_c(f_2) + 2L_c(f_2) \leq \Phi_c(h_2) + 2L_c(h_2)$ holds.

Proof of (iv). We only prove the result for h_1, which we shall denote by h. It follows from (2.7) and (2.8) that

$$U(z) \;\equiv\; \int_0^1 \frac{r}{2}\left(h_r^2 + \frac{\sin^2 h}{r^2} - G_b(r)\right)\Big|_z \,\mathrm{d}\,r \geq 0 \quad \text{if } z \geq z_1,$$

where z_1 is defined by (2.2). Since $\int_{z_1}^{\infty} \mathrm{e}^{cz}\, U(z)\,\mathrm{d}\,z \leq \Phi_c(h) + \frac{2b^2\,\mathrm{e}^{cz_1}}{c(1+b^2)} < \infty$, there exists a sequence $z_n \to \infty$ such that $U(z_n) \to 0$ as $n \to \infty$. Hence, by Theorem A.7, $h(r,z_n) \to 2\arctan(br)$ uniformly with respect to $r \in [0,1]$ as $n \to \infty$.
By standard Schauder estimates, for any $\rho > 0$ the function $V_\rho(z) \equiv \int_\rho^1 h_z^2(r,z)\,\mathrm{d}\,r$ is Lipschitz continuous in $\mathbb{R}$.
On the other hand, the inequality $\int_{z_1}^{\infty}\mathrm{d}\,z\int_0^1 \frac{r\,\mathrm{e}^{cz}}{2} h_z^2\,\mathrm{d}\,r \;\leq \Phi_c(h) + \frac{2b^2\,\mathrm{e}^{cz_1}}{c(1+b^2)}$ implies

$$\int_{z_1}^{\infty} V_\rho(z)\,\mathrm{e}^{cz}\;\mathrm{d}\,z < \infty$$

and then $V_\rho(z)\,\mathrm{e}^{\frac{c}{2}z} \to 0$ as $z \to \infty$.
By Schauder estimates, from here follows the existence of $K, \delta > 0$ such that $\|h_z(\cdot,z)\|_{L^\infty(\rho,1)} \leq K\,\mathrm{e}^{-\delta z}$. Hence $\lim\limits_{z\to\infty} h(r,z)$ exists for all $r \in (0,1]$ and it is equal to $\lim\limits_{n\to\infty} h(r,z_n) = 2\arctan(br)$. Obviously, for any $\rho > 0$,

$$\lim_{z\to\infty} h(r,z) = 2\arctan(br) \quad \text{uniformly with respect to } r \in [\rho,1]. \tag{2.24}$$

It remains to show that the limit is uniform with respect to $r \in (0,1]$. In the next section we shall show that we may assume that h is decreasing with respect to z (in the proof we shall use (2.24)). Hence the uniform convergence follows at once from the uniform convergence along the subsequence $\{z_n\}$. ∎

2.4 Monotonicity properties of minimizers

In this section we shall show that our two variational problems have solutions which are decreasing with respect to z. We use a onedimensional monotone rearrangement technique ([20]) applied to the variable $x = \mathrm{e}^{cz}$.

Let T be the operator induced by the transformation $x = \mathrm{e}^{cz}$ as introduced in section 2.2 . If f_1 and f_2 are the functions defined by

$$f_1 \equiv T(h_1) \quad \text{and} \quad f_2 \equiv T(h_2)\,, \tag{2.25}$$

then, in view of Theorem 2.7, (2.24) and standard Schauder estimates applied to equation (7), f_1 and f_2 satisfy properties (P1)-(P4) of Appendix B, with $\ell(r) = 2\arctan(br)$. If we denote by f_1^* and f_2^* the one-dimensional decreasing rearrangements of f_1 and f_2 with respect to the x variable (see Appendix B for the exact definition), then by using some results collected in Appendix B we can prove that

Theorem 2.8. *The functions $T^{-1}(f_1^*)$ and $T^{-1}(f_2^*)$ are solutions of, respectively, the first and second variational problem.*

Proof: It follows at once from Propositions B.7 and B.8 that $\Psi_c(f_i^*) \leq \Psi_c(f_i)$, and hence, by (2.14), $\Phi_c(T^{-1}(f_i^*)) \leq \Phi_c(h_i)$ for $i = 1, 2$.

In view of (2.15) it remains to prove that $\mathcal{L}_c(f_2^*) = \mathcal{L}_c(f_2)$. By Theorem 2.7(iii) and Propositions 2.5 and 2.6, this is equivalent to proving that, for $\sigma > 0$ small enough,

$$\lim_{r\to 0^+} \mu(\{x>0; \pi-\sigma \leq f_2^*(r,x) < \pi+\sigma\}) = \lim_{r\to 0^+} \mu(\{x>0; \pi-\sigma \leq f_2(r,x) < \pi+\sigma\}).$$

The latter equality follows at once from (B.1). $\square$

Corollary 2.9. *We may assume that the functions h_1 and h_2, defined in Theorem 2.7, are strictly decreasing with respect to z in $(0,1)\times\mathbb{R}$, and that for all $\rho > 0$*

$$h_1(r,z) \to \pi + 2\arctan(a_1 r) \ \textit{uniformly with respect to } r \in [\rho, 1] \ \textit{as } z \to -\infty. \tag{2.26}$$

The first part of Corollary 2.9 follows at once from Theorem 2.8 and the monotonicity of the rearranged functions. The monotonicity of h_1 implies the existence of the limit in (2.26), which we denote by $v(r)$. It easily follows that v is a solution of the equation $v_{rr} + \frac{1}{r}v_r - \frac{\sin(2v)}{2r^2} = 0$ in the interval $(0,1)$, with boundary condition $v(1) = g(-\infty) = \pi + 2\arctan a_1$. In addition it follows from Theorem 2.7(ii) that $2\arctan(br) \leq v(r) \leq \pi + 2\arctan(a_1 r)$ in $(0,1)$. The only function v satisfying all these conditions is the function $\pi + 2\arctan(a_1 r)$. It follows at once from Schauder estimates that the convergence is uniform in the sets $[\rho, 1]$ for $\rho > 0$, which completes the proof of Corollary 2.9.

We observe that, arguing as before, we need the condition that $a_2 \geq 0$ to obtain a result similar to (2.26) for the function h_2:

$$h_2(r,z) \to 2\pi + 2\arctan(a_2 r) \text{ uniformly with respect to } r \in [\rho, 1] \text{ as } z \to -\infty. \tag{2.27}$$

Indeed, if $a_2 < 0$ the same procedure leads to two possible limit functions in (2.27): $2\pi + 2\arctan(a_2 r)$ and $\pi - 2\arctan\left(\frac{r}{a_2}\right)$. Only in section 2.5 we shall be able to exclude the latter possibility.

2.5 Existence of a point singularity

By Theorems 2.7 and 2.8, both variational problems have a minimizer which is strictly decreasing with respect to z in $(0,1)\times\mathbb{R}$. In this section we complete the proofs of Theorems 2.1 and 2.2. In particular we shall prove that both minimizers have exactly one singular point at the axis $r=0$ and we shall determine the behavior of the minimizers as $z\to-\infty$.

Theorem 2.10. *Let h_1 and h_2 be a minimizer of, respectively, the first and second variational problem which is strictly decreasing with respect to z for all $0<r<1$.*
(i) There exists $\bar z_1\in\mathbb{R}$ such that $h_1(0,z)=\pi$ if $z<\bar z_1$ and $h_1(0,z)=0$ if $z>\bar z_1$.
(ii) $h_1(r,z)\to\pi+2\arctan(a_1 r)$ uniformly with respect to $r\in[0,1]$ as $z\to-\infty$, where a_1 is defined by $\pi+2\arctan a_1=A$.
(iii) There exists $\bar z_2\in\mathbb{R}$ such that $h_2(0,z)=2\pi$ if $z<\bar z_2$ and $h_2(0,z)=0$ if $z>\bar z_2$.
(iv) $h_2(r,z)\to 2\pi+2\arctan(a_2 r)$ uniformly with respect to $r\in[0,1]$ as $z\to-\infty$, where a_2 is defined by $2\pi+2\arctan a_2=A$.
(v) h_i is continuous in $[0,1]\times\mathbb{R}\setminus\{(0,\bar z_i)\}$ and real analytic in $[0,1)\times\mathbb{R}\setminus\{(0,\bar z_i)\}$ ($i=1,2$).

The proof of (i) is based on the following lemma. We omit its proof, which is based on straightforward computations and estimates.

Lemma 2.11. *Let $p<q$ and $\alpha\in C^1((p,q])$ be such that*

$$\alpha>0 \text{ in } (p,q], \quad \frac{(\alpha')^2}{\alpha^3}\in L^1(p,q), \quad \alpha(z)\to\infty \text{ and } \frac{\alpha'(z)}{\alpha^2(z)}\to 0 \text{ as } z\to p^+.$$

Then the function $v\in C^1((0,1]\times(p,q))$, defined by

$$v(r,z) \;=\; 2\arctan\left(\frac{\alpha(z)r^2}{r+1}\right) \quad \text{for } (r,z)\in(0,1]\times(p,q],$$

satisfies
(i) $\int_0^1 rv_r^2(r,z)\,\mathrm{d}r\le 12$, $\int_0^1\frac{\sin^2 v(r,z)}{r}\,\mathrm{d}r\le 6$, $\int_0^1 rv_z^2(r,z)\,\mathrm{d}r\le 8(\alpha'(z))^2\alpha^{-3}(z)$ for $p<z\le q$;
(ii) $v_z\in L^2((p,q);L^2_r(0,1))$;
(iii) for all $0<\rho<1$, $v(r,z)\to\pi$ and $v_r(r,z),v_z(r,z)\to 0$ uniformly in $[\rho,1]$ as $z\to p^+$.

Proof of Theorem 2.10(i). By (2.9), $h_1(0,z)=k(z)\pi$ for some integer $k(z)$ for a.e. z. By Theorem 2.7(ii) and (iv), $k(z)=0$ or $k(z)=1$ for a.e. z, and $k(z)=0$ for z large enough. Since h_1, and hence also k, is nonincreasing with respect to z, it remains to show that $k\not\equiv 0$ in $\mathbb{R}$. We argue by contradiction and suppose that $h_1(0,z)=0$ for all $z\in\mathbb{R}$.

Given $n\in\mathbb{N}$ and $0<r_n<1$, by (2.26) there exists $q_n\le z_0$ such that $h_1(r,z)\ge\pi$ for $z\le q_n$ and $r\in[r_n,1]$. We define $p_n=q_n-\frac{1}{n}$, $\alpha_n(z)=(z-p_n)^{-2}$ and

$$h_{1,n}(r,z) \;=\; \begin{cases} h_1(r,z) & z>q_n \\ \max\{h_1(r,z),v_n(r,z)\} & z\in(p_n,q_n] \\ \max\{\pi,h_1(r,z)\} & z\le p_n, \end{cases}$$

where

$$v_n(r,z) = 2\arctan\left(\frac{\alpha_n(z)r^2}{r+1}\right).$$

Choosing $r_n = \frac{b}{n^2-b}$, which is a root of the equation $v_n(r,q_n) = 2\arctan(br)$, it follows easily from Lemma 2.11 and the definition of p_n and q_n that $h_{1,n} \in \mathcal{W}^c$.

We claim that $\Phi_c(h_{1,n}) < \Phi_c(h_1)$ for n large enough, which is a contradiction since h_1 is a minimizer of Φ_c in $\mathcal{W}^c$.

Given a measurable set $S \subset (0,1)\times\mathbb{R}$ and $f \in \mathcal{W}$, we set

$$E_S(f) := \iint_S \frac{r}{2}\,\mathrm{e}^{cz}\left(f_r^2 + f_z^2 + \frac{\sin^2 f}{r^2}\right)\mathrm{d}\,r\,\mathrm{d}\,z.$$

Then $\Phi_c(h_{1,n}) - \Phi_c(h_1) = I_{1,n} - I_{2,n}$, where

$$I_{1,n} = E_{[0,1]\times(p_n,q_n)}(h_{1,n}) - E_{[0,1]\times(p_n,q_n)}(h_1),$$

$$I_{2,n} = E_{[0,1]\times(-\infty,p_n)}(h_{1,n}) - E_{[0,1]\times(-\infty,p_n)}(h_1).$$

By Lemma 2.11, $I_{1,n} \le E_{[0,1]\times(p_n,q_n)}(v_n) \le \frac{25}{c}\,\mathrm{e}^{cp_n}(\mathrm{e}^{\frac{c}{n}}-1)$.

We define $\rho(z) = \inf\{r \in (0,1]; h_1(r,z) \ge \pi\}$ for $z \le z_0$. Then $0 < \rho(z) < 1$, since $h_1(0,z) = 0$ and $h_1(1,z) = \pi + 2\arctan a_1$ if $z \le z_0$. We set

$$A_n = \{(r,z); 0 < r < \rho(z), z < p_n\} \text{ and } B_n = \{(r,z); \rho(z) < r < 1, z < p_n\}.$$

Since $h_{1,n} = \pi$ in A_n, it follows from Lemma A.1 that

$$E_{A_n}(h_{1,n}) - E_{A_n}(h_1) = -E_{A_n}(h_1) \le -2\int_{-\infty}^{p_n} \mathrm{e}^{cz}\,\mathrm{d}\,z = -\frac{2}{c}\,\mathrm{e}^{cp_n}.$$

Since $|(h_{1,n})_r| \le |h_{1r}|$, $|(h_{1,n})_z| \le |h_{1z}|$ and $|\sin h_{1,n}| \le |\sin h_1|$ in B_n, this implies that $I_{2,n} \le -\frac{2}{c}\,\mathrm{e}^{cp_n}$.

We conclude that $\Phi_c(h_{1,n}) - \Phi_c(h_1) \le \frac{\mathrm{e}^{cp_n}}{c}(25\,\mathrm{e}^{\frac{c}{n}} - 25 - 2) < 0$ for n large enough, and we have proved our claim. $\square$

Proof of Theorem 2.10(ii). The uniform convergence follows at once from (2.26), Theorem 2.10(i), the monotonicity in z and the upper bound in Theorem 2.7(ii). $\square$

In the proof of part (iii) we shall use an auxiliary lemma which is based on the following proposition.

Proposition 2.12. *Let h_i be as in Theorem 2.10, I be an open nonempty interval and $k \in \mathbb{Z}$ a constant such that $h_i(0,z) = k\pi$ for $z \in I$. Then h_i is real analytic in $[0,1)\times I$.*

Proof: The argument proving the thesis is the same as for Proposition 1.24 (replace θ by h_i). $\square$

Lemma 2.13. *Let h_i, I and k be as in Proposition 2.12. Then there exists $\tilde{z} \in I$ such that $(h_i)_r(0,\tilde{z}) \neq 0$.*

Proof: Omitting the subscript i and arguing by contradiction we suppose that $h_r(0,z) = 0$ for all $z \in I$. We claim that for all positive integers α

$$\frac{\partial^\alpha h}{\partial r^\alpha}(0,z) \equiv 0 \quad \text{for } z \in I. \tag{2.28}$$

This leads immediately to a contradiction: by Proposition 2.12 and (2.28) h is constant in $(0,1)\times I$, which is impossible since h is strictly decreasing with respect to z in $(0,1)\times\mathbb{R}$.

In order to prove (2.28) we argue by induction. We know that (2.28) is true for $\alpha = 1$. Suppose that it is true for $\alpha = 1,...,\beta$ for some $\beta \geq 1$. Using a Taylor expansion we obtain that for all $z \in I$ and $\alpha = 1,...,\beta$

$$\frac{\partial^\alpha}{\partial r^\alpha}\Big(\sin(2h)\Big)\Big|_{r=0} = 0 \quad \text{and} \quad \frac{\partial^{\beta+1}}{\partial r^{\beta+1}}\Big(\sin(2h)\Big)\Big|_{r=0} = 2\frac{\partial^{\beta+1}h}{\partial r^{\beta+1}}(0,z),$$

$$\begin{aligned}
h_{rr}(r,z) &= \frac{1}{(\beta-1)!}\frac{\partial^{\beta+1}h}{\partial r^{\beta+1}}(0,z)r^{\beta-1} + O(r^\beta),\\
h_r(r,z)r^{-1} &= \frac{1}{\beta!}\frac{\partial^{\beta+1}h}{\partial r^{\beta+1}}(0,z)r^{\beta-1} + O(r^\beta),\\
\frac{1}{2}\sin(2h(r,z))r^{-2} &= \frac{1}{(\beta+1)!}\frac{\partial^{\beta+1}h}{\partial r^{\beta+1}}(0,z)r^{\beta-1} + O(r^\beta),\\
h_z(r,z) &= O(r^{\beta+1}),\ h_{zz}(r,z) = O(r^{\beta+1}).
\end{aligned}$$

Substituting these equalities in equation (7), we find that (2.28) holds for $\alpha = \beta+1$. □

Proof of Theorem 2.10(iii). The proof consists of two steps. In the first one we exclude the possibility that $h_2(0,z) = 0$ for all $z \in \mathbb{R}$. In the second one we show that $h_2(0,z) \neq \pi$ for a.e. $z \in \mathbb{R}$. Since h_2 is nonincreasing with respect to z, the proof is then completed by Theorem 2.7(iii).

Step 1. We only give the proof in the case that $a_2 < 0$ (if $a_2 \geq 0$ the proof can be considerably simplified). As in the proof of part (i) we argue by contradiction and suppose that $h_2(0,z) = 0$ for all $z \in \mathbb{R}$.

Given $n \in \mathbb{N}$ and $\rho_n = \frac{b}{n^2-b}$, the statement which follows formula (2.27) (which treats the case $a_2 < 0$) implies that there exists $q_n \leq z_0$ such that

$$h_2(r,z) \geq \pi + 2\arctan\left(\frac{r}{2|a_2|}\right) \qquad \text{if } z \leq q_n \text{ and } \rho_n \leq r \leq 1. \tag{2.29}$$

We set

$$p_n = q_n - \frac{1}{n}, \quad z_n = p_n - 1, \quad r_n \in [\rho_n, 1],$$

and we define for all $0 \leq r \leq 1$

$$\begin{aligned}
&v_n(r,z) = 2\arctan\left(\tfrac{\alpha_n(z)r^2}{r+1}\right), && p_n < z \leq q_n,\\
&w_n(r,z) = \pi + 2\arctan(\beta_n(z)r), && z_n \leq z \leq p_n,\\
&\omega_n(r,z) = \max\Big\{2\pi - 2\arctan(\gamma_n(z)r), \pi + 2\arctan\Big(\tfrac{r}{2|a_2|}\Big)\Big\}, && z_n - r_n \leq z < z_n,\\
&\chi_n(r) = \omega_n(r, z_n - r_n),
\end{aligned}$$

where

$$\alpha_n(z) = \frac{1}{(z-p_n)^2}, \quad \beta_n(z) = \frac{p_n - z}{2|a_2|}, \quad \gamma_n(z) = \frac{2|a_2|}{(z_n - z)^2}.$$

Finally, we set, for $0 \le r \le 1$,

$$h_{2,n}(r,z) = \begin{cases} h_2(r,z) & \text{if } z > q_n \\ \max\{h_2(r,z), v_n(r,z)\} & \text{if } z \in (p_n, q_n] \\ \max\{h_2(r,z), w_n(r,z)\} & \text{if } z \in [z_n, p_n] \\ \max\{h_2(r,z), \omega_n(r,z)\} & \text{if } z \in [z_n - r_n, z_n) \\ \max\{h_2(r,z), \chi_n(r)\} & \text{if } z < z_n - r_n. \end{cases}$$

It is easy to show that $h_{2,n}$ is locally Lipschitz continuous in $(0,1] \times \mathbb{R}$ and belongs to $\mathcal{W}^c$. To obtain a contradiction it is enough to show that

$$\Phi_c(h_{2,n}) + 2L_c(h_{2,n}) < \Phi_c(h_2) + 2L_c(h_2) \quad \text{for } n \text{ large enough.} \tag{2.30}$$

Defining $E_S(f)$ as in the proof of part (i), we write

$$\Phi_c(h_{2,n}) - \Phi_c(h_2) = I_{1,n} + I_{2,n} + I_{3,n} + I_{4,n},$$

where

$$\begin{aligned} I_{1,n} &:= E_{(0,1)\times(p_n,q_n)}(h_{2,n}) - E_{(0,1)\times(p_n,q_n)}(h_2), \\ I_{2,n} &:= E_{(0,1)\times(z_n,p_n)}(h_{2,n}) - E_{(0,1)\times(z_n,p_n)}(h_2), \\ I_{3,n} &:= E_{(0,1)\times(z_n-r_n,z_n)}(h_{2,n}) - E_{(0,1)\times(z_n-r_n,z_n)}(h_2), \\ I_{4,n} &:= E_{(0,1)\times(-\infty,z_n-r_n)}(h_{2,n}) - E_{(0,1)\times(-\infty,z_n-r_n)}(h_2). \end{aligned}$$

By Lemma 2.11,

$$I_{1,n} \le E_{(0,1)\times(p_n,q_n)}(v_n) \le \frac{25}{c} \, \mathrm{e}^{cp_n} \left(\mathrm{e}^{\frac{c}{n}} - 1 \right). \tag{2.31}$$

Since $w_n(r,z) \le \pi + 2\arctan(r/(2|a_2|))$ if $0 < r < 1$ and $z_n < z < p_n$, it follows from (2.29) that

$$I_{2,n} \le E_{(0,\rho_n)\times(z_n,p_n)}(h_{2,n}) - E_{(0,\rho_n)\times(z_n,p_n)}(h_2).$$

Hence, by Corollary A.3 and a straightforward calculation,

$$\begin{aligned} I_{2,n} &\le \textstyle\int_{z_n}^{p_n} \mathrm{e}^{cz} \left(-2 + \int_0^{\rho_n} \frac{1}{2} r (h_{2,n})_z^2 \, \mathrm{d}\, r \right) \mathrm{d}\, z \\ &= \textstyle\int_{z_n}^{p_n} \mathrm{e}^{cz} \left(-2 + \left(\frac{\beta_n'(z)}{\beta_n^2(z)} \right)^2 \left(\log(1 + \beta_n^2 \rho_n^2) - 1 + \frac{1}{1+\beta_n^2\rho_n^2} \right) \right) \mathrm{d}\, z, \end{aligned}$$

and there exists a constant $C_2 > 0$ which does not depend on n such that

$$I_{2,n} \le \frac{-2 + C_2 \rho_n^4}{c} (\mathrm{e}^{cp_n} - \mathrm{e}^{cz_n}). \tag{2.32}$$

Since $\gamma_n(z) r \ge 2|a_2|/r_n$ for all $r \ge r_n (\ge \rho_n)$, it follows from (2.29) that

$$2\pi - 2\arctan(\gamma_n(z) r) \le \pi + 2\arctan\left(\frac{r}{2|a_2|} \right) \le h_2(r,z) \quad \text{if } r \ge r_n.$$

Hence

$$I_{3,n} = E_{(0,r_n)\times(z_n-r_n,z_n)}(h_{2,n}) - E_{(0,r_n)\times(z_n-r_n,z_n)}(h_2),$$

and, by Corollary A.3,

$$\begin{aligned} I_{3,n} &\leq \int_{z_n-r_n}^{z_n} \mathrm{e}^{cz}\,\mathrm{d}\,z \int_0^{r_n} \tfrac{1}{2}r((h_{2,n})_z^2 - (h_2)_z^2)\,\mathrm{d}\,r \\ &\leq \int_{z_n-r_n}^{z_n} \mathrm{e}^{cz}\,\mathrm{d}\,z \int_0^{r_n} r(\arctan(\gamma_n(z)r))_z^2\,\mathrm{d}\,r \\ &= \int_{z_n-r_n}^{z_n} \mathrm{e}^{cz} \left(\tfrac{\gamma_n'(z)}{\gamma_n^2(z)}\right)^2 \left(\log(1+\gamma_n^2 r_n^2) - 1 + \tfrac{1}{1+\gamma_n^2 r_n^2}\right)\mathrm{d}\,z. \end{aligned}$$

Since $\log(1+s^2) \leq 4\sqrt{s}$ for $s > 0$, it follows easily that there exists a constant $C_3 > 0$ which does not depend on n such that

$$I_{3,n} \leq C_3 r_n^2 \sqrt{r_n}\,\mathrm{e}^{cz_n}\,. \tag{2.33}$$

Since $\chi_n(r) = \pi + 2\arctan(r/(2|a_2|))$ for $r \geq r_n \geq \rho_n$,

$$I_{4,n} = E_{(0,r_n)\times(-\infty,z_n-r_n)}(h_{2,n}) - E_{(0,r_n)\times(-\infty,z_n-r_n)}(h_2).$$

On the other hand, $\chi_n(r) = 2\pi - 2\arctan(2|a_2|r/r_n^2)$ for $r \leq r_n$, and hence, setting $S_- := \{(r,z) \in (0,r_n)\times(-\infty,z_n-r_n); h_2(r,z) < \chi_n(r)\}$,

$$\begin{aligned} I_{4,n} &\leq \iint_{S_-} \tfrac{1}{2}r\,\mathrm{e}^{cz}\left(\left((\chi_n)_r^2 + \tfrac{\sin^2\chi_n}{r^2}\right) - \left((h_{2,n})_r^2 + \tfrac{\sin^2 h_{2,n}}{r^2}\right)\right)\mathrm{d}\,r\,\mathrm{d}\,z \\ &\leq \int_{-\infty}^{z_n-r_n} \mathrm{e}^{cz}(J_{1,n}(z) - J_{2,n}(z))\,\mathrm{d}\,z, \end{aligned}$$

where

$$J_{1,n}(z) := \int_0^{r_n} \frac{1}{2}r\left((\chi_n)_r^2 + \frac{\sin^2\chi_n}{r^2}\right)\mathrm{d}\,r = 2 - \frac{2r_n^2}{r_n^2 + 4|a_2|^2},$$

and

$$J_{2,n}(z) := \int_0^{\rho(z)} \frac{1}{2}r\left((h_{2,n})_r^2 + \frac{\sin^2 h_{2,n}}{r^2}\right)\mathrm{d}\,r$$

with $\rho(z) := \inf\{r \in [0,r_n]; h_2(r,z) \geq \pi\}$. By Lemma A.1 $J_{2,n}(z) \geq 2$, and hence there exists a constant $C_4 > 0$ which does not depend on n such that

$$I_{4,n} \leq -C_4 r_n^2\,\mathrm{e}^{c(z_n-r_n)}\,. \tag{2.34}$$

Since $L_c(h_{2,n}) = \frac{\mathrm{e}^{cp_n} - \mathrm{e}^{cz_n}}{c}$ (by Theorem 2.3), it follows from (2.32), (2.33) and (2.34) that there exists $\delta > 0$ such that if $\rho_n \leq r_n \leq \delta$ then

$$I_{2,n} + I_{3,n} + I_{4,n} + 2L_c(h_{2,n}) \leq -\frac{1}{2}C_4 r_n^2\,\mathrm{e}^{c(z_n-r_n)}\,.$$

Hence, by (2.31), we can choose n so large that

$$\Phi_c(h_{2,n}) + 2L_c(h_{2,n}) < \Phi_c(h_2),$$

and (2.30) follows.

Step 2. We argue by contradiction and suppose that there exist $p < q$ such that $h_2(0,z) = \pi$ if $p < z < q$. In view of Lemma 2.13 and the monotonicity of h_2 with respect to z, we may assume, without loss of generality, that for some $k_0 > 0$ either

$$(h_2)_r(0,z) \geq (h_2)_r(0,q) > k_0 > 0 \quad \text{if } z < q \tag{2.35}$$

or

$$(h_2)_r(0,z) \le (h_2)_r(0,p) < -k_0 < 0 \quad \text{if } p < z. \tag{2.36}$$

One way to obtain a contradiction is to modify the proof of a more general result in [18]. Alternatively, we can use the approach used in the proof of part (iii): if (2.35) holds, we can construct a function h_2^* such that $h_2^*(0,z) = 2\pi$ if $z < q$ and $\Phi_c(h_2^*) + 2L_c(h_2^*) < \Phi_c(h_2) + 2L_c(h_2)$; if (2.36) holds, a similar function h_2^* exists such that $h_2^*(0,z) = 0$ if $z > p$. For example, in the first case we can choose h_2^* of the type

$$h_2^*(r,z) = \begin{cases} h_2(r,z) & \text{if } 0<r<1, z\ge q \text{ or } r^*<r<1, z<q \\ \max\{h_2(r,z), \omega(r,z)\} & \text{if } 0<r<r^*, z \in [q-z^*, q) \\ \max\{h_2(r,z), \omega(r, q-z^*)\} & \text{if } 0<r<r^*, z<q-z^*, \end{cases}$$

where $\omega(r,z) = 2\pi - 2\arctan(\gamma(z)r)$, $\gamma_n(z) = C^*(q-z)^{-2}$, and r^*, z^* and C^* are constants to be chosen appropriately. We leave the details to the interested reader. □

Proof of Theorem 2.10(iv). The uniform convergence follows at once from formula (2.27) (which holds only if $a_2 \ge 0$) and the sentence immediately after (2.27) (which holds if $a_2 < 0$), Theorem 2.10(iii), the monotonicity in z and the upper bound in Theorem 2.7(iii). □

Proof of Theorem 2.10(v). The proof is an immediate consequence of Proposition 2.12. □

2.6 Position of the singularity when $c \to \infty$

Let $c > 0$ and let h_1 and h_2 be the solutions given by, respectively, Theorems 2.1 and 2.2 with a point singularity in $(0, \bar z_1)$ and $(0, \bar z_2)$. In this section we consider the behavior of $\bar z_i$ as $c \to \infty$. We shall often add the subscript c and use the notation $h_{i,c}$ and $\bar z_{i,c}$ $(i = 1, 2)$.

We first give a heuristic argument and set

$$\tau = -\frac{z}{c}, \quad \tau_{i,c} = -\frac{\bar z_{i,c}}{c} \quad \text{and} \quad q_{i,c}(r,\tau) = h_{i,c}(r, -c\tau). \tag{2.37}$$

Then $q_{i,c}$ is smooth in $[0,1] \times \mathbb{R} \setminus \{(0, \tau_{i,c})\}$ and is a solution of the equation

$$q_\tau = \frac{q_{\tau\tau}}{c^2} + q_{rr} + \frac{q_r}{r} - \frac{\sin(2q)}{2r^2} \quad \text{in } (0,1) \times \mathbb{R}. \tag{2.38}$$

In addition $q_{i,c}$ satisfies the properties:

$$\begin{cases} q_{i,c}(r,\infty) = i\pi + 2\arctan(a_i r) & r \in [0,1] \\ q_{i,c}(r,-\infty) = 2\arctan(br) & r \in [0,1] \\ q_{i,c}(1,\tau) = g(-c\tau) & \tau \in \mathbb{R} \\ q_{i,c}(0,\tau) = 0 & \tau < \tau_{i,c} \\ q_{i,c}(0,\tau) = i\pi & \tau > \tau_{i,c}. \end{cases} \tag{2.39}$$

If $q_{i,c}$ converges to some limit function q_i as $c \to \infty$, it is plausible that q_i satisfies the parabolic equation

$$q_\tau = q_{rr} + \frac{q_r}{r} - \frac{\sin(2q)}{2r^2} \quad \text{in } (0,1) \times \mathbb{R} \tag{2.40}$$

with the following conditions at $\tau = -\infty$ and $r = 1$:

$$\begin{cases} q_i(r, -\infty) = 2\arctan(br) & r \in [0,1] \\ q_i(1,\tau) \ = \ g(\infty) = B & \tau < 0 \\ q_i(1,\tau) \ = \ g(-\infty) = A & \tau > 0. \end{cases} \tag{2.41}$$

So q_i is a solution of the harmonic map flow on the unit disk, with τ playing the role of time. The problem for q_i can be easily split up in two separate problems: one for $\tau < 0$, with the trivial solution

$$q_i(r,\tau) = 2\arctan(br) \quad \text{if } 0 \le r \le 1, \tau < 0, \tag{2.42}$$

and the other one for $\tau > 0$ with an initial condition at $\tau = 0$ inherited from (2.42):

$$\begin{cases} q_\tau \ = \ q_{rr} + \frac{q_r}{r} - \frac{\sin(2q)}{2r^2} & 0 < r < 1, \tau > 0 \\ q(r,0) = 2\arctan(br) & 0 < r < 1 \\ q(1,\tau) \ = \ g(-\infty) = A & \tau > 0. \end{cases} \tag{2.43}$$

Since $A > \pi$ it is known (see [7]) that (2.43) has a classical solution q which blows up after finite time $\bar{\tau} > 0$, satisfying

$$q(0,\tau) = 0 \text{ if } \tau < \bar{\tau} \text{ and } q(0,\bar{\tau}) = \pi. \tag{2.44}$$

In [3],[29] it has been shown that this solution can be continued for $\tau > \bar{\tau}$ in at least 2 different ways: for $\tau > \bar{\tau}$, q satisfies either $q(0,\tau) = \pi$ or $q(0,\tau) = 2\pi$. The latter property explains the difference between the limit functions q_1 and q_2. In particular we claim that $\bar{z}_{1,c}$ and $\bar{z}_{2,c}$ have the same limiting behavior as $c \to \infty$:

Theorem 2.14. *Let $h_{1,c}$ and $h_{2,c}$ be the solutions constructed in Theorems 2.1 and 2.2, and let $(0, \bar{z}_{1.c})$ and $(0, \bar{z}_{2,c})$ be their singularities. Then*

$$\bar{z}_{i,c} = -\bar{\tau}c(1 + o(1)) \to -\infty \quad \text{as } c \to \infty \quad (i = 1,2), \tag{2.45}$$

where $\bar{\tau} > 0$ is defined by (2.44).

The rigorous proof of this result is quite lengthy, and below we only sketch its structure.

It is not difficult to show that for all compact subsets Ω of $(0,1) \times \mathbb{R}$ there exists a constant $K = K(\Omega)$ which does not depend on c such that such that for all $c \ge 1$

$$\iint_\Omega \left(\left| \frac{\partial q_{i,c}}{\partial r} \right|^2 + \left| \frac{\partial q_{i,c}}{\partial \tau} \right|^2 \right) \mathrm{d}\,\tau \,\mathrm{d}\,r \le K.$$

Hence there exist $q_i \in H^1_{\mathrm{loc}}((0,1) \times \mathbb{R})$ such that, up to subsequences,

$$q_{i,c} \rightharpoonup q_i \quad \text{in } H^1_{\mathrm{loc}}((0,1) \times \mathbb{R}) \text{ as } c \to \infty.$$

By standard regularity theory, q_i is a smooth solution of equation (2.40) in $(0,1) \times \mathbb{R}$. In addition q_i is increasing with respect to τ and satisfies $2\arctan(br) < q_i(r,\tau) < i\pi + 2\arctan(a_i r)$.

Using that

$$\Phi_c(h_{i,c}) \le \Phi_c(2\arctan(br) + (g(z) - 2\arctan b)r) \le \frac{K}{c}\,\mathrm{e}^{cz_1}$$

for some K which does not depend on c one can prove that, for any $M > 0$ and $\varepsilon > 0$,

$$\int_{-M-\varepsilon}^{-M} f_{i,c}(\tau)\,\mathrm{d}\,\tau \le \frac{K}{c^2}\,\mathrm{e}^{cz_1 - c^2 M} \to 0 \quad \text{as } c \to \infty, \tag{2.46}$$

where

$$f_{i,c}(\tau) := \int_0^1 \frac{r}{2}\left((q_{i,c})_r^2 + \frac{\sin^2(q_{i,c})}{r^2} - G_b(r)\right)\mathrm{d}\,r.$$

By Lemma A.6 $f_{i,c}(\tau) \ge 0$ in $(-\infty, -\frac{z_1}{c})$. Using the monotonicity with respect to τ, it follows easily from (2.46) and Theorem A.7 that $q_{i,c} \to 2\arctan(br)$ uniformly in $[0,1] \times (-\infty, -M]$ for all $M > 0$, and (2.42) follows.

The rest of the proof is based on some detailed information about the minimal solution, $q_{min}(r,\tau)$ $(\tau \ge 0)$, of (2.43). In particular q_{min} satisfies (2.44), $q_{min}(0,\tau) = \pi$ if $\tau \ge \bar\tau$, and q_{min} is increasing with respect to τ (since the initial function is a subsolution). Lap-number theory (see [26]) implies that for all $0 < \tau < \bar\tau$ there exists a unique $r(\tau)$ such that $q_{min}(r(\tau),\tau) = \pi$. In addition $r(\tau)$ is decreasing with respect to τ and $r(\tau) \to 0$ as $\tau \to \bar\tau$. Finally $q_{min} > \pi$ in $(0,1) \times (\bar\tau, \infty)$ and $(q_{min})_r(0,\tau) > 0$ if $\tau > \bar\tau$.

Arguing by contradiction, we use these properties and the fact that h_i is a minimizer to prove that

(i) $q_i = q_{min}$ in $(0,1) \times (0,\bar\tau)$ and for all $\varepsilon > 0$ there exists $c_{\varepsilon,1}$ such that $-\bar z_{i,c} > (\bar\tau - \varepsilon)c$ for all $c > c_{\varepsilon,1}$;

(ii) for all $\varepsilon > 0$ there exists $c_{\varepsilon,2}$ such that $-\bar z_{i,c} < (\bar\tau + \varepsilon)c$ for all $c > c_{\varepsilon,2}$.

The proofs of (i) and (ii) are based on the construction of functions which are similar to the ones used in the previous section (the functions $h_{1,n}$ and $h_{2,n}$). We omit their construction, which is rather delicate and lengthy.

Chapter 3

A simple application

In this chapter we shall use our traveling waves to study nonuniqueness properties for axially symmetric solutions of

$$\begin{cases} u_t - \Delta u \;=\; |\nabla u|^2 u & \text{in } \Omega \times \mathbb{R}^+ \\ u(x,0) \;=\; u_0(x) & \text{in } \Omega \\ u(x,t) \;=\; u_0(x) & \text{in } \partial\Omega \times \mathbb{R}^+ \end{cases}$$

when $\Omega = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 < 1\} \subset \mathbb{R}^3$, the initial-boundary data u_0 is itself axially symmetric:

$$u_0(x_1, x_2, x_3) \;=\; \Big(\frac{x_1}{r} \sin h_0(r, x_3), \frac{x_2}{r} \sin h_0(r, x_3), \cos h_0(r, x_3)\Big) \qquad \Big(r := \sqrt{x_1^2 + x_2^2}\Big)$$

and the function h_0 satisfies suitable conditions, which we shall specify later. We know that, if we denote by $h = h(r, x_3, t)$ the angle function of an axially symmetric solution

$$u(x_1, x_2, x_3, t) \;=\; \Big(\frac{x_1}{r} \sin h, \frac{x_2}{r} \sin h, \cos h\Big)$$

of the previous problem, then h is a solution to the scalar problem

$$\begin{cases} h_t \;=\; h_{rr} + h_{x_3x_3} + \frac{h_r}{r} - \frac{\sin(2h)}{2r^2} & \text{for } 0 < r < 1, x_3 \in \mathbb{R}, t \in \mathbb{R}^+ \\ h(r, x_3, 0) \;=\; h_0(r, x_3) & \text{for } 0 < r < 1, x_3 \in \mathbb{R} \\ h(1, x_3, t) \;=\; h_0(1, x_3) & \text{for } x_3 \in \mathbb{R}, t \geq 0\,. \end{cases} \tag{3.1}$$

If h_0 is bounded and smooth in the strip $[0,1] \times \mathbb{R}$ with $h_0(0, x_3) \equiv 0$, then Problem (3.1) has a unique bounded classical solution h in the maximal time interval $[0, T)$, where $T \in (0, \infty]$ is a value depending on h_0, which satisfies the condition $h(0, x_3, t) \equiv 0$ for every $t \in [0, T)$. In this chapter we show that, by choosing as h_0 a suitable subsolution of

$$h_{rr} + h_{x_3x_3} + \frac{h_r}{r} - \frac{\sin(2h)}{2r^2} = 0 \qquad \text{for } 0 < r < 1, x_3 \in \mathbb{R},$$

T is finite and the classical solution h blows up at $t = T$, that is to say

$$\limsup_{t \to T^-} \|\nabla h(\cdot,\, \cdot,\, t)\|_\infty = \infty\,.$$

Of course, for $t > T$ it does not make sense to look for classical solutions of Problem (3.1), but only for weak ones. We shall prove that, with our choice of h_0, there is no uniqueness of weak solutions, but there exist infinitely many weak solutions of Problem (3.1) attaining different values on the line $\{r = 0\}$.

After introducing some preliminary technical results, the chapter starts by specifying how the subsolution $h_0 \geq 0$ must be chosen to make the classical solution of (3.1) blow up in finite time. Then, it continues with the construction of a non-negative weak solution h_m such that

$$h_m(0, x_3, t) \equiv 0 \quad \text{if } t < T, \qquad h_m(0, x_3, t) = \begin{cases} 0 & \text{if } |x_3| > \zeta(t) \\ \pi & \text{if } |x_3| < \zeta(t) \end{cases} \quad \text{if } t \geq T\,,$$

where $\zeta : [T, \infty) \to \mathbb{R}^+$ is an increasing function having the property that

$$\zeta(\tau) = \lim_{t \to \tau^+} \zeta(t)$$

for every $\tau \geq T$. Afterwards, for every $M \geq 0$ we construct a different non-negative weak solution $h_M \geq h_m$ such that, for a suitable constant $S > 0$ independent of M,

1. $h_M(0, x_3, t) = \pi$ for every $t > 0$ and $|x_3| \leq M$,
2. $h_M(0, x_3, t) = 0$ for every $t > 0$ and $|x_3| > S + M + t$.

This construction proves the existence of infinitely many weak solutions to Problem (3.1), which are distinguished by attaining different values on the line $\{r = 0\}$. In the last section we briefly discuss the results obtained in this Chapter.
In what follows we shall always use z in place of x_3.

3.1 Some technical results

Let H_1, H_2 be the functions defined by

$$H_1(r, z, t) = 2 \arctan\left(\frac{r}{\lambda(z, t)}\right), \quad H_2(r) = Br^\alpha \qquad r \in [0, 1], z \in \mathbb{R}, t \geq 0, \tag{3.2}$$

where $B \in \mathbb{R}$, $\alpha \in \mathbb{R}^+$ and $\lambda : \mathbb{R} \times [0, T) \longrightarrow \mathbb{R}^+$ is a smooth function (here $T \in (0, \infty]$). We shall prove that

Lemma 3.1. *If $B > 0$ and $\alpha \in (\sqrt{2}, 3]$, then there exists a constant*

$$C(\alpha) \in [(\alpha^2 - 2)/2, \alpha^2 - 2]$$

such that if λ satisfies the differential inequality

$$\lambda_t - \lambda_{zz} \geq -BC\lambda^{\alpha - 1} \qquad \text{in } \mathbb{R} \times (0, T)\,, \tag{3.3}$$

then $H(r, z, t) := H_1(r, z, t) + H_2(r)$ is a subsolution to

$$h_t \;=\; h_{rr} + h_{zz} + \frac{h_r}{r} - \frac{\sin(2h)}{2r^2} \tag{3.4}$$

in the open set $(0, 1) \times \mathbb{R} \times (0, T)$.

Proof: A straightforward computation shows that

$$H_{rr}+\frac{H_r}{r}-\frac{\sin(2H)}{2r^2}=\frac{(rH_r)_r}{r}-\frac{\sin(2H)}{2r^2}=\frac{1}{r}\frac{\partial}{\partial r}\Big(r(H_1)_r\Big)+\frac{1}{r}\frac{d}{dr}\Big(r(H_2)_r\Big)-\frac{\sin(2H_1+2H_2)}{2r^2}$$
$$=\frac{\sin(2H_1)+\alpha^2 2H_2-\sin(2H_1+2H_2)}{2r^2}$$
$$=\frac{\sin(2H_1)(1-\cos(2H_2))+\alpha^2 2H_2-\cos(2H_1)\sin(2H_2)}{2r^2}$$
$$=\frac{H_2}{r^2}\left(\sin(2H_1)\frac{\sin^2 H_2}{H_2}+\alpha^2-\frac{\cos(2H_1)\sin(2H_2)}{2H_2}\right)\geq\frac{H_2}{r^2}(\alpha^2-2)=Br^{\alpha-2}(\alpha^2-2) \tag{3.5}$$

due to the inequality $|\sin x/x|\leq 1$ forall $x\neq 0$. At the same time

$$H_t-H_{zz}=-\frac{\sin H_1}{\lambda}\left(\lambda_t-\lambda_{zz}+(1+\cos H_1)\frac{\lambda_z^2}{\lambda}\right)\leq-\frac{\sin H_1}{\lambda}\left(\lambda_t-\lambda_{zz}\right) \tag{3.6}$$

since $\sin H_1\geq 0$. From the inequalities (3.5) and (3.6) we deduce that

$$Br^{\alpha-2}(\alpha^2-2)\geq-\frac{\sin H_1}{\lambda}\left(\lambda_t-\lambda_{zz}\right) \tag{3.7}$$

is a sufficient condition in order that H is a subsolution of (3.4). Since

$$\sin H_1=\frac{2\lambda r}{\lambda^2+r^2}>0$$

in $(0,1)\times\mathbb{R}\times(0,T)$, we can rewrite (3.7) as

$$\lambda_t-\lambda_{zz}\geq-B(\alpha^2-2)\frac{\lambda^2+r^2}{2r^{3-\alpha}}\,. \tag{3.8}$$

Thanks to Lemma 4.5 (replace B with λ and set $\delta=2-\alpha$ for $\alpha\in(\sqrt{2},2]$, $\delta=\alpha-2$ for $\alpha\in(2,3]$) we have that

$$\frac{2r^{3-\alpha}}{\lambda^2+r^2}=\frac{2r^{3-\alpha}\lambda^{\alpha-1}}{\lambda^2+r^2}\lambda^{1-\alpha}\leq C(\alpha)\lambda^{1-\alpha}$$

with $C(\alpha)\in[1,2]$. Then

$$-B(\alpha^2-2)\frac{\lambda^2+r^2}{2r^{3-\alpha}}\leq\frac{-B(\alpha^2-2)}{C(\alpha)}\lambda^{\alpha-1}$$

and, up to redefining $C(\alpha)$ as $(\alpha^2-2)/C(\alpha)$, from (3.8) we obtain that (3.3) is a sufficient condition in order that H is a subsolution of (3.4). □

Let H_1, H_2 be the same function as before with $\lambda:(\zeta,\infty)\times(0,T)\longrightarrow\mathbb{R}^+$ smooth and positive (here $\zeta\in[-\infty,\infty)$ and $T\in(0,\infty]$).

Lemma 3.2. *If $B < 0$ and $\alpha \in (\sqrt{2}, 3]$, then there exists a constant*

$$C(\alpha) \in [(\alpha^2 - 2)/2, \alpha^2 - 2]$$

such that if λ satisfies the differential inequality

$$\lambda_t - \lambda_{zz} + 2\,\frac{\lambda_z^2}{\lambda} \le |B| C \lambda^{\alpha-1} \tag{3.9}$$

in $(\zeta, \infty) \times (0, T)$, then $H(r, z, t) := H_1(r, z, t) + H_2(r)$ is a supersolution to

$$h_t \ = \ h_{rr} + h_{zz} + \frac{h_r}{r} - \frac{\sin(2h)}{2r^2}$$

in the open set $(0, 1) \times (\zeta, \infty) \times (0, T)$.

Proof: Just as in the proof of Lemma 3.1 we have that

$$H_{rr} + \frac{H_r}{r} - \frac{\sin(2H)}{2r^2} = \frac{H_2}{r^2}\left(\sin(2H_1)\frac{\sin^2 H_2}{H_2} + \alpha^2 - \frac{\cos(2H_1)\sin(2H_2)}{2H_2}\right).$$

Since the expression within parentheses is always greater or equal than $\alpha^2 - 2$ and $\frac{H_2}{r^2} = Br^{\alpha-2} < 0$, we deduce that

$$H_t - H_{zz} \ge (\alpha^2 - 2) B r^{\alpha-2} \tag{3.10}$$

is a sufficient condition in order that H is a supersolution of (3.4). But

$$H_t - H_{zz} = -\frac{\sin H_1}{\lambda}\left(\lambda_t - \lambda_{zz} + (1 + \cos H_1)\frac{\lambda_z^2}{\lambda}\right)$$

and (3.10) can be rewritten as

$$\lambda_t - \lambda_{zz} + (1 + \cos H_1)\frac{\lambda_z^2}{\lambda} \le \frac{\lambda}{\sin H_1}(\alpha^2 - 2)|B| r^{\alpha-2}.$$

Since

$$\frac{\lambda}{\sin H_1}(\alpha^2 - 2)|B| r^{\alpha-2} = \frac{\lambda^2 + r^2}{2r^{3-\alpha}}(\alpha^2 - 2)|B|$$

and, as in the proof of Lemma 3.1,

$$\frac{2r^{3-\alpha}}{\lambda^2 + r^2} \le C(\alpha)\lambda^{1-\alpha}$$

for a suitable constant $C(\alpha) \in [1, 2]$, we deduce that

$$\lambda_t - \lambda_{zz} + (1 + \cos H_1)\frac{\lambda_z^2}{\lambda} \le \frac{\lambda^{\alpha-1}}{C(\alpha)}|B|(\alpha^2 - 2)$$

implies (3.10). The thesis then follows up to redefining $C(\alpha)$ as $(\alpha^2 - 2)/C(\alpha)$. $\square$

Remark 3.3. By comparing the proofs of Lemma 3.1 and Lemma 3.2 the reader can easily verify that the constant $C(\alpha)$ appearing in their statements is the same.

If we look for a subsolution $\xi = \xi(r, z, t)$ of equation (3.4) in the form

$$\xi(r, z, t) = 2 \arctan\left(\frac{r}{\lambda(z, t)}\right) + Br^{\alpha},$$

where $B > 0, \alpha \in (\sqrt{2}, 3]$ are given constants and $\lambda : \ \mathbb{R} \times [0, T) \longrightarrow \mathbb{R}^+$ is a smooth function, then, due to Lemma 3.1, we can reduce the problem to construct a such subsolution to the choice of a suitable function λ. In particular we want to choose λ such that

$$\xi(0, z, t) = 0 \ \ \forall t \in [0, T), z \in \mathbb{R} \qquad \text{and} \qquad \lim_{r \to 0^+}\left(\lim_{t \to T^-} \xi(r, z, t)\right) = \begin{cases} \pi & \text{if } |z| \leq \sigma \\ 0 & \text{if } |z| > \sigma \end{cases} \tag{3.11}$$

for a suitable $\sigma \geq 0$ and a finite time $T > 0$.

Let $\phi = \phi(t) \in C^{\infty}([0, T])$ be a function such that $\phi(0) = \phi_0 > 0$, $\phi(T) = 0$ and $\phi'(t) < 0$ in $[0, T]$. For every $\sigma > 0$ we set

$$\mu_{\sigma}(z) = \begin{cases} |z - \sigma|^4 & \text{if } z > \sigma \\ 0 & \text{if } z \in [-\sigma, \sigma] \\ |z + \sigma|^4 & \text{if } z < -\sigma\,. \end{cases}$$

Lemma 3.4. *Let $B > 0, \alpha \in (\sqrt{2}, 2)$ and let λ be the function*

$$\lambda(z, t) = K\, \mathrm{e}^{\frac{-Q}{\phi(t) + \mu_{\sigma}(z)}} \qquad z \in \mathbb{R}, t \in [0, T]$$

with K, Q and σ positive constants. There exists $\bar{Q} = \bar{Q}(B, K, \alpha, \|\phi'\|) > 0$, where $\|\phi'\| = \max_{t \in [0,T]} |\phi'(t)|$, such that if $Q \geq \bar{Q}$, then λ satisfies (3.3) for any $\sigma > 0$.

Proof: A straightforward computation shows that

$$\lambda_t - \lambda_{zz} = KQ\, \mathrm{e}^{\frac{-Q}{\phi(t) + \mu_{\sigma}(z)}} \left(\frac{\phi'(t) - \mu_{\sigma}''(z)}{(\phi(t) + \mu_{\sigma}(z))^2} + \frac{2(\mu_{\sigma}'(z))^2}{(\phi(t) + \mu_{\sigma}(z))^3} - \frac{Q(\mu_{\sigma}'(z))^2}{(\phi(t) + \mu_{\sigma}(z))^4}\right).$$

Since

$$|\mu_{\sigma}'(z)| \equiv 4\mu_{\sigma}(z)^{3/4}, \qquad \mu_{\sigma}''(z) \equiv 12\mu_{\sigma}(z)^{1/2}$$

and $\phi \geq 0$ in $[0, T]$, we deduce that

$$\lambda_t - \lambda_{zz} \geq KQ\, \mathrm{e}^{\frac{-Q}{\phi(t) + \mu_{\sigma}(z)}} \left(\frac{\phi'(t) - 12\mu_{\sigma}(z)^{1/2}}{(\phi(t) + \mu_{\sigma}(z))^2} - \frac{16Q\mu_{\sigma}(z)^{3/2}}{(\phi(t) + \mu_{\sigma}(z))^4}\right) \geq$$

$$KQ\, \mathrm{e}^{\frac{-Q}{\phi(t) + \mu_{\sigma}(z)}} \left(-\frac{\|\phi'\|}{(\phi(t) + \mu_{\sigma}(z))^2} - \frac{12}{(\phi(t) + \mu_{\sigma}(z))^{3/2}} - \frac{16Q}{(\phi(t) + \mu_{\sigma}(z))^{5/2}}\right).$$

Then

$$Q\lambda^{2-\alpha}\left(\frac{\|\phi'\|}{(\phi(t) + \mu_{\sigma}(z))^2} + \frac{12}{(\phi(t) + \mu_{\sigma}(z))^{3/2}} + \frac{16Q}{(\phi(t) + \mu_{\sigma}(z))^{5/2}}\right) \leq BC \tag{3.12}$$

is a sufficient condition in order that (3.3) is true. From the elementary inequality

$$\forall k, \beta > 0 \qquad x^\beta \,\mathrm{e}^{-kx} \le \left(\frac{\beta}{k\,\mathrm{e}}\right)^\beta \qquad \forall x \ge 0$$

follows that (take $k = Q(2-\alpha)$ and $x = (\phi(t) + \mu_\sigma(z))^{-1}$) the left-hand side of (3.12) is less than or equal to

$$K^{2-\alpha} Q \left(\|\phi'\| \left(\frac{2}{Q(2-\alpha)\,\mathrm{e}}\right)^2 + 12 \left(\frac{3/2}{Q(2-\alpha)\,\mathrm{e}}\right)^{3/2} + 16Q \left(\frac{5/2}{Q(2-\alpha)\,\mathrm{e}}\right)^{5/2} \right) =$$

$$= K^{2-\alpha} \mathscr{C}(\alpha) \left(\frac{\|\phi'\|}{Q} + \frac{1}{\sqrt{Q}} \right)$$

where $\mathscr{C}(\alpha)$ is a positive constant depending on α. Since the constant C in (3.12) depends on α too, we obtain the thesis. $\square$

As a direct consequence of Lemma 3.1 and Lemma 3.4 we obtain the following

Proposition 3.5. *Let $B, T, \sigma, Q, K > 0$, $\alpha \in (\sqrt{2}, 2)$ and let ξ be the function*

$$\xi(r, z, t) = 2 \arctan\left(\frac{r}{K} \,\mathrm{e}^{\frac{Q}{T - t + \mu_\sigma(z)}} \right) + B r^\alpha \,.$$

There exists $\bar{Q} = \bar{Q}(B, \alpha, K) > 0$ such that if $Q \ge \bar{Q}$ then ξ is a subsolution of (3.4) satisfying the conditions (3.11).

Remark 3.6. From the proof of Lemma 3.4 it follows that

1. $\bar{Q} \to \infty$ when $K \to \infty$,
2. $\bar{Q}$ is a decreasing function of B.

Lemma 3.7. *Let $B < 0$, $\alpha \in (\sqrt{2},\, 2)$ and let $\gamma = \gamma(z, t)$ be a function of $z \in (\zeta, \infty)$ and $t \in (0, T)$ ($\zeta \in \mathbb{R}, T > 0$) such that*

$$\gamma > 0, \;\; |\gamma_t| \le M_1, \;\; |\gamma_z| \le M_2, \;\; |\gamma_{zz}| \le M_{22} \qquad \text{in } (\zeta, \infty) \times (0, T) \tag{3.13}$$

for some constants M_1, M_2, $M_{22} > 0$. There exists $\hat{Q} = \hat{Q}(\alpha, |B|, M_1, M_2, M_{22}) > 0$ such that for every $Q \ge \hat{Q}$ the function

$$\lambda(z, t) := \mathrm{e}^{-\frac{Q}{\gamma(z,t)}}$$

satisfies (3.9) in $(\zeta, \infty) \times (0, T)$.

Proof: A straightforward computation shows that

$$\lambda_t - \lambda_{zz} + 2\,\frac{\lambda_z^2}{\lambda} = \frac{Q\lambda}{\gamma^2} \left(\gamma_t - \gamma_{zz} + \frac{Q}{\gamma^2}\gamma_z^2 + \frac{2}{\gamma}\gamma_z^2 \right)$$

and then (3.9) is equivalent to

$$\frac{Q\lambda^{2-\alpha}}{\gamma^2} \left(\gamma_t - \gamma_{zz} + \frac{Q}{\gamma^2}\gamma_z^2 + \frac{2}{\gamma}\gamma_z^2 \right) \le |B| C \,. \tag{3.14}$$

Due to (3.13) and since

$$\forall k, \beta > 0 \qquad x^\beta \, \mathrm{e}^{-kx} \le \left(\frac{\beta}{k\,\mathrm{e}}\right)^\beta \qquad \forall x \ge 0\,,$$

we have that

$$\frac{Q\lambda^{2-\alpha}}{\gamma^2}\left(\gamma_t - \gamma_{zz} + \frac{Q}{\gamma^2}\gamma_z^2 + \frac{2}{\gamma}\gamma_z^2\right) \le \mathrm{e}^{-\frac{(2-\alpha)Q}{\gamma}}\left(\frac{M_1 Q}{\gamma^2} + \frac{M_{22}Q}{\gamma^2} + \frac{Q^2 M_2^2}{\gamma^4} + \frac{2M_2^2 Q}{\gamma^3}\right) \le$$
$$\le \mathcal{G}\left(\frac{M_1 + M_{22}}{Q} + \frac{3M_2^2}{Q^2}\right)$$

for a suitable constant $\mathcal{G}$ depending on α. Here we have used that $(2-\alpha)Q > 0$. The thesis then follows from (3.14) since C is a constant depending on α. □

As a direct consequence of Lemma 3.2 and Lemma 3.7 we obtain the following assertion:

Proposition 3.8. *Let $\phi \in C^1([0,T))$ ($T \in [0,\infty)$) and $\mu \in C^2([\zeta,\infty))$ ($\zeta \in \mathbb{R}$) be two functions satisfying the conditions:*

1. *$\phi > 0$ in $(0,T)$,*
2. *there exist $A_0, A_1 > 0$ such that $|\phi| \le A_0$ and $|\phi'| \le A_1$ in $[0,T)$,*
3. *$\mu > 0$ in (ζ,∞),*
4. *there exist $M_0, M_1, M_2 > 0$ such that $|\mu| \le M_0$, $|\mu'| \le M_1$ and $|\mu''| \le M_2$ in $[\zeta,\infty)$.*

If $\alpha \in (\sqrt{2}, 2)$ and $B < 0$, then there exists $\hat{Q} = \hat{Q}(\alpha, |B|, A_0, A_1, M_0, M_1, M_2) > 0$ such that for every $Q \ge \hat{Q}$ the function

$$\psi(r,z,t) = 2\arctan\left(r\,\mathrm{e}^{\frac{Q}{\phi(t)\mu(z)}}\right) + Br^\alpha$$

is a supersolution to the equation

$$h_t \;=\; h_{rr} + h_{zz} + \frac{h_r}{r} - \frac{\sin(2h)}{2r^2}$$

in the open set $(0,1)\times(\zeta,\infty)\times(0,T)$.

We conclude this section with a proposition which will be used at the end of the chapter. Its proof requires two lemmas.

Lemma 3.9. *Let $\alpha \in (\sqrt{2}, 3]$, $B \in \mathbb{R}^+$ and let h be the function*

$$h(r,t) = 2\arctan\left(\frac{\lambda(z,t)}{r}\right) + Br^\alpha$$

with $\lambda : (-\zeta,\zeta)\times(0,T) \longrightarrow \mathbb{R}^+$ regular function ($\zeta, T > 0$).
There exists $C(\alpha) \in [(\alpha^2-2)/2, \alpha^2-2]$ such that if λ is a solution of

$$\lambda_t - \lambda_{zz} + 2\frac{\lambda_z^2}{\lambda} \le BC\lambda^{\alpha-1} \qquad \text{in } (-\zeta,\,\zeta)\times(0,T), \tag{3.15}$$

then h is a subsolution of (3.4) in $(0,1)\times(-\zeta,\,\zeta)\times(0,T)$.

Proof: Let H be the function defined by

$$H(r,z,t) = 2\arctan\left(\frac{r}{\lambda(z,t)}\right) - Br^{\alpha}$$

for $r \in [0,1]$, $z \in (-\zeta,\zeta)$ and $t \in (0,T)$. Since $\alpha \in (\sqrt{2},3]$ and $-B < 0$, by the same arguments used in the proof of Lemma 3.2 we can show that there exists $C(\alpha) \in [(\alpha^2-2)/2, \alpha^2-2]$ such that if λ is a solution of (3.9), then H is a supersolution of (3.4) in $(0,1)\times(-\zeta,\zeta)\times(0,T)$. On the other hand, for every $(r,z,t) \in (0,1)\times(-\zeta,\zeta)\times(0,T)$ we have trivially that

$$h(r,z,t) = \pi - 2\arctan\left(\frac{r}{\lambda(z,t)}\right) + Br^{\alpha} = \pi - H(r,z,t)\,.$$

Hence the thesis. $\square$

Lemma 3.10. *Let $\alpha \in (\sqrt{2},2)$, $B,\zeta,T > 0$ and let $\gamma : (-\zeta,\zeta)\times(0,T) \longrightarrow \mathbb{R}^+$ be a positive function of $z \in (-\zeta,\zeta)$ and $t \in (0,T)$ such that*

$$|\gamma_t| \le M_1,\ |\gamma_z| \le M_2,\ |\gamma_{zz}| \le M_{22} \qquad in (-\zeta,\zeta)\times(0,T)$$

for some constants $M_1, M_2, M_{22} > 0$. There exists $\tilde{Q} = \tilde{Q}(\alpha,B,M_1,M_2,M_{22}) > 0$ such that for every $Q \ge \tilde{Q}$ the function

$$\lambda := \mathrm{e}^{-\frac{Q}{\gamma}}$$

satisfies (3.15).

Proof: The proof is formally identical to that one of Lemma 3.7 and can be omitted. $\square$

Let $\{\gamma_\zeta\}_{\zeta\ge 0} \subset C^\infty(\mathbb{R})$ be a family of functions such that

$$\gamma_\zeta(z) \equiv \gamma_\zeta(-z), \quad \gamma_\zeta(z) = 1 \ \ \forall z \in [0,\zeta], \quad \gamma_\zeta(z) = 0 \ \ \forall z \ge \zeta+1,$$

$$-N_1 \le \frac{d\gamma_\zeta}{dz} < 0\,, \quad \left|\frac{d\gamma_\zeta}{dz}\right|^2 \gamma_\zeta^{-1} \le N_2\,, \quad \text{and} \quad \left|\frac{d^2\gamma_\zeta}{dz^2}\right| \le N_3 \quad \forall z \in (\zeta,\zeta+1) \tag{3.16}$$

where the constants $N_1, N_2, N_3 \in \mathbb{R}^+$ do not depend on ζ. We remark that for every $\zeta \ge 0 \quad \gamma_\zeta(z) \in (0,1)$ if $z \in (\zeta,\zeta+1)$.

Proposition 3.11. *Let $\alpha \in (\sqrt{2},2)$, $B,T \in \mathbb{R}^+$. There exists $\mathcal{Q} = \mathcal{Q}(\alpha,B,T) > 0$ such that for every $Q \ge \mathcal{Q}$ and for every $\zeta \ge 0$ the function*

$$h(r,z,t) = \begin{cases} 2\arctan\left(\dfrac{\mathrm{e}^{-\frac{Q}{t\gamma_\zeta(z)}}}{r}\right) + Br^{\alpha} & r \in (0,1],\ t \in (0,T],\ |z| < \zeta+1 \\ Br^{\alpha} & r \in (0,1],\ |z| \ge \zeta+1 \ \textit{or}\ t = 0 \end{cases}$$

is a subsolution of (3.4) in $(0,1)\times\mathbb{R}\times(0,T)$.

Proof: We can write h as

$$2\arctan\left(\frac{\lambda(z,t)}{r}\right) + Br^\alpha$$

where

$$\lambda(z,t) = \begin{cases} \mathrm{e}^{-\frac{Q}{t\gamma_\zeta(z)}} & t \in (0,T],\ |z| < \zeta+1 \\ 0 & |z| \geq \zeta+1 \text{ or } t = 0\,. \end{cases}$$

Since $\lambda \in C^\infty(\mathbb{R}\times(0,T))$ we have that $h \in C^\infty((0,1)\times\mathbb{R}\times(0,T))$ and so is for

$$\mathscr{L}(h) := h_t - h_{rr} - h_{zz} - \frac{h_r}{r} + \frac{\sin(2h)}{2r^2}\,.$$

If $|z| > \zeta+1$ then

$$\mathscr{L}(h) = \frac{\sin(2h) - \alpha^2 2h}{2r^2} \leq \frac{\sin(2h) - 2h}{2r^2} \leq 0$$

while, by Lemma 3.9 and Lemma 3.10, $\mathscr{L}(h) \leq 0$ in the open set $(0,1)\times(-\zeta,\zeta)\times(0,T)$ provided that $Q \geq \mathcal{Q}$ for a suitable constant $\mathcal{Q} = \mathcal{Q}(\alpha, B, T)$. $\square$

3.2 Assumptions on h_0 and construction of h_m

Let $\alpha \in (\sqrt{2}, 2), B \in (0, \pi/2), \mathcal{T} > 0$ three constants arbitrarily chosen. Let $Z \in [\sqrt[4]{\mathcal{T}}, \infty)$, $K \in [8/(\pi - 2B), \infty)$ and let $Q > 0$ be a constant such that

$$\mathscr{B} := \arctan\left(\frac{\mathrm{e}^{\frac{Q}{\mathcal{T}}}}{K}\right) + \frac{B-\pi}{2} > 0$$

and $Q \geq \bar{Q}(\mathscr{B}, \alpha, K) \geq \bar{Q}(B, \alpha, K)$. If we define

$$\xi(r,z,t) = 2\arctan\left(\frac{r}{K}\,\mathrm{e}^{\frac{Q}{\mathcal{T}-t+\mu_Z(z)}}\right) + Br^\alpha$$

and, for $\mathscr{Z} = Z - \sqrt[4]{\mathcal{T}}$,

$$\tilde{\xi}(r,z,t) = 2\arctan\left(\frac{r}{K}\,\mathrm{e}^{\frac{Q}{\mathcal{T}-t+\mu_{\mathscr{Z}}(z)}}\right) + \mathscr{B}r^\alpha\,,$$

then, thanks to Proposition 3.5, we have that $\xi, \tilde{\xi}$ are subsolutions of (3.4) satisfying conditions (3.11) for $T = \mathcal{T}$ and $\sigma = Z, \mathscr{Z}$ respectively. Since $B > 2\mathscr{B}$ and $Z \geq \mathscr{Z}$, which implies $\mu_Z \leq \mu_{\mathscr{Z}}$, it is obvious that

$$\xi(r,z,0) \geq \tilde{\xi}(r,z,0) + \mathscr{B}r^\alpha \qquad \forall (r,z) \in [0,1]\times\mathbb{R}. \tag{3.17}$$

Moreover we have that

Lemma 3.12.

$$\xi(1,z,0) \geq \tilde{\xi}(1,z,t) + \mathscr{B}, \qquad \forall z \in \mathbb{R}, t \in [0,\mathcal{T})\,. \tag{3.18}$$

Proof: For every $z \in [-Z, Z]$ we have that

$$\xi(1,z,0) = 2\arctan\left(\frac{e^{\frac{Q}{\mathcal{T}}}}{K}\right) + B = \pi + 2\mathscr{B} \geq \tilde{\xi}(1,z,t) + \mathscr{B} \qquad \forall t \in [0,\mathcal{T})$$

by definition of $\mathscr{B}$ and $\tilde{\xi}$. On the other hand, if $z \geq Z$ then

$$\mu_{\mathscr{Z}}(z) = (z-\mathscr{Z})^4 = ((z-Z)+(Z-\mathscr{Z}))^4 \geq (z-Z)^4 + (Z-\mathscr{Z})^4 = \mu_Z(z) + \mathcal{T}$$

and if $z \leq -Z \quad \mu_{\mathscr{Z}}(z) = \mu_{\mathscr{Z}}(|z|) \geq \mu_Z(|z|) + \mathcal{T} = \mu_Z(z) + \mathcal{T}$. Therefore, for every $z \in \mathbb{R} \setminus [-Z, Z]$ and $t \in [0, \mathcal{T})$

$$\frac{Q}{\mathcal{T} - t + \mu_{\mathscr{Z}}(z)} \leq \frac{Q}{\mu_Z(z) + \mathcal{T}}$$

which, together with $B > 2\mathscr{B}$, implies $\tilde{\xi}(1,z,t) + \mathscr{B} \leq \xi(1,z,0)$. $\square$

At last, since $K \geq 8/(\pi - 2B)$, it turns out that

Lemma 3.13. *There exist $\bar{z} = \bar{z}(Z,Q) > 0$ and $b = b(\pi/2 - B) \in (0,1)$ such that*

$$\xi(r,z,0) \leq 2\arctan(br) \qquad \forall r \in [0,1],\ |z| \geq \bar{z}$$

Proof: Let $\bar{z} = \bar{z}(Z,Q) > 0$ be a constant such that for every $z \in (-\infty, -\bar{z}] \cup [\bar{z}, \infty)$

$$e^{\frac{Q}{\mathcal{T}+\mu_Z(z)}} \leq 3/2\,.$$

Therefore, for every $z \in (-\infty, -\bar{z}] \cup [\bar{z}, \infty)$ and $r \in [0,1]$

$$\xi(r,z,0) \leq 2\arctan\left(\frac{3r}{2K}\right) + Br^{\alpha} \leq (3/K + B)r \leq \left(\pi/2 - \frac{\pi/2 - B}{4}\right) r\,.$$

Since $A := \pi/2 - (\pi/2 - B)/4 \in (0, \pi/2)$, $Ar \leq 2\arctan(2Ar/\pi)$ for all $r \in [0,1]$. Then we can obtain the thesis by defining $b = \frac{2A}{\pi}$. $\square$

If q is the function defined by $q(r,z) = \xi(r,z,0)$, then

1. $$q_{rr} + q_{zz} + \frac{q_r}{r} - \frac{\sin(2q)}{2r^2} \geq \xi_t(r,z,0) = \frac{2Kr\,e^{\frac{Q}{\mathcal{T}+\mu_Z(z)}}}{K^2 + r^2\,e^{\frac{2Q}{\mathcal{T}+\mu_Z(z)}}}\,\frac{Q}{(\mathcal{T}+\mu_Z(z))^2} > 0$$

   in $(0,1] \times \mathbb{R}$,

2. $q(0,z) = 0 \quad \forall z \in \mathbb{R}$,

3. $2\arctan(r/K) \leq q(r,z) \leq \pi + 2\arctan(ar)$ for some $a > 0$, and there exist $\bar{z} > 0$, $b \in (0,1)$ such that $q(r,z) \leq 2\arctan(br)$ if $|z| \geq \bar{z}$,

4. $q(r,-z) \equiv q(r,z)$, $q_z(r,z) \leq 0$ for $r \in [0,1]$, $z \geq 0$, and

5. $q(r,z) \geq \tilde{\xi}(r,z,0) + \mathscr{B}r^{\alpha}$, $q(1,z) \geq \tilde{\xi}(1,z,t) + \mathscr{B}$ for all $r \in [0,1]$, $z \in \mathbb{R}$ and $t \in [0,\mathcal{T})$.

Since the function $\theta_b(r) := 2\arctan(br)$ is a subsolution of (3.4) and the maximum of two subsolutions is itself a subsolution, up to a regularization we can find a function $h_0 \in C^\infty((0,1] \times \mathbb{R})$, Lipschitz continuous on $[0,1] \times \mathbb{R}$, such that

(P1)
$$(h_0)_{rr} + (h_0)_{zz} + \frac{(h_0)_r}{r} - \frac{\sin(2h_0)}{2r^2} \geq 0 \qquad \text{in } (0,1) \times \mathbb{R},$$

(P2) $h_0(0,z) = 0 \quad \forall z \in \mathbb{R}$,

(P3) $2\arctan(br) \leq h_0(r,z) \leq \pi + 2\arctan(ar)$ for some $a > 0$, $b \in (0,1)$, and there exists $\bar{z} > 0$ such that $h_0(r,z) \equiv 2\arctan(br)$ if $|z| \geq \bar{z}$,

(P4) $h_0(r,-z) \equiv h_0(r,z)$, $(h_0)_z(r,z) \leq 0$ for $r \in [0,1], z \geq 0$, and

(P5) $h_0(r,z) \geq \tilde{\xi}(r,z,0)$, $h_0(1,z) \geq \tilde{\xi}(1,z,t)$ for all $r \in [0,1]$, $z \in \mathbb{R}$ and $t \in [0,\mathcal{T})$.

In order to construct the weak solution h_m of (3.1), for every $n \in \mathbb{N}, n \geq 2$ we consider the domain $A_n = (1/n,\, 1) \times (-n,n)$ and the differential problem

$$(P_n) \quad \begin{cases} h_t = h_{rr} + h_{zz} + \frac{h_r}{r} - \frac{\sin(2h)}{2r^2} & (r,z) \in A_n,\ t > 0 \\ h(r,z,0) = h_0(r,z) & (r,z) \in \bar{A}_n \\ h(1,z,t) = h_0(1,z) & z \in [-n,n],\ t > 0 \\ h(1/n,z,t) = h_0(1/n,z) & z \in [-n,n],\ t > 0 \\ h(r,\pm n,t) = h_0(r,\pm n) & r \in [1/n,1],\ t > 0 \end{cases}$$

By standard solvability, comparison and regularity results for parabolic problems (see [21]), by properties (P1),(P3),(P4) and since the functions $\theta_b(r) = 2\arctan(br)$, $\theta_a(r) := \pi + 2\arctan(ar)$ both solve (3.4), we can say that for every $n \in \mathbb{N}$ Problem (P_n) has a unique classical solution $h_n \in C^\infty(\bar{A}_n \times \mathbb{R}^+) \cap C^0(\bar{A}_n \times [0,\infty))$ and, for $(r,z) \in \bar{A}_n$, $t' \geq t \geq 0$:

$$\begin{aligned} 2\arctan(br) \leq h_n(r,z,t) \leq \pi + 2\arctan(ar), &\quad h_0(r,z) \leq h_n(r,z,t) \leq h_n(r,z,t'), \\ h_n(r,-z,t) = h_n(r,z,t), &\quad (h_n)_z(r,z,t) \leq 0 \quad \text{if } z \geq 0, \\ (h_n)_r, (h_n)_z \text{ are Hölder continuous in } (r,z,t), &\quad h_{n+1}(r,z,t) \geq h_n(r,z,t)\,. \end{aligned} \tag{3.19}$$

Moreover, if we define

$$E_n(\cdot) := \iint_{A_n} \frac{r}{2}\left((\cdot)_r^2 + (\cdot)_z^2 + \frac{\sin^2(\cdot)}{r^2}\right) \mathrm{d}\,r\,\mathrm{d}\,z\,,$$

we have that

Proposition 3.14. *For every $n \geq 2$ and $T > 0$*

$$E_n(h_n(\cdot\,,\cdot\,,T)) + \iiint_{A_n \times [0,T]} r(h_n)_t^2 \,\mathrm{d}\,r\,\mathrm{d}\,z\,\mathrm{d}\,t \;=\; E_n(h_0)\,.$$

Proof: Let n be an arbitrary integer value with $n \geq 2$. For sake of simplicity, we shall denote by h the function h_n, so omitting the subscript n. Given $0 < \tau < T$, if we multiply by rh_t the differential equation of h and then we integrate the resulting equation over $A_n \times [\tau, T]$, we find that

$$\iiint_{A_n \times [\tau,T]} r h_t^2 \,\mathrm{d}\, r \,\mathrm{d}\, z \,\mathrm{d}\, t = \iiint_{A_n \times [\tau,T]} \left((rh_r)_r + (rh_z)_z - \frac{\sin(2h)}{2r} \right) h_t \,\mathrm{d}\, r \,\mathrm{d}\, z \,\mathrm{d}\, t \,.$$

Since $h \in C^{2,1}(\bar{A}_n \times [\tau, T])$ we may integrate by parts and obtain that

$$\iiint_{A_n \times [\tau,T]} r h_t^2 \,\mathrm{d}\, r \,\mathrm{d}\, z \,\mathrm{d}\, t = - \iiint_{A_n \times [\tau,T]} \left(\frac{r}{2} \left(h_r^2 + h_z^2 + \frac{\sin^2 h}{r^2} \right) \right)_t \mathrm{d}\, r \,\mathrm{d}\, z \,\mathrm{d}\, t$$

which we can rewrite as

$$\iiint_{A_n \times [\tau,T]} r h_t^2 \,\mathrm{d}\, r \,\mathrm{d}\, z \,\mathrm{d}\, t + E_n(h(\cdot, \cdot, T)) = E_n(h(\cdot, \cdot, \tau)) \,.$$

Since $(h_n)_r$ and $(h_n)_z$ are Hölder continuous in r, z and t, from the last equality we obtain the thesis by simply letting $\tau \to 0$. $\square$

Thanks to properties (3.19) we can define the function

$$h_m(r, z, t) := \lim_{n \to \infty} h_n(r, z, t) = \sup_{n \geq 2} h_n(r, z, t) \qquad (r, z, t) \in (0, 1] \times \mathbb{R} \times [0, \infty)$$

and say that, for $(r, z) \in (0, 1] \times \mathbb{R}$, $t' \geq t \geq 0$

$$\begin{aligned} \theta_b(r) \leq h_m(r, z, t) \leq \theta_a(r) \,, \quad & h_0(r, z) \leq h_m(r, z, t) \leq h_m(r, z, t'), \\ h_m(r, -z, t) = h_m(r, z, t), \quad & h_m(r, z', t) \leq h_m(r, z, t) \text{ if } z' \geq z \geq 0 \,. \end{aligned} \tag{3.20}$$

By standard regularity results, $h_m \in C^\infty((0, 1] \times \mathbb{R} \times \mathbb{R}^+) \cap C^0((0, 1] \times \mathbb{R} \times [0, \infty))$ and it is a classical solution of (3.1). Actually, h_m is a "minimal" solution for Problem (3.1), in the sense specified by the next statement:

Proposition 3.15. *Let $h \in L^\infty((0, 1) \times \mathbb{R} \times \mathbb{R}^+)$ be a weak solution of (3.1) with $h \geq h_0$. Then $h \geq h_m$.*

Proof: Due to parabolic Schauder-type estimates, h is smooth out of $\{r = 0\}$ and is a supersolution of problem (P_n) for every $n \geq 2$. Then the thesis directly follows from the parabolic maximum principle and the definition of h_m. $\square$

Proposition 3.16. *For every $\zeta > \bar{z}$ and $T > 0$*

$$\int_{-\zeta}^{\zeta} \mathrm{d}\, z \int_0^1 \frac{r}{2} \left((h_m)_r^2 + (h_m)_z^2 + \frac{\sin^2(h_m)}{r^2} \right) \Bigg|_{t=T} \mathrm{d}\, r + \iiint_{(0,1) \times \mathbb{R} \times [0,T]} r (h_m)_t^2 \,\mathrm{d}\, r \,\mathrm{d}\, z \,\mathrm{d}\, t \leq$$

$$\leq \int_{-\zeta}^{\zeta} \mathrm{d}\, z \int_0^1 \frac{r}{2} \left((h_0)_r^2 + (h_0)_z^2 + \frac{\sin^2(h_0)}{r^2} \right) \mathrm{d}\, r < \infty \,.$$

Proof: For every $\zeta > 0$, $n \in \mathbb{N}$ with $n \geq 2$, let $E_{n,\zeta}$ be the functional defined by

$$E_{n,\zeta}(\cdot) = \int_{-\zeta}^{\zeta} \mathrm{d}\, z \int_{1/n}^{1} \frac{r}{2}\left((\cdot)_r^2 + (\cdot)_z^2 + \frac{\sin^2(\cdot)}{r^2}\right) \mathrm{d}\, r\,.$$

In particular, $E_{n,n}$ coincides with the energy functional E_n previously defined. Let $\zeta > \bar{z}$ and $T > 0$ be two values arbitrarily chosen. Due to Proposition 3.14, for every $n \geq 2$

$$E_{n,n}(h_n(\cdot,\,\cdot,\,T)) + \iiint_{A_n \times [0,T]} r(h_n)_t^2 \,\mathrm{d}\, r \,\mathrm{d}\, z \,\mathrm{d}\, t = E_{n,n}(h_0)\,. \tag{3.21}$$

Since $h_0(r,z) = \theta_b(r)$ if $|z| \geq \bar{z}$, when $|z| > \bar{z}$ we have that

$$\int_{1/n}^{1} \frac{r}{2}\left((h_0)_r^2 + (h_0)_z^2 + \frac{\sin^2(h_0)}{r^2}\right) \mathrm{d}\, r = \int_{1/n}^{1} \frac{r}{2}\left((\theta_b)_r^2 + \frac{\sin^2(\theta_b)}{r^2}\right) \mathrm{d}\, r\,.$$

At the same time, given any $n > \bar{z}$, if $|z| \geq \bar{z}$ then $h_n(1/n, z, T) = h_0(1/n, z) = \theta_b(1/n)$, $h_n(1, z, T) = h_0(1, z) = \theta_b(1)$, and by Corollary A.3,

$$\int_{1/n}^{1} \frac{r}{2}\left((h_n)_r^2(r,z,T) + (h_n)_z^2(r,z,T) + \frac{\sin^2 h_n(r,z,T)}{r^2}\right) \mathrm{d}\, r \geq \int_{1/n}^{1} \frac{r}{2}\left((\theta_b)_r^2 + \frac{\sin^2(\theta_b)}{r^2}\right) \mathrm{d}\, r\,.$$

So, for every $n \geq \zeta$

$$E_{n,\zeta}(h_n(\cdot,\,\cdot,\,T)) + \iiint_{A_n \times [0,T]} r(h_n)_t^2 \,\mathrm{d}\, r \,\mathrm{d}\, z \,\mathrm{d}\, t \leq E_{n,\zeta}(h_0) \leq$$

$$\leq \iint_{[0,1]\times[-\zeta,\zeta]} \frac{r}{2}\left((h_0)_r^2 + (h_0)_z^2 + \frac{\sin^2(h_0)}{r^2}\right) \mathrm{d}\, r \,\mathrm{d}\, z\,.$$

Passing to the limit as $n \to \infty$ and using the Fatou's Lemma we obtain the thesis. $\square$

The previous proposition allows to say that h_m is a weak solution of (3.1), since

$$(h_m)_t \in L_r^2((0,1) \times \mathbb{R} \times \mathbb{R}^+), \qquad \frac{\sin h_m}{r},\ \nabla h_m \in L^\infty(\,\mathbb{R}^+;\, L_r^2((0,1) \times (-\zeta,\zeta))\,)$$

for every $\zeta > 0$. Hence (see [30]) for all $t \geq 0$ and for a.e. $z \in \mathbb{R}$ there exists

$$h_m(0,z,t) := \lim_{r \to 0^+} h_m(r,z,t) = k\pi \qquad \text{for some } k = k(z,t) \in \mathbb{Z}.$$

Due to the inequality $\theta_b(r) \leq h_m(r,z,t) \leq \theta_a(r)$, we can say that, if

$$I_0(t) := \{z \in \mathbb{R} \mid h_m(0,z,t) = 0\} \text{ and } I_1(t) := \{z \in \mathbb{R} \mid h_m(0,z,t) = \pi\} \text{ for } t \geq 0,$$

then for every $t \geq 0$ $\mathbb{R} \setminus (I_0(t) \cup I_1(t))$ is a set of zero Lebesgue measure. We remark that, since $h_m(r,-z,t) \equiv h_m(r,z,t)$ and $h_m(r,z',t) \leq h_m(r,z,t)$ for $r \in (0,1], t \geq 0$ and $z' \geq z \geq 0$, for every $t \geq 0$ there exists $\zeta(t) \in [0,\infty]$ such that

$$I_1(t) = (-\zeta(t), \zeta(t)),\, I_0(t) = (-\infty, -\zeta(t)) \cup (\zeta(t), \infty),$$

up to the values $\pm\zeta(t)$. Moreover, for every $t' \geq t \geq 0$ one has $\zeta(t') \geq \zeta(t)$, because h_m is an increasing function of t.

Theorem 3.17. *(i) There exists a constant $S \in \mathbb{R}^+$ such that*

$$\zeta(t) \le S + t \qquad \forall t \ge 0 \,.$$

(ii) There exists a time $T > 0$ such that for every $t \in [0,T)$ $I_0(t) = \mathbb{R} \Rightarrow \zeta(t) = 0$.

(iii) If $\tau > 0$ is a value such that $I_0(\tau) = \mathbb{R}$, then there exist $\rho > 0$ and $C > 0$ such that

$$h_m(r,z,t) \le 2\arctan(Cr) \qquad \forall r \in [0,\rho], z \in \mathbb{R}, t \in [0,\tau] \,.$$

(iv) If $\tau > 0$ is a value such that $I_0(\tau) \neq \mathbb{R}$, then for every $\varepsilon > 0$ there exist $\rho \in (0,1)$ and $C > 0$ (both ρ and C depend on ε) such that

$$h_m(r,z,t) \le 2\arctan(Cr) \qquad r \in [0,\rho], |z| \ge \zeta(\tau) + \varepsilon, t \in [0,\tau] \,.$$

Proof: *(i)* Let $g = g(y) \in C^\infty(\mathbb{R})$ a function such that

$$g(y) = 2\pi \ \text{ if } y < -1, \quad g(y) = 2\arctan b \ \text{ if } y > 1, \quad g' \le 0 \,.$$

Thanks to Theorem 2.2, there exists a function $\psi = \psi(r,y)$ which is smooth in $(0,1] \times \mathbb{R}$, satisfies $\psi(1,y) = g(y)$ for all $y \in \mathbb{R}$ and

$$\psi_{rr} + \frac{\psi_r}{r} + \psi_{yy} + \psi_y - \frac{\sin(2\psi)}{2r^2} = 0$$

in the open set $(0,1) \times \mathbb{R}$. Moreover, there exists $\bar{y} \in \mathbb{R}$ such that ψ is real analytic in $[0,1) \times \mathbb{R} \setminus \{(0,\bar{y})\}$, $\psi(0,y) = 0$ if $y > \bar{y}$, $\psi(0,y) = 2\pi$ if $y < \bar{y}$. At last, ψ is non increasing with respect to y and

$$\psi(r,y) \to \theta_b(r) \ \text{ as } y \to \infty, \qquad \psi(r,y) \to 2\pi \ \text{ as } y \to -\infty$$

uniformly with respect to $r \in [0,1]$. Therefore, since $h_0 \le \pi + 2\arctan(a) < 2\pi$ and $h_0(r,z) = \theta_b(r)$ for $z \ge \bar{z}$, there exists $\sigma > 0$ such that

$$\psi(r, z - \sigma) \ge h_0(r,z) \,.$$

(it is sufficient to take $\sigma > 0$ such that $\psi(r, \bar{z} - \sigma) \ge \pi + 2\arctan(a)$). If we define

$$h(r,z,t) = \psi(r, z - t - \sigma) \,,$$

then h solves equation (3.4) and for all $(r,z,t) \in (0,1] \times \mathbb{R} \times [0,\infty)$

$$h(r,z,t) \ge h(r,z,0) = \psi(r, z-\sigma) \ge h_0(r,z).$$

So, h is a supersolution of Problem (P_n) for every $n \ge 2$ and, by parabolic comparison principle, $h \ge h_n$. Consequently $h \ge h_m$ and, given any $t \ge 0$, one has that

$$0 \le h_m(0,z,t) \le \psi(0, z - t - \sigma) = 0 \qquad \text{if } z > \bar{y} + \sigma + t \,.$$

Hence the thesis by taking $S = \bar{y} + \sigma$.
(ii) Let $L > 0$ be a constant such that $0 \le h_0(r,0) \le Lr$ for all $r \in [0,1]$. A such value

surely exists because $h_0(0,0)=0$ and h_0 is Lipschitz continuous in $[0,1]\times\mathbb{R}$. If $\rho\in(0,1)$ is a value less or equal to $\frac{\pi}{2L}$, then from the trivial inequality

$$x \le 2\arctan\left(\frac{2}{\pi}x\right) \qquad x\in[0,\pi/2] \tag{3.22}$$

follows that for every $r\in[0,\rho]$

$$Lr \le 2\arctan\left(\frac{2}{\pi}Lr\right)$$

and therefore, in view of property (P4),

$$h_0(r,z)\le 2\arctan(Cr) \qquad \forall r\in[0,\rho],\ z\in\mathbb{R}\,, \tag{3.23}$$

if $C=\frac{2}{\pi}L$.

Since h_m is continuous in $(0,1]\times\mathbb{R}\times[0,\infty)$ and $h_m(\rho,0,0)=h_0(\rho,0)\le 2\arctan(C\rho)\le \pi/2$, there must be a value $T>0$ such that $A:=h_m(\rho,0,T)<\pi$, whence, due to (3.20), we deduce that $h_m(\rho,z,t)\le A<\pi$ for all $z\in\mathbb{R}$ and $t\in[0,T]$. Up to redefining C as the maximum between $2L/\pi$ and $\tan(A/2)\rho^{-1}$, we can say that for all $n>1/\rho$, $z\in[-n,n]$ and $t\in[0,T]$

$$h_n(\rho,z,t)\le h_m(\rho,z,t)\le 2\arctan(C\rho)\,. \tag{3.24}$$

Since the function $2\arctan(Cr)$ is a solution of (3.4), by (3.23), (3.24) and the parabolic maximum principle we obtain that for all $n>1/\rho$

$$h_n(r,z,t)\le 2\arctan(Cr) \qquad r\in[1/n,\rho], z\in[-n,n], t\in[0,T].$$

Passing to the limit as $n\to\infty$, it follows that for all $z\in\mathbb{R}$, $t\in[0,T]$

$$h_m(r,z,t)\le 2\arctan(Cr) \quad \forall r\in(0,\rho] \quad\Rightarrow\quad h_m(0,z,t)=0\,.$$

(iii) As in the proof of *(ii)*, if $\rho\in(0,1)$ is a value less or equal to $\frac{\pi}{2L}$ and $C=\frac{2}{\pi}L$, then inequality (3.23) is satisfied. Since $\lim\limits_{r\to0^+} h_m(r,0,\tau)=0$, up to redefining the constant ρ, we may always assume that $h_m(\rho,0,\tau)\le A<\pi$ and then, in view of (3.20), $h_m(\rho,z,t)\le A<\pi$ for all $z\in\mathbb{R}$ and $t\in[0,\tau]$. Up to redefining C as the maximum between $2L/\pi$ and $\tan(A/2)\rho^{-1}$, we can say that for all $n>1/\rho$, $z\in[-n,n]$ and $t\in[0,\tau]$ inequality (3.24) is satisfied. Since the function $2\arctan(Cr)$ is a solution of (3.4), by (3.23), (3.24) and the parabolic maximum principle we obtain that for all $n>1/\rho$

$$h_n(r,z,t)\le 2\arctan(Cr) \qquad r\in[1/n,\rho], z\in[-n,n], t\in[0,\tau].$$

Passing to the limit as $n\to\infty$, it follows that for all $r\in(0,\rho]$, $z\in\mathbb{R}$, $t\in[0,\tau]$

$$h_m(r,z,t)\le 2\arctan(Cr)\,,$$

whence the thesis follows.

(iv) Given any $\varepsilon > 0$, let $\tilde{z} = \zeta(\tau) + \varepsilon/2$ and let $\mu = \mu(z)$ be the function defined by

$$\mu(z) = \frac{z - \tilde{z}}{1 + z - \tilde{z}} \qquad z \in [\tilde{z}, \infty)\,.$$

If ψ is the function

$$\psi(r,z,t) := \begin{cases} 2\arctan\left(r\,\mathrm{e}^{\frac{Q}{t\mu(z)}}\right) - r^{3/2} & z > \tilde{z}, t > 0 \\ \pi - r^{3/2} & z = \tilde{z} \text{ or } t = 0\,, \end{cases}$$

where Q is a value greater or equal to the constant $\hat{Q}(\tau+1,\mu)$ of Proposition 3.8, then ψ is a supersolution of (3.4) in the open set $(0,1) \times (\tilde{z},\infty) \times (0,\tau+1)$. As in the proof of (ii), let $L > 0$ be a value such that $h_0(r,0) \leq Lr \quad \Rightarrow$

$$h_0(r,z) + r^{3/2} \leq (L+1)r \ \ \forall r \in [0,1], z \in \mathbb{R}\,.$$

If $\rho \in (0,1)$ is a value less or equal to $\pi/(2L+2)$, then, as in the proof of (ii), we have that for every $r \in [0,\rho] \quad (L+1)r \leq 2\arctan\left(\frac{2}{\pi}(L+1)r\right)$, whence we deduce that

$$h_0(r,z) + r^{3/2} \leq 2\arctan(Cr) \qquad r \in [0,\rho],\, z \in \mathbb{R} \tag{3.25}$$

if $C = (2L+2)/\pi$. Since $\lim_{r\to 0^+} h_m(r,\tilde{z},\tau) = 0$, up to redefining ρ as a smaller positive value, we may always assume that $h_m(r,\tilde{z},\tau) \leq \pi/2$ for all $r \in [0,\rho]$ and hence, by (3.20),

$$h_m(r,z,t) \leq \pi/2 \qquad r \in [0,\rho],\, z \geq \tilde{z},\, t \in [0,\tau]\,.$$

At the same time, $\psi(r,\tilde{z},t) = \pi - r^{3/2} \geq \pi - r > \pi/2$ and $\mathrm{e}^{\frac{Q}{\tau}} \geq C$,

$$\psi(\rho,z,t) \geq 2\arctan\left(\rho\,\mathrm{e}^{\frac{Q}{\tau}}\right) - \rho^{3/2} > \pi/2 \qquad z \geq \tilde{z}, t \in [0,\tau]$$

if we choose Q sufficiently large (so, the final value of Q also depends on L and ρ). Therefore, we can say that

$$h_0(r,z) \leq 2\arctan(Cr) - r^{3/2} \leq 2\arctan(r\,\mathrm{e}^{\frac{Q}{\tau}}) - r^{3/2} \leq \psi(r,z,t) \quad r \in [0,\rho], z \geq \tilde{z}, t \in [0,\tau], \tag{3.26}$$

and

$$\begin{aligned} \psi(r,\tilde{z},t) &\geq h_m(r,\tilde{z},t) \qquad r \in [0,\rho], t \in [0,\tau]\,, \\ \psi(\rho,z,t) &\geq h_m(\rho,z,t) \qquad z \geq \tilde{z}, t \in [0,\tau]\,. \end{aligned} \tag{3.27}$$

Since $h_n \leq h_m$ in $\bar{A}_n \times [0,\infty)$ for every integer $n \geq 2$, by (3.26), (3.27) and parabolic maximum principle we obtain that for every $n > 1/\rho$

$$h_n(r,z,t) \leq \psi(r,z,t) \qquad r \in [1/n,\rho],\, z \in [\tilde{z},n],\, t \in [0,\tau]\,.$$

Passing to the limit as $n \to \infty$, we deduce that

$$h_m(r,z,t) \leq \psi(r,z,t) \qquad r \in (0,\rho],\, z \geq \tilde{z},\, t \in [0,\tau]\,.$$

In particular, for every $r \in [0,\rho]$, $z \geq \zeta(\tau) + \varepsilon = \tilde{z} + \varepsilon/2$

$$h_m(r,z,\tau) \leq 2\arctan\left(r\,\mathrm{e}^{\frac{Q(2+\varepsilon)}{\tau\varepsilon}}\right).$$

The thesis then follows in view of (3.20). □

The first interesting consequence of the previous Theorem, namely of its points (i) and (iv), is given by the following

Corollary 3.18. *For every $T > 0$*

$$h_m(r, z, t) \to \theta_b(r) \text{ as } |z| \to \infty$$

uniformly with respect to $r \in [0, 1]$ and $t \in [0, T]$.

Proof: By parabolic Schauder-type estimates, there exists a function $h_\infty = h_\infty(r, t)$ such that for every $\sigma \in (0, 1)$, $\tau > 0$:

$$h_m(r, z, t) \to h_\infty(r, t) \text{ as } |z| \to \infty$$

in $C^{2,1}([\sigma, 1] \times [\tau, \infty))$ other than in $C^0([\sigma, 1] \times [0, \infty))$. Fixed arbitrarily a value $T > 0$, by (i) and (iv) of Theorem 3.17, there exist $\zeta \in \mathbb{R}^+$, $\rho \in (0, 1)$ and $C > 0$ such that for all $r \in [0, \rho]$, $z \in \mathbb{R} \setminus (-\zeta, \zeta)$ and $t \in [0, T]$ $\;0 \le h_m(r, z, t) \le 2 \arctan(Cr)$. Hence we deduce that $h_\infty(0, t) = 0$ for all $t \in [0, T]$ and $h(r, z, t) \to h_\infty(r, t)$, as $|z| \to \infty$, uniformly with respect to $r \in [0, 1]$ and $t \in [0, T]$. Therefore, $h_\infty \in C^{2,1}((0, 1) \times \mathbb{R}^+) \cap C^0([0, 1] \times [0, \infty))$ and solves the problem

$$\begin{cases} h_t = h_{rr} + \frac{h_r}{r} - \frac{\sin(2h)}{2r^2} & r \in (0, 1), t > 0 \\ h(r, 0) = \lim\limits_{|z| \to \infty} h_0(r, z) = \theta_b(r) & r \in (0, 1) \\ h(0, t) = 0, \quad h(1, t) = \theta_b(1) & t \ge 0 \,. \end{cases}$$

Since the unique classical solution of this problem is $\theta_b(r)$, we find out that $h_\infty \equiv \theta_b$. □

Let $T \in [0, \infty]$ be the *first time of blow-up* for h_m, i.e.

$$T := \sup\{t > 0 \mid h_m(0, z, t) = 0 \;\; \forall z \in \mathbb{R}\} \,.$$

By Theorem 3.17, (ii), T is greater than zero. Moreover, in view of the point (iii) of the same theorem, **if $T < \infty$, then $I_0(T) \neq \mathbb{R}$.** By contradiction, if $I_0(T) = \mathbb{R}$, then there exist $\rho > 0$ and $C > 0$ such that

$$h_m(r, z, t) \le 2 \arctan(Cr) \qquad \forall r \in [0, \rho], z \in \mathbb{R}, t \in [0, T].$$

Since h_m is smooth in $(0, 1] \times \mathbb{R} \times \mathbb{R}^+$, there must be $\delta > 0$ and $A \in (0, \pi)$ such that

$$h_m(\rho, z, t) \le A \qquad \forall z \in \mathbb{R}, t \in [0, T + \delta] \,.$$

At the same time, since $h_0(r, z) \le Lr$ for a suitable $L > 0$ and in view of (3.22), up to redefining ρ as a smaller positive value, we may always assume that

$$h_0(r, z) \le 2 \arctan\left(\frac{2L}{\pi} r\right) \qquad \forall r \in [0, \rho], z \in \mathbb{R} \,.$$

If now we take $D = \max(2L/\pi, \tan(A/2)\rho^{-1})$, then

$$h_0(r, z) \le 2 \arctan(Dr), \quad h_m(\rho, z, t) \le 2 \arctan(D\rho)$$

for all $r \in [0, \rho], z \in \mathbb{R}, t \in [0, T + \delta]$. By the same argument used to conclude the proof of the statement (iii) of Theorem 3.17 we obtain that

$$h_m(r, z, t) \le 2 \arctan(Dr) \qquad r \in (0, \rho], z \in \mathbb{R}, t \in [0, T + \delta]$$

and therefore for all $t \in [0, T+\delta]$

$$\lim_{r \to 0^+} h_m(r, z, t) = 0 \quad \forall z \in \mathbb{R}$$

But this implies that $I_0(T+\delta) = \mathbb{R}$, which is clearly absurd by definition of T.

Let us suppose that $T < \infty$. Since $I_0(T) \neq \mathbb{R}$, one of the two following cases occurs:

(A) $\limsup_{r \to 0^+} h_m(r, 0, T) > 0$ while $h_m(0, z, T) = 0$ for $z \neq 0$,

(B) there exists $\zeta \in (0, \infty)$ such that $h_m(0, z, T) = \pi$ for $|z| < \zeta$ and $h_m(0, z, T) = 0$ for $|z| > \zeta$.

In both cases, if $\nabla h_m = ((h_m)_r, (h_m)_z)$ is the gradient of h_m with respect to the variables r and z, and

$$\|\nabla h_m\|_\infty(t) = \sup_{(r,z) \in (0,1] \times \mathbb{R}} |\nabla h_m|(r, z, t) \qquad t \geq 0\,,$$

we have that

$$\|\nabla h_m\|_\infty(T) = \infty\,.$$

On the other hand, if $\tau < T$, then $I_0(\tau) = \mathbb{R}$ and, due to the regularity of h out of $\{r = 0\}$ and to Theorem 3.17, (iii),

$$\sup_{t \in [0,\tau]} \|\nabla h_m\|_\infty(t) < \infty.$$

This explains why T is called *first time of blow-up* for the function h_m.

In view of what is already known, it could also occur that $T = \infty$. In the next section we prove that actually $T < \infty$ and the function

$$\begin{array}{rccc} \zeta: & [T, \infty) & \longrightarrow & [0, \infty) \\ & t & \longrightarrow & |I_1(t)|/2\,, \end{array} \tag{3.28}$$

where $|I_1(t)|$ is the one-dimensional Lebesgue measure of the interval $I_1(t)$, is right continuous (in addition to being sublinear, according to Theorem 3.17, (i)).

3.3 Blow up of h_m in finite time

We start this section with a comparison principle which has been obtained by slightly modifying a similar result contained in [14].

Lemma 3.19. *Let ψ, ξ be respectively a regular super- and subsolution to Problem (3.1) on a time interval $[0, \mathscr{T})$ ($\mathscr{T} > 0$). If, for every $\tau \in (0, \mathscr{T})$,*

(h1)

$$\xi(r, z, t)\,, \psi(r, z, t) \longrightarrow 0 \qquad \text{as } r \to 0^+$$

uniformly with respect to $z \in \mathbb{R}$ and $t \in [0, \tau]$, and

(h2)

$$\limsup_{|z| \to \infty} \sup_{r \in [0,1], t \in [0,\tau]} (\xi(r, z, t) - \psi(r, z, t)) \leq 0\,,$$

then $\xi \le \psi$ *on* $[0,1]\times\mathbb{R}\times[0,\mathscr{T})$.

Proof: Let $\eta := \xi - \psi$. Then $\eta \le 0$ on $[0,1]\times\mathbb{R}\times\{0\}$ and on $\{0,1\}\times\mathbb{R}\times[0,\mathscr{T})$. At the same time, η satisfies the differential inequality

$$\eta_t - \Delta\eta - \frac{\eta_r}{r} + \frac{f}{2r^2}\eta \le 0 \tag{3.29}$$

on $(0,1)\times\mathbb{R}\times(0,\mathscr{T})$, where $\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian in (r,z) coordinates, and

$$f(r,z,t) := \int_0^1 \cos(2(s\xi(r,z,t) + (1-s)\psi(r,z,t)))\,\mathrm{d}\,s$$

is bounded on $[0,1]\times\mathbb{R}\times[0,\mathscr{T})$. If the thesis is false, then there exists a time $\tau \in (0,\mathscr{T})$ such that

$$\sup_{[0,1]\times\mathbb{R}\times[0,\tau]} \eta > 0\,. \tag{3.30}$$

In view of (h1) there exists $\rho \in (0,1)$ such that for every $r \in [0,\rho]$, $z \in \mathbb{R}$ and $t \in [0,\tau]$

$$|\xi(r,z,t)|, |\psi(r,z,t)| \le \pi/4 \tag{3.31}$$

and therefore $f \ge 0$ in $[0,\rho]\times\mathbb{R}\times[0,\tau]$. Let $M > 0$ be a constant such that $|f| < M$. Multiplying (3.29) by $\mathrm{e}^{-\frac{Mt}{2\rho^2}}$ and introducing

$$h(r,z,t) := \mathrm{e}^{-\frac{Mt}{2\rho^2}}\,\eta$$

we have that

$$h_t - \Delta h - \frac{h_r}{r} + h\left(\frac{f}{2r^2} + \frac{M}{2\rho^2}\right) \le 0\,. \tag{3.32}$$

Since $\mathrm{e}^{-\frac{M\tau}{2\rho^2}}\,\eta \le h \le \eta$ on $[0,1]\times\mathbb{R}\times[0,\tau]$, from (3.30) we deduce that

$$s := \sup_{[0,1]\times\mathbb{R}\times[0,\tau]} h > 0\,,$$

from (h2) that

$$\limsup_{|z|\to\infty} \sup_{r\in[0,1],t\in[0,\tau]} h(r,z,t) \le 0\,. \tag{3.33}$$

But (3.33) implies the existence of $\zeta > 0$ such that

$$\sup_{r\in[0,1],\,|z|>\zeta,\,t\in[0,\tau]} h(r,z,t) < s/2$$

and then

$$s = \max_{[0,1]\times[-\zeta,\zeta]\times[0,\tau]} h\,,$$

i.e. h attains a positive maximum on $[0,1]\times\mathbb{R}\times[0,\tau]$. On the other hand, just like η, h is non positive on $\{0,1\}\times\mathbb{R}\times[0,\tau]$ and on $(0,1)\times\mathbb{R}\times\{0\}$. Therefore, the positive maximum is achieved on $(0,1)\times\mathbb{R}\times(0,\tau]$, say at $(\bar r,\bar z,\bar t)$. By the regularity of h we obtain that

$$h_t(\bar r,\bar z,\bar t) \ge 0,\ \Delta h(\bar r,\bar z,\bar t) \le 0,\ h_r(\bar r,\bar z,\bar t) = 0\,,$$

and, using (3.32), that

$$\frac{f(\bar r,\bar z,\bar t)}{2\bar r^2}+\frac{M}{2\rho^2}\le 0. \tag{3.34}$$

But if $\bar r\le\rho$, then $f(\bar r,\bar z,\bar t)\ge 0$ and the previous inequality is false. Otherwise, if $\bar r>\rho$, then

$$\frac{f(\bar r,\bar z,\bar t)}{2\bar r^2}>-\frac{M}{2\rho^2}$$

and (3.34) is again false. □

Remark 3.20. The hypothesis (h1) in the statement of the previous lemma can be replaced by

(h3) $\xi\ge 0$ on $[0,1]\times\mathbb{R}\times[0,\mathscr{T})$ and for every $\tau\in(0,\mathscr{T})$

$$\limsup_{r\to 0^+}\sup_{z\in\mathbb{R},t\in[0,\tau]}\psi(r,z,t)\le 0\,.$$

Actually (h1) is only used to establish (3.31) on a strip $[0,\rho]\times\mathbb{R}\times[0,\tau]$ ($\rho\in(0,1)$) and we can obtain these inequalities also starting from (3.30), (h2) and (h3). First from (3.30) and (h2) we derive that

$$S:=\sup_{[0,1]\times\mathbb{R}\times[0,\tau]}\eta=\max_{[0,1]\times[-z^*,\,z^*]\times[0,\tau]}\eta>0$$

for some $z^*>0$. Since η is uniformly continuous in $[0,1]\times[-z^*,z^*]\times[0,\tau]$, the function

$$\mathcal{M}(t):=\max_{[0,1]\times[-z^*,\,z^*]\times[0,t]}\eta$$

is continuous on $[0,\tau]$ with $\mathcal{M}(\tau)=S$, $\mathcal{M}(0)\le 0$. Then, we can always assume that $S\le\frac{\pi}{8}$. Hence

$$\xi-\psi\le\pi/8\quad\text{on }[0,1]\times\mathbb{R}\times[0,\tau]. \tag{3.35}$$

On the other hand, (h3) implies the existence of $\rho\in(0,1)$ such that for every $r\in[0,\rho]$, $z\in\mathbb{R}$ and $t\in[0,\tau]$

$$\psi(r,z,t)\le\pi/8\,. \tag{3.36}$$

Putting together $\xi\ge 0$, (3.35) and (3.36) we deduce that

$$|\psi|\le\pi/8,\ |\xi|\le\pi/4\quad\Rightarrow\quad f\ge 0$$

on $[0,\rho]\times\mathbb{R}\times[0,\tau]$. Once obtained this inequality the proof continues in the same way we have already shown.

Theorem 3.21. *The first time of blow up of h_m is less than or equal to $\mathcal{T}$.*

Proof: We know that h_m is a solution (smooth in $(0,1)\times\mathbb{R}\times\mathbb{R}^+$ and continuous in $[0,1]\times\mathbb{R}\times[0,T)$) of Problem (3.1). At the same time, $\tilde\xi$ is a subsolution of (3.4) in the open set $(0,1)\times\mathbb{R}\times(0,\mathcal{T})$. Hence, by property (P5) of h_0, $\tilde\xi$ is a subsolution of Problem (3.1) in the time interval $[0,\mathcal{T})$. By contradiction, let $T>\mathcal{T}$. Then, $I_0(\mathcal{T})=\emptyset$ and, by Theorem 3.17, (iii), there exist $\rho>0$ and $C>0$ such that

$$h_m(r,z,t)\le 2\arctan(Cr)\qquad\forall r\in[0,\rho],z\in\mathbb{R},t\in[0,\mathcal{T}]\,.$$

Therefore, given any $\tau \in (0, \mathcal{T})$, the functions h_m and $\tilde{\xi}$ satisfy the hypothesis (h1) of Lemma 3.19. Due to (P5) and to (3.20), for every $r \in [0,1], z \in \mathbb{R}$ and $t \in [0, \tau]$ we have that

$$\tilde{\xi}(r,z,t) - h_m(r,z,t) = \tilde{\xi}(r,z,t) - \tilde{\xi}(r,z,0) + \tilde{\xi}(r,z,0) - h_m(r,z,t) \le \tilde{\xi}(r,z,t) - \tilde{\xi}(r,z,0)$$

and, if we denote by $\tilde{\lambda}$ the function defined by

$$\tilde{\lambda}(z,t) = \frac{1}{K}\, \mathrm{e}^{\frac{Q}{\mathcal{T} - t + \mu_{\mathscr{Z}}(z)}},$$

$$\begin{aligned}\tilde{\xi}(r,z,t) - \tilde{\xi}(r,z,0) = 2\arctan(r\tilde{\lambda}(z,t)) - 2\arctan(r\tilde{\lambda}(z,0)) \le 2r(\tilde{\lambda}(z,t) - \tilde{\lambda}(z,0)) \le \\ \le 2(\tilde{\lambda}(z,\tau) - \tilde{\lambda}(z,0))\,.\end{aligned}$$

Since this last function goes to 0 as $|z| \to \infty$, h_m and $\tilde{\xi}$ also satisfy the hypothesis (h2) of Lemma 3.19. Then must be

$$\tilde{\xi} \le h_m \quad \text{in } [0,1] \times \mathbb{R} \times [0, \mathcal{T}).$$

Since h_m is continuous in $(0,1] \times \mathbb{R} \times [0, \infty)$, the previous inequality implies that for every $r \in (0,1]$, $z \in [-\mathscr{Z}, \mathscr{Z}]$

$$\pi + \mathscr{B} r^{\alpha} = \lim_{t \to \mathcal{T}^-} \tilde{\xi}(r,z,t) \le h_m(r,z,\mathcal{T}).$$

Then, for every $z \in [-\mathscr{Z}, \mathscr{Z}]$ must be $h_m(0,z,\mathcal{T}) = \pi \;\; \Rightarrow I_0(\mathcal{T}) \ne \mathbb{R}$, which contradicts the assumption $T > \mathcal{T}$. $\square$

Now we have to show that the function ζ defined by (3.28) is right continuous. In order to do this, we need a generalization of Theorem 2.2. By repeating the same construction of chapter 2 in the strip $(0, R) \times \mathbb{R}$ ($R > 0$) rather than in $(0,1) \times \mathbb{R}$, it is possible to show that

Theorem 3.22. *Given $c > 0$ and a function $g = g(z)$ satisfying*

$$g \in C^4(\mathbb{R}),\;\; g' \le 0 \textit{ in } \mathbb{R},\;\; g = A \textit{ in } (-\infty, z_0),\;\; g = B \textit{ in } (z_1, \infty).$$

for some $z_0 < z_1$ and

$$\pi < A < 3\pi \quad \textit{and} \quad 0 < B < \pi/2,$$

there exists a function $\psi : [0,R] \times \mathbb{R} \to \mathbb{R}$ which is smooth in $(0,R] \times \mathbb{R}$ and satisfies equations (7) and

$$\psi(R,z) = g(z)\,.$$

In addition the following properties are satisfied:
(i) there exists $\hat{z}$ such that ψ is continuous in $\{(0,z) : z \ne \hat{z}\}$, $\psi(0,z) = 0$ if $z > \hat{z}$ and $\psi(0,z) = 2\pi$ if $z < \hat{z}$;
(ii) $\psi(r,z)$ is nonincreasing with respect to z;
(iii) $\psi(r,z) \to 2\arctan(\beta r/R)$ uniformly with respect to $r \in [0,R]$ as $z \to \infty$, where β is defined by $2\arctan\beta = B$;
(iv) $\psi(r,z) \to 2\pi + 2\arctan(\alpha r/R)$ uniformly with respect to $r \in [0,R]$ as $z \to -\infty$, where α is defined by $2\pi + 2\arctan\alpha = A$;
(v) ψ is real analytic in $[0,R) \times \mathbb{R} \setminus \{(0,\hat{z})\}$.

We remark that, up to a translation in the z variable, it is always possible to make $\hat{z} = 0$.

Since h_m is monotone increasing with respect to the time variable t and so is ζ, to prove the right-continuity of ζ is sufficient to show that

Theorem 3.23. *For every $\tau \in [T, \infty)$*

$$\limsup_{t \to \tau^+} \zeta(t) \leq \zeta(\tau).$$

Proof: Let τ be a value greater or equal to T. Thanks to Theorem 3.17,(iv), for every $\varepsilon > 0$ there exist $\rho \in (0,1)$ and $C > 0$ (both ρ and C depend on ε) such that

$$h_m(r,z,t) \leq 2\arctan(Cr) \qquad r \in [0,\rho], |z| \geq \zeta(\tau) + \varepsilon, t \in [0,\tau].$$

Hence, given any $\varepsilon > 0$, there exists $R \in (0, \rho]$ such that

$$h_m(r,z,\tau) \leq \arctan\left(\frac{br}{R}\right) \qquad \forall r \in [0,R],\ \forall z \geq \zeta(\tau) + \varepsilon \tag{3.37}$$

(it is sufficient to take $R = \min(\rho, b/(4C))$). By Theorem 3.22, there exists a function

$$\begin{array}{rccc} \psi: & [0,R] \times \mathbb{R} \setminus \{(0,0)\} & \longrightarrow & \mathbb{R} \\ & (r,y) & \longrightarrow & \psi(r,y) \end{array}$$

which satisfies all the following properties:

(p1) ψ is smooth (C^∞) in its domain,

(p2) ψ is non increasing with respect to y,

(p3) $\psi \to 2\arctan(br/R)$ as $y \to \infty$, and $\psi \to 2\pi$ as $y \to -\infty$ uniformly with respect to $r \in [0, R]$,

(p4) ψ solves the problem

$$\begin{cases} \psi_{yy} + \psi_y + \psi_{rr} + \frac{\psi_r}{r} - \frac{\sin(2\psi)}{2r^2} = 0 & (0,R) \times \mathbb{R} \\ \psi(0,y) = \begin{cases} 0 & \text{if } y > 0 \\ 2\pi & \text{if } y < 0 \end{cases} & \\ \psi(R,y) = g(y) & y \in \mathbb{R} \end{cases}$$

for a suitable function $g \in C^\infty(\mathbb{R})$ such that $g' \leq 0$, $g(y) = 2\arctan(b)$ for every $y \geq 1$ and $g(y) = 2\pi$ for every $y \leq -1$.

In view of (p1), (p2), (p4) and of inequality $h_m(r,z,t) \leq \theta_a(r)$, we obtain that, for a suitable $\mathcal{R} \in (0, R]$,

$$\psi(r,y) \geq 3\pi/2 \geq h_m(r,z,t) + \pi/4 \qquad \forall r \in [0,\mathcal{R}],\ y \leq -\varepsilon \text{ and } \forall (z,t) \in \mathbb{R} \times \mathbb{R}^+. \tag{3.38}$$

At the same time, from (3.37) follows that

$$h_m(r,z,\tau) + \arctan(br/R) \leq 2\arctan(br/R) \leq \psi(r,y) \qquad \forall r \in [0,\mathcal{R}],\ z \geq \zeta(\tau) + \varepsilon,\ y \in \mathbb{R}. \tag{3.39}$$

Let $C = \zeta(\tau) + 2\varepsilon - \tau$ and let w be the function

$$w(r, z, t) = \psi(r, z - t - C) \qquad r \in [0, R],\ z \in \mathbb{R}, t \geq 0\,.$$

w is a solution of equation (3.4) which satisfies the following properties:

1. thanks to (3.39), $\forall r \in [0, \mathcal{R}]$, $\forall z \geq \zeta(\tau) + \varepsilon$

$$h_m(r, z, \tau) + \arctan(br/R) \leq w(r, z, \tau),$$

2. $\forall r \in [0, \mathcal{R}]$, $z \leq \zeta(\tau) + \varepsilon$

$$w(r, z, \tau) = \psi(r, z - \tau - C) \geq \psi(r, \zeta(\tau) + \varepsilon - \tau - C) = \psi(r, -\varepsilon) \geq h_m(r, z, \tau) + \pi/4$$

   by the monotonicity of ψ and (3.38).

Therefore, $\forall r \in [0, \mathcal{R}]$ and $z \in \mathbb{R}$

$$w(r, z, \tau) - h_m(r, z, \tau) \geq \min(\pi/4, \arctan(br/R)) = \arctan(br/R)\,.$$

Hence follows that

$$w(\mathcal{R}, z, \tau) - h_m(\mathcal{R}, z, \tau) \geq \arctan(b\mathcal{R}/R) > 0 \qquad \forall z \in \mathbb{R}$$

and then, since w is increasing with respect to t and, by parabolic Schauder-type estimates, $h_m(\mathcal{R}, z, t)$ is continuous in $t \in J$ uniformly with respect to $z \in \mathbb{R}$ for a suitable neighborhood J of τ, there exists $\delta > 0$ such that

$$w(\mathcal{R}, z, t) \geq h_m(\mathcal{R}, z, t) \qquad \forall z \in \mathbb{R},\ t \in [\tau, \tau + \delta].$$

Since w is increasing with respect to t and $\forall n \in \mathbb{N}$, $n \geq 2$

$$h_0(r, z) \leq h_n(r, z, t) \leq h_m(r, z, t) \qquad \forall (r, z) \in \bar{A}_n,\ t \geq 0,$$

we deduce that for every $n \in \mathbb{N}$ with $n > 1/\mathcal{R}$ the function w is a supersolution of the differential problem

$$\begin{cases} h_t \;=\; h_{rr} + h_{zz} + \frac{h_r}{r} - \frac{\sin(2h)}{2r^2} & \text{in } (1/n, \mathcal{R}) \times (-n, n) \times (\tau, \infty) \\ h(r, z, \tau) \;=\; h_n(r, z, \tau) & \text{for } 1/n < r < \mathcal{R}, z \in (-n, n) \\ h(r, \pm n, t) = h_0(r, \pm n) & \text{for } r \in [1/n, \mathcal{R}],\ t \geq \tau \\ h(1/n, z, t) \;=\; h_0(1/n, z) & \text{for } z \in [-n, n], t \geq \tau \\ h(\mathcal{R}, z, t) \;=\; h_n(\mathcal{R}, z, t) & \text{for } z \in [-n, n], t \geq \tau\,. \end{cases}$$

in the time interval $[\tau, \tau + \delta]$.

Since h_n is the solution of the previous problem, we then obtain that

$$w(r, z, t) \geq h_n(r, z, t) \qquad \forall r \in [1/n, \mathcal{R}],\ z \in [-n, n],\ t \in [\tau, \tau + \delta]\,.$$

Passing to the limit as $n \to \infty$ we deduce that

$$w(r, z, t) \geq h_m(r, z, t) \qquad \forall r \in (0, \mathcal{R}],\ z \in \mathbb{R},\ t \in [\tau, \tau + \delta]\,.$$

Hence for every $t \in [\tau, \tau + \delta]$ and for all $z > t + \zeta(\tau) + 2\varepsilon - \tau$

$$0 \leq \lim_{r \to 0^+} h_m(r, z, t) \leq \psi(0,\ z - t - \zeta(\tau) - 2\varepsilon + \tau) = 0.$$

Therefore, for every $t \in [\tau, \tau + \delta]$

$$\zeta(t) = \inf\{z \in \mathbb{R} \mid h_m(0, z, t) = 0\} \leq t + \zeta(\tau) + 2\varepsilon - \tau$$

and so

$$\limsup_{t \to \tau^+} \zeta(t) \leq \zeta(\tau) + 2\varepsilon\,.$$

The thesis then follows from the arbitrariness of $\varepsilon > 0$. □

3.4 Construction of h_M

This last section is devoted to the construction of a weak solution h_M of Problem (3.1) such that $h_M \geq h_m$ and, for every $t > 0$,

$$\lim_{r \to 0^+} h_M(r, z, t) \;=\; \pi \quad \text{if } |z| \leq M, \qquad \lim_{r \to 0^+} h_M(r, z, t) \;=\; 0 \quad \text{if } |z| > S + M + t,$$

where M is a non-negative value arbitrarily chosen and $S > 0$ is a constant independent from M. For every $n \in \mathbb{N}, n \geq 2$ let ω_n be the function defined as

$$\omega_n(r, z) = \begin{cases} \pi & r = 0,\ |z| < M + 2 \\ 0 & r = 0,\ |z| \geq M + 2 \\ 2\arctan\left(\frac{1+b}{1-b}\frac{\gamma_{M+1}(z)}{nr^2}\right) & r \in (0, 1],\ z \in \mathbb{R}, \end{cases}$$

where γ_{M+1} belongs to the family of functions given in (3.16). Then the following assertion results to be true:

Proposition 3.24. *There exists a constant $C > 0$ such that*

$$\int_0^1 \frac{r}{2}\left((\omega_n)_r^2 + (\omega_n)_z^2 + \frac{\sin^2(\omega_n)}{r^2}\right) \mathrm{d}\,r \;\leq\; C$$

for every $n \in \mathbb{N}, n \geq 2$ and for every $z \in \mathbb{R}$.

Proof: For sake of simplicity we fix $n \in \mathbb{N}$, $n \geq 2$ and we denote by ω the function ω_n and by γ the function

$$\frac{1+b}{1-b}\frac{\gamma_{M+1}}{n},$$

so that

$$\omega(r, z) = 2\arctan(\gamma(z)/r^2), \qquad \frac{\sin\omega}{r} = \frac{2\gamma(z)r}{r^4 + \gamma^2(z)}$$

and

$$\omega_r = \frac{-4\gamma(z)r}{r^4 + \gamma^2(z)}, \quad \omega_z = \frac{2\gamma'(z)r^2}{r^4 + \gamma^2(z)}\,.$$

Therefore, for every fixed $z \in \mathbb{R}$

$$\int_0^1 \frac{r}{2}\left(\omega_r^2 + \frac{\sin^2\omega}{r^2}\right) \mathrm{d}\, r = \int_0^1 \frac{10\gamma^2 r^3}{(r^4+\gamma^2)^2} \mathrm{d}\, r = \frac{5}{2}\gamma^2 \int_{\gamma^2}^{1+\gamma^2} \frac{\mathrm{d}\, s}{s^2} = \frac{5}{2(1+\gamma^2)} \leq 5/2\,.$$

If $|z| \leq M+1$ or $|z| \geq M+2$, then

$$\int_0^1 \frac{r}{2}\omega_z^2 \,\mathrm{d}\, r = 0,$$

else

$$\int_0^1 \frac{r}{2}\omega_z^2 \,\mathrm{d}\, r = \frac{1}{2}\left|\frac{d\gamma}{dz}\right|^2 \int_0^1 \frac{4r^5}{(r^4+\gamma^2)^2} \mathrm{d}\, r = \frac{1}{2}\left|\frac{d\gamma}{dz}\right|^2 \int_{\gamma^2}^{1+\gamma^2} \frac{\sqrt{s-\gamma^2}}{s^2} \mathrm{d}\, s \leq$$

$$\frac{1}{2}\left|\frac{d\gamma}{dz}\right|^2 \int_{\gamma^2}^{1+\gamma^2} s^{-3/2} \,\mathrm{d}\, s = \gamma^{-1}\left|\frac{d\gamma}{dz}\right|^2 \frac{\sqrt{1+\gamma^2}-\gamma}{\sqrt{1+\gamma^2}} \leq \gamma^{-1}\left|\frac{d\gamma}{dz}\right|^2 =$$

$$= \frac{1+b}{1-b}\frac{1}{n}\gamma_{M+1}^{-1}\left|\frac{d\gamma_{M+1}}{dz}\right|^2 \leq \frac{1+b}{1-b}N_2$$

where N_2 is the constant of (3.16). □

For every $n \in \mathbb{N}, n \geq 2$ we consider the differential problem

$$(\mathscr{P}_n)\begin{cases} h_t \;=\; h_{rr} + h_{zz} + \frac{h_r}{r} - \frac{\sin(2h)}{2r^2} & (r,z,t) \in A_n \times \mathbb{R}^+ \\ h(r,z,0) = h_{0n}(r,z) & (r,z) \in \bar{A}_n \\ h(1/n,z,t) = h_{0n}(1/n,z) & (z,t) \in [-n,n] \times \mathbb{R}^+ \\ h(1,z,t) = h_{0n}(1,z) & (z,t) \in [-n,n] \times \mathbb{R}^+ \\ h(r,\pm n,t) = h_{0n}(r,\pm n) & (r,t) \in [1/n,1] \times \mathbb{R}^+ \end{cases}$$

where A_n is the rectangle $(1/n,\, 1) \times (-n,n)$ and $h_{0n} = \max(\omega_n,\, h_0)$.

The functions h_{0n} $(n \in \mathbb{N},\, n \geq 2)$ satisfy the following properties:

1. $\theta_b(r) \leq h_{0n}(r,z) \leq \theta_a(r)$, $h_{0n}(r,z) \equiv \theta_b(r)$ if $|z| \geq \max(\bar{z}, M+2)$,
2. There exists a constant $C = C(h_0) > 0$ such that for every fixed $z \in \mathbb{R}$
$$\int_0^1 \frac{r}{2}\left((h_{0n})_r^2 + (h_{0n})_z^2 + \frac{\sin^2(h_{0n})}{r^2}\right) \mathrm{d}\, r \leq C,$$
3. $h_{0n} \geq h_0$, $h_{0n}(r,-z) \equiv h_{0n}(r,z)$, $(h_{0n})_z(r,z) \leq 0$ for $r \in [0,1], z \geq 0$ and
4. for every $n \geq 2$ the function h_{0n} is Lipschitz continuous in $[1/n,1] \times \mathbb{R}$.

Moreover, up to a regularization, we can also assume that $h_{0n} \in C^\infty([1/n,1] \times \mathbb{R})$. By standard solvability, comparison and regularity results for parabolic problems (see [21]) and since the functions $\theta_b(r) = 2\arctan(br)$, $\theta_a(r) := \pi + 2\arctan(ar)$ both solve

(3.4), we can say that for every $n \in \mathbb{N}$ Problem $(\mathscr{P}_n)$ has a unique classical solution $H_n \in C^\infty(\bar{A}_n \times \mathbb{R}^+) \cap C^0(\bar{A}_n \times [0,\infty))$ and

$$\begin{gathered} 2\arctan(br) \leq H_n(r,z,t) \leq \pi + 2\arctan(ar), \\ H_n(r,-z,t) = H_n(r,z,t), \\ (H_n)_r, (H_n)_z \text{ are Hölder continuous in } (r,z,t). \end{gathered} \tag{3.40}$$

Moreover, due to property (P1) and because $h_{0n} \geq h_0$, we have that h_0 is a subsolution of Problem $(\mathscr{P}_n)$ and then

$$H_n(r,z,t) \geq h_0(r,z) \qquad \forall (r,z,t) \in \bar{A}_n \times [0,\infty). \tag{3.41}$$

At last, in view of (3.40) and properties of h_{0n}, for every $n \in \mathbb{N}$ with $n \geq \max(\bar{z}, M+2)$ the function $(H_n)_z$ is a subsolution of the problem

$$\begin{cases} \psi_t \,=\, \psi_{rr} + \psi_{zz} + \frac{\psi_r}{r} - \frac{\cos(2h)}{r^2}\psi & (r,z,t) \in (1/n,1) \times (0,n) \times \mathbb{R}^+ \\ \psi(r,z,0) = 0 & (r,z) \in [1/n,1] \times [0,n] \\ \psi(1/n,z,t) = 0 & (z,t) \in [0,n] \times \mathbb{R}^+ \\ \psi(1,z,t) = 0 & (z,t) \in [0,n] \times \mathbb{R}^+ \\ \psi(r,0,t) = 0 & (r,t) \in [1/n,1] \times \mathbb{R}^+ \\ \psi(r,n,t) = 0 & (r,t) \in [1/n,1] \times \mathbb{R}^+ . \end{cases}$$

From the parabolic comparison principle follows that for every $n \in \mathbb{N}$ with $n \geq \max(\bar{z}, M+2)$

$$(H_n)_z(r,z,t) \leq 0 \qquad \text{if } z \geq 0\,. \tag{3.42}$$

If we define the functional E_n just as in section 3.2, then

Proposition 3.25. *For every $n \geq 2$ and $T > 0$*

$$E_n(H_n(\cdot\,,\cdot\,,T)) + \iiint_{A_n \times [0,T]} r(H_n)_t^2 \,\mathrm{d}\,r\,\mathrm{d}\,z\,\mathrm{d}\,t \;=\; E_n(h_{0n})\,.$$

Proof: The proof is formally identical to that one of Proposition 3.14 and so it can be omitted. □

Let $E_{n,\zeta}$ $(n \in \mathbb{N}, n \geq 2,\ \zeta > 0)$ be the functional defined in the proof of Proposition 3.16. Then

Proposition 3.26. *There exists a constant $C = C(h_0) > 0$ such that for every $\zeta > \max(\bar{z},\ M+2)$ and $T > 0$*

$$E_{n,\zeta}(H_n(\cdot,\ \cdot,\ T)) + \iiint_{A_n \times [0,T]} r(H_n)_t^2 \,\mathrm{d}\,r\,\mathrm{d}\,z\,\mathrm{d}\,t \leq 2C(h_0)\zeta$$

for all $n \geq \zeta$.

Proof: Let $T > 0$, $\zeta > \max(\bar{z},\ M+2)$ be arbitrarily fixed. From Proposition 3.25 follows that $\forall n \in \mathbb{N}, n \geq \zeta$

$$E_{n,n}(H_n(\cdot\,,\cdot\,,T)) + \iiint_{A_n \times [0,T]} r(H_n)_t^2 \,\mathrm{d}\,r\,\mathrm{d}\,z\,\mathrm{d}\,t \;=\; E_{n,n}(h_{0n})\,. \tag{3.43}$$

Since $h_{0n}(r,z)=\theta_b(r)$ if $|z|\geq \max(\bar{z},M+2)$, when $|z|>\max(\bar{z},M+2)$ we have that

$$\int_{1/n}^{1}\frac{r}{2}\left((h_{0n})_r^2+(h_{0n})_z^2+\frac{\sin^2(h_{0n})}{r^2}\right)\mathrm{d}\,r=\int_{1/n}^{1}\frac{r}{2}\left((\theta_b)_r^2+\frac{\sin^2(\theta_b)}{r^2}\right)\mathrm{d}\,r\,.$$

At the same time, if $|z|\geq \max(\bar{z},M+2)$ then $H_n(1/n,z,T)=h_{0n}(1/n,z)=\theta_b(1/n)$, $H_n(1,z,T)=h_{0n}(1,z)=\theta_b(1)$, and by Corollary A.3,

$$\int_{1/n}^{1}\frac{r}{2}\left((H_n)_r^2(r,z,T)+(H_n)_z^2(r,z,T)+\frac{\sin^2 H_n(r,z,T)}{r^2}\right)\mathrm{d}\,r\geq\int_{1/n}^{1}\frac{r}{2}\left((\theta_b)_r^2+\frac{\sin^2(\theta_b)}{r^2}\right)\mathrm{d}\,r\,.$$

Therefore from (3.43) follows that $\forall n\in\mathbb{N}, n\geq\zeta$

$$E_{n,\zeta}(H_n(\cdot,\,\cdot,\,T))+\iiint_{A_n\times[0,T]}r(H_n)_t^2\,\mathrm{d}\,r\,\mathrm{d}\,z\,\mathrm{d}\,t\leq E_{n,\zeta}(h_{0n})\leq$$

$$\leq\iint_{[0,1]\times[-\zeta,\zeta]}\frac{r}{2}\left((h_{0n})_r^2+(h_{0n})_z^2+\frac{\sin^2(h_{0n})}{r^2}\right)\mathrm{d}\,r\,\mathrm{d}\,z\leq 2C\zeta,$$

where $C=C(h_0)>0$ is the constant such that

$$\int_0^1\frac{r}{2}\left((h_{0n})_r^2+(h_{0n})_z^2+\frac{\sin^2(h_{0n})}{r^2}\right)\mathrm{d}\,r\leq C$$

for every $n\in\mathbb{N}$ and every fixed $z\in\mathbb{R}$. □

By parabolic Schauder type estimates (see [21]), we can say that, up to a subsequence, as $n\to\infty$

$$H_n\to h_M\quad\text{in } C^{2,1}([\rho,1]\times\mathbb{R}\times[\sigma,\tau])\cap C^0([\rho,1]\times\mathbb{R}\times[0,\tau])$$

for every $\rho\in(0,1), 0<\sigma<\tau$. Thanks to Proposition 3.26 and Fatou's Lemma, we can say that

$$(h_M)_t\in L^2_r((0,1)\times\mathbb{R}\times\mathbb{R}^+),$$

and

$$\frac{\sin h_M}{r},\ \nabla h_M\in L^\infty(\,\mathbb{R}^+;\ L^2_r((0,1)\times(-\zeta,\zeta))\,)$$

for every $\zeta>0$. Then h_M is a weak solution of Problem (3.1) and, by (3.40), (3.42), it satisfies

$$\theta_b(r)\leq h_M(r,z,t)\leq\theta_a(r),\quad h_M(r,-z,t)\equiv h_M(r,z,t),$$
$$h_M(r,z',t)\leq h_M(r,z,t)\quad\text{if } z'\geq z\geq 0. \tag{3.44}$$

Moreover, passing to the limit in (3.41) as $n\to\infty$, we obtain that $h_M\geq h_0$ and therefore, by Proposition 3.15, $h_M\geq h_m$.

By using Proposition 3.11 we can prove that

Theorem 3.27. *For all $t>0$ and $z\in[-M,M]$*

$$\lim_{r\to 0^+}h_M(r,z,t)=\pi\,.$$

Proof: Let $\mathscr{H}$ be the function defined by

$$\mathscr{H}(r,z,t) = \begin{cases} 2\arctan\left(\dfrac{e^{-\frac{Q}{t\gamma_M(z)}}}{r}\right) + br^{3/2} & r \in (0,1],\ t \in (0,T],\ |z| < M+1 \\ br^{3/2} & r \in (0,1],\ |z| \geq M+1 \text{ or } t = 0\,, \end{cases}$$

where γ_M belongs to the family of functions given in (3.16) and $Q > 0$ is a positive constant satisfying a condition that we shall specify later. Since $b \in (0,1)$, for every $n \in \mathbb{N}$, $n \geq M+1$ we have that

(*) $\mathscr{H}(r,z,0) = br^{3/2} \leq br \leq 2\arctan(br) \leq h_{0n}(r,z) \qquad \forall (r,z) \in (0,1] \times \mathbb{R}$,

(**) $\mathscr{H}(r,\pm n,t) = br^{3/2} \leq br \leq 2\arctan(br) \leq h_{0n}(r,\pm) \qquad \forall (r,t) \in (0,1] \times \mathbb{R}^+$,

(***) for every $t > 0$, if $|z| \geq M+1$ then

$$\mathscr{H}(1/n,z,t) = b(1/n)^{3/2} \leq b/n \leq 2\arctan(b/n) \leq h_{0n}(1/n,z),$$

else

$$\mathscr{H}(1/n,z,t) = 2\arctan\left(n\, e^{-\frac{Q}{t\gamma_M(z)}}\right) + bn^{-3/2} < 2\arctan(n) + 2\arctan(b/n) =$$

$$= 2\arctan\left(\frac{n + b/n}{1-b}\right) < 2\arctan\left(\frac{1+b}{1-b}\,n\right) = \omega_n(1/n,z) \leq h_{0n}(1/n,z)\,.$$

Therefore, $\mathscr{H}(1/n,z,t) \leq h_{0n}(1/n,z)\ \forall z \in \mathbb{R},\ t > 0$.

We remark that the identity

$$2\arctan(n) + 2\arctan(b/n) = 2\arctan\left(\frac{n+b/n}{1-b}\right)$$

follows from the elementary one

$$x + y = \arctan\left(\frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}\right) \qquad \forall x, y \geq 0 \text{ such that } x + y < \pi/2\,.$$

Let T be a positive value arbitrarily fixed. For every $t \in (0,T]$, we have that if $|z| \geq M+1$ then $\mathscr{H}(1,z,t) = b \leq 2\arctan(b) \leq h_{0n}(1,z)$, else

$$\mathscr{H}(1,z,t) = 2\arctan\left(e^{-\frac{Q}{t\gamma_M(z)}}\right) + b \leq 2\arctan\left(e^{-\frac{Q}{T}}\right) + b \leq 2\arctan(b) \leq h_{0n}(1,z) \tag{3.45}$$

provided that $Q \geq K$, where $K > 0$ is a suitable constant depending on b and T. If we take $Q \geq \max(\mathcal{Q}, K)$, where $\mathcal{Q}$ is the same constant as in Proposition 3.11, whose value only depends on b and T, then, due to (*), (**), (***), (3.45) and to Proposition 3.11, $\mathscr{H}$ is a subsolution of $(\mathscr{P}_n)$ in the time interval $[0,T)$ for every $n \in \mathbb{N}, n \geq M+1$. Hence follows that

$$\mathscr{H} \leq H_n \qquad \text{in } \bar{A}_n \times [0,T)$$

by the parabolic comparison principle, and, passing to the limit as $n \to \infty$, that

$$\mathscr{H}(r,z,t) \leq h_M(r,z,t) \qquad \forall r \in (0,1],\ z \in \mathbb{R} \text{ and } t \in [0,T).$$

Together with the inequality $h_M \leq \theta_a$ this implies

$$\lim_{r\to 0^+} h_M(r,z,t) = \pi$$

for every $z \in [-M, M]$ and $t \in (0,T)$. The thesis then follows from the arbitrariness of $T > 0$. $\square$

Theorem 3.28. *There exists a constant $S > 0$ such that*

$$\lim_{r\to 0^+} h_M(r,z,t) = 0$$

for every $t > 0$ and $|z| > S + M + t$.

Proof: Let $\psi = \psi(r,y)$ the same function as in proof of Theorem 3.17, (i), and let $Z > 0$ be a value such that

$$\pi + 2\arctan(a) \leq \psi(r,z) \qquad \forall r \in [0,1],\ z \leq -Z\,.$$

We remark that, due to properties of ψ, it must be $-Z \leq \bar{y}$, i.e. $\bar{y} + Z \geq 0$. Since $h_{0n} \leq \pi + 2\arctan(a)$ and $h_{0n}(r,z) = \theta_b(r)$ for $z \geq \max(\bar{z}, M+2)$, if we take

$$\sigma = \max(\bar{z}, M+2) + Z,$$

so that $\psi(r, \max(\bar{z}, M+2) - \sigma) = \psi(r, -Z) \geq \pi + 2\arctan(a)$, then

$$\psi(r, z-\sigma) \geq h_{0n}(r,z)$$

for all $n \in \mathbb{N}$, $n \geq 2$. If we define

$$h(r,z,t) = \psi(r, z - t - \sigma)\,,$$

then h solves equation (3.4) and for every $(r,z,t) \in (0,1] \times \mathbb{R} \times [0,\infty)$

$$h(r,z,t) \geq h(r,z,0) = \psi(r, z-\sigma) \geq h_{0n}(r,z)$$

for all $n \in \mathbb{N}$, $n \geq 2$. So, h is a supersolution of Problem $(\mathscr{P}_n)$ for every $n \geq 2$ and, by parabolic comparison principle, $h \geq H_n$. Consequently $h \geq h_M$ and, given any $t \geq 0$, one has that

$$0 \leq h_M(0,z,t) \leq \psi(0, z-t-\sigma) = 0 \qquad \text{if } z > \bar{y} + \sigma + t\,.$$

Therefore, taking into account that $h_M(r,-z,t) \equiv h_M(r,z,t)$, the thesis is verified by choosing

$$S = \bar{y} + \bar{z} + 2 + Z \quad \Rightarrow \quad S + M \geq \bar{y} + \sigma.$$

$\square$

Remark 3.29. Let $\mathcal{M}_2 > \mathcal{M}_1 \geq 0$. From the definition of ω_n follows that the initial and boundary data of Problem $(\mathscr{P}_n)$ when $M = \mathcal{M}_1$ is less than the data corresponding to the choice $M = \mathcal{M}_2$. By parabolic comparison principle we then obtain $h_{\mathcal{M}_1} \leq h_{\mathcal{M}_2}$.

Remark 3.30. Since for every $M \geq 0$ the function h_M is a weak solution of Problem (3.1) satisfying (3.44), by the same arguments used for h_m it is possible to prove that if we define

$$I_0^M(t) := \{z \in \mathbb{R} \mid h_M(0,z,t) = 0\} \text{ and } I_1^M(t) := \{z \in \mathbb{R} \mid h_M(0,z,t) = \pi\} \text{ for } t \geq 0,$$

then for every $t \geq 0$ there exists $\zeta_M(t) \in [0, \infty]$ such that

$$I_1^M(t) = (-\zeta_M(t), \zeta_M(t)), \quad I_0^M(t) = (-\infty, -\zeta_M(t)) \cup (\zeta_M(t), \infty),$$

up to the values $\pm\zeta_M(t)$. In terms of the function ζ_M the last two theorems can be restated by saying that

$$\exists\, S > 0 \text{ (not depending on } M) \text{ such that } M \leq \zeta_M(t) \leq S + M + t \qquad \forall t > 0\,.$$

3.5 Concluding remarks

The results we have proved in this chapter confirm the connection between nonuniqueness of axially symmetric solutions for the harmonic map heat flow and occurrence of point singularities in the solutions. We have shown that, by choosing as initial and boundary data a suitable smooth function h_0, identically equal to 0 on the x_3-axis, Problem (3.1) has a "minimal" solution h_m which is regular until a time $T > 0$. In addition there exist infinitely many weak solutions that, at any positive time t, attain the value π on segments of the x_3-axis which can be chosen arbitrarily large. If we argue in terms of vector fields rather than in terms of angle functions, i.e. returning from Problem (3.1) to its original formulation, Problem (5), we have found an axially symmetric director field u_0, smooth and identically equal to the north pole $\mathbf{N} = (0,0,1)$ on the vertical axis of the cylinder $\Omega = \{(x_1,x_2,x_3) : x_1^2 + x_2^2 < 1\} \subset \mathbb{R}^3$, for which Problem (5) does not possess a global classical solution. At the same time, for this special choice of u_0 Problem (5) has infinitely many weak solutions: a weak solution u_m, corresponding to the angle function h_m, which is smooth in a finite time interval $[0, T)$, and infinitely many weak solutions that, immediately after the initial time, are attaining the value $\mathbf{S} = (0,0,-1)$ on segments of the vertical axis of Ω which can be chosen arbitrarily large.

Roughly speaking we can speed up the natural development of singularities which can be observed in the vector field u_m, meaning that there is quite an amount of freedom to prescribe the actual position of the singular points along the vertical axis.

A similar remark can be found in [2] and in [27], where the same Problem (5) is studied when the spatial domain Ω is given by the unit ball in $\mathbb{R}^3$. In particular, if $u_0(x) = x/|x|$, then for every function $\zeta_0(t) : [0, \infty) \longrightarrow (-1, 1)$ there exists an axially symmetric solution of (5) which is regular in Ω except of the set $\{(x_1,x_2,x_3,t) = (0,0,\zeta_0(t),t), t \geq 0\}$. Thus, for every point $P = (0,0,z) \in \Omega \setminus \{0,0,0\}$ one can find a solution u of Problem (5) that instantaneously moves its singular point, at the initial time located in the origin, in the point P. This implies that it is possible both to expand and shrink the region of the vertical axis where the value $\mathbf{S}$ is attained.

With respect to this situation of "total" freedom to prescribe the position of the singularities, at first sight it could seem that in our context there is less degree of freedom. Indeed, the minimality of h_m and the finiteness of the blow-up time T mean that $h(0,z,t)$

cannot keep the value 0 for all z and t. We should not forget however that also the value 2π for h corresponds to $\mathbf{N}$ and we could imagine that for almost every $t > 0$ it is possible to prescribe the values $u = \mathbf{N}$ and $u = \mathbf{S}$ in almost every point of the vertical axis. This problem is completely open and the results obtained in this chapter only indicate that the use of traveling wave solutions as barrier functions may be useful to shed some light on this question.

Chapter 4

Nonuniqueness of the wave speed

In this final chapter we reconsider the traveling wave problem

$$(\mathrm{I}_{c,R}) \qquad \begin{cases} \theta_{rr} + \dfrac{1}{r}\theta_r + \theta_{zz} + c\theta_z - \dfrac{\sin(2\theta)}{2r^2} = 0 & \text{in } (0,R)\times\mathbb{R} \\ \theta(R,z) = 2\arctan(bR) & \text{for } z \in \mathbb{R} \\ \theta(r,\pm\infty) = \theta_\pm(r) & \text{for } 0 < r < R \end{cases}$$

where $b > 0$, $R > 0$ and $bR > 1$. In Chapter 1 we have used a variational technique to show that for a certain wave speed $c_R > 0$ Problem $(\mathrm{I}_{c_R,R})$ has a solution θ_R with a singular point at $(0,0)$. In the present chapter we use an entirely different technique to show that Problem $(\mathrm{I}_{c,R})$ has a solution for any $c \in \mathbb{R}$. For this purpose, if $c \neq c_R$ is a prescribed value and θ_R is the solution of $(\mathrm{I}_{c_R,R})$ defined by Theorem 1.1, we consider the initial-boundary value problem

$$\begin{cases} \vartheta_t = \vartheta_{zz} + c\vartheta_z + \vartheta_{rr} + \frac{\vartheta_r}{r} - \frac{\sin(2\vartheta)}{2r^2} & \text{in } (0,R)\times\mathbb{R}\times\mathbb{R}^+ \\ \vartheta(r,z,0) = \theta_R(r,z) & \text{in } [0,R]\times\mathbb{R} \\ \vartheta(R,z,t) = \theta_+(R) & \forall z\in\mathbb{R}, t>0\,. \end{cases} \tag{4.1}$$

Since the initial function θ_R solves the equation

$$\theta_{zz} + c_R\theta_z + \theta_{rr} + \frac{\theta_r}{r} - \frac{\sin(2\theta)}{2r^2} = 0\,,$$

a "trivial" solution of Problem 4.1 is given by

$$\vartheta(r,z,t) = \theta_R(r, z-(c_R-c)t)\,.$$

Obviously this solution satisfies

$$\forall t>0 \qquad \vartheta(0,z,t) = \begin{cases} 0 & \text{if } z > (c_R-c)t \\ \pi & \text{if } z < (c_R-c)t, \end{cases}$$

i.e. ϑ has, at time t, a singularity at the point $(0,(c_R-c)t)$. But the nonuniqueness results in [2] and [27] suggest there may be different solutions of the same initial-boundary value problem 4.1 corresponding to different evolutions of the singular point. In this chapter we shall prove that we may keep the singular point fixed at the origin:

Theorem 4.1. *Let $b > 0$, $R > 0$ and $bR > 1$. Let θ_R be the solution of problem $(\mathrm{I}_{c_R,R})$ defined by Theorem 1.1, and let $c \neq c_R$. Then Problem 4.1 has a solution $\vartheta_c = \vartheta_c(r,z,t) \in C^\infty((0,R] \times \mathbb{R} \times [0,\infty))$ such that*

1. *$\vartheta_c(\cdot,\cdot,t) \in C^0([0,R] \times \mathbb{R} \setminus \{(0,0)\})$ uniformly with respect to $t \geq 0$,*
2. *$\forall t > 0$*
$$\vartheta(0,z,t) = \begin{cases} 0 & \text{if } z > 0 \\ \pi & \text{if } z < 0, \end{cases} \tag{4.2}$$
3. $$\lim_{z\to+\infty} \vartheta_c(r,z,t) = \theta_+(r), \qquad \lim_{z\to-\infty} \vartheta_c(r,z,t) = \theta_-(r)$$
uniformly with respect to $r \in [0,R]$ and $t \geq 0$,
4. *ϑ_c is decreasing with respect to z,*
5. *ϑ_c is monotone with respect to t, decreasing if $c > c_R$, increasing if $c < c_R$,*
6. *$\vartheta_c(r,z,t)$ is decreasing with respect to c for $(r,z,t) \in (0,R] \times \mathbb{R} \times [0,\infty)$.*

By Theorem 4.1, point 5, we may define

$$\theta_c(r,z) := \lim_{t\to\infty} \vartheta_c(r,z,t) \qquad \text{if } (r,z) \in [0,R] \times \mathbb{R} \setminus \{(0,0)\}\,. \tag{4.3}$$

We shall prove that θ_c is actually a solution of Problem $(\mathrm{I}_{c,R})$:

Theorem 4.2. *Let the hypotheses of Theorem 4.1 be satisfied, and let θ_c be defined by (4.3). Then θ_c is a solution of Problem $(\mathrm{I}_{c,R})$ which satisfies:*

1. *$\theta_c \in C^\infty([0,R] \times \mathbb{R} \setminus \{(0,0)\})$,*
2. $$\theta_c(0,z) \;=\; \begin{cases} 0 & \text{if } z > 0 \\ \pi & \text{if } z < 0\,, \end{cases}$$
3. *$(\theta_c)_z < 0$ in $(0,R) \times \mathbb{R}$,*
4. *$\theta_c(r,z)$ is strictly decreasing with respect to c for $(r,z) \in (0,R) \times \mathbb{R}$, and*
5. *$\theta_c(r,z) \to \theta_\pm(r)$ as $z \to \pm\infty$ uniformly with respect to $r \in [0,R]$.*

As we shall see in section 4.1, Theorem 4.1 is based on a rather straightforward construction of barrier functions, and in this sense it supplies a relatively simple example of nonuniqueness for the flow of director fields:

Corollary 4.3. *Let u_0 be the director field defined by*

$$u_0(x_1,x_2,x_3) = \Big(\frac{x_1}{r}\sin\theta_R(r,x_3), \frac{x_2}{r}\sin\theta_R(r,x_3), \cos\theta_R(r,x_3)\Big) \qquad \left(r = \sqrt{x_1^2+x_2^2}\right), \tag{4.4}$$

for $(x_1,x_2,x_3) \in \Omega = \{(x_1,x_2,x_3) : x_1^2+x_2^2 < R^2\} \subset \mathbb{R}^3$ *and consider the initial-boundary value problem*

$$\begin{cases} u_t - \Delta u \;=\; |\nabla u|^2 u & \text{in } \Omega \times \mathbb{R} \\ u(x,0) \;=\; u_0(x) & \text{in } \Omega \\ u(x,t) \;=\; u_0(x) & \text{in } \partial\Omega \times \mathbb{R}^+ . \end{cases} \tag{4.5}$$

Then (4.5) has infinitely many solutions, defined by

$$u_c(x_1,x_2,x_3,t) = \begin{cases} \Big(\frac{x_1}{r}\sin\theta_R(r,x_3-c_Rt), \frac{x_2}{r}\sin\theta_R(r,x_3-c_Rt), \cos\theta_R(r,x_3-c_Rt)\Big) & \text{if } c = c_R \\ \\ \Big(\frac{x_1}{r}\sin\vartheta_c(r,x_3-ct,t), \frac{x_2}{r}\sin\vartheta_c(r,x_3-ct,t), \cos\vartheta_c(r,x_3-ct,t)\Big) & \text{if } c \neq c_R . \end{cases}$$

In addition, u_c *converges to a traveling wave of speed* c *as* $t \to \infty$, *in the sense that for all* $(x_1,x_2,x_3) \neq (0,0,0)$

$$u_c(x_1,x_2,x_3+ct,t) \longrightarrow \Big(\frac{x_1}{r}\sin\theta_c(r,x_3), \frac{x_2}{r}\sin\theta_c(r,x_3), \cos\theta_c(r,x_3)\Big)$$

as $t \to \infty$.

We shall conclude this chapter with a discussion of these results. In particular, in the last section we shall formulate and explain a conjecture about the local behavior of θ_c near the point singularity for $c \neq c_R$ and we shall discuss its possible consequences concerning the nonuniqueness of the flow of director fields.

4.1 Proof of Theorems 4.1 and 4.2

Proof of Theorem 4.1:

For every $\rho \in (0,R)$, we consider the problem

$$(P_{c,\rho}) \begin{cases} \vartheta_t = \vartheta_{zz} + c\vartheta_z + \vartheta_{rr} + \frac{\vartheta_r}{r} - \frac{\sin(2\vartheta)}{2r^2} & \text{in } (\rho,R)\times\mathbb{R}\times\mathbb{R}^+ \\ \vartheta(r,z,0) = \theta_R(r,z) & \text{in } [\rho,R]\times\mathbb{R} \\ \vartheta(R,z,t) = \theta_+(R) & \text{for } z\in\mathbb{R}, t>0 \\ \vartheta(\rho,z,t) = \theta_R(\rho,z) & \text{for } z\in\mathbb{R}, t>0 . \end{cases}$$

Since $\theta_R(R,z) = \theta_+(R)$, it is obvious that this problem has a unique classical solution $\vartheta_{c,\rho} \in C^\infty((\rho,R]\times\mathbb{R}\times[0,\infty)) \cap C^0([\rho,R]\times\mathbb{R}\times[0,\infty))$. It is easy to check that

(a1) θ_+ and θ_- are, respectively, a sub- and a supersolution to $(P_{c,\rho})$.

(a2) Since $(\theta_R)_z < 0$ in $(0,R)\times\mathbb{R}$, θ_R is a supersolution of $(P_{c,\rho})$ if $c > c_R$ and a subsolution if $c < c_R$.

(a3) The inequality $(\theta_R)_z < 0$ in $(0,R)\times\mathbb{R}$ also implies that $\frac{\partial\vartheta_{c,\rho}}{\partial z}$ is a subsolution of the differential problem

$$\begin{cases} \psi_t = \psi_{zz} + c\psi_z + \psi_{rr} + \frac{\psi_r}{r} - \frac{\cos(2\vartheta_{c,\rho})}{r^2}\psi & \text{in } (\rho,R)\times\mathbb{R}\times\mathbb{R}^+ \\ \psi(r,z,0) = 0 & \text{for } r\in(0,R), z\in\mathbb{R} \\ \psi(\rho,z,t) = \psi(R,z,t) = 0 & \text{for } z\in\mathbb{R}, t>0 \end{cases} \tag{4.6}$$

and therefore $\frac{\partial\vartheta_{c,\rho}}{\partial z} \le 0$.

(a4) If $c > c_R$, then $\vartheta_{c,\rho}(\cdot,\cdot,t) \le \theta_R$ for all $t \ge 0$ and therefore $\frac{\partial \vartheta_{c,\rho}}{\partial t}$ is a subsolution of (4.6). Hence, $\frac{\partial \vartheta_{c,\rho}}{\partial t} \le 0$ if $c > c_R$. Analogously, $\frac{\partial \vartheta_{c,\rho}}{\partial t} \ge 0$ if $c < c_R$.

(a5) Let $0 < \rho_1 < \rho_2 < R$. If $c > c_R$ then, in view of (a2), ϑ_{c,ρ_1} is a subsolution of (P_{c,ρ_2}) and therefore $\vartheta_{c,\rho_1} \le \vartheta_{c,\rho_2}$. Analogously, $\vartheta_{c,\rho_1} \ge \vartheta_{c,\rho_2}$ if $c < c_R$.

(a6) In view of (a3), given $c_1, c_2 \in \mathbb{R} \setminus \{c_R\}$, if $c_2 > c_1$, then for every $\rho \in (0,R)$ $\vartheta_{c_1,\rho}$ is a supersolution of $(P_{c_2,\rho})$ and therefore $\vartheta_{c_1,\rho} \ge \vartheta_{c_2,\rho}$.

Assertions (a1) and (a5) imply that for every $(r,z,t) \in (0,R] \times \mathbb{R} \times [0,\infty)$ there exists

$$\vartheta_c(r,z,t) := \lim_{\rho \to 0+} \vartheta_{c,\rho}(r,z,t) \in [\theta_+(r), \theta_-(r)]\,. \tag{4.7}$$

We shall prove Theorem 4.1 by showing that this function has all the desired properties. Assertions (a2) and (a6) directly imply point 6 of Theorem 4.1. By parabolic Schauder-type estimates, for every $\delta \in (0,R)$ there exists $C = C(\delta,c) > 0$ such that for every $\rho \in (0,\delta/2)$

$$\|\vartheta_{c,\rho}\|_{C^{3,2}([\delta,R]\times\mathbb{R}\times[0,\infty))} \le C\,.$$

Hence ϑ_c is smooth in $(0,R]\times\mathbb{R}\times[0,\infty)$ and solves the differential equation of problem 4.1. Moreover, we have trivially $(\vartheta_c)_z \le 0$, $(\vartheta_c)_t \le 0$ if $c > c_R$ and $(\vartheta_c)_t \ge 0$ if $c < c_R$, $\vartheta_c(r,z,0) = \theta_R(r,z)$ for all $(r,z) \in (0,R] \times \mathbb{R}$, $\vartheta_c(R,z,t) = \theta_+(R)$ for all $z \in \mathbb{R}$ and $t > 0$, and we have obtained points 4 and 5 of Theorem 4.1.

We claim that for all $t > 0$

$$\lim_{r \to 0^+} \vartheta_c(r,z,t) \; = \begin{cases} 0 & \text{if } z > 0 \\ \pi & \text{if } z < 0\,. \end{cases} \tag{4.8}$$

The proof is based on the construction of appropriate barrier functions:

Lemma 4.4. *(i) If $c > c_R$, for every $\varepsilon > 0$ there exists a smooth function*

$$\sigma_\varepsilon : [0,R] \times \mathbb{R} \setminus \{(0,-\varepsilon)\} \longrightarrow \mathbb{R}$$

such that:

(p1) σ_ε is a subsolution of problem $(P_{c,\rho})$ for every $\rho \in (0,R)$,

(p2) $\sigma_\varepsilon(0,z) = \pi$ for every $z < -\varepsilon$.

(ii) If $c < c_R$, for every $\varepsilon > 0$ there exists a smooth function

$$\Sigma_\varepsilon : [0,R] \times \mathbb{R} \setminus \{(0,\varepsilon)\} \longrightarrow \mathbb{R}$$

such that:

(p3) Σ_ε is a supersolution of problem $(P_{c,\rho})$ for every $\rho \in (0,R)$,

(p4) $\Sigma_\varepsilon(0,z) = 0$ for every $z > \varepsilon$.

We postpone the proof of this key result to section 4.2 and complete the proof of Theorem 4.1.

If $c > c_R$, we obtain from Lemma 4.4 the inequalities

$$\sigma_\varepsilon(r,z) \le \vartheta_{c,\rho}(r,z,t) \le \theta_R(r,z)$$

which are satisfied for every $\rho \in (0,R)$ and every $(r,z,t) \in [\rho,R] \times \mathbb{R} \times [0,\infty)$. Passing to the limit $\rho \to 0$ we obtain that

$$\sigma_\varepsilon(r,z) \le \vartheta_c(r,z,t) \le \theta_R(r,z) \tag{4.9}$$

and hence Claim 4.8 follows from property (p2), Theorem 1.1, (ii) and the arbitrariness of $\varepsilon > 0$. Analogously, if $c < c_R$, Claim 4.8 follows from Lemma 4.4, (ii) with (4.9) replaced by

$$\theta_R(r,z) \le \vartheta_c(r,z,t) \le \Sigma_\varepsilon(r,z) \tag{4.10}$$

The monotonicity of ϑ_c with respect to t and the definition of θ_c imply the inequalities

$$\begin{array}{ll} \theta_c(r,z) \le \vartheta_c(r,z,t) \le \theta_R(r,z) & \text{if } c > c_R \\ \theta_R(r,z) \le \vartheta_c(r,z,t) \le \theta_c(r,z) & \text{if } c < c_R, \end{array}$$

Assuming that θ_c satisfies Theorem 4.2, as we shall prove below, we obtain points 1 and 3 of Theorem 4.1. This completes the proof of Theorem 4.1. □

Proof of Theorem 4.2: By (4.3) and parabolic Schauder-type estimates, the function θ_c is smooth out of $\{r = 0\}$, solves

$$\theta_{rr} + \frac{\theta_r}{r} + \theta_{zz} + c\theta_z - \frac{\sin(2\theta)}{2r^2} = 0 \qquad \text{in } (0,R) \times \mathbb{R}$$

and satisfies the condition $\theta(R,z) = \theta_+(R)$. It follows at once from (4.3), (4.9), (4.10) and Lemma 4.4 that

$$\lim_{r \to 0^+} \theta_c(r,z) = \begin{cases} 0 & \text{if } z > 0 \\ \pi & \text{if } z < 0\,. \end{cases} \tag{4.11}$$

Obviously θ_c satisfies $(\theta_c)_z \le 0$ and $\theta_+(r) \le \theta_c(r,z) \le \theta_-(r)$, and it follows from (4.11) that $\theta_c \in C^0([0,R] \times \mathbb{R} \setminus \{(0,0)\})$. By the strong maximum principle, $(\theta_c)_z < 0$ in $(0,R) \times \mathbb{R}$. Since $\theta_+(r) \le \vartheta_c(r,z,t) \le \theta_R(r,z)$ if $c > c_R$, and $\theta_R(r,z) \le \vartheta_c(r,z,t) \le \theta_-(r)$ if $c < c_R$, it follows from the strong maximum principle that

$$\begin{array}{ll} \theta_+ < \theta_c < \theta_R & \text{if } c > c_R, \\ \theta_R < \theta_c < \theta_- & \text{if } c < c_R \end{array}$$

in the open set $(0,R) \times \mathbb{R}$. Using the same argument as in the proof of Proposition 2.12, the continuity of θ_c in $[0,R] \times \mathbb{R} \setminus \{(0,0)\}$ implies that θ_c is C^∞ in this set.

Since θ_c is strictly decreasing with respect to z and bounded, there exist

$$\theta_{c,+}(r) := \lim_{z \to +\infty} \theta_c(r,z), \quad \theta_{c,-}(r) := \lim_{z \to -\infty} \theta_c(r,z).$$

Standard Schauder type estimates imply that $\theta_{c,+}, \theta_{c,-} \in C^2((0,R])$ and solve the problem

$$\begin{cases} \psi_{rr} + \frac{\psi_r}{r} - \frac{\sin(2\psi)}{2r^2} = 0 & \text{in } (0,R) \\ \psi(R) = \theta_+(R)\,. \end{cases} \tag{4.12}$$

Because of the bounds $\theta_-(r) \leq \theta_c(r,\cdot) \leq \theta_+(r)$ for $r \in [0,R]$, $\theta_{c,+}, \theta_{c,-} \in C^0([0,R])$ and

$$\theta_{c,+}(0) = 0,\ \theta_{c,-}(0) = \pi\,.$$

A straightforward computation implies that $\theta_{c,+} \equiv \theta_+$ and $\theta_{c,-} \equiv \theta_-$. We remark that by Schauder estimates, monotonicity in z and the bounds $\theta_-(r) \leq \theta_c(r,\cdot) \leq \theta_+(r)$, the convergence of θ_c to $\theta_\pm$ when $z \to \pm\infty$ is uniform with respect to $r \in [0,R]$. Finally, by point 6 of Theorem 4.1, $\theta_c(r,z)$ is decreasing with respect to c for $(r,z) \in (0,R] \times \mathbb{R}$. Since $(\theta_c)_z < 0$ in $(0,R) \times \mathbb{R}$ for every $c \in \mathbb{R}$, it follows from the strong maximum principle that, given $c_1, c_2 \in \mathbb{R}$, $\theta_{c_2} < \theta_{c_1}$ in $(0,R) \times \mathbb{R}$ if $c_2 > c_1$. We conclude that θ_c is a solution of problem $(I_{c,R})$ satisfing properties 1-4 of Theorem 4.2. $\square$

4.2 Barrier functions

In this section we prove Lemma 4.4. We need the following trivial result.

Lemma 4.5. *Let $B > 0, \delta \in [0,1]$. If*

$$q(r) := \frac{2B^{1-\delta} r^{1+\delta}}{r^2 + B^2} \quad \textit{for } r > 0,$$

then for every $r \in \mathbb{R}^+$

$$0 \leq q(r) \leq \begin{cases} (1-\delta)^{\frac{1-\delta}{2}} (1+\delta)^{\frac{1+\delta}{2}} & \textit{if } \delta \in [0,1) \\ 2 & \textit{if } \delta = 1\,. \end{cases}$$

For every $C, D > 0$ we define the functions

$$B_C(z) = \begin{cases} C\,\mathrm{e}^{1/z} & \text{if } z < 0 \\ 0 & \text{if } z \geq 0\,, \end{cases}$$

$$A_D(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ D\,\mathrm{e}^{-1/z} & \text{if } z > 0\,. \end{cases}$$

It is well known that $B_C, A_D \in C^\infty(\mathbb{R})$.

In the following we consider c as prescribed.

Lemma 4.6. *For every $\mu \in (0,1)$ there exists $\bar{C} = \bar{C}(\mu) \in (0,R]$ such that for every $C \in (0,\bar{C}]$ the function*

$$\sigma_C(r,z) = 2\arctan\left(\frac{B_C(z)}{r}\right) + 2\arctan\left(\mu\left(\frac{r}{R}\right)^{3/2}\right), \quad (r,z) \in [0,R] \times \mathbb{R}\,,$$

satisfies the differential inequality

$$\mathscr{L}(\sigma) := \sigma_{zz} + c\sigma_z + \sigma_{rr} + \frac{\sigma_r}{r} - \frac{\sin(2\sigma)}{2r^2} \geq 0$$

in the open set $(0,R) \times \mathbb{R}$.

Proof: Let $\mu \in (0,1)$ arbitrarily fixed. For sake of simplicity, we shall denote σ_C by σ. In the open set $(0,R)\times(0,\infty)$ $\sigma(r,z)\equiv\gamma(r)$, where $\gamma(r):=2\arctan(\mu(r/R)^{3/2})$, and

$$\mathscr{L}(\sigma)(r,z)=\gamma''(r)+\frac{\gamma'(r)}{r}-\frac{\sin(2\gamma)}{2r^2}=\frac{5}{4}\frac{\sin(2\gamma)}{2r^2}\geq 0 \tag{4.13}$$

because $\gamma(r)\leq 2\arctan(\mu)\in(0,\pi/2)$. Since $\mathscr{L}(\sigma)$ is continuous in $(0,R)\times\mathbb{R}$, (4.13) holds up to $z=0, r\in(0,R)$.

A straightforward computation shows that in the open set $(0,R)\times(-\infty,0)$

$$\mathscr{L}(\sigma)=\phi'(z)\sin(f)+\phi^2(z)\frac{\sin(2f)}{2}+c\phi(z)\sin(f)+\frac{\sin(2f)+9/4\sin(2\gamma)-\sin(2f+2\gamma)}{2r^2}$$

where $\phi(z)\equiv\frac{B'_C(z)}{B_C(z)}\equiv-(1/z)^2$ and $f=f(r,z)=2\arctan(\frac{B_C(z)}{r})$.
If we take $C\in(0,R]$, we have

$$0<B_C(z)\leq C,\quad 0<\mu<R/C$$

and then

$$\sigma(r,z)=f(r,z)+\gamma(r)\leq\pi-2\arctan\left(\frac{r}{B_C(z)}\right)+2\arctan\left(\mu\frac{r}{R}\right)\leq$$

$$\leq\pi-2\arctan\left(\frac{r}{B_C(z)}\right)+2\arctan\left(\frac{r}{C}\right)\leq\pi\,.$$

Of course also $0\leq\gamma\leq\sigma$. Therefore, by using standard trigonometric identities we find

$$\sin(2f)+9/4\sin(2\gamma)-\sin(2f+2\gamma)=9/4\sin(2\gamma)-2\cos(2f+\gamma)\sin(\gamma)=$$

$$=2\sin(\gamma)\big(5/4\cos(\gamma)+2\sin(\sigma)\sin(f)\big)\geq 5/4\sin(2\gamma)\Rightarrow$$

$$\mathscr{L}(\sigma)\geq\sin(f)(\phi'+\phi^2\cos(f)+c\phi)+\frac{5}{4}\frac{\sin(2\gamma)}{2r^2}\geq\sin(f)(\phi'-\phi^2+|c|\phi)+\frac{5}{4}\frac{\sin(2\gamma)}{2r^2} \tag{4.14}$$

in $(0,R)\times(-\infty,0)$. Since $\mu\in(0,1)$ we have

$$\cos(\gamma)=\frac{1-\mu^2(r/R)^3}{1+\mu^2(r/R)^3}\geq\frac{1-\mu^2}{1+\mu^2}>0$$

and, at the same time,

$$\sin(\gamma)=\frac{2\mu(r/R)^{3/2}}{1+\mu^2(r/R)^3}\geq\frac{2\mu}{1+\mu^2}\left(\frac{r}{R}\right)^{3/2}.$$

Since $\phi(z)=-(1/z)^2$, it turns out that $\phi'-\phi^2+|c|\phi<0$. Finally, substituting $\delta=1/2$ in Lemma 4.5,

$$\sin(f(r,z))=\frac{2B_C(z)r}{r^2+B_C(z)^2}\leq\frac{\sqrt[4]{27}}{2}\sqrt{\frac{B_C(z)}{r}}\leq\frac{5}{4}\sqrt{\frac{B_C(z)}{r}}\,.$$

It follows from (4.14) that in $(0,R)\times(-\infty,0)$

$$\mathscr{L}(\sigma) \geq \frac{5r^{-1/2}}{4}\left((\phi' - \phi^2 + |c|\phi)\sqrt{B_C} + \frac{2\mu(1-\mu^2)}{(1+\mu^2)^2}R^{-3/2}\right).$$

Since there exists $Q = Q(c) > 0$ such that

$$(\phi' - \phi^2 + |c|\phi)\sqrt{B_C} \geq -Q\sqrt{C}$$

we obtain that in the open set $(0,R)\times(-\infty,0)$

$$\mathscr{L}(\sigma) \geq \frac{5r^{-1/2}}{4}\left(-Q\sqrt{C} + \frac{2\mu(1-\mu^2)}{(1+\mu^2)^2}R^{-3/2}\right).$$

Therefore, there exists $\bar{C} = \bar{C}(\mu, c, R) \in (0, R]$ such that by choosing $C \in (0, \bar{C}]$ $\mathscr{L}(\sigma) \geq 0$ in $(0,R)\times\mathbb{R}$. □

Remark 4.7. As shown by the proof of Lemma 4.6, $\bar{C}$ depends also on c and R. However, since these are given constants, we have made explicit only the dependence on μ.

We omit the proof of the next result, which is formally identical to the previous one.

Lemma 4.8. *For every $\mu > 1$ there exists $\bar{D} = \bar{D}(\mu) \in (0, R]$ such that for every $D \in (0, \bar{D}]$ the function*

$$\Sigma_D(r,z) = 2\arctan\left(\mu\left(\frac{R}{r}\right)^{3/2}\right) - 2\arctan\left(\frac{A_D(z)}{r}\right), \quad (r,z) \in [0,R]\times\mathbb{R},$$

satisfies the differential inequality

$$\mathscr{L}(\Sigma) := \Sigma_{zz} + c\Sigma_z + \Sigma_{rr} + \frac{\Sigma_r}{r} - \frac{\sin(2\Sigma)}{2r^2} \leq 0$$

in the open set $(0,R)\times\mathbb{R}$.

We now assume $c > c_R$ to show how to construct the family of functions σ_ε for $\varepsilon > 0$. From the properties of θ_R (see Theorem 1.1) we know that for every $\varepsilon > 0$ there exist $\hat{C}(\epsilon) > 0$ and $\hat{\mu}(\epsilon) \in (0,1)$ such that for every $C \in (0, \hat{C}(\varepsilon)]$ and $\mu \in (0, \hat{\mu}(\varepsilon)]$

$$2\arctan(C/r) + 2\arctan(\mu(r/R)^{3/2}) \leq \theta_R(r, -\varepsilon), \quad \forall r \in [0,R].$$

Given $\varepsilon > 0$, if we take $\mu = \hat{\mu}(\varepsilon)$, $C \in (0, \min\{\bar{C}(\mu), \hat{C}(\varepsilon)\}]$ and define $\sigma_\varepsilon(r,z) = \sigma_C(r, z+\varepsilon)$, then we have

1. σ_ε is a subsolution of

$$\vartheta_t = \vartheta_{zz} + c\vartheta_z + \vartheta_{rr} + \frac{\vartheta_r}{r} - \frac{\sin(2\vartheta)}{2r^2},$$

2. for every $z < -\varepsilon, r \in [0,R]$

$$\sigma_\varepsilon(r,z) \leq 2\arctan(C/r) + 2\arctan(\mu(r/R)^{3/2}) \leq \theta_R(r,z),$$

since θ_R is decreasing with respect to z, and

3. for every $z \geq -\varepsilon, r \in [0, R]$

$$\sigma_\varepsilon(r, z) = 2\arctan(\mu(r/R)^{3/2}) \leq 2\arctan(\mu(r/R)) \leq \theta_+(r) \leq \theta_R(r, z)$$

since $\mu < 1 < bR$.

Since $\sigma_C(0, z) = 0$ for $z > 0$, $\sigma_C(0, z) = \pi$ for $z < 0$, we conclude that σ_ε satisfies property (p1) and (p2).

Analogously, if $c < c_R$, we construct the family $\{\Sigma_\varepsilon\}_{\varepsilon>0}$. From the properties of θ_R (see Theorem 1.1) we know that for every $\varepsilon > 0$ there exist $\tilde{D}(\epsilon) > 0$ and $\tilde{\mu}(\epsilon) > 1$ such that for every $D \in (0, \tilde{D}(\varepsilon)]$ and $\mu \geq \tilde{\mu}(\varepsilon)$

$$2\arctan(\mu(R/r)^{3/2}) - 2\arctan(D/r) \geq \theta_R(r, \varepsilon), \quad \forall r \in [0, R].$$

Given $\varepsilon > 0$, if we take $\mu = \tilde{\mu}(\varepsilon)$, $D \in (0, \min\{\bar{D}(\mu), \tilde{D}(\varepsilon)\}]$ and define $\Sigma_\varepsilon(r, z) = \Sigma_D(r, z - \varepsilon)$, then Σ_ε satisfies (p3) and (p4).

4.3 A conjecture

In Chapter 1 we have shown that the "variational" solution θ_R behaves near its singular point $(0, 0)$ as

$$\frac{\pi}{2} - \arctan\left(\frac{z}{r}\right).$$

Equivalently, the corresponding traveling wave solution for the director field, u_{c_R}, behaves near its singular point $x_R(t) := (0, 0, c_R t)$ as

$$\frac{x - x_R(t)}{|x - x_R(t)|},$$

which, for each fixed $t > 0$, is a harmonic map from $\mathbb{R}^3$ to $\mathbb{S}^2$. A straightforward computation shows that this harmonic map is an element of a 1-parameter family of axially symmetric harmonic maps with a given singular point at the vertical axis. To fix the ideas, if this singular point is the origin, the corresponding angle function of the harmonic map is of the form

$$\theta(r, z) = 2\arctan\left(A\tan\left(\frac{\pi}{4} - \frac{\arctan(z/r)}{2}\right)\right),$$

where $A \in \mathbb{R}^+$ represents the parameter. Observe that, if $A = 1$, we obtain the local behavior of the variational solution θ_R.

Conjecture. Let $c \in \mathbb{R}$ and let θ_c be the solution defined by Theorem 4.2. Then there exists a continuous and strictly decreasing map from $\mathbb{R}$ to $\mathbb{R}^+$, $c \mapsto A_c$, such that θ_c behaves near the origin as

$$2\arctan\left(A_c\tan\left(\frac{\pi}{4} - \frac{\arctan(z/r)}{2}\right)\right),$$

$A_{c_R} = 1$, and

$$A_c \to \begin{cases} 0 & \text{as } c \to +\infty \\ +\infty & \text{as } c \to -\infty. \end{cases}$$

In particular, the conjecture implies that

$$A_c \begin{cases} < 1 & \text{if } c > c_R \\ > 1 & \text{if } c < c_R\,. \end{cases} \tag{4.15}$$

In order to understand the basis for our conjecture, below we give a heuristic explanation for the inequalities (4.15). For this purpose, we introduce the functions

$$f_R(x,\varphi) = \log\left(\frac{\tan(\theta_R(\mathrm{e}^x\cos\varphi, \mathrm{e}^x\sin\varphi)/2)}{\tan(\pi/4-\varphi/2)}\right) \quad x \in (-\infty, \log(R)],\ \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

and

$$f_c(x,\varphi,t) = \log\left(\frac{\tan(\vartheta_c(\mathrm{e}^x\cos\varphi, \mathrm{e}^x\sin\varphi, t)/2)}{\tan(\pi/4-\varphi/2)}\right) \quad x \in (-\infty, \log(R)], \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), t \geq 0$$

where ϑ_c is the solution defined by Theorem 4.1. We emphasize that $f_c \leq f_R$ if $c > c_R$, $f_c \geq f_R$ if $c < c_R$, and, by Theorem 1.3, $f_R(x,\varphi) \to 0$ as $x \to -\infty$ loc. uniformly in $[-\pi/2, \pi/2]$. Moreover, we know from Theorem 4.1, point (5) that there exists $f_{c,\infty} := \lim_{t\to\infty} f_c(\cdot,\cdot,t)$ and

$$f_{c,\infty}(x,\varphi) = \log\left(\frac{\tan(\theta_c(\mathrm{e}^x\cos\varphi, \mathrm{e}^x\sin\varphi)/2)}{\tan(\pi/4-\varphi/2)}\right) \quad \text{for } x \in (-\infty, \log(R)],\ \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\,. \tag{4.16}$$

By the given definitions,

$$\mathscr{G}_c(x,\varphi,t) := \vartheta_c(\mathrm{e}^x\cos\varphi, \mathrm{e}^x\sin\varphi, t) \equiv 2\arctan(\mathrm{e}^{f_c(x,\varphi,t)}\tan(\pi/4-\varphi/2)),$$

$$\mathscr{G}_R(x,\varphi) := \theta_R(\mathrm{e}^x\cos\varphi, \mathrm{e}^x\sin\varphi) \equiv 2\arctan(\mathrm{e}^{f_R(x,\varphi)}\tan(\pi/4-\varphi/2)),$$

and f_c, f_R respectively solve the differential equation

$$\mathrm{e}^{2x} f_t = f_{xx} + f_x + \frac{(\cos^3\varphi\, f_\varphi)_\varphi}{\cos^3\varphi} + \gamma_c f_\varphi + \cos(\mathscr{G}_c)|\nabla f|^2 + c\,\mathrm{e}^x(f_x\sin\varphi + f_\varphi\cos\varphi - 1) \tag{4.17}$$

and

$$f_{xx} + f_x + \frac{(\cos^3\varphi\, f_\varphi)_\varphi}{\cos^3\varphi} + \gamma_R f_\varphi + \cos(\mathscr{G}_R)|\nabla f|^2 + c_R\,\mathrm{e}^x(f_x\sin\varphi + f_\varphi\cos\varphi - 1) = 0 \tag{4.18}$$

where

$$\gamma_c := 2\frac{\sin\varphi - \cos\mathscr{G}_c}{\cos\varphi}\,, \quad \gamma_R := 2\frac{\sin\varphi - \cos\mathscr{G}_R}{\cos\varphi}\,.$$

Since $\theta_R \in C^\infty([0,R]\times\mathbb{R}\setminus\{(0,0)\})$ and, as one can easily show, $\vartheta_c \in C^\infty([0,R]\times\mathbb{R}\times\mathbb{R}^+\setminus\{(0,0)\}\times\mathbb{R}^+)$, it follows from Taylor expansion that f_c, f_R can be extended up to $\varphi = \pm\pi/2$ and

$$\frac{\partial f_c}{\partial\varphi}(x,\pm\pi/2,t) = 0, \quad \frac{\partial f_R}{\partial\varphi}(x,\pm\pi/2) = 0 \quad \text{for } x \in (-\infty, \log(R)],\ t \geq 0. \tag{4.19}$$

Therefore, given an arbitrary $M \in (-\infty, \log(R))$, if we consider the differential problem

$$\begin{cases} \mathrm{e}^{2x} f_t = f_{xx} + f_x + \frac{(\cos^3\varphi f_\varphi)_\varphi}{\cos^3\varphi} + \gamma_c f_\varphi + \cos(\mathscr{G}_c)|\nabla f|^2 + c\,\mathrm{e}^x(f_x \sin\varphi + f_\varphi\cos\varphi - 1) & \\ & \text{in } (-M, \log(R)) \times (-\pi/2, \pi/2) \times \mathbb{R}^+ \\ f(x,\varphi,0) \;=\; f_R(x,\varphi) & \text{in } [-M, \log(R)] \times [-\pi/2, \pi/2] \\ f_\varphi(x, \pm\pi/2, t) = 0 & \text{for } x \in [-M, \log(R)], t > 0 \\ f(M,\varphi,t) \;=\; f_R(M,\varphi) & \text{for } \varphi \in (-\pi/2, \pi/2), t > 0 \\ f(\log(R),\varphi,t) \;=\; f_R(\log(R),\varphi) & \text{for } \varphi \in (-\pi/2, \pi/2), t > 0\,, \end{cases} \tag{4.20}$$

f_c is a subsolution of this problem if $c > c_R$, a supersolution if $c < c_R$. We conjecture that, by using properties of θ_R, from (4.18) and (4.19) follows that

$$f_R(x,\varphi) \approx \mathrm{e}^x \;\text{ as } x \to -\infty. \tag{4.21}$$

If this is true, then, under the assumption $c > c_R$, we are able to prove that:

1. for every $M \in (-\infty, \log(R))$ problem 4.20 has a supersolution F_M which is decreasing with respect to t,
2. the sequence F_M is decreasing with respect to M and, denoted by F its limit as $M \to -\infty$,
3. $$\limsup_{x\to-\infty} \lim_{t\to\infty} f_c(x,\varphi,t) \le \limsup_{x\to-\infty} \lim_{t\to\infty} F(x,\varphi,t) \le -K$$

   where $K > 0$ is a constant (the first inequality follows from the maximum principle).

Then, if (4.21) is true and $c > c_R$, $\limsup_{x\to-\infty} f_{c,\infty}(x,\varphi) \le -K$ for every $\varphi \in (-\pi/2, \pi/2)$, which, together with (4.16), explains the inequality (4.15) in the case $c > c_R$. Of course, there exists an analogous argument for the case $c < c_R$.

If our conjecture turns out to be true, it suggests that, at least in the class of axially symmetric solutions, the nonuniqueness phenomena for several initial value problems are directly related to the local structure of the solution in a neighborhood of its singularities. The structure of the traveling waves suggests that in a neighborhood of each singularity, the solution behaves as a harmonic map from $\mathbb{R}^3$ to $\mathbb{S}^2$, but the instantaneous speed of the singular point is related to *which* harmonic map represents the local behavior. So it is natural to ask whether the sort of nonuniqueness observed in [27] and [2], caused by the degree of freedom to prescribe the evolution of a singular point, could be explained, alternatively, by the degree of freedom to prescribe the harmonic map which describes the local behavior of the solution near a singular point.

Appendix A

Some energetic inequalities concerning harmonic maps

Lemma A.1. *For all* $w \in H^1_{\mathrm{loc}}(0,\infty) \subset C^0((0,\infty))$ *and* $0 < \rho_1 < \rho_2$

$$\int_{\rho_1}^{\rho_2} \frac{r}{2}\left(\frac{\sin^2 w}{r^2} + \left|\frac{dw}{dr}\right|^2\right) \mathrm{d}\, r \geq |\cos(w(\rho_2)) - \cos(w(\rho_1))| \,.$$

Proof:

$$|\cos(w(\rho_2)) - \cos(w(\rho_1))| = \left|\int_{\rho_1}^{\rho_2} -\sin(w)\frac{dw}{dr}\,\mathrm{d}\, r\right| \leq$$

$$\leq \int_{\rho_1}^{\rho_2} \frac{r}{2}\left(2\left|\frac{\sin w}{r}\right|\left|\frac{dw}{dr}\right|\right) \mathrm{d}\, r \leq \int_{\rho_1}^{\rho_2} \frac{r}{2}\left(\frac{\sin^2 w}{r^2} + \left|\frac{dw}{dr}\right|^2\right) \mathrm{d}\, r.$$

□

Lemma A.2. *For all* $W \in H^1_{\mathrm{loc}}(-\pi/2,\pi/2) \subset C^0((-\pi/2,\pi/2))$ *and* $-\pi/2 \leq \varphi_1 < \varphi_2 \leq \pi/2$

$$\int_{\varphi_1}^{\varphi_2} \frac{\cos(\varphi)}{2}\left(\left|\frac{dW}{d\varphi}\right|^2 + \frac{\sin^2(W)}{\cos^2(\varphi)}\right) \mathrm{d}\, \varphi \geq |\cos(W(\varphi_2)) - \cos(W(\varphi_1))| \,.$$

Proof:

$$|\cos(W(\varphi_2)) - \cos(W(\varphi_1))| = \left|\int_{\varphi_1}^{\varphi_2} -\sin(W)\frac{dW}{d\varphi}\,\mathrm{d}\, \varphi\right| \leq$$

$$\int_{\varphi_1}^{\varphi_2} \frac{\cos(\varphi)}{2}\left(2\left|\frac{dW}{d\varphi}\right|\frac{|\sin(W)|}{\cos(\varphi)}\right) \mathrm{d}\, \varphi \leq \int_{\varphi_1}^{\varphi_2} \frac{\cos(\varphi)}{2}\left(\left|\frac{dW}{d\varphi}\right|^2 + \frac{\sin^2(W)}{\cos^2(\varphi)}\right) \mathrm{d}\, \varphi \,.$$

□

A straightforward calculation leads to the following consequences:

Corollary A.3. *Let* $0 < \alpha < \beta$, $k \in \mathbb{Z}$ *and* $b \in \mathbb{R}$. *Let*

$$E_\alpha^\beta(w) = \int_\alpha^\beta \frac{r}{2}\left(\left|\frac{dw}{dr}\right|^2 + \frac{\sin^2 w}{r^2}\right) \mathrm{d}\, r \quad \textit{for } w \in H^1(\alpha,\beta). \tag{A.1}$$

Then

$$E_\alpha^\beta(w) \geq E_\alpha^\beta(k\pi + 2\arctan(br)) \;=\; \frac{2}{1+b^2\alpha^2} - \frac{2}{1+b^2\beta^2}$$

for all $w \in H^1(\alpha,\beta)$ *satisfying* $w(\alpha) \;=\; k\pi + 2\arctan(b\alpha)$ *and* $w(\beta) \;=\; k\pi + 2\arctan(b\beta)$.

Corollary A.4. *Let* $-\pi/2 < \alpha < \beta < \pi/2$, $k \in \mathbb{Z}$ *and* $A \in \mathbb{R}$. *Let*

$$F_\alpha^\beta(W) \;=\; \int_\alpha^\beta \frac{\cos(\varphi)}{2}\left(\left|\frac{dW}{d\varphi}\right|^2 + \frac{\sin^2(W)}{\cos^2(\varphi)}\right) \mathrm{d}\,\varphi \quad \textit{for } W \in H^1(\alpha,\beta). \tag{A.2}$$

Then

$$\begin{aligned} F_\alpha^\beta(W) &\geq F_\alpha^\beta\left(k\pi + 2\arctan\left(A\tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)\right)\right) \;=\\ &= \frac{1 - A^2 + \sin(\beta)(1+A^2)}{1 + A^2 + \sin(\beta)(1-A^2)} - \frac{1 - A^2 + \sin(\alpha)(1+A^2)}{1 + A^2 + \sin(\alpha)(1-A^2)} \end{aligned}$$

for all $W \in H^1(\alpha,\beta)$ *satisfying* $W(\alpha) \;=\; k\pi + 2\arctan(A\tan(\pi/4 - \alpha/2))$ *and* $W(\beta) \;=\; k\pi + 2\arctan(A\tan(\pi/4 - \beta/2))$.

Lemma A.5. *Let* $0 < \alpha < \beta$, $w \in H^1(\alpha,\beta)$ *and let* $E_\alpha^\beta(w)$ *be defined by (A.1). If* k_1, k_2 *are integers satisfying* $w(\alpha) \in [k_1\pi, (k_1+1)\pi)$, $w(\beta) \in [k_2\pi, (k_2+1)\pi)$, *then*

$$E_\alpha^\beta(w) \geq \begin{cases} 2(k_2-k_1-1) + |\cos(w(\beta)) - (-1)^{k_2}| + |(-1)^{k_1+1} - \cos(w(\alpha))| & \textit{if } k_2 > k_1 \\ |\cos(w(\beta)) - \cos(w(\alpha))| & \textit{if } k_2 = k_1 \\ 2(k_1-k_2-1) + |\cos(w(\alpha)) - (-1)^{k_1}| + |(-1)^{k_2+1} - \cos(w(\beta))| & \textit{if } k_2 < k_1. \end{cases}$$

Proof: If $k_2 = k_1$ the conclusion follows directly from Lemma A.1. If $k_2 > k_1$ and so $w(\beta) > w(\alpha)$, it is sufficient to apply Lemma A.1 to the partition $\alpha < R_0 < \ldots < R_{k_2-k_1-1} < \beta$ of $[\alpha,\beta]$, where $w(R_j) = (k_1+1+j)\pi$ for all $j = 0, 1, \ldots, k_2 - k_1 - 1$. The case $k_2 < k_1$ is similar. $\square$

Theorem A.6. *Let* $R > 0$, $0 < b < 1$ *and* $w \in H^1_{\text{loc}}((0,R])$. *If* $w(R) \;=\; 2\arctan b$ *then*

$$E_0^R(w) = \int_0^R \frac{r}{2}\left(\frac{\sin^2 w}{r^2} + \left|\frac{dw}{dr}\right|^2\right) \mathrm{d}\,r \geq E_0^R\left(2\arctan\left(\frac{br}{R}\right)\right) = \frac{2b^2}{1+b^2}\ .$$

Proof: The latter equality is trivial. To prove the inequality, we observe that, since $0 < b < 1$, if $\lim_{\rho\to 0^+} w(\rho) = k\pi$ for some $k \in \mathbb{Z}$, Lemma A.1 implies that $E_0^R(w) \geq \frac{2}{1+b^2} > \frac{2b^2}{1+b^2}$ if k is odd, and $E_0^R(w) \geq \frac{2b^2}{1+b^2}$ if k is even. It is easy to prove that in all other cases $E_0^R(w) = \infty$. Indeed, if $\lim_{\rho\to 0^+} w(\rho)$ exists and is finite but not equal to a multiple of π, then $\frac{\sin^2 w}{r}$ is not integrable at $r = 0$; if $\lim_{\rho\to 0^+} w(\rho)$ is infinite or does not exist it is enough to apply (repeatedly in the latter case) Lemma A.5. $\square$

Setting

$$\mathcal{S}_b(R) \;=\; \left\{ w \in H^1_r(0,R)\,;\, \frac{\sin w}{r} \in L^2_r(0,R), w(R) \;=\; 2\arctan b \right\},$$

Theorem A.6 implies that if $0 < b < 1$ the function $2\arctan\left(\frac{br}{R}\right)$ is a minimum of the functional $E_0^R(w)$ on $\mathcal{S}_b(R)$. Since any minimum satisfies the Euler-Lagrange equation $w_{rr} + \frac{1}{r}w_r - \frac{\sin(2w)}{2r^2} = 0$, it is easy to show that $2\arctan\left(\frac{br}{R}\right)$ is the unique minimum. Using the estimates obtained in this appendix it is very easy to show a slightly sharper result, of which we omit the proof:

Theorem A.7. *Let $R > 0$, $0 < b < 1$ and let $\{w_n\}$ be a minimizing sequence for $E_0^R(w)$ on $\mathcal{S}_b(R)$. Then $w_n(0) = 0$ for n large enough and $w_n(r) \to 2\arctan\left(\frac{br}{R}\right)$ uniformly in $[0, R]$ as $n \to \infty$.*

Appendix B

Onedimensional monotone rearrangements

Throughout this section $f(r,x)$ will denote a C^1-function defined in $(0,1)\times\mathbb{R}^+$ satisfying the following four properties:

(P1) for all $r\in(0,1)$, $C\in\mathbb{R}$ and $0<\alpha<\beta$ the sets $\{x\in[\alpha,\beta]\,;\, f(r,x) = C\}$ and $\{x\in[\alpha,\beta]\,;\, f_x(r,x) = 0\}$ are finite;

(P2) $f_r\in L^\infty((\rho,1)\times\mathbb{R}^+)$ and $f_x\in L^\infty((\rho,1)\times(\rho,\infty))$ for all $\rho>0$;

(P3) $f\in L^\infty((0,1)\times\mathbb{R}^+)$ and

$$\ell(r) \equiv \inf_{x>0} f(r,x) < L(r) \equiv \sup_{x>0} f(r,x) \quad \text{for } 0<r<1;$$

(P4) for any $\rho>0$, $\lim_{x\to\infty} f(r,x) = \ell(r)$ uniformly with respect to $r\in[\rho,1)$.

Given f, we set

$$\Omega_{d,r} = \{x>0\,;\, f(r,x)\ge d\} \quad \text{for } d\in\mathbb{R}\,,\ 0<r<1\,,$$

$$f^*(r,x) = \sup\{d\in\mathbb{R}\,|\, x\le\mu(\Omega_{d,r})\} \quad \text{for } 0<r<1\,,\ x>0\,,$$

where μ is the onedimensional Lebesgue measure. By construction, the rearrangement f^* of f is nonincreasing with respect to x, for every $r\in(0,1)$ $\sup_{x>0} f^*(r,x) = L(r)$, $\lim_{x\to\infty} f^*(r,x) = \ell(r)$ uniformly with respect to $r\in[\rho,1)$ for $\rho>0$, and for all $0<r<1$, $d_1<d_2$

$$\mu(\{x>0\,;\, d_1\le f^*(r,x)<d_2\}) = \mu(\{x>0\,;\, d_1\le f(r,x)<d_2\}). \tag{B.1}$$

We define, for $0<r<1$, $d\in\mathbb{R}$, and $0<\sigma<\tau$, the sets

$$\Omega_{\sigma,d,r} = \{x\ge\sigma\,;\, f(r,x)\ge d\}, \qquad \Omega^\tau_{d,r} = \{x\in(0,\tau]\,;\, f(r,x)\ge d\},$$
$$\Omega^\tau_{\sigma,d,r} = \{x\in[\sigma,\tau]\,;\, f(r,x)\ge d\},$$

and, in $(0,1)\times\mathbb{R}^+$, the rearranged functions

$$f^{*\sigma}(r,x) = \begin{cases} \sup\{d\in\mathbb{R}\,;\, \mu(\Omega_{\sigma,d,r})>0\} & \text{if } x\le\sigma \\ \sup\{d\in\mathbb{R}\,;\, x-\sigma\le\mu(\Omega_{\sigma,d,r})\} & \text{if } x>\sigma\,, \end{cases}$$

$$f^*_\tau(r,x) \;=\; \begin{cases} \sup\{d\in\mathbb{R}\,;\; x \le \mu(\Omega^\tau_{d,r})\} & \text{if } x \le \tau \\ \sup\{d\in\mathbb{R}\,;\; \tau \le \mu(\Omega^\tau_{d,r})\} & \text{if } x > \tau\,, \end{cases}$$

$$f^{*\sigma}_\tau(r,x) \;=\; \begin{cases} \sup\{d\in\mathbb{R}\,;\; \mu(\Omega^\tau_{\sigma,d,r}) > 0\} & \text{if } x \le \sigma \\ \sup\{d\in\mathbb{R}\,;\; x-\sigma \le \mu(\Omega^\tau_{\sigma,d,r})\} & \text{if } x \in (\sigma,\tau] \\ \sup\{d\in\mathbb{R}\,;\; \tau-\sigma \le \mu(\Omega^\tau_{\sigma,d,r})\} & \text{if } x > \tau\,. \end{cases}$$

It follows at once from the previous definitions that $f^{*\sigma}, f^*_\tau$ and $f^{*\sigma}_\tau$ are non increasing with respect to x and

1. $\lim\limits_{x\to\infty} f^{*\sigma}(r,x) = \ell(r)\,,$
2. for all $x \le \sigma$
$$f^{*\sigma}(r,x) \;=\; L_{\sigma,r} \;:=\; \sup_{x\ge\sigma} f(r,x), \quad f^{*\sigma}_\tau(r,x) \;=\; \sup_{x\in[\sigma,\tau]} f(r,x)\,,$$
3. $\sup\limits_{x>0} f^*_\tau(r,x) = \sup\limits_{x\in(0,\tau]} f(r,x)$ and for all $x \ge \tau$
$$f^*_\tau(r,x) \;=\; \inf_{x\in(0,\tau]} f(r,x), \quad f^{*\sigma}_\tau(r,x) \;=\; \inf_{x\in[\sigma,\tau]} f(r,x)\,.$$

The proofs of the following propositions are based on standard techniques for onedimensional rearrangements (see [20]). In particular we remind the reader that it is well-known that f^*, $f^{*\sigma}$, f^*_τ and $f^{*\sigma}_\tau$ are continuous and a.e. differentiable in $(0,1)\times\mathbb{R}^+$, and that, for all $0<\rho<1$ and $\tau>\sigma>0$,

$$\begin{aligned}&\|(f^*)_r\|_{L^\infty(R_\rho)}, \|(f^{*\sigma})_r\|_{L^\infty(R_\rho)}, \|(f^*_\tau)_r\|_{L^\infty(R_\rho)}, \|(f^{*\sigma}_\tau)_r\|_{L^\infty(R_\rho)} \le \|f_r\|_{L^\infty(R_\rho)},\\ &\|(f^*)_x\|_{L^\infty(R_{\rho,\sigma})}, \|(f^{*\sigma})_x\|_{L^\infty(R_\rho)}, \|(f^*_\tau)_x\|_{L^\infty(R^\tau_{\rho,\sigma})}, \|(f^{*\sigma}_\tau)_x\|_{L^\infty(R_\rho)} \le \|f_x\|_{L^\infty(R_{\rho,\sigma})}\end{aligned}$$

where $R_\rho = (\rho,1)\times\mathbb{R}^+$, $R_{\rho,\sigma} = (\rho,1)\times(\sigma,\infty)$ and $R^\tau_{\rho,\sigma} = (\rho,1)\times(\sigma,\tau-\sigma)$.

Proposition B.1. *For all* $0<\sigma<\tau$

$$f^*(r,x) \le f^{*\sigma}(r,x) \le f^*(r,x-\sigma) \qquad \textit{if } 0<r<1 \textit{ and } x>\sigma\,,$$

$$f^*_\tau(r,x) \le f^{*\sigma}_\tau(r,x) \le f^*_\tau(r,x-\sigma) \quad \textit{if } 0<r<1 \textit{ and } \sigma \le x \le \tau.$$

In particular $f^{*\sigma} \to f^*$ *uniformly on* $[\alpha,1)\times[\alpha,\infty)$ *(*$\alpha>0$*) as* $\sigma\to0^+$*, and* $f^{*\sigma}_\tau \to f^*_\tau$ *in* $C_{\rm loc}((0,1)\times(0,\tau))$ *as* $\sigma\to0^+$*.*

Proposition B.2. *For any* $\rho\in(0,1/2)$ *and* $M>0$ *there exists* $\tau(\rho,M)$ *such that* $f^*_\tau = f^*$ *in* $[\rho,1-\rho]\times(0,M]$ *if* $\tau>\tau(\rho,M)$*. Then* $f^*_\tau\to f^*$ *and* $(f^*_\tau)_r \to (f^*)_r$ *locally uniformly on* $(0,1)\times\mathbb{R}^+$

Proof: The thesis follows easily from property (P4), which holds true for f^*. □

Proposition B.3. *Let* $F:(0,1)\times\mathbb{R}\to\mathbb{R}$ *be continuous and nonnegative, and let* $G:[0,\infty)\to[0,\infty)$ *be convex and nondecreasing (and hence continuous). Then, for all* $0<\sigma<\tau$ *and* $0<\rho<1$*,*

$$\int_\sigma^\tau F(\rho, f^{*\sigma}_\tau(\rho,x))G\left(|(f^{*\sigma}_\tau)_r(\rho,x)|\right)\mathrm{d}\,x \le \int_\sigma^\tau F(\rho,f(\rho,x))G\left(|f_r(\rho,x)|\right)\mathrm{d}\,x\,. \qquad \text{(B.2)}$$

For the proof it is sufficient to apply lemma 2.6 and remark 2.22 of [20] to the function $f(\rho, x)$, with $x \in [\sigma, \tau]$. However, for convenience of the reader we sketch the proof of this proposition.

Proof: Fixed any $\rho \in (0,1)$, we denote by $\mathcal{G}$ and $\mathcal{A}$ the sets:

$$\mathcal{G} = \{x \in [\sigma, \tau] \text{ such that } f_x(\rho, x) = 0\}, \quad \mathcal{A} = \{f(\rho, x) \mid x \in \mathcal{G}\}.$$

If $\mathcal{A} = \emptyset$ and so $\mathcal{G} = \emptyset$, then $f(\rho, \cdot)$ is monotone, decreasing or increasing, and equality holds in (B.2): if $f(\rho, x)$ is a decreasing function of x, then $f_\tau^{*\sigma}(\rho, x) \equiv f(\rho, x)$, else if $f(\rho, x)$ is an increasing function of x, then $f_\tau^{*\sigma}(\rho, x) \equiv f(\rho, \sigma + \tau - x)$. Therefore, we may assume

$$\mathcal{A} = \{a_1, \dots, a_N\}$$

with $a_i > a_{i+1}$ for every $i = 1, \dots, N-1$. We define

$$D_0 = \left\{ x \in [\sigma, \tau] \mid f(\rho, x) \in \left(a_1, \sup_{x \in [\sigma, \tau]} f(\rho, x) \right) \right\},$$

$$D_N = \left\{ x \in [\sigma, \tau] \mid f(\rho, x) \in \left(\inf_{x \in [\sigma, \tau]} f(\rho, x), a_N \right) \right\}$$

and

$$D_i = \{x \in [\sigma, \tau] \mid f(\rho, x) \in (a_{i+1}, a_i)\}$$

for $i = 1, \dots, N-1$. Analogously are defined $D_0^*, D_1^*, \dots, D_N^*$ with f replaced by $f_\tau^{*\sigma}$. It is obvious that

- $D_0^* = \emptyset \Leftrightarrow a_1 = \sup_{x \in [\sigma, \tau]} f(\rho, x) \Leftrightarrow D_0 = \emptyset$;
- $D_N^* = \emptyset \Leftrightarrow a_N = \inf_{x \in [\sigma, \tau]} f(\rho, x) \Leftrightarrow D_N = \emptyset$;
- $D_0, D_1, \dots, D_N$ are disjoint open (i.e. relatively open in $[\sigma, \tau]$) sets and

$$\int_\sigma^\tau f(\rho, f(\rho, x)) G(|f_r|(\rho, x)) dx = \sum_{i=0}^N \int_{D_i} f(\rho, f(\rho, x)) G(|f_r|(\rho, x)) dx ;$$

- $D_0^*, D_1^*, \dots, D_N^*$ are disjoint open sets and

$$\int_\sigma^\tau F(\rho, f_\tau^{*\sigma}(\rho, x)) G(|(f_\tau^{*\sigma})_r|(\rho, x)) dx = \sum_{i=0}^N \int_{D_i^*} F(\rho, f_\tau^{*\sigma}(\rho, x)) G(|(f_\tau^{*\sigma})_r|(\rho, x)) dx .$$

Therefore, we can obtain the thesis by showing that for any i

$$\int_{D_i^*} F(\rho, f_\tau^{*\sigma}(\rho, x)) G(|(f_\tau^{*\sigma})_r|(\rho, x)) dx \le \int_{D_i} F(\rho, f(\rho, x)) G(|f_r|(\rho, x)) dx . \qquad \text{(B.3)}$$

We shall limit to prove the previous inequality for $i = 1, \dots, N-1$, since the proofs for $i = 0$ and $i = N$ (when D_0 or/and D_N are not empty) are simple adaptations of the following general argumentation.

We remark that $\forall\lambda \in [\inf_{x\in[\sigma,\tau]} f(\rho,x), \sup_{x\in[\sigma,\tau]} f(\rho,x)]$ there exists one and only one $x^*(\rho,\lambda)$ such that $f_\tau^{*\sigma}(\rho, x^*(\rho,\lambda)) = \lambda$. Actually, x^* is a function defined for every $r \in (0,1)$ and every $\lambda \in [\inf_{x\in[\sigma,\tau]} f(r,x), \sup_{x\in[\sigma,\tau]} f(r,x)]$ and by construction

$$x^*(r,\lambda) = \sigma + \mu(\Omega^\tau_{\sigma,\lambda,r}) = \sigma + \mu(\{x \in [\sigma,\tau] \mid f(r,x) \geq \lambda\}), \tag{B.4}$$

x^* is strictly decreasing with respect to λ and $x^*(\rho,\lambda) \in D_i^*$ if $a_{i+1} < \lambda < a_i$.

Fixed any $i = 1, \ldots, N-1$, we can write

$$D_i = \bigsqcup_{j=1}^{n(i)} \gamma_{ij}$$

where $\gamma_{i1}, \ldots, \gamma_{in(i)}$ are subintervals of $[\sigma,\tau]$, relatively open in $[\sigma,\tau]$, where f_x is always positive or negative. If we denote by γ_{i1} the subinterval closest to σ and then we enumerate the γ_{ij}'s depending on their distance from σ, then we have that $\forall j = 1, \ldots, n(i)$

$$\mathrm{sign}(f_x(\rho,\cdot)\big|_{\gamma_{ij}}) = (-1)^{j-1}\mathrm{sign}(f_x(\rho,\cdot)\big|_{\gamma_{i1}}) .$$

Naturally, for any $\lambda \in (a_{i+1}, a_i)$ there exists one and only one $x_{ij} = x_{ij}(\rho,\lambda) \in \gamma_{ij}$ such that $f(\rho, x_{ij}(\rho,\lambda)) = \lambda$. By using some theorems of elementary analysis (mainly the implicit function theorem) it is easy to see that there exists an open neighborhood I of $\rho \times (a_{i+1}, a_i)$ such that all the x_{ij} are defined and smooth over I and $\forall (r,\lambda) \in I$ $f(r, x_{ij}(r,\lambda)) = \lambda$. Moreover, by using formula (B.4) one easily sees that $\forall (r,\lambda) \in I$

$$x^*(r,\lambda) = \begin{cases} x_{i2} - x_{i1} + x_{i4} - x_{i3} + \cdots + \tau - x_{i\,n(i)} & \text{if } f_x(r, x_{i1}(r,\lambda)) > 0, \\ & n(i) \text{ odd} \\ x_{i2} - x_{i1} + x_{i4} - x_{i3} + \cdots + x_{i\,n(i)} - x_{i.n(i)-1} & \text{if } f_x(r, x_{i1}(r,\lambda)) > 0, \\ & n(i) \text{ even} \\ x_{i1} - \sigma + x_{i3} - x_{i2} + \cdots + \tau - x_{i\,n(i)} & \text{if } f_x(r, x_{i1}(r,\lambda)) < 0, \\ & n(i) \text{ even} \\ x_{i1} - \sigma + x_{i3} - x_{i2} + \cdots + x_{i\,n(i)} - x_{i.n(i)-1} & \text{if } f_x(r, x_{i1}(r,\lambda)) < 0, \\ & n(i) \text{ odd} \end{cases}$$

In any case x^* is a smooth function over I and

$$\left|\frac{\partial x^*}{\partial r}\right| = \left|\sum_{j=1}^{n(i)} (-1)^j \frac{\partial x_{ij}}{\partial r}\right| \tag{B.5}$$

$$\left|\frac{\partial x^*}{\partial \lambda}\right| = \left|\sum_{j=1}^{n(i)} (-1)^j \frac{\partial x_{ij}}{\partial \lambda}\right| . \tag{B.6}$$

Some simple computations show that

(a) $(f_\tau^{*\sigma})_x(\rho, x^*(\rho,\lambda)) = (x^*_\lambda(\rho,\lambda))^{-1}$;

(b) $(f_\tau^{*\sigma})_r(\rho, x^*(\rho,\lambda)) = -x^*_r(\rho,\lambda)(x^*_\lambda(\rho,\lambda))^{-1}$;

(c) $f_x(\rho, x_{ij}(\rho,\lambda)) = (\frac{\partial x_{ij}}{\partial \lambda}(\rho,\lambda))^{-1}$;

(d) $f_r(\rho, x_{ij}(\rho,\lambda)) = -\frac{\partial x_{ij}}{\partial r}(\rho,\lambda)(\frac{\partial x_{ij}}{\partial \lambda}(\rho,\lambda))^{-1}$;

for any $\lambda \in (a_{i+1}, a_i)$ and $j = 1, \dots, n(i)$. Then for any $j = 1, \dots, n(i)$ and $\lambda \in (a_{i+1}, a_i)$

$$\mathrm{sign}\left(\frac{\partial x_{ij}}{\partial \lambda}(\rho,\lambda)\right) = \mathrm{sign}\left(f_x(\rho, x_{ij}(\rho,\lambda))\right) =$$

$$= (-1)^{j-1}\mathrm{sign}\left(f_x(\rho, x_{i1}(\rho,\lambda))\right) = (-1)^{j-1}\mathrm{sign}\left(\frac{\partial x_{i1}}{\partial \lambda}(\rho,\lambda)\right)$$

and so from (B.6) we deduce that

$$\left|\frac{\partial x^*}{\partial \lambda}(\rho,\lambda)\right| = \sum_{j=1}^{n(i)}\left|\frac{\partial x_{ij}}{\partial \lambda}(\rho,\lambda)\right| . \tag{B.7}$$

By using (B.5), (B.7) and (a)-(c) we obtain through a change of variable ($x = x^*(\rho,\lambda)$ or $x = x_{ij}(\rho,\lambda)$) that

$$\int_{D_i^*} F(\rho, f_\tau^{*\sigma}(\rho,x))G(|(f_\tau^{*\sigma})_r|(\rho,x))dx =$$

$$= \int_{a_{i+1}}^{a_i} F(\rho,\lambda)G\left(\frac{\left|\sum_{j=1}^{n(i)}(-1)^j\frac{\partial x_{ij}}{\partial r}\right|}{\sum_{j=1}^{n(i)}\left|\frac{\partial x_{ij}}{\partial \lambda}\right|}\right)\left(\sum_{j=1}^{n(i)}\left|\frac{\partial x_{ij}}{\partial \lambda}\right|\right)d\lambda$$

and

$$\int_{D_i} F(\rho, f(\rho,x))G(|f_r|(\rho,x))dx = \sum_{j=1}^{n(i)}\int_{\gamma_{ij}} F(\rho, f(\rho,x))G(|f_r|(\rho,x))dx =$$

$$= \int_{a_{i+1}}^{a_i} F(\rho,\lambda)\sum_{j=1}^{n(i)}\left(G\left(\frac{\left|\frac{\partial x_{ij}}{\partial r}\right|}{\left|\frac{\partial x_{ij}}{\partial \lambda}\right|}\right)\left|\frac{\partial x_{ij}}{\partial \lambda}\right|\right)d\lambda .$$

Of course the partial derivatives of x^* and x_{ij} are all computed in (ρ,λ). Since F is non-negative, in order to prove (B.3) it is sufficient to show that for any $\lambda \in (a_{i+1}, a_i)$

$$G\left(\frac{\left|\sum_{j=1}^{n(i)}(-1)^j\frac{\partial x_{ij}}{\partial r}\right|}{\sum_{k=1}^{n(i)}\left|\frac{\partial x_{ik}}{\partial \lambda}\right|}\right)\left(\sum_{k=1}^{n(i)}\left|\frac{\partial x_{ik}}{\partial \lambda}\right|\right) \le \sum_{j=1}^{n(i)}\left(G\left(\frac{\left|\frac{\partial x_{ij}}{\partial r}\right|}{\left|\frac{\partial x_{ij}}{\partial \lambda}\right|}\right)\left|\frac{\partial x_{ij}}{\partial \lambda}\right|\right) : \tag{B.8}$$

If for every $j = 1, \dots, n(i)$ we set

$$\alpha_j = \frac{\left|\frac{\partial x_{ij}}{\partial \lambda}\right|}{\sum_{k=1}^{n(i)}\left|\frac{\partial x_{ik}}{\partial \lambda}\right|} ,$$

then we have that $\alpha_j > 0$ and $\sum_{j=1}^{n(i)} \alpha_j = 1$ at any point $\lambda \in (a_{i+1}, a_i)$. By using the monotonicity and the convexity of G we find that

$$G\left(\frac{\left|\sum\limits_{j=1}^{n(i)} (-1)^j \frac{\partial x_{ij}}{\partial r}\right|}{\sum\limits_{k=1}^{n(i)} \left|\frac{\partial x_{ik}}{\partial \lambda}\right|}\right) \leq G\left(\frac{\sum\limits_{j=1}^{n(i)} \left|\frac{\partial x_{ij}}{\partial r}\right|}{\sum\limits_{k=1}^{n(i)} \left|\frac{\partial x_{ik}}{\partial \lambda}\right|}\right) = G\left(\sum_{j=1}^{n(i)} \frac{\left|\frac{\partial x_{ij}}{\partial r}\right|}{\left|\frac{\partial x_{ij}}{\partial \lambda}\right|}\alpha_j\right) \leq$$

$$\leq \sum_{j=1}^{n(i)} \alpha_j G\left(\frac{\left|\frac{\partial x_{ij}}{\partial r}\right|}{\left|\frac{\partial x_{ij}}{\partial \lambda}\right|}\right),$$

and (B.8) follows from the definition of α_j. This completes the proof. $\square$

Proposition B.4. *Let $P(x)$ be a nonnegative and nondecreasing function defined for $x > 0$. Then, for all $\sigma > 0$ and $0 < \rho < 1$,*

$$\int_0^\infty P(x)(f^{*\sigma})_x^2(\rho, x)\,\mathrm{d}\,x \leq \int_0^\infty P(x) f_x^2(\rho, x)\,\mathrm{d}\,x\,. \tag{B.9}$$

Proof: We fix $\sigma > 0$ and $0 < \rho < 1$ and set

$$\mathcal{A} = \{f(\rho, x)\,;\, x \geq \sigma \text{ and } f_x(\rho, x) = 0\}.$$

In view of the properties of f the set $\mathcal{A}$ is either finite or countable. We give the proof only in the latter case. So we assume that $A = \{a_n\}$, where $a_n > a_{n+1}$ for all $n \geq 0$ and $\lim\limits_{n\to\infty} a_n = \ell(\rho)$. Of course, $\sup\limits_{n\in\mathbb{N}} a_n = a_0 \leq L_{\sigma,\rho} := \sup\limits_{x\geq\sigma} f(\rho, x)$.

We define the open sets $D_n = \{x > \sigma\,;\, a_{n+1} < f(\rho, x) < a_n\}$ and $D_n^* = \{x > \sigma\,;\, a_{n+1} < f^{*\sigma}(\rho, x) < a_n\}$. For each n we can decompose D_n in a finite number, k_n, of disjoint open intervals $\gamma_{n,j}$ $(j = 1, ..., k_n)$ on each of which $f_x(\rho, \cdot)$ is either strictly positive or strictly negative. We label these intervals according to their distance to the origin by taking $\gamma_{n,1}$ as the farthest one. Then $\mathrm{sgn}(f_x(\rho, \cdot)) = (-1)^j$ on $\gamma_{n,j}$ for all $j = 1, ..., k_n$, and there exists for all $j = 1, ..., k_n$ and $\lambda \in (a_{n+1}, a_n)$ a unique $x_j = x_j(\rho, \lambda) \in \gamma_{n,j}$ such that $f(\rho, x_j(\rho, \lambda)) = \lambda$. By the implicit function theorem, x_j can be thought as a smooth function defined in an open set containing $\{\rho\} \times (a_{n+1}, a_n)$. Similarly there exists for all $\lambda \in (\ell(\rho), L_{\sigma,\rho}]$ a unique $x^*(\rho, \lambda) \geq \sigma$ such that $f^{*\sigma}(\rho, x^*(\rho, \lambda)) = \lambda$. By construction, $x^*(\rho, \lambda) = \mu(\Omega_{\sigma,\lambda,r}) + \sigma$, x^* is strictly decreasing with respect to λ, and $x^*(\rho, \lambda) \in D_n^*$ if $a_{n+1} < \lambda < a_n$. It is easy to check that

$$x^*(\rho, \lambda) = p(k_n)\sigma + \sum_{j=1}^{k_n} (-1)^{j+1} x_j(\rho, \lambda), \tag{B.10}$$

where $p(k_n) = 0$ if k_n is odd and $p(k_n) = 1$ if k_n even. A simple computation yields that for every n and for almost all $\lambda \in (a_{n+1}, a_n)$

$$|(f^{*\sigma})_x(\rho, x^*(\rho, \lambda))| = \left(\sum_{j=1}^{k_n} |(x_j)_\lambda(\rho, \lambda)|\right)^{-1},$$

$$|f_x(\rho, x_j(\rho, \lambda))| \;=\; |(x_j)_\lambda(\rho, \lambda)|^{-1} \quad j \;=\; 1, ..., k_n.$$

These equalities imply that for every n

$$\int_{D_n} P(x) f_x^2(\rho, x)\,\mathrm{d}\,x = \sum_{j=1}^{k_n} \int_{\gamma_{n,j}} P(x) f_x^2(\rho, x)\,\mathrm{d}\,x = \int_{a_{n+1}}^{a_n} \left(\sum_{j=1}^{k_n} \frac{P(x_j(\rho, \lambda))}{|(x_j)_\lambda(\rho, \lambda)|} \right) \mathrm{d}\,\lambda \qquad \text{(B.11)}$$

and

$$\int_{D_n^*} P(x) (f^{*\sigma})_x^2(\rho, x)\,\mathrm{d}\,x \;=\; \int_{a_{n+1}}^{a_n} \frac{P(x^*(\rho, \lambda))}{\sum_{j=1}^{k_n} |(x_j)_\lambda(\rho, \lambda)|}\,\mathrm{d}\,\lambda. \qquad \text{(B.12)}$$

On the other hand we know that $x^*(\rho, \lambda) \;=\; \mu(\Omega_{\sigma,\lambda,r}) + \sigma \;\le\; x_1(\rho, \lambda)$, since $\Omega_{\sigma,\lambda,r} \subseteq [\sigma, x_1(\rho, \lambda)]$ by the definition of $x_1(\rho, \lambda)$. Hence it follows from (B.11) and (B.12) that

$$\int_{D_n^*} P(x) (f^{*\sigma})_x^2(\rho, x)\,\mathrm{d}\,x \;\le\; \int_{D_n} P(x) f_x^2(\rho, x)\,\mathrm{d}\,x\,. \qquad \text{(B.13)}$$

We remind that $a_0 = \max_{n \in \mathbb{N}} a_n$ and $L_{\sigma,\rho} = \sup_{x \ge \sigma} f(\rho, x)$. If $a_0 = L_{\sigma,\rho}$, then we have that

$$(\sigma, \infty) \setminus \bigcup_{n \in \mathbb{N}} D_n \qquad \text{and} \qquad (\sigma, \infty) \setminus \bigcup_{n \in \mathbb{N}} D_n^* \qquad \text{(B.14)}$$

are sets of zero Lebesgue measure, and the inequality (B.9) follows at once from (B.13) (we have used that $f^{*\sigma}(\rho, \cdot)$ is constant for $x \in (0, \sigma]$).

It remains to consider the case in which $a_0 < L_{\sigma,\rho}$. Then there exists $\bar{x} > \sigma$ such that $\{x \ge \sigma\,;\; a_0 < f(\rho, x) < L_{\sigma,\rho})\} \;=\; (\sigma, \bar{x})$ and $f_x(\rho, \cdot) < 0$ in $(\sigma, \bar{x})$. Hence $f^{*\sigma}(\rho, x) \;=\; f(\rho, x)$ for all $\sigma \le x \le \bar{x}$. Arguing now as in (B.14) with (σ, ∞) replaced by $(\bar{x}, \infty)$, we obtain (B.9). $\square$

Remark B.5. Even if we stated all the properties and the results of this chapter for functions defined in $(0, 1) \times \mathbb{R}^+$, it is simple to check that all of them can be reformulated for functions defined in $(0, R) \times \mathbb{R}^+$, where R is an arbitrary positive value.

In what follows we shall denote by f_1 and f_2 the functions defined in (2.25) and by x_0 and x_1 the values e^{cz_0} and e^{cz_1} respectively, where z_0 and z_1 are the same as in (2.2). We remark that f_1 and f_2 satisfy properties (P1)-(P4) with $\ell(r) = 2 \arctan(br)$ for a suitable constant $b \in (0, 1)$.

Proposition B.6. *For $i = 1, 2$ and $\tau \ge x_1$,*

$$f_{i\,\tau}^*(1, x) \equiv f_i^*(1, x) \equiv f_i(1, x) \equiv g(\log(x)/c)\,.$$

Proof: We omit the subscript i, since the argument is the same for f_1 and f_2. Since f is Lipschitz continuous in r uniformly with respect to x in $[\rho, 1] \times \mathbb{R}^+$ with $\rho \in (0, 1)$, we can say that for every $x \in \mathbb{R}^+$ there exists

$$f_\tau^*(1, x) \equiv \lim_{r \to 1^-} f_\tau^*(r, x)$$

and

$$f_\tau^*(1, x) \;=\; \begin{cases} \sup(\{d \in \mathbb{R} \,|\, x \le \mu(\Omega_{d,1}^\tau)\}) & \text{if } x \in (0, \tau] \\ \sup(\{d \in \mathbb{R} \,|\, \tau \le \mu(\Omega_{d,1}^\tau)\}) & \text{if } x > \tau \end{cases}$$

where $\Omega^\tau_{d,1} = \{x \in (0,\tau] \mid f(1,x) \geq d\}$. We know that $f(1,x)$ is smooth, strictly decreasing in (x_0,x_1), identically equal to A in $(0,x_0]$ and to B in $[x_1,\infty)$, where A, B are the constant values in (2.2). Therefore, for every $\tau \geq x_1$ and $d \in \mathbb{R}$

$$\Omega^\tau_{d,1} \;=\; \begin{cases} \emptyset & \text{if } d > A \\ (0,x_0] & \text{if } d = A \\ (0,(f(1,\cdot))^{-1}(d)] & \text{if } d \in (B,A) \\ (0,\tau] & \text{if } d \leq B \end{cases}$$

and $f^*_\tau(1,x) \equiv f(1,x)$. A similar argumentation can be used to show that $f^*(1,x) \equiv f(1,x)$. $\square$

From (B.2) and (B.9) we deduce the following results.

Proposition B.7. *For $i = 1,2$,*

$$\int_{\mathbb{R}^+} \mathrm{d}\,x \int_0^1 r x^2 (f^*_i)^2_x \,\mathrm{d}\,r \leq \int_{\mathbb{R}^+} \mathrm{d}\,x \int_0^1 r x^2 (f_i)^2_x \,\mathrm{d}\,r < \infty.$$

Proof: Since $h_i = T^{-1}(f_i)$ is a minimizer (see Chapter 2, section 2.1), it follows from (2.14) that the latter integral is finite. We omit the subscript i. It is sufficient to prove that for any $\rho \in (0,1)$

$$\int_{\mathbb{R}^+} x^2 (f^*)^2_x(\rho,x)\,\mathrm{d}\,x \leq \int_{\mathbb{R}^+} x^2 f^2_x(\rho,x)\,\mathrm{d}\,x. \tag{B.15}$$

Without loss of generality we may assume that the right hand side is finite, i.e. $xf_x \in L^2(\mathbb{R}^+)$. Let $\sigma_n \to 0^+$. By (B.9) (with $P(x) = x^2$) the sequence $\{x(f^{*\sigma_n})_x\}$ is bounded in $L^2(\mathbb{R}^+)$ and, up to a subsequence, there exists $v \in L^2(\mathbb{R}^+)$ such that $x(f^{*\sigma_n})_x \to v$ weakly in $L^2(\mathbb{R}^+)$. It follows easily from Proposition B.1 and the regularity properties of $f^{*\sigma}$ and f^* that $v(\rho,x) = x(f^*)_x(\rho,x)$ for a.e. $x \in \mathbb{R}^+$. Hence (B.15) follows from (B.9). $\square$

Proposition B.8. *Let $G_b = G_b(r)$ be the function given by (2.6). For $i = 1,2$ and for every $M \geq x_1$,*

$$\begin{gathered} \textstyle\int_0^M \mathrm{d}\,x \int_0^1 r \left((f^*_i)^2_r + \frac{\sin^2 f^*_i}{r^2} - G_b(r)\right) \mathrm{d}\,r \leq \\ \textstyle\int_0^\infty \mathrm{d}\,x \int_0^1 r \left((f_i)^2_r + \frac{\sin^2 f_i}{r^2} - G_b(r)\right) \mathrm{d}\,r < \infty. \end{gathered}$$

Proof: Since $h_i = T^{-1}(f_i)$ is a minimizer (see Chapter 2, section 2.1), it follows from (2.14) that the latter integral is finite. We omit the subscript i. For any $\tau > 0$ we set

$$q_\tau(x) \;=\; \int_0^1 r\left((f^*_\tau)^2_r(r,x) + \frac{\sin^2 f^*_\tau(r,x)}{r^2} - G_b(r)\right)\mathrm{d}\,r \quad \text{for } x > 0.$$

Observe that q_τ is a measurable function with values in $\mathbb{R} \cup \{\infty\}$ and that, by Proposition B.6 and Theorem A.6, $q_\tau(x) \geq 0$ for a.e. $x > x_1$ if $\tau \geq x_1$. Similarly, the function

$$q(x) \;=\; \int_0^1 r\left((f^*)^2_r(r,x) + \frac{\sin^2 f^*(r,x)}{r^2} - G_b(r)\right)\mathrm{d}\,r \quad \text{for } x > 0$$

is nonnegative a.e. in (x_1, ∞). By Proposition B.2 and by Fatou's lemma, $q(x) \leq \liminf_{\tau\to\infty} q_\tau(x)$ for all $x > 0$. In particular

$$\int_0^{x_1} q(x)\,\mathrm{d}\,x \leq \liminf_{\tau\to\infty} \int_0^{x_1} q_\tau(x)\;\mathrm{d}\,x.$$

Since $q, q_\tau \geq 0$ a.e. in $[x_1, \infty)$ if $\tau \geq x_1$, it follows again from Fatou's lemma that for all $M \geq x_1$

$$\int_{x_1}^{M} q(x)\,\mathrm{d}\,x \leq \int_{x_1}^{\infty} q(x)\,\mathrm{d}\,x \leq \liminf_{\tau\to\infty} \int_{x_1}^{\tau} q_\tau(x)\;\mathrm{d}\,x.$$

The proof is complete if we show that, for every $\tau > 0$,

$$\int_0^\tau \mathrm{d}\,x \int_0^1 \frac{\sin^2 f_\tau^*}{r}\,\mathrm{d}\,r \leq \int_0^\tau \mathrm{d}\,x \int_0^1 \frac{\sin^2 f}{r}\,\mathrm{d}\,r\,, \tag{B.16}$$

$$\int_0^\tau \mathrm{d}\,x \int_0^1 r(f_\tau^*)_r^2\,\mathrm{d}\,r \leq \int_0^\tau \mathrm{d}\,x \int_0^1 r f_r^2\,\mathrm{d}\,r\,. \tag{B.17}$$

Inequality (B.16) follows at once from (B.2), with $G \equiv 1$ and $F = \sin^2(f)r^{-1}$, Proposition B.1 and Fatou's lemma. Applying (B.2) with $F = r$ and $G(v) = v^2$ we find that for all $0 < \sigma < \tau$

$$\int_0^1 \int_\sigma^\tau r(f^{*\sigma}\tau)_r^2\,\mathrm{d}\,x\;\mathrm{d}\,r \leq \int_0^1 \int_\sigma^\tau r f_r^2\,\mathrm{d}\,x\;\mathrm{d}\,r. \tag{B.18}$$

Letting $\sigma \to \infty$ and arguing as in the previous proof we easily obtain (B.17). □

Now, let $\bar{h}$ and ϑ be the functions defined in (1.29). Then $\bar{h}, \vartheta : (0, R) \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ satisfy properties (P1)-(P4) with $\ell(r) \equiv 0$ and $\ell(r) \equiv 2\arctan(br)$, for a suitable constant $b > 0$, respectively. Since ϑ is Lipschitz continuous in r uniformly with respect to x in $[\rho, R] \times \mathbb{R}^+$ with $\rho \in (0, R)$, we can say that for every $x \in \mathbb{R}^+$ there exist

$$\vartheta_\tau^*(R, x) \equiv \lim_{r\to R^-} \vartheta_\tau^*(r, x), \quad \vartheta^*(R, x) \equiv \lim_{r\to R^-} \vartheta^*(r, x)\,.$$

Moreover, since $\vartheta(R, x) \equiv 2\arctan(bR)$, it is immediate to conclude that $\vartheta_\tau^*(R, x) \equiv \vartheta^*(R, x) \equiv 2\arctan(bR)$ ($\tau > 0$ is arbitrary).

From (B.2), applied with $G \equiv 1$ and $F(r, f) = rf^2$, and Propositions B.1, B.2 follows that

Proposition B.9.

$$\int_{\mathbb{R}^+} \mathrm{d}\,x \int_0^R r\left|\bar{h}^*\right|^2 \mathrm{d}\,r \leq \int_{\mathbb{R}^+} \mathrm{d}\,x \int_0^R r\left|\bar{h}\right|^2 \mathrm{d}\,r < \infty$$

Moreover, arguing as in the proof of Proposition B.7 one can prove that

Proposition B.10.

$$\int_{\mathbb{R}^+} \mathrm{d}\,x \int_0^R rx^2(\vartheta_x^*)^2\,\mathrm{d}\,r \leq \int_{\mathbb{R}^+} \mathrm{d}\,x \int_0^R rx^2\vartheta_x^2\,\mathrm{d}\,r < \infty.$$

The analogue of Proposition B.8 requires however a slightly different proof.

Lemma B.11. *Let*

$$M = \int_{\mathbb{R}^+} \mathrm{d}\,x \int_0^R r\left|\bar{h}\right|^2 \mathrm{d}\,r\,.$$

There exist a constant $C(M) > 0$ and $B \subset \mathbb{R}^+$ such that $\forall x \in \mathbb{R}^+ \setminus B$ $\vartheta^*(r(x), x) = \pi/2$ for some $r(x) \in (0, R)$ and $\mu(B) \leq C(M)$, where μ is the onedimensional Lebesgue measure.

Proof: Since $bR > 1$ (see chapter 1), $\vartheta^*(R, x) = 2\arctan(bR) > \frac{\pi}{2}$. Let $\rho_b \in (0, R)$ be such that $\theta_+(\rho_b) = \frac{\pi}{3}$. For every $x \in \mathbb{R}^+$ we define $A_x = \{r \in [3^{-1/2}\rho_b, \rho_b]\ ;\ |\bar{h}^*(r, x)| > \frac{\pi}{6}\}$ and

$$B = \left\{x \in \mathbb{R}^+ : \mu(A_x) \geq \rho_b\left(1 - \frac{1}{\sqrt{3}}\right)\right\} = \left\{x \in \mathbb{R}^+ : \mu(A_x) = \rho_b\left(1 - \frac{1}{\sqrt{3}}\right)\right\}.$$

A simple computation shows that $\|\bar{h}^*\| \geq \frac{\pi^2\rho_b^2}{36\sqrt{3}}(1 - 3^{-1/2})\mu(B)$, where $\|\cdot\|$ is the norm in $L^2_r((0, R)\times\mathbb{R}^+)$. Thanks to Lemma B.9 we deduce that $\mu(B) \leq Kb^2 M$ for some $K > 0$. On the other hand, if $x \notin B$ there exists $\rho(x) \in [\rho_b 3^{-1/2}, \rho_b]$ such that $|\bar{h}^*(\rho(x), x)| \leq \frac{\pi}{6}$, and therefore $|\vartheta^*(\rho(x), x)| \leq \pi/2$. Since $\vartheta^*(R, x) > \pi/2$, there exists $r(x) \in [0, R]$ such that $\vartheta^*(r(x), x) = \pi/2$. □

In the following we shall denote by X a fixed value greater than $C(M)$.

Proposition B.12. *There exists $\rho \in (0, R)$ such that for all $x \in [X, \infty)$ $\vartheta^*_\tau(\rho, x), \vartheta^*(\rho, x) \leq \pi/2$ if $\tau \geq X$.*

Proof: Since $X > C(M)$, there exists $\xi \in (0, X)$ such that $\xi \notin B$ and then $\vartheta^*(r(\xi), \xi) = \pi/2$ for some $r(\xi) \in (0, R)$. Since for every $\tau > 0$ one has $\vartheta^*_\tau(r, x) \leq \vartheta^*(r, x)$ for every $r \in (0, R)$ and $x \leq \tau$, must be $\vartheta^*_\tau(r(\xi), \xi) \leq \pi/2$ if $\tau \geq X$. If we define $\rho = r(\xi)$, then the thesis follows from the monotonicity of ϑ^* and ϑ^*_τ with respect to x. □

Proposition B.13. *Let $G_b = G_b(r)$ be the function given by (1.28). For every $M \geq x_1$,*

$$\begin{aligned}&\textstyle\int_0^M \mathrm{d}\,x \int_0^R r\left((\vartheta^*_r)^2 + \frac{\sin^2(\vartheta^*)}{r^2} - G_b(r)\right)\mathrm{d}\,r \leq \\ &\textstyle\int_0^\infty \mathrm{d}\,x \int_0^R r\left(\vartheta_r^2 + \frac{\sin^2(\vartheta)}{r^2} - G_b(r)\right)\mathrm{d}\,r < \infty.\end{aligned}$$

Proof: Since $h = T^{-1}(\vartheta - \theta_+)$ is a minimizer to problem (MP) (see chapter 1), it follows from (1.27) that the latter integral is finite. For any $\tau \geq X$ we set

$$q_\tau(x) \;=\; \int_0^R r\left((\vartheta^*_\tau)^2_r(r, x) + \frac{\sin^2(\vartheta^*_\tau(r, x))}{r^2} - G_b(r)\right)\mathrm{d}\,r \quad \text{for } x > 0.$$

Observe that q_τ is a measurable function with values in $\mathbb{R}\cup\{\infty\}$ and that, by Lemma A.1, Proposition B.12 and identity $\vartheta^*_\tau(R, x) \equiv 2\arctan(bR)$, $q_\tau(x) \geq 0$ for $x \geq X$. Similarly, the function

$$q(x) \;=\; \int_0^R r\left((\vartheta^*_r)^2(r, x) + \frac{\sin^2(\vartheta^*(r, x))}{r^2} - G_b(r)\right)\mathrm{d}\,r \quad \text{for } x > 0$$

is nonnegative for $x \in [X, \infty)$. By Proposition B.2 and by Fatou's lemma, $q(x) \leq \liminf_{\tau\to\infty} q_\tau(x)$ for all $x > 0$. In particular

$$\int_0^X q(x)\,\mathrm{d}\,x \leq \liminf_{\tau\to\infty} \int_0^X q_\tau(x)\ \mathrm{d}\,x.$$

Since $q, q_\tau \geq 0$ in $[X, \infty)$ if $\tau \geq X$, it follows again from Fatou's lemma that for all $M \geq X$

$$\int_X^M q(x)\,\mathrm{d}\,x \leq \int_X^\infty q(x)\,\mathrm{d}\,x \leq \liminf_{\tau\to\infty} \int_X^\tau q_\tau(x)\ \mathrm{d}\,x.$$

The proof is complete if we show that, for every $\tau > 0$,

$$\int_0^\tau \mathrm{d}\,x \int_0^R \frac{\sin^2(\vartheta_\tau^*)}{r}\,\mathrm{d}\,r \leq \int_0^\tau \mathrm{d}\,x \int_0^R \frac{\sin^2(\vartheta)}{r}\,\mathrm{d}\,r\,, \tag{B.19}$$

$$\int_0^\tau \mathrm{d}\,x \int_0^R r(\vartheta_\tau^*)_r^2\,\mathrm{d}\,r \leq \int_0^\tau \mathrm{d}\,x \int_0^R r\vartheta_r^2\,\mathrm{d}\,r\,. \tag{B.20}$$

Inequality (B.19) follows at once from (B.2), with $G \equiv 1$ and $F(r, f) = \sin^2(f)r^{-1}$, Proposition B.1 and Fatou's lemma. Applying (B.2) with $F = r$ and $G(v) = v^2$ we find that for all $0 < \sigma < \tau$

$$\int_0^R \int_\sigma^\tau r(\vartheta_\tau^{*\sigma})_r^2\,\mathrm{d}\,x\ \mathrm{d}\,r \leq \int_0^R \int_\sigma^\tau \vartheta_r^2\,\mathrm{d}\,x\ \mathrm{d}\,r. \tag{B.21}$$

Letting $\sigma \to \infty$ and arguing as in the proof of Proposition B.7 we easily obtain (B.20). $\square$

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