Giorgio Fabbri

# First order HJB equations in Hilbert spaces and applications

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# First order HJB equations in Hilbert spaces and applications

Giorgio Fabbri

Relatore: prof. Fausto Gozzi

Dipartimento di matematica - Istituto Guido Castelnuovo
Università degli studi di Roma - *La Sapienza*
**Dottorato in Matematica**

Only free software has been used to write this thesis

# Contents

# Introduction

This thesis is mainly devoted to develop the Dynamic Programming approach to study a class of optimal control problems in infinite dimension mainly motivated from economic applications (see Chapter 7 and 8). We will focus our attention in particular[1] into HJB related to optimal control problems governed by linear differential equation of the form

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t}(t) = Ax(t) + Bu(t) \\ x(t) = x \end{cases}$$

where $A$ is the generator of a $C_0$ semigroup of contractions and $B$ is an unbounded operator, that arises, for example, in the infinite dimensional formulation of PDE with boundary control or DDE with delay in the control. We assume to have a objective functional of the form

$$J(t, x, u(\cdot)) = \int_t^T L(s, x(s), u(s))\mathrm{d}s + h(x(T)).$$

The HJB related to the described optimal control problem is (formally)

$$-\partial_t w(t,x) - \langle Ax, \nabla w(t,x)\rangle - \inf_{u \in U}\Big( \langle Bu, \nabla w(t,x)\rangle + L(t,x,u)\Big) = 0$$

Such class of problems has not been extensively treated in the literature. The difficulties arise in particular from the unboundedness of the term $B$, the non-analyticity of the semigroup $A$ and the non-convexity of the functional $J(t, x, u(\cdot))$[2]. Other difficulties arise from the presence of state-control and state constraint, that we introduce in very specific case in Chapters 4, 6 and in the applied models presented in Chapters 7 and 8.

In the thesis we do not give a complete theory but various results on the subject that may contributes to have a better picture of the argument (a scheme of the contributions is given below). In particular we limit our study to the case of HJB related to optimal control problem governed by transport equation with boundary (and distributed) control (Chapter 3) and to HJB arising in optimal control problem governed by linear DDEs with delay in the control (Chapter 4, 6, 7, 8).

[1] Except for Chapter 5 and Appendix A.

[2] We use convexity assumption in Chapter 6.

As we said above the main motivation for studying such problems comes from economic applications, in particular from models of capital accumulation with heterogeneous capital[3], see Subsection 3.1.1, Section 7.2 and Section 8.1.

**Dynamic programming approach.** The Dynamic Programming (DP) approach to optimal control problems can be summarized in four main steps :

(i) letting the initial data vary, calling *value function* the infimum (or the supremum) of the objective functional and writing an equation whose candidate solution is the value function: the so-called DP Principle, together with its infinitesimal version, the HJB equation;

(ii) solving (whenever possible) the HJB equation to find the value function, if it is not possible at least give existence and uniqueness theorem;

(iii) proving that the present value of the optimal control strategy can be expressed as a function of the present value of the optimal state trajectory: a so-called closed loop (or feedback) relation for the optimal control;

(iv) studying the Closed Loop Equation, i.e. the state equation where the control is replaced by the closed loop relation: the solution is the optimal state trajectory and the optimal control strategy is consequently derived from the closed loop relation.

We can give a description of the contributions presented in this thesis through such scheme: In Chapters 3, 4, 6 we present three optimal control problems with the related HJB equations proving that the value function is a solution (unique in Chapter 3) of the related HJB equation, so we are on step (i) and (ii). In Chapter 5 and in Appendix A we present two verifications results that allows to use a solution of the HJB equation to find a optimal feedback (or to verify if a given pair is optimal). So we are on step (iii). In chapter 4 also a verification result is proved. Eventually in specific models treated in Chapter 7 and 8 we can write explicitly the HJB equation (step (i)), find an explicit solution (step (ii)) and find an explicit optimal feedback (step (iii)). The delay differential equation found replacing the control with the closed loop is not solvable (and so we cannot find explicitly the optimal trajectory) but we can anyway study existence and uniqueness of the solution and study some properties and the behaviour of such solution (step (iv)).

## The plan of the thesis

We divide the thesis in three parts in which we use different approaches for the study of the HJB: in first part we use the tools of the viscosity solution approach to study some particular cases, in the second part we refer, for a

[3]Such models are used to study for example depreciation and obsolescence of physical capital, geographical difference in growth, innovation and R&D.

particular problem, to strong solutions method and in third part we examine two specific applied cases in which it is possible to find an explicit solution of the HJB.

**Part I: Viscosity solution approach**:

**Chapter 2** is devoted to a brief description of existing literature on viscosity solution for first order HJB in Hilbert spaces, underlining in particular the relations between optimal control problems in Hilbert spaces and solutions of HJB equations. Differently from the finite dimensional case, where a certain number of review and books on the theme are available, in the infinite dimensional case there are not exhaustive summaries about viscosity solutions approach for HJ in literature.

In **Chapter 3** we describe the results obtained in [Faba] related to viscosity solution approach for HJB equation related to optimal control problems governed by a family of linear transport equations with boundary (and distributed) control of the form

$$\begin{cases} \frac{\partial}{\partial s}x(s,r)+\beta\frac{\partial}{\partial r}x(s,r)=-\mu x(s,r)+\tilde{u}(s,r) \quad (s,r)\in(0,+\infty)\times(0,\bar{s}) \\ x(s,0)=u(s) \quad\quad if \;\; s>0 \\ x(0,r)=x^0(r) \quad\quad if \;\; r\in[0,\bar{s}] \end{cases}$$

The state equation in $L^2(0,\bar{s})$ appears as

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s}x(s)=Ax(s)-\mu x(s)+\tilde{u}(s)+\beta\delta_0 u(s) \\ x(0)=x^0 \end{cases}$$

where the unbounded term $\beta\delta_0 u(s)$ arises from the boundary control $u(s)$.

Our main problem is to write a suitable definition of viscosity solution, so that an existence and uniqueness theorem can be derived for such a solution. The main difficulties we encounter, with respect to the existing literature, is in dealing with the boundary term and the non-analyticity of the semigroup. We substantially follow the original idea of Crandall and Lions ([CL90] and [CL91]) - with some changes, as the reader will rate in Definition 3.15 and Definition 3.16 - of writing test functions as the sum of a "good part" as it is a regular function with differential in $D(A^*)$ and a "bad part" represented by some radial function. The main problems arise in the evaluation of the boundary term on the radial part.

In order to write a working definition in our case, some further requirements are needed, like a $C^2$ regularity of the test functions, the presence of a "remainder term" in the definition of sub/super solution and the $P$-Lipschitz continuity (see Definition 3.10) of the solution. This last feature guarantees that the maxima and the minima in the definition of sub/super solution remain in $D(A^*)$ (see Proposition 3.23).

In **Chapter 4** we present the results obtained in the work [Fab06]. Here we consider a class of optimal control problems with state-control constraints, where the state equation is a linear delay differential equation with delay in the control.

The main results are that the value function is a viscosity solution of the associated HJB equation and a verification theorem.

Due to the difficulties of the problem the result that we find in the chapter are not very strong and may be considered as a first step of the study of such problem. Indeed we use a quite small set of test functions and we can only prove an existence result without any comparison statement. The DDE we consider in the chapter is a a general homogeneous linear DDE of the following form:

$$\begin{cases} \dot\theta(s) = N(\theta_s) + B(u_s) \quad for\ s \in [t,T] \\ (\theta(t), \theta_t, u_t) = (\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2(-R,0) \times L^2(-R,0) \end{cases}$$

in which $\theta_s, u_s$ are the histor of the state and of the control (the notation $_s$ is introduced in Section 1.3) and $N$ and $B$ are linear continuous applications $N, B\colon C([-R,0];\mathbb{R}) \to \mathbb{R}$.

The presence of the delay in the control yields an unbounded term in the HJB equation. Moreover in the state equation as reformulated in $M^2$ a non-analytic semigroup appears.

We consider two cases: a first case in which we use only state-control constraint (that generalize the constraints required in the applied examples), in which we prove an existence result and a verification theorem and a second case in which we introduce also a (simple) state constraint. In such case the continuity of the value function is not guaranteed but an existence result is anyway proved. We present three main examples that show why we study the problem.

The results of **Chapter 5** are from [FGŚ]. We present two results: a verification theorem within the framework of viscosity solution and a method to construct $\varepsilon$-optimal piecewise constant controls. As we stressed above a verification results represents a key step in the dynamic programming approach to optimal control problems. A verification theorem is a tool to check whether a given admissible control is optimal and, more importantly, suggests a way of constructing optimal feedback control.

As observed in Chapter 2 different works treat viscosity solutions in Hilbert spaces using different approaches and definitions. The verification results depend on the approach and on the definition of solution we use. We prove a verification results for the approach of Crandall and Lions. The main difficulty we have to deal with is the fact that in the infinite dimensional setting not all regular functions that "touch" the candidate-solution of the HJB equation are test functions but only particular ones. The test functions that are considered by Crandall and Lions are sum of two parts: one regular and compatible with the generator of semigroup that appears in the state equation of the system ("test1"), and one radial ("test2"). The differentials of such functions do not span all the super (or sub) differential of the candidate-solution so we cannot reformulate the definition in terms of super(sub)differentials as in finite dimensional case. The two families *test*1 and $test2$ have different role in the definition of super/sub-solution (see Definitions 5.5) and so they have to be treated in different way when we prove the verification theorem.

In order to construct the $\epsilon$-optimal controls we have first to approximate the viscosity sub and supersolution of the HJB using sub and superconvolution and then prove that the approximating functions solve (in viscosity sense) suitable approximating HJB equations.

Under suitable hypotheses we prove that the subsolutions (resp. supersolutions) of the HJB satisfy the suboptimal (resp. super-optimal) principle

$$w(t,x) \leq \inf_{w(\cdot)\in\mathcal{U}[t,T]} \left\{ \int_t^{t+h} L(s,x(s),u(s))\mathrm{d}s + w(t+h,x(t+h)) \right\}.$$

(resp.

$$w(t,x) \geq \inf_{u(\cdot)\in\mathcal{U}[t,T]} \left\{ \int_t^{t+h} L(s,x(s),u(s))\mathrm{d}s + w(t+h,x(t+h)) \right\} \quad )$$

**Part II: Strong solutions approach**:

The second part of the thesis (**Chapter 6**) is devoted to strong solution approach to first order HJB equations in Hilbert spaces. The first two sections (Section 6.1 and Section 6.2) are devoted to the description of the existing literature and in particular to the introduction of the notions of strong and weak solution of the HJB when the state equation has an unbounded linear term.

In the other sections the results obtained in [FGF] are presented. They are very preliminary and a more in-depth studies is needed in the future, but, as we have already stressed, we have decided to devote them a whole Part of this thesis to give to the reader a more complete image of the techniques used to study first order HJB equation in Hilbert spaces. The concept of ultra-weak solution as limit of weak solutions is introduced. We will proceed first showing some motivating examples and then proving that in such examples the value function is an ultra-weak solution of the HJB.

**Part III: Special applied cases**:

In **Chapter 7** we present the results of [FGa] where the explicit solution of a first order HJB equation in the Hilbert space $M^2$ is used to study a vintage capital model.

We denote by $k(t)$ the stock of capital at time $t$; $i(t)$ and $c(t)$ are the investment and the consumption at time $t$. All of them are nonnegative. The aggregate production at time $t$ is denoted by $y(t)$ and it satisfies, for

$t \geq 0$

$$y(t) = a \int_{t-R}^{t} i(s)\mathrm{d}s \qquad a > 0.$$

We have the following accounting relation, for $t \geq 0$

$$ak(t) = y(t) = i(t) + c(t)$$

so the non-negativity of all variables is equivalent to ask that, for $t \geq 0$

$$i(t), c(t) \in [0, y(t)] = [0, ak(t)].$$

The equilibrium is the solution of the problem of maximizing, over all investment-consumption strategies that satisfy the above constraints (7.1), (7.2), (7.3), the functional of CRRA (Constant Relative Risk Aversion) type

$$\int_0^{+\infty} e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} \mathrm{d}t$$

where $\rho > 0$, $\sigma > 0$ (and $\sigma \neq 1$).

Standard techniques are used to rewrite the problem in the Hilbert space $M^2 = \mathbb{R} \times L^2(-R, 0)$. The associated HJB equation cannot be treated with the results of the existing literature. This is due to the presence of the state/control constraint (i.e. $i(t) \in [0, ak(t)]$), to the unboundedness of the control operator and the non-analyticity of the semigroup generated by the operator $A$. To overcome these difficulties we give a suitable definition of solution, find an explicit solution of the HJB and use it to give an explicit form for the optimal feedback. This allows to determine an equation for the optimal trajectories of the capital stock and of the investment. Long run equilibrium of the discounted paths is also explicitly given.

In **Chapter 8** we present a contribution based a model presented by Boucekkine DelRio and Martinez [BdRM]. We use the explicit solution of a first order HJB equation in $M^2$ is used study a model for obsolescence and physical depreciation. The main ideas are similar to those we have seen in Chapter 7 (a class of similar problems is studied in [Fab06]) but the different explicit form of the HJB equation and of the the state equation needs to adapt the proofs for the new case.

**A note on the organization of the work**:

A Ph.D. thesis has two different purposes: the first is to produce a presentation of a research problem, the second is to explain how the research of the candidate has developed during the Ph.D. period. These two different requirements suggest two different approaches and it is not always easy satisfy both. We will try to describe the main motivations of the problem and to give, briefly, a description of existing literature and, at the same time, we will try to focus our attention on the new contributions. The result is a mix that sometimes try to be a presentation of the the research argument and sometimes favors the report function of the thesis. For example we have chosen to dedicate a whole part of the thesis (part 2) to strong solutions approach that is a key-argument in the literature on HJB equations in Hilbert spaces but is not central in the studies of the candidate and for the new contributions presented. With the same aim we have written a section in which we describe the problem in the regular case (Section 1.2) and where we recall some existing results (Section 1.3) to explain which is the background of the research. On the other hand the chapters that present new contributions (Chapter 3, 4, 5, 6, 7, 8) are almost self-contained and maintain the structure of original works[4] to give a clearer presentation of research activity. Sometimes this choice produce some repetitions in the text.

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Thanks to Silvia Faggian for the many invaluable advices (in particular for Chapters 3, 4 and 6), for her patience and for the so big amount of time she spent helping me.

[4] Except for Chapter 6

# Notation

| | |
|---|---|
| $X$ | real separable Hilbert space |
| $\Omega$ | open or closed subset of $X$ |
| $Y$ | real Banach space |
| $\langle \cdot, \cdot \rangle$ | scalar product on $X$ |
| $B_R(\bar{x})$ | $\{x \in X \ : \ \lvert x - \bar{x} \rvert \leq R\}$ |
| $B_R$ | $\{x \in X \ : \ \lvert x \rvert \leq R\}$ |
| $O$ | an open set in $\mathbb{R}^n$ |
| $L^p(O;Y)$ | the set of $L^p$ functions $: O \to Y$ |
| $L^p(O)$ | the set $L^p(O;\mathbb{R})$ |
| $L^p(a,b)$ | $L^p((a,b);\mathbb{R})$ |
| $L^p_{loc}([a,+\infty);Y)$ | the set $\left\{f \colon [a,+\infty) \to Y \ : \ f_{\lvert (a,b)} \in L^p((a,b);Y) \ \forall b > a\right\}$ |
| $L^p_{loc}[a,+\infty)$ | $L^p_{loc}([a,+\infty);\mathbb{R})$ |
| $W^{s,p}(O;Y)$ | with $s \in \mathbb{N}$ and $p \geq 1$ the real Sobolev space given by $\{f \in L^p(O;Y) \ : \ \forall \lvert\alpha\rvert \leq s \partial_\alpha f \in L^p(O;Y)\}$ |
| $W^{s,p}(O)$ | $W^{s,p}(O;\mathbb{R})$ |
| $H^s(O;Y)$ | the real Sobolev space of real index $s \in \mathbb{R}$ |
| $H^s_{loc}([a,+\infty),Y)$ | (for $s \geq 0$) the set $\left\{f \colon [a,+\infty) \to Y \ : \ f_{\lvert (a,b)} \in H^s((a,b);Y) \ \forall b > a\right\}$ |
| $H^s(a,b)$ | $H^s((a,b);\mathbb{R})$ |
| $H^s_{loc}[a,+\infty)$ | $H^s_{loc}([a,+\infty),\mathbb{R})$ |
| $C(\Omega;Y)$ | the set of continuous functions $: \Omega \to Y$ |
| $C(\Omega)$ | the set of continuous functions $: \Omega \to \mathbb{R}$ (that is $C(\Omega;\mathbb{R})$) |
| $C^{0,s}(\Omega;Y)$ | the set of continuous $s$-Hoelder functions ($s > 0$) |
| $C^{0,s}(\Omega)$ | $C^{0,s}(\Omega;\mathbb{R})$ |
| $C^1(\Omega;Y)$ | ($\Omega \subseteq X$ open) the set of continuous and continuously differentiable function $: \Omega \to Y$ |
| $C^1(\Omega)$ | ($\Omega \subseteq X$ open) $C(\Omega;\mathbb{R})$ |
| $C_c(\Omega;Y)$ | the set $\{f \in C(\Omega;Y) \ : \ supp(f) \Subset \Omega\}$ |
| $UC(\Omega;Y)$ | the set of uniformly continuous function $: \Omega \to Y$ |
| $UC(\Omega)$ | the set of uniformly continuous function $: \Omega \to \mathbb{R}$ |
| $BUC(\Omega)$ | the set of bounded uniformly continuous function $: \Omega \to \mathbb{R}$ |
| $C_\omega(\Omega)$ | the set of weakly continuous function $: \Omega \to \mathbb{R}$ |
| $LSC(\Omega)$ | the set of lower semicontinuous function $: \Omega \to \mathbb{R}$ |
| $USC(\Omega)$ | the set of upper semicontinuous function $: \Omega \to \mathbb{R}$ |

| | |
|---|---|
| $Lip(\Omega;Y)$ | the set of Lipschitz functions endowed with the natural seminorm that is $\left\{ f:\Omega\to Y \ : \ [f]_L := \sup_{x,y\in\Omega,\ x\neq y} \frac{\lvert f(x)-f(y)\rvert_Y}{\lvert x-y\rvert_X} < +\infty \right\}$ |
| $Lip(\Omega)$ | $Lip(\Omega;\mathbb{R})$ |
| $C^1_{Lip}(\Omega)$ | ($\Omega\subseteq X$ open) the set $\left\{ f\in C^1(\Omega) \ : \ [f']_L < +\infty \right\}$ |
| $C_p(\Omega;Y)$ | the set $\left\{ f:\Omega\to Y \ : \ \lvert f\rvert_{C_p} := \sup_{x\in\Omega} \frac{\lvert f(x)\rvert_Y}{1+\lvert x\rvert_X^p} < +\infty \right\}$ |
| $C_p(\Omega)$ | the set $\left\{ f:\Omega\to \mathbb{R} \ : \ \lvert f\rvert_{C_p} := \sup_{x\in\Omega} \frac{\lvert f(x)\rvert}{1+\lvert x\rvert_X^p} < +\infty \right\}$ that is $C_p(\Omega;\mathbb{R})$ |
| $\Sigma_0(\Omega)$ | $\left\{ w\in C_2(\Omega) \ : \ w \text{ is convex, } w\in C^1_{Lip}(\Omega) \right\}$ |
| $\mathcal{Y}([0,T]\times\Omega)$ | $\Big\{ w:[0,T]\times\Omega\to\mathbb{R} \ : \ w\in C([0,T];C_2(\Omega)),$ $w(t,\cdot)\in\Sigma_0(\Omega),\ \nabla w\in C([0,T];C_1(\Omega;X'))\Big\}$ |
| $D^+u(x)$ | the superdifferential of the continuous function $u\colon\Omega\to\mathbb{R}$ at $x$ that is $D^+u(x) \overset{def}{=} \left\{ p\in X \ : \ \overline{\lim}_{\substack{y\in\Omega\\ y\to x}} \left( \frac{u(x)-u(y)-\langle p, y-x\rangle}{\lvert y-x\rvert} \right) \leq 0 \right\}$ |
| $D^-u(x)$ | the subdifferential of the continuous function $u\colon\Omega\to\mathbb{R}$ at $x$ that is $D^-u(x) \overset{def}{=} \left\{ p\in X \ : \ \overline{\lim}_{\substack{y\in\Omega\\ y\to x}} \left( \frac{u(x)-u(y)-\langle p, y-x\rangle}{\lvert y-x\rvert} \right) \geq 0 \right\}$ |
| $u^*(\cdot)$ | the upper semicontinuous envelopes of a function $u\colon\Omega\to\mathbb{R}$ at that is $u^*(x) \overset{def}{=} \limsup_{r\downarrow 0} \{u(y) \ : \ y\in\Omega \ \lvert y-x\rvert\leq r\}$ |
| $u_*(\cdot)$ | the lower semicontinuous envelopes of a function $u\colon\Omega\to\mathbb{R}$ at that is $u_*(x) \overset{def}{=} \liminf_{r\downarrow 0} \{u(y) \ : \ y\in\Omega \ \lvert y-x\rvert\leq r\}$ |
| $\sigma(\cdot)$ | a modulus of continuity (page 17) |
| $\sigma(\cdot,\cdot)$ | a local modulus of continuity (page 17) |
| $A$ | the generator of $C_0$ semigroup on $X$ |
| $e^{tA}$ or $T(t)$ | the semigroup generated by $A$ |

CHAPTER 1

# First order HJB equations in Hilbert spaces and optimal control problems

In this chapter we collect different "utilities" that we will use in the other parts of the thesis: we first present the notation (Section 1.1), then we describe the dynamic programming approach to optimal control problem , seeing how it works when the HJB equation admits a classical solution (Section 1.2), and eventually we describe some mathematical tools that we will use in the sequel (Section 1.3). The description of the regular case contained in Section 1.2 is useful to make the thesis self-contained and to explain the background and the motivations of our studies. It can be skipped by the reader that already knows the method.

## 1.1. Definitions: the optimal control problem and the HJB equation

This section is devoted to the description of the notation that will be used. The hypotheses on the object we consider (like the functions $w(\cdot)$, $f(\cdot,\cdot)$, $L(\cdot,\cdot)$, the generator $A$, the space $U$...) will be specified from time to time during the exposition.

We describe first the problem characterized by homogeneous state equation and infinite horizon functional: we consider a state equation in the Hilbert space $X$ of the form

$$\begin{cases} \dot{x}(s) = Ax(s) + f(x(s), u(s)) \\ x(0) = x \end{cases} \tag{1.1}$$

where $A$ is the generator of a $C_0$-semigroup, $U$ is a metric space and $f\colon X \times U \to X$.

Remark 1.1. *We will always work using hypotheses that guarantee existence and uniqueness of the solution. For example $f\colon [0,+\infty) \times X \times U \to X$ continuous and Lipschitz continuous in $x$ uniformly in $(t,u) \in [0,T] \times U$.*

Remark 1.2. *The solution $x(s)$ depends on the initial data and on the control, so we should write $x_{u(\cdot),x}(\cdot)$. We will use simply the form $x(\cdot)$ when there are not possibilities of misunderstandings.*

We want to minimize the infinite horizon functional

$$J(x, u(\cdot)) = \int_0^\infty e^{-\rho s} L(x_{u(\cdot),x}(s), u(s)) \mathrm{d}s \tag{1.2}$$

among the controls of the set

$$\mathcal{U} = \{u\colon [0,+\infty) \to U \ :\ u \text{ locally integrable}\} \tag{1.3}$$

(where $U$ is a metric space). We ask that $L$ does not depend (directly) on the time so the Hamiltonians will not depend on the time.

Notation 1.3. *We call admissible pair a couple $(x_{u(\cdot),x}(\cdot), u(\cdot))$ where $u(\cdot) \in \mathcal{U}$. When there are not possibilities of misunderstandings we will write simply $(x(\cdot), u(\cdot))$*

The current value Hamiltonian of the system is

$$H_{CV}(x,p,u) = \langle f(x,u), p\rangle + L(x,u) \tag{1.4}$$

The maximal value Hamiltonian (or simply Hamiltonian) is

$$H(x,p) = \inf_{u\in U} \left(\langle f(x,u), p\rangle + L(x,u)\right) \tag{1.5}$$

The HJB equation is defined as

$$\rho w(x) - \langle Ax, \nabla w(x)\rangle - H(x, \nabla w(x)) = 0, \tag{1.6}$$

and the value function of the problem is

$$V(x) = \inf_{u(\cdot)\in\mathcal{U}} \left(J(x,u(\cdot))\right) \tag{1.7}$$

REMARK 1.4. *We will work under assumptions that guarantee that $|V(x)| < \infty$ for every $x \in X$. For example in Chapter 4 we will assume $L$ bounded and in Chapter 7 we will assume $\rho > r$ where $r$ is the maximal growth of $|x(\cdot)|$.*

NOTATION 1.5. *An admissible pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ will be called optimal pair if*

$$V(x) = J(x, \bar{u}(\cdot))$$

The finite horizon problem is characterized by a state equation of the form

$$\begin{cases} \dot{x}(s) = Ax(s) + f(s, x(s), u(s)) \\ x(t) = x \end{cases} . \tag{1.8}$$

NOTATION 1.6. *We will denote by $x_{u(\cdot),t,x}(\cdot)$ the solution of (1.8) when we want to emphasize the initial data $(t,x)$ and the control $u(\cdot)$.*

We consider a target functional (with finite horizon)

$$J(t,x,u(\cdot)) = \int_t^T L(s, x_{u(\cdot),x}(s), u(s))\mathrm{d}s + h(x(T)) \tag{1.9}$$

and a set of controls

$$\mathcal{U}_t = L^1(t,T,U). \tag{1.10}$$

The current value Hamiltonian of the problem is

$$H_{CV}(t,x,p,u) = \langle f(t,x,u), p\rangle + L(t,x,u) \tag{1.11}$$

and the (maximal value) Hamiltonian is

$$H(t,x,p) = \inf_{u\in U} \left(\langle f(t,x,u), p\rangle + L(t,x,u)\right). \tag{1.12}$$

The HJB equation of the system is defined as

$$\begin{cases} -\partial_t w(t,x) - \langle Ax, \nabla w(t,x)\rangle - H(t,x,\nabla w(t,x)) = 0 \\ w(T,x) = h(x) \end{cases} \tag{1.13}$$

and the value function is

$$V(t,x) = \inf_{u(\cdot)\in\mathcal{U}_t} \left(J(t,x,u(\cdot))\right) \tag{1.14}$$

REMARK 1.7. *We will work under assumptions that guarantee the uniqueness of the solution of the (1.8) and that $V(t,x)| < \infty$ for all $(t,x) \in [0,T]\times X$.*

NOTATION 1.8. *We call admissible pair at $(t,x)$ a couple $(x_{u(\cdot),t,x}(\cdot), u(\cdot))$ where $u(\cdot) \in \mathcal{U}_t$. When there are not possibilities of misunderstandings we will write simply $(x(\cdot), u(\cdot))$. An admissible pair at $(t,x)$ we be said an optimal pair at $(t,x)$ if*

$$V(t,x) = J(t,x,u(\cdot))$$

NOTATION 1.9. *In the following we will refer to equations (1.6) and (1.13) both in the cases in which an optimal control problems like (1.1, 1.2) and (1.8, 1.9) are involved or not. In other words we will refer to equations (1.6) and (1.13) also when the function $H$ is not of the forms (1.5, 1.12).*

Eventually we give the definition of the terms "modulus" and "local modulus" that we will often use in the thesis:

DEFINITION 1.10 (**Modulus, from** [CL85]). *A function $\sigma(\cdot)\colon [0,+\infty) \to [0,+\infty)$ will be called a* modulus *if it is continuous, nondecreasing, nonnegative, sub-additive and satisfies $\sigma(0) = 0$*

DEFINITION 1.11 (**Local modulus, from** [CL85]). *A function $\sigma(\cdot,\cdot)\colon [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ will be called a* local modulus *if $r \mapsto \sigma(r,R)$ is a modulus for each $R \geq 0$ and $\sigma(\cdot,\cdot)$ is continuous and non-decreasing in both variables.*

## 1.2. The dynamic programming in the regular case

In this section we will describe the dynamic programming approach in the regular case. So we will describe how solving the HJB equation of the problem can be useful to find an optimal control for the optimization problem. All the results of this section are well known in literature and so it can be skipped. We wrote it to introduce the background in which our researches live. The section can be skipped We refer to the finite horizon problem described in Section 1.1. We will state and prove the dynamic programming principle ("Bellman optimality principle" , Theorem 1.13), we will see that the value function solves the HJB equation (Theorem 1.17 and 1.19 and that such fact can be used to find an optimal control in feedback form (a "verification theorem" , Theorem 1.20). So we develop the steps of the dynamic programming presented in the introduction in the regular case: writing and solving the HJB equation and using the solution to obtain an optimal control.

HYPOTHESIS 1.12. *We will assume that $f, L$ and $h$ are continuous and that $f\colon [0,T] \times X \times U \to X$ is Lipschitz continuous in $x$ uniformly in $(t,u) \in [0,T] \times U$.*

THEOREM 1.13. *If the value function is everywhere finite (so that $V\colon [t,T] \times X \to \mathbb{R}$) then, for every $s \in [t,T]$ it solves the following integral equation:*

$$V(t,x) = \inf_{u(\cdot)\in\mathcal{U}_t} \left\{ \int_t^s L(r,x(r),u(r))\mathrm{d}r + V(s,x(s)) \right\}$$

PROOF. We will show the two inequalities:
($\leq$): We take $u(\cdot) \in \mathcal{U}_t$. If $u_1 \in \mathcal{U}_s$ then the function

$$u_{u_1}(r) = \begin{cases} u(r) & if\ r \in [t,s] \\ u_1(r) & if\ r \in [s,T] \end{cases}$$

is in $\mathcal{U}_t$ For the definition of $V(t,x)$ we have

$$V(t,x) \leq \int_t^s L(r,x(r),u_{u_1}(r))\mathrm{d}r + \int_s^T L(r,x(r),u_{u_1}(r))\mathrm{d}r + h(x(T))$$

taking the infimum on $u_1(\cdot)$ we find

$$V(t,x) \leq \int_t^s L(r,x(r),u(r))\mathrm{d}r + V(s,x(s))$$

and taking the infimum on $u(\cdot)$ we find:

$$V(t,x) \leq \inf_{u(\cdot)\in\mathcal{U}_t} \left\{ \int_t^s L(r,x(r),u(r))\mathrm{d}r + V(s,x(s)) \right\}$$

($\geq$): We take $u_2(\cdot) \in \mathcal{U}_t$. We have

(1.15)
$$\int_t^T L(r,x(r),u_2(r))\mathrm{d}r + h(x(T)) \geq \inf_{u(\cdot)\in\mathcal{U}_t} \left\{ \int_t^T L(r,x(r),u(r))\mathrm{d}r + h(x(T)) \right\} \geq$$
$$\geq \inf_{u(\cdot)\in\mathcal{U}_t} \left\{ \int_t^s L(r,x(r),u(r))\mathrm{d}r + \inf_{u_1(\cdot)\in\mathcal{U}_s} \left\{ \int_s^T L(r,x(r),u_{u_1}(r))\mathrm{d}r + h(x(T)) \right\} \right\} =$$
$$= \inf_{u(\cdot)\in\mathcal{U}_t} \left\{ \int_t^s L(r,x(r),u(r))\mathrm{d}r + V(s,x(s)) \right\}$$

taking the infimum on $u_2$ we have

$$V(t,x) \geq \inf_{u(\cdot)\in\mathcal{U}_t} \left\{ \int_t^s L(r,x(r),u(r))\mathrm{d}r + V(s,x(s)) \right\}$$

□

DEFINITION 1.14 (**Classical solution**). *We will say that* $w\colon [0,T]\times X \to \mathbb{R}$ *is a classical solution of the HJB equation if it is in* $C^1([t,T]\times X)$ *and*

$$\begin{cases} -\partial_t w(t,x) - \langle Ax, \nabla w(x)\rangle - H(t,x,\nabla w(t,x)) = 0 & \text{in } [0,T]\times D(A) \\ w(T,x) = h(x) \end{cases}$$

We recall now a standard lemma that we will often use in the sequel:

LEMMA 1.15. *Given a generator* $A$ *of a* $C_0$ *semigroup* $T(s)$ *on* $X$ *we can extend* $T(s)$ *to a* $C_0$ *semigroup* $T^{(E)}(s)$ *on the space* $D(A^*)'$ *defining for all* $f \in D(A^*)'$ $T^{(E)}(s)f \in D(A^*)'$ *the element such that*

$$\left\langle T^{(E)}(s)f, v \right\rangle_{D(A^*)'\times D(A)} = \left\langle f, e^{sA^*}v \right\rangle_{D(A^*)'\times D(A)} \quad \text{for all } x \in D(A^*).$$

*If we call* $A^{(E)}$ *the generator of* $T^{(E)}(s) = e^{sA^{(E)}}$ *we have that* $A^{(E)}$ *extends* $A$ *(that is* $Ax = A^{(E)}x$ *for all* $x \in D(A)$*) and* $X \subseteq D(A^{(E)})$.

PROOF. See [Fag02] Proposition 4.5 page 59. □

NOTATION 1.16. *In the sequel we will use the notation* $A^{(E)}$ *to denote the extension described in Lemma 1.15. Sometimes, if the context is not ambiguous, we will abuse of the notation and we will call* $A$ *the extended generator.*

THEOREM 1.17. *Assume that the value function* $V\colon [t,T]\times X \to \mathbb{R}$ *is in* $C^1([t,T]\times X)$. *Then it is a classical solution of the HJB equation.*

PROOF. (This proof is from [LY95]: Proposition 1.2 page 225) By definition $V(T,x) = h(x)$. Let us fix $u \in U$ and $x \in D(A)$. By Theorem 1.13 we have that for $s > t$

$$(1.16) \quad 0 \leq V(s,x(s)) - V(t,x) + \int_t^s L(r,x(r),u)\mathrm{d}r =$$
$$= V_t(t,x)(s-t) + \langle \nabla V(t,x), x(s)-x\rangle + \int_t^s L(r,x(r),u)\mathrm{d}r + o(|s-t|+|x(s)-x|)$$

Since $x \in D(A)$ we have

(1.17)
$$\frac{1}{s-t}(x(s)-x) = \frac{1}{s-t}(e^{A(s-t)}-I)x + \frac{1}{s-t}\int_t^s e^{A(s-t)} f(r,x(r),u)\mathrm{d}r \xrightarrow{s\downarrow t}$$
$$\xrightarrow{s\downarrow t} Ax + f(t,x,u)$$

Hence, dividing by $(s-t)$ in (1.16) and sending $s \downarrow t$ we obtain

$$-V_t(t,x) - \langle \nabla V(t,x), Ax + f(t,x,u)\rangle - L(t,x,u) \leq 0 \quad \forall u \in U$$

Taking the infimum in $u$ we find

$$-V_t(t,x) - \langle \nabla V(t,x), Ax\rangle - H(t,x,\nabla V(t,x)) \leq 0.$$

On the other hand, let $x \in D(A)$ be fixed. For any $\varepsilon > 0$ and $s > t$ by Theorem 1.13 there exist a $u(\cdot) \in \mathcal{U}_t$ such that

$$\begin{aligned}
\varepsilon(s-t) \geq V(s,x(s)) - V(t,x) + \int_t^s L(r,x(r)u(r))\mathrm{d}r = \\
= V_t(t,x)(s-t) + \left\langle \nabla V(t,x), \left(e^{A(s-t)} - I\right)x\right\rangle + \\
\left\langle \nabla V(t,x), \int_t^s e^{A(s-r)} f(r,x(r),u(r))\mathrm{d}r\right\rangle + \int_t^s L(r,x(r),u(r))\mathrm{d}r + o(|s-t|) = \\
= V_t(t,x)(s-t) + \left\langle \nabla V(t,x), \left(e^{A(s-t)} - I\right)x\right\rangle + \\
+ \int_t^s \langle \nabla V(t,x), f(r,x,u(r))\rangle + L(r,x,u(r))\mathrm{d}r + \\
\left(\int_t^s \langle \nabla V(t,x), f(r,x(r),u(r)) - f(r,x,u(r))\rangle \,\mathrm{d}r + \right.\\
\left. + \int_t^s L(r,x(r),u(r)) - L(r,x,u(r))\mathrm{d}r\right) + o(|s-t|) \geq \\
\geq V_t(t,x)(s-t) + \left\langle \nabla V(t,x), \left(e^{A(s-t)} - I\right)x\right\rangle + H(t,x,\nabla V(t,x)) + o(|s-t|)
\end{aligned} \tag{1.18}$$

and dividing by $(s-t)$ and letting $(s-t) \to 0$ we obtain

$$-V_t(t,x) - \langle \nabla V(t,x), Ax\rangle - H(t,x,\nabla V(t,x)) \geq -\varepsilon$$

and so we have the thesis. □

Another definition, in which we check the solution in all the points of $[0,T] \times X$, is possible if we ask that $\nabla w \in C([0,T] \times X; D(A^*))$:

Definition 1.18 (**Strict solution**). *We will say that $w\colon [0,T] \times X \to \mathbb{R}$ is a strict solution of the HJB equation if it is in $C^1([t,T] \times X)$, $\nabla w \in C([0,T] \times X; D(A^*))$ and*

$$\begin{cases} -\partial_t w(t,x) - \langle x, A^* \nabla w(x)\rangle - H(t,x,\nabla w(t,x)) = 0 & \text{in } [0,T] \times X \\ w(T,x) = h(x) \end{cases}$$

We could prove with the same techniques used in the proof of Theorem 1.17 the following

Theorem 1.19. *Assume that the value function $V\colon [t,T] \times X \to \mathbb{R}$ is in $C^1([t,T] \times X)$ and that $V \in C([0,T] \times X; D(A^*))$. Then it is a strict solution of the HJB equation.*

Eventually using the solution of the HJB equation we can state a verification result:

Theorem 1.20. *If $v \in C^1([0,T] \times X)$ is a strict solution of the HJB equation then $v(t,x) \leq V(t,x)$ for every $(t,x) \in [0,T] \times X$. Moreover if we have an admissible pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ such that*

$$\bar{u}(s) \in argmin_{u \in U} H_{CV}(s, \bar{x}(s), \nabla v(s, \bar{x}(s)), u) \qquad \text{a.e. in } [t,T] \tag{1.19}$$

*Then the couple $(\bar{x}(\cdot), \bar{u}(\cdot))$ is optimal at $(t,x)$.*

PROOF. We take a generic couple $(x(\cdot), u(\cdot))$ that solves (1.8), since $\nabla v \in C([0,T] \times X; D(A^*))$ we can state that

$$\frac{\mathrm{d}}{\mathrm{d}s} v(s, x(s)) = v_t(s, x(s)) + \langle A^* \nabla v(s, x(s)), x(s) \rangle + \\ + \langle \nabla v(s, x(s)), +f(s, x(s), u(s)) \rangle \quad a.e.\ in\ [t, T] \tag{1.20}$$

so we have that

$$v(t,x) - J(x, u(\cdot)) = v(T, x(T)) - \int_t^T v_t(s, x(s)) + \\ + \langle A^* \nabla v(s, x(s)), x(s) \rangle + \langle \nabla v(s, x(s)), +f(s, x(s), u(s)) \rangle - \\ - h(T, x(T)) - \int_t^T L(s, x(s), u(s)) \mathrm{d}s = \tag{1.21}$$

since $v(T, x(T)) = h(T, x(T))$:

$$v(t,x) - J(x, u(\cdot)) = - \int_t^T v_t(s, x(s)) + \\ \langle A^* \nabla v(s, x(s)), x(s) \rangle + \langle \nabla v(s, x(s)), +f(s, x(s), u(s)) \rangle + L(s, x(s), u(s)) \mathrm{d}s = \tag{1.22}$$

using that $v$ is a solution of the HJB equation

$$v(t,x) - J(x, u(\cdot)) = \int_t^T H(s, x(s), \nabla v(s, x(s))) - \\ - \langle \nabla v(s, x(s)), f(s, x(s), u(s)) \rangle - L(s, x(s), u(s)) \mathrm{d}s = \\ \int_t^T H(s, x(s), \nabla v(s, x(s))) - H_{CV}(s, x(s), \nabla v(s, x(s)), u(s)) \mathrm{d}s \leq 0 \tag{1.23}$$

Taking the infimum over the set of admissible controls in identity (1.23) we see that $v(t,x) \leq V(t,x)$ for all $(t,x) \in [0,T] \times X$. Moreover, since the minimization of $J(t, x, u(\cdot))$ over $u(\cdot)$ is equivalent to the maximization of $v(t,x) - J(t, x, u(\cdot))$ over $u(\cdot)$ if we have a control such that $v(t,x) - J(x, u(\cdot)) = 0$ then it is an optimal control. $(\bar{x}(\cdot), \bar{u}(\cdot))$ satisfies by hypothesis such condition and so it is optimal. $\square$

REMARK 1.21. *Equation (1.23) is called* fundamental identity. *It implies the following: if we already know that $v(t,x) = V(t,x)$ for some $(t,x) \in [0,T] \times X$ then the condition (1.19) is also a necessary condition to make an admissible pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ optimal at $(t,x)$.*

## 1.3. Infinite dimensional setting for delay differential equation

In this section we will recall some general results on linear differential equation (DDE) and on related Hilbert space approach that can be found e.g. in [BDPDM92]. Such results will be used in the thesis, the reader that already knows the method can skip the section.

The idea of writing delay system using a Hilbert space setting was first due to Delfour and Mitter [DM72], [DM75]. Variants and improvements were proposed by Delfour and Manitius [DM77], Ichikawa [Ich77], Delfour [Del86], [Del80], [Del84], Vinter and Kwong [VK81], (see also the precise systematization of the argument in chapter 4 of Bensoussan, Da Prato, Delfour and Mitter [BDPDM92]).

The optimal control problem in the (linear) quadratic case is studied in Vinter, Kwong [VK81], Ichikawa [Ich82], Delfour, McCalla and Mitter [DMM75]. In that case the HJB equation reduces to the Riccati equation.

We will concentrate our attention on a particular class of linear DDE. We will treat the problem using the techniques introduced by Vinter and Kwong ([VK81]).

We will use the notation of the book of Bensoussan, Da Prato, Delfour and Mitter [BDPDM92].

The space of the states we will use (the one introduced by Vinter and Kwong) is the Hilbert $M^2 = \mathbb{R} \times L^2(-R,0)$. So the evolution of the delay system will be described using $x(\cdot)\colon [0,T] \to M^2$. The real part of $x(t)$ (the first component) will be the value of the solution of the delay equation at time $t$. The second will be a function of the "histories" of the state and of the control (see 1.37 for details). In this section we will briefly recall the method that allow to give description of the system in $M^2$, in order to obtain such result some tools are needed and we have to rewrite more then once the initial delay equation in different equivalent forms. This will be done in Subsection 1.3.1.

The state we used is not the only one introduced in the literature. It is, for example, also possible to use an extended state $\tilde{x}$ in $M^2 \times L^2(-R,0)$ of three components: the first is the (real) solution of the DDE, the second is the "history" of the solution and the third is the "history" of the control. The space is bigger but the state is more intuitive. See [Ich82], and ([BDPDM92] chapter 4) for details.

**1.3.1. Writing and re-writing the DDE.** In this subsection we present the class of DDE we will work on, we introduce some notations and then we will rewrite the DDE in a form that will allow, in the next subsection, to formulate the DDE as an equation in a suitable Hilbert space.

We introduce first some notations. We will call $L^2((a,b);\mathbb{R})$ (or simply $L^2(a,b)$) the set of the square integrable functions : $(a,b) \to \mathbb{R}$, $L^2_{loc}([a,+\infty);\mathbb{R})$ (or simply $L^2_{loc}[a,+\infty)$) the locally square integrable functions : $[a,+\infty) \to \mathbb{R}$

Given $R > 0$, $T > 0$ and $z \in L^2(-R,T)$ (or $z \in L^2_{loc}[-R,+\infty)$), for every $t \in [0,T]$ we call $z_t \in L^2(-R,0)$ the function

$$\begin{cases} z_t\colon [-R,0] \to \mathbb{R} \\ z_t(s) \stackrel{def}{=} z(s+t) \end{cases} \tag{1.24}$$

Given $N$, $B$ two continuous linear functionals

$$N, B\colon C([-R,0]) \to \mathbb{R}$$

with norm respectively $\|N\|$ and $\|B\|$, we define $\mathcal{N}$ and $\mathcal{B}$ be the following applications

$$\begin{aligned} \mathcal{N}, \mathcal{B}\colon C_c((-R,T);\mathbb{R}) &\to L^2(0,T) \\ \mathcal{N}(\phi)\colon t &\mapsto N(\phi_t) \\ \mathcal{B}(\phi)\colon t &\mapsto B(\phi_t) \end{aligned} \tag{1.25}$$

THEOREM 1.22. *$\mathcal{N}, \mathcal{B}\colon C_c((-R,T);\mathbb{R}) \to L^2(0,T)$ have continuous linear extensions $L^2(-R,T) \to L^2(0,T)$ with norm $\le \|N\|$ and $\le \|B\|$.*

PROOF. See [BDPDM92] Theorem 3.3 page. 217. □

Given a function $f \in L^2_{loc}[0,+\infty)$ and a control $u \in L^2_{loc}[0,+\infty)$ we consider the the following delay differential equation (where $\theta_0, \theta_t, u_0, u_t$ have the sense described in (1.24):

$$\begin{cases} \dot{\theta}(t) = N\theta_t + Bu_t + f(t) \\ (\theta(0), \theta_0, u_0) = (\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2(-R,0) \times L^2(-R,0) \end{cases} \tag{1.26}$$

In the delay setting the initial data are a triple, whose first component is the value of the state variable at initial time, the second and third are respectively the history of the state and the history of the control up to time 0 (more precisely, on the interval $[-R,0]$).

EXAMPLE 1.23. *Here we consider and example to illustrate the meaning of the operators we use in the section. We consider $N = \delta_{-r}$ where $-r \in [-R, 0]$ and $B = \delta_{-s}$ where $-s \in [-R, 0]$. $\delta_{-r}$ and $\delta_{-s}$ are the delta measure in $-r$ and $-s$. We choose $f \equiv 0$, so the DDE we obtain is:*

$$\begin{cases} \dot{\theta}(t) = \theta(t-r) + u(t-s) \\ (\theta(0), \theta_0, u_0) = (\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2(-R, 0) \times L^2(-R, 0) \end{cases} \tag{1.27}$$

*The explicit form of $\mathcal{N}$ and $\mathcal{B}$ in this case are:*

$$\begin{aligned} &\mathcal{N}, \mathcal{B} \colon L^2(-R, T) \to L^2(0, T) \\ &\mathcal{N}(\theta) \colon t \mapsto N(\theta_t) = \theta(t-r) \\ &\mathcal{B}(u) \colon t \mapsto B(u_t) = u(t-s) \end{aligned} \tag{1.28}$$

We state now an existence result and an estimate on the solution:

THEOREM 1.24. *Given an initial condition $(\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2(-R, 0) \times L^2(-R, 0)$ and a control $u \in L^2_{loc}[0, +\infty)$ there exists a unique solution $\theta(\cdot)$ of (1.26) in $H^1_{loc}[0, \infty)$. Moreover for all $T > 0$ there exists a constant $c(T)$ depending only on $R, T, \|N\|$ and $\|B\|$ such that*

$$|\theta|_{H^1(0,T)} \leq c(T)\Big(|\phi^0| + |\phi^1|_{L^2(-R,0)} + |\omega|_{L^2(-R,0)} + |u|_{L^2(0,T)} + |f|_{L^2(0,T)}\Big) \tag{1.29}$$

PROOF. see [BDPDM92] Theorem 3.3 page 217 for the first part and Theorem 3.3 page 217, Theorem 4.1 page. 222 and page 255 for the second statement. □

In view of the continuous embedding $H^1(0, T) \hookrightarrow C([0, T])$ we have:

COROLLARY 1.25. *For all $T > 0$ there exists a constant $c(T)$ such that*

$$|\theta|_{L^\infty(0,T)} \leq c(T)\Big(|\phi^0| + |\phi^1|_{L^2(-R,0)} + |\omega|_{L^2(-R,0)} + |u|_{L^2(0,T)} + |f|_{L^2(0,T)}\Big) \tag{1.30}$$

DEFINITION 1.26 ($\mathbf{e^s_+ u}$, $\mathbf{e^s_- u}$). *Let $a$ and $b$, $a < b$, two real number. Let $\mathcal{F}(a, b)$ be a set of functions from $[a, b]$ to $\mathbb{R}$. For each $u$ in $\mathcal{F}(a, b)$ and all $s \in [a, b]$, define the functions $e^s_- u$ and $e^s_+ u$ as follows*

$$e^s_- u \colon [a, +\infty) \to \mathbb{R}, \quad e^s_- u(t) = \begin{cases} u(t) & t \in [a, s] \\ 0 & t \in (s, +\infty) \end{cases}$$
$$e^s_+ u \colon (-\infty, b) \to \mathbb{R}, \quad e^s_+ u(t) = \begin{cases} 0 & t \in (-\infty, s] \\ u(t) & t \in (s, b] \end{cases}$$

REMARK 1.27. *We will use $e^s_-$ and $e_+ s$ to divide the function $\theta(\cdot)$ in two parts: the part "before zero" that belongs to the initial datum and the part "after zero" that is the solution of the DDE. In the same way $e^s_-$ and $e_+ s$ will be useful to distinguish the part of $u(\cdot)$ that belongs to the initial datum and the control.*

Using the $\mathcal{N}$ and $\mathcal{B}$ notation we can rewrite the (1.26) as

$$\begin{cases} \dot{\theta}(t) = \mathcal{N}\theta + \mathcal{B}u + f \\ (\theta(0), \theta_0, u_0) = (\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2(-R, 0) \times L^2(-R, 0) \end{cases} \tag{1.31}$$

Using $e^s_-$ and $e_+ s$ we can decompose $\theta(\cdot)$ and $u(\cdot)$ as $\theta = e^0_+ \theta + e^0_+ \phi^1$ and $u = e^0_+ u + e^0_+ \omega$. So we can separate the solution $\theta(t)$, $t \geq 0$ and the control $u(t)$, $t \geq 0$ from the initial functions $\phi^1$ and $\omega$:

$$\begin{cases} \dot{\theta}(t) = \mathcal{N} e^0_+ \theta + \mathcal{B} e^0_+ u + \mathcal{N} e^0_- \phi^1 + \mathcal{B} e^0_- \omega + f \\ \theta(0) = \phi^0 \in \mathbb{R} \end{cases} \tag{1.32}$$

Now we are ready to describe the key-step in order to obtain $\mathbb{R} \times L^2(-R, 0)$ as state space. The system (1.32) does not directly use the initial function $\phi^1$ and $\omega$ but only the sum of their images $\mathcal{N}e^0_-\phi^1 + \mathcal{B}e^0_-\omega$. We need a last step before we can write the delay equation in Hilbert space. We introduce two operators

$$\begin{cases} \overline{N}\colon L^2(-R,0) \to L^2(-R,0) \\ (\overline{N}\phi^1)(\alpha) \overset{def}{=} (\mathcal{N}e^0_-\phi^1)(-\alpha) \qquad \alpha \in (-R,0) \end{cases}$$

and

$$\begin{cases} \overline{B}\colon L^2(-R,0) \to L^2(-R,0) \\ (\overline{B}\omega)(\alpha) \overset{def}{=} (\mathcal{B}e^0_-\omega)(-\alpha) \qquad \alpha \in (-R,0) \end{cases}$$

The operators $\overline{N}$ and $\overline{B}$ are continuous (see [BDPDM92] page 235). We note that

$$\mathcal{N}e^0_+\phi^1(t) + \mathcal{B}e^0_+\omega(t) = (e^{-R}_+(\overline{N}\phi^1 + \overline{B}\omega))(-t) \qquad for\ t \geq 0$$

So, if we call

$$\xi^1 = (\overline{N}\phi^1 + \overline{B}\omega) \tag{1.33}$$

and $\xi^0 = \phi^0$, we can rewrite the (1.32) (and then the (1.26) as

$$\begin{cases} \dot{\theta}(t) = (\mathcal{N}e^0_+\theta)(t) + (\mathcal{B}e^0_+u)(t) + (e^{-R}_+\xi^1)(-t) + f(t) \\ \theta(0) = \xi^0 \in \mathbb{R} \end{cases} \tag{1.34}$$

where $\mathbb{R} \times L^2(-R,0) \ni \xi \overset{def}{=} (\xi^0, \xi^1)$. The (1.34) makes sense for all $\xi \in \mathbb{R} \times L^2(-R,0)$ also when $\xi^1$ is not of the form (1.33). So we have embedded the original system (1.26) in a family of systems of the form (1.34).

EXAMPLE 1.28. *In the case $N = \delta_{-r}$ and $B = \delta_{-s}$ we have the following explicit expression for $\overline{N}$ and $\overline{B}$:*

$$\overline{N}(\phi^1)(\alpha) = \mathcal{N}(e^-_0\phi^1)(-\alpha) = \begin{cases} 0 & \alpha < -r \\ \phi^1(-\alpha - r) & \alpha \geq -r \end{cases}$$

$$\overline{B}(\omega)(\alpha) = \mathcal{B}(e^-_0\omega)(-\alpha) = \begin{cases} 0 & \alpha < -s \\ \omega(-\alpha - s) & \alpha \geq -s \end{cases}$$

REMARK 1.29. *This remark is very (too?) informal but can be useful to understand the steps we are describing: one can ask why we take the inverse of $\alpha$ in the definition of $\overline{N}$ and the inverse of $t$ in the term $e^{-R}_+\xi(-t)$. The advantage of this approach will be clearer when we will see the state equation in the Hilbert space, indeed this setting allows to have a state equation in which the unbounded operator that acts on the control is exactly $B^*$ (without any transformations). The "inversion" of the time we are considering now will become an "inversion" in the state equation (1.39) in $M^2$ where the generator of the semigroup considered is not the "natural one" $S^*(t)$ but its adjoint $S(t)$.*

**1.3.2. The infinite dimensional setting.** We consider from now on the case $f = 0$.

The scalar product between two elements of $M^2 \overset{def}{=} \mathbb{R} \times L^2(-R,0)$ $\phi = (\phi^0, \phi^1)$ and $\xi = (\xi^0, \xi^1)$ is $\langle \phi, \xi \rangle_{M^2} \overset{def}{=} \langle \phi^1, \xi^1 \rangle_{L^2} + \phi^0\xi^0$.

We now introduce the generator of the semigroup that will appear in the equation in the Hilbert space. We consider the homogeneous system[1]

$$\begin{cases} \dot{y}(t) = (\mathcal{N}y)(t) \\ (y(0), y_0) = \phi \in M^2 \end{cases}$$

[1] Such homogeneous system can be written in the standard way as

$$\begin{cases} \dot{y}(t) = Ny_t \\ (y(0), y_0) = (\phi^0, \phi^1) \end{cases}.$$

We can construct the following semigroup of continuous linear transformations on $M^2$:

$$\begin{cases} S^*(t)\colon M^2 \to M^2 \\ \phi \mapsto S^*(t)\phi \stackrel{def}{=} (y(t), y_t) \end{cases}$$

$S^*(t)$ for $t \geq 0$ is a $C_0$ semigroup on $M^2$ and its generator is characterized by

$$\begin{cases} D(A^*) = \left\{(\phi^0, \phi^1) \in M^2 \ : \ \phi^1 \in H^1(-R,0) \ and \ \phi^0 = \phi^1(0)\right\} \\ A^*(\phi^0, \phi^1) = (N\phi^1, D\phi^1) \end{cases} \tag{1.35}$$

where $D\phi^1$ is the first derivative of $\phi^1$. The second component of the elements of $D(A^*)$ (endowed with the graph norm) is included in $C([-R,0])$ so, abusing notation, we can restrict $B$ on $D(A^*)$ in the following way

$$\begin{cases} B\colon D(A^*) \to \mathbb{R} \\ B(\phi^0, \phi^1) = B\phi^1 \end{cases} \tag{1.36}$$

Moreover we call $j$ the continuous inclusion $D(A^*) \hookrightarrow M^2$. So the operators $A^*$ and $j$ are continuous from $D(A^*)$ (endowed with the graph norm) in $M^2$ and $B$ is continuous from $D(A^*)$ in $\mathbb{R}$. We call $A$, $j^*$ and $B^*$ their adjoints. Identifying $M^2$ and $\mathbb{R}$ with their duals we have that

$$\begin{aligned} &A\colon M^2 \to D(A^*)' \\ &j^*\colon M^2 \to D(A^*)' \\ &B^*\colon \mathbb{R} \to D(A^*)' \end{aligned}$$

are linear continuous.

Definition 1.30 (**Structural state**). *The* structural state $x(t)$ *at time* $t \geq 0$ *is defined by*

$$x(t) \stackrel{def}{=} (\theta(t), \overline{N}(e^0_+\theta)_t + \overline{B}(e^0_+u)_t + \Xi(t)\xi^1) \tag{1.37}$$

*where* $\Xi(t)$ *is the right translation operator defined as*

$$(\Xi(t)\xi^1)(r) = (e^{-R}_+\xi^1)(r-t) \qquad r \in [-R,0] \tag{1.38}$$

Example 1.31. *In the case* $N = \delta_{-r}$ *and* $B = \delta_{-s}$ *we have the following:*

$$\overline{N}(e^0_+\theta)_t(\alpha) = \mathcal{N}(e^0_-(e^0_+\theta)_t)(-\alpha) = \begin{cases} 0 & \alpha \in [-R,-r) \\ \theta(t-\alpha-r) & \alpha \in [-r,-r+t) \\ 0 & \alpha \in [-r+t, 0] \end{cases}$$

$$\overline{B}(e^0_+u)_t(\alpha) = \mathcal{B}(e^0_-(e^0_+u)_t)(-\alpha) = \begin{cases} 0 & \alpha \in [-R,-s) \\ u(t-\alpha-s) & \alpha \in [-s,-s+t) \\ 0 & \alpha \in [-s+t, 0] \end{cases}$$

Example 1.32. *See Section 6.5 to see another example of explicit structural state in a applied case.*

Theorem 1.33. *Assume that* $\theta(t)$ *is the solution of system (1.34) for* $\xi \in M^2$, $f \in L^2_{loc}[0,+\infty)$ *and* $u \in L^2_{loc}[0,+\infty)$ *and let* $x(t)$ *be the structural state constructed from* $x$ *in (1.37). Then for each* $T > 0$, *the state* $x$ *is the unique solution in*

$$\left\{ z \in C([0,T]; M^2) \ : \ \frac{\mathrm{d}}{\mathrm{d}t} j^* z \in L^2((0,T); D(A^*)') \right\}$$

*to the following equation*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} j^* x(t) = Ax(t) + B^*u(t) \\ x(0) = \xi \end{cases} \tag{1.39}$$

Notation 1.34. *In the following we will understand the* $j^*$ *and we will write simply* $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t) + B^*u(t)$.

Remark 1.35. *The equation (1.39) is an equation in* $D(A^*)'$*: if we use the extension described in Lemma 1.15, observing that the term* $B^*$ *is a constant in* $D(A^*)'$*, the solution of (1.39) can be written in mild form :*

$$x(t) = e^{tA^{(E)}}\xi + \int_0^t u(t)e^{(t-r)A^{(E)}}B^*\mathrm{d}r$$

# Part 1

# Viscosity solution approach

CHAPTER 2

# A survey on viscosity solutions for first order Hamilton-Jacobi in Hilbert spaces

As we have already stressed in the Introduction, the study of an optimal control problem and of the related Hamilton-Jacobi-Bellman equations (from now simply HJB equation) is strongly connected. The concept of viscosity solutions (that was introduced for Hamilton-Jacobi (HJ) equations in $\mathbb{R}^n$ in [CL83, CEL84]) seems to be an appropriate tool to implement such connection[1]. Differently from the finite dimensional case, where a certain number of review and books on the theme are available (see for example [FR75], [YZ99] and particularly [BCD97]), in the infinite dimensional case there are not exhaustive summaries about viscosity solutions approach for HJ in literature (the only one, to the best of our knowledge, is a chapter in the book [LY95] that present the method used in [CL90, CL91]). In this first chapter we will present a brief survey of the literature about viscosity solution of first order HJB equations in Hilbert spaces whereas in the next ones we will present some our original contributions. In order to try to make an overview some choices are needed: different authors treat different formulations of the problem, different hypotheses are required, different spaces are used and the results are not always comparable. In particular a lot of different definitions of viscosity solutions are given. If we want not only to make a list of works but to try to organize them we have to choose and underline some "main" differences and contributions and to uniform some details. The distinction between *main differences* and *details* is part of the work of classification and is of course debatable.

**Acknowledgements** Thanks to professor Andrzej Święch that suggested me many interesting works about viscosity solutions.

**A note on the notation** Often in the literature the same symbol represents different mathematical objects. Writing a thesis some choices are required and the result is not always pleasure. In particular the letter $B$ is often used to indicate both the boundary terms and the operator introduced in [CL90]. We choose to call $P$ this second operator. This choice gives some unpleasant effects. For example the $B$-continuity becomes the $P$-continuity. We apologize with the reader.

NOTATION 2.1. *Sometimes we use the name* HJ equation*, sometime* HJB equation*. The distinction is not always clear but we will try to use the expression* HJB equation *when the equation is directly related to an optimal control problem and the expression HJ equation when we consider a generic HJ.*

## 2.1. The first works

Viscosity solutions for HJ equations in Hilbert spaces were first treated by Crandall and Lions in a series of seven works [CL85, CL86a, CL86b, CL90, CL91,

[1]"Appropriate" means, for example, that using viscosity solutions we can prove in a important class of cases that the value function of the optimal control problem is the only viscosity solution of the HJB.

CL94a, CL94b]. The first two pioneering works ([CL85, CL86a]) deal with a definition of viscosity solution that extends in natural way the definition given by the same authors for the HJ equations in $\mathbb{R}^n$ in [CL83] that is:

Definition 2.2 (**Viscosity solutions in** [CL85, CL86a]). *Given an HJ equation of the form*

$$\rho w(x) - H(x, \nabla w(x)) = 0 \ \ in\ \Omega \tag{2.1}$$

*where $H\colon X \times \mathbb{R} \times X \to \mathbb{R}$ is a continuous function, $w \in C(\Omega)$ is a viscosity subsolution of (2.1) on $\Omega$ if*

$$\rho w(x) - H(x,p) \leq 0 \text{ for every } x \in \Omega \text{ and } p \in D^+ w(x)$$

*similarly $w \in C(\Omega)$ is a viscosity supersolution of (2.1) on $\Omega$ if*

$$\rho w(x) - H(x,p) \geq 0 \text{ for every } x \in \Omega \text{ and } p \in D^- w(x)$$

*$w \in C(\Omega)$ is a viscosity solution of (2.1) on $\Omega$ if it is both subsolution and supersolution.*

Similar definitions are given in [CL85, CL86a] for HJB equation of the form:

$$-w_t(x) - H(x, \nabla w(x)) = 0 \ \ in\ \Omega. \tag{2.2}$$

Remark 2.3. *For this first case the authors prove that there exists an equivalent formulation that uses regular tests functions instead of sub and super differentials. For example the definition of viscosity subsolution can be given in the following equivalent form: $w \in C(\Omega)$ is a viscosity subsolution of (2.1) on $\Omega$ if for every $\varphi \in C(\Omega)$ and every point $x$ of maximum of $w - \varphi$ at which $\varphi$ is differentiable we have:*

$$\rho w(x) - H(x, \nabla \varphi(x)) \leq 0$$

*This formulation is different from that obtained in the finite dimensional case where (see for example* [YZ99] *page 173) we can take $\varphi \in C^1(\Omega)$ and obtain the equivalence.*

*In the subsequent literature more elaborated definitions will be given and the equivalence between a sub/superdifferential formulation and the related formulation that uses the test function will be hard to prove. The main contributions will favor the formulations based on test functions.*

[CL85] is devoted to the proof of comparison theorem for HJB of the form (2.1) and (2.2). The hypotheses required are similar to that required in the finite dimensional case. Existence results are studied in [CL86a] proving first an existence theorem for a class of HJ related to differential games and then proving existence theorem in general case using approximating techniques (the same scheme will be used in [CL91]). Existence for (2.1) and (2.2) can be also proved via Perron's method as seen in [Ish87].

The main limit of Definitions 2.1 and 2.2 is the absence of the unbounded term $\langle Ax, \nabla w(x) \rangle$ in the HJB equation. This fact implies that, using such definition of solution, we can only treat HJB equation related to optimal control problem where the state equation does not contains a generator of a semigroup $A$ that does appear in many interesting cases. The subsequent literature will try to solve the problem. The first two works in which a complete HJB equations of the form (1.6) or (1.13) is considered are [CL86b] (in Section II.3) and [Son88]. In both the HJB equation of the form (1.13) arising in the optimal control problem (1.8, 1.9) is studied.

In [CL86b] the authors treat the case in which $A$ is the generator of a $C_0$ semigroup of linear operator, $U = X$, $f(t,x,u) = u$, $L(t,x,u) = F^*(u)$ where $F^*$ is the conjugate convex function of coercive convex function $F(p)$. We call the value function of such optimization problem $V_{[\text{CL86b}]}$. The related HJB equation is

$$\begin{cases} -w_t(t,x) - \langle Ax, \nabla w(x)\rangle - F(\nabla w(t,x)) = 0 \\ w(T,x) = h(x) \end{cases} \tag{2.3}$$

We expect, "calculating formally", that, if $w$ solves (2.3) and $w(x,t) = v(e^{(T-t)A}x, t)$, then $v$ solves

$$-v_t(t,x) - \tilde{F}(t, \nabla v(t,x)) = 0 \tag{2.4}$$

where $\tilde{F}(t,p) = F(e^{(T-t)A^*}p)$ The authors prove that:

PROPOSITION 2.4. *If* $U = X$, $f(t,x,u) = u$, $L(t,x,u) = F^*(u)$, $h \in UC(X)$, $F \colon X \to \mathbb{R}$ *is continuous coercive convex and*

$$|F(p) - F(q)| \le \sigma(|p-q|, R) \quad \textit{for } p,\, q \textit{ in } B_R$$

*then there is a unique viscosity solution* $v$ *(in sense of Definition 2.2) of the equation*

$$\begin{cases} -v_t(t,x) - \tilde{F}(\nabla v(t,x)) = 0 \\ v(T,x) = h(x) \end{cases}$$

*where* $\tilde{F}(t,p) = F(e^{(T-t)A^*}p)$. *Moreover we have that* $V_{[\text{CL86b}]}(x,t) = v(e^{(T-t)A}x, t)$

Then the authors say, *by definition*, that $w$ is a solution of (2.3) if and only if $w(x,t) = v(e^{(T-t)A}x, t)$ and $v$ solves the (2.4) and so $V_{[\text{CL86b}]}(x,t)$ is a solution of the equation (2.3).

Analogous change of variables methods are also used, for different HJB equations, in the more recent works (see for example [CT96c, CT96a, CC04],).

The method used by the author in [Son88] allow to treat families of infinite dimensional HJB equations coming from finite dimensional delay or parabolic problems proving that the value function of the problem is the unique solution of the HJB. Some cases in which the generator $A$ depends on the control are considered also. The idea is to check the (1.13) only in the points of a growing family of (also non-dense) sets $E_n$ with $E_n \subseteq E_{n+1} \subseteq \ldots \subseteq D(A)$. The author, called $w$ the solution and $\varphi$ a regular test function , consider the maxima (or the minima) of the restriction $(w - \varphi)_{|E_n}$.

The main limits of such approach are two: the first is the difficulty to find, in a quite general case, a sequence of set $E_n$ that satisfy the requirements, the second (and more important) is the fact that the definition of solution depend on the choice of the sequence $E_n$. Indeed the methodology used in [Son88] was not more used after the publishing of [CL90] and [CL91].

The notion of *generalized viscosity solution* was proposed by Cannarsa and Da Prato in the works [CDP89, CDP90, CDP89] in which the authors treat the the unbounded term using Yosida approximation and defining the *generalized viscosity solution* of the unbounded HJ equation as the pointwise limit (on $D(A)$) of the viscosity solution of the approximating HJ equations. The approximating HJ equations do not contain unbounded term and so Definition 2.2 and results contained in [CL85, CL86a] may be used. In particular in [CDP89] the authors study the HJB related to the optimal control problem characterized by a state equation of the form

$$\begin{cases} \dot{x}(s) = Ax(s) + F(x(s)) + Bu(s)) \\ x(0) = x \end{cases} \tag{2.5}$$

(where $B\colon U \to X$ is linear continuous and $U$, $F$, $A$ satisfy suitable hypotheses) and by an objective functional

$$\int_0^\infty g(x(s)) + h(u(s))\mathrm{d}s.$$

An existence and uniqueness result is proved. HJB equations of a system governed by a semilinear dynamics are also treated in [CDP90].

REMARK 2.5. *In* [Bar86] *(and also* [BBJ88]*) the author present a notion of viscosity solution for the unbounded case in which the solution is checked only in the point of $D(A)$ so the term $\langle Ax, \nabla\phi(x)\rangle$ makes sense. Such definition is quite "weak" and it is difficult to prove uniqueness results.*

REMARK 2.6. *A definition similar to Definition 2.2 was also used in the work* [CGS91] *in which the authors study existence of a viscosity solution an HJ equation of the form*

$$v(x) + H(x, \nabla v(x)) = 0$$

*with a boundary condition (arising in exit time or state constraint problem) of the form*

$$v = g \quad \text{or} \quad H(x, \nabla v(x)) = 0 \qquad \text{on } \partial\Omega$$

*In such case a suitable definition, involving boundary condition, have to be used.*

## 2.2. Viscosity solution in [CL90] and [CL91]

Besides [CL90, CL91] there is a certain number of works that study viscosity solutions for equations (1.6) and (1.13) in general form (we will examine the main contributions in the following sections). The definitions of *viscosity solution* that various authors give are not always equivalent. We choose to begin describing in detail the works [CL90, CL91] because they are the first that consider a large family of cases and their approach can be considered the most standard and classical.

Crandall and Lions in [CL90, CL91] prove existence and uniqueness results for two different sets of hypotheses: "strong" and "weak". We begin describing results for the "strong" case[2]. In Subsection 2.2.2 we will see the modifications needed in the weak case.

Some remarks are needed in order to understand the definition given by Crandall and Lions. The first easy remark is that if we use a small set of test functions we can easily prove the existence of a solution and it is more difficult to prove the uniqueness of such solution. On the other hand if we use a bigger set of test function the proof of comparison result (and then uniqueness) becomes simpler and the existence result becomes more difficult.

If we try to use the Definition 2.2 in the equation (1.6), that is

$$\rho w(x) - \langle Ax, \nabla w(x)\rangle - H(x, \nabla w(x)) = 0,$$

the first problem that we see is that it makes sense only if $x \in D(A)$. So the first idea of Crandall and Lions is to take the adjoint of $A$, to write the equation as

$$\rho w(x) - \langle x, A^*\nabla w(x)\rangle - H(x, \nabla w(x)) = 0,$$

and to consider regular test functions $\varphi$ such that $\nabla\varphi(x) \in D(A^*)$, so we have to check that

$$\rho w(x) - \langle x, A^*\nabla\varphi(x)\rangle - H(x, \nabla\varphi(x)) \lesseqgtr 0,$$

and it makes sense. Nevertheless if we consider a set of test functions composed only on regular function with differential in $D(A^*)$ such set is too small to prove a

[2]The Hypotheses of "weak" case are not weaker, they are simply different. Strong case is not a particular case of "weak" case.

comparison result (we cannot "localize" the problem and we cannot find a maximum in the uniqueness proof) so we need to enlarge our test functions set. So, as we will see, the authors will use also a class of radial functions.

The second idea that we find in [CL90] is the following: if we assume that $-A^*$ is monotone we have

$$- \langle A^* x, x \rangle \geq 0$$

for all $x \in D(A^*)$ and then if we consider a growing radial test function $g(x) = g_0(|x|)$, its differential is $\nabla g(x) = g_0'(|x|)\frac{x}{|x|}$, and we have, on $D(A^*)$,

$$- \langle A^* \nabla g(x), x \rangle = -\frac{g_0'(|x|)}{|x|} \langle A^* x, x \rangle \geq 0.$$

So if we consider test functions that are sum of two functions, the first $\varphi$ regular with differential in $D(A^*)$ and the second $g$ growing radial function, seems reasonable, when $w - \varphi - g$ attains a maximum, to ask that

$$\rho w(x) - \langle x, A^* \nabla \varphi(x) \rangle - H(x, \nabla \varphi(x) + \nabla g(x)) \leq 0.$$

We can now pass to a more rigorous description. We assume the following Hypotheses:

HYPOTHESIS 2.7. *$A$ is the generator of a $C_0$ semigroup of linear contraction on separable real Hilbert space $X$, so $-A$ is a densely defined, maximal monotone operator.*

HYPOTHESIS 2.8. *$H$ is uniformly continuous on bounded sets and there exists a modulus of continuity $\sigma(\cdot)$ such that*

$$|H(y,p) - H(x,p)| \leq \sigma(|x-y|(1+|p|)) \tag{2.6}$$

*for every $x, y, p \in X$. Moreover there exists a radial function $\mu \in C^1(X, \mathbb{R})$ and local modulus of continuity $\sigma(\cdot,\cdot)$ such that $\lim_{|x|\to\infty} \mu(x) = \infty$ and for every $\lambda > 0$*

$$\begin{aligned} \max \{ &|H(x,p) - H(x, p + \lambda \nabla \mu(x))|, \\ &|H(x,p) - H(x, p - \lambda \nabla \mu(x))| \} \leq \sigma(\lambda, |p|) \end{aligned} \tag{2.7}$$

*for every $x, p \in X$.*

HYPOTHESIS 2.9. *There exists $P$ a linear bounded positive selfadjoint operator on $X$ such that $A^*P$ is a bounded operator on $X$ and*

$$\text{there exists } c_0 \in \mathbb{R} \text{ such that for } x \in X \ \ \langle (A^*P + c_0 P)x, x \rangle \geq |x|^2 \tag{2.8}$$

REMARK 2.10 (**On the existence of** $P$). *In* [Ren95] *some results for the existence of operator $P$ is proved. In Remark 2.24 we will summarize it.*

We can now give the definition of sub/super-solution contained in [CL90]:

DEFINITION 2.11 (**Viscosity solution in** [CL90]). *Hypotheses 2.7, 2.8, 2.9. Let $w \in C(\Omega)$. $w$ is a subsolution (respectively supersolution) of (1.6) if for every $\varphi \in C^1(\Omega)$ such that*

*$\varphi$ is weakly sequentially lower semicontinuous*[3]
*$\nabla \varphi(x) \in D(A^*)$ for all $x \in \Omega$*
*$A^* \nabla \varphi$ is continuous*

[3]The hypothesis of weak sequential continuity is used in [CL90] in the existence proof that uses an approximation argument, see the discussion at page 35, equation (2.10). If we prove we existence for the problem related to optimal control problem, using Bellman's optimality principle, we can avoid such assumption.

*and every $g \in C^1(\Omega)$ radial non-decreasing and every local maximum (respectively minimum) $x \in \Omega$ of $w - \varphi - g$ (respectively $w + \varphi + g$) we have*

$$\rho w(x) - \langle x, A^* \nabla \varphi(x) \rangle - H(x, \nabla \varphi(x) + \nabla g(x)) \leq 0,$$

*(respectively*

$$\rho w(x) + \langle x, A^* \nabla \varphi(x) \rangle - H(x, -\nabla \varphi(x) - \nabla g(x)) \geq 0)$$

THEOREM 2.12 (**Existence and uniqueness, from** [CL90] **Theorem 1.2**). *Let Hypotheses 2.7, 2.8, 2.9 be satisfied. Let $w(\cdot), v(\cdot) \in UC(X)$ bounded and weak continuous. Let $w(\cdot)$ and $v(\cdot)$ be respectively a subsolution and a supersolution of (1.6) with $w$ bounded above and $v$ bounded below. Then*

*Comparison: $w \leq v$ in $X$.*

*Existence: If $P$ is compact and $H(\cdot, 0)$ bounded on $X$ then there exists a unique solution $z \in BUC(X) \cap C_\omega(X)$*

REMARK 2.13. *The theorem we gave above is less general than the one of* [CL90]*, in particular the boundedness hypothesis can be relaxed.*

REMARK 2.14 (**The non-autonomous case**). *We present here only the results related to autonomous/infinite horizon case (1.6), but also the finite horizon case is treated in* [CL90] *(Theorem 1.3 and 1.5)*

We can now make some remarks on the hypotheses and on the statements of Theorem 2.12. Equation (2.6) and boundedness hypothesis on $w$ and $v$ are quite standard also in the finite dimensional case (they can be be weakened, but anyway similar hypotheses are needed). Hypotheses 2.7 and 2.9 are typical of the infinite dimensional setting and become void for finite dimensional case. Definition 2.11 become equivalent to Definition 2.2 when $A$ is continuous and it becomes the traditional definition (see [BCD97]) when $X = \mathbb{R}^n$. We can now observe the role of the test functions $g$ in the uniqueness (comparison) part of the theorem: it is proved similarly to the finite dimensional case writing first

$$\Psi(x,y) = w(x) - v(y) - \frac{\langle P(x-y), (x-y) \rangle}{2\varepsilon} - \lambda(\mu(x) + \mu(y)) \tag{2.9}$$

and proving that $\Psi(x,y)$ has a maximum on $X \times X$. So we can understand the meaning of some of our hypotheses:

The boundedness of $w$ and $v$, the weak continuity and the presence of the term $\lambda(\mu(x) + \mu(y))$ (and then the use of the family of radial test functions ) are needed in order to prove the existence of a maximum point of $\Psi$

If the maximum point is attained in $(\bar{x}, \bar{y})$ then we use the definition of viscosity subsolution noting that

$$x \mapsto w(x) - \frac{\langle P(x-\bar{y}), (x-\bar{y}) \rangle}{2\varepsilon} - \lambda(\mu(x))$$

attains a maximum in $\bar{x}$, so Hypothesis 2.9 is needed in order to guarantee that

$$x \mapsto \frac{\langle P(x-\bar{y}), (x-\bar{y}) \rangle}{2\varepsilon} + \lambda(\mu(x))$$

is a test function[4]. The comparison result follows using the definition of sub and supersolution and letting $\lambda$ and $\varepsilon$ to zero.

[4] $\lambda\mu(\cdot)$ is the radial part and the hypotheses on $P$ guarantee that

$$\varphi \colon x \mapsto \frac{\langle P(x-\bar{y}), (x-\bar{y}) \rangle}{2\varepsilon}$$

have the properties required in Definition 2.11.

Remark 2.15 (**The existence of the maximum in comparison proof**). *The existence of the maximum for the function $\Psi$ defined in (2.9) is one of the difficulties in the proofs of the comparison results. In* [CL90] *the weak continuity of the sub and super solutions is used (see the hypotheses of Theorem 2.12). In* [CL91] *a perturbation technique is used and so we do not need the weak continuity hypothesis (see Remark 2.18 for more details). We will see in Subsection 2.3 how the perturbation method is used in* [Tat92b, Tat94, CL94a].

Finally a remark on the proof of existence: it is difficult, using the definition given in [CL90, CL91], to use Perron's method to prove existence result. Indeed different arguments are used by Crandall and Lions. In [CL90] a family of finite dimensional HJ equations that converges to (1.6) is used. In particular the authors consider the projection $P_n$ on the the part of $X$ on which $P \geq \frac{1}{n}$ and consider the approximating equations:

$$w_n(x) + \langle A_n x, \nabla w_n(x)\rangle + H(Px, \nabla w_n(x)) = 0 \tag{2.10}$$

where $A_n = (P_n A^* P_n)^*$. The techniques used in [CL91] are described in Subsection 2.2.3.

### 2.2.1. The operator $P$ .

The operator $P$, that appears in Hypothesis 2.9 is crucial in the arguments of [CL90, CL91] but it does not appear in the definition of viscosity solution (Definition 2.11) and in in Theorem 2.12. Nevertheless using the operator $P$ it is possible to define more precise functional spaces than can be used to improve existence and uniqueness results finding more regularity for the solution and proving uniqueness in larger space:

Definition 2.16 (**P-continuity, from** [CL91]). *Given a function $w\colon \Omega \to \mathbb{R}$ we will say that $w(\cdot)$ is $P$-continuous on $\Omega$ if $w(x_n) \to w(x)$ whenever $x_n$ is a sequence in $\Omega$ such that $x_n \rightharpoonup x$ and $Px_n \to Px$.*

If $w$ is $P$-continuous then $w$ is continuous, and if $P$ is compact the $P$-continuity is equivalent to weak continuity.

Notation 2.17. *Using $P$ (that is bounded, positive and selfadjoint), for any $\alpha > 0$ we can define the norm*

$$|x|_{-\alpha} = \langle P^{\alpha} x, x\rangle^{1/2}$$

*We call $X_{-\alpha}$ the completion of $X$ in the $_{-\alpha}$ norm. In the natural way $P^{\alpha/2}$ is an isometry from $X_{-\alpha}$ onto $X$. $D(P^{-\alpha/2})$ may be isometrically identified with the dual of $X_{-\alpha}$ using the norm*

$$|x|_{\alpha} = |P^{-\alpha/2} x|$$

*We denote this dual by $X_{\alpha}$ . $P^{-\alpha/2}$ is an isometry from $X_{\alpha}$ onto $X$.*

Remark 2.18. *In order to obtain a maximum point for $\Psi$ (see Remark 2.15) in* [CL91] *the authors use the Ekeland-Lebourg lemma (see* [EL77] *and* [LY95] *page 245*[5]*). The comparison result is proved on the set of $P$-continuous function and it is possible to find a perturbation "small" with respect to the $|\cdot|_2$ norm. More precisely the author prove that one can find $p$ and $q$ with $|q|, |p| \leq \varepsilon$ (in the $X$ norm) such that the function*

$$(x, y) \mapsto w(x) - v(y) - \frac{\langle P(x-y), (x-y)\rangle}{2\varepsilon} - \lambda(\mu(x) + \mu(y)) + \langle Pp, x\rangle + \langle Pq, y\rangle$$

*admits a maximum. The linear function $x \to \langle Pp, x\rangle$ has differential equal to $Pp$. This term has $D(A^*)$ norm less then $(\|A^*P\| + \|P\|)\varepsilon$ and then can be easily treated.*

[5]A version of the theorem can be also found (without proof) in Theorem 3.29, it is less general than the result showed in [EL77].

**2.2.2. The "weak" case .** Here we briefly describe the Hypotheses and the results for the "weak" case. The requirement on operator $P$ is weakened but a stronger assumption is needed on $H$:

HYPOTHESIS 2.19. *There exists $P$ a linear bounded positive selfadjoint operator on $X$ such that $A^*P$ is a bounded operator on $X$ and*

$$\text{There exists } c_0 \in \mathbb{R} \text{ such that for } x \in X \;\; \langle (A^*P + c_0P)x, x\rangle \geq 0 \tag{2.11}$$

HYPOTHESIS 2.20. *$H$ is uniformly continuous on bounded sets and there exists a modulus of continuity $\sigma(\cdot)$ such that*

$$H(y, P(x-y)) - H(x, P(x-y)) \leq \sigma(|x-y|_{-1}(1 + \lambda|x-y|_{-1})) \tag{2.12}$$

*for every $x, y, p \in X$. Moreover there exists a radial function $\mu \in C^1(X, \mathbb{R})$ and local modulus of continuity $\sigma(\cdot,\cdot)$ such that* $\lim_{|x|\to\infty} \mu(x) = \infty$ *and for every $\lambda > 0$*

$$\max\{|H(x,p) - H(x, p + \lambda\nabla\mu(x))|, \\ |H(x,p) - H(x, p - \lambda\nabla\mu(x))|\} \leq \sigma(\lambda, |p|) \tag{2.13}$$

*for every $x, p \in X$.*

REMARK 2.21. *Hypothesis 2.20 is the same of 2.8 except for the fact that we changed we change (2.6) with (2.12).*

The following is the theorem proved in [CL90] for the autonomous weak case:

THEOREM 2.22 (**Existence and uniqueness, from** [CL90] **Theorem 1.4**). *Let Hypotheses 2.7, 2.19 and 2.20 hold.*
*Comparison: Let $w, v \in UC(X_{-1})$ and weakly continuous. Let $w$ and $v$ be respectively a subsolution and a supersolution of (1.6). If either $w$ and $-v$ are bounded above then $w \leq v$ on $X$.*
*Existence: Let $P$ compact and $H(\cdot, 0)$ bounded on $X$, then there is a unique solution $w \in BUC(X_{-1})$ of (1.6).*

REMARK 2.23. *The main contribution of* [CL91] *was to give more precise results on the existence and uniqueness of the solutions using the $P$-continuity. Indeed the authors prove that the comparison result among all $P$-continuous functions (in the Theorem 2.12 the comparison result was proved only among weak continuous functions). Moreover they prove the existence of a $P$-continuous solution without the Hypothesis of compactness of the operator $P$ (that was needed in Theorem 2.12 and Theorem 2.22).*

REMARK 2.24 (**The existence for operator** $P$). *In* [Ren95] *Renardy prove that the operator $P = (A^*A)^{-1/2}$ satisfies in the general case Hypothesis 2.19. Moreover if*

$$[D(A), X]_{\frac{1}{2}} = [D(A^*), X]_{\frac{1}{2}} =: W$$

*and there exists a $\delta > 0$ such that $\langle Ax, x\rangle \leq \delta \langle x, x\rangle$ for all $x \in D(A)$ then $P = (A^*A)^{-1/2}$ satisfies also Hypothesis 2.9. ($[\cdot,\cdot]_{\frac{1}{2}}$ is the complex interpolation, see* [LM72b] *for a beautiful reference on the argument).*

**2.2.3. [CL91], optimal control and differential games.** In [CL91] the authors prove the existence result exploiting the HJ associated to an optimal control problem and a differential game[6]: they first show that the HJ equation they study is the HJB equation related to a certain optimal control problems or the Hamilton-Jacobi-Isaac (HJI) equations related to some games and then prove that the value function of the system is a solution of the (1.6). We will recall the use of differential

[6]In particular they can prove existence using optimal control in a sub-family of the problem treatable with differential games.

games in Subsection 2.7.2, here we make some remarks on the relation with optimal control that is more related to the argument of the thesis. The proof of the fact that the value function $V$ is a viscosity solution of the (1.6) is quite general: $V$ has to be well defined on $X$ and continuous. The idea for the proof is the same of the finite dimensional case (see [YZ99], [FR75]) and it is the same we will use in Theorem 3.28 and Theorem 4.19, it is based on the Bellman's principle: the value function satisfies, under quite genera assumptions, the integral equation:

$$V(t,x) = \inf_{u \in \mathcal{U}_t} \left\{ \int_t^s L(r, x(r), u(r)) \mathrm{d}r + V(s, x(s)) \right\}$$

for every $s \geq t$. It is, roughly speaking, the integral version of the HJB equation. It can be seen the proof of Theorem 3.28 for an example of how the proof go on in a viscosity case. The idea in the use of differential games is not very different and we will describe it in Subsection 2.7.2. Now we state some of the results of [CL91], note that here the compactness of $P$ is not required:

Suppose to study the optimal control problem subject to state equation (1.1) and functional (1.2), then we have the following

Theorem 2.25 (**Existence, from** [CL91] **Theorem 4.1 and 4.3**). *Let $A$ satisfy Hypothesis 2.7. Moreover let $f$ and $L$ satisfy the following hypotheses:*

$$\begin{array}{l} |f(x,u) - f(y,u)| \leq C_0 |x-y| \ \textit{ for } x, y \in X \textit{ and } u \in U \\ |f(x,u)| \leq C_1 \ \textit{ for } x \in X \textit{ and } u \in U \\ |L(x,u)| \leq C \ \textit{ for } x \in X \textit{ and } u \in U \\ |L(x,u) - L(y,u)| \leq \sigma(|x-y|) \ \textit{ for } x, y \in X \textit{ and } u \in U \end{array} \tag{2.14}$$

*Then the value function $V$ (defined in (1.7) is $P$-continuous (Theorem 4.3* [CL91]*) and it is a viscosity solution of (1.6) (Theorem 4.1* [CL91]*).*

Remark 2.26 (**Non-bounded case**). *The $P$-continuity in* [CL91] *is proved also using different sets of Hypotheses for example assuming polynomial growth of $L$ and $f$ instead of the boundedness requirements of (2.14).*

Remark 2.27. *Also in* [CL91] *the authors distinguish between "weak" and "strong" case (as in* [CL90]*). We have described the strong case but analogous results are proved for the weak one.*

## 2.3. Problems with non-linear semigroups

In the works [Tat92a, Tat92b, Tat94, Ish92, CL94a] the authors study the problem in the case in which the operator $A$ is non-linear. In particular here we describe the works [Tat92b, Tat94, CL94a] that introduce two main improvements with respect to [CL91]:

- The use of nonlinear operator
- The absence of the auxiliary operator $P$

making the theory more general. Indeed to prove the comparison result between a subsolution $w$ and a supersolution $v$ it is only needed the upper semicontinuity of $u$ and the lower semicontinuity of $v$ (without $P$-continuity assumptions). $A$ is assumed to be $m$-dissipative (see below).

Here we briefly recall some results for nonlinear operator in Banach spaces and then we will describe the results proved in [CL94a], where the authors revisit the results of [Tat92b, Tat94] using a simplified notation and rend the results more accessible[7].

[7] In [KSŚ97] the authors prove that the definitions of [Tat92b] and [CL94a] are not always equivalent, but weak hypotheses are needed in order to obtain equivalence.

**2.3.1. Some results for nonlinear operator in Hilbert spaces.** References for the theory of nonlinear operator in Banach spaces are [DP76, Bar76, Bar93, Bré73], we refer in particular to [Bar93].

Let $K$ and $Y$ be Banach spaces, A *multivalued operator* $A$ is a subset of $K \times Y$. We call

$$\begin{array}{l} Ax = \{y \in Y \ : \ (x,y) \in A\} \subseteq Y \\ D(A) = \{x \in K \ : \ Ax \neq \emptyset\} \subseteq K \\ R(A) = \bigcup_{x \in D(A)} Ax \subseteq Y \\ A^{-1} = \{(y,x) \ : \ (x,y) \in A\} \subseteq Y \times K \end{array}$$

The duality map is a multivalued operator $\Delta \subseteq K \times K^*$ defined as

$$\Delta x = \left\{ x^* \in K^* \ : \ \langle x, x^* \rangle = |x|_K^2 = |x^*|_{K^*}^2 \right\}$$

A multivalued operator $A \subseteq K \times K$ is said *dissipative* if for all $(x_1, y_1), (x_2, y_2) \in A$ there exists a $f \in \Delta(x_1 - x_2)$ such that $\langle y_1 - y_2, f \rangle \leq 0$. A dissipative operator $A$ is said *$m$-dissipative* if $R(I - A) = K$. $A$ is said *accreative* (resp. $m$-accreative) if $-A$ is dissipative (resp. $m$-dissipative).

PROPOSITION 2.28. *Let $A$ be a $m$-dissipative nonlinear operator and $y_0 \in \overline{D(A)}$. Then the Cauchy problem*

$$\begin{cases} \frac{\mathrm{d}y(t)}{\mathrm{d}t} - Ay \ni 0 \\ y(0) = y_0 \end{cases}$$

*has a unique mild solution $y(\cdot)$. We write $\mathrm{e}^{tA} y_0$ to mean such solution. We have*

$$e^{tA} = \lim_{n \to \infty} \left( I + \frac{t}{n} A \right)^{-n} y_0$$

*uniformly in $t$ on compact intervals.*

PROOF. See [Bar93] page 204. ☐

**2.3.2. Results for the non-linear setting.** In the sequel we will assume that $\overline{D(A)} = X$. This fact, that in the linear case is a consequence of hypothesis that $A$ is a generator of a $C_0$-semigroup, is not guaranteed in the non-linear case. This assumption is not assumed in [CL94a] and in the Tataru's works but is useful to make the description easier. In [Tat92a, Tat92b, Tat94] the unbounded term $\langle Ax, \nabla w(x) \rangle$ is treated using a limit and a new perturbation technique is used in order to obtain the maxima in the comparison proof:

We consider an HJB equation of the form (1.6):

$$\rho w(x) - \langle Ax, \nabla w(x) \rangle - H(x, \nabla w(x)) = 0,$$

In order to give meaning to the term $\langle Ax, \nabla w(x) \rangle$ we consider the liminf (the limsup in the supersolution definition) of the quantity

$$\frac{w(e^{hA}y) - w(y)}{h}$$

for $h \to 0$ and $y \to x$. Actually in the definition the function $w$ will be replaced with a regular test function $\Phi$ so that we will consider a term of the form

$$\liminf_{\substack{h \downarrow 0 \\ y \to x}} \frac{\Phi(e^{hA}y) - \Phi(y)}{h},$$

that can be computed also when $\nabla \Phi(x) \notin D(A^*)$.

The test functions we will consider are made of two part: $\Phi = \varphi + \psi$ where $\varphi$ is in $C^1(X)$ and $\psi \in Lip(X)$ (in the sequel when will write $\Phi = \varphi + \psi$ we will refer to this decomposition). The regular part $\varphi$ will be used in the comparison proof to localize the problem and to penalize the doubling, whereas the Lipschitz term will be used as perturbation to create the maximum. We set

$$H_\lambda(x,p) \stackrel{def}{=} \inf_{|q|\le\lambda} H(x,p+q)$$

and

$$H^\lambda(x,p) \stackrel{def}{=} \sup_{|q|\le\lambda} H(x,p+q)$$

and we will call $[\psi]_L$ the Lipschitz constant of $\psi$.

Now we can describe the hypotheses and give the definition of viscosity solution of HJ (1.6) in the setting of [Tat92b, Tat94, CL94a]:

Hypothesis 2.29. *A is an m-dissipative operator with* $\overline{D(A)} = X$[8].

Hypothesis 2.30. *H is continuous and there exists a modulus $\sigma(\cdot)$ and a local modulus $\sigma(\cdot,\cdot)$ such that*

$$|H(x,q) - H(x,p)| \le \sigma(|p-q|, |p|+|q|) \tag{2.15}$$

*for every $x,q,p \in X$. Moreover*

$$|H(x,\alpha(x-y)) - H(y,\alpha(x-y))| \le \sigma(|x-y| + \alpha|x-y|^2)$$

*for every $x,y \in X$, $\alpha \ge 0$.*

Definition 2.31 (**Viscosity subsolutions in** [CL94a]). *Let $\Omega$ an open set of $X$. An upper semicontinuous function $w\colon \Omega \to \mathbb{R}$ is a viscosity subsolution of (1.6) on $\Omega$ if for every test function $\Phi = \varphi + \psi \in C^1(X) + Lip(X)$ and every local maximum $x$ of $w - \Phi$ we have*

$$\liminf_{\substack{h\downarrow 0\\ y\to x}} \frac{\Phi(e^{hA}y) - \Phi(y)}{h} + H_{[\psi]_L}(x, \nabla\varphi(x)) \le 0$$

Definition 2.32 (**Viscosity supersolutions in** [CL94a]). *Let $\Omega$ an open set of $X$. A lower semicontinuous function $v\colon \Omega \to \mathbb{R}$ is a viscosity supersolution of (1.6) on $\Omega$ if for every test function $\Phi = \varphi + \psi \in C^1(X) + Lip(X)$ and every local minimum $x$ of $v - \Phi$ we have*

$$\limsup_{\substack{h\downarrow 0\\ y\to x}} \frac{\Phi(e^{hA}y) - \Phi(y)}{h} + H^{[\psi]_L}(x, \nabla\varphi(x)) \ge 0$$

Definition 2.33 (**Viscosity solutions in** [CL94a]). *Let $\Omega$ an open set of $X$. A lower continuous function $w\colon \Omega \to \mathbb{R}$ is a viscosity solution of (1.6) if it is both a subsolution and a supersolution.*

An improvement with respect to [CL91] is that here the authors require only the continuity of the viscosity solution. This improvement adds some difficulties in the comparison proof. A key step of the comparison proof is to find a maximum of the function

$$(x,y) \mapsto w(x) - v(y) - \frac{|(x-y)|^2}{2\varepsilon} - \lambda(\mu(x)+\mu(y)) + \alpha Pert(x,y) \tag{2.16}$$

where $\alpha Pert(x,y)$ is a perturbation that allow to "generate maxima" and $\mu(x)$ (here equal to $\frac{|x|^2}{2}$) localizes the problem. In [CL91], as we emphasized in Remark 2.18 the authors used the $P$-continuity of the sub and of the super solution to find a perturbation with differential small in the $D(A^*)$ norm. Here we do not have $P$-continuity and then new techniques have to be developed. In particular the so called Tataru distance as perturbation function will be used. We will see in the next subsection some details of such method.

[8] This second hypothesis is not used in [Tat92b, Tat94, CL94a].

### 2.3.3. Distance space and Tataru's distance.

DEFINITION 2.34 (**Distant space, from** [CL94a]). *A distance space is a couple* $(\mathcal{M}, d)$ *where* $\mathcal{M}$ *is a set and* $d$ *is a function*

$$d \colon \mathcal{M} \times \mathcal{M} \to \mathbb{R}$$

*such that*

$$\begin{array}{ll} (i) & d(x,y) \leq d(x,z) + d(z,y) \\ (ii) & d(x,y) = 0 \Longleftrightarrow (x = y) \end{array}$$

This $d$ is almost a metric but it is non-symmetric

REMARK 2.35. *The notion of distant space, and the related result, will be generalized by Suzuki (*[Suz01], [Suz06]*) that will introduce the notion of $\tau$-spaces.*

DEFINITION 2.36 (**Complete distant space**). *A distant space* $(\mathcal{M}, d)$ *is said complete if whenever* $\{x_n\}$ *is a sequence in* $\mathcal{M}$ *such that*

$$\sum_{n=1}^{\infty} d(x_{n+1}, x_n) < \infty$$

*then there exists* $x \in \mathcal{M}$ *such that* $d(x, x_n) \to 0$.

DEFINITION 2.37 (**d-upper semicontinuity**). *If* $x, x_n \in \mathcal{M}$ *for* $n = 1, 2...$ *we define* $x_n \to_d x$ *if* $\sum_n d(x_{n+1}, x_n) < \infty$ *and* $d(x, x_n) \to 0$. *We say that*

$$w \colon \mathcal{M} \to [-\infty, +\infty]$$

*is* $d$-upper semicontinuous *if*

$$(x_n \to_d x) \Longrightarrow \limsup_{n \to \infty} w(x_n) \leq w(x)$$

PROPOSITION 2.38 (**Ekeland Lemma for distant space**, [Tat92a]). *Let* $(\mathcal{M}, d)$ *a complete distant space and* $w \colon \mathcal{M} \to [-\infty, +\infty)$ *a bounded above and d-upper semicontinuous function. Then for any* $x_0 \in \mathcal{M}$ *and* $\lambda > 0$ *there exists* $x_1 \in \mathcal{M}$ *such that*

$$\begin{array}{ll} (i) & d(x_0, x_1) \leq \frac{w(x_1) - w(x_0)}{\lambda} \\ (ii) & w(x) - \lambda d(x, x_1) \leq w(x_1) \text{ for all } x \in \mathcal{M} \end{array}$$

PROOF. See [CL94a] page 64. □

DEFINITION 2.39 (**Tataru distance, from** [CL94a]). *Given a nonlinear m-dissipative operator A, for any* $x \in X$ *and* $y \in \overline{D(A)}$ *we define the Tataru's distance* $d_A$ *as*

$$d_A(x, y) = \inf_{t \geq 0} \left[ t + |x - e^{tA} y| \right]$$

PROPOSITION 2.40. *The couple* $(\overline{D(A)}, d_A)$ *is a complete distance space. Moreover given* $\{x_n\}, x \in \overline{D(A)}$

$$(x_n \to_{d_A} x) \Longrightarrow (x_n \to x)$$

*so any function* $w \colon \overline{D(A)} \to [-\infty, +\infty]$ *which is upper-semicontinuous in the norm topology is also d-upper semicontinuous.*

PROOF. See [CL94a] page 65. □

The last step we had to do in order to find a maximum in (2.16) is to find a "right" perturbation. Using Proposition 2.38 and Proposition 2.40 we can take

$$Pert(x, y) = d_A(x, \hat{x}) + d_A(y, \hat{y})$$

where $\hat{x}$ and $\hat{y}$ are suitable points of $X$.

THEOREM 2.41 (**Comparison, from** [CL94a] **page 66**). *Let Hypothesis 2.29 and 2.30 be satisfied. If $w(\cdot)$ is a subsolution and $v(\cdot)$ a supersolution and there exist two real constant $C_1$ and $C_2$ such that*

$$w(x) \le C_1|x| + C_2 \quad \text{and} \quad -v(x) \le C_1|x| + C_2$$

*then $w(x) \le v(x)$ for all $x \in X$.*

## 2.4. Existence via Perron's method

In the work [Ish92] Perron's method for HJ equations in Hilbert spaces with unbounded term[9] appears for the first time. Perron's method was already used for the finite dimensional case (see [Ish87, BCD97]) but the definition of viscosity solution given in [CL90, CL91] made it difficult to generalize it to the infinite dimensional case. The definition of viscosity solution given in [Ish92] is referred to the case in which the unbounded operator $-A$ is maximal monotone and it is of the form $-D^-\phi$ for a lower semicontinuous convex function $\phi\colon X \to (-\infty, +\infty]$. Such definition with $A$ of that form, in the best of our knowledge, was in the sequel used only in [Ish93, Kel02] but it have the advantage to make possible the use Perron's method in Hilbert case. Perron's method can be also used in in the setting of [Tat92b, Tat94, CL94a]. To uniform our exposition we refer here to [CL94a] and we cite some key results of the Perron's method in the setting of Definition 2.33 so we refer to definitions and assumptions of Section 2.3.

LEMMA 2.42 ([CL94a] **page 71**). *If $H$ is locally uniformly continuous and $\mathcal{F}$ a non-empty family of upper semicontinuous subsolutions of (1.6) (with respect to Definition 2.31) on an open set $\Omega$. Let*

$$U(x) = \sup_{w \in \mathcal{F}} w(x) \quad \text{for } x \in \Omega$$

*If the upper semicontinuous envelopes $U^* < \infty$ on $\Omega$, then $U^*$ is a subsolution of (1.6) on $\Omega$*[10]

LEMMA 2.43 (([CL94a] **page 72**). *Let $H$ be locally uniformly continuous, $v \in LSC(\Omega)$ a supersolution of (1.6) (with respect to Definition 2.32) and*

$$\mathcal{F} = \{w \ : \ w^* \text{ is a subsolution of (1.6) on } \Omega \text{ and } w \le v\}$$

*nonempty. If*

$$\omega(x) = \sup_{\omega \in \mathcal{F}} w(x)$$

*then $\omega_*$ is a supersolution of (1.6) (with respect to Definition 2.32).*

THEOREM 2.44 (([CL94a] **page 73**). *Let $H$ be locally uniformly continuous. Assume $w$ is an upper semicontinuous subsolution and $v < \infty$ a lower semicontinuous supersolution of (1.6) in $\Omega$ (with respect to Definition 2.32). Moreover assume that given any subsolution $U \in USC(\Omega)$ and supersolution $V \in LSC(\Omega)$ with $w_* \le U$ and $V \le v^*$ we have $U \le V$. Suppose*

$$\mathcal{F} = \{w \in USC(\Omega) \ : \ w \text{ is a subsolution of (1.6) on } \Omega \text{ and } w \le v\}$$

*is nonempty. If*

$$Z(x) = \sup_{w \in \mathcal{F}} w(x)$$

*then $Z \in C(\Omega)$ and $Z$ is the only continuous solution of (1.6) (with respect to Definition 2.33) on $\Omega$ satisfying $w_* \le Z \le v^*$.*

[9]The case without unbounded term (that is $A = 0$) can be treated using the results of [Ish87].

[10]Similar results for finite dimensional case can be found in [CDL90, CEL84, Ish87].

Here we have to assume the comparison between sub and super solution because we are only assuming $H$ locally uniformly continuous. As showed in [CL94a] page 74 it is not difficult to construct a sub and super solution where $H$ satisfies Hypothesis 2.30 with $\lim_{r\to\infty}\frac{\sigma(r)}{r} < 1$ and prove the unique existence of a bounded uniformly continuous solution of (1.6).

## 2.5. Applications of the classical theory to optimal control problem subject to PDE

One of the main uses of optimal control problem in infinite dimensional is the study of optimal control problems governed by PDE. We briefly recall typical problems that can be treated in a infinite dimensional formulation using "classical" literature (that is the literature we have describe in first four sections).

The main examples for the use of strong case of the definition introduced in [CL90] and [CL91] are sub-families of the set of elliptic operators of the form (see [CL90] page 244):

$$\begin{cases} D(A) = \left\{ v \in H_0^1(O) \ : \ Av \in L^2(O) \right\} \\ Av = -\left( -\sum_{i,j} \partial_i(\alpha_{i,j}\partial_j) + \sum_i b_i \partial_i(\beta_i) + c \right) v \end{cases}$$

where $O \subseteq \mathbb{R}^n$ is a bounded domain, $a_{i,j} = a_{j,i}$ and $a_{i,j}, b_i, \beta_i, c \in L^\infty(O)$, $-A$ is monotone and (ellipticity condition)

$$\exists \gamma > 0 \ \ such\ that \quad -\sum_{i,j} a_{i,j}\zeta_i\zeta_j \geq \gamma |\zeta|^2_{\mathbb{R}^n} \quad for\ \zeta \in \mathbb{R}^n$$

We recall now how to obtain operator $P$ (satisfying "strong" condition) for some particular cases:

- If $b_i = \beta_i = 0$ for all $i$ then $A$ is selfadjoint and we can take $P = (I - A)^{-1}$.
- In the case $b_i = 0$ we let $A_0$ be the operator obtained by setting $b_i = \beta_i = c = 0$ and $P_0 = (I - A_0)^{-1}$. The strong condition is satisfied by a multiple of $P_0$.
- If $\beta_i = 0$ we let $A_0$ be the operator obtained by setting $b_i = \beta_i = 0$ and replacing $c$ by a constant $\hat{c}$ large enough so that $A_0$ is monotone. The strong condition is satisfied by $P_0 = (I - A_0)^{-1}$.
- In case $b_i = 0$ and $a_{i,j} \in W^{1,\infty}(O)$ we can choose $P = \gamma \hat{A}^{-1}$ where
$$\begin{cases} D(\hat{A}) = \left\{ v \in H_0^1 \ : \ \Delta v \in L^2(O) \right\} \\ \hat{A}(v) = \Delta v \end{cases}$$
and $\gamma \in \mathbb{R}$ is large enough

In the weak case, as we have seen in Remark 2.24, the operator $P$ can be found for every $A$ such that there exists a constant $\gamma$ such that $(A - \gamma I)$ is the generator of a $C_0$-semigroup of contractions. In particular state equations of delay problems, rewritten in the Hilbert space $M^2$ following the theory described in 1.3, are treatable with this theory if there is not delay in the control.

## 2.6. Works on specific HJB equations with unbounded terms

The general theory developed in the "classical works" of Crandall and Lions, Tataru and Ishii allows to treat a certain number of cases of optimal control problem governed by PDEs. The subsequent literature tried to extend the definition of viscosity solution to study the cases that are non-treatable with classical works, particular attention was devoted to the study of HJB equations related to optimal control problems governed by PDE with boundary control. The presence of boundary control in the PDE gives a non-bounded term in the state equation in Hilbert space (another unbounded term besides the generator of the semigroup). The works presented in this thesis except for Chapter 5 are referred to HJB equations in which this second unbounded term appears.

A key-point in the choice of the definition for the "new" HJB equations is the relation with optimal control problems: a good definition of viscosity solution for an HJB equations allows to prove that the value function of the related optimal control problem is the only solution.

In the last work of the Crandall and Lions' series ([CL94b]) the authors treat the case in which the set $U$ is bounded with respect to the $X_{-1}$ norm (see Notation 2.17) and $L$ is continuous with respect to the $X_1$ norm (moreover they assume that $L$ depends only on the state $x$ and $f(t,x,u) = Ax + u$ with $A$ selfadjoint). In the case $A = \Delta$, $X = L^2(O)$, if we consider $P = (I - A)^{-1}$, such description covers for example the case in which the controls are bounded on $H^{-1}(O)$ and $L$ is continuous on $H^1_0(O)$. In [CL94b] a definition similar to the one used in [CL90] and [CL91] is used. The authors prove that the value function is a viscosity subsolution of the HJB equations (Theorem II.2 of [CL94b]) and that it is grater to every $P$-continuous and uniformly continuous subsolution of the HJB equation[11]. A comparison result between $P$-continuous sub- and super- solutions is also proved.

**2.6.1. HJB related to parabolic equations with boundary control.** [12] [CGS93] is the first work in which an existence and uniqueness result is proved for HJB equation related to boundary control problems of parabolic type. The authors consider the state equation with Neumann boundary condition given by

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s} z(s,r) = \Delta_r z(s,r) + g(z(s,r)) & on\ (0,+\infty) \times O \\ z(0,r) = z_0(r) & on\ O \\ \frac{\partial}{\partial n} z(s,r) = \nu(s,r) & on\ (0,+\infty) \times \partial O \end{cases}$$

As recalled in Section 2 of [CGS93] such equation can be written in abstract form as

$$\begin{cases} \dot{x}(s) = Ax(s) + f(x(t)) + (-A)^{\beta} B(\nu(s)) \\ x(0) = z_0 \end{cases}$$

where $\beta \in (\frac{1}{4}, \frac{1}{2})$, the state space is $X = L^2(O)$, $A$ is defined as

$$\begin{cases} D(A) = \left\{ \phi \in H^2(O) \ : \ \frac{\partial \phi}{\partial n} = 0 \right\} \\ Ax = \Delta x - x \end{cases}$$

and $B\colon L^2(\partial O) \to L^2(O)$ is linear and continuous and it is given by $(-A)^{1-\beta} N$ where $N$ is the Neumann map (see the references contained in [CGS93] and in particular [LM72a]). Also in this case $A$ is selfadjoint. The HJB equation of the system (we take a functional of the form (1.2)) is

$$\rho v(x) - \langle Ax + f(x), \nabla v(x) \rangle - H(x, \mathcal{C} \nabla v(x)) = 0$$

where $\mathcal{C} = (-A)^{\beta}$. The definition of sub and super solution is new and quite particular but some ideas are not much different from that of [Tat92b, Ish92, CL94a] so the kind of approach is quite different from [CL94b][13]. In the definition a class of regularizing convolution operators appears and the limit of the regularizations is considered. Sub and supersolutions are required to be bounded uniformly continuous and weakly continuous so it is possible to find minima without the use of

[11] Indeed they do not prove that the value function is $P$-continuous.

[12] Unboundedness similar to the ones originated by boundary control problems can be encountered in the study of point control, see for instance [LT00, Las93].

[13] Note that [CGS93] is earlier than [CL94b], we have described it now for exposition reasons.

perturbations. In [CT94a] (see also [CT94b]) the finite horizon version is studied with similar techniques.[14]

In the work [CT96c] the problem for Dirichlet boundary conditions is solved. Indeed the tools introduced in [CGS93] are not enough to prove existence and uniqueness of the solution of the HJB equation related to Dirichlet boundary control case. In the Dirichlet case, that is

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s} z(s,r) = \Delta_r z(s,r) + g(z(s,r)) & \textit{on } (0,+\infty)\times O \\ z(0,r) = z_0(r) & \textit{on } O \\ z(s,r) = \nu(s,r) & \textit{on } (0,+\infty)\times \partial O \end{cases},$$

the state equation can be written in abstract form as

$$\begin{cases} \dot{x}(s) = Ax(s) + f(x(t)) + (-A)D(\nu(s)) \\ x(0) = z_0 \end{cases}$$

where the state space is $X = L^2(O)$, $A$ is defined as

$$\begin{cases} D(A) = H^2(O) \cap H_0^1(O) \\ Ax = \Delta x \end{cases}$$

and $D : L^2(\partial O) \to H^{1/2}(O)$ is the Dirichlet map that is continuous and linear (see the references contained in [CT96c] and in particular [LM72a]). Taking $\beta \in (\frac{3}{4}, 1]$ we have that $D\colon L^2(\partial O) \to D(A^{1-\beta})$ is continuous and we can rewrite the state equation as

$$\begin{cases} \dot{x}(s) = Ax(s) + f(x(t)) + (-A)^{\beta} D_\beta(\nu(s)) \\ x(0) = z_0 \end{cases}$$

where $D_\beta = A^{1-\beta}D\colon L^2(\partial O) \to L^2(O)$ is continuous. Using the change of variable

$$y(t) = A^{-\beta}x(t)$$

and calling $B = D_\beta$ the state equation becomes

$$\begin{cases} \dot{x}(s) = Ax(s) + (-A)^{-\beta} f((-A)^{\beta}x(t)) + B(\nu(s)) \\ x(0) = z_0 \end{cases}$$

The HJB equation related to such state equation and functional (1.2) is

$$\rho v(x) - \left\langle Ax + (-A)^{-\beta} f((-A)^{\beta}x(t)), \nabla v(x) \right\rangle - H((-A)^{\beta}x, \nabla v(x)) = 0. \tag{2.17}$$

In [CT96c] the authors prove that the value function (of the transformed problem) is the only viscosity solution of the HJB equation (2.17). The definition (that requires the weakly sequential continuity of the solution as in [CGS93]) is new (the authors state that it uses a certain number of ideas from [Ish92] but they seems quite different). The definition is given using regular test functions $\varphi$ but it is checked only in the points of maximum (or minimum) that are in $D(A)$. A comparison result is also proved. In [CT96a] the authors treat, with the same techniques, the finite horizon version of the same problem. See also [CT96b].

Other improvements in the study of HJB equations arising in semilinear parabolic problems with boundary control are presented in [GJ06] where existence (via value function) and comparison theorem is proved for HJB equations related to a family of optimal control problems including some problems with nonlinear boundary conditions or the case of nonlinearity of Burgers type in two dimension, for example

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s} z(s,r) = \Delta_r z(s,r) + z(s,r) + g(s,r) & \textit{on } (0,+\infty)\times O \\ z(0,r) = z_0(r) & \textit{on } O \\ \frac{\partial}{\partial n} z(s,r) = \hat{h}(z(s,r)) + \nu(s,r) & \textit{on } (0,+\infty)\times \partial O \end{cases}$$

[14] In [QT] the Bolza's problem is treated, the absence of $L$ gives an HJB equation without unbounded term and so the author can use the definition of [CL85].

where $\hat{h}$ is a regular nondecreasing function with $\hat{h}(0) = 0$. The definition used is not very intuitive.

**2.6.2. State constraints.** In [KS98] the authors consider the problem

$$\begin{cases} \dot{x}(s) = Ax(s) + f(x(t), u(t)) \\ x(0) = x \end{cases}$$

with cost functional

$$J(x, u(\cdot)) = \int_0^\infty L(x(s)) + \frac{1}{2}|u(s))|^2 \mathrm{d}s$$

and impose a state constraints defining $L = +\infty$ out of an admissible set $K$. Note that $L$ in the previous literature was taken continuous and assumptions on its growth was imposed and so the new setting presents new difficulties. The author choose a definition of viscosity solution similar to the one of [Tat92b, CL94a] and prove that the value function is a viscosity solution of the HJB equation and that it is the minimal supersolution.

State constraint was treated also in [CGS91] (see Remark 2.6) and in [AIL00] where the authors consider the problem with constraints given by

$$\begin{cases} H(x, w, Dw) \geq 0 & in\ \overline{\Omega} \\ H(x, w, Dw) \leq 0 & in\ \Omega \end{cases}$$

for a continuous Hamiltonian $H$ and an open subset of $\Omega \subseteq X$. Such kind of HJ equation appears in ergodic control. The continuity of $H$ allows to use the definition of viscosity solution given in [CL85].

REMARK 2.45. *The results related to state (and state-control) constraints are interesting especially for the problems that directly come from applied example. We treat particular problems with state-control constraints in Chapter 6, 7 and 8*

## 2.7. Other items

**2.7.1. HJB related to Navier-Stokes equation and other HJB with unbounded term acting on the state.** In [GSŚ02] the HJB equation related to a two-dimensional Navier-Stokes equation is treated. We recall that, given a domain $O \subseteq \mathbb{R}^n$, called $z(s, r)$ the (vectorial) field of the velocities and $p(s, r)$ the scalar field of the pressures, the usual Navier-Stokes equations have the form:

$$\begin{cases} \frac{\partial}{\partial s} z(s, r) = \Delta_r z(s, r) - (z(s, r) \cdot \nabla) z(s, r) - \\ \qquad\qquad -\nabla p(s, r) + g(s, r, u(s)) \ \ on\ (0, T) \times O \\ \nabla \cdot z(s, r) = 0 \ \ on\ [0, T] \times O \\ z(s, r) = 0 \ \ on\ [0, T] \times \partial O \\ z(0, r) = z_0(r) \ \ on\ O \end{cases}$$

where the first equation express the conservation of momentum, the second have the meaning of irrepressibility, the third and the fourth are boundary and initial conditions. $z$ and $p$ are the unknowns. Such equation can be rewritten in abstract form (see [Tem77]) as

$$\begin{cases} \dot{x}(s) = Ax(s) + B(x(s), x(s)) + f(s, u(s)) \\ x(0) = z_0 \end{cases}$$

In such formulation the unknowns $p$ disappear. This is because it is obtained through a projection $P_H$ on the subspace obtained by closing the set of regular compact supported field with divergence equal to 0 and then the term $\nabla p$ disappears. So the (2.7.1) allows to describe the behavior of the velocities $z$. In the equation $A = P_H \Delta$ and the unbounded term $B$ is given by $B(x,y) = P_H((x \cdot \nabla)y)$. The HJB equation becomes:

$$\rho v(x) - \langle A(x) + B(x,x), \nabla v(x) \rangle - H(x, \nabla v(x)) = 0$$

In the Navier-Stokes case the unbounded term $B$ is not given by a boundary control but it comes simply from the form of the equation. The definition used by the author reminds the definition used in [CT96c] (but it is not the same) and allows to prove existence (via value function) and uniqueness in a quite large set of functions (the solution is only required to be continuous). Otherwise the generator $A$ is very good: it is positive selfadjoint and $A^{-1}$ is compact. Viscosity solution for HJB equation related to two dimensional flow problems are also used in [FS94] but there the definition of viscosity solution is very large and it is not possible to prove any comparison results.

A classes of HJB equations arising in optimal control governed by three dimensional Navier-Stokes equations can be studied using the tools introduced by Shimano in [Shi02]. He uses a setting similar to [Ish92], in which $A = (-D^- \phi)$ for a lower semicontinuous convex function $\phi \colon X \to (-\infty, +\infty]$. Existence (via value function) and uniqueness of the solution of the HJB equation is proved. In such setting it is also possible to treat the $p$-Laplacian problem:

$$\begin{cases} \frac{\partial z}{\partial s} = \sum_{i=1}^n \frac{\partial}{\partial r_i} \left( \left| \frac{\partial z}{\partial r_i} \right|^{p-2} \frac{\partial z}{\partial r_i} \right) + |z|^\gamma + g(s,r,u(s)) \;\; on \; [0,T] \times O \\ z(s,r) = 0 \;\; on \; [0,T] \times \partial O \\ z(0,r) = z_0(r) \end{cases}$$

where $O$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $p \geq 2$ and $0 \leq \gamma < p-2$ if $p < n$ or $\gamma \geq 0$ if $n < p$. The results of [Shi02] cannot be used to study HJB equations with unbounded term arising in boundary control because the unbounded term $B$ acts on the state $x$

**2.7.2. Relation with differential games.** As we have already said in [CL91] the existence results are proved interpreting the HJ equation studied as the HJB equation or the HJI equation related to an optimal control problems or games. Relations between the viscosity solution of the HJ equations and differential games is in-deeply studied in finite dimension (see [ES84, Sou85, LS85, LS86]) and it is also the subject of a certain number of works related to HJ equations in Hilbert spaces (see for example [Tat92a, Tat92b, KSŚ97, KS00, GNMR03, GS03, GS04]).

Following the existent literature for the finite dimensional case, the main point of the proof presented in [CL91] is to find a differential game that admits as HJI equation the equation we want to study. It can be done as below when the Hamiltonian $H$ is Lipschitz continuous with respect to the second variable[15], so that (consider equation (1.6))

$$|H(x,p) - H(x,q)| \leq C|p-q|$$

Using such hypothesis we can state that

$$\begin{aligned} (2.18) \quad H(x,p) &= \inf_{|q| \leq C} (H(x,q) + C|p-q|) = \\ &= \inf_{|q| \leq C} \sup_{|z| \leq C} (- \langle p, z \rangle + \langle q, z \rangle + H(x,q)) \end{aligned}$$

and then it associated to the differential game characterized by state equation

$$\begin{cases} \dot{x}(s) = Ax(s) + z(q(t)) \\ x(t) = x \end{cases}$$

[15]Indeed, often, when such condition is not satisfied, the first step of the proof to reduce the problem to such case.

and cost functional

$$J(x, z(\cdot), q(\cdot)) = \int_t^{+\infty} e^{-s}(\langle q(s), z(q(s))\rangle + H(x(s), q(s)))\mathrm{d}s$$

We call $M(s, x(s), z(s), q(s)) = (\langle q(t), z(q(t))\rangle + H(x(t), q(t)))$ so the cost functional becomes

$$J(x, z(\cdot), q(\cdot)) = \int_t^{+\infty} e^{-s} M(s, x(s), z(s), q(s))\mathrm{d}s$$

Its lower value function is

$$U(t, x) = \inf_{z(\cdot)\in\mathcal{S}_t} \sup_{q(\cdot)\in\mathcal{R}_t} J(x, z(\cdot), q(\cdot))$$

where

$$\mathcal{R}_t \stackrel{def}{=} \{q\colon [t, +\infty) \to B_C \ : \ q \text{ mesurable}\}$$

and

$$\mathcal{S}_t \stackrel{def}{=} \{z\colon \mathcal{R}_t \to \mathcal{R}_t \ : \ z \text{ non anticipating } \}$$

where non anticipating means that if $q_1 = q_2$ almost everywhere on an interval $[0, T]$ then $z(q_1) = z(q_2)$ almost everywhere on $[0, T]$. The method used to prove that the lower value function is a viscosity solution of the (1.6) is similar to the one used in finite dimensional one (see for example [ES84, Sou85]). It is based (like in the optimal control problem) on an integral equation satisfied by lower value function:

$$U(t, x) = \inf_{z(\cdot)\in\mathcal{S}} \sup_{q(\cdot)\in\mathcal{R}} \left\{ \int_t^{t+\tau} M(s, x(s), z(s), q(s))\mathrm{d}s + U(t+\tau, x(t+\tau)) \right\}$$

We want in particular to cite the work [KSŚ97] in which the authors explore the role of lower value function in the context of [CL94a]: We consider the two players differential games characterized by the dynamical system

$$\begin{cases} \dot{x}(s) = Ax(s) + f(z(s), q(t)) \\ x(0) = x \end{cases}$$

and payoff given by

$$\int_0^{\infty} e^{-s} M(x(s), z(s), q(s))\mathrm{d}s$$

The related HJI equation is

$$v(x) - \langle Ax, \nabla v(x)\rangle + H^+(x, \nabla v(x)) = 0$$

where

$$H^+(x, p) = \sup_z \inf_q \{\langle -f(x, q, z), p\rangle - M(x, q, z)\}$$

Then

THEOREM 2.46 ([KSŚ97] page 401). *If $f$ and $M$ are continuous and there exists $C > 0$ such that for all $(x_i, q, z) \in X \times Q \times Z$, $i = 1, 2$*

$$\begin{aligned} &|f(x_1, q, z)| \le C(1 + |x_1|) \\ &|M(x_1, q, z)| \le C \\ &|f(x_1, q, z) - f(x_2, q, z)| \le C|x_1 - x_2| \\ &|M(x_1, q, z) - M(x_2, q, z)| \le \sigma(|x_1 - x_2|) \end{aligned}$$

*Then the lower value function is a viscosity solution of the HJI equation (with respect to the Definition 2.33) and it is the unique solution in the class of the bounded uniformly continuous function on $X$.*

Analogous results are proved in [CL91] for the Definition 2.11, in [Tat92b] for the definition of viscosity solution given there (that is almost the same of [CL94a] but not exactly the same). Other improvements to the theory can be found in [KS00, GNMR03, GS03, GS04]

**2.7.3. Exit time problems.** In [Bar91] the author prove, using a definition a bit stronger than that introduced in [CL90] existence (through minimal time function) and uniqueness for a class of HJ associated with the time-optimal control problem for a semilinear evolution equation

$$\begin{cases} \dot{x}(s) = Ax(s) + F(x(s)) + Bu(s)) \\ x(0) = x \end{cases} . \tag{2.19}$$

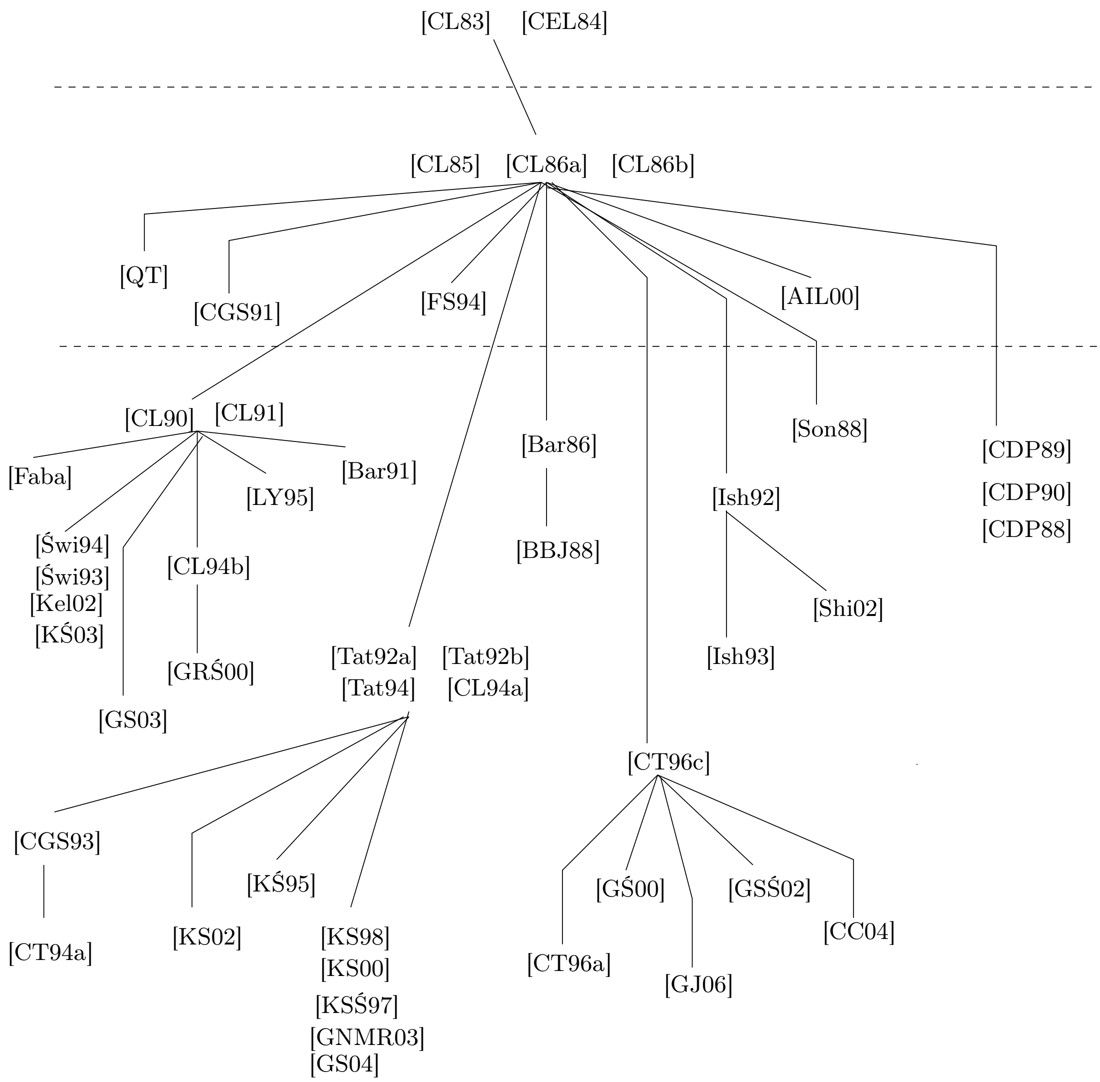
The generator is assumed to be analytical and $F$ dissipative. Moreover the author assume the reachable set is all the space.

In [CC04] a more general semilinear problem is studied ($F$ is assumed to be Lipschitz continuous, $A$ strictly dissipative and the reachable set is admitted to be a subset of $X$). The authors prove an existence and uniqueness result (the only solution is the minimal time function) using a Kruzkov-like transformation and a definition of viscosity solution similar to that of [CT96c]. See also [CS97, Car00].

REMARK 2.47. *In the finite dimensional case is well developed there are results that connect maximum principle and viscosity solution of the HJB equation (see for example* [BJ86, BJ90, BJ91]*). For the infinite dimensional, in the best of our knowledge, only the results of the work* [BBJ88] *are available.*

## 2.8. A genealogy of the definitions

In the following graph we try to summarize the genealogy of the used definitions. Of course when an author write a paper he does not refer only to a precise work but to all existing literature. So our scheme can be very debatable. Anyway we think it can be useful to orient in the existing contributions. We have also inserted some works that refer to second order HJB equations. It is not the field we have tried to explore in this Chapter, but it is a related problem and it can be interesting to see the connections between the definitions in the two cases.



CHAPTER 3

# A viscosity solution approach to the infinite dimensional HJB equation related to boundary control problem in transport equation

In the present chapter we describe the results of the work [Faba] in which the HJB equation related to the infinite dimensional formulation of an optimal control problem subject to a PDE of transport type is studied.

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## 3.1. Introduction

We consider the PDE

$$\begin{cases} \frac{\partial}{\partial s}x(s,r) + \beta\frac{\partial}{\partial r}x(s,r) = -\mu x(s,r) + \tilde{u}(s,r) & (s,r) \in (0,+\infty)\times(0,\bar{s}) \\ x(s,0) = u(s) \quad if \;\; s>0 \\ x(0,r) = x^0(r) \quad if \;\; r\in[0,\bar{s}] \end{cases} \tag{3.1}$$

where $\bar{s}, \beta$ are positive constants, $\mu \in \mathbb{R}$, the initial data $x^0$ is in $L^2(0,\bar{s})$, and we consider two controls: a boundary control $u$ is in $L^2_{loc}[0,+\infty)$ and a distributed control $\tilde{u} \in L^2_{loc}([0,+\infty)\times[0,\bar{s}];\mathbb{R})$.[1]

Using the approach and the references described in Section 3.2, the above equation can be written as an ordinary differential equation in the Hilbert space $X = L^2(0,\bar{s})$ as follows

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s}x(s) = Ax(s) - \mu x(s) + \tilde{u}(s) + \beta\delta_0 u(s) \\ x(0) = x^0 \end{cases} \tag{3.2}$$

where $A$ is the generator of a suitable $C_0$ semigroup and $\delta_0$ is the Dirac delta in 0. Such an unbounded contribution in the Hilbert formulation comes from the presence in the PDE of a boundary control (see [BDPDM92]). Besides we consider the problem of minimizing the cost functional

$$J(x,\tilde{u}(\cdot),u(\cdot)) = \int_0^\infty e^{-\rho s} L(x(s),\tilde{u}(s),u(s))\mathrm{d}s \tag{3.3}$$

where $\rho > 0$ and $L$ is globally bounded and satisfies some Lipschitz-type condition, as better described in Section 3.2. The HJB equation related to the control problem with state equation (3.2) and target functional (3.3) is

$$\begin{aligned} \rho w(x) - \langle \nabla w(x), Ax\rangle - \langle \nabla w(x), -\mu x\rangle_{L^2(0,\bar{s})} - \\ - \inf_{(\tilde{u},u)\in\tilde{U}\times U} \Big( \langle \beta\delta_0(\nabla w(x)), u\rangle_{\mathbb{R}} + \langle \nabla w(x), \tilde{u}\rangle_{L^2(0,\bar{s})} + L(x,\tilde{u},u)\Big) = 0. \end{aligned} \tag{3.4}$$

[1] We write "$-\mu x$" instead of "$\mu x$" because it is the standard way to write the equation in the economic literature where $-\mu$ has the meaning of a depreciation factor (and only the case $\mu \geq 0$ is used). Here we consider a generic $\mu \in \mathbb{R}$.

The sets $U$ and $\tilde{U}$ will be introduced in Section 3.2, they are suitable subset respectively of $\mathbb{R}$ and $X$. If we define the value function of the control problem as

$$V(x) \stackrel{def}{=} \inf_{(\tilde{u}(\cdot),u(\cdot))\in\tilde{\mathcal{U}}\times\mathcal{U}} J(x,\tilde{u}(\cdot),u(\cdot)),$$

we wish to prove that $V(\cdot)$ is the unique solution, in a suitable sense, of the HJB equation.

We use the viscosity approach. Our main problem is to write a suitable definition of viscosity solution, so that an existence and uniqueness theorem can be derived for such a solution. The main difficulties we encounter, with respect to the existing literature, is in dealing with the boundary term and the non-analyticity of the semigroup. We substantially follow the original idea of Crandall and Lions ([CL90] and [CL91]) - with some changes, as the reader will rate in Definition 3.15 and Definition 3.16 - of writing test functions as the sum of a "good part" as it is a regular function with differential in $D(A^*)$ and a "bad part" represented by some radial function. The main problems arise in the evaluation of the boundary term on the radial part.

In order to write a working definition in our case, some further requirements are needed, like a $C^2$ regularity of the test functions, the presence of a "remainder term" in the definition of sub/super solution, and the $P$-Lipschitz continuity (see Definition 3.10) of the solution. This last feature guarantees that the maxima and the minima in the definition of sub/super solution remain in $D(A^*)$ (see Proposition 3.23). Some other comments on the definition of solution (Definition 3.15 and 3.16) need some technical details and can be found in Remark 3.18.

The used technique cannot be easily extended to treat a general non-linear problem because we use the explicit form of the PDE that we give in (3.6). A non-trivial generalization would be also that of replacing the constant $\mu$ with a function $\mu(r)$ in $L^\infty(0,\bar{s})$ (see Remark 3.32 for details). Nevertheless the problem remains challenging.

**3.1.1. A motivating economic problem.** Transport equations are used to model a large variety of phenomena, from age-structured population models (see for instance [Ian95, Ani00, IMM05]) to population economics ([FPV04]), from epidemiologic studies to socio-economic science and transport phenomena in physics.

Problems such as (3.1) can be used to describe, in economics, capital accumulation processes where an heterogeneous capital is involved, and this is the reason why the study of infinite dimensional control problem is of growing interest in the economic fields. For instance in the vintage capital models $x(t,s)$ may be regarded as the stock of capital goods differentiated with respect to the time $t$ and the age $s$. Heterogeneous capital, both in the finite and infinite dimensional approach, is used to study depreciation and obsolescence of physical capital, geographical difference in growth, innovation and R&D.

Regarding problems modeled by a transport equation where an infinite dimensional setting is used we cite the following papers: [BG98] and [BG01] on optimal technology adoption in a vintage capital context (in the case of quadratic cost functional), [HKVF03] on capital accumulation, [BG99] on optimal advertising and [Fag] [Fag05b] on the case of general objective convex functional with strong solutions approach. See also [FG04].

Moreover, we mention that the infinite dimensional approach may apply to problems such as issuance of public debt (see [AAB+04] for a description of the problem). In that problem a stochastic setting and simple state-control constraints appear, but hopefully the present work can be a first step in this direction.

The work is organized as follows: in the first section we remind some results on the state equation, we introduce some preliminary remarks on the main operators involved in the problem, we explain some notations, we define the HJB equation and we give the definition of solution. The second section regards some properties of the value function (in particular some regularity properties) that we will be used in the third section to prove that it is the unique (viscosity) solution of the HJB equation.

## 3.2. Notation and preliminary results

**3.2.1. State equation.** In this subsection we will see some properties of the state equation: we write it in three different (and equivalent) forms that point out different properties of the solution. We will use all the three forms in the following proofs.

We consider the PDE on $[0,+\infty)\times[0,\bar{s}]$ given by

$$\begin{cases} \frac{\partial}{\partial s}x(s,r)+\beta\frac{\partial}{\partial r}x(s,r)=-\mu x(s,r)+\tilde{u}(s,r) & (s,r)\in(0,+\infty)\times(0,\bar{s})\\ x(s,0)=u(s) \quad if \;\; s>0 \\ x(0,r)=x^0(r) \quad if \;\; r\in[0,\bar{s}] \end{cases} \tag{3.5}$$

Given an initial datum $x^0\in L^2(0,\bar{s})$ (from now simply $L^2(0,\bar{s})$), a boundary control $u(\cdot)\in L^2_{loc}([0,+\infty);\mathbb{R})$ and a distributed control $\tilde{u}(\cdot)\in L^2_{loc}([0,+\infty)\times[0,\bar{s}];\mathbb{R})$ the (3.5) has a unique solution in $L^2_{loc}([0,+\infty)\times[0,\bar{s}];\mathbb{R})$ given by

$$x(s,r)=\begin{cases} e^{-\mu s}x^0(r-\beta s)+\int_0^s e^{-\mu\tau}\tilde{u}(s-\tau,r-\beta\tau)\mathrm{d}\tau & r\in[\beta s,\bar{s}]\\ e^{\frac{-\mu}{\beta}r}u(s-r/\beta)+\int_0^{r/\beta}e^{-\mu\tau}\tilde{u}(s-\tau,r-\beta\tau)\mathrm{d}\tau & r\in[0,\beta s) \end{cases} \tag{3.6}$$

In the following $x(s,r)$ is the function defined in (3.6).

REMARK 3.1. *To avoid confusion if $x\in L^2(0,\bar{s})$ we will use $[\cdot]$ to denote the pointwise evaluation, so $x[r]$ is the value of $x$ in $r\in[0,\bar{s}]$. On the other hand $x(s)$ will denote the evolution of the solution of the state equation (in the Hilbert space) at time $s$ (as in (3.7)). That is, $x(s)$ is an element of $X$ while $x[r]$ is a real number.*

We can rewrite such equation in a suitable Hilbert space setting. We take the Hilbert space $X\overset{def}{=}L^2(0,\bar{s})$ and the $C_0$ semigroup $S(t)$ given by

$$S(s)f[r]\overset{def}{=}\begin{cases} f(r-\beta s) & for\;\; r\in[\beta s,\bar{s}]\\ 0 & for\;\; r\in[0,\beta s) \end{cases}$$

The generator of $S(s)$ is the operator $A$ given by

$$\begin{cases} D(A)=\{f\in H^1(0,\bar{s})\;:\;f(0)=0\}\\ A(f)[r]=-\beta\frac{\mathrm{d}}{\mathrm{d}r}f(r) \end{cases}$$

(see [BG01] for a proof in the case $\beta=1$, the proof in our case can be obtained simply taking $s'=\beta s$). In the following we will use the notation $e^{sA}$ instead of $S(s)$.

We want to write an infinite dimensional formulation of (3.5) but in $L^2(0,\bar{s})$ it should appear like

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s}x(s)=Ax(s)-\mu x(s)+\tilde{u}(s)+\beta\delta_0u(s)\\ x(0)=x^0 \end{cases} \tag{3.7}$$

where $\tilde{u}(s)\in L^2(0,\bar{s})$ is the function $r\mapsto\tilde{u}(s,r)$. Such expression does not make sense in $L^2(0,\bar{s})$ for the presence of the unbounded term $\beta\delta_0u(s)$ . We can anyway apply formally the variation of constants method to (3.7) and obtain a mild form of (3.7) that is continuous from $[0,+\infty)$ to $L^2(0,\bar{s})$. This is what we do in the next definition.

DEFINITION 3.2 (**Mild solution**). *Given* $x^0 \in L^2(0,\bar{s})$, $u(\cdot) \in L^2_{loc}([0,+\infty);\mathbb{R})$ *and*
$\tilde{u}(\cdot) \in L^2_{loc}([0,+\infty);L^2(0,\bar{s}))$ *the function in* $C([0,+\infty);L^2(0,\bar{s}))$ *given by*

$$x(s) = e^{-\mu s}e^{sA}x^0 - A\int_0^s e^{-\mu(s-\tau)}e^{(s-\tau)A}(u(\tau)\nu)\mathrm{d}\tau + \\ + \int_0^s e^{-\mu(s-\tau)}e^{(s-\tau)A}\tilde{u}(\tau)\mathrm{d}\tau \tag{3.8}$$

*where*

$$\begin{aligned} \nu &\colon [0,\bar{s}] \to \mathbb{R} \\ \nu &\colon r \mapsto e^{-\frac{\mu}{\beta}r} \end{aligned}$$

*is called* mild solution *of (3.7).*

REMARK 3.3. *We could include the term* $-\mu x$ *in the generator of the semigroup* $A$ *taking a* $\tilde{A} = A - \mu\mathbb{1}$ *as done in* [BG01] *The problem of this approach is that often we will use, in the estimates, the dissipativity of the generator and* $\tilde{A}$ *is dissipative only if* $\mu \geq 0$.

PROPOSITION 3.4. *Taken* $x(s)$ *the function from* $\mathbb{R}^+$ *to* $L^2(0,\bar{s})$ *given by (3.8) and* $x(s,r)$ *the function from* $\mathbb{R}^+\times[0,\bar{s}]$ *to* $\mathbb{R}$ *given by (3.6) we have* $x(s)[r] = x(s,r)$.

PROOF. See [BG01]. □

Eventually we observe that (3.7) can be rewritten in a precise way in a larger space in which $\beta\delta_0$ belongs. To this extent, we consider the adjoint operator $A^*$, whose explicit expression is given by

$$\begin{cases} D(A^*) \overset{def}{=} \{f \in H^1(0,\bar{s}) \ : \ f(\bar{s}) = 0\} \\ A^*(f)[r] = \beta\frac{\mathrm{d}}{\mathrm{d}r}f(r) \end{cases}$$

We observe that the Dirac's measure $\delta_0 \in D(A^*)'$ and we use the $A^{(E)}$ notation introduce in Lemma 1.15:

PROPOSITION 3.5. *Given* $T > 0$, $x^0 \in L^2(0,\bar{s})$, $u(\cdot) \in L^2(0,T)$, $\tilde{u}(\cdot) \in L^2((0,T);L^2(0,\bar{s}))$, *(3.8) is the unique solution of*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s}x(s) = A^{(E)}x(s) - \mu x(s) + \tilde{u}(s) + \beta\delta_0 u(s) \\ x(0) = x^0 \end{cases} \tag{3.9}$$

*in* $W^{1,2}((0,T);D(A^*)') \cap C([0,T];X)$.

PROOF. See [BDPDM92] Chapter 3.2 (in particular Theorem 3.1 page 173). □

**3.2.2. The definition of the operator $P$.** In this subsection we give the definition of the operator $P$ that will have a fundamental role. We could use an abstract approach using the properties of the operators $A$ and $A^*$, anyway in this case we can also follow a more direct approach that allows to find the explicit form of the operator we are going to introduce.

We note that $A^*$ and $A$ are negative operators we take $\phi \in D(A^*)$, so that $\phi(\bar{s}) = 0$, then

$$\langle A^*\phi, \phi\rangle = \int_0^{\bar{s}} \beta\phi'(r)\phi(r)\mathrm{d}r = \frac{-\beta\phi(0)^2}{2}$$

and for $\phi \in D(A)$ (so that $\phi(0) = 0$)

$$\langle A\phi, \phi\rangle = \int_0^{\bar{s}} -\beta\phi'(r)\phi(r)\mathrm{d}r = \frac{-\beta\phi(\bar{s})^2}{2}$$

Therefore, given a $\lambda > 0$, the operators $(A - \lambda \mathrm{I})$ and $(A^* - \lambda \mathrm{I})$ are strongly negative: $\langle (A - \lambda \mathrm{I})x, x\rangle \leq -\lambda |x|_X^2$ for all $x \in D(A)$ and $\langle (A^* - \lambda \mathrm{I})x, x\rangle \leq -\lambda |x|_X^2$ for all $x \in D(A^*)$.

We can also directly prove that

$$(A - \lambda \mathrm{I})^{-1} \colon X \to D(A)$$

is a continuous negative linear operator whose explicit expression is given by

$$(A - \lambda \mathrm{I})^{-1}(\phi)[r] = \frac{1}{\beta}\left(-e^{-\frac{\lambda}{\beta}r}\int_0^r e^{\frac{\lambda}{\beta}\tau}\phi(\tau)\mathrm{d}\tau\right)$$

The continuity can be proved directly with not difficult estimates and the negativity can be proved directly using an integration by part argument.

In the same way we can prove that

$$(A^* - \lambda \mathrm{I})^{-1} \colon X \to D(A^*)$$

is a continuous and negative linear operator and that and its explicit expression is given by

$$(A^* - \lambda \mathrm{I})^{-1}(\phi)[r] = \frac{1}{\beta}\left(-e^{\frac{\lambda}{\beta}r}\int_r^{\bar{s}} e^{-\frac{\lambda}{\beta}\tau}\phi(\tau)\mathrm{d}\tau\right)$$

Eventually we can define $P \stackrel{def}{=} (A^* - \lambda \mathrm{I})^{-1}(A - \lambda \mathrm{I})^{-1} = ((A - \lambda \mathrm{I})^{-1})^*(A - \lambda \mathrm{I})^{-1}$ that is continuous, positive and selfadjoint[2]. Moreover

$$(A^* - \lambda \mathrm{I})P = (A - \lambda \mathrm{I})^{-1} \leq 0$$

and so

$$A^*P = (A - \lambda \mathrm{I})^{-1} + \lambda P \leq \lambda P$$

if we choose $\lambda < 1$ we have that $A^*P$ is bounded and

$$A^*P \leq P \tag{3.10}$$

Thus $P$ satisfies all requirements of the so called "weak case" of [CL90] (see Remark 2.24).

Remark 3.6. *We note that $P^{1/2}$ is a particular case of the operator that Renardy found in more generality in* [Ren95] *and so $P^{1/2} \colon X \to D(A^*)$ continuously and in particular $\mathcal{R}(P^{1/2}) \subseteq D(A^*)$.*

Notation 3.7. *For every $x \in X$ we will indicate with $|x|_P$ the $P$-norm that is $\sqrt{\langle Px, x\rangle_X}$. We will write $X_P$ for the completion of $X$ with respect to the $P$-norm.*

Remark 3.8. *Thanks to the definition of $A^*$, the graph norm on $D(A^*)$ is equivalent to the $H^1(0, \bar{s})$ norm. In particular $D(A^*)$ is the the completion of*

$$K = \{f|_{[0,\bar{s}]} \ : \ f \in C_c^\infty(\mathbb{R}) \text{ with } supp(f) \subseteq (-\infty, \bar{s})\}$$

*with respect to the $H^1(0, \bar{s})$ norm. So, since $H^1(0, \bar{s}) \hookrightarrow C([0, \bar{s}])$, we can apply $\beta\delta_0$ on the elements of $D(A^*)$.*

Notation 3.9. *The notation $\langle x, y\rangle_H$ will indicate the inner product in the Hilbert space $H$ (for example $H = X \equiv L^2(0, \bar{s})$ or $H = H^1(0, \bar{s})$ or $D(A^*)$...). Otherwise if $Z$ is an Banach space (possibly an Hilbert space) and $Z'$ its dual the notation $\langle x, y\rangle_{Z\times Z'}$ will indicate the duality pairing. In a few words, a single index means inner product, a double one indicates duality. Eventually $\langle x, y\rangle \equiv \langle x, y\rangle_{L^2(0,\bar{s})}$.*

[2] See [Yos95] Proposition 2 page 273 for a proof of the equality $(A^* - \lambda \mathrm{I})^{-1} = ((A - \lambda \mathrm{I})^{-1})^*$.

**3.2.3. The control problem and the HJB equation.** In this subsection we describe the optimal control problem, state the hypotheses, define the HJB equation of the system and give a suitable definition of solution of the HJB equation.

We consider the optimal control problem governed by the state equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s}x(s) = Ax(s) - \mu x(s) + \tilde{u}(s) + \beta\delta_0 u(s) \\ x(0) = x \end{cases} \tag{3.11}$$

that has a unique solution in the sense described in Section 3.2.1. Given two compact subsets $U$ and $\Lambda$ of $\mathbb{R}$, we consider the set of admissible boundary controls given by

$$\mathcal{U} \stackrel{def}{=} \{u\colon [0,+\infty) \to U \subseteq \mathbb{R} \ : \ u(\cdot) \ \textit{is measurable}\}.$$

Moreover we call

$$\tilde{U} \stackrel{def}{=} \{\gamma\colon [0,\bar{s}] \to \Lambda \subseteq \mathbb{R} \ : \ \gamma(\cdot) \ \textit{is measurable}\}.$$

In view of the compactness of $\Lambda$ we have that $\tilde{U} \subseteq L^2(0,\bar{s})$. We define the set of admissible distributed controls as

$$\tilde{\mathcal{U}} \stackrel{def}{=} \left\{\tilde{u}\colon [0,+\infty) \to \tilde{U} \subseteq L^2(0,\bar{s}) \ : \ \tilde{u}(\cdot) \ \textit{is measurable}\right\}.$$

In view of the compactness of $U$ and $\Lambda$, we have $\mathcal{U} \subseteq L^2_{loc}([0,+\infty);\mathbb{R})$ and $\tilde{\mathcal{U}} \subseteq L^2_{loc}([0,+\infty) \times [0,\bar{s}];\mathbb{R})$. We call $\|U\| \stackrel{def}{=} \sup_{u\in U}(|u|)$, $\|\Lambda\| \stackrel{def}{=} \sup_{b\in\Lambda}(|b|)$ and $\|\tilde{U}\| \stackrel{def}{=} \sup_{\tilde{u}\in\tilde{U}}(|\tilde{u}|_{X=L^2(0,\bar{s})})$ (they are bounded thanks to the boundedness of $U$ and $\Lambda$).

We call *admissible control* a couple $(\tilde{u}(\cdot), u(\cdot)) \in \tilde{\mathcal{U}} \times \mathcal{U}$. The cost functional will be of the form

$$J(x, \tilde{u}(\cdot), u(\cdot)) = \int_0^\infty e^{-\rho s} L(x(s), \tilde{u}(s), u(s))\mathrm{d}s$$

where $L$ is uniformly continuous and satisfies the following conditions: there exists a $C_L \geq 0$ with

(L1) $|L(x,\tilde{u},u) - L(y,\tilde{u},u)| \leq C_L \langle P(x-y), (x-y)\rangle \ \forall(\tilde{u},u) \in \tilde{U} \times U$
(L2) $|L| \leq C_L < +\infty$

We define formally the HJB equation of the system as

$$\rho w(x) - \langle \nabla w(x), Ax\rangle - \langle \nabla w(x), -\mu x\rangle - H(x, \nabla w(x)) = 0 \tag{3.12}$$

where $H$ is the Hamiltonian of the system and is defined as:

$$\begin{cases} H\colon X \times D(A^*) \to \mathbb{R} \\ H(x,p) \stackrel{def}{=} \inf_{(\tilde{u},u)\in\tilde{U}\times U} \left(\langle \beta\delta_0(p), u\rangle_{\mathbb{R}} + \langle p, \tilde{u}\rangle_X + L(x,\tilde{u},u)\right) \end{cases}$$

(according to Notation 3.9 $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ is the usual real product).

Before introducing a suitable definition of (viscosity) solution of the HJB equation we give some preliminary definitions.

DEFINITION 3.10 (**P-Lipschitz continuity**). *A function $v \in C(X)$ is Lipschitz with respect to the $P$-norm, or $P$-Lipschitz, if there exists a constant $C$ such that*

$$|v(x) - v(y)| \leq C|(x-y)|_P \stackrel{def}{=} C|P^{1/2}(x-y)|_X$$

*for every choice of $x$ and $y$ in $X$. In the same way we can give the definition of a locally $P$-Lipschitz function.*

DEFINITION 3.11 (**P-continuity**). *A function $v \in C(X)$ is said to be $P$-*continuous at a point $x \in X$ *if for every $x_n \in X$ with $x_n \rightharpoonup x$ and $|P(x_n - x)| \to 0$, it holds that $v(x_n) \to v(x)$.*

DEFINITION 3.12 (**P-lower semicontinuity**). *A function $v \in C(X)$ is said to be $P$-*lower semicontinuous continuous at a point *$x \in X$ if for every $x_n \in X$ with $x_n \rightharpoonup x$ and $|P(x_n - x)| \to 0$, it holds that $v(x) \le \liminf_{n\to\infty} v(x_n)$.*

DEFINITION 3.13 (**test1**). *We say that a function $\varphi$ such that $\varphi \in C^1(X)$ and $\varphi$ is $P$-lower semicontinuous is a test function of type 1, and we write $\varphi \in test1$, if $\nabla\varphi(x) \in D(A^*)$ for all $x \in X$ and $A^*\nabla\varphi \colon X \to X$ is continuous.*

DEFINITION 3.14 (**test2**). *We say that $g \in C^2(X)$ is a test function of type 2, and we write $g \in test2$, if $g(x) = g_0(|x|)$ for some nondecreasing function $g_0 \colon \mathbb{R}^+ \to \mathbb{R}$.*

DEFINITION 3.15 (**Viscosity subsolution**). *A function $w \in C(X)$ bounded and Lipschitz with respect to the $P$-norm, is a* subsolution *of the HJB equation (or simply a "*subsolution*") if for every $\varphi \in test1$, $g \in test2$, local maximum point $x$ of $w - (\varphi + g)$ we have*

$$\rho w(x) - \langle A^*\nabla\varphi(x), x\rangle - \langle \nabla\varphi(x) + \nabla g(x), -\mu x\rangle - \\ - \inf_{(\tilde u, u)\in \tilde U\times U} \Big( \langle \beta\delta_0(\nabla\varphi(x), u\rangle_{\mathbb{R}} + \langle \nabla\varphi(x) + \nabla g(x), \tilde u\rangle_X + L(x,\tilde u, u)\Big) \le \\ \le \frac{g_0'(|x|)}{|x|}\beta\frac{\|U\|^2}{2} \tag{3.13}$$

DEFINITION 3.16 (**Viscosity supersolution**). *A function $v \in C(X)$ bounded and Lipschitz with respect to the $P$-norm is a* supersolution *of the HJB equation (or simply a "*supersolution*") if for every $\varphi \in test1$, $g \in test2$, local minimum point $x$ of $v + (\varphi + g)$ we have*

$$\rho v(x) + \langle A^*\nabla\varphi(x), x\rangle + \langle \nabla\varphi(x) + \nabla g(x), -\mu x\rangle - \\ - \inf_{(\tilde u, u)\in \tilde U\times U} \Big( - \langle \beta\delta_0(\nabla\varphi(x), u\rangle_{\mathbb{R}} - \langle \nabla\varphi(x) + \nabla g(x), \tilde u\rangle_X + L(x,\tilde u, u)\Big) \ge \\ \ge -\frac{g_0'(|x|)}{|x|}\beta\frac{\|U\|^2}{2} \tag{3.14}$$

DEFINITION 3.17 (**Viscosity solution**). *A function $v \in C(X)$ bounded and Lipschitz with respect to the $P$-norm is a* solution *of the HJB equation if it is at the same time a supersolution and a subsolution.*

REMARK 3.18. *In the definition of viscosity solution we have used two kinds of test functions: those in test1 and those in test2 which, as usual in the literature, play a different role. In view of their properties and their regularity the functions of the first set (test1) represent the "good part" . More difficulties is to deal with the functions of the set test2, that have the role of localizing the problem. A difficulty of our case is to treat the term*

$$\frac{g(x(s)) - g(x)}{s} \tag{3.15}$$

*(where $x(s)$ is a trajectory starting from $x$). The idea then to consider only $P$-Lipschitz solution so that the maxima/minima considered in Definition 3.15 and Definition 3.16 are in $D(A^*)$. If the starting point $x$ is in $D(A^*)$ there are some advantages in the estimate of (3.15) but some problems remain: in such case we will prove in Proposition 3.26 that (if $\tilde u(\cdot)$ is continuous).*

$$\left| \frac{g(x(s)) - g(x)}{s} - \langle \nabla g(x), -\mu x + \tilde u(0)\rangle \right| \le \frac{g_0'(|x|)}{|x|}\beta\frac{\|U\|^2}{2} + O(s)$$

*where the rest $O \xrightarrow{s\to 0} 0$ and does not depend on the control. So the most challenging case is the one described in the definition.*

## 3.3. The value function and its properties

The value function is, as usual, the candidate unique solution of the HJB equation. In this section we define the value function $V(\cdot)$ of the problem and then we verify that it has the regularity properties required to be a solution. Namely we will check that $V(\cdot)$ is $P$-Lipschitz (Proposition 3.22). To obtain such result we prove an approximation result (Proposition 3.19) and then a suitable estimate for the solution of the state equation (Proposition 3.21).

The value function of our problem is defined as:

$$V(x) \stackrel{def}{=} \inf_{(\tilde{u}(\cdot),u(\cdot))\in\tilde{\mathcal{U}}\times\mathcal{U}} J(x,\tilde{u}(\cdot),u(\cdot))$$

We consider the functions

$$\begin{cases} \eta_n\colon [0,\bar{s}] \to \mathbb{R} \\ \eta_n(r) \stackrel{def}{=} [2n - 2n^2 r]^+ \end{cases}$$

(where $[\cdot]^+$ is the positive part). We then define

$$\begin{cases} \mathcal{C}_n^*\colon \mathbb{R} \to X \\ \mathcal{C}_n^*\colon \gamma \mapsto \gamma\eta_n \end{cases}$$

Such functions are linear and continuous and their adjoints are

$$\begin{cases} \mathcal{C}_n\colon X \to \mathbb{R} \\ \mathcal{C}_n\colon x \mapsto \langle x, \eta_n\rangle \end{cases} \tag{3.16}$$

$\mathcal{C}_n$ "approximate the delta measure". The approximating equations we consider are

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s} x_n(s) = Ax_n(s) - \mu x_n(s) + \tilde{u}(s) + \beta\mathcal{C}_n^* u(s) \\ x_n(0) = x \end{cases} \tag{3.17}$$

Together with (3.8) we use the mild form of the approximating state equations (that can be found in [Paz83] page 105 equation (2.3)):

$$x_n(s) = e^{-\mu s}e^{sA}x + \int_0^s e^{-(s-\tau)\mu}e^{(s-\tau)A}\tilde{u}(\tau)\mathrm{d}\tau + \\ + \int_0^s e^{-(s-\tau)\mu}e^{(s-\tau)A}\beta\mathcal{C}_n^* u(\tau)\mathrm{d}\tau \tag{3.18}$$

PROPOSITION 3.19. *For $T > 0$ and $(\tilde{u}(\cdot), u(\cdot)) \in \tilde{\mathcal{U}} \times \mathcal{U}$*

$$\lim_{n\to\infty} \sup_{s\in[0,T]} |x_n(s) - x(s)|_X = 0$$

PROOF. Using the mild expressions we find

$$|x(s) - x_n(s)| = \Big| -A\int_0^s e^{-(s-\tau)\mu}e^{(s-\tau)A}(u(\tau)\nu)\mathrm{d}\tau - \\ - \int_0^s e^{-(s-\tau)\mu}e^{(s-\tau)A}\beta\mathcal{C}_n^*(u(\tau))\mathrm{d}\tau \Big| \tag{3.19}$$

To estimate such expression we will use the explicit expression of the two terms (as two-variable function). We simplify the notation (only in this proof!) taking an "extension" of $u(\cdot)$ to the whole $\mathbb{R}$ obtained by putting $u(\cdot)$ identically 0 on $\mathbb{R}^-$. So

$$y(s,r) \stackrel{def}{=} \left(-A\int_0^s e^{-(s-\tau)\mu}e^{(s-\tau)A}(u(\tau)\nu)\mathrm{d}\tau\right)[r] = e^{-\frac{\mu}{\beta}r}u(s - r/\beta)$$

$$(3.20)\quad y_n(s,r) \stackrel{def}{=} \left(\int_0^s e^{-(s-\tau)\mu} e^{(s-\tau)A} \beta \mathcal{C}_n^*(u(\tau)) \mathrm{d}\tau\right)[r] = \\ = \int_0^{r\wedge(1/n)} e^{-\frac{\mu}{\beta}(r-\theta)} [2n - 2n^2\theta]^+ u\left(\frac{\theta - r}{\beta} + s\right) \mathrm{d}\theta$$

Now for all $s \in [0,T]$

$$(3.21)\quad |y(s,\cdot) - y_n(s,\cdot)|^2_{X=L^2(0,\bar{s})} \leq \\ \leq \left(\int_{1/n}^{\bar{s}} \left| e^{-\frac{\mu}{\beta}r} u(s - r/\beta) - \int_0^{1/n} e^{-\frac{\mu}{\beta}(r-\theta)} [2n - 2n^2\theta]^+ u\left(\frac{\theta - r}{\beta} + s\right) \mathrm{d}\theta \right|^2 \mathrm{d}r\right) + \\ + \left(\int_0^{1/n} \left| e^{-\frac{\mu}{\beta}r} u(s - r/\beta) - \int_0^{r} e^{-\frac{\mu}{\beta}(r-\theta)} [2n - 2n^2\theta]^+ u\left(\frac{\theta - r}{\beta} + s\right) \mathrm{d}\theta \right|^2 \mathrm{d}r\right) \leq$$

(for $\bar{s} \leq T$)

$$(3.22)\quad \leq \left(e^{|\mu| s} \int_0^T \left| e^{-\mu(\frac{r}{\beta} - s)} u(s - r/\beta) - \right.\right. \\ \left.\left. - \int_0^{1/n} e^{-\mu(\frac{r-\theta}{\beta} - s)} [2n - 2n^2\theta]^+ u\left(s + \frac{\theta - r}{\beta}\right) \mathrm{d}\theta \right|^2 \mathrm{d}r\right) + \left(\frac{1}{n} e^{|\mu|/\beta T} 2\|U\|\right).$$

Such estimate does not depends on $s$, the integral term goes to zero because it is the convolution of a function in $L^2(0,T)$ with an approximate unit and the second goes to zero for $n \to \infty$. □

PROPOSITION 3.20. *Let $\varphi \in C^1(X)$ be such that $\nabla\varphi \colon X \to D(A^*)$ ($D(A^*)$ is endowed, as usual, with the graph norm) is continuous. Then, for an admissible control $(\tilde{u}(\cdot), u(\cdot))$, if we call $x(\cdot)$ the trajectory starting from $x$ and subject to the control $(\tilde{u}(\cdot), u(\cdot))$, we have that, for every $s > 0$,*

$$(3.23)\quad \varphi(x(s)) = \varphi(x) + \int_0^s [\langle A^* \nabla\varphi(x(\tau)), x(\tau)\rangle + \langle \beta\delta_0(\nabla\varphi(x(\tau))), u(\tau)\rangle_{\mathbb{R}} + \\ + \langle \nabla\varphi(x(\tau)), \tilde{u}(\tau)\rangle + \langle \nabla\varphi(x(\tau)), -\mu x(\tau)\rangle] \,\mathrm{d}\tau$$

PROOF. In the approximating state equation (3.17) the unbounded term $\beta\delta_0$ does not appear ($\beta\mathcal{C}_n^*$ are continuous) and then (see [LY95] Proposition 5.5 page 67) for every $\varphi(\cdot) \in C^1(X)$ such that $A^*\nabla\varphi(\cdot) \in C(X)$ we have

$$(3.24)\quad \varphi(x_n(s)) = \varphi(x) + \int_0^s [\langle A^* \nabla\varphi(x_n(\tau)), x_n(\tau)\rangle + \langle \nabla\varphi(x_n(\tau)), \beta\mathcal{C}_n^* u(\tau)\rangle + \\ + \langle \nabla\varphi(x_n(\tau)), \tilde{u}(\tau)\rangle + \langle \nabla\varphi(x_n(\tau)), -\mu x_n(\tau)\rangle] \,\mathrm{d}\tau.$$

In view of the continuity of the operator $\mathcal{C}_n^*$ we can pass to its adjoint (see (3.16) for an explicit form of the operator $\mathcal{C}_n$) and we obtain:

$$(3.25)\quad \varphi(x_n(s)) = \varphi(x) + \int_0^s [\langle A^* \nabla\varphi(x_n(\tau)), x_n(\tau)\rangle + \langle \beta\mathcal{C}_n \nabla\varphi(x_n(\tau)), u(\tau)\rangle + \\ + \langle \nabla\varphi(x_n(\tau)), \tilde{u}(\tau)\rangle + \langle \nabla\varphi(x_n(\tau)), -\mu x_n(\tau)\rangle] \,\mathrm{d}\tau.$$

Now we prove that every integral term of the (3.25) converges to the corresponding term of the (3.23). This fact, together with the pointwise convergence of ($\varphi(x_n(s)) \xrightarrow{n\to\infty} \varphi(x(s))$ due to Proposition 3.19) prove the claim.

First we note that, in view of Proposition 3.19 and of the continuity of $x$, $x_n(\tau)$ is bounded uniformly in $n$ and $\tau \in [0,s]$ and, in view of the continuity of $\nabla\varphi$, $\nabla\varphi(x_n(r))$ is bounded uniformly in $n$ and $\tau \in [0,s]$ So we can apply the Lebesgue theorem (the pointwise convergence is given by Proposition 3.19 and $|\tilde{u}(\tau)| \leq \|\tilde{U}\|$) and we prove that

$$\int_0^s [\langle \nabla\varphi(x_n(\tau)), \tilde{u}(\tau)\rangle + \langle \nabla\varphi(x_n(\tau)), -\mu x_n(\tau)\rangle]\, \mathrm{d}\tau \xrightarrow{n\to\infty} \tag{3.26}$$

$$\xrightarrow{n\to\infty} \int_0^s [\langle \nabla\varphi(x(\tau)), \tilde{u}(\tau)\rangle + \langle \nabla\varphi(x(\tau)), -\mu x(\tau)\rangle]\, \mathrm{d}\tau$$

Now we observe that, in view of the continuity of $A^*\nabla\varphi$ and of the of Proposition 3.19, the term $A^*\nabla\varphi(x_n(\tau))$ is bounded uniformly in $n$ and $\tau \in [0, s]$ so the same is true for

$$|A^*\nabla\varphi(x_n(\tau)) - A^*\nabla\varphi(x(\tau))|.$$

Therefore we can use the Lebesgue theorem (the pointwise convergence is given by Proposition 3.19) to conclude that

$$\int_0^s \langle A^*\nabla\varphi(x_n(\tau)), x_n(\tau)\rangle\, \mathrm{d}\tau \to \int_0^s \langle A^*\nabla\varphi(x(\tau)), x(\tau)\rangle\, \mathrm{d}\tau$$

We have now to prove that

$$\int_0^s \langle \beta\mathcal{C}_n\nabla\varphi(x_n(\tau)), u(\tau)\rangle\, \mathrm{d}\tau \to \int_0^s \langle \beta\delta_0(\nabla\varphi(x(\tau))), u(\tau)\rangle_{\mathbb{R}}\, \mathrm{d}\tau \tag{3.27}$$

We first note that $\mathcal{C}_n \xrightarrow{n\to\infty} \delta_0$ in $H^{-1}(0, \bar{s}) \hookrightarrow D(A^*)'$. Indeed given $z \in H^1(0, \bar{s})$ we have

$$|(\mathcal{C}_n - \delta_0)z| = \left| \int_0^{\bar{s}} z[\tau]\eta_n[\tau]\mathrm{d}\tau - z[0] \right| = \tag{3.28}$$

$$= \left| \int_0^{1/n} \left( z[0] + \int_0^\tau \partial_\omega z[r]\mathrm{d}r \right) \eta_n[\tau]\mathrm{d}\tau - z[0] \right| =$$

($\partial_\omega z$ is the weak derivative of z) integrating by part

$$= \left| \left( z[0] + \int_0^{1/n} \partial_\omega z[r]\mathrm{d}r \right) \left( \int_0^{1/n} \eta_n[r]\mathrm{d}r \right) - \right. \tag{3.29}$$

$$\left. - \int_0^{1/n} \partial_\omega z[\tau] \int_0^\tau \eta_n[r]\mathrm{d}r\mathrm{d}\tau - z[0] \right| \leq$$

writing $\eta_n$ in explicit form and making computation (using $\int_0^{1/n} \eta_n[r]\mathrm{d}r = 1$)

$$\leq \left| \int_0^{\bar{s}} \chi_{[0,1/n]}[\tau]|\partial_\omega z[\tau]|\mathrm{d}\tau \right| \leq \frac{1}{\sqrt{n}}\|z\|_{H^1(0,\bar{s})}$$

Summarizing: by Proposition 3.19 $x_n(\cdot) \xrightarrow{n\to\infty} x(\cdot)$ in $C([0,T];X)$, then (by hypothesis on $\varphi$) $\nabla\varphi(x_n(\cdot)) \xrightarrow{n\to\infty} \nabla\varphi(x(\cdot))$ in $C([0,T];D(A^*))$ and then, by the last estimate $\beta\mathcal{C}_n(\nabla\varphi(x_n(\cdot))) \xrightarrow{n\to\infty} \beta\delta_0(\nabla\varphi(x(\cdot)))$ in $C([0,T])$. Then (3.27) follows by Cauchy-Schwartz inequality (it is the scalar product in $L^2(0,s)$). □

PROPOSITION 3.21. *Given $T > 0$ and a control $(\tilde{u}(\cdot), u(\cdot)) \in \tilde{\mathcal{U}} \times \mathcal{U}$ there exists $c_T$ such that for every $x, y \in X$*

$$\sup_{s\in[0,T]} |x_x(s) - x_y(s)|_P^2 \leq c_T|x - y|_P^2,$$

*where $x_y(\cdot)$ is the solution of*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s}x(s) = Ax(s) + \tilde{u}(s) - \mu x(s) + \beta\delta_0 u(s) \\ x(0) = y \end{cases}$$

*and $x_x(\cdot)$ the solution with initial data $x$*

PROOF. We use Proposition 3.20 with $\varphi(x) = \langle Px, x\rangle$. So $\nabla\varphi(x) = 2Px$. We observe that $x_x(\cdot) - x_y(\cdot)$ satisfies the equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s}(x_x(s) - x_y(s)) = A(x_x(s) - x_y(s)) - \mu(x_x(s) - x_y(s)) \\ (x_x - x_y)(0) = x - y \end{cases}$$

(the one of Proposition 3.20 with control identically 0) and then by (3.10)

(3.30)

$$\begin{aligned} |x_x(s) - x_y(s)|_P^2 = |x-y|_P^2 + 2\int_0^s \langle A^*P(x_x(r) - x_y(r)), (x_x(r) - x_y(r))\rangle - \\ - \mu \langle P(x_x(s) - x_y(s)), x_x(s) - x_y(s)\rangle \,\mathrm{d}r \le \\ \le |x-y|_P^2 + 2(1+|\mu|)\int_0^s \langle P(x_x(r) - x_y(r)), (x_x(r) - x_y(r))\rangle \,\mathrm{d}r \end{aligned}$$

now we can use the Gronwall's lemma and obtain the claim. □

PROPOSITION 3.22. *Let $L$ satisfy (L1) and (L2). Then the value function $V$ is Lipschitz with respect to the $P$-norm*

PROOF. Assume $V(y) > V(x)$. Then we take $(\tilde{u}(\cdot), u(\cdot)) \in \tilde{\mathcal{U}} \times \mathcal{U}$ an $\varepsilon$-optimal control for $x$. We have:

$$V(y) - V(x) - \varepsilon \le \int_0^\infty e^{-\rho t}|L(x_y(s), \tilde{u}(s), u(s)) - L(x_x(s), \tilde{u}(s), u(s))|\mathrm{d}s =$$

If we look the explicit for of $x_x(\cdot)$ and $x_y(\cdot)$ as two-variables functions we see that they depend on the initial data only for $s \in [0, \frac{\bar{s}}{\beta}]$. After this period they depends only on the control. So for $s > \frac{\bar{s}}{\beta}$ $x_x(s) = x_y(s)$ and so the previous integral is equal to

$$= \int_0^{\bar{s}/\beta} e^{-\rho t}|L(x_y(s), \tilde{u}(s), u(s)) - L(x_x(s), , \tilde{u}(s), u(s))|\mathrm{d}s \le$$

(by (L1) and Proposition 3.21)

$$\le \int_0^{\bar{s}} e^{-\rho t} C_L |x_y(s) - x_x(s)|_P \mathrm{d}s \le \bar{s} c_{\bar{s}} C_L |x-y|_P$$

Letting $\varepsilon \to 0$ we have claim. □

## 3.4. Existence and uniqueness of solution

In this section we will prove that the value function is a viscosity solution of the HJB equation (Theorem 3.28) and that the HJB equation admits at most one solution (Theorem 3.31).

We remind that we use $X_P$ to denote the completion of $X$ in the $P$-norm. This notation will be used in the next propositions.

PROPOSITION 3.23. *Let $u \in C(X)$ be a locally $P$-Lipschitz function. Let $\psi \in C^1(X)$, and let $x$ be a local maximum (or a local minimum) of $u - \psi$. Then $\nabla\psi(x) \in \mathcal{R}(P^{1/2}) \subseteq D(A^*)$.*

PROOF. We do the proof only in the case in which $x$ is a local maximum (the other case is similar).

We take $\omega \in X$ with $|\omega| = 1$ and $h \in (0,1)$. Then for every $h$ small enough

$$\frac{(u(x-h\omega) - \psi(x-h\omega))}{h} \le \frac{u(x) - \psi(x)}{h}$$

so

$$\frac{\psi(x)-\psi(x-h\omega)}{h} \leq C|w|_P$$

and passing to the limit we have $\langle \nabla\psi(x), \omega\rangle \leq C|\omega|_P$. Likewise

$$\frac{(u(x+h\omega)-\psi(x+h\omega))}{h} \leq \frac{u(x)-\psi(x)}{h}$$

so

$$\frac{\psi(x)-\psi(x+h\omega)}{h} \leq C|w|_P$$

and passing to the limit we have $-\langle \nabla\psi(x), \omega\rangle \leq C|\omega|_P$.

Putting together these two remarks we have

$$|\langle \nabla\psi(x), \omega\rangle| \leq C|\omega|_P$$

for all $\omega \in X$. So we can consider the linear extension of the continuous linear functional $\omega \mapsto \langle \nabla\psi(x), \omega\rangle$ to $X_P$; we will call such extension $\Phi_x$ and by Riesz representation theorem we can find $z_x \in X_P$ such that

$$\Phi_x(\omega) = \langle z_x, \omega\rangle_{X_P} \quad \forall \omega \in X_P$$

however

$$\begin{aligned}\langle z_x, \omega\rangle_{X_P} &= \left\langle P^{1/2}(z_x), P^{1/2}(\omega)\right\rangle_X = \\ &= \left\langle P^{1/2}(P^{1/2}(z_x)), \omega\right\rangle_{(X_P)'\times(X_P)} = \left\langle P^{1/2}(m_x), \omega\right\rangle_{(X_P)'\times(X_P)}\end{aligned} \tag{3.31}$$

where $m_x \stackrel{def}{=} (P^{1/2}(z_x)) \in X$. Now for $\omega \in X$

$$\left\langle P^{1/2}(m_x), \omega\right\rangle_{(X_P)'\times(X_P)} = \left\langle P^{1/2}(m_x), \omega\right\rangle_X$$

Therefore $\nabla\psi(x) = P^{1/2}(m_x) \in \mathcal{R}(P^{1/2}) \subseteq D(A^*)$ where the last inclusion follows from Remark 3.6. □

**3.4.1. Existence.** In this subsection we will prove that the value function is a solution of the HJB equation. In the next subsection we will prove that such solution is unique. We start with a lemma and two propositions. We will use the notation introduced in Remark 3.1 on "$x(s)$" and "$x[r]$". Moreover we will continue to use the symbol $\delta_0$ in the text so that $x[0] = \delta_0 x$ if $x \in D(A^*)$.

We have not found a simple reference for the following lemma so we prove it:

LEMMA 3.24. *Let $x$ be a function of $H^1(0,\bar{s})$ then*

$$(i) \quad \lim_{s\to 0^+}\left(\int_s^{\bar{s}} \frac{(x[r]-x[r-s])^2}{s}\mathrm{d}r\right) = 0 \tag{3.32}$$

$$(ii) \quad \lim_{s\to 0^+}\left(\int_s^{\bar{s}-s} \frac{(x[r+s]-x[r])}{s}x[r]\mathrm{d}r\right) = \frac{x^2[\bar{s}]-x^2[0]}{2} \tag{3.33}$$

PROOF. **part (i)**

$$\int_s^{\bar{s}} \frac{(x[r]-x[r-s])^2}{s}\mathrm{d}r = \int_0^{\bar{s}} \psi_s[r]\mathrm{d}r$$

where $\psi_s\colon [0,\bar{s}] \to \mathbb{R}$ is defined in the following way:

$$\psi_s[r] = \begin{cases} 0 & if\ r \in [0,s) \\ \frac{(x[r]-x[r-s])^2}{s} & if\ r \in [s,\bar{s}] \end{cases}$$

In order to prove the claim we want to apply the Lebesgue theorem. First we will see the a.e. convergence of the $\psi_s$ to zero: for $r > 0$ we take $s < r$:

$$\psi_s[r] \leq \frac{\left|\int_{r-s}^{r} \partial_\omega x[\tau] \mathrm{d}\tau\right|}{s} |x[r] - x[r-s]|$$

where $\partial_\omega x$ is the weak derivative of $x$ ($x$ is in $H^1$ for hypothesis). Now almost every $r$ is a Lebesgue point and then

$$\frac{\left|\int_{r-s}^{r} \partial_\omega x(\tau) \mathrm{d}\tau\right|}{s} \xrightarrow{s\to 0^+} |\partial_\omega x[r]| \qquad a.e.\ in\ r \in (0, \bar{s}]$$

while the part $|x[r] - x[r-s]|$ goes uniformly to 0.

In order to dominate the convergence we note that by Morrey's theorem ([Eva98] Theorem 4 page 266) every $x \in H^1(0, \bar{s})$ is 1/2-Holder then there exists a positive $C$ such that for every $s \in (0, \bar{s}]$ and every $r \in [s, \bar{s}]$ we have

$$\frac{|x[r] - x[r-s]|}{\sqrt{s}} \leq C$$

and then

$$\frac{|x[r] - x[r-s]|^2}{s} \leq C^2$$

this allows to dominate $\psi_s$ with the constant $C^2$, use the Lebesgue theorem and obtain the claim.

**part (ii):**

$$I(s) \stackrel{def}{=} \int_s^{\bar{s}-s} \frac{(x[r+s] - x[r])}{s} x[r] \mathrm{d}r = \tag{3.34}$$
$$= \int_s^{\bar{s}-s} \frac{(x[r+s]x[r])}{s} \mathrm{d}r - \int_0^{\bar{s}-2s} \frac{(x[r+s]x[r+s])}{s} \mathrm{d}r =$$
$$= -\int_s^{\bar{s}-2s} \frac{(x[r+s] - x[r])}{s} x[r+s] \mathrm{d}r + \int_{\bar{s}-2s}^{\bar{s}-s} \frac{(x[r+s]x[r])}{s} \mathrm{d}r +$$
$$+ \int_0^s -\frac{(x[r+s])^2}{s} \mathrm{d}r \stackrel{def}{=} -I_1(s) + I_2(s) + I_3(s)$$

By the continuity of $x$ we see that:

$$I_2(s) \xrightarrow{s\to 0^+} x^2[\bar{s}]$$

and

$$I_3(s) \xrightarrow{s\to 0^+} -x^2[0]$$

Moreover, using similar arguments that in (i) we find that

$$\lim_{s\to 0^+} (I(s) - I_1(s)) = \lim_{s\to 0^+} \int_s^{\bar{s}-2s} -\frac{(x[r+s] - x[r])^2}{s} \mathrm{d}r + \tag{3.35}$$
$$+ \lim_{s\to 0^+} \int_{\bar{s}-2s}^{\bar{s}-s} \frac{(x[r+s] - x[r])}{s} x[r] \mathrm{d}r = 0$$

so the limit $\lim_{s\to 0^+} I(s)$ exist if and only if there exist the limit $\lim_{s\to 0^+} \frac{I_1(s)+I(s)}{2}$ and in such case they have the same value. But

$$\frac{I_1(s) + I(s)}{2} = \frac{I_2(s) + I_3(s)}{2} \xrightarrow{s\to 0^+} \frac{x^2[\bar{s}] - x^2[0]}{2}$$

and then $\lim_{s\to 0^+} \left(\int_s^{\bar{s}-s} \frac{(x[r+s]-x[r])}{s} x[r] \mathrm{d}r\right) = \frac{x^2[\bar{s}]-x^2[0]}{2}$. □

LEMMA 3.25. *Given $x \in D(A^*)$ there exists a real function $O(s)$ such that $O(s) \xrightarrow{s\to 0} 0$ and such that for every control $(\tilde{u}(\cdot), u(\cdot)) \in \tilde{\mathcal{U}} \times \mathcal{U}$ we have that*

$$|x(s) - x| \leq O(s)$$

*(where we called $x(s)$ the trajectory that starts from $x$ and subject to the control $(\tilde{u}(\cdot), u(\cdot))$). Note that $O(s)$ is independent of the control.*

PROOF. We consider $s \in (0,1]$. This is an arbitrary choice but we are interested only in the behavior of $x(\cdot)$ near to 0 so we can assume it without problems. We use the explicit expression of $x(s,r)$:

$$\begin{aligned}
(3.36)\quad & \|x(s) - x\|^2_{X=L^2(0,\bar{s})} = \\
&= \int_{\beta s}^{\bar{s}} \left| e^{-\mu s} x[r - \beta s] + \int_0^s e^{-\mu\tau} \tilde{u}(s-\tau, r - \beta\tau) \mathrm{d}\tau - x[r] \right|^2 \mathrm{d}r + \\
&+ \int_0^{\beta s} \left| e^{-\frac{\mu}{\beta} r} u(s - r/\beta) + \int_0^{r/\beta} e^{-\mu\tau} u(s-\tau, r-\beta\tau) \mathrm{d}\tau - x[r] \right|^2 \mathrm{d}r \leq \\
&\leq 2\int_{\beta s}^{\bar{s}} \left| e^{-\mu s} x[r-\beta s] - x[r] \right|^2 \mathrm{d}r + 2\int_{\beta s}^{\bar{s}} \left| \int_0^s e^{|\mu|} \|U\| \mathrm{d}\tau \right|^2 \mathrm{d}r + \\
&+ \int_0^{\beta s} \left| e^{|\mu|}\|U\| + \int_0^{r/\beta} e^{|\mu|} \|\Lambda\| \mathrm{d}\tau + |x|_{L^\infty(0,\bar{s})} \right|^2 \mathrm{d}r \leq
\end{aligned}$$

(We have used that $x \in D(A^*) \subseteq H^1(0,\bar{s})$ so it is continuous and $|x|_{L^\infty(0,\bar{s})} < +\infty$)

$$\begin{aligned}
(3.37)\quad & \leq 2\int_0^{\bar{s}} \left| e^{-\mu s} x[(r-\beta s)\wedge 0] - x[r] \right|^2 \mathrm{d}r + 2s^2\bar{s}\left(e^{|\mu|}\|U\|\right)^2 + \\
&+ s\beta\left(e^{|\mu|}\|U\| + |x|_{L^\infty} + s e^{|\mu|}\|\Lambda\|\right)^2
\end{aligned}$$

Observe that in this estimate the control $(\tilde{u}(\cdot), u(\cdot))$ does not appear. The second and the third terms goes to zero for $s \to 0$. In the first we can use Lebesgue theorem observing that

$$\left| e^{-\mu s} x[(r-\beta s)\wedge 0] - x[r] \right| \leq \left( e^{|\mu|}|x|_{L^\infty} + |x|_{L^\infty} \right) \quad \forall (s,r) \in (0,1] \times [0,\bar{s}]$$

and that $|e^{-\mu s} x[(r-\beta s)\wedge 0] - x[r]| \xrightarrow{s\to 0} 0$ pointwise. So the statement is proved. □

PROPOSITION 3.26. *Given $x \in D(A^*)$ and $g \in test2$ there exists a real function $O(s)$ such that $O(s) \xrightarrow{s\to 0} 0$ and such that for every control $(\tilde{u}(\cdot), u(\cdot)) \in \tilde{\mathcal{U}} \times \mathcal{U}$ with $u(\cdot)$ continuous we have that*

$$\left| \frac{g(x(s)) - g(x)}{s} - \frac{\int_0^s \langle \nabla g(x), \tilde{u}(r) \rangle}{s} - \langle \nabla g(x), -\mu x \rangle \right| \leq \frac{g_0'(|x|)}{|x|} \beta \frac{\|U\|^2}{2} + O(s)$$

*(where we called $x(s)$ the trajectory that starts from $x$ and subject to the control $(\tilde{u}(\cdot), u(\cdot))$). Note that $O(s)$ is independent of the control.*

PROOF. First we write

$$\begin{aligned}
(3.38)\quad & \frac{g(x(s)) - g(x)}{s} - \langle \nabla g(x), -\mu x \rangle - \frac{\int_0^s \langle \nabla g(x), \tilde{u}(r) \rangle}{s} = \\
&= \frac{g(x(s)) - g(y(s)) + g(y(s)) - g(x)}{s} - \langle \nabla g(x), -\mu x \rangle - \frac{\int_0^s \langle \nabla g(x), \tilde{u}(r) \rangle}{s}
\end{aligned}$$

where $y(\cdot)$ is the solution of

$$\begin{cases} \dot{y}(s) = Ay(s) + \beta\delta_0 u(s) \\ y(0) = x \end{cases} \tag{3.39}$$

(that is our system when $\mu = 0$ and $\tilde{u}(\cdot) = 0$). $x(\cdot)$ satisfies the mild equation[3]

$$x(s) = e^{sA}x - A\int_0^s e^{(s-\tau)A}(u(\tau)\nu)\mathrm{d}\tau + \int_0^s e^{(s-\tau)A}(\tilde{u}(\tau) - \mu x(\tau))\mathrm{d}\tau \tag{3.40}$$

The term $\left(e^{sA}x - A\int_0^s e^{(s-\tau)A}(u(\tau)\nu)\mathrm{d}\tau\right)$ is the mild solution of $y(\cdot)$ and

$$x(s) - y(s) = \int_0^s e^{(s-\tau)A}(\tilde{u}(\tau) - \mu x(\tau))\mathrm{d}\tau.$$

Now we come back to (3.38), we have

$$\left| \frac{g(x(s)) - g(x)}{s} - \frac{\int_0^s \langle \nabla g(x), \tilde{u}(r)\rangle\,\mathrm{d}r}{s} - \langle \nabla g(x), -\mu x\rangle \right| \leq \tag{3.41}$$

$$\leq \left| \frac{g(x(s)) - g(y(s))}{s} - \frac{\int_0^s \langle \nabla g(x), \tilde{u}(r)\rangle\,\mathrm{d}r}{s} - \langle \nabla g(x), -\mu x\rangle \right| + \left| \frac{g(y(s)) - g(x)}{s} \right|.$$

In order to estimate the first addendum we use the Taylor expansion as follows:

$$\frac{g(x(s)) - g(y(s))}{s} = \left\langle \nabla g(y(s)), \frac{x(s) - y(s)}{s} \right\rangle + \tag{3.42}$$

$$+ \left\langle \nabla g(\xi(s)) - \nabla g(y(s)), \frac{x(s) - y(s)}{s} \right\rangle =$$

where $\xi(s)$ is a point of the line segment connecting $x(s)$ and $y(s)$

$$= \left\langle \nabla g(y(s)), \frac{\int_0^s e^{(s-\tau)A}(\tilde{u}(\tau) - \mu x(\tau))\mathrm{d}\tau}{s} \right\rangle + \tag{3.43}$$

$$+ \left\langle \nabla g(\xi(s)) - \nabla g(y(s)), \frac{\int_0^s e^{(s-\tau)A}(\tilde{u}(\tau) - \mu x(\tau))\mathrm{d}\tau}{s} \right\rangle$$

We know by Lemma 3.25 that $x(s) \xrightarrow{s\to 0} x$ $y(s) \xrightarrow{s\to 0} x$ uniformly in the control $(\tilde{u}(\cdot), u(\cdot))$, and so $\nabla g(y(s)) \xrightarrow{s\to 0} \nabla g(x)$ uniformly in the control and $|\nabla g(y(s)) - \nabla g(\xi(s))| \xrightarrow{s\to 0} 0$ uniformly in the control. Moreover, in view of boundedness of the control and of the fact that $x(s) \xrightarrow{s\to 0} x$ uniformly in the control (Lemma 3.25) we can prove that the term

$$\left| \frac{\int_0^s e^{(s-\tau)A}(\tilde{u}(\tau) - \mu x(\tau))\mathrm{d}\tau}{s} \right|_X$$

is bounded uniformly in the control and $s \in (0, \bar{s}]$ and we conclude that the second term of the (3.43) goes to zero uniformly in $(\tilde{u}(\cdot), u(\cdot))$ and that

$$\left| \frac{g(x(s)) - g(y(s))}{s} - \frac{\int_0^s \langle \nabla g(x), \tilde{u}(r)\rangle\,\mathrm{d}r}{s} - \langle \nabla g(x), -\mu x\rangle \right| \leq O(s) \tag{3.44}$$

where $O(s) \xrightarrow{s\to 0} 0$ and it does not depend on the control.

[3]We have already written an explicit mild form of the solution in (3.8), the form we use here is different, indeed it is not explicit because the $x$ appears also in the second term. The only difference between the two formula is the following: equation (3.8) is the equation we obtain if we include the term $-\mu x$ in the generator of the semigroup, equation (3.40) is the form we obtain if we maintain the term $-\mu x$ out of the generator of the semigroup. The two forms are equivalent.

So we have now to estimate the second term of the (3.41), namely $\left|\frac{g(y(s))-g(x)}{s}\right|$. If we prove that it is smaller then $\frac{g_0'(|x|)}{|x|}\beta\frac{\|U\|^2}{2}+O(s)$ where $O(s)$ does not depend on the control we have proved the proposition.

We first note that

$$\nabla g(x) = g_0'(|x|)\frac{x}{|x|}$$

and

$$D^2 g(x) = g_0''(|x|)\frac{x}{|x|}\otimes\frac{x}{|x|} + g_0'(|x|)\left(\frac{\mathrm{I}}{|x|}-\frac{x\otimes x}{|x|^3}\right)$$

We consider the Taylor's expansion of $g$ at $x$:

$$\begin{aligned}
(3.45)\quad \frac{g(y(s))-g(x)}{s} &= \frac{\langle\nabla g(x), y(s)-x\rangle}{s} + \frac{1}{2}\frac{(y(s)-x)^T(D^2 g(x))(y(s)-x)}{s} + \\
&\quad + \frac{o(|y(s)-x|^2)}{s} = \\
&= \frac{g_0'(|x|)}{|x|}\left(\left\langle x, \frac{y(s)-x}{s}\right\rangle + \frac{1}{2}\frac{\langle y(s)-x, y(s)-x\rangle}{s}\right) + \\
&\quad + \frac{1}{2}\left(\frac{g_0''(|x|)}{|x|^2}-\frac{g_0'(|x|)}{|x|^3}\right)\frac{\langle x, y(s)-x\rangle^2}{s} + \frac{o(|y(s)-x|^2)}{s} \stackrel{def}{=} \\
&\stackrel{def}{=} P1 \;+\; P2 \;+\; P3
\end{aligned}$$

First we prove that $P2$ and $P3$ go to zero uniformly in $(\tilde{u}(\cdot), u(\cdot))$ and then we will estimate $P1$. We proceed in two steps:

**step 1**: *There exists a constant $C$ such that for every admissible control $(\tilde{u}(\cdot), u(\cdot)) \in \tilde{\mathcal{U}}\times\mathcal{U}$ with $u(\cdot)$ continuous and every $s\in(0,1]$*[4]

$$\left|\frac{\langle x, y(s)-x\rangle}{s}\right| \leq C$$

(as before the choice of the interval (0,1] it is not essential: we are interested in the behavior near zero). We observe first that the explicit solution of $y(s)[r]$ can be found taking $\mu=0$ and $\tilde{u}=0$ in (3.6). We have:

$$y(s,r) = \begin{cases} x(r-\beta s) & r\in[\beta s, \bar{s}] \\ u(s-r/\beta) & r\in[0,\beta s) \end{cases}$$

so

(3.46)

$$\begin{aligned}
\frac{\langle x, y(s)-x\rangle}{s} &= \int_{\beta s}^{\bar{s}} x[r]\frac{(x[r-\beta s]-x[r])}{s}\mathrm{d}r + \frac{\int_0^{\beta s} x[r](u(s-r/\beta)-x[r])\mathrm{d}r}{s} = \\
&= \int_{\beta s}^{\bar{s}-\beta s} x[r]\frac{(x[r+\beta s]-x[r])}{s}\mathrm{d}r + \frac{\int_{\bar{s}-\beta s}^{\bar{s}} -x^2[r]\mathrm{d}r}{s} + \frac{\int_0^{\beta s} x[r]x[r+\beta s]\mathrm{d}r}{s} + \\
&\quad + \frac{\int_0^{\beta s} x[r]u(s-r/\beta)\mathrm{d}r}{s} - \frac{\int_0^{\beta s} x^2[r]\mathrm{d}r}{s}
\end{aligned}$$

The third and the fifth part have opposite limits, the second goes to zero thanks to the fact that $x\in D(A^*)$ and then $x$ is continuous and $x(\bar{s})=0$. The first part goes

[4]In the expression of $y(\cdot)$ the distributed control $\tilde{u}(\cdot)$ does not appear, so we will speak from now only of the boundary control $u(\cdot)$.

to $-\frac{\beta}{2}x^2[0] = \langle A^*x, x\rangle$ in view of Lemma 3.24. The only term in which the control appears is the fourth but we can estimate it as follows:

$$\left|\frac{\int_0^{\beta s} x[r]u(s-r/\beta)\mathrm{d}r}{s}\right| \leq \frac{\int_0^{\beta s} |x[r]|\|U\|\mathrm{d}r}{s} \leq \beta \max_{r\in[0,\bar{s}]} |x[r]|\|U\|$$

**step 2**: *There exists a constant $C$ such that for every admissible control $u(\cdot) \in \mathcal{U}$ with $u(\cdot)$ continuous and every $s \in (0,1]$*

$$\frac{|y(s)-x|^2}{s} \leq C$$

Indeed

$$\left|\frac{\langle y(s)-x, y(s)-x\rangle}{s}\right| = \left|\int_{\beta s}^{\bar{s}} \frac{(x[r-\beta s]-x[r])^2}{s}\mathrm{d}r\right| + \\ + \left|\frac{\int_0^{\beta s}(u(s-r/\beta)-x[r])^2\mathrm{d}r}{s}\right| \tag{3.47}$$

in view of the fact that $x \in D(A^*) \subseteq H^1(0,\bar{s})$ and of the Lemma 3.24 the first part goes to zero. Moreover, since $x \in H(0,\bar{s}) \subseteq L^\infty(0,\bar{s})$, the second part is less or equal to

$$\frac{\int_0^{\beta s} \|U\|^2\mathrm{d}r}{s} + \frac{\int_0^{\beta s} 2|x[r]|\|U\|\mathrm{d}r}{s} + \frac{\int_0^{\beta s} |x[r]|^2\mathrm{d}r}{s} \leq C. \tag{3.48}$$

This completes step 2.

From step 2 it follows that

$$\frac{o(|y(s)-x|^2)}{s} = \frac{o(|y(s)-x|^2)}{|y(s)-x|^2}\frac{|y(s)-x|^2}{s} \xrightarrow{s\to 0^+} 0$$

uniformly in $u(\cdot)$. Thus $|P3| \xrightarrow{s\to 0} 0$ uniformly in $u(\cdot)$. Moreover

$$\frac{\langle x, y(s)-x\rangle^2}{s} \leq \frac{|\langle x, y(s)-x\rangle|}{s}|x||y(s)-x|$$

and so, from step 1 and Lemma 3.25, $|P2| \xrightarrow{s\to 0} 0$ uniformly in $u(\cdot)$.

**step 3**: *Conclusion*

We now estimate $P1$. We can write a more explicit form of $P1$ as in the proofs of step 1 and step 2 ((3.46), (3.47) and (3.48)) and using the same arguments we can see that there exists a rest $o(1)$ (depending only on $x$) with $o(1) \xrightarrow{s\to 0} 0$ such that for every control $u(\cdot)$ continuous

$$P1 = \frac{g_0'(|x|)}{|x|}\left(\langle A^*x, x\rangle + \frac{\int_0^{\beta s} x[s]u(s-r/\beta)\mathrm{d}r}{s} + \frac{1}{2}\frac{\int_0^{\beta s}(u(s-r/\beta))^2\mathrm{d}r}{s} + \right. \\ \left. + \frac{1}{2}\frac{\int_0^{\beta s} x^2[r]\mathrm{d}r}{s} + \frac{1}{2}\frac{\int_0^{\beta s} -2x[r]u(s-r/\beta)\mathrm{d}r}{s}\right) + o(1) \tag{3.49}$$

The fourth part of the above, that does not depend on the control, goes to $\beta\frac{x[0]^2}{2}$ that is the opposite of the first part. The second and the fifth part are opposite. So we have that

$$P1 = o(1) + \frac{g_0'(|x|)}{|x|}\left(\frac{1}{2}\frac{\int_0^{\beta s}(u(s-r/\beta))^2\mathrm{d}r}{s}\right) \leq o(1) + \frac{1}{2}\frac{g_0'(|x|)}{|x|}\beta\|U\|^2$$

Now, using the estimates on $P1$, $P2$ and $P3$ we see that

$$\left|\frac{g(y(s)) - g(x)}{s}\right| \leq O(s) + \frac{1}{2}\frac{g_0'(|x|)}{|x|}\beta\|U\|^2.$$

Using this fact and equation (3.44) in (3.41) we have proved the proposition. □

PROPOSITION 3.27. *If $x \in D(A^*)$ and $\varphi \in test1$ then there exists a real function $O(s)$ such that $O(s) \xrightarrow{s\to 0} 0$ and such that for every control $(\tilde{u}(\cdot), u(\cdot)) \in \tilde{\mathcal{U}} \times \mathcal{U}$ with $u(\cdot)$ continuous we have that*

$$\begin{aligned}(3.50)\quad \Bigg| \frac{\varphi(x(s)) - \varphi(x)}{s} - \frac{\int_0^s \langle \nabla\varphi(x), \tilde{u}(r)\rangle \,\mathrm{d}r}{s} - \langle \nabla\varphi(x), -\mu x\rangle - \\ - \langle A^*\nabla\varphi(x), x\rangle - \frac{\int_0^s \langle \beta\delta_0(\nabla\varphi(x)), u(r)\rangle_{\mathbb{R}} \,\mathrm{d}r}{s}\Bigg| \leq O(s)\end{aligned}$$

*(where we called $x(s)$ the trajectory that starts from $x$ and subject to the control $(\tilde{u}(\cdot), u(\cdot))$). Note that $O(s)$ is independent of the control.*

PROOF. We proceed as in the proof of Proposition 3.26 observing that

$$\frac{\varphi(x(s)) - \varphi(x)}{s} = \frac{\varphi(x(s)) - \varphi(y(s))}{s} + \frac{\varphi(y(s)) - \varphi(x)}{s}$$

where $y(\cdot)$ is the solution of (3.39). It is possible to prove, using exactly the same arguments used in the proof of Proposition 3.26 that

$$\left| \frac{\varphi(x(s)) - \varphi(y(s))}{s} - \langle \nabla\varphi(x), -\mu x\rangle - \frac{\int_0^s \langle \nabla\varphi(x), \tilde{u}(r)\rangle \,\mathrm{d}r}{s}\right| \leq O(s)$$

where $O(s) \xrightarrow{s\to 0} 0$ and does not depend on the control. So we have to prove that

$$\left| \frac{\varphi(y(s)) - \varphi(x)}{s} - \langle A^*\nabla\varphi(x), x\rangle - \frac{\int_0^s \beta\,\langle \delta_0\nabla\varphi(x), u(r)\rangle_{\mathbb{R}} \,\mathrm{d}r}{s}\right| \leq O(s)$$

where $O(s) \xrightarrow{s\to 0} 0$ and does not depend on the control.

We write

$$\begin{aligned}(3.51)\quad \frac{\varphi(y(s)) - \varphi(x)}{s} = I_0 + I_1 \overset{def}{=} \left\langle \nabla\varphi(x), \frac{y(s) - x}{s}\right\rangle + \\ + \left\langle \nabla\varphi(\xi(s)) - \nabla\varphi(x), \frac{y(s) - x}{s}\right\rangle\end{aligned}$$

where $\xi(s)$ is a point of the line segment connecting $x$ and $y(s)$. In view of Lemma 3.25, $|y(s) - x| \xrightarrow{s\to 0} 0$ uniformly in the control, so $|\xi(s) - x| \xrightarrow{s\to 0} 0$ uniformly in $u(\cdot)$. By hypothesis

$$\nabla\varphi\colon X \to D(A^*) \quad \textit{and it is continuous}$$

($D(A^*)$ is endowed with the graph norm). Then

$$(3.52)\qquad |\nabla\varphi(\xi(s)) - \nabla\varphi(x)|_{D(A^*)} \xrightarrow{s\to 0} 0$$

uniformly in $u(\cdot)$.

If we read equation (3.39) in $D(A^*)'$ it appears as an equation of the form

$$\begin{cases} \dot{u}(t) = A^{(E)}u(t) + f(t) \\ u(0) = x \end{cases}$$

where $f(t)$ is a bounded measurable function ($|f(t)|_{D(A^*)'} \leq \beta|\delta_0|_{D(A^*)'}\|U\|$) we can choose a constant $C$ that depends on $x$ such that, for all admissible control $u(\cdot)$ continuous and all $s \in (0,1]$,

$$\frac{|y(s)-x|_{D(A^*)'}}{s} \leq C \tag{3.53}$$

Thus by (3.52) and (3.53), we can say that $|I_1| \xrightarrow{s\to 0} 0$ uniformly in $u(\cdot)$. Therefore

$$\left| \frac{\varphi(y(s))-\varphi(x)}{s} - \frac{\langle \nabla\varphi(x), y(s)-x\rangle}{s} \right| \xrightarrow{s\to 0} 0$$

uniformly in $u(\cdot)$. We now write

$$\begin{aligned}\frac{\langle \nabla\varphi(x), y(s)-x\rangle}{s} &= \int_{\beta s}^{\bar{s}} \nabla\varphi(x)[r]\frac{(x[r-\beta s]-x[r])}{s}\mathrm{d}r + \\ &+ \frac{\int_0^{\beta s} \nabla\varphi(x)[r](u(s-r/\beta)-x[r])\mathrm{d}r}{s} = \\ = \int_{\beta s}^{\bar{s}-\beta s} x[r]\frac{\nabla\varphi(x)[r+\beta s]-\nabla\varphi(x)[r]}{s}\mathrm{d}r &+ \int_{\bar{s}-\beta s}^{\bar{s}} \frac{(-\nabla\varphi(x)[r]x[r])}{s}\mathrm{d}r + \\ + \frac{\int_0^{\beta s}(\nabla\varphi(x)[r+\beta s]x[r])\mathrm{d}r}{s} &+ \frac{\int_0^{\beta s} \nabla\varphi(x)[r]u(s-r/\beta)\mathrm{d}r}{s} + \\ &+ \frac{\int_0^{\beta s} -\nabla\varphi(x)[r]x[r]\mathrm{d}r}{s}\end{aligned} \tag{3.54}$$

The third and the fifth terms, that do not depend on the control, have opposite limits, the second goes to zero because $\nabla\varphi(x)$ and $x$ are in $D(A^*)$ and then $x[\bar{s}] = 0 = \nabla\varphi(x)[\bar{s}]$. The first term goes to $\langle A^*\nabla\varphi(x), x\rangle$. Finally we observe that the only term that depends on the control is the fourth and

$$\left| \frac{\int_0^{\beta s} \nabla\varphi(x)[r]u(s-r/\beta)\mathrm{d}r}{s} - \beta\frac{\int_0^{s} \nabla\varphi(x)[0]u(s-r')\mathrm{d}r'}{s} \right| \xrightarrow{s\to 0} 0$$

uniformly in $u(\cdot)$ and, since $\varphi(x)[0]$ is a constant,

$$\beta\frac{\int_0^{s} \nabla\varphi(x)[0]u(s-r)\mathrm{d}r}{s} = \frac{\int_0^{s} \langle \beta\delta_0\nabla\varphi(x), u(r)\rangle_{\mathbb{R}}\,\mathrm{d}r}{s}$$

This complete the proof. ☐

We can now prove that the value function is a solution of the HJB equation equation.

THEOREM 3.28. *Let $L$ satisfy (L1) and (L2) let $U$ and $\Lambda$ be a compact subsets of $\mathbb{R}$. Then the value function $V$ is bounded, $P$-Lipschitz and is a solution of the HJB equation.*

PROOF. The boundedness of $V$ follows from the boundedness of $L$ (assumption (L2)). The $P$-Lipschitz property is the result of Proposition 3.22. It remains to verify that $V$ is a solution of the HJB equation.

**Subsolution:**

Let $x$ be a local maximum of $V-(\varphi+g)$ for $\varphi \in test1$ and $g \in test2$. Thanks to Proposition 3.23 we know that $\nabla(\varphi+g)(x) \in D(A^*)$. Moreover we know that $\nabla\varphi(x) \in D(A^*)$ for the definition of the set test1. So $\nabla g(x) = g_0'(|x|)\frac{x}{|x|} \in D(A^*)$ and this implies that $x \in D(A^*)$. We can assume that $V(x)-(\varphi+g)(x) = 0$. We consider the constant control $(\tilde{u}(\cdot), u(\cdot)) \equiv (\tilde{u}, u) \in \tilde{U}\times U$ and $x(s)$ the trajectory starting from $x$ and subject to $(\tilde{u}, u)$. Then for $s$ small enough

$$V(x(s)) - (\varphi+g)(x(s)) \leq V(x) - (\varphi+g)(x)$$

and thanks to the Bellman principle of optimality we know that

$$V(x) \leq e^{-\rho s} V(x(s)) + \int_0^s e^{-\rho r} L(x(r), \tilde{u}, u) \mathrm{d}r$$

Then

$$\frac{1-e^{-\rho s}}{s} V(x(s)) - \frac{\varphi(x(s)) - \varphi(x)}{s} - \frac{g(x(s)) - g(x)}{s} - \\ - \frac{\int_0^s e^{-\rho r} L(x(r), \tilde{u}, u) \mathrm{d}r}{s} \leq 0. \tag{3.55}$$

Using Proposition 3.26 and Proposition 3.27 we can now pass to the limsup as $s \to 0$ to obtain

$$\rho V(x) - \langle \nabla\varphi(x), -\mu x\rangle - \langle \nabla g(x), -\mu x\rangle - \\ - \Big( \langle A^* \nabla\varphi(x), x\rangle + \langle \beta\delta_0(\nabla\varphi(x)), u\rangle_{\mathbb{R}} + \langle \nabla\varphi(x), \tilde{u}\rangle + \langle \nabla g(x), \tilde{u}\rangle + L(x, \tilde{u}, u) \Big) \leq \\ \leq \frac{g_0'(|x|)}{|x|} \beta \frac{\|U\|^2}{2}. \tag{3.56}$$

Taking the $\inf_{(\tilde{u},u)\in\tilde{U}\times U}$ we obtain the subsolution inequality.

**Supersolution:**
Let $x$ be a minimum for $V + (\varphi + g)$ and such that $V + (\varphi + g)(x) = 0$. As in the subsolution proof we obtain that $x \in D(A^*)$. For $\varepsilon > 0$ take $(\tilde{u}_\varepsilon(\cdot), a_\varepsilon(\cdot))$ an $\varepsilon^2$-optimal strategy. We can assume $u(\cdot)$ continuous (it is not hard to see). We call $x(s)$ the trajectory starting from $x$ and subject to $(\tilde{u}_\varepsilon(\cdot), u_\varepsilon(\cdot)$. Now for $s$ small enough

$$V(x(s)) + (\varphi + g)(x(s)) \geq V(x) + (\varphi + g)(x)$$

and thanks to the $\varepsilon^2$-optimality and the Bellman principle we know that

$$V(x) + \varepsilon^2 \geq e^{-\rho s} V(x(s)) + \int_0^s e^{-\rho r} L(x(r), \tilde{u}_\varepsilon(r), u_\varepsilon(r)) \mathrm{d}r$$

We take $s = \varepsilon$. Then

$$\frac{1-e^{-\rho\varepsilon}}{\varepsilon} V(x(\varepsilon)) + \frac{\varphi(x(\varepsilon)) - \varphi(x)}{\varepsilon} + \frac{g(x(\varepsilon)) - g(x)}{\varepsilon} - \\ - \frac{\int_0^\varepsilon e^{-\rho r} L(x(r), \tilde{u}_\varepsilon(r), u_\varepsilon(r)) \mathrm{d}r}{\varepsilon} + \frac{\varepsilon^2}{\varepsilon} \geq 0 \tag{3.57}$$

in view of Proposition 3.26 and Proposition 3.27 we can choose, independently of the control $(\tilde{u}_\varepsilon(\cdot), u_\varepsilon(\cdot))$, a $o(1)$ with $o(1) \xrightarrow{\varepsilon\to 0} 0$ such that:

$$\rho V(x) + \langle A^* \nabla\varphi(x), x\rangle + \langle \nabla\varphi(x) + \nabla g(x), -\mu x\rangle - \\ - \left( \frac{\int_0^\varepsilon \langle -\beta\delta_0(\nabla\varphi(x), u_\varepsilon(r)\rangle_{\mathbb{R}} + e^{-\rho r} L(x(r), \tilde{u}_\varepsilon(r), u_\varepsilon(r)) \mathrm{d}r}{\varepsilon} - \right. \\ \left. - \frac{\int_0^\varepsilon \langle \nabla\varphi(x) + \nabla g(x), \tilde{u}_\varepsilon(r)\rangle \, \mathrm{d}r}{\varepsilon} \right) \geq o(1) - \frac{g_0'(|x|)}{|x|} \beta \frac{\|U\|^2}{2} \tag{3.58}$$

we now take inf over $u$ and $\tilde{u}$ inside the integral and let $\varepsilon \to 0$ to obtain that

$$\rho V(x) + \langle A^*\nabla\varphi(x), x\rangle + \langle \nabla\varphi(x) + \nabla g(x), -\mu x\rangle - \tag{3.59}$$
$$- \inf_{(\tilde{u},u)\in\tilde{U}\times U} \Big( - \langle \beta\delta_0(\nabla\varphi(x)), u\rangle_{\mathbb{R}} + L(x,\tilde{u},u) - \langle \nabla\varphi(x) + \nabla g(x), \tilde{u}\rangle \Big) \geq$$
$$\geq -\frac{g_0'(|x|)}{|x|}\beta\frac{\|U\|^2}{2}.$$

(we observe again that the fact that $o(1) \xrightarrow{\varepsilon\to 0} 0$ uniformly in the control is essential). Therefore $V$ is a solution of the HJB equation. $\square$

**3.4.2. Uniqueness.** In the proof of uniqueness result we will use the following theorem proved in general case in [EL77]:

THEOREM 3.29. *Let $Y$ be a uniformly continuous Banach space. Let $D$ be a bounded closed subset of $X$ and $f\colon D \to (-\infty, +\infty]$ be a proper lower semicontinuous function bounded from below. Then, for any $\varepsilon > 0$, there exists a $p \in Y^*$ with $|p|_* < \varepsilon$ such that the map $x \mapsto f(x) + \langle p, x\rangle_{Y^*\times Y}$ attains its minimum over $D$ at some point $x_0 \in D$.*

PROOF. See [LY95] page 245. $\square$

Moreover, recalling the notation introduced in Notation 2.17, we will use the following

LEMMA 3.30. *Any convex, bounded, closed set $S \subseteq X$ is convex, bounded and closed in $X_{-2}$.*

PROOF. See [LY95] page 250. $\square$

Now we can prove a uniqueness result: we prove the result in the case $\mu \neq 0$. The case $\mu = 0$ is simpler and can be proved with small changes in the proof.

THEOREM 3.31. *Let $L$ satisfy (L1) and (L2) let $U$ and $\Lambda$ be compact subsets of $\mathbb{R}$. Then given a supersolution $v$ of the HJB equation and a subsolution $w$ we have*

$$w(x) \leq v(x) \ \ for\ every\ x \in X$$

*In particular there exist at most one solution of the HJB equation*

PROOF. We will proceed by contradiction. Assume that $w$ is a subsolution of the HJB equation and $v$ a supersolution and suppose that there exists $\check{x} \in X$ and $\gamma > 0$ such that

$$(w(\check{x}) - v(\check{x})) > \frac{3\gamma}{\rho} > 0$$

We take $\gamma < 1$. So, taken $\vartheta > 0$ small enough we have

$$w(\check{x}) - v(\check{x}) - \vartheta|\check{x}|^2 > \frac{2\gamma}{\rho} > 0 \tag{3.60}$$

We consider $\varepsilon > 0$ and $\Psi\colon X \times X \to \mathbb{R}$ given by

$$\Psi(x,y) \stackrel{def}{=} w(x) - v(y) - \frac{1}{2\varepsilon}|P^{1/2}(x-y)|^2 - \frac{\vartheta}{2}|x|^2 - \frac{\vartheta}{2}|y|^2.$$

Thanks to the boundedness of $w$ and $v$, chosen $\vartheta > 0$, there exist $R_\vartheta > 0$ such that

$$\Psi(0,0) \geq \left( \sup_{(|x|\geq R_\vartheta)\ or\ (|y|\geq R_\vartheta)} (\Psi(x,y)) \right) + 1 \tag{3.61}$$

We set

$$S = \{(x,y) \in X \times X \ : \ |x| \leq R_\vartheta \ and\ |y| \leq R_\vartheta\}$$

If we choose $R_\vartheta$ big enough $\breve{x} \in S$. The next step is to perturbate $\Psi(\cdot,\cdot)$ to obtain a maxima[5]: We first observe that $S$ is bounded convex and closed and so in view of Lemma 3.30 it is bounded convex and closed in $X^2_{-2}$. Next we note that $\Psi(\cdot,\cdot)$ is upper semicontinuous on $S$ with respect to the topology of $X^2_{-2}$. We take $(x_k, y_k) \in S \times S$ and $(x, y)$ such that $|P(x_k - x)| \to 0$ and $|P(y_k - y)| \to 0$. Since $S$ is bounded on $X$ we may assume $x_k \rightharpoonup \tilde{x}$ and $y_k \rightharpoonup \tilde{y}$ in $X$. Since $P$ is continuous and selfadjoint we have $Px_k \rightharpoonup P\tilde{x}$ and $Py_k \rightharpoonup P\tilde{y}$ and for the uniqueness of the weak limit $\tilde{x} = x$, $\tilde{y} = y$. We have $|x_k - x|_{-1} = \langle P(x_k - x), (x_k - x)\rangle^{1/2} \longrightarrow 0$ and the same for $y_k$. So, since $w(\cdot)$ and $v(\cdot)$ are $P$-Lipschitz continuous,

$$w(x_k) \longrightarrow w(x) \quad and \quad v(y_k) \longrightarrow v(y).$$

Eventually we use the $P$-continuity of the function $(x, y) \mapsto |P^{1/2}(x - y)|^2$ and the lower weak semicontinuity of $x \mapsto |x|^2$ to conclude that $\Psi(\cdot,\cdot)$ is upper semicontinuous on $S$ with respect to the topology of $X^2_{-2}$. Moreover $\Psi(\cdot,\cdot)$ is bounded on $S$ that is bounded closed and convex in $X_{-2}$. So we can use Theorem 3.29 and state that there exist $\tilde{p}$ and $\tilde{q}$ in $(X_{-2})' = X_2$ such that $|\tilde{p}|_2, |\tilde{q}|_2 \le \sigma$ and

$$(x, y) \mapsto \varphi(x, y) - \langle \tilde{p}, x\rangle_{X_2 \times X_{-2}} - \langle \tilde{q}, y\rangle_{X_2 \times X_{-2}}$$

attains a maximum over $S$ in $(\bar{x}, \bar{y})$. But for definition of $X_2$ there exist $p, q \in X$ with $\tilde{q} = Pq$, $\tilde{p} = Pp$ and $|p| = |\tilde{p}|_2 \le \sigma$, $|q| = |\tilde{q}|_2 \le \sigma$ and so

$$(x, y) \mapsto \varphi(x, y) - \left\langle \tilde{P}p, x\right\rangle - \left\langle \tilde{P}q, y\right\rangle$$

attains a maximum over $S$ in $(\bar{x}, \bar{y})$. If we choose $\sigma$ small enough (for example such that $\sigma\|P\|R_\vartheta < \frac{1}{4}\frac{\gamma}{\rho}$) we know by (3.61) that such maximum is in the interior of $S$ and, thanks to (3.60), that

$$\Psi(\bar{x}, \bar{y}) - \langle Pp, \bar{x}\rangle - \langle Pq, \bar{y}\rangle > \frac{3\gamma}{2\rho}.$$

Moreover

$$\Psi(\bar{x}, \bar{y}) > \frac{\gamma}{\rho} \quad and\ so \quad w(\bar{x}) - v(\bar{y}) > \frac{\gamma}{\rho}. \tag{3.62}$$

We now make some preliminary estimates that we will use in the following:

**Estimates 1 (on $\varepsilon$):**

We observe that

$$\begin{cases} M\colon (0,1] \to \mathbb{R} \\ M\colon \varepsilon \mapsto \sup_{(x,y)\in X\times X} \left(w(x) - v(y) - \frac{1}{2\varepsilon}\left|P^{1/2}(x-y)\right|^2\right) \end{cases}$$

is non-decreasing and bounded and so it admits a limit for $\varepsilon \to 0^+$. So there exists a $\bar{\varepsilon} > 0$ such that, for every $0 < \varepsilon_1, \varepsilon_2 \le \bar{\varepsilon}$ we have that

$$|M(\varepsilon_1) - M(\varepsilon_2)| < \left(\frac{\gamma}{16(1 + |\mu|)}\right)^2 \tag{3.63}$$

We choose now $\varepsilon$, that will be fixed in the sequel of the proof:

$$\varepsilon := min\left\{\bar{\varepsilon}, \frac{1}{32C_L^2}\right\} \tag{3.64}$$

($C_L$ is the constant introduced in hypotheses (L1) and (L2)). Now we state and prove a claim that we will use in the following:

[5]We use an approach similar to the one of [LY95] (page 252) with some differences due to the different properties of the functions $w$ and $v$.

**Claim**
*If $\tilde{x} \in X$ and $\tilde{y} \in X$ satisfy*

$$w(\tilde{x}) - v(\tilde{y}) - \frac{1}{2\varepsilon}\left|P^{1/2}(\tilde{x}-\tilde{y})\right|^2 \geq M(\varepsilon) - \left(\frac{\gamma}{16(1+|\mu|)}\right)^2 \tag{3.65}$$

*then*

$$\frac{1}{\varepsilon}\left|P^{1/2}(\tilde{x}-\tilde{y})\right|^2 \leq \frac{1}{32}\left(\frac{\gamma}{(1+|\mu|)}\right)^2 \tag{3.66}$$

proof of the claim:
(We follow the idea used in Lemma 3.2 of [CL94a])

$$\begin{aligned} M(\varepsilon/2) &\geq w(\tilde{x}) - y(\tilde{y}) - \frac{1}{4\varepsilon}\left|P^{1/2}(\tilde{x}-\tilde{y})\right|^2 = \\ &= w(\tilde{x}) - y(\tilde{y}) - \frac{1}{2\varepsilon}\left|P^{1/2}(\tilde{x}-\tilde{y})\right|^2 + \frac{1}{4\varepsilon}\left|P^{1/2}(\tilde{x}-\tilde{y})\right|^2 \geq \\ &\geq M(\varepsilon) - \left(\frac{\gamma}{16(1+|\mu|)}\right)^2 + \frac{1}{4\varepsilon}\left|P^{1/2}(\tilde{x}-\tilde{y})\right|^2 \end{aligned} \tag{3.67}$$

So

$$\begin{aligned} \frac{1}{4\varepsilon}\left|P^{1/2}(\tilde{x}-\tilde{y})\right|^2 &\leq M(\varepsilon/2) - M(\varepsilon) + \left(\frac{\gamma}{16(1+|\mu|)}\right)^2 \leq \\ &\leq \left(\frac{\gamma}{16(1+|\mu|)}\right)^2 + \left(\frac{\gamma}{16(1+|\mu|)}\right)^2 = 2\left(\frac{\gamma}{16(1+|\mu|)}\right)^2 \end{aligned} \tag{3.68}$$

where the inequality $M(\varepsilon/2) - M(\varepsilon) < \left(\frac{\gamma}{16(1+|\mu|)}\right)^2$ follows from the definition of $\varepsilon$ (3.64) that implies $\varepsilon \leq \bar{\varepsilon}$ and then the (3.63). The claim follows. ∎

From (3.64) we have

$$\frac{1}{\sqrt{\varepsilon}} \geq 4\sqrt{2}C_L$$

and then if $\tilde{x}$, $\tilde{y}$ satisfy the hypothesis (3.65) of the claim we have

$$C_L\left|\tilde{x}-\tilde{y}\right|_P \leq \frac{\gamma}{32(1+|\mu|)} \tag{3.69}$$

**Estimates 2 (on $\sigma$):**
We have already imposed $\sigma < \frac{\gamma/\rho}{4\|P\|R_\vartheta}$, we take from now

$$\sigma = min\left\{\frac{\gamma}{8\rho\|P\|R_\vartheta}, \vartheta, \frac{\vartheta}{R_\vartheta}\right\} \tag{3.70}$$

so that

$$\sigma \xrightarrow{\vartheta\to 0} 0 \tag{3.71}$$

and

$$\sigma R_\vartheta \xrightarrow{\vartheta\to 0} 0 \tag{3.72}$$

We recall that we have already fixed $\varepsilon$ in (3.64). From the choice of $\sigma$ (3.70) follows that

$$|\langle Pp, \bar{x}\rangle| \leq \|P\|\sigma R_\vartheta \xrightarrow{\vartheta\to 0} 0, \qquad |\langle Pq, \bar{y}\rangle| \leq \|P\|\sigma R_\vartheta \xrightarrow{\vartheta\to 0} 0 \tag{3.73}$$

Moreover, in view of the continuity of the linear operator $A^*P\colon X \to X$ that has norm $\|A^*P\|$, we have

$$|\langle A^*Pp, \bar{x}\rangle| \leq \|A^*P\|\sigma R_\vartheta \xrightarrow{\vartheta\to 0} 0, \qquad |\langle A^*Pq, \bar{y}\rangle| \leq \|P\|\sigma R_\vartheta \xrightarrow{\vartheta\to 0} 0 \tag{3.74}$$

**Estimates 3 (on $\vartheta$)**: Fixed $\varepsilon$, we have

$$\vartheta |\bar{x}|^2 \xrightarrow{\vartheta\to 0} 0, \quad \vartheta |\bar{y}|^2 \xrightarrow{\vartheta\to 0} 0 \tag{3.75}$$

$$\lim_{\vartheta\to 0} (\Psi(\bar{x},\bar{y}) - \langle Pp, \bar{x}\rangle - \langle Pq, \bar{y}\rangle) = \tag{3.76}$$
$$= \sup_{(x,y)\in X\times X} \left( x(x) - v(y) - \frac{1}{2\varepsilon}\left| P^{1/2}(x-y)\right|^2 \right) > 2\frac{\gamma}{\rho}$$

(where the last inequality follows from the (3.60)). In (3.64) we fixed $\varepsilon$, in (3.70) we chose $\sigma$ as function of $\vartheta$. Now we will fix $\vartheta$. We begin taking

$$\vartheta < \frac{\gamma}{64\beta\|U\|^2}$$

so that

$$\beta\vartheta\|U\|^2 < \frac{\gamma}{64} \tag{3.77}$$

We know from (3.73) and (3.74) that if we choose $\vartheta$ small enough we have

$$\begin{aligned} &|\mu|\,|\langle Pp, \bar{x}\rangle| < \tfrac{\gamma}{16}, \qquad |\mu|\,|\langle Pq, \bar{y}\rangle| < \tfrac{\gamma}{16} \\ &|\langle A^* Pp, \bar{x}\rangle| < \tfrac{\gamma}{16}, \qquad |\langle A^* Pq, \bar{y}\rangle| < \tfrac{\gamma}{16} \end{aligned} \tag{3.78}$$

From (3.75) we know that if we choose $\vartheta$ small enough we have

$$|\mu|\vartheta\,|\bar{x}|^2 < \frac{\gamma}{32}, \quad |\mu|\vartheta\,|\bar{y}|^2 < \frac{\gamma}{32} \tag{3.79}$$

Moreover the (3.75) implies also that

$$\vartheta\,|\bar{x}| \xrightarrow{\vartheta\to 0} 0, \quad \vartheta\,|\bar{y}| \xrightarrow{\vartheta\to 0} 0$$

and then if we choose $\vartheta$ small enough we have

$$\vartheta\|\tilde{U}\|\,(|\bar{x}| + |\bar{y}|) < \frac{\gamma}{32} \tag{3.80}$$

Moreover, in view of (3.76) we know that that if we choose $\vartheta$ small enough, $\bar{x}$ and $\bar{y}$ satisfy the hypothesis (3.65) of the Claim and then, from the (3.66), we have

$$\frac{1}{\varepsilon}\left|P^{1/2}(\bar{x}-\bar{y})\right|^2 \le \frac{1}{32}\left(\frac{\gamma}{(1+|\mu|)}\right)^2 \le \frac{1}{32}\frac{\gamma}{(1+|\mu|)} \le \frac{\gamma}{32} \tag{3.81}$$

(where we uses that if $0 < a < 1$ then $a^2 < a$, we recall that we took $0 < \gamma < 1$). From the (3.66) in the same way we obtain

$$\frac{|\mu|}{\varepsilon}\left|P^{1/2}(\bar{x}-\bar{y})\right|^2 \le \frac{\gamma}{32} \tag{3.82}$$

and, from (3.69),

$$C_L\left|P^{1/2}(\bar{x}-\bar{y})\right| \le \frac{\gamma}{32} \tag{3.83}$$

Eventually, from (3.71) if we choose $\vartheta$ small enough we have

$$2\|P\||p|\|\tilde{U}\| \le 2\|P\|\sigma\|\tilde{U}\| \le \frac{\gamma}{64} \tag{3.84}$$

and

$$2\sigma\beta\|\delta_0 \circ P\|\|U\| \le \frac{\gamma}{32} \tag{3.85}$$

where we have called $\|\delta_0 \circ P\|$ the norm of the linear continuous functional $\delta_0 \circ P\colon X \to \mathbb{R}$.

We choose $\vartheta > 0$ small enough to satisfy (3.78), (3.79), (3.80), (3.81), (3.82) (3.83), (3.84), (3.85). ∎

We have finished our preliminary estimates and we come back to the main part of the proof of the theorem. The map

$$x \mapsto w(x) - \left( \frac{1}{2\varepsilon} |P^{1/2}(x - \bar{y})|^2 + \frac{\vartheta}{2}|x|^2 + \langle Pp, x\rangle \right)$$

attains a maximum at $\bar{x}$ and

$$y \mapsto v(y) + \left( \frac{1}{2\varepsilon} |P^{1/2}(\bar{x} - y)|^2 + \frac{\vartheta}{2}|y|^2 + \langle Pq, y\rangle \right)$$

attains a minimum at $\bar{y}$.

Thanks to Proposition 3.23 $\bar{x}$ and $\bar{y}$ are in $D(A^*)$. We can now use the definition of sub- and super-solution (page 57) to obtain that

$$\begin{aligned} (3.86)\quad & \rho w(\bar{x}) - \frac{1}{\varepsilon}\langle A^* P(\bar{x}-\bar{y}), \bar{x}\rangle - \frac{1}{\varepsilon}\langle P(\bar{x}-\bar{y}), -\mu\bar{x}\rangle - \langle A^* Pp, \bar{x}\rangle - \\ & - \langle Pp, -\mu\bar{x}\rangle - \vartheta\langle \bar{x}, -\mu\bar{x}\rangle - \inf_{(\tilde{u},u)\in\tilde{U}\times U} \left( \frac{1}{\varepsilon}\langle \beta\delta_0(P(\bar{x}-\bar{y})), u\rangle_{\mathbb{R}} + \langle \beta\delta_0(Pp), u\rangle_{\mathbb{R}} + \right. \\ & \left. + \vartheta\langle \bar{x}, \tilde{u}\rangle + \frac{1}{\varepsilon}\langle P(\bar{x}-\bar{y}), \tilde{u}\rangle + \langle Pp, \tilde{u}\rangle + L(\bar{x}, \tilde{u}, u) \right) \le \frac{\vartheta\beta\|U\|^2}{2} \end{aligned}$$

and

$$\begin{aligned} (3.87)\quad & \rho v(\bar{y}) - \frac{1}{\varepsilon}\langle A^* P(\bar{x}-\bar{y}), \bar{y}\rangle - \frac{1}{\varepsilon}\langle P(\bar{x}-\bar{y}), -\mu\bar{y}\rangle + \langle A^* Pq, \bar{y}\rangle + \\ & + \langle Pq, -\mu\bar{y}\rangle + \vartheta\langle \bar{y}, -\mu\bar{y}\rangle - \inf_{(\tilde{u},u)\in\tilde{U}\times U} \left( \frac{1}{\varepsilon}\langle \beta\delta_0(P(\bar{x}-\bar{y})), u\rangle_{\mathbb{R}} - \langle \beta\delta_0(Pq), u\rangle_{\mathbb{R}} - \right. \\ & \left. - \vartheta\langle \bar{y}, \tilde{u}\rangle + \frac{1}{\varepsilon}\langle P(\bar{x}-\bar{y}), \tilde{u}\rangle - \langle Pq, \tilde{u}\rangle + L(\bar{y}, \tilde{u}, u) \right) \ge -\frac{\vartheta\beta\|U\|^2}{2} \end{aligned}$$

Subtracting the above we obtain

$$\begin{aligned} (3.88)\quad & \rho w(\bar{x}) - \rho v(\bar{y}) - \frac{1}{\varepsilon}\langle A^* P(\bar{x}-\bar{y}), (\bar{x}-\bar{y})\rangle - \\ & - \frac{1}{\varepsilon}\langle P(\bar{x}-\bar{y}), -\mu(\bar{x}-\bar{y})\rangle - \langle A^* Pp, \bar{x}\rangle - \langle A^* Pq, \bar{y}\rangle - \\ & - \langle Pp, -\mu\bar{x}\rangle - \langle Pq, -\mu\bar{y}\rangle - \vartheta\langle \bar{x}, -\mu\bar{x}\rangle - \vartheta\langle \bar{y}, -\mu\bar{y}\rangle - \\ & - \inf_{(\tilde{u},u)\in\tilde{U}\times U} \left( \frac{1}{\varepsilon}\langle \beta\delta_0(P(\bar{x}-\bar{y})), u\rangle_{\mathbb{R}} + \langle \beta\delta_0(Pp), u\rangle_{\mathbb{R}} + \right. \\ & \left. + \vartheta\langle \bar{x}, \tilde{u}\rangle + \frac{1}{\varepsilon}\langle P(\bar{x}-\bar{y}), \tilde{u}\rangle + \langle Pp, \tilde{u}\rangle + L(\bar{x}, \tilde{u}, u) \right) + \\ & + \inf_{(\tilde{u},u)\in\tilde{U}\times U} \left( \frac{1}{\varepsilon}\langle \beta\delta_0(P(\bar{x}-\bar{y})), a\rangle_{\mathbb{R}} - \langle \beta\delta_0(Pq), u\rangle_{\mathbb{R}} - \right. \\ & \left. - \vartheta\langle \bar{y}, \tilde{u}\rangle + \frac{1}{\varepsilon}\langle P(\bar{x}-\bar{y}), \tilde{u}\rangle - \langle Pq, \tilde{u}\rangle + L(\bar{y}, \tilde{u}, u) \right) \le \\ & \le \beta\vartheta\|U\|^2 \end{aligned}$$

We now note that:
**(A)**: from (3.10) $A^*P \le P$ and then

$$-\frac{1}{\varepsilon}\langle A^* P(\bar{x}-\bar{y}), (\bar{x}-\bar{y})\rangle \ge -\frac{1}{\varepsilon}\langle P(\bar{x}-\bar{y}), (\bar{x}-\bar{y})\rangle = -\frac{1}{\varepsilon}|\bar{x}-\bar{y}|_P^2$$

**(B)**: We have

$$
\begin{aligned}
(3.89)\quad & -\inf_{(\tilde u,u)\in\tilde U\times U}\Big(\frac{1}{\varepsilon}\left\langle \beta\delta_0(P(\bar x-\bar y)),u\right\rangle_{\mathbb{R}}+\left\langle \beta\delta_0(Pp),u\right\rangle_{\mathbb{R}}+\\
&+\vartheta\left\langle \bar x,\tilde u\right\rangle+\frac{1}{\varepsilon}\left\langle P(\bar x-\bar y),\tilde u\right\rangle+\left\langle Pp,\tilde u\right\rangle+L(\bar x,\tilde u,u)\Big)+\\
&+\inf_{(\tilde u,u)\in\tilde U\times U}\Big(\frac{1}{\varepsilon}\left\langle \beta\delta_0(P(\bar x-\bar y)),u\right\rangle_{\mathbb{R}}-\left\langle \beta\delta_0(Pq),u\right\rangle_{\mathbb{R}}-\\
&-\vartheta\left\langle \bar y,\tilde u\right\rangle+\frac{1}{\varepsilon}\left\langle P(\bar x-\bar y),\tilde u\right\rangle-\left\langle Pq,\tilde u\right\rangle+L(\bar y,\tilde u,u)\Big)\geq\\
&\geq\inf_{(\tilde u,u)\in\tilde U\times U}\Big(-\left\langle \beta\delta_0(Pp),u\right\rangle_{\mathbb{R}}-\left\langle \beta\delta_0(Pq),u\right\rangle_{\mathbb{R}}+L(\bar y,\tilde u,u)-L(\bar x,\tilde u,u)-\\
&-\vartheta\left\langle \bar y,\tilde u\right\rangle-\vartheta\left\langle \bar x,\tilde u\right\rangle-\left\langle Pq,\tilde u\right\rangle-\left\langle Pp,\tilde u\right\rangle\Big)\geq\\
&\geq\inf_{(\tilde u,u)\in\tilde U\times U}\Big(L(\bar y,\tilde u,u)-L(\bar x,\tilde u,u)\Big)-\\
&-\sup_{(\tilde u,u)\in\tilde U\times U}\Big(\left\langle \beta\delta_0(Pp),u\right\rangle_{\mathbb{R}}+\left\langle \beta\delta_0(Pq),u\right\rangle_{\mathbb{R}}\Big)-\\
&-\sup_{(\tilde u,u)\in\tilde U\times U}\Big(\vartheta\left\langle \bar y,\tilde u\right\rangle+\vartheta\left\langle \bar x,\tilde u\right\rangle\Big)-\sup_{(\tilde u,u)\in\tilde U\times U}\Big(\left\langle Pq,\tilde u\right\rangle+\left\langle Pp,\tilde u\right\rangle\Big)\geq\\
&\geq -C_L|\bar x-\bar y|_P-2\sigma\beta\|\delta_0\circ P\|\|U\|-\|\tilde U\|\vartheta(|\bar x|+|\bar y|)-2\|P\|\sigma\|\tilde U\|
\end{aligned}
$$

Thus using (A) and (B) in (3.88) we have

$$
\begin{aligned}
(3.90)\quad & \rho\Big(w(\bar x)-v(\bar y)\Big)-\frac{1}{\varepsilon}|\bar x-\bar y|_P^2-\\
&-\frac{\mu}{\varepsilon}\left\langle P(\bar x-\bar y),-(\bar x-\bar y)\right\rangle-\left\langle A^*Pp,\bar x\right\rangle-\left\langle A^*Pq,\bar y\right\rangle-\\
&-\left\langle Pp,-\mu\bar x\right\rangle-\left\langle Pq,-\mu\bar y\right\rangle-\vartheta\left\langle \bar x,-\mu\bar x\right\rangle-\vartheta\left\langle \bar y,-\mu\bar y\right\rangle-\\
&-C_L|\bar x-\bar y|_P-2\sigma\beta\|\delta_0\circ P\|\|U\|-\|\tilde U\|\vartheta(|\bar x|+|\bar y|)-2\|P\|\sigma\|\tilde U\|-\beta\vartheta\|U\|^2\leq 0
\end{aligned}
$$

using (3.81), (3.82), (3.78), (3.79), (3.83), (3.85), (3.80), (3.84), (3.77) we obtain

(3.91)

$$
\rho(w(\bar x)-v(\bar y))-2\left(\frac{\gamma}{32}\right)-4\left(\frac{\gamma}{16}\right)-2\left(\frac{\gamma}{32}\right)-\frac{\gamma}{32}-\frac{\gamma}{32}-\frac{\gamma}{32}-\frac{\gamma}{64}-\frac{\gamma}{64}\leq 0
$$

that is

$$
\rho(w(\bar x)-v(\bar y))-\frac{1}{2}\gamma\leq 0 \tag{3.92}
$$

but from the (3.62) we have $\rho\big(w(\bar x)-v(\bar y)\big)>\gamma$ and then we obtain from the (3.92)

$$
\frac{1}{2}\gamma=\gamma-\frac{1}{2}\gamma<\rho(w(\bar x)-v(\bar y))-\frac{1}{2}\gamma\leq 0
$$

that is a contradiction because $\gamma>0$ and so the theorem is proved. □

REMARK 3.32. *Now we can better explain the remark in the introduction saying that it is difficult to treat the case of non-constant coefficient. We can estimate the term* $\frac{1}{\varepsilon}\left\langle P(\bar x-\bar y),-\mu(\bar x-\bar y)\right\rangle$ *because we use the term* $\frac{1}{\varepsilon}|x-y|_P^2$ *to penalize the doubling of the variables with respect to the* $P$*-norm. If* $\mu$ *is a function of* $r$ *such term is replaced by* $\frac{1}{\varepsilon}\left\langle P(\bar x-\bar y),-\mu(\cdot)(\bar x-\bar y)\right\rangle$ *where* $-\mu(\cdot)(\bar x-\bar y)$ *is the pointwise product of the* $L^\infty(0,\bar s)$ *function* $\mu(\cdot)$ *and the* $L^2(0,\bar s)$ *function* $(\bar x-\bar y)$*, which cannot be estimated by using similar arguments.*

CHAPTER 4

# Existence of viscosity solution for a family of HJB equations related to economic problems with delay

In this chapter we present the results obtained in the work [Fab06] in which we prove that, under suitable hypotheses, the value function is a viscosity solution of the first order HJB equation in the Hilbert space $M^2$ related to optimal control problem subject to linear delay differential equation .

The result that we find in this work is not very strong. Indeed we use a quite small set of test functions and we can only prove an existence result without any comparison statement. Anyway the existence of a viscosity solution, that is proved both in state-control constraint and in state constraint case, is sufficient to prove a verification theorem.

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## 4.1. Introduction

The present work is strictly connected to the studies presented in [FGF] that we will describe in Chapter 6 of this thesis. We treat a class of delay optimal control problems arising from economics, using the equivalent formulation of the problems in $M^2$ described in Section 1.3 and then using the dynamic programming approach.

We concentrate on three main examples: The first is an AK model with vintage capital, taken from [BLPdR05] (see also [BdlCL04] and [FGa]), we refer to Chapter 7 for description of the model. The second is an advertising model with delay effects (see [GM04, GMS06]), we refer to Section 6.3 for a brief description. The third is an AK model for obsolescence and depreciation (from [BdRM]) and can be found in Chapter 8). We study a general problem that includes, as particular cases, the three main examples.

The examples are used to understand which can be, from an applied point of view, the more reasonable hypotheses that have to be introduced in the study. Nevertheless the results we found are applicable independently from the form of the linear delay differential equation that governs the system.

The economic nature of the example suggest to treat the maximization problem instead of the minimization one. So the value function will be defined as the supremum, over all admissible control strategies, of the utility functional and the Hamiltonian will change too.

In [FGF] we study the HJB equation using an approximation method with techniques similar to the ones used in [Fag05b, FG04, Fag05a] for other classes of problems (see Chapter 6 for details). Here we treat a more general case studying the existence of viscosity solutions for the HJB equation. Indeed the use of viscosity solutions in the study of the HJB equation allows to avoid the concavity assumption for the Hamiltonian and the target functional of the system. Moreover, using the viscosity solution approach, we do not need a current objective function in which the control and the state appears de-coupled (see Subsection 4.2.2 and in particular Remark 4.3).

**4.1.1. The main "technical" difficulties of the problem. the state equation:** we consider a general homogeneous linear DDE, in which the derivative of the state $\theta$ depends both on the history of the state $\theta_s$ (the notation $_s$ was introduced in Section 1.3) and on the history of the control $u_s$:

$$\begin{cases} \dot\theta(s) = N(\theta_s) + B(u_s) \quad for\ s \in [t,T] \\ (\theta(t), \theta_t, u_t) = (\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2(-R,0) \times L^2(-R,0) \end{cases}$$

The presence of the delay in the control yields a unbounded term. There are similar terms in the papers [CGS93, CT94a, CT96a, CT96c, GSŚ02, Faba] that study viscosity solution for HJB equation related to optimal control problems governed by specific PDEs and whose results do not apply to our case. Moreover in our state equation as reformulated in $M^2$ a non-analytic semigroup appears. The only work, as far we know, that treat viscosity solution for HJB equation with boundary term with non-analytic semigroup is [Faba] (presented in Chapter 3), but only a very specific transport PDE is treated there.

**the constraints:** we consider both state-control constraints (see Hypothesis 4.4 for a precise definition) and state constraints ($\theta \geq 0$ in Section 4.6).

**the target functional:** Here we consider a functional of the form

$$\int_t^T L_0(s, \theta(s), u(s)) \mathrm{d}s + h_0(\theta(T)) \tag{4.1}$$

where we assume $L_0$ and $h_0$ continuous. In [BLPdR05, FGa, Fab06] a CRRA utility function is considered and in [FGF] a concave utility function is used.

We include the state constraint $\theta \geq 0$ only in the last section. In Section 4.2 we describe the general delay problem and we show how to specify the problem to obtain our main examples. In Section 4.3 we briefly show how the techniques presented in Section 1.3 can be used in our case to rewrite the optimal control problem subject to DDE as a an optimal control problem subject to ODE in $M^2$. In Section 4.4 we present the definition of viscosity solution of the HJB equation (Definition 4.12, Definition 4.13, Definition 4.14) and then we prove that the value function of the problem is a viscosity solution of the HJB equation (Theorem 4.19). In Section 4.5 we give a verification result that can be used to verify if a given control is optimal and to find optimal controls in feedback form. In Section 4.6 we consider the constraint $\theta \geq 0$: giving a definition of viscosity solution in such a case (Definition 4.28) and we prove that the value function is a viscosity solution of the HJB equation according to the new definition (Theorem 4.30).

## 4.2. The Problem

**4.2.1. The problem - the delay state equation .** From now on we consider a fixed $R > 0$. Given $T > t \geq 0$, $z \in L^2(t-R, T)$ and an admissible control $u(\cdot) \in L^2(-R,T)$ we consider the the following delay differential equation:

$$\begin{cases} \dot\theta(s) = N(\theta_s) + B(u_s) \quad for\ s \in [t,T] \\ (\theta(t), \theta_t, u_t) = (\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2(-R,0) \times L^2(-R,0) \end{cases} \tag{4.2}$$

where $\theta_t$ and $u_t$ are interpreted by means of the Definition (1.24) and

HYPOTHESIS 4.1. $N, B \colon C([-R,0];\mathbb{R}) \to \mathbb{R}$ *are continuous linear functionals*

The equation (4.2) is a general form that includes our three main examples. Namely:

- In [BLPdR05, FGa] the control variable is called $i$ (as "investment") instead of $u$ and the state variable is called $k$ (as "capital"), $N = 0$ and

$B = \delta_0 - \delta_R$, the state equation is representing the stock of capital $k$ at time $s$ is

$$k(s) = \int_{s-R}^{s} i(r)\mathrm{d}r \tag{4.3}$$

- In [GMS06, GM04] the definitions of $N$ and $B$ are respectively

$$\begin{array}{l} N\colon C([-R,0]) \to \mathbb{R} \\ N\colon \gamma \mapsto a_0\gamma(0) + \int_{-R}^{0} \gamma(r) da_1(r) \end{array} \tag{4.4}$$

$$\begin{array}{l} B\colon C([-R,0]) \to \mathbb{R} \\ B\colon \gamma \mapsto b_0\gamma(0) + \int_{-R}^{0} \gamma(r) db_1(r) \end{array} \tag{4.5}$$

  where $a_0$, $b_0$ are real constants and $a_1$, $b_1$ are function with bounded variation.
- In [BdRM] $N = 0$ and

$$\begin{array}{l} B\colon C([-R,0]) \to \mathbb{R} \\ B\colon \gamma \mapsto (\Omega - \eta)\gamma(0) - \delta\Omega \int_{-R}^{0} e^{\delta r}\gamma(r)\mathrm{d}r \end{array} \tag{4.6}$$

  where $\Omega$, $\eta$ and $\delta$ are constants.

**4.2.2. The problem - the target functional.** We consider a target functional to be maximized of the form

$$\int_t^T L_0(s, \theta(s), u(s))\mathrm{d}s + h_0(\theta(T)) \tag{4.7}$$

where

$$\begin{array}{l} L_0\colon [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ h_0\colon \mathbb{R} \to \mathbb{R} \end{array} \tag{4.8}$$

are continuous functions.

Remark 4.2. *In our main examples the functional are the following*

- *In* [BLPdR05, FGa] *the horizon is infinite and the objective functional was CRRA:*

$$\int_0^{+\infty} \frac{(Ak(s) - i(s))^{1-\sigma}}{1-\sigma}\mathrm{d}s \tag{4.9}$$

  *where $A$ is a positive constant and $\sigma > 0$. In such a work the production function is linear and given by $y = Ak$, while $(Ak - i)$ is the consumption.*
- *In* [BdRM] *the state is called $y$ (it represents the production net of maintenance and repair costs), the control variable is $i$ (investment). The functional is CRRA:*

$$\int_0^{+\infty} \frac{(y(s) - i(s))^{1-\sigma}}{1-\sigma}\mathrm{d}s \tag{4.10}$$

Remark 4.3. *The generality of the objective functional is one of the improvements due to the viscosity solutions approach, indeed in* [FGF] *(in* [GMS06, GM04] *a similar stochastic case is studied) the authors considered only objective functional of the form*

$$\int_t^T e^{-\rho s} l_0(c(s))\mathrm{d}s + m_0(\theta(T)) \tag{4.11}$$

*where $l_0$ and $m_0$ are concave functions, and the utility function $l_0$ depends only on the consumption (that is the control) $c$.*

**4.2.3. The problem - the constraints.** The last thing to choose to define the optimization problem is the choice of the set of the admissible trajectories. In our main examples a lower bound on the control variable is assumed. In [BLPdR05, FGa] the constraint $u \geq 0$ is assumed and the same is done in [BdRM]. Here we assume a more general constraint:

$$u \geq \Gamma_-(\theta) \tag{4.12}$$

where $\Gamma_- \colon \mathbb{R} \to (-\infty, 0]$ is a continuous function (see Hypothesis 4.4 for other assumptions on $\Gamma_-$). (In [BLPdR05, FGa] $\Gamma_-(\theta) = 0$).

Moreover we assume another state-control constraint that is a generalization of the constraints imposed in [BLPdR05, FGa, BdRM]: the control cannot be greater than some number depending on the state. For example in [BLPdR05, FGa] the investment $i$ cannot be greater then the production $Ak(t)$, in [BdRM] we have $i \leq y$. Here we impose

$$u \leq \Gamma_+(\theta) \tag{4.13}$$

where $\Gamma_+ \colon \mathbb{R} \to [0, +\infty)$ is a continuous function. (In [BLPdR05, FGa] $\Gamma_+(\theta) = A\theta$, in [BdRM] $\Gamma_+(\theta) = \theta$)

**The constraint** $\theta \geq 0$ The state constraint $\theta \geq 0$ is quite natural in our examples: $y$ is the stock of capital, the net production, the stock of advertising goodwill of the product to be launched. In the advertising model it is required ([FGF] and the stochastic version [GM04, GMS06]), whereas it is not directly needed in [BLPdR05] where it is consequence of the positivity of investment. The constraint $\theta \geq 0$ needs to be treated with more attention in the viscosity solution setting, so we introduce it only in Section 4.6 and we ignore it before.

## 4.3. The problem in Hilbert space

Following the steps we presented in Section 1.3 we obtain an equivalence of our optimal control problem governed by (4.2) into the following problem in the Hilbert space $M^2 = \mathbb{R} \times L^2(-R, 0)$ governed by the state equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s} x(s) = Ax(s) + B^* u(s) \\ x(t) = x. \end{cases} \tag{4.14}$$

where $A$ is the generator introduced in Section 1.3 and used in (1.39) and $B^*$ is the adjoint[1] of $B$ used in (4.2). From Theorem 1.33 we know that (4.14) admits a unique solution $x(\cdot)$ such that $x(s) = (x^0(s), x^1(s)) \in \mathbb{R} \times L^2(-R, 0)$ where $x^0(\cdot)$ is the unique absolutely continuous solution $\theta(\cdot)$ of (4.2).

In the next Hypothesis we formalize the state-control constraint above as $u \in [\Gamma_-(\theta), \Gamma_+(\theta)]$:

Hypothesis 4.4. *If we consider a control $u(\cdot)$ and the related state trajectory $x(\cdot) = (x^0(\cdot), x^1(\cdot))$ we impose the state-control constraint*

$$\Gamma_-(x^0(s)) \leq u(s) \leq \Gamma_+(x^0(s)) \tag{4.15}$$

*where $\Gamma_-$ and $\Gamma_+$ are locally Lipschitz continuous functions:*

$$\begin{aligned} &\Gamma_+ \colon \mathbb{R} \to [0, +\infty) \\ &\Gamma_- \colon \mathbb{R} \to (-\infty, 0] \end{aligned} \tag{4.16}$$

*such that $|\Gamma_-(t)| \leq a + b|t|$ and $|\Gamma_+(t)| \leq a + b|t|$ for two positive constant $a$ and $b$.*

[1] We abuse notation as described in (1.36).

The set of admissible controls is

$$\mathcal{U}_{t,x} \stackrel{def}{=} \{u(\cdot) \in L^2(t,T) \ : \ \Gamma_-(x^0_{u(\cdot),t,x}(s)) \leq u(s) \leq \Gamma_+(x^0_{u(\cdot),t,x}(s))\} \tag{4.17}$$

The target functional (4.7) written in the new variables is

$$\int_t^T L_0(s, x^0(s), u(s))\mathrm{d}s + h_0(x^0(T)).$$

So we rewrite it as follows

$$J(t,x,u(\cdot)) = \int_t^T L(s, x(s), u(s))\mathrm{d}s + h(x(T)) \tag{4.18}$$

where

$$\begin{cases} L\colon [0,T] \times M^2 \times \mathbb{R} \to \mathbb{R} \\ L\colon (s,x,u) \mapsto L_0(s,x^0,u) \end{cases} \tag{4.19}$$

$$\begin{cases} h\colon M^2 \to \mathbb{R} \\ h\colon x \mapsto h_0(x^0) \end{cases} \tag{4.20}$$

and so $L$ and $h$ are continuous functions. Moreover we ask that

HYPOTHESIS 4.5. *$L$ and $h$ are uniformly continuous and*

$$|L(s,x,u) - L(s,y,u)| \leq \sigma(|x-y|) \quad \textit{for all } (s,u) \in [0,T] \times \mathbb{R} \tag{4.21}$$

*where $\sigma$ is a modulus of continuity.*

The original optimization problem is equivalent to the optimal control problem in $M^2$ with state equation (4.14) and target functional given by (4.18).

LEMMA 4.6. *Assuming Hypothesis 4.4, given an initial datum $(\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2(-R,0) \times L^2(-R,0)$, equation (4.2) has a unique solution $\theta(\cdot)$ in $H^1(t,T)$. It is bounded in the interval $[t,T]$ uniformly in the control $u(\cdot) \in \mathcal{U}_{t,x}$ and in the initial time $t \in [0,T)$. We call $K$ a constant such that $|\theta(s)| \leq K$ for any $t \in [0,T)$, any control $u(\cdot) \in \mathcal{U}_{t,x}$ and any $s \in [t,T]$.*

PROOF. The existence of the solution follows from Theorem 1.24. It can be proved (see (1.34)) that the solution of (4.2) is also the solution of the equation

$$\begin{cases} \dot{\theta}(s) = N(e^t_+\theta)_s + B(e^t_+u)_s + (e^{-R}_+\xi^1)(-t) \qquad for\ s \geq t \\ \theta(t) = \phi^0 \in \mathbb{R} \end{cases} \tag{4.22}$$

where $\xi^1 = (\overline{N}\phi^1 + \overline{B}\omega)$. So, using Hypothesis 4.4 we can state that, for every control $u(\cdot) \in \mathcal{U}_{t,x}$ and related trajectory $y(\cdot)$, the solution $\theta_M$ of the following ODE satisfies $|\theta(s)| \leq |\theta_M(s-t)|$ for all $s \in [t,T]$:

$$\begin{cases} \dot{\theta}_M(s) = \|N\|\theta_M(s) + \|B\|(a + b\theta_M(s)) + (e^{-R}_+\xi^1)(-t) \qquad for\ s \geq 0 \\ \theta_M(0) = |\phi^0| \in \mathbb{R} \end{cases} \tag{4.23}$$

and $\theta_M$ is bounded on $[0,T]$ and this complete the proof. $\square$

REMARK 4.7. *Using the Hypothesis 4.4 such result implies $u(s) \leq a + bK$ for all the controls in $\mathcal{U}_{t,x}$.*

LEMMA 4.8. *If Hypothesis 4.4 holds, calling $x(s)$ the solution of (4.14),*

$$|x(s) - x|_{M^2} \xrightarrow{s \to t^+} 0 \tag{4.24}$$

*uniformly in the control $u(\cdot) \in \mathcal{U}_{t,x}$*

Proof. We have to prove that $|x(s)-x|_{M^2} \xrightarrow{s\to t^+} 0$ uniformly in $u(\cdot)\in\mathcal{U}_{t,x}$, so it is enough to show that $|x^0(s)-x^0|_{\mathbb{R}} \xrightarrow{s\to t^+} 0$ uniformly in $u(\cdot)\in\mathcal{U}_{t,x}$ and that $|x^1(s)-x^1|_{L^2} \xrightarrow{s\to t^+} 0$ uniformly in $u(\cdot)\in\mathcal{U}_{t,x}$. The first fact is a corollary of the proof of Lemma 4.6 (because $|x^0(s)-x^0|\le\theta_M(s-t)$ defined in (4.23), for the second, using the expression (1.37):

$$\begin{aligned}\left|x^1(s)-x^1\right|_{L^2} &\le \left|\Xi(s)x^1-x^1\right|_{L^2}+\left|\overline{N}(e^0_+\theta)_s\right|_{L^2}+\left|\overline{B}(e^0_+u)_s\right|_{L^2}\le\\ &\le \left|\Xi(s)x^1-x^1\right|_{L^2}+\|\overline{N}\|(s-t)^{\frac12}K+\|\overline{B}\|(s-t)^{\frac12}(a+Kb)\end{aligned}\tag{4.25}$$

where $a$ e $b$ are the constants of Hypothesis 4.4 and $K$ the constant of Lemma 4.6 and Remark 4.7 ($\Xi(\cdot)$ is defined in (1.38). Now we observe $\left|\Xi(s)x^1-x^1\right|_{L^2}\xrightarrow{s\to0}0$ for the continuity of the translation with respect to the $L^2$ norm and such limit does not depend on the control, the other two term are given by a constant multiplied by $(s-t)^{1/2}$ and so they go to zero uniformly in the control. □

The value function of the problem is defined as

$$V(t,x)=\sup_{u(\cdot)\in\mathcal{U}_{t,x}} J(t,x,u(\cdot))\tag{4.26}$$

Proposition 4.9. *The value function $V\colon[0,T]\times M^2\to\mathbb{R}$ is continuous*

Proof. We consider $[0,T]\times M^2\ni(t_n,x_n)\xrightarrow[\mathbb{R}\times M^2]{n\to\infty}(t,x)$. We have to estimate the terms

$$|V(t,x)-V(t,x_n)| \quad \textit{and} \quad |V(t_n,x_n)-V(t_n,x)|\tag{4.27}$$

the difficulties are similar, we analyze the term $|V(t,x)-V(t,y)|$, the other can be treated using similar steps. Using arguments similar to the ones of Lemma 4.6[2] we can state that there exists a $M>0$ such that, for every admissible control,

$$|x_n(s)|\le M \quad \text{for every } s\in[t_n,T],\ n\in\mathbb{N}$$

in particular $|x^0_n(s)|\le M$. In view of Hypothesis 4.4 the restrictions of $\Gamma_+$ and $\Gamma_-$ in $[-M,M]$ are Lipschitz continuous for some Lipschitz constant $Z$. Suppose that $V(t,x)\ge V(t,x_n)$, then we take an $\varepsilon$-optimal control $u^\varepsilon(\cdot)$ for $V(t,x)$. The problem is that $u^\varepsilon(\cdot)$ could not be in the set $\mathcal{U}_{t,x_n}$. So we consider the approximating control given in feedback form:

$$u^\varepsilon_n(s)\stackrel{def}{=}\begin{cases} u^\varepsilon(s) & \text{if } u^\varepsilon(s)\in[\Gamma_-(x_{n\varepsilon}(s)),\Gamma_+(x_{n\varepsilon}(s))]\\ \Gamma_-(x_{n\varepsilon}(s)) & \text{if } u^\varepsilon(s)\in[\Gamma_-(x_n(s)),\Gamma_-(x_{n\varepsilon}(s))]\\ \Gamma_+(x_{n\varepsilon}(s)) & \text{if } u^\varepsilon(s)\in[\Gamma_+(x_{n\varepsilon}(s)),\Gamma_+(x_n(s))]\end{cases}\tag{4.28}$$

where $x_{n\varepsilon}(\cdot)$ the solution of

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}s}x_{n\varepsilon}(s)=Ax_{n\varepsilon}(s)+B^*u^\varepsilon_n(s)\\ x_{n\varepsilon}(t)=x_n.\end{cases}\tag{4.29}$$

The definition of $u^\varepsilon(s)$ implies that it is bounded, measurable, and then $L^2[0,T]$. We call $x_\varepsilon(\cdot)$ the solution of

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}s}x_\varepsilon(s)=Ax_\varepsilon(s)+B^*u^\varepsilon(s)\\ x_\varepsilon(t)=x.\end{cases}\tag{4.30}$$

and we call $y(\cdot)\stackrel{def}{=}x_\varepsilon(\cdot)-x_{n\varepsilon}(\cdot)$, By definition of $u^\varepsilon_n(\cdot)$ we know that

$$|u^\varepsilon(s)-u^\varepsilon_n(s)|\le Z|y^0(s)|\tag{4.31}$$

[2]Using that $(e_+^{-R}\overline{N}\phi^1+\overline{B}\omega)(\cdot)$ is continuous with respect to the initial data.

where $y^0(s)$ is the first component of $y(s)$. Moreover $y^0(\cdot)$ solves the following DDE (using the notation of (1.34):

$$\begin{cases} \dot{y}^0(s) = (\mathcal{N}e^0_+ y^0)(s) + (\mathcal{B}e^0_+(u^\varepsilon(s) - u^\varepsilon_n))(s) + e^{-R}_+(x^1 - x^1_n)(-s) \\ y^0(t) = x^0 - x^0_n \end{cases}$$

Arguing as in the proof of Lemma 4.6 and using (4.31) we can state that $|y^0(s)| \leq \theta_M(s)|$ where $\theta_M$ is the solution of the ODE

$$\begin{cases} \dot{\theta}_M(s) = \|N\|\theta_M(s) + \|B\|\theta_M(s) + e^{-R}_+|x^1 - x^1_n|(-s) \\ \theta_M(t) = |x^0 - x^0_n| \end{cases} .$$

We have

(4.32)

$$\theta_M(s) = |x^0 - x^0_n|e^{(\|N\|+\|B\|)(s-t)} + \int_s^t e^{(\|N\|+\|B\|)(s-\tau)} e^{-R}_+ |x^1 - x^1_n|(-\tau)\mathrm{d}\tau \leq$$
$$\leq C\|x - x_n\|_{M^2}$$

for all $s \in [t, T]$ so,

$$|x^0_\varepsilon(s) - {x_n}^0_\varepsilon(s)| \leq C\|x - x_n\|_{M^2} \quad \text{for all } s \in [t, T]$$

and

$$|u^\varepsilon(s) - u^\varepsilon_n(s)| \leq ZC\|x - x_n\|_{M^2} \quad \text{for all } s \in [t, T]$$

So, by the uniform continuity of the $L$ we can conclude that

$$|L(s, x^0_\varepsilon(s), u^\varepsilon(s)) - L(s, {x_n}^0_\varepsilon(s), u^\varepsilon_n(s)) \leq \sigma(\|x - x_n\|_{M^2}) \quad \text{for all } s \in [t, T]$$

So, for the continuity of $h$ we have (using $\sigma(\cdot)$ for a generic modulus),

$$J(t, x, u^\varepsilon(\cdot)) - J(t, x_n, u^\varepsilon_n(\cdot)) \leq \sigma(\|x - x_n\|_{M^2})$$

and then

$$|V(t, x) - V(t, x_n)| = V(t, x) - V(t, x_n) \leq \varepsilon + \sigma(\|x - x_n\|_{M^2})$$

We conclude for the arbitrariness of $\varepsilon$. □

## 4.4. Viscosity solutions for HJB

The HJB of the system is defined as

(4.33)
$$\begin{cases} \partial_t w(t, x) + \langle \nabla w(t, x), Ax \rangle + H(t, x, \nabla w(t, x)) = 0 \\ w(T, x) = h(x) \end{cases}$$

where $H$ is defined as follows

(4.34)
$$\begin{cases} H : [0, T] \times D(A^*) \to \mathbb{R} \\ H(t, x, p) \overset{def}{=} \sup_{u \in [\Gamma_-(x^0), \Gamma_+(x^0)]} \{uB(p) + L(t, x, u)\} \end{cases}$$

We refer to $H$ as to the *Hamiltonian* of the system

REMARK 4.10. *In this chapter we define the value function as supremum of the objective functional. Moreover the HJB (4.33) is formally different from (1.13) in which the terms appear with negative sign. So we will use also different definition for sub and super solution. Such formulation are equivalent.*

### 4.4.1. Definition and preliminary lemma.

DEFINITION 4.11 (**Test**). *We say that a function $\varphi \in C^1([0,T] \times M^2)$ is a test function and we will write $\varphi \in$ TEST if $\nabla\varphi(s,x) \in D(A^*)$ for all $(s,x) \in [0,T] \times M^2$ and $A^*\nabla\varphi \colon [0,T] \times M^2 \to \mathbb{R}$ is continuous. This means that $\nabla\varphi \in C([0,T] \times M^2; D(A^*))$ where $D(A^*)$ is endowed with the graph norm.*

DEFINITION 4.12 (**Viscosity subsolution**). *$w \in C([0,T] \times M^2)$ is a viscosity subsolution of the HJB equation (or simply a "subsolution") if $w(T,x) \le h(x)$ for all $x \in M^2$ and for every $\varphi \in$ TEST and every local minimum point $(t,x)$ of $w - \varphi$ we have*

$$\partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + H(t,x,\nabla\varphi(t,x)) \le 0 \tag{4.35}$$

DEFINITION 4.13 (**Viscosity supersolution**). *$w \in C([0,T] \times M^2)$ is a viscosity supersolution of the HJB equation (or simply a "supersolution") if $w(T,x) \ge h(x)$ for all $x \in M^2$ and for every $\varphi \in$ TEST and every local maximum point $(t,x)$ of $w - \varphi$ we have*

$$\partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + H(t,x,\nabla\varphi(t,x)) \ge 0 \tag{4.36}$$

DEFINITION 4.14 (**Viscosity solution**). *$w \in C([0,T] \times M^2)$ is a viscosity solution of the HJB equation if it is, at the same time, a supersolution and a subsolution.*

PROPOSITION 4.15. *Given $(t,x) \in [0,T] \times M^2$ and $\varphi \in$ TEST there exists a real continuous function $O(s)$ such that $O(s) \xrightarrow{s\to t^+} 0$ and such that for every admissible control $u(\cdot) \in \mathcal{U}_{t,x}$ we have that*

$$\left| \frac{\varphi(s,x(s)) - \varphi(t,x)}{s-t} - \partial_t\varphi(t,x) - \langle A^*\nabla\varphi(t,x), x\rangle - \right. \\ \left. - \frac{\int_t^s \langle B(\nabla\varphi(t,x)), u(r)\rangle_{\mathbb{R}}\, \mathrm{d}r}{s-t} \right| \le O(s) \tag{4.37}$$

*(where we called $x(s)$ the trajectory that starts at time $t$ from $x$ and subject to the control $u(\cdot)$).*

*Moreover if $u(\cdot) \in \mathcal{U}_{t,x}$ is continuous in $t$ we have that*

$$\frac{\varphi(s,x(s)) - \varphi(t,x)}{s-t} \xrightarrow{s\to t^+} \\ \xrightarrow{s\to t^+} \partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + \langle B(\nabla\varphi(t,x)), u(t)\rangle_{\mathbb{R}} \tag{4.38}$$

PROOF. We write

$$\frac{\varphi(s,x(s)) - \varphi(t,x)}{s-t} = I_t + I_0 + I_1 \stackrel{def}{=} \partial_t\varphi(\xi^t(s),\xi^x(s)) + \\ + \left\langle \nabla\varphi(t,x), \frac{x(s)-x}{s-t} \right\rangle + \left\langle \nabla\varphi(\xi^t(s),\xi^x(s)) - \nabla\varphi(t,x), \frac{x(s)-x}{s-t} \right\rangle \tag{4.39}$$

where $[t,T] \times M^2 \ni \xi(s) = (\xi^t(s), \xi^x(s))$ is a point of the line segment connecting $(t,x)$ and $(s,x(s))$. In view of Lemma 4.8, $|x(s) - x|_{M^2} \xrightarrow{s\to t^+} 0$ uniformly in $u(\cdot) \in \mathcal{U}_{t,x}$, so $|\xi(s) - (t,x)|_{\mathbb{R}\times M^2} \xrightarrow{s\to t^+} 0$ uniformly in $u(\cdot) \in \mathcal{U}_{t,x}$ and in particular

$$|\xi^x(s) - x|_{M^2} \xrightarrow{s\to t^+} 0 \text{ uniformly in } u(\cdot) \in \mathcal{U}_{t,x} \tag{4.40}$$

and then

$$|\xi(s) - (t,x)|_{[t,T]\times M^2} \le |s-t| + |\xi^x(s) - x|_{M^2} \xrightarrow{s\to t^+} 0 \quad \text{uniformly in } u(\cdot) \in \mathcal{U}_{t,x}. \tag{4.41}$$

By definition of test function we have that

$$\nabla\varphi\colon [0,T]\times M^2 \to D(A^*) \quad \textit{and it is continuous.} \tag{4.42}$$

Then

$$|\nabla\varphi(\xi^t(s), \xi^x(s)) - \nabla\varphi(t,x)|_{D(A^*)} \xrightarrow{s\to t^+} 0 \tag{4.43}$$

uniformly in $u(\cdot) \in \mathcal{U}_{t,x}$.

As observed in Lemma 1.15 the state equation (4.14) may be extended to an equation in $D(A^*)'$ of the form

$$\begin{cases} \dot{x}(s) = A^{(E)}x(s) + B^*u(s) \\ x(t) = x \end{cases} \tag{4.44}$$

and, in view of Lemma 4.6 and Remark 4.7, $|B^*u(s)|_{D(A^*)'} \le |B|_{D(A^*)'}|a + bK|$, where $a$ and $b$. The solution of (4.44) in $D(A^*)'$ can be expressed in mild form [Paz83] as described in Remark 1.35:

$$x(s) = e^{(s-t)A^{(E)}}x + \int_t^s e^{(s-r)A^{(E)}}B^*u(r)\mathrm{d}r \tag{4.45}$$

So, since $x \in X \subseteq D(A^{(E)})$ we can choose a constant $C$ that depends on $x$ such that, for all admissible controls and all $s \in [t,T]$,

$$\frac{|x(s) - x|_{D(A^*)'}}{s-t} \le C \tag{4.46}$$

So by (4.43) and (4.46), we can say that $|I_1| \xrightarrow{s\to t^+} 0$ uniform in $u(\cdot) \in \mathcal{U}_{t,x}$. Thanks to the uniformly (in $u(\cdot) \in \mathcal{U}_{t,x}$) convergence $\xi(s) \to (t,x)$ we can also state that $I_t = \partial_t\varphi(\xi^t(s), \xi^x(s)) \xrightarrow{s\to t^+} \partial_t\varphi(t,x)$ uniformly in $u(\cdot) \in \mathcal{U}_{t,x}$. So to prove the thesis it remains to show that

$$\begin{aligned} &\left| \frac{\langle \nabla\varphi(t,x), x(s) - x\rangle}{s-t} - \langle A^*\nabla\varphi(t,x), x\rangle - \right. \\ &\qquad\qquad \left. - \frac{\int_t^s \langle B(\nabla\varphi(t,x)), u(r)\rangle_{\mathbb{R}}\, \mathrm{d}r}{s-t} \right| = \\ &= \left| \left\langle \nabla\varphi(t,x), \left( \frac{x(s)-x}{s-t} - A^{(E)}x - \frac{\int_t^s B^*u(r)\mathrm{d}r}{s-t} \right) \right\rangle_{D(A^*)\times D(A^*)'} \right| \le O(s) \end{aligned} \tag{4.47}$$

uniformly in $u(\cdot) \in \mathcal{U}_{t,x}$.

We can use (4.45) and write down explicitly the expression $\frac{x(s)-x}{s-t}$ in $D(A^*)'$:

$$\frac{x(s)-x}{s-t} = \frac{(e^{(s-t)A^{(E)}} - \mathrm{I})x}{s-t} + \frac{\int_t^s e^{(s-r)A^{(E)}}B^*u(r)\mathrm{d}r}{s-t} \tag{4.48}$$

So we need to estimate:

$$\left|\frac{x(s)-x}{s-t} - A^{(E)}(x) - \frac{\int_t^s B^* u(r)\mathrm{d}r}{s-t}\right|_{D(A^*)'} = \tag{4.49}$$
$$= \left|\frac{(e^{sA^{(E)}}-\mathrm{I})x}{s-t} - A^{(E)}(x) + \frac{\int_t^s \left(e^{(s-r)A^{(E)}}-\mathrm{I}\right) B^* u(r)\mathrm{d}r}{s-t}\right|_{D(A^*)'}$$

where the term $\frac{(e^{sA}-\mathrm{I})x}{s-t} - A^{(E)}(x) \xrightarrow[D(A^*)']{s\to t^+} 0$ because $x \in M^2 \in D(A^{(E)})$ (the convergence is uniform in $u(\cdot) \in \mathcal{U}_{t,x}$ because it does not depend on $u(\cdot)$) and the second term can be estimated, using Lemma 4.6 and Remark 4.7, with

$$\frac{\int_t^s |u(r)| \left|\left(e^{(s-r)A^{(E)}}-\mathrm{I}\right) B\right|_{D(A^*)'} \mathrm{d}r}{s-t} \le (aK+b) \sup_{r\in[t,s]} \left|\left(e^{(s-r)A^{(E)}}-\mathrm{I}\right) B\right|_{D(A^*)'} \tag{4.50}$$

that goes to zero (the estimate is uniform in the control). Then since $\nabla\varphi(t,x) \in D(A^*)$, the proof is complete.

The (4.38), with $u(\cdot)$ continuous, is a simple corollary of the proof of the first part. Indeed if $u(\cdot)$ is continuous we have that

$$\frac{\int_t^s \langle B(\nabla\varphi(t,x)), u(r)\rangle_{\mathbb{R}}\,\mathrm{d}r}{s-t} \to \langle B(\nabla\varphi(t,x)), u(t)\rangle_{\mathbb{R}} \tag{4.51}$$

□

REMARK 4.16. *We want to emphasize that $O(s)$ is independent of the control, this fact will be crucial when we prove that the value function is a viscosity supersolution of the HJB equation.*

COROLLARY 4.17. *Given $(t,x) \in [0,T]\times M^2$ and $\varphi \in$ TEST and an admissible control $u(\cdot) \in \mathcal{U}_{t,x}$ we have that*

$$\varphi(s,x(s)) - \varphi(t,x) = \tag{4.52}$$
$$= \int_t^s \partial_t\varphi(r,x(r)) + \langle A^*\nabla\varphi(r,x(r)), x(r)\rangle + \langle B(\nabla\varphi(r,x(r))), u(r)\rangle_{\mathbb{R}}\,\mathrm{d}r$$

*(where we called $x(s)$ the trajectory that starts at time $t$ from $x$ and subject to the control $u(\cdot)$).*

#### 4.4.2. The value function as viscosity solution of HJB equation.

PROPOSITION 4.18. **(Bellman's optimality principle)** *The Value function $V$, defined in (4.26) satisfies for all $s > t$:*

$$V(t,x) = \sup_{u(\cdot)\in\mathcal{U}_{t,x}} \left( V(s,x(s)) + \int_t^s L(r,x(r),u(r))\mathrm{d}r \right) \tag{4.53}$$

*where $x(s)$ is the trajectory at time $s$ starting from $x$ subject to control $u(\cdot) \in \mathcal{U}_{t,x}$.*

PROOF. This is a standard result. It can be done as in Theorem 1.13. □

We can now prove that the value function is a viscosity solution of the HJB equation.

THEOREM 4.19. *The value function $V$ is a viscosity solution of the HJB equation.*

Proof. **Subsolution:**
Let $(t,x)$ be a local minimum of $V-\varphi$ for $\varphi \in$ Test. We can assume that $(V-\varphi)(t,x)=0$. We choose $u \in [\Gamma_-(x^0), \Gamma_+(x^0)]$. We consider a continuous control $u(\cdot) \in \mathcal{U}_{t,x}$ such that $u(t)=u$[3]. We call $x(s)$ the trajectory starting from $(t,x)$ and subject to $u(\cdot) \in \mathcal{U}_{t,x}$. Then for $s>t$ with $s-t$ small enough we have

$$V(s,x(s)) - \varphi(s,x(s)) \geq V(t,x) - \varphi(t,x) \tag{4.54}$$

and thanks to the Bellman principle of optimality we know that

$$V(t,x) \geq V(s,x(s)) + \int_t^s L(r,x(r),u(r))\mathrm{d}r. \tag{4.55}$$

Then

$$\varphi(s,x(s)) - \varphi(t,x) \leq V(s,x(s)) - V(t,x) \leq -\int_t^s L(r,x(r),u(r))\mathrm{d}r, \tag{4.56}$$

which implies, dividing by $(t-s)$,

$$\frac{\varphi(s,x(s)) - \varphi(t,x)}{s-t} \leq -\frac{\int_t^s L(r,x(r),u(r))\mathrm{d}r}{s-t}. \tag{4.57}$$

Using Proposition 4.15 we pass to the limit as $s \to t^+$ and obtain

$$\partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + \langle B(\nabla\varphi(t,x)), u(t)\rangle_{\mathbb{R}} \leq -L(t,x,u) \tag{4.58}$$

so

$$\partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + (\langle B(\nabla\varphi(t,x)), u\rangle_{\mathbb{R}} + L(t,x,u)) \leq 0 \tag{4.59}$$

Taking the $\sup_{u\in[\Gamma_-(x^0),\Gamma_+(x^0)]}$ we obtain the subsolution inequality:

$$\partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + H(t,x,\nabla\varphi(t,x)) \leq 0 \tag{4.60}$$

**Supersolution:**
Let $(t,x)$ be a maximum for $V-\varphi$ and such that $(V-\varphi)(t,x)=0$. For $\varepsilon>0$ we take $u(\cdot) \in \mathcal{U}_{t,x}$ an $\varepsilon^2$-optimal strategy. We call $x(s)$ the trajectory starting from $(t,x)$ and subject to $u(\cdot) \in \mathcal{U}_{t,x}$. Now for $(s-t)$ small enough

$$V(t,x) - V(s,x(s)) \geq \varphi(t,x) - \varphi(s,x(s)) \tag{4.61}$$

and from $\varepsilon^2$ optimality we know that

$$V(t,x) - V(s,x(s)) \leq \varepsilon^2 + \int_t^s L(r,x(r),u(r))\mathrm{d}r \tag{4.62}$$

so

$$\frac{\varphi(s,x(s)) - \varphi(t,x)}{s-t} \geq \frac{-\varepsilon^2 - \int_t^s L(r,x(r),u(r))\mathrm{d}r}{s-t} \tag{4.63}$$

We take $(s-t)=\varepsilon$ so that

$$\frac{\varphi(t+\varepsilon, x(t+\varepsilon)) - \varphi(t,x)}{\varepsilon} \geq -\varepsilon - \frac{\int_t^{t+\varepsilon} -L(r,x(r),u(r))\mathrm{d}r}{\varepsilon} \tag{4.64}$$

and in view of Proposition 4.15 we can choose, independently on the control $u(\cdot) \in \mathcal{U}_{t,x}$, a $O(\varepsilon)$ with $O(\varepsilon) \xrightarrow{\varepsilon\to 0} 0$ such that:

$$\begin{aligned}\partial_t\varphi(t,x) &+ \langle A^*\nabla\varphi(t,x), x\rangle + \\ &+ \frac{\int_t^{t+\varepsilon} \langle B(\nabla\varphi(t,x)), u(r)\rangle_{\mathbb{R}} + L(r,x(r),u(r))\mathrm{d}r}{\varepsilon} \geq -\varepsilon + O(\varepsilon).\end{aligned} \tag{4.65}$$

[3]It exists: for example if $u>0$ the control $u(s) = \frac{u}{\Gamma_+(x^0)}\Gamma_+(x^0(s))$ until $\Gamma_+(x^0(s)>0$ and then equal to 0: since $\Gamma_+$ is locally Lipschitz and sublinear all works.

We now take the supremum over $u$ inside the integral and let $\varepsilon \to 0$ and obtain that

$$\partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + H(t,x,\nabla\varphi(t,x)) \geq 0 \tag{4.66}$$

Then $V$ is a supersolution of the HJB equation. So $V$ is both a viscosity supersolution and a viscosity subsolution of the HJB equation and then, by definition, it is a viscosity solution of the HJB equation. □

## 4.5. A verification result

We use the following lemma

LEMMA 4.20. *Let $f \in C([0,T])$. Extend $f$ to a $g$ on $(-\infty,+\infty)$ with $g(t) = g(T)$ for $t > T$ and $g(t) = g(0)$ for $t < 0$. Suppose there is a $\rho \in L^1(0,T;\mathbb{R})$ such that*

$$\left|\liminf_{h\to 0} \frac{g(t+h)-g(t)}{h}\right| \leq \rho(t) \quad a.e.\ t \in [0,T] \tag{4.67}$$

*Then*

$$g(\beta) - g(\alpha) \geq \int_\alpha^\beta \liminf_{h\to 0} \frac{g(t+h)-g(t)}{h} \mathrm{d}t \quad \forall\, 0 \leq \alpha \leq \beta \leq T \tag{4.68}$$

PROOF. The proof can be found in [YZ99] page 270. □

We first introduce a set related with a subset of the subdifferential of a function in $C([0,T]\times M^2)$. Its definition is suggested by the definition of sub/super solution. We define

DEFINITION 4.21 ($\mathbf{Ev(t,x)}$). *Given $v \in C([0,T]\times M^2)$ and $(t,x) \in [0,T]\times M^2$ we define $Ev(t,x)$ as*

$$\begin{aligned} Ev(t,x) = \{(q,p) \in \mathbb{R}\times D(A^*) \ : \ & \exists \varphi \in \text{TEST}, s.t. \\ & v-\varphi \textit{ attains a loc. min. in } (t,x), \\ & \partial_t\varphi(t,x) = q,\ \nabla\varphi(t,x) = p, \\ & \textit{and } v(t,x) = \varphi(t,x)\} \end{aligned} \tag{4.69}$$

REMARK 4.22. *$Ev(t,x)$ is a subset of the subdifferential of $v$ defined at page 13.*

We can now pass to formulate and prove a verification theorem:

THEOREM 4.23. *Let $(t,x) \in [0,T]\times M^2$ an initial datum ($x(t) = x$). Let $u(\cdot) \in \mathcal{U}_{t,x}$ and $x(\cdot)$ be the relate trajectory. Let $q \in L^1(t,T;\mathbb{R})$, $p \in L^1(t,T;D(A^*))$ be such that*

$$(q(s),p(s)) \in EV(t,x_{t,y}(s)) \quad \textit{for almost all } s \in (t,T) \tag{4.70}$$

*Moreover if $u(\cdot)$ satisfies*

$$\begin{aligned} \int_t^T \langle A^*p(s), x(s)\rangle_{M^2} + \langle Bp(s), u(s)\rangle_{\mathbb{R}} + q(s)\ \mathrm{d}s \geq \\ \geq \int_t^T -L(s,x(s),u(s))\ \mathrm{d}s \end{aligned} \tag{4.71}$$

*then $u(\cdot)$ is an optimal control at $(t,x)$.*

PROOF. The function

$$\begin{cases} \Psi\colon [t,T] \to \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \\ \Psi\colon s \mapsto (\langle A^*p(s),x(s)\rangle_{M^2}, \langle Bp(s),u(s)\rangle_{\mathbb{R}}, q(s), L(s,x(s),u(s))) \end{cases} \tag{4.72}$$

in view of Lemma 4.6 is in $L^1(t,T;\mathbb{R}^4)$. So the set of the right-Lebesgue point is of full measure. We choose $\bar{s}$ a point in such a set. We can continue to choose $\bar{s}$ in a full measure set if we assume that (4.70) is satisfied at $\bar{s}$. We set $\bar{x} := x(\bar{s})$ and we consider a functions $\varphi \equiv \varphi^{\bar{s},\bar{x}} \in \textsc{Test}$ such that $V \geq \varphi$ in a neighborhood of $(\bar{s},\bar{x})$, $V(\bar{s},\bar{x}) - \varphi(\bar{s},\bar{x}) = 0$ and $(\partial_t)(\varphi)(\bar{s},\bar{x})) = q(\bar{s})$, $\nabla\varphi(\bar{s},\bar{x}) = p(\bar{s})$. Then for $\tau \in (\bar{s},T]$ and $(\tau - \bar{s})$ small enough we have

$$\frac{V(\tau,x(\tau)) - V(\bar{s},\bar{x})}{\tau - \bar{s}} \geq \frac{\varphi(\tau,x(\tau)) - \varphi(\bar{s},\bar{x})}{\tau - \bar{s}} \geq \tag{4.73}$$

for Proposition 4.15

$$\geq \partial_t\varphi(\bar{s},\bar{x}) + \frac{\int_{\bar{s}}^{\tau} \langle B\nabla\varphi(\bar{s},\bar{x}), u(r)\rangle_{\mathbb{R}}\, \mathrm{d}r}{\tau - \bar{s}} + \langle A^*\nabla\varphi(\bar{s},\bar{x}), x\rangle + O(\tau - \bar{s}) \tag{4.74}$$

In view of the choice of $\bar{s}$ we know that

$$\frac{\int_{\bar{s}}^{\tau} \langle B\nabla\varphi(\bar{s},\bar{x}), u(r)\rangle_{\mathbb{R}}\, \mathrm{d}r}{\tau - \bar{s}} \xrightarrow{\tau\to\bar{s}^+} \langle B\nabla\varphi(\bar{s},\bar{x}), u(\bar{s})\rangle_{\mathbb{R}} \tag{4.75}$$

So that for almost every $\bar{s}$ in $[t,T]$ we have

$$\begin{aligned}\liminf_{\tau\downarrow\bar{s}} \frac{V(\tau,x(\tau)) - V(\bar{s},x(\bar{s}))}{\tau - \bar{s}} &\geq \\ &\geq \langle B\nabla\varphi(\bar{s},x(\bar{s})), u(\bar{s})\rangle_{\mathbb{R}} + \\ &+ \partial_t\varphi(\bar{s},x(\bar{s})) + \langle A^*\nabla\varphi(\bar{s},x(\bar{s})), x(\bar{s})\rangle = \\ &= \langle Bp(\bar{s}), u(\bar{s})\rangle_{\mathbb{R}} + q(\bar{s}) + \langle A^*\nabla p(\bar{s}), x(\bar{s})\rangle\end{aligned} \tag{4.76}$$

then we can use Lemma 4.20 and find that

$$\begin{aligned}V(T,x(T)) - V(t,x) &\geq \\ &\geq \int_t^T \langle Bp(\bar{s}), u(\bar{s}))\rangle_{\mathbb{R}} + q(\bar{s}) + \langle A^*\nabla p(\bar{s}), x(\bar{s})\rangle\, \mathrm{d}\bar{s} \geq\end{aligned} \tag{4.77}$$

using (4.71)

$$\geq \int_t^T -L(r,x(r),u(r))\mathrm{d}r \tag{4.78}$$

Hence

$$\begin{aligned}V(t,x) \leq V(T,x(T)) + \int_t^T L(r,x(r),u(r))\mathrm{d}r &= \\ &= h(x(T)) + \int_t^T L(r,x(r),u(r))\mathrm{d}r\end{aligned} \tag{4.79}$$

and then $(x(\cdot),u(\cdot))$ is an optimal pair. □

Remark 4.24. *Such result is not a consequence of Theorem 5.15, indeed here we have the unbounded term $B$ and different test functions.*

## 4.6. The constraint $\theta \geq 0$

In this section we add the constraint $\theta \geq 0$ (that is, in the Hilbert setting, $x^0 \geq 0$) to the problem. The state equation in Hilbert space remains (4.14) as in Section 4.3, indeed we are considering the same system as in previous sections but we admit a smaller set of controls. So we can use Lemma 4.8, Lemma 4.6 and Remark 4.7 without changes.

In order to impose the condition $\theta \geq 0$, we use the same method we applied in [FGF] (the same used in [Fag05b, Fag05a, Fag06]): we add to the target functional a function $g$ that assumes the value 0 inside the admissible set and the value $-\infty$ on the "forbidden" set so that we do not need do modify the set of admissible controls:

$$\begin{aligned} & g\colon \mathbb{R} \to \overline{\mathbb{R}} \\ & g(r) = \begin{cases} 0 & if\ r \geq 0 \\ -\infty & if\ r < 0 \end{cases} \end{aligned} \tag{4.80}$$

The set of admissible controls is still

$$\mathcal{U}_{t,x} \stackrel{def}{=} \{u(\cdot) \in L^2(t,T)\ :\ \Gamma_-(x^0(s)) \leq u(s) \leq \Gamma_+(x^0(s))\ \forall s \in [t,T]\} \tag{4.81}$$

but the target functional becomes

$$J(t,x,u(\cdot)) = \int_t^T L(s,x(s),u(s)) + g(x^0(s))\mathrm{d}s + h(x(T)) \tag{4.82}$$

The value function of the problem is defined as

$$V(t,x) = \sup_{u(\cdot)\in\mathcal{U}_{t,x}} J(t,x,u(\cdot)) \tag{4.83}$$

If we had imposed the constraint $\theta \geq 0$ (and then $x^0 \geq 0$) in the definition of the set of admissible controls $\mathcal{U}_{t,x}$, there would have been some points of $[0,T] \times M^2$ where $\mathcal{U}_{t,x} = \emptyset$. It means that there are some initial data $(\phi^0, \phi^1, \omega)$ whose trajectories, for every control $u(\cdot)$ with $u \in [\Gamma_-(\theta), \Gamma_+(\theta)]$, violate the constraint $\theta \geq 0$. It is the case of initial data with $\phi^0 < 0$ but there may be less simple examples in the same line of Remark 7.2 where an example in the case $\Gamma_- = -\infty$ is given. Here we allow such points knowing that they will be excluded in the optimization process. Indeed the value function $V$ is a function in extended real set: $V\colon [0,T] \times M^2 \to \overline{\mathbb{R}}$ such that $V = -\infty$ if a trajectory stays for a while in the "forbidden zone".

Otherwise if the set $\mathcal{U}_{t,x}$ contains at least an element with $x^0(\cdot) \geq 0$ then, taken an initial datum $x$, by Lemma 4.6 and Remark 4.7,

$$|x^0(t)|,\ |u(t)| \leq C_0(|x|) \quad \forall t \in [0,T]$$

and so, by the continuity of $L$ and $h$ we have can state that

$$C_1(|x|) \leq J(t,x,u(\cdot)) \leq C_2(|x|).$$

So $V > -\infty$. This argument shows that

PROPOSITION 4.25. *If we call $D(V)$ (the* domain *of $V$) the set of the points in which $V > -\infty$, the restriction $V_{|D(V)}$ is locally bounded.*

Moreover we note that the value function is upper semicontinuous in $(t,x)$ because it is the supremum of the upper semicontinuous functions $J(t,x,u(\cdot))$ over $u(\cdot)$. This fact, together with Proposition 4.25 shows that $D(V)$ is closed.

The HJB equation of the system is defined as

$$\begin{cases} \partial_t w(t,x) + \langle \nabla w(t,x), Ax \rangle + g(x) + H(t,x,\nabla w(t,x)) = 0 \\ w(T,x) = h(x) \end{cases}, \tag{4.84}$$

where $H$ is defined as before

$$\begin{cases} H\colon [0,T] \times D(A^*) \to \mathbb{R} \\ H(t,x,p) \stackrel{def}{=} \sup_{u\in[\Gamma_-(x^0),\Gamma_+(x^0)]} \{uB(p) + L(t,x,u)\} \end{cases}. \tag{4.85}$$

**4.6.1. Viscosity solution.** Now we give a definition of viscosity solution that takes into account the constraint $\theta \geq 0$. We are interested only in the points of the set $D(V)$ because on the other points all the trajectories have utility $-\infty$. The definition will be the same as before in the interior of $D(V)$. Instead, a good definition on the boundary $\partial D(V)$ should allow only those controls that keep the trajectory on $D(V)$. So we consider the following set

$$U(t,x) = \begin{cases} [\Gamma_-(x^0), \Gamma_+(x^0)] & \text{if } (t,x) \text{ is in the interior of } D(V) \\ \left.\begin{cases} u \in [\Gamma_-(x^0), \Gamma_+(x^0)] & : \text{there exist } u(\cdot) \in \mathcal{U}_{t,x} \\ & \quad \text{continuous} \\ & \quad \text{with } u(t) = u \\ & \quad \text{s.t. } J(t,x,u(\cdot)) > -\infty \end{cases}\right\} & \text{if } (t,x) \in \partial D(V) \end{cases} \tag{4.86}$$

and we change the definition of viscosity solution in the "subsolution" part, while the definition of supersolution remains the same. We change the set where we compute from $[\Gamma_-(x^0), \Gamma_+(x^0)]$ to $U(t,x)$, so, when $x$ is an interior point of $D(V)$ the definition is the same, when $x$ is a boundary point it changes.

We have already observed that the value function is upper semicontinuous in $(t,x)$ anyway it could be non-continuous and then we have to introduce a definition of viscosity solution for a family of non-continuous function.

Given a locally bounded function $w\colon D(V) \to \mathbb{R}$ we call ${w^*}_{D(V)}(\cdot)$ and ${w_*}_{D(V)}(\cdot)\colon D(V) \to \mathbb{R}$ the functions:

$${w^*}_{D(V)}(t,x) \stackrel{def}{=} \limsup_{\substack{(s,y)\to(t,x) \\ (s,y)\in D(V)}} w(s,y)$$

and

$${w_*}_{D(V)}(t,x) \stackrel{def}{=} \liminf_{\substack{(s,y)\to(t,x) \\ (s,y)\in D(V)}} w(s,y)$$

Definition 4.26 (**Viscosity subsolution**). *A locally bounded function $w\colon (D(V)) \to \mathbb{R}$ is a viscosity subsolution of the HJB equation (or simply a "subsolution") if $w^*_{D(V)}(T,x) \leq h(x)$ for all $x \in M^2$ and for every $\varphi \in$ Test and a local minimum point $(t,x) \in D(V)$ of $w^*_{D(V)} - \varphi_{|D(V)}$ we have*

$$\partial_t \varphi(t,x) + \langle A^* \nabla\varphi(t,x), x\rangle + \sup_{u \in U(t,x)} \{uB(\nabla\varphi(t,x)) + L(t,x,u)\} \leq 0 \tag{4.87}$$

Definition 4.27 (**Viscosity supersolution**). *A locally bounded function $w\colon (D(V)) \to \mathbb{R}$ is a viscosity supersolution of the HJB equation (or simply a "supersolution") if ${w_*}_{D(V)}(T,x) \geq h(x)$ for all $x \in M^2$ and for every $\varphi \in$ Test and a local maximum point $(t,x) \in D(V)$ of ${w_*}_{D(V)} - \varphi_{|D(V)}$ we have*

$$\partial_t \varphi(t,x) + \langle A^* \nabla\varphi(t,x), x\rangle + H(t,x,\nabla\varphi(t,x)) \geq 0 \tag{4.88}$$

Definition 4.28 (**Viscosity solution**). *A locally bounded function $w\colon (D(V)) \to \mathbb{R}$ is a viscosity solution of the HJB equation if it is at the same time a supersolution and a subsolution.*

Remark 4.29. *The boundary condition $w(T,x) \geq h(x)$ for all $x \in M^2$ (or $w(T,x) \leq h(x)$ for all $x \in M^2$) makes sense because $\{T\} \times M^2 \subseteq D(V)$.*

Theorem 4.30. *The restriction to $D(V)$ of the value function is a viscosity solution of HJB equation.*

Proof. In the proof we will call the restriction $V_{|D(V)}$ simply $\omega$.

**Subsolution:**
Since $V$ is upper semicontinuous $\omega^*_{D(V)} = \omega$ Let $(t,x)$ be a local minimum of $(\omega - \varphi_{|D(V)})$ for $\varphi \in$ TEST. We can assume that $(\omega - \varphi_{|D(V)})(t,x) = 0$. If $(t,x)$ is in the interior of $D(V)$ the proof is the same as in Theorem 4.19. Otherwise we choose $u \in U(t,x)$ and we consider a continuous control $u(\cdot) \in \mathcal{U}_{t,x}$ such that $u(t) = u$ and $J(t,x,u(\cdot) > -\infty$ (it exists for the definition of $U(t,x)$). We call $x(s)$ the trajectory starting from $(t,x)$ and subject to $u(\cdot)$. Then for $(s-t)$ small enough

$$\omega(s,x(s)) - \varphi_{|D(V)}(s,x(s)) \geq \omega(t,x) - \varphi_{|D(V)}(t,x) \tag{4.89}$$

and thanks to the Bellman principle of optimality we know that

$$\omega(t,x) \geq \omega(s,x(s)) + \int_t^s L(r,x(r),u(r))\mathrm{d}r \tag{4.90}$$

So

$$\varphi(s,x(s)) - \varphi(t,x) \leq \omega(s,x(s)) - \omega(t,x) \leq -\int_t^s L(r,x(r),u(r))\mathrm{d}r \tag{4.91}$$

Then, dividing by $(t-s)$,

$$\frac{\varphi(s,x(s)) - \varphi(t,x)}{s-t} \leq -\frac{\int_t^s L(r,x(r),u(r))\mathrm{d}r}{s-t} \tag{4.92}$$

Using Proposition 4.15 we can now pass to the limit as $s \to t^+$ and obtain

$$\partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + \langle B(\nabla\varphi(t,x)), u(t)\rangle_{\mathbb{R}} \leq -L(t,x,u) \tag{4.93}$$

so

$$\partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + (\langle B(\nabla\varphi(t,x)), u\rangle_{\mathbb{R}} + L(t,x,u)) \leq 0 \tag{4.94}$$

Taking the $\sup_{u\in U(t,x)}$ we obtain the subsolution inequality:

$$\partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + \sup_{u\in U(t,x)} \{uB(p) + L(t,x,u)\} \leq 0 \tag{4.95}$$

**Supersolution:**
We take $(t,x) \in D(V)$ and a test function $\varphi$ such that $\varphi(t,x) = \omega_{*D(V)}(t,x)$ and $\omega_{*D(V)} \leq \varphi$ in a neighborhood $B_r(t,x) \cap D(V)$ of $(t,x)$. We assume by contradiction that

$$\partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + H(t,x,\nabla\varphi(t,x)) < 0$$

By continuity we have

$$+\partial_t\varphi(s,y) + \langle A^*\nabla\varphi(s,y), y\rangle + H(s,y,\nabla\varphi(s,y)) < -\sigma \tag{4.96}$$

for some $0 < \sigma < r$ and for all the points in $B_\sigma(t,x)$.

By definition of $\omega_{*D(V)}$ we can choose $(t_n,x_n) \in B_{\frac{\sigma}{2}}(t,x) \cap D(V)$ such that $(t_n,x_n) \xrightarrow[n\in\mathbb{N}]{n\to\infty} (t,x)$ and such that

$$\omega_{*D(V)} - V(t_n,x_n) \in (-1/n, 1/n). \tag{4.97}$$

$\{(t_n,x_n)\}_{n\in\mathbb{N}}$ converges to $(t,x)$ and then it is compact, so, thanks to the Frechet-Kolmogorov theorem (see [Yos95] page 275) we have that $x^1_n$

$$\sup_{n\in\mathbb{N}} |\Xi(s)x^1_n - x^1_n|_{L^2(-R,0)} \xrightarrow{s\to 0} 0.$$

and then, arguing in way similar to Lemma 4.8, we can find $\zeta > 0$ such that for every $n$ and every admissible control $u(\cdot)$ we have $x_{u(\cdot),t_n,x_n}(t_n+\tau) \in B_\sigma(t,x)$ for every $\tau \in [0,\zeta]$.

Moreover, for the (4.97) and the continuity of $\varphi$ we can suppose

$$V(t_n,x_n) - \varphi(t_n,x_n) \in (-2/n, 2/n). \tag{4.98}$$

Given a $\delta > 0$ we can choose an admissible control $u(\cdot)$ (remaining in $D(V)$) such that

(4.99)

$$V(t_n, x_n) \leq \delta + V(\zeta + t_n, x_{u(\cdot),x_n,t_n}(\zeta + t_n)) + \int_{t_n}^{\zeta+t_n} L(s, x_{u(\cdot),t_n,x_n}(s), u(s))\mathrm{d}s$$

Since $(t,x)$ is a minimum for $V_* - \varphi$ and thanks to the choice of $\zeta$ we have

$$\varphi(\zeta + t_n, x_{u(\cdot),t_n,x_n}(\zeta + t_n)) - V_*(\zeta + t_n, x_{u(\cdot),t_n,x_n}(\zeta + t_n)) \geq 0$$

so

$$\varphi(\zeta + t_n, x_{u(\cdot),t_n,x_n}(\zeta + t_n)) - V(\zeta + t_n, x_{u(\cdot),t_n,x_n}(\zeta + t_n)) \geq 0 \tag{4.100}$$

So, from (4.98), (4.99) and (4.100) we have

$$-\delta + \int_{t_n}^{\zeta+t_n} -L(s, x_{u(\cdot),t_n,x_n}(s), u(s))\mathrm{d}s - 2/n \leq \varphi(\zeta+t_n, x_{u(\cdot),t_n,x_n}(\zeta+t_n)) - \varphi(t_n, x_n)$$

and using Corollary 4.17 we have

(4.101)

$$-\delta + \int_{t_n}^{\zeta+t_n} -L(s, x_{u(\cdot),t_n,x_n}(s), u(s))\mathrm{d}s - 2/n \leq \int_{t_n}^{\zeta+t_n} \partial_t\varphi(s, x_{u(\cdot),t_n,x_n}(s), u(s)) +$$
$$+ \left\langle A^*\nabla\varphi(s, x_{u(\cdot),t_n,x_n}(s)), x_{u(\cdot),t_n,x_n}(s)\right\rangle + \left\langle B(\nabla\varphi(s, x_{u(\cdot),t_n,x_n}(s)), u(r)\right\rangle_{\mathbb{R}} \mathrm{d}s$$

so, using (4.96) we find

$$-\delta - 2/n \leq -\zeta\sigma$$

Since the choices of $\delta$ was arbitrary and the estimate is uniform in $n$ we find

$$\zeta\sigma \leq 0$$

that is a contradiction. □

CHAPTER 5

# A verification result within the framework of viscosity solutions in infinite dimensions and construction of $\epsilon$-optimal strategies

In this chapter we present the results of [FGŚ]: a verification results within the framework of viscosity solution and a result on $\varepsilon$-optimal controls. In Appendix A we describe a verification result for the stochastic case (indeed it is the only stochastic optimal control in Hilbert space we treat in this thesis).

As we stressed in the Introduction (see page 5) a verification results represents a key step of the dynamic programming approach to optimal control problems. A verification theorem is a tool to check whether a given admissible control is optimal and, more importantly, suggests a way of constructing optimal feedback controls.

In finite dimensional case the verification theorem for smooth solutions $V$ can be found for example in [Zab92]), and, in the viscosity case in [BCD97, YZ99].

We have already observed in Chapter 2 that different papers use different approaches and definitions and prove that the value function of the problem is the only viscosity solution of the HJB equation equation. To "complete" the dynamic programming approach also in the infinite dimensional case it is necessary to prove, as in finite dimensional case, some verification theorems. The verification results depend on the approach and on the definition of solution we use. We prove a verification results for the approach of Crandall and Lions in [CL90, CL91] in deterministic case (and in Appendix A for the approach due to Święch (in [Świ94]) for the stochastic case). The reason of such choice can be understood (at least for deterministic case) reading Chapter 2: they are key-works, and they can be considered (the most) "classical".

The main difficulty we have to deal with is the fact that in the infinite dimensional setting not all regular functions that "touch" the candidate-solution of the HJB equation can be test function in the definition of viscosity solution but only particular ones, indeed the test functions that are considered in [CL90] (see Section 2.2) are sum of two parts: one regular and compatible with the generator of semigroup that appears in the state equation of the system (5.1) ("test1"), and one radial ("test2"). The differentials of such functions do not span all the super (or sub) differential of the candidate-solution so we cannot reformulate the definition in terms of super(sub)differentials as in finite dimensional case.

The two families $test1$ and $test2$ have different role in the definition of super/sub-solution (see Definitions 5.5) and so they have to be treated in different way when we prove the verification theorem.

## 5.1. Notation, definitions and background

We are interested in the optimal control problem in a real separable Hilbert space $X$ characterized by the state equation

$$\begin{cases} \dot{x}(s) = Ax(s) + f(s, x(s), u(s)) \\ x(0) = x \end{cases} \tag{5.1}$$

and cost functional

$$J(y;u(\cdot)) = \int_0^T L(s,x(s),u(s))\mathrm{d}s + h(x(T)) \tag{5.2}$$

where the set of admissible controls is given by

$$\mathcal{U}[0,T] = \{u\colon [0,T] \to U:\ u \text{ is measurable}\},$$

and $U$ is a metric space.

We assume that:

Hypothesis 5.1. *$A$ is linear, densely defined and $-A$ is maximal monotone , and then $A$ generates a $C_0$ semigroup of contractions $e^{sA}$, that is*

$$\|e^{sA}\| \le 1 \quad \textit{for all } s \ge 0$$

Remark 5.2. *If we replace $A$ and $b$ with $\tilde{A} = A - \omega I$ and $f(t,x,u)$ with $\tilde{f}(t,x,u) = f(t,x,u) + \omega x$ Hypothesis 5.1 covers a more general case*

$$\|e^{sA}\| \le e^{\omega s} \quad \textit{for all } s \ge 0$$

*for some $\omega \ge 0$*

Hypothesis 5.3.

$$f\colon [0,T] \times X \times U \to X \textit{ is continuous}$$

*and there exist a constant $M > 0$ and a local modulus of continuity $\omega(\cdot,\cdot)$ such that*

$$\begin{aligned} &|f(t,x,u) - f(s,y,u)| \le M|x-y| + \omega(|t-s|, |x| \vee |y|) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \textit{for all } t,s \in [0,T],\ u \in U\ x,y \in X \\ &|f(t,0,u)| \le M \ \textit{ for all } (t,u) \in [0,T] \times U \end{aligned}$$

Hypothesis 5.4.

$$L\colon [0,T] \times X \times U \to \mathbb{R} \quad \textit{and} \quad h\colon X \to \mathbb{R} \quad \textit{are continuous}$$

*and there exist $M > 0$ and a local modulus of continuity $\omega(\cdot,\cdot)$ such that*

$$\begin{aligned} &|L(t,x,u) - L(s,y,u)|,\ |h(x) - h(y)| \le \omega(|x-y| + |t-s|, |x| \vee |y|) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \textit{for all } t,s \in [0,T],\ u \in U\ x,y \in X \\ &|L(t,0,u)|, |h(0)| \le M \ \textit{ for all } (t,u) \in [0,T] \times U \end{aligned}$$

Following the dynamic programming approach we consider a family of problems for every $t \in [0,T], y \in X$

$$\begin{cases} \dot{x}_{t,x}(s) = Ax(s) + f(s,x(s),u(s)) \\ x(t) = x \end{cases} \tag{5.3}$$

We consider the functional

$$J(t,x;u(\cdot)) = \int_t^T L(s,x(s),u(s))\mathrm{d}s + h(x(T)), \tag{5.4}$$

where $u(\cdot)$ is in the set of admissible controls

$$\mathcal{U}[t,T] = \{u\colon [t,T] \to U:\ u \text{ is measurable}\}.$$

The associated value function $V\colon [0,T] \times X \to \mathbb{R}$ is defined by

$$V(t,x) = \inf_{u(\cdot) \in \mathcal{U}[t,T]} J(t,x;u(\cdot)). \tag{5.5}$$

The HJB related to such optimal control problems is

$$\begin{cases} v_t(t,x) + \langle Dv(t,x), Ax\rangle + H(t,x,Dv(t,x)) = 0 \\ v(T,x) = h(x), \end{cases} \tag{5.6}$$

where

$$\begin{cases} H\colon [0,T]\times X\times X\to\mathbb{R}, \\ H(t,x,p)=\inf_{u\in U}\left(\langle p,f(t,x,u)\rangle + L(t,x,u)\right)\end{cases}$$

The solution of the above HJB equation is understood in the viscosity sense of Crandall and Lions [CL90, CL91] which is slightly modified here. We consider two sets of functions:

$$test1=\{\varphi\in C^1([0,T)\times X)\ :\ \varphi \text{ is weakly sequentially lower semicontinuous and } A^*D\varphi\in C([0,T)\times X)\}$$

and

$$test2=\{g\in C^1([0,T]\times X)\ :\ \exists g_0,: [0,+\infty)\to[0,+\infty), \textit{ and } \eta\in C^1([0,T]) \text{ positive } \textit{s.t.}\ g_0\in C^1([0,+\infty)),\ g_0'(r)\geq 0\ \forall r\geq 0,\ g_0'(0)=0 \textit{ and } g(t,x)=\eta(t)g_0(|x|)\ \forall (t,x)\in[0,T]\times X\}$$

We use test2 functions that are a little different from the ones used in [CL90]. The extra term $\eta(\cdot)$ in test2 functions is added to deal with polynomially growing solutions.

We can now give the definitions of viscosity sub/super-solutions.

Definition 5.5 (**Viscosity subsolution**). *$v\in C([0,T]\times X)$ is a (viscosity)* subsolution *of the HJB (5.6) if*

$$v(T,x)\leq h(x)\quad \textit{for all } x\in X$$

*and whenever $v-\varphi-g$ has a local maximum at $(\bar t,\bar x)\in[0,T)\times X$ for $\varphi\in test1$ and $g\in test2$, we have*

$$\varphi_t(\bar t,\bar x)+g_t(\bar t,\bar x)+\langle A^*D\varphi(\bar t,\bar x),\bar x\rangle + H(\bar t,\bar x, D\varphi(\bar t,\bar x)+Dg(\bar t,\bar x))\geq 0. \tag{5.7}$$

Definition 5.6 (**Viscosity supersolution**). *$v\in C([0,T]\times X)$ is a (viscosity)* supersolution *of the HJB (5.6) if*

$$v(T,x)\geq h(x)\quad \textit{for all } x\in X$$

*and whenever $v+\varphi+g$ has a local minimum at $(\bar t,\bar x)\in[0,T)\times X$ for $\varphi\in test1$ and $g\in test2$, we have*

$$-\varphi_t(\bar t,\bar x)-g_t(\bar t,\bar x)-\langle A^*D\varphi(\bar t,\bar x),\bar x\rangle + H(\bar t,\bar x,-D\varphi(\bar t,\bar x)-Dg(\bar t,\bar x))\leq 0.$$

Definition 5.7 (**Viscosity solution**). *$v\in C([0,T]\times X)$ is a (viscosity)* solution *of the HJB (5.6) if it is at the same time a subsolution and a supersolution.*

Lemma 5.8. *Let Hypotheses 5.1, 5.3 and 5.4 hold. Let $\varphi\in test1$ and $(t,x)\in[0,T]\times X$. Then the following convergence holds uniformly in $u(\cdot)\in\mathcal{U}[t,T]$*

$$\lim_{s\downarrow t}\left(\frac{1}{s-t}\left(\varphi(s,x(s))-\varphi(t,x)\right)-\varphi_t(t,x)-\langle A^*D\varphi(t,x),x\rangle - \frac{1}{s-t}\int_t^s \langle D\varphi(t,x),f(t,x,u(r))\rangle\,\mathrm{d}r\right)=0 \tag{5.8}$$

*Moreover we have*

$$\varphi(s,x(s))-\varphi(t,x)=\int_t^s \varphi_t(r,x(r))+\langle A^*D\varphi(r,x(r)),x(r)\rangle + \langle D\varphi(r,x(r)),f(r,x(r),u(r))\rangle\,\mathrm{d}r \tag{5.9}$$

Proof. See [LY95] Lemma 3.3 page 240 and Proposition 5.5. page 67 (the first sentence is the same of Proposition 3.27. □

LEMMA 5.9. *Let Hypotheses 5.1, 5.3 and 5.4 hold. Let $g \in test2$ and $(t,x) \in [0,T] \times X$. The following holds*

$$\frac{1}{s-t}\left(g(s,x(s)) - g(t,x)\right) \leq g_t(t,x) + \frac{1}{s-t}\int_t^s \langle Dg(t,x), f(t,x,u(r))\rangle \, \mathrm{d}r + o(1) \tag{5.10}$$

*where $o(1)$ is uniform in $u(\cdot) \in \mathcal{U}[t,T]$*

PROOF. To prove the statement we use the fact that (see [LY95] page 241, equation (3.11)):

$$|x(s)| \leq |x| + \int_t^s \left\langle \frac{x}{|x|}, f(t,x,u(r)) \right\rangle \mathrm{d}r + o(s-t)$$

So we have

$$\begin{aligned} g(s,x(s)) - g(t,x) &= \eta(s)g_0(|x(s)|) - \eta(t)g_0(|x|) = 0 \leq \\ &\leq \eta(s)g_0\left(|x| + \int_t^s \left\langle \frac{x}{|x|}, f(t,x,u(r)) \right\rangle \mathrm{d}r + o(s-t)\right) - \eta(t)g_0(|x|) = 0 \leq \\ &\leq \eta'(t)g_0(|x|)(s-t) + \eta(t)g_0'(|x|)\left(\int_t^s \left\langle \frac{x}{|x|}, f(t,x,u(r)) \right\rangle \mathrm{d}r\right) + o(s-t) = \\ &= g_t(t,x)(s-t) + \int_t^s \langle Dg(t,x), f(t,x,u(r))\rangle \, \mathrm{d}r + o(s-t) \end{aligned} \tag{5.11}$$

where $o(s-t)$ can be choose uniform in $u(\cdot)$. □

THEOREM 5.10. *Let Hypotheses 5.1, 5.3 and 5.4 hold. Then the value function $V$ (defined in (5.5)) is a viscosity solution of the HJB.*

PROOF. The proof is quite standard and can be obtained with small changes (due to the small differences in the definition of test2 functions) from the proof of Theorem 3.2 Chapter 6 of [LY95]. □

We need other assumptions to guarantee a comparison result, and then a uniqueness of the solution of the HJB. There are some different possibilities in the literature, two different sufficient set of hypotheses can be already found in [CL90]. What we need in the proof of the verification theorem is the comparison result in itself, so all different set of hypotheses that guarantee the comparison result are good for us. So we choose to assume a form of verification theorem as hypothesis.

HYPOTHESIS 5.11. *There exists a $\mathcal{G} \subseteq C([0,T] \times X)$ such that:*

(i) *The value function $V$ is in $\mathcal{G}$*
(ii) *If $v_1, v_2 \in \mathcal{G}$, $v_1$ is a subsolution of the HJB and $v_2$ is a supersolution of the HJB then $v_1 \leq v_2$*

From $(i)$ and $(ii)$ we know that $V$ is the only solution of the HJB in $\mathcal{G}$.
We will use the following lemma whose proof can be found in [YZ99], page 270.

LEMMA 5.12. *Let $f \in C([0,T];\mathbb{R})$. We extend $f$ to a function $g$ on $(-\infty,+\infty)$ by setting $g(t) = g(T)$ for $t > T$ and $g(t) = g(0)$ for $t < 0$. Suppose there is a function $\rho \in L^1((0,T);\mathbb{R})$ such that*

$$\limsup_{h \to 0^+} \frac{g(t+h) + g(t)}{h} \leq \rho(t) \quad a.e.\ t \in [0,T].$$

*Then*

$$g(\beta) - g(\alpha) \leq \int_\alpha^\beta \limsup_{h\to 0} \frac{g(t+h)+g(t)}{h} \mathrm{d}t \quad \forall\, 0 \leq \alpha \leq \beta \leq T.$$

## 5.2. The verification theorem

We first introduce a set related with a subset of the superdifferential of a function in $C([0,T]\times X)$. Its definition is suggested by the definition of sub/super solution.

DEFINITION 5.13 ($\mathbf{Ev(t,x)}$). *Given $v \in C([0,T]\times X)$ and $(t,x)\in[0,T]\times X$ we define $E^{1,+}v(t,x)$, or simply $Ev(t,x)$ as*

$$\begin{array}{lcl} Ev(t,x) = \{(q,p_1,p_2) \in & : & \exists \varphi \in test1,\ g\in test2 \ s.t. \\ \qquad \in \mathbb{R}\times D(A^*)\times X & & v-\varphi-g \text{ attains a local} \\ & & \text{maximum at } (t,x), \\ & & \partial_t(\varphi+g)(t,x) = q, \\ & & D\varphi(t,x)=p_1,\ \ Dg(t,x)=p_2 \text{ and} \\ & & v(t,x)=\varphi(t,x)+g(t,x)\} \end{array}$$

REMARK 5.14. *If we call*

$$E_1v(t,x) = \{(q,p)\in\mathbb{R}\times X \ :\ p=p_1+p_2 \text{ with } (q,p_1,p_2)\in Ev(t,x)\}$$

*then $E_1v(t,x) \subseteq D^{1,+}v(t,x)$ and in the finite dimensional case we have $E_1v(t,x) = D^{1,+}v(t,x)$. We use $Ev(t,x)$ instead of $E_1v(t,x)$ to underline the different role of $g$ and $\varphi$. We will need this fact in the proofs in the sequel.*

We can now state and prove the verification theorem.

THEOREM 5.15. *Let Hypotheses 5.1, 5.3, 5.4 and 5.11 hold. Let $(x(\cdot),u(\cdot))$ be an admissible pair at $(t,x)$, let $v\in\mathcal{G}$ be a subsolution of the HJB (see Definition 5.5) such that*

$$v(T,x) = h(x) \quad for\ all\ x\ in\ X. \tag{5.12}$$

*Let $q\in L^1((t,T);\mathbb{R})$, $p_1\in L^1((t,T);D(A^*))$ and $p_2\in L^1((t,T);X)$ be such that*

$$(q(s),p_1(s),p_2(s))\in Ev(s,x(s)) \quad for\ almost\ all\ s\in(t,T). \tag{5.13}$$

*Moreover we assume that*

$$\int_t^T \langle p_1(s)+p_2(s), f(s,x(s),u(s))\rangle + q(s) + \langle A^*p_1(s), x(s)\rangle\,\mathrm{d}t \leq \tag{5.14}$$

$$\leq \int_t^T -L(s,x(s),u(s))\mathrm{d}s$$

*Then*

(a) $v(t,x)\leq V(t,x)\leq J(t,x,u(\cdot))\ \ \forall (t,x)\in[0,T]\times X,\ u(\cdot)\in\mathcal{U}[t,T],$
(b) *$(x(\cdot),u(\cdot))$ is an optimal pair at $(t,x)$.*

PROOF. Part $(a)$ follows from Hypothesis 5.11. We go to $(b)$. The function

$$\begin{cases} [t,T]\to X\times\mathbb{R} \\ s\mapsto (f(s,x(s),u(s)), L(s,x(s),u(s)) \end{cases}$$

in view of Hypotheses 5.3 and 5.4 is in $L^1((t,T);X\times\mathbb{R})$ (in fact it is bounded). So the set of the right-Lebesgue point of this function that in addition satisfy (5.13) is of full measure. We choose $r$ a point in such set. We will call $y \overset{def}{=} x(r)$.

We consider now two functions $\varphi^{r,y}\in test1$ and $g^{r,y}\in test2$ such that (we will avoid the index $^{r,y}$ in the sequel) $v\leq\varphi+g$ in a neighborhood of $(r,y)$, $v(r,y)-$

$\varphi(r, y) - g(r, y) = 0$ and $\partial_t(\varphi + g)(r, y) = q(r)$, $D\varphi(r, y) = p_1(r)$ and $Dg(r, y) = p_2(r)$. So for $\tau \in (r, T]$ and $(\tau - r)$ small enough we have by Lemmas 5.8 and 5.9

$$\frac{v(\tau, x(\tau)) - v(r, y)}{\tau - r} \leq \frac{g(\tau, x(\tau)) - g(r, y)}{\tau - r} + \frac{\varphi(\tau, x(\tau)) - \varphi(r, y)}{\tau - r}$$

$$\leq g_t(r, y) + \frac{\int_r^\tau \langle Dg(r, y), f(r, y, u(s))\rangle \, \mathrm{d}s}{\tau - r} + \tag{5.15}$$
$$+ \varphi_t(r, y) + \frac{\int_r^\tau \langle D\varphi(r, y), f(r, y, u(s))\rangle \, \mathrm{d}s}{\tau - r} + \langle A^* D\varphi(r, y), y\rangle + o(1)$$

In view of the choice of $r$ we know that

$$\frac{\int_r^\tau \langle Dg(r, y), f(r, y, u(s))\rangle \, \mathrm{d}s}{\tau - r} \xrightarrow{\tau \to r} \langle Dg(r, y), f(r, y, u(r))\rangle$$

and

$$\frac{\int_r^\tau \langle D\varphi(r, y), f(r, y, u(s))\rangle \, \mathrm{d}s}{\tau - r} \xrightarrow{\tau \to r} \langle D\varphi(r, y), f(r, y, u(r))\rangle$$

So for almost every $r$ in $[t, T]$ we have

$$\limsup_{\tau \downarrow r} \frac{v(\tau, x(\tau)) - v(r, x(r)))}{\tau - r} \tag{5.16}$$
$$\leq \langle Dg(r, x(r)) + D\varphi(r, x(r)), f(r, x(r), u(r))\rangle$$
$$+ g_t(r, x(r)) + \varphi_t(r, x(r)) + \langle A^* D\varphi(r, x(r)), x(r)\rangle$$
$$= \langle p_1(r) + p_2(r), f(r, x(r), u(r))\rangle + q(r) + \langle A^* p_1(r), x(r)\rangle \, .$$

Then we can use Lemma 5.12 and (5.14) to obtain

$$v(T, x(T)) - v(t, x) \tag{5.17}$$
$$\leq \int_t^T \langle p(r), f(r, x(r), u(r))\rangle + q(r) + \langle A^* p_1(r), x(r)\rangle \, \mathrm{d}r$$
$$\leq \int_t^T -L(r, x(r), u(r))\mathrm{d}r$$

Therefore, using (a), we finally arrive at

$$V(T, x(T)) - V(t, x) = h(x(T)) - V(t, x) \leq h(x(T)) - v(t, x) \tag{5.18}$$
$$= v(T, x(T)) - v(t, x) \leq \int_t^T -L(r, x(r), u(r))\mathrm{d}r$$

and so $(x(\cdot), u(\cdot))$ is an optimal pair at $(t, x)$. □

REMARK 5.16. *In the book* [LY95] *(page 262 Theorem 5.4) the authors prove a verification theorem in which it is required that the trajectories of the system remain in the domain of $A$ a.e. for every admissible control $u(\cdot)$. They prove (under hypothesis similar to Hypotheses 5.1, 5.3 and 5.4) that a couple $x(\cdot), u(\cdot))$ is optimal if and only if*

$$u(s) \in \tag{5.19}$$
$$\left\{ u \in U \, : \, \lim_{\delta \to 0} \frac{V((s + \delta), x(s) + \delta(Ax(s) + f(s, x(s), u))) - V(s, x(s))}{\delta} = \right.$$
$$\left. = -L(s, x(s), u) \right\}$$

*for almost every $s \in [t, T]$, where $V$ is the value function.*

## 5.3. Sub- and super-optimality principles and construction of $\epsilon$-optimal controls

In this section we will work using the weak $P$ condition introduced in [CL90], namely we assume (besides Hypotheses 5.1, 5.3, 5.4) that

HYPOTHESIS 5.17. *There exists a linear bounded positive self-adjoint operator $P$ on $X$ such that $A^*P$ is a bounded operator on $X$ and there exists a real constant $c_0$ such that*

$$\langle (A^*P + c_0 P)x, x\rangle \leq 0 \qquad \textit{for all } x \in X$$

and

HYPOTHESIS 5.18. *There exist a constant $M > 0$ and a local modulus of continuity $\bar{\sigma}(\cdot,\cdot)$ such that:*

$$|f(t,x,u) - f(s,y,u)| \leq M|x-y|_{-1} + \bar{\sigma}(|t-s|, |x| \vee |y|)$$

*and*

$$|L(t,x,u) - L(s,y,u)| \leq \bar{\sigma}(|x-y|_{-1} + |t-s|, |x| \vee |y|)$$

(the $|\cdot|_{-1}$ nor was introduced in Notation 2.17).

Let $m \geq 2, K > 0$. For $\epsilon, \beta, \lambda > 0$ define

$$w^{\lambda,\epsilon,\beta}(t,x) = \sup_{(s,y)\in(0,T)\times H} \left\{ w(s,y) - \frac{|x-y|_{-1}^2}{2\epsilon} - \frac{(t-s)^2}{2\beta} - \lambda e^{K(T-s)}|y|^m \right\},$$

$$w_{\lambda,\epsilon,\beta}(t,x) = \inf_{(s,y)\in(0,T)\times H} \left\{ w(s,y) + \frac{|x-y|_{-1}^2}{2\epsilon} + \frac{(t-s)^2}{2\beta} + \lambda e^{K(T-s)}|y|^m \right\}.$$

LEMMA 5.19. *Let $v$ be such that for every $R > 0$ there exists a modulus $\sigma_R$ such that*

$$|v(t,x) - v(s,y)| \leq \sigma_R(|t-s| + |x-y|_{-1}) \quad \textit{for } t,s \in (0,T], |x|, |y| \leq R \tag{5.20}$$

*and*

$$v(t,x) \leq C(1+|x|^k) \quad (\textit{respectively, } v(t,x) \geq -C(1+|x|^k)) \tag{5.21}$$

*on $(0,T]\times H$ for some $k \geq 0$. Let $m > k$. Then:*

(i) *For every $R > 0$ there exists $M_{R,\epsilon,\beta}$ such that if $v = w^{\lambda,\epsilon,\beta}$ (respectively, $v = w_{\lambda,\epsilon,\beta}$) then*

$$|v(t,x) - v(s,y)| \leq M_{R,\epsilon,\beta}(|t-s| + |x-y|_{-2}) \quad \textit{on } (0,T)\times B_R \tag{5.22}$$

(ii) *The function*

$$w^{\lambda,\epsilon,\beta}(t,x) + \frac{|x|_{-1}^2}{2\epsilon} + \frac{t^2}{2\beta}$$

*is convex (respectively,*

$$w_{\lambda,\epsilon,\beta}(t,x) - \frac{|x|_{-1}^2}{2\epsilon} - \frac{t^2}{2\beta}$$

*is concave). In other terms $w^{\lambda,\epsilon,\beta}$ is semi-convex and $w_{\lambda,\epsilon,\beta}$ is semi-concave.*

(iii) *If $v = w^{\lambda,\epsilon,\beta}$ (respectively, $v = w_{\lambda,\epsilon,\beta}$) and $v$ is differentiable at $(t,x) \in (0,T)\times B_R$ then $|v_t(t,x)| \leq M_{R,\epsilon,\beta}$, and $Dv(t,x) = Pq$, where $|q| \leq M_{R,\epsilon,\beta}$*

PROOF. (Some stuff of the proof are from [CL91] page 446).

**(i)**: consider the case $v = w^{\lambda,\epsilon,\beta}(t,x)$. Observe first that

$$w^{\lambda,\epsilon,\beta}(t,x) = \tag{5.23}$$
$$= \sup_{(s,y)\in(0,T),\ |y|\le Z} \left\{ w(s,y) - \frac{|x-y|_{-1}^2}{2\epsilon} - \frac{(t-s)^2}{2\beta} - \lambda e^{K(T-s)}|y|^m \right\},$$

where $Z$ depends only on $R$ and $\lambda$.

Now suppose $w^{\lambda,\epsilon,\beta}(t,x) \ge w^{\lambda,\epsilon,\beta}(s,y)$. We choose a small $\sigma > 0$ and $(\tilde{t},\tilde{x})$ such that

$$w^{\lambda,\epsilon,\beta}(t,x) \le \sigma + w(\tilde{t},\tilde{x}) - \frac{|x-\tilde{x}|_{-1}^2}{2\epsilon} - \frac{(t-\tilde{t})^2}{2\beta} - \lambda e^{K(T-\tilde{t})}|\tilde{x}|^m$$

So

$$|w^{\lambda,\epsilon,\beta}(t,x) - w^{\lambda,\epsilon,\beta}(s,y)| \le \sigma + \frac{|x-\tilde{x}|_{-1}^2}{2\epsilon} - \frac{(t-\tilde{t})^2}{2\beta} - \frac{|\tilde{x}-y|_{-1}^2}{2\epsilon} - \frac{(\tilde{t}-s)^2}{2\beta} \le \tag{5.24}$$
$$\le \sigma + \frac{\langle P(x-y), x+y\rangle}{2\varepsilon} + \frac{\langle P(x-y), \tilde{x}\rangle}{2\varepsilon} + \frac{(2\tilde{t}-t-s)(t-s)}{2\beta} \le$$
$$\le \frac{(2R+Z)}{2\varepsilon}|P(x-y)| + \frac{2T}{2\beta}|t-s| + \sigma$$

for the arbitrariness of $\sigma$ we conclude. The case $w_{\lambda,\epsilon,\beta}$ is similar.

**(ii)** It is a standard fact, see for example [CIL92] (in Appendix).

**(iii)** It is a consequence of (5.22). □

LEMMA 5.20. *Let Hypotheses 5.18 be satisfied. Let $w$ be a viscosity subsolution (respectively, supersolution) of (5.6) satisfying (5.20) and (5.21) Let $m > k$. Then for every $R, \delta > 0$ there exists a non-negative function $\gamma_{R,\delta}(\lambda,\epsilon,\beta)$, where*

$$\lim_{\lambda\to 0} \limsup_{\epsilon\to 0} \limsup_{\beta\to 0} \gamma_{R,\delta}(\lambda,\epsilon,\beta) = 0 \tag{5.25}$$

*such that $w^{\lambda,\epsilon,\beta}$ (respectively, $w_{\lambda,\epsilon,\beta}$) is a viscosity subsolution (respectively, supersolution) of*

$$v_t(t,x) + \langle Dv(t,x), Ax\rangle + H(t,x,Dv(t,x)) = -\gamma_{R,\delta}(\lambda,\epsilon,\beta) \quad \text{in } (\delta, T-\delta)\times B_R \tag{5.26}$$

*(respectively,*

$$v_t(t,x) + \langle Dv(t,x), Ax\rangle + H(t,x,Dv(t,x)) = \gamma_{R,\delta}(\lambda,\epsilon,\beta) \quad \text{in } (\delta, T-\delta)\times B_R) \tag{5.27}$$

*for $\beta$ sufficiently small.*

PROOF. Some of the ideas of this proof are from [CL91] Proposition 5.3. Let $(t_0,x_0) \in (\delta, T-\delta)\times X$ be a maximum of $w^{\lambda,\epsilon,\beta} - \varphi - g$. We can assume the maximum is global (see Remark 2.5 of [CL91]) and strict (see Proposition 2.4 of [CL91]). Moreover we can assume, without loss of generality, that $\varphi$ is bounded from above and $\frac{g_0(|x|)}{|x|} \xrightarrow{|x|\to\infty} \infty$. In view of this fact, of the (5.20) and of the (5.23) we can choose $S > 0$, depending on $\lambda$ (and $g$) such that, for all $|x| + |y| > S - 1$ and $s, t \in [0,T]$,

$$w(s,y) - \frac{1}{2\epsilon}|P^{1/2}(x-y)|^2 - \frac{(t-s)^2}{2\beta} - \lambda e^{K(T-s)}|y|^m - \varphi(t,x) - g(t,x) \le \tag{5.28}$$
$$\le w(t_0,x_0) - \lambda e^{K(T-t_0)}|x_0|^m - \varphi(t_0,x_0) - g(t_0,x_0) - 1.$$

We can use Ekeland-Lebourg Lemma (see [EL77] a version is Theorem 3.29) and state that for every $\alpha > 0$ we can find $p, q \in X$ and $a, b \in \mathbb{R}$ with $|p|, |q| \leq \alpha$ and $|a|, |b| \leq \alpha$ such that the function

$$\Psi(t,x,s,y) \stackrel{def}{=} w(s,y) - \frac{1}{2\epsilon}|P^{1/2}(x-y)|^2 - \frac{(t-s)^2}{2\beta} - \lambda e^{K(T-s)}|y|^m - \\ - g(t,x) - \varphi(t,x) - \langle Pp, y\rangle - \langle Pq, x\rangle - at - bs \tag{5.29}$$

attains a local maximum $(\bar{t}, \bar{x}, \bar{s}, \bar{y})$ on $[0,T] \times \overline{B_S} \times [0,T] \times \overline{B_S}$. For what we have seen $|\bar{x}|, |\bar{y}| \leq S - 1$.

Now, since we can choose at the beginning $S$ such that $B_S$ contains a maximization sequence (bounded for (5.23)) for

$$\sup_{(s,y)\in(0,T),\ |y|\leq Z} \left\{ w(s,y) - \frac{|x_0 - y|_{-1}^2}{2\epsilon} - \frac{(t_0-s)^2}{2\beta} - \lambda e^{K(T-s)}|y|^m \right\},$$

we have

$$\Psi(\bar{t}, \bar{x}, \bar{s}, \bar{y}) \geq w^{\lambda,\epsilon,\beta}(t_0, x_0) - \varphi(t_0, x_0) - g(t_0, x_0) - C\alpha$$

where the constant $C$ does not depend on $\alpha > 0$. Moreover

$$\Psi(\bar{t}, \bar{x}, \bar{s}, \bar{y}) \leq w^{\lambda,\epsilon,\beta}(\bar{t}, \bar{x}) - \varphi(\bar{t}, \bar{x}) - g(\bar{t}, \bar{x}) + C\alpha.$$

So, since $(t_0, x_0)$ is a strict maximum, we have that $(\bar{t}, \bar{x}) \xrightarrow{\alpha\downarrow 0} (t_0, x_0)$. We choose $\alpha$ so that $\bar{t} \in (\delta, T - \delta)$

We fix now an upper-bound for $\beta$ (as required in the claim of the lemma): we choose $\beta$ so that for all $t \in (\delta, T-\delta)$ and every $|y| \leq S - 1$ we have

$$\sup_{s\in(0,\frac{\delta}{2})\cup(T-\frac{\delta}{2},T)} \left( w(s,y) - \frac{(t-s)^2}{2\beta} - \lambda e^{K(T-s)}|y| \right) \leq \\ \leq -1 - w(t,y) - \lambda e^{K(T-t)}|y| \tag{5.30}$$

(such bound depends on $\lambda$ and $\delta$), so $\bar{s} \in [\delta/2, T - \delta/2]$.

For (5.21), since $m > k$, we have that

$$|\lambda e^{K(T-\bar{s})}|\bar{y}|^m| \leq \sigma(\lambda). \tag{5.31}$$

Moreover, since we have

$$w(\bar{s}, \bar{y}) - \frac{1}{2\epsilon}|P^{1/2}(\bar{x}-\bar{y})|^2 - \frac{(\bar{t}-\bar{s})^2}{2\beta} - \lambda e^{K(T-\bar{s})}|y|^m \geq w^{\lambda,\beta,\epsilon}(\bar{t}, \bar{x}) - C\alpha$$

and then

$$\frac{1}{2\epsilon}|P^{1/2}(\bar{x}-\bar{y})|^2 - \frac{(\bar{t}-\bar{s})^2}{2\beta} \leq w(\bar{s}, \bar{y}) - w^{\lambda,\beta,\epsilon}(\bar{t}, \bar{x}) + C\alpha - \lambda e^{K(T-\bar{s})}|y|^m$$

we find, using standard arguments (see for example [CL90] page 250),

$$\left( \frac{|\bar{s}-\bar{t}|^2}{2\beta} + \frac{|\bar{x}-\bar{y}|_{-1}^2}{2\epsilon} \right) \leq \sigma_\lambda(\epsilon + \beta) + +C_\lambda \alpha = \sigma_\lambda(\epsilon+\beta) + \sigma_\lambda(\alpha) \tag{5.32}$$

for some positive constant $C_\lambda$.

Now we use the fact that $w$ is a subsolution:

$$-\frac{(\bar{t}-\bar{s})}{\beta} - \lambda K e^{K(T-\bar{s})}|\bar{y}|^m + b - \frac{\langle A^* P(\bar{x}-\bar{y}), \bar{y}\rangle}{\epsilon} + \langle A^* Pp, \bar{y}\rangle + \\ + H\left( \bar{s}, \bar{y}, \frac{1}{\epsilon}P(\bar{y}-\bar{x}) + \lambda e^{K(T-\bar{s})}|y|^{m-1}\frac{y}{|y|} + Pp \right) \geq 0. \tag{5.33}$$

Now we note that

$$-\frac{(\bar{t}-\bar{s})}{\beta} = \varphi_t(\bar{t}, \bar{x}) + g_t(\bar{t}, \bar{x}) + a$$

and

$$\frac{1}{\epsilon}P(\bar{y}-\bar{x}) = D\varphi(\bar{t},\bar{x}) + Dg(\bar{t},\bar{x}) + Pq$$

So

$$\begin{aligned}(5.34)\quad & \varphi_t(\bar{t},\bar{x}) + g_t(\bar{t},\bar{x}) + \langle \bar{x}, A^* D\varphi(\bar{t},\bar{x})\rangle + H\left(\bar{t},\bar{x},D\varphi(\bar{t},\bar{x}) + Dg(\bar{t},\bar{x})\right) \geq \\ & \geq \lambda K e^{K(T-\bar{s})}|\bar{y}|^m - \langle A^*Pp, \bar{y}\rangle - a - b - \\ & - \left\langle (\bar{y}-\bar{x}), A^*\frac{1}{\epsilon}P(\bar{y}-\bar{x})\right\rangle - \langle \bar{x}, A^*Dg(\bar{t},\bar{x}) + A^*Pq)\rangle + \\ & + H\left(\bar{t},\bar{x},\frac{1}{\epsilon}P(\bar{y}-\bar{x}) - Pq\right) - H\left(\bar{s},\bar{y},\frac{1}{\epsilon}P(\bar{y}-\bar{x}) + \lambda e^{K(T-\bar{s})}|y|^{m-1}\frac{y}{|y|} + Pp\right) \geq\end{aligned}$$

Using this and the fact that we can take $\varphi$ such that $P^{-1}D\varphi \in C(X)$ (see [CL91] proposition 5.4) we have that $Dg(\bar{t},\bar{x}) \in D(P^{-1})$ and thus $\langle \bar{x}, A^*Dg(\bar{t},\bar{x})\rangle \leq 0$. Moreover we use Hypothesis 5.17 and (5.31) to obtain

$$\begin{aligned}(5.35)\quad & \geq \left[-\sigma(\lambda) - \sigma_\lambda(\alpha) + c_0\frac{1}{\epsilon}|P^{1/2}(\bar{x}-\bar{y})|^2\right] - \sigma_\lambda(\alpha) - \\ & - \left| H\left(\bar{t},\bar{x},\frac{1}{\epsilon}P(\bar{y}-\bar{x}) - Pq\right) - H\left(\bar{t},\bar{x},\frac{1}{\epsilon}P(\bar{y}-\bar{x}) + \lambda e^{K(T-\bar{s})}|y|^{m-1}\frac{y}{|y|} + Pp\right)\right| - \\ & \left(\text{calling } J = \tfrac{1}{\epsilon}P(\bar{y}-\bar{x}) + \lambda e^{K(T-\bar{s})}|y|^{m-1}\tfrac{y}{|y|} + Pp\right) \\ & - \left|H\left(\bar{t},\bar{x},J\right) - H\left(\bar{s},\bar{x},J\right)\right| - \left|H\left(\bar{s},\bar{x},J\right) - H\left(\bar{s},\bar{y},J\right)\right| \geq\end{aligned}$$

using (5.31), (5.32) and Hypothesis 5.18

$$\begin{aligned}(5.36)\quad & \\ & \geq \left[-\sigma(\lambda) - \sigma_\lambda(\alpha) - \sigma_\lambda(\epsilon+\beta) - \sigma_\lambda(\alpha)\right] - -\sigma_\lambda(\alpha) - \sigma(\lambda) - \sigma(\alpha) - \sigma(\epsilon+\beta) - \\ & - \sigma_\lambda(\epsilon+\beta) - \sigma_\lambda(\alpha) - \sigma(\lambda) = \sigma_\lambda(\alpha) - \sigma_\lambda(\epsilon+\beta) - \sigma(\lambda)\end{aligned}$$

where the modules depend on $R$. Letting $\alpha \to 0$ we have the claim. The proof for $w_{\lambda,\beta,\epsilon}$ is similar. □

LEMMA 5.21. *Let the assumptions of Lemma 5.20 be satisfied. Then:*

(a) *If* $(a,p) \in D^{1,-}w^{\lambda,\epsilon,\beta}(t,x)$ *for* $(t,x) \in (\delta, T-\delta)\times B_R$ *then*

$$(5.37)\qquad a + \langle A^*p, x\rangle + H(t,x,p) \geq -\gamma_{R,\delta}(\lambda,\epsilon,\beta)$$

*for* $\beta$ *sufficiently small.*

(b) *If in addition* $U$ *is compact and* $(a,p) \in D^{1,+}w_{\lambda,\epsilon,\beta}(t,x)$ *for* $(t,x) \in (\delta, T-\delta)\times B_R$ *is such that* $Dw_{\lambda,\epsilon,\beta}(t_n,x_n) \rightharpoonup p$ *for some* $(t_n,x_n) \to (t,x)$, $(t_n,x_n) \in (\delta,T-\delta)\times B_R$, *then*

$$a + \langle A^*p, x\rangle + H(t,x,p) \leq \gamma_{R,\delta}(\lambda,\epsilon,\beta)$$

*for* $\beta$ *sufficiently small.*

PROOF. **(a)- Step 1**: At points of differentiability, it follows from Lemma 5.19(iii) and the semiconvexity of $w^{\lambda,\epsilon,\beta}$ that there exists a test1 function $\varphi$ such that $w^{\lambda,\epsilon,\beta} - \varphi$ has a local maximum and the result then follows from Lemma 5.20.

**Step 2**: For $v = w^{\lambda,\epsilon,\beta}$ and $(a,p) \in D^{1,-}v(t,x)$ we recall first that for a convex/concave function $\nu$ its sub/super-differential at a point $z$ is equal to

$$\overline{\text{conv}}\{p : D\nu(z_n) \rightharpoonup p, z_n \to z\}.$$

that is the weak closure of the convex hull of $\{p : D\nu(z_n) \rightharpoonup p, z_n \to z\}$ (see [Pre90] page 319).

Consider the case $Dv(t_n,x_n) \rightharpoonup p$ with $(t_n,x_n) \to (t,x)$. For Lemma 5.19 (iii) $Dv(t_n,x_n) = Pq_n$ with $|q_n| \leq M_{R,\epsilon,\beta}$, so, it is always possible to extract a subsequence $q_{n_k} \rightharpoonup q$ for some $q \in X$. For the boundedness and the selfadjointness of $P$ we know that $Dv(t_n, x_n) = Pq_{n_k} \rightharpoonup Pq$ and for the uniqueness of the weak limit $Pq = p$. So, using that $A^*P$ is continuous, we have

$$\langle A^* Pq_{n_k}, x_n \rangle = \langle q_{n_k}, (A^*P)^* x_n \rangle \longrightarrow \langle q, (A^*P)^* x \rangle = \langle A^* Pq, x \rangle = \langle A^* p, x \rangle$$

Moreover, $H$ is concave in $p$ and then it is upper weakly semicontinuous so we have

$$H(t, x, p) \geq \lim_{n \to +\infty} H(t, x, Dv(t_n, x_n))$$

So the (a) from **Step 1**.

**Step 3**: If $p$ is a generic point in $\overline{\text{conv}}\{p : Dv(t_n, x_n) \rightharpoonup p, (t_n, x_n) \to (t, x)\}$ we can conclude using the concavity of

$$p \mapsto \langle A^* p, x \rangle + H(t, x, p)$$

and arguments used in Step 2.

**(b)**: As in (a) for the points of differentiability the claim follows from Lemma 5.19 and Lemma 5.20. Thanks to the compactness of $U$ the infimum in the definition of the Hamiltonian is a minimum. We call $v = w_{\lambda,\epsilon,\beta}$. If $Dv(t_n, x_n) \rightharpoonup p$ for some $(t_n, x_n) \to (t, x)$, $(t_n, x_n) \in (\delta, T - \delta) \times B_R$ we have that

$$\begin{aligned} &\langle A^* Dv(t_n, x_n), x_n \rangle + H(t_n, x_n, Dv(t_n, x_n)) = \\ &\qquad = \langle A^* Dv(t_n, x_n), x_n \rangle + \langle Dv(t_n, x_n), f(t_n, x_n, u_n^*) \rangle + L(t_n, x_n, u_n^*) \end{aligned} \tag{5.38}$$

for some $u_n^* \in U$, and, since the thesis is true at the points of differentiability we have

$$\begin{aligned} a + \langle A^* Dv(t_n, x_n), x_n \rangle + \langle Dv(t_n, x_n), f(t_n, x_n, u_n^*) \rangle + L(t_n, x_n, u_n^*) \leq \\ \leq \gamma_{R,\delta}(\lambda, \epsilon, \beta) \end{aligned} \tag{5.39}$$

we pass to a subsequence $u_{n_k}^* \longrightarrow \bar{u}$, and we pass to the limit in the above expression for $n_k \longrightarrow \infty$, observing that we can prove as in Step 2 of (a) that

$$\langle A^* Dv(t_n, x_n), x_n \rangle \longrightarrow \langle A^* p, x \rangle$$

and so we have

$$a + \langle A^* p, x \rangle + \langle p, f(t, x, \bar{u}) \rangle + L(t, x, \bar{u}) \leq \gamma_{R,\delta}(\lambda, \epsilon, \beta),$$

passing to the infimum in $\bar{u}$ we have the thesis. □

THEOREM 5.22. *Let the assumptions of Lemma 5.20 be satisfied. Then:*

(a) *Let $w$ be a viscosity subsolution of (5.6) satisfying (5.20) and (5.21) Let $m > k$. Then for every $0 < t < t + h < T$ and $x \in H$*

$$w(t, x) \leq \inf_{w(\cdot) \in \mathcal{U}[t,T]} \left\{ \int_t^{t+h} L(s, x(s), u(s)) \mathrm{d}s + w(t + h, x(t + h)) \right\}. \tag{5.40}$$

(b) *Assume in addition that $U$ is compact and for every $(t, x)$ there exists a modulus $\sigma_{t,x}$ such that*

$$|x_{t,x}(s_2) - x_{t,x}(s_1)| \leq \sigma_{t,x}(s_2 - s_1) \tag{5.41}$$

*for all $t \leq s_1 \leq s_2 \leq T$ and all $u(\cdot) \in \mathcal{U}[t, T]$, where $x_{t,x}(\cdot)$ is the solution of (5.3). Let $w$ be a viscosity supersolution of (5.6) satisfying (5.20) and (5.21) Let $m > k$. Then for every $0 < t < t + h < T, x \in H$, and $\nu > 0$ there exists a piecewise constant control $u_\nu \in \mathcal{U}[t, T]$ such that*

$$w(t, x) \geq \int_t^{t+h} L(s, x(s), u_\nu(s)) \mathrm{d}s + w(t + h, x(t + h)) - \nu. \tag{5.42}$$

*In particular we obtain the superoptimality principle*

$$w(t,x) \geq \inf_{u(\cdot)\in\mathcal{U}[t,T]} \left\{ \int_t^{t+h} L(s,x(s),u(s))\mathrm{d}s + w(t+h,x(t+h)) \right\} \tag{5.43}$$

*and if $u$ is the value function $V$ we have existence (together with the explicit construction) of piecewise constant $\nu$-optimal controls .*

PROOF. We will only prove $(b)$ as the proof of $(a)$ follows the same strategy for a fixed control $u(\cdot)$ and is much easier. We follow the ideas of [Ś96] (that treat the finite dimensional case).

**Step 1**. Let $n \geq 1$. We approximate $w$ by $w_{\lambda,\epsilon,\beta}$.

**Step 2**. Take any $(a,p) \in D^{1,+}w_{\lambda,\epsilon,\beta}(t,x)$ as in Lemma 5.21$(b)$ (we can do this because $w_{\lambda,\epsilon,\beta}$ is semi-concave). Then there exists $u_1 \in U$ such that

$$a + \langle A^*p, x\rangle + \langle p, f(t,x,u_1)\rangle + L(t,x,u_1) \leq \gamma_{R,\delta}(\lambda,\epsilon,\beta) + \frac{1}{n^2}. \tag{5.44}$$

By semiconcavity of $w_{\lambda,\epsilon,\beta}$

$$w_{\lambda,\epsilon,\beta}(s,y) - w_{\lambda,\epsilon,\beta}(t,x) \leq +a(s-t) + \langle p, y-x\rangle + \frac{|x-y|^2_{-1}}{\epsilon} + \frac{(t-s)^2}{\beta}. \tag{5.45}$$

But the right hand side of the above is a test1 function so if $s \geq t$ and $y = x_{t,x}(s)$ with constant control $u(s) = u_1$ we can use Lemma 5.8 and write

$$\begin{aligned} &\left| \frac{a(s-t) + \langle p, x(s)-x\rangle + \frac{|x(s)-x|^2_{-1}}{\epsilon} + \frac{(s-t)^2}{\beta}}{s-t} - \right. \\ &\qquad\qquad \left. - \left(a + \langle p, f(t,x,u_1)\rangle + \langle A^*p, x\rangle\right) \right| \leq \\ &\qquad \leq \left| \frac{\int_t^s 2\frac{(r-t)}{\beta}\mathrm{d}r}{s-t} \right| + \left| \frac{\int_t^s \langle A^*p, x(r)-x\rangle\,\mathrm{d}r}{s-t} \right| + \\ &+ \left| \frac{\int_t^s \langle p, f(r,x(r),u_1) - f(t,x,u_1)\rangle\,\mathrm{d}r}{s-t} \right| + \left| \frac{\int_t^s 2\,\langle A^*P(x(r)-x), x(r)\rangle\,\mathrm{d}r}{\epsilon(s-t)} \right| + \\ &+ \left| \frac{\int_t^s \langle P(x(r)-x), f(r,x(r),u(r))\rangle\,\mathrm{d}r}{\epsilon(s-t)} \right| \leq \sigma(|s-t| + |x(s)-x|) \leq \sigma'_{t,x}(s-t) \end{aligned} \tag{5.46}$$

where we used in the third term the fact that $f$ is Lipschitz with respect to the variable $x$. $\sigma'_{t,x}$ is modulus (different from the one of (5.41)) that depends on $(t,x)$. We can now use (5.44), (5.45), (5.46) to estimate

$$\begin{aligned} &\frac{w_{\lambda,\epsilon,\beta}(t+\frac{h}{n}, x(t+\frac{h}{n})) - w_{\lambda,\epsilon,\beta}(t,x)}{h/n} \leq \\ &\qquad\qquad \leq \sigma'_{t,x}(\frac{h}{n}) + \gamma_{R,\delta}(\lambda,\epsilon,\beta) + \frac{1}{n^2} - L(t,x,u_1) \end{aligned} \tag{5.47}$$

**Step 3**. We repeat this procedure $n$ times to arrive at the final estimate. Note that we can choose $\sigma'_{t,x}$ depending only on the initial point $(t,x)$. Thus we obtain the controls $u_i$ on the intervals $[t+\frac{i-1}{n}h, t+\frac{i}{n}h]$ and the control $u^{(n)}$ obtained by gluing the $u_i$. We obtain

$$\begin{aligned} &\frac{w_{\lambda,\epsilon,\beta}(t+h, x(t+h)) - w_{\lambda,\epsilon,\beta}(t,x)}{h/n} \leq \\ &\qquad\qquad \leq \sigma'_{t,x}(\frac{h}{n})n + \gamma_{R,\delta}(\lambda,\epsilon,\beta)n + \frac{n}{n^2} - \sum_i L(t_i,x_i,u_i) \end{aligned} \tag{5.48}$$

and then

$$(5.49)\quad w_{\lambda,\epsilon,\beta}(t+h,x(t+h)) - w_{\lambda,\epsilon,\beta}(t,x) \le$$
$$\le \sigma'_{t,x}(\frac{h}{n})T + \gamma_{R,\delta}(\lambda,\epsilon,\beta)T + \frac{T}{n^2} - \int_t^{t+h} L(r,x(r),u^{(n)})\mathrm{d}r + \sigma''_{t,x}(\frac{h}{n})T$$

where we used Hypothesis 5.18 to estimate how the sum converges to the integral. So choosing $\beta, \lambda, \epsilon, \frac{1}{n}$ small enough and using that

$$(5.50)\quad |w_{\lambda,\epsilon,\beta}(t,x) - w(s,y)| \le \sigma_{R,\lambda}(\epsilon+\beta, |t-s| + |x-y|_{-1})$$
$$\text{for } \ t,s \in (0,T], |x|,|y| \le R$$

we we have the (5.42). □

**An example** Condition (5.41) holds for example if $A = A^*$, it generates a differentiable semigroup , and $\|Ae^{tA}\| \le C/t^\delta$ for some $\delta < 2$. We then have

$$|(A+I)^{\frac{1}{2}}x(s)| \le |(A+I)^{\frac{1}{2}}e^{(s-t)A}x| + \int_t^s |(A+I)^{\frac{1}{2}}e^{(s-\tau)A}f(\tau,x(\tau),u(\tau))|d\tau$$

However for every $y \in H$ and $0 \le \tau \le T$

$$|(A+I)^{\frac{1}{2}}e^{\tau A}y|^2 \le |(A+I)e^{\tau A}y||y| \le \frac{C_1}{\tau^\delta}|y|^2.$$

This yields

$$\|(A+I)^{\frac{1}{2}}e^{\tau A}\| \le \frac{\sqrt{C_1}}{\tau^{\frac{\delta}{2}}}$$

and therefore

$$|(A+I)^{\frac{1}{2}}x(s)| \le C_2\left(\frac{1}{(s-t)^{\frac{\delta}{2}}} + (s-t)^{1-\frac{\delta}{2}}\right) \le \frac{C_3}{(s-t)^{\frac{\delta}{2}}}.$$

Thus to show (5.41) it is enough to prove that for every $\epsilon > 0$ and $t+\epsilon \le s_1 < s_2 \le T$, there exists a modulus $\sigma_\epsilon$ (also depending on $x$) such that $|e^{(s_2-s_1)A}x(s_1) - x(s_1)| \le \sigma_\epsilon(s_2-s_1)$. But this is now clear since

$$e^{(s_2-s_1)A}x(s_1) - x(s_1) = \int_0^{s_2-s_1} Ae^{sA}x(s_1)ds$$
$$= \int_0^{s_2-s_1}(A+I)^{\frac{1}{2}}e^{sA}(A+I)^{\frac{1}{2}}x(s_1)ds - \int_0^{s_2-s_1} e^{sA}x(s_1)ds.$$

Thus

$$|e^{(s_2-s_1)A}x(s_1) - x(s_1)| \le |(A+I)^{\frac{1}{2}}x(s_1)| \int_0^{s_2-s_1}\frac{\sqrt{C_1}}{s^{\frac{\delta}{2}}}ds + (s_2-s_1)|x(s_1)|$$
$$\le \frac{C_4}{\epsilon^{\frac{\delta}{2}}}(s_2-s_1)^{1-\frac{\delta}{2}} + C_5(s_2-s_1).$$

# Part 2

# Strong solutions approach

CHAPTER 6

# Strong solutions for first order HJB equations in Hilbert spaces arising in economic models governed by DDEs

This Chapter is devoted to the description of the results obtained in [FGF]. As we have already stated in Chapter 4 the optimal control problem treated is similar to the one treated in [Fabb] (Chapter 4). The main difference is in the approach: there we study the HJB equation using a viscosity solutions approach, here we look for the existence of strong solutions.

The results of [FGF] are very preliminary and a more in-depth studies are needed in the future, but, as we have already stressed in the introduction, we have decided to devote them a whole Part of this thesis to give to the reader a more complete image of the techniques used to study first order HJB equation in Hilbert spaces.

Before showing the new results we include two short sections: the first is devoted to a brief description of the literature on strong solutions for first order HJB equations in Hilbert spaces (Section 6.1) with special attention to boundary control, the second (Section 6.2) contains some of the results obtained by Faggian in [Fag05b] and [Fag06], regarding strong and weak solutions of HJB equations, that we will use in the sequel.

As in Chapter 4 we will use an applied approach and we will refer in particular to two main models. The first is the vintage capital model recalled in Subsection 6.3.1 and better described in Chapter 7 and the second is an advertising model due to [GM04, GMS06]) that is presented in Section 6.3.

In Section 6.5 we write the state equation of such problems as an ODE in $M^2$. We have already described the general technique in Section 1.3, here we see how the structural state becomes in our particular cases, concentrating on the first example, as the second can be rephrased similarly. In Section 6.6 we show our main result: the value function is an ultra-weak solution of the HJB equation.

## 6.1. A brief overview on literature

The idea of strong solutions method is to define the solution of the HJ as the limit of the solutions of a suitable family of approximating HJ.

Strong solutions for first order HJ equations in Hilbert space were first studied by Barbu and Da Prato in [BDP81] and then in [BDP83, BDP85a, BDP85b].

In [BDP83] a class of HJ equations arising in optimal control problem in Hilbert spaces with convex cost functional is treated. Existence and uniqueness of strong solution are proved for regular convex cost functional . A verification theorem is proved. In [BDP83, BDP85a, BDP85b] and [DB86] the authors study an HJ equation of the form

$$\begin{cases} v_t(t,x) + H(\nabla v(t,x)) - \langle Ax, \nabla v(t,x)\rangle = g(x) \\ v(0,x) = h(x) \text{ for all } x \in X \end{cases}$$

where $g, h, H: X \to \mathbb{R}$ are convex, continuous and bounded on bounded sets. In particular in [BDPP83] the authors study local existence for the case $H(p) = \frac{1}{2}|Rp|^2$ for a linear continuous operator $R$ and the case $H(p) = \frac{1}{2}|p|^2$ is in-deeply treated. In [BDP85a] the authors study to the case $H(p) = \eta(|p|^2)$, where $\eta \in W^{2,\infty}_{\mathrm{loc}}$ is convex, increasing and $\lim_{r\to+\infty} \eta(r) = +\infty$. In [DB86] the author studies existence and uniqueness of a global weak solution when $H$ is continuously Frechet differentiable and $H'$ is bounded. See also [BP86].

In [Goz88, Goz91] (see also [Goz89]) the author considers the semilinear optimal control and prove global existence for problem with small perturbations.

In [DB91] the author studies the case in which $B$ is linear and continuous and $U$ is allowed to be a proper subset of the control space and is supposed to be bounded closed and convex. The theory is applied to the optimal control of a parabolic partial differential equation with homogeneous Neumann boundary condition in the infinite horizon case.

In [CDB95] the authors study the convex control problem with state constraints in a prescribed convex set possibly with empty interior. If every initial state admits an admissible control the authors prove the existence and the uniqueness of the (weak) solution, it is in fact the value function of the control problem. An explicit feedback law is also found.

See [BDPDM92] for the LQ case.

**The boundary control case:** For the case of linear systems and quadratic costs (where HJB equation reduces to the operator Riccati equation) the reader is referred to [LT00] and to [BDPDM92]. In [AFT91] and [AT96, AT99, AT00] the authors study the non-autonomous linear-quadratic case. The convex case is treated in [Fag05a, Fag05b] (see [Fag04] and [BP86] for a maximum principle approach to the same problem). More precisely the optimal control problem studied in [Fag05a, Fag05b] is governed by a state equation of the form

$$\begin{cases} x(s) = Ax(s) + Bu(s) & for\ s \in [t,T] \\ x(t) = x \end{cases}$$

where $B$ is an unbounded linear operator and the cost functional is "decoupled":

$$J(x, u(\cdot)) = \int_0^T [L_1(x(s)) + L_2(u(s))]\mathrm{d}s + h(x(T))$$

where $g, f$ and $h$ are convex . In [Fag06] the constrained case is treated. See [FG04] for a more applied approach. See [FGb] for the infinite horizon case.

## 6.2. Some results from [Fag05a] and [Fag06]

Let $X$ be a separable Hilbert space, $U$ an real Hilbert space, $A$ is the generator of a strongly continuous semigroup of operators on $X$, and $D(A^*)$ is endowed with the scalar product $\langle v, w\rangle_{D(A^*)} := \langle v, w\rangle_X + \langle A^*v, A^*w\rangle_X$. Let $D(A^*)'$ be its dual space endowed with the operator norm.

We consider the following state equation in $D(A^*)'$

$$\begin{cases} \dot{x}(s) = A^{(E)}x(s) + B^*u(s), & s \in [t,T] \\ x(t) = x \in D(A^*)' \end{cases} \tag{6.1}$$

with control operator $B^* \in \mathscr{L}(U, D(A^*)')$ (although $B^* \notin \mathscr{L}(U, X)$), where $U$ is the control space and $u(\cdot) \in L^2((t,T); U)$ the control. Such equation may be readily expressed in mild form as

$$x(s) = e^{(s-t)A^{(E)}}x + \int_t^s e^{(s-r)A^{(E)}}B^*u(r)\mathrm{d}r. \tag{6.2}$$

We consider first a target functional $J$, associated to the state equation, of type

$$J(t,x,u(\cdot)) = \int_t^T \left[L_1\left(s,x(s)\right) + L_2\left(s,u(s)\right)\right] ds + h(x(T)) \tag{6.3}$$

with $L_2(t,\cdot)$ real, convex, *l.s.c.*, coercive, and $L_1(t,\cdot)$ and $h$ real, convex, and $C^1(D(A^*)')$ (respectively, *l.s.c.* in $D(A^*)'$) in the $x$ variable. The problem is that of minimizing $J(t,x,\cdot)$ over the set of admissible controls $L^2((t,T);U)$.

REMARK 6.1. *Indeed, in the applications, the target functional is rather of type*

$$J_0(t,x,u(\cdot)) = \int_t^T \left[\xi\left(s,x(s)\right) + \eta\left(s,u(s)\right)\right] ds + \nu(x(T))$$

*with $\eta(t,\cdot)$ real, convex, l.s.c., coercive, and $\xi(t,\cdot)$ and $\nu$ real, convex, and $C^1(X)$ (respectively, l.s.c. in $X$) in the $x$ variable, defined on $X$, but not necessarily on $D(A^*)'$. Then we need to* assume *that $\xi$ and $\nu$ allow $C^1$ (respectively, l.s.c.) extensions $L_1(t,\cdot)$ and $h$ on the space $D(A^*)'$. The existence of such extensions is a strong assumption, see* [Fag05b] *for details and comments upon this matter.*

REMARK 6.2. *Actually the framework of* [Fag05b] *and* [Fag06] *allows to study the a more general case in which we have an Hilbert space $V$ such that $V \subseteq X \subseteq V'$. The case $V = D(A^*)$ is included in such framework.*

The value function is defined as

$$V(t,x) = \inf_{u(\cdot)\in L^2((t,T);U)} J(t,x,u(\cdot)), \tag{6.4}$$

The HJB equation in $[0,T]\times D(A^*)'$ associated to the problem is

$$\begin{cases} v_t(t,x) - H(t,B\nabla v(t,x)) + \langle A^{(E)}x, \nabla v(t,x)\rangle + L_1(t,x) = 0, \\ v(T,x) = h(x), \end{cases} \tag{6.5}$$

for all $t\in[0,T]$ and $x\in D(A^{(E)})$ where

$$H(t,q) = [L_2(t,\cdot)]^*(-q) = \sup_{u\in U}\{\langle -u,q\rangle_U - L_2(t,u)\}.$$

$H(t,Bp)$ is well defined only for $p$ in $D(A^*)$, that is a proper subspace of $X$, to which $\nabla v(t,x)$ (the spatial gradient of $v$) should belong.

With such a problem in mind, it is natural to investigate existence and uniqueness for the following HJB equation

$$\begin{cases} w_t(t,x) + F(t,\nabla w(t,x)) - \langle A^{(E)}x, \nabla w(t,x)\rangle = L_1(T-t,x) \quad (t,x)\in[0,T]\times D(A^*)' \\ w(0,x) = h(x). \end{cases} \tag{6.6}$$

Note in fact that such a HJB equation is the forward version of (6.5) once one have set

$$F(t,p) := H(t,Bp) = \sup_{u\in U}\{\langle -B^*u,p\rangle_U - L_2(t,u)\}.$$

**6.2.1. Regular data and strong solutions of HJB equations.** This is the case of regular data, from which the notion of strong solution originates. We recall some notation: If $\Omega$ is an open set subset of $X$ we define $C_p(\Omega)$ is the set

$$C_p(\Omega) = \left\{ f:\Omega\to\mathbb{R} \;:\; |f|_{C_p} := \sup_{x\in\Omega}\frac{|f(x)|}{1+|x|_X^p} < +\infty \right\}$$

Moreover

$$\Sigma_0(\Omega) = \left\{ w\in C_2(\Omega) \;:\; w \text{ is convex, } w\in C^1_{Lip}(\Omega)\right\}$$

($C^1_{Lip}(\Omega)$ is defined at page 13) and

$$(6.7) \quad \mathcal{Y}([0,T]\times\Omega) \stackrel{def}{=} \Big\{ w : [0,T]\times\Omega \to \mathbb{R} \ : \ w \in C([0,T];C_2(\Omega)), \\ w(t,\cdot)\in\Sigma_0(\Omega),\ \nabla w \in C([0,T];C_1(\Omega;X)) \Big\}$$

HYPOTHESIS 6.3. *Assume that*

1. $A^{(E)} : D(A^{(E)}) \subset D(A^*)' \to D(A^*)'$ *is the infinitesimal generator of a strongly continuous semigroup* $\{e^{sA^{(E)}}\}_{s\geq 0}$ *on* $D(A^*)'$*;*
2. $B^* \in \mathscr{L}(U, D(A^*)')$*;*
3. *there exists* $w > 0$ *such that* $|e^{\tau A^{(E)}}x|_{D(A^*)'} \leq Me^{w\tau}|x|_{D(A^*)'}$, $\forall \tau \geq 0$*;*
4. $F \in \mathcal{Y}([0,T]\times D(A^*))$, $F(t,0)=0$, $\sup_{t\in[0,T]}[F_p(t,\cdot)]_L < +\infty$*;*
5. $L_1 \in \mathcal{Y}([0,T]\times D(A^*)')$, $t\mapsto [L_{1x}(t,\cdot)]_L \in L^1(0,T)$
6. $h \in \Sigma_0(D(A^*)')$*;*
7. $L_2(t,\cdot)$ *is convex, lower semi-continuous,* $D^-L_2(t,\cdot)$ *is injective for all* $t\in[0,T]$.
8. $H \in \mathcal{Y}([0,T]\times U)$, $H(t,0)=0$, *and* $\sup_{t\in[0,T]}[\partial_q H(t,\cdot)]_L < +\infty$.

DEFINITION 6.4 (**Strong solution**). *Let Assumptions 6.3 be satisfied. We say that* $w \in C([0,T], C_2(D(A^*)'))$ *is a* strong *solution of* (6.6) *if there exists a family* $\{w^\varepsilon\}_\varepsilon \subset C([0,T], C_2(D(A^*)'))$ *such that:*

$(i)$ $w^\varepsilon(t,\cdot)\in C^1_{Lip}(D(A^*)')$ *and* $w^\varepsilon(t,\cdot)$ *is convex for all* $t\in[0,T]$*;* $w^\varepsilon(0,x) = h(x)$ *for all* $x\in D(A^*)'$.

$(ii)$ *there exist constants* $\Gamma_1,\Gamma_2 > 0$ *such that*

$$\sup_{t\in[0,T]}[\nabla w^\varepsilon(t)]_L \leq \Gamma_1,\ \sup_{t\in[0,T]}|\nabla w^\varepsilon(t,0)|_{D(A^*)} \leq \Gamma_2,\ \forall\varepsilon>0;$$

$(iii)$ *for all* $x\in D(A^{(E)})$, $t\mapsto w^\varepsilon(t,x)$ *is continuously differentiable;*

$(iv)$ $w^\varepsilon \to w$, *as* $\varepsilon\to 0^+$, *in* $C([0,T], C_2(D(A^*)'))$*;*

$(v)$ *there exists* $L_{1\varepsilon}\in C([0,T];C_2(D(A^*)'))$ *such that, for all* $t\in[0,T]$ *and* $x\in D(A^{(E)})$,

$$w^\varepsilon_t(t,x) - F(t,\nabla w^\varepsilon(t,x)) + \langle A^{(E)}x, \nabla w^\varepsilon(t,x)\rangle v = L_{1\varepsilon}(T-t,x)$$

*with* $L_{1\varepsilon}(t,x)\to L_1(t,x)$ *pointwise, and* $\int_0^T |L_{1\varepsilon}(s)-L_1(s)|_{C_2}\mathrm{d}s \to 0$, *as* $\varepsilon\to 0^+$.

The main result contained in [Fag05b] is the following.

THEOREM 6.5. *Let Assumptions 6.3 be satisfied. There exists a unique strong solution* $w$ *of* (6.6) *in the class* $C([0,T], C_2(D(A^*)'))$ *with the following properties:*

$(i)$ *for all* $x\in D(A^{(E)})$, $w(\cdot,x)$ *is Lipschitz continuous;*

$(ii)$ $w(t,\cdot)\in\Sigma_0(D(A^*)')$, *for all* $t\in[0,T]$.

Regarding applications to the optimal control problem, in [Fag00] the following theorem is proved:

THEOREM 6.6. *Let Assumptions 6.3 be satisfied, with* $F(t,p) := H(t,Bp)$. *Let* $V$ *be the value function of the control problem, and let* $w$ *be the strong solution of* (6.6) *described in Theorem 6.5. Then*

$$V(t,x) = w(T-t,x),\ \forall t\in[0,T],\ \forall x\in D(A^*)',$$

*that is, the value function* $V$ *of the optimal control problem is the unique strong solution of the HJB equation* (6.5).

**6.2.2. Semicontinuous data and weak solutions of HJB equations.** We then treat the case of merely semicontinuous data, from which the notion of *weak* solution originates.

HYPOTHESIS 6.7. *If $\Upsilon$ is a convex closed subset of $D(A^*)'$, define*

$$\Sigma_\Upsilon \equiv \Sigma_\Upsilon(D(A^*)') := \tag{6.8}$$
$$:= \Big\{ w : D(A^*)' \to (-\infty, +\infty] \; : \; w \text{ is convex and } l.s.c.,\ \Upsilon \subset D(w) \Big\}$$

*where $D(w) = \{x \in D(A^*)' \; : \; w(x) < +\infty\}$, and assume:*

1. *$A^{(E)} : D(A^{(E)}) \subset D(A^*)' \to D(A^*)'$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{sA^{(E)}}\}_{s\geq 0}$ on $D(A^*)'$;*
2. *$B^* \in \mathscr{L}(U, D(A^*)')$;*
3. *there exists $w > 0$ such that $|e^{sA^{(E)}} x|_{D(A^*)'} \leq e^{ws}|x|_{D(A^*)'}$, $\forall s \geq 0$;*
4. *$F \in \mathcal{Y}([0,T] \times D(A^*))$, $F(t,0) = 0$, $\sup_{t\in[0,T]}[F_p(t,\cdot)]_L < +\infty$;*
5. *$L_1(t,\cdot) \in \Sigma_\Upsilon(D(A^*)')$, for all $t \in [0,T]$; $L_1(\cdot, x)$ l.s.c. and $L^1(0,T)$ for all $x \in D(A^*)'$;*
6. *$h \in \Sigma_\Upsilon(D(A^*)')$;*
7. *$L_2(t,\cdot)$ is convex, lower semi-continuous, $D^- L_2(t,\cdot)$ is injective for all $t \in [0,T]$; moreover $L_2(t,u) \geq a(t)|u|_U^2 + b(t)$, with $a(t) \geq c > 0$, $a, b \in L^1(0,T;\mathbb{R})$ (and $c$ a real constant).*
8. *$H \in \mathcal{Y}([0,T] \times U)$, $H(t,0) = 0$, and $\sup_{t\in[0,T]}[\partial_q H(t,\cdot)]_L < +\infty$.*

DEFINITION 6.8 (**Weak solution**). *Let Assumptions 6.7 be satisfied. Let $\Upsilon \subset D(A^*)'$ be a closed convex set, and let $h \in \Sigma_\Upsilon$ and $L_1(t,\cdot) \in \Sigma_\Upsilon$ for all $t$ in $[0,T]$. Then $w : [0,T] \times D(A^*)' \to (-\infty, +\infty]$ is a* weak *solution of $(HJB)$ if:*

*$(i)$ $w(t,\cdot) \in \Sigma_\Upsilon$, $\forall t \in [0,T]$;*

*$(ii)$ there exist sequences $\{h_n\}_n \subset \Sigma_0$, and $\{L_{1n}\} \subset \mathcal{Y}([0,T] \times D(A^*)')$, such that*

$$h_n(x) \uparrow h(x),\ L_{1n}(t,x) \uparrow L_1(t,x),\ \forall x \in D(A^*)',\ \forall t \in [0,T],\ as\ n \to +\infty,$$

*and moreover, if $w_n$ is the unique strong solution of*

$$\begin{cases} w_t(t,x) + F(t, \nabla w(t,x)) - \langle A^{(E)} x, \nabla w(t,x)\rangle = L_{1n}(t,x) & (t,x) \in [0,T] \times D(A^*)' \\ w(0,x) = h_n(x) \end{cases}$$

*in $C([0,T], C_2(D(A^*)'))$, then*

$$w_n(t,x) \uparrow w(t,x),\ \forall (t,x) \in [0,T] \times D(A^*)'.$$

REMARK 6.9. *Since strong solution were proved in* [Fag05b] *to be Lipschitz with respect to the time variable and $C^1$ with respect to the space variable, and the weak solution $w$ is a sup-envelop of strong solutions $w_n$, then $w$ is lower semi-continuous in $[0,T] \times D(A^*)'$. For the same reason $w_n$ convex in the $x$ variable implies that $w$ is convex in $x$ as well.*

THEOREM 6.10. *Let Assumptions 6.7 be satisfied. Let also $L_1$ and $L_2$ be of the following type*

$$L_1(t,x) = e^{-\rho t} L_{10}(x), \quad L_2(t,u) = e^{-\rho t} L_{20}(u)$$

*. Then the following properties are equivalent:*

*$(i)$ there exists a unique weak solution of (6.6);*

*$(ii)$ At each $(t,x) \in [0,T] \times \Upsilon$ there exists an admissible control.*

*Moreover if $(i)$ or $(ii)$ holds, there exists an optimal pair $(u^*, x^*)$ and*

$$w(T - t, x) = J(t, x, u^*(\cdot)).$$

**Strong, weak and ultra-weak solutions**
One of the advantages in the introduction of the notion of weak solutions is the possibility to treat optimal control problems with state constraints. Indeed a way to impose state constraints is to add a penalizing function in the target functional (we use such technique in (6.16). The properties of the penalizing function do not allow to treat it using strong solutions and then a more general definition is needed so the definition of weak solution is introduced. So, if the data satisfy certain assumptions (involving convexity, semicontinuity, and coercivity of $L_2$), then the value function of an optimal control problem with state constraints of type (6.17) is indeed the unique *weak* solution to the HJB equation (6.5).

Some coercivity for the function $L_2$ is indeed lacking for some applied example, as the prototype of $L_{20}$ is $\frac{u^{1-\sigma}}{1-\sigma}$ as mentioned in Section 6.3 which is sublinear on the positive real axis. This causes the Hamiltonian of the problem - that is related to the Legendre transform of $L_{20}$ - to be possibly non-regular, so that all previous definition of solutions do not apply. (Note indeed that, as more precisely stated in the Section 6.2, a weak solution is limit of strong solutions of approximating equations, while a strong solution is itself limit of classical solutions of approximating equations. All of these notions require the Hamiltonian to be differentiable with respect to the co-state variable $p$.)

Here we are about to define a *ultra-weak* solution as limit of weak solutions to (6.5). The concept of solution is indeed generalized, although not in the same direction as before, due to the presence of possibly non-regular Hamiltonians.

We will proceed first showing some motivating examples and then using ultra-weak solution for the case of HJB arising in optimal control problem governed by linear delay equations. Such results are preliminary and a more in-depth study are needed in the future.

## 6.3. Two examples

We present the two applied problems motivating this work.

**6.3.1. An AK model with vintage capital.** We consider here an optimal control problem related to a generalization of the model presented by Boucekkine, Puch, Licandro and Del Rio in [BLPdR05]. The model is in-deeply described in Chapter 7, here the formulation is equivalent but a bit different because the control variable is the consumption instead of investment. We assume that the system is ruled by the same evolution law as the one in [BLPdR05], but we consider the finite horizon problem with a (more) general concave target functional, as specified later. The state equation of the model is

$$\dot{\theta}(s) = a\theta(s) - a\theta(s-R) - u(s) + u(s-R), \quad s \in [t, T]$$

where the state variable $\theta$ is the stock of capital at time $t$ and the control $u$ is the consumption.

The social planner has to maximize the following functional

$$\int_t^T e^{-\rho s} l_{20}(c(s))\mathrm{d}s + h_0(k(T)) \tag{6.9}$$

where $h_0$ and $l_{20}$ are concave u.s.c. utility functions.

We assume that the capital at time $s$ (and consequently the production) and the consumption at time $s$ cannot be negative:

$$\theta(s) \geq 0, \qquad u(s) \geq 0, \quad \forall s \in [t, T] \tag{6.10}$$

These constraints are different from the more restrictive ones of [BLPdR05], where also the investment path $i(\cdot)$ was assumed positive.

**6.3.2. An advertising model with delay effects.** The model we describe was presented in the stochastic case in the papers [GMS06, GM04], and, in deterministic one, in [FG04] (see also [FHS94] and the references therein for related models)[1].

Let $t \geq 0$ be an initial time, and $T > t$ a terminal time ($T < +\infty$ here). Moreover let $\theta(s)$, with $0 \leq t \leq s \leq T$, represents the stock of advertising goodwill of the product to be launched. Then the general model for the dynamics is given by the following controlled Delay Differential Equation (DDE) with delay $R > 0$ where $z$ models the intensity of advertising spending:

(6.11)
$$\begin{cases} \dot{\theta}(s) = a_0\theta(s) + \int_{-R}^{0} \theta(s+\xi)da_1(\xi) + b_0 u(s) + \int_{-R}^{0} u(s+\xi)db_1(\xi) \quad s \in [t,T] \\ \\ \theta(t) = \phi^0; \quad \theta(r) = \phi^1(r),\ u(r) = \sigma(r) \ \ \forall \xi \in [t-R,t], \end{cases}$$

with the following assumptions:

- $a_0$ is a constant factor of image deterioration in absence of advertising, $a_0 \leq 0$;
- $a_1(\cdot)$ is the distribution of the forgetting time, $a_1(\cdot)$ is a bounded variation function;
- $b_0$ is a constant advertising effectiveness factor, $b_0 \geq 0$;
- $b_1(\cdot)$ is the density function of the time lag between the advertising expenditure $u$ and the corresponding effect on the goodwill level, $b_1(\cdot)$ is a bounded variation function;
- $\phi^0$ is the level of goodwill at the beginning of the advertising campaign, $\phi^0 \geq 0$;
- $\phi^1(\cdot)$ and $\omega(\cdot)$ are respectively the goodwill and the spending rate before the beginning, $\phi^1(\cdot) \geq 0$, with $\phi^1(0) = \phi^0$, and $\omega(\cdot) \geq 0$.

When $a_1(\cdot)$, $b_1(\cdot)$ are identically zero, equation (6.11) reduces to the classical model contained in the paper by Nerlove and Arrow (1962). We assume that the goodwill and the investment in advertising at each time $s$ cannot be negative:

$$\theta(s) \geq 0, \qquad u(s) \geq 0, \quad \forall s \in [t,T]. \tag{6.12}$$

Finally, we define the objective functional, to be maximized, as

$$J(t,(\phi^0,\phi^1,\omega),u(\cdot)) = h_0(\theta(T)) - \int_t^T e^{-\rho s} l_{20}(u(s))\, ds, \tag{6.13}$$

where $h_0$ is a concave utility function, $l_{20}$ is a convex cost function, and the dynamic of $\theta$ is determined by (6.11). The functional $J$ has to be maximized over some set of admissible controls $\mathcal{U}$, for instance $L^2((t,T);\mathbb{R}^+)$, the space of square integrable nonnegative functions.

## 6.4. The state equation in the infinite dimensional setting.

To obtain the state equation in $M^2$ of our control problem we need to specify the general form considered in Section 1.3 considering the continuous linear application $B$ given by

$$B\colon C([-R,0]) \to \mathbb{R}$$
$$B\colon \varphi \mapsto -\varphi(0) + \varphi(-R)$$

and the continuous linear application $N$ with norm $\|N\|$ given by

$$N\colon C([-R,0]) \to \mathbb{R}$$
$$N\colon \varphi \mapsto a\varphi(0) - a\varphi(-R)$$

[1] We observe that also other models of delay type arising in economic theory can be treated with our tools (see *e.g.* the paper by [BdlCL04]).

Following the steps describe in Section 1.3 we define the structural state in our specific case: In this particular case it can be explicitly found: If we call $\tilde{\theta}_s, \tilde{u}_s \in L^2(-R, 0)$ the applications

$$\tilde{\theta}_s\colon \lambda \mapsto -\theta(s-R-\lambda)$$
$$\tilde{u}_s\colon \lambda \mapsto -u(s-R-\lambda)$$

the structural state can be written as

$$x(s) \stackrel{def}{=} (\theta(s), a\tilde{\theta}_s - \tilde{u}_s + \Xi(s)x^1). \tag{6.14}$$

($\Xi(\cdot)$ is defined in (1.38)).

It solves, as better describe in Section 1.3 the equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}x(s) = Ax(s) + B^*u(s) \\ x(t) = x. \end{cases} \tag{6.15}$$

where $A$ is defined in (1.35).

**6.4.1. The state equation of the advertising model in the Hilbert setting.** Similar arguments can be used for the advertising model. Following Section 1.3, if we call $N$, $B$ the continuous linear functionals given by

$$N\colon C([-R,0]) \to \mathbb{R}$$
$$N\colon \varphi \mapsto a_0\varphi(0) + \textstyle\int_{-r}^{0} \varphi(\xi)da_1(\xi)$$

$$B\colon C([-R,0]) \to \mathbb{R}$$
$$B\colon \varphi \mapsto b_0\varphi(0) + \textstyle\int_{-r}^{0} \varphi(\xi)db_1(\xi)$$

we obtain that

- The structural state in the advertising model will have the following expression ($e^0_+$, $e^0_-$, $\Xi(s)$, $\overline{N}$ and $\overline{B}$ are the same of Section 1.3):

  $$x(t) = (x^0(s), x^1(s)) \stackrel{def}{=} (\theta(s), \overline{N}(e^0_+\theta)_s - \overline{B}(e^0_+u)_s + \Xi(s)x^1)$$

  where $x_1 = \overline{N}(\phi^1) - \overline{B}(\omega)$.
- The state equation becomes

  $$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}x(s) = Ax(s) + B^*u(s) \\ x(t) = x. \end{cases}$$

## 6.5. The target functional and the HJB equation

We now rewrite the profit functional for the first example in abstract terms, noting that a similar reformulation holds for the target functional of the second example. We consider a control system governed by the linear equation (6.15). We assume that the set of admissible controls is defined by

$$\mathcal{U} \stackrel{def}{=} \{u(\cdot) \in L^2(t,T) \ : \ u(\cdot) \geq 0 \ and \ x^0(\cdot) \geq 0\}$$

In order to apply the results contained in [Fag06] and recalled in Section 6.2, we reformulate the maximization problem as a minimization problem and we take the constraints into account by modifying the target functional as follows. If $l_{20}$ and $h^0$ are the concave *u.s.c.* functions appearing in (6.9), then we define

$$L_{20}\colon \mathbb{R} \to \overline{\mathbb{R}}$$
$$L_{20}(u) = \begin{cases} -l_{20}(u) & if\ u \geq 0 \\ +\infty & if\ u < 0 \end{cases}$$

(6.16)
$$
\begin{aligned}
&h\colon \mathbb{R} \to \overline{\mathbb{R}} \\
&h(r) = \begin{cases} -h_0(r) & if\ r \geq 0 \\ +\infty & if\ r < 0 \end{cases}
\end{aligned}
$$

Moreover we set

$$
\begin{aligned}
&L_{10}\colon \mathbb{R} \to \overline{\mathbb{R}} \\
&L_{10}(r) = \begin{cases} 0 & if\ r \geq 0 \\ +\infty & if\ r < 0 \end{cases}
\end{aligned}
$$

Both $L_2$, $h$ and $L_1$ are convex *l.s.c.* functions on $\mathbb{R}$. Then we define the *target functional* as

$$J(t,x,u(\cdot)) = \int_t^T e^{-\rho s}[L_{20}(u(s)) + L_{10}(x^0(s))]\mathrm{d}s + h(x^0(T))$$

with $u$ varying in the *set of admissible controls* $L^2(t,T)$. It is easy to check that the problem of maximizing (6.9) in the class $\mathcal{U}$ is equivalent to that of minimizing $J$ on the whole space $L^2(t,T)$. Then the original maximization problem for the AK-model can be reformulated as minimization problem:

$$\inf\{J(t,x,u(\cdot)) \ : \ u \in L^2(t,T), \text{ and } x \text{ satisfies (6.15)}\}, \tag{6.17}$$

The HJB equation naturally associated to such minimization problem by DP is

$$\begin{cases} \partial_t v(t,x) + \langle \nabla v(t,x), Ax\rangle - H(t, \nabla v(t,x)) + e^{-\rho t}L_1(x) = 0 \\ v(T,x) = h(x) \end{cases}$$

with $H$ defined as follows

$$\begin{cases} H\colon [0,T] \times D(A^*) \to \mathbb{R} \\ H(t,p) \stackrel{def}{=} \sup_{u \geq 0}\{-B(p)u - e^{-\rho t}L_{20}(u)\} = e^{-\rho t}{L_2}^*(-e^{\rho t}B(p)) \end{cases}$$

where ${L_2}^*$ is the Legendre transform of the convex function $L_2$. We refer to $H$ as to the *Hamiltonian* of the system

## 6.6. The value function as ultra-weak solution of HJB equation

We define the value function of the optimal control problem described in the previous sections as

$$V(t,x) \stackrel{def}{=} \inf_{u(\cdot)\in L^2(t,T)} J(t,x,u(\cdot)).$$

Our objective here is to provide a suitable definition of solution of the HJB equation, so that the value function $V$ is a solution, in such sense.

DEFINITION 6.11 (**Ultra-weak solution**). *We say that a function $W$ is a ultra-weak solution to*

$$\begin{cases} \partial_t v(t,x) + \langle \nabla v(t,x), Ax\rangle - H(t, \nabla v(t,x)) + e^{-\rho t}L_1(x) = 0 \\ v(T,x) = (x) \end{cases}$$

*if there exists a sequence $\{H_n\}_n$ of functions in the space $\mathcal{Y}([0,T] \times D(A^*))$, such that $H_n \uparrow H$ pointwise, and*

$$W(t,x) = \lim_{n\to+\infty} W_n(t,x) = \inf_{n\geq 0} W_n(t,x)$$

*with $W_n$ the unique weak solutions to*

$$\begin{cases} \partial_t v(t,x) + \langle \nabla v(t,x), Ax\rangle - H_n(t, \nabla v(t,x)) + e^{-\rho t}L_1(x) = 0 \\ v(T,x) = h(x) \end{cases}$$

Any weak solution $V$ is convex in the state variable $x$, but not necessarily $l.s.c$ in $(t,x)$. We are able to prove an existence result for equation (6.5) by proving that the value function of the control problem set in the previous section is a ultra-weak solution.

THEOREM 6.12. *The value function $V$ of the optimal control problem (6.17) is an ultra-weak solution of (6.5).*

*Proof.* First of all we need to construct a sequence of Hamiltonians $H_n$ having the properties required by the definition above. We choose

$$H_n(t,p) := e^{-\rho t} L_{2n}^*(-e^{\rho t}B(p))$$

with

$$L_{2n}(u) = L_{20}(u) + \frac{1}{2n}|u|^2,\ \ n \in \mathbb{N}.$$

Indeed if we denote with $S_n f(x) = \inf_{y\in\mathbb{R}}\left\{f(y) + \frac{n}{2}|x-y|^2\right\}$ the Yosida approximation of a function $f$, then it is easy to check that $[S_n f]^*(x) = f^*(x) + \frac{1}{2n}|x|^2$, so that

$$L_{2n}^*(u) = S_n(L_{20}^*)(u).$$

Being $L_{2n}^*$ the Yosida approximations of a *l.s.c.* convex function, they result to be Frechét differentiable with Lipschitz gradient, with Lipschitz constant $[(L_{2n}^*)']_L \le n$. Moreover, as $L_{2n}$ is a decreasing sequence, $H_n$ is then increasing, as required by Definition 6.11. Hence the assumptions in Theorem 6.10 are satisfied for the problem of minimizing the functional

$$J_n(t,x,u(\cdot)) = J(t,x,u(\cdot)) + \frac{1}{2n}\int_t^T e^{-\rho s}|u(s)|^2 \mathrm{d}s$$

in $L^2(t,T)$, and we derive as a consequence the following result:

LEMMA 6.13. *Let*

$$V_n(t,x) \stackrel{def}{=} \inf_{u\in L^2(t,T)} J_n(t,x,u(\cdot)),$$

*be the value functions of the approximating optimal control problem. Then $V_n$ is convex in $x$ and l.s.c. in $x$ and $t$, and it is the unique weak solution of*

$$\begin{cases} \partial_t v(t,x) + \langle \nabla v(t,x), Ax\rangle - H_n(t,\nabla v(t,x)) + e^{-\rho t}L_1(x) = 0 \\ v(T,x) = h(x) \end{cases}$$

*Moreover there exists $u_n^* \in L^2(t,T)$ optimal for the approximating problems, i.e. $V_n(t,x) = J_n(t,x,u_n^*(\cdot))$.*

To complete the proof we need to show that $V_n(t,x) \downarrow V(t,x)$.

LEMMA 6.14. *The value function of (6.17) is given by*

$$V(t,x) = \lim_{n\to\infty} V_n(t,x) = \inf_n V_n(t,x).$$

*Proof.* By definition of $J_n$, for all $t$, $x$ and $n$ we have $J_n(t,x,u(\cdot)) \ge J_{n+1}(t,x,u(\cdot))$ for all admissible controls $u$, so that

$$V_n(t,x) \ge V_{n+1}(t,x),$$

and $\{V_n(t,x)\}_n$ is a decreasing sequence. As a consequence, a ultra-weak solution $V$ of HJB equation exists, and it is given by

$$W(t,x) \stackrel{def}{=} \lim_{n\to\infty} V_n(t,x) = \inf_{n\in\mathbb{N}} V_n(t,x).$$

Next we show that a solution $W$ built this way necessarily coincides with $V$. We note that

$$J(t,x,u(\cdot)) \le J_n(t,x,u(\cdot)),\quad \forall u \in L^2(t,T),$$

so that by taking the infimum and then passing to limits, we obtain

$$V(t,x) \le W(t,x). \tag{6.18}$$

We then prove the reverse inequality. Let $\varepsilon > 0$ be arbitrarily fixed, and $u_\varepsilon$ be an $\varepsilon$-optimal control for the problem, that is $V(t,x) + \varepsilon > J(t,x,u_\varepsilon(\cdot))$. By passing to limits as $n \to +\infty$ in

$$V_n(t,x) \le J_n(t,x,u_\varepsilon(\cdot))$$

one obtains

$$W(t,x) \le J(t,x,u_\varepsilon(\cdot)) < V(t,x) + \varepsilon,$$

which implies, together with (6.18), the thesis.

Doing so we proved the lemma and Theorem 6.12.

REMARK 6.15. *Note that we do not derive any uniqueness result for ultra-weak solutions.*

# Part 3

# Special applied cases

CHAPTER 7

# Vintage Capital in the AK growth model

In this Chapter we present the results of [FGa] where the explicit solution of a first order HJB equation in the Hilbert space $M^2$ is used to study a vintage capital model. The attention is mainly on the economic side[1]. The reader not interested in the economic problem can skip the introduction and the presentation of the model starting from Section 7.3. We have chosen to maintain the Chapter (almost) self-contained so the reader will find some concepts he already found in other parts of the thesis. In particular in Section 7.4 some statements described in Section 1.3 will be revisited in the particular case of the AK model with vintage capital.

## 7.1. Introduction

We develop the Dynamic Programming approach to study a continuous time endogenous growth model with vintage capital. We focus on the AK model proposed by Boucekkine, Puch, Licandro and Del Rio in [BLPdR05] (see e.g. [BdlCL04], [Ben91] for related models) which is summarized in Section 7.2.

In the literature continuous time endogenous growth models with vintage capital are treated by using the Maximum Principle. Here we develop the Dynamic Programming approach to the representative model of [BLPdR05] getting sharper results. The improvements we obtain mainly come from the fact that we are able to find the value function and solve the optimal control problem in closed loop form, a key feature of the Dynamic Programming approach.

We stress the fact that the novelty of this work is mainly on the methodological side, i.e. we provide an example of the power of the Dynamic Programming approach in the analysis of endogenous growth models.

In our opinion the Dynamic Programming approach to continuous time optimal control problems arising in economic theory has not been exploited in its whole power. This is especially true when the model presents some features like the presence of Delay Differential Equations and/or Partial Differential Equations and/or state-control constraints. However the presence of such features is needed when we want to look at problems with vintage capital, see for instance the quoted papers [BLPdR05, BdlCL04, Ben91], and also [BG98, BG01], [FHKV06],[Fag05a, Fag05b] on optimal technology adoption and capital accumulation.

The main methodological issues treated in this work are the following.

(I) (*Explicit form of solutions*).
Providing solutions in explicit form, when possible, helps the analysis of the model. In [BLPdR05] it is shown that the optimal consumption path has a specific form (i.e. it is an exponential multiplied by a constant $\Lambda$) but none is said about the form of $\Lambda$, the explicit expression of the capital stock and investment trajectories. Moreover existence of

[1]The notation emphasizes this choice indeed the names of the variables are the ones used in the economic literature: $k$, for "capital", is the name of the state variable, $i$, as investment, is the name of control variable. Moreover we use the Hamiltonian with "supremum" instead of "infimum".

a long run equilibrium for the discounted paths is established but none is said about its form.

Here, using the fact that we can calculate explicitly the value function, we show a more precise result on the optimal consumption path determining the constant $\Lambda$. So we explicitly determine an equation for the optimal trajectories of the capital stock and of the investment. This allows to find explicitly the long run equilibrium of the discounted paths. So the study of their properties can be performed in a more efficient way: in particular we can give more precise analysis of the presence of oscillations in the capital and investment stock and in the growth rates and we can make a precise comparison with the standard AK model with zero depreciation rate of capital. See Section 7.6.1 for further explanations.

(II) (*Admissibility of candidate solutions*).

When state/control constraints are present the necessary conditions of Maximum Principle are difficult to solve. Often in studying growth models one considers the problem without such constraints and then checks if the optimal path for the unconstrained problem satisfy them. This may be a difficult task and in some cases may even be not true. Indeed, in [BLPdR05] it is not proved that the candidate optimal trajectory of capital and investment is admissible (see the discussion in Section 4.3, p. 60 of [BLPdR05]) so a nontrivial gap remains in the theoretical analysis of the model.

Here we prove that the candidate optimal trajectory is admissible, so fixing such gap: such difficult task is accomplished by changing the point of view used in [BLPdR05] (and in many papers on continuous time endogenous growth models) to find the optimal trajectory. See Section 7.6.2 for further explanations.

(III) (*Wider parameter set*).

We work under more general assumptions on the parameters that includes cases which may be still interesting from the economic point of view. These cases are not included in [BLPdR05] and for this reason the set of parameters for which their theory applies can be empty for some values of $\sigma \in (0, 1)$. See Section 7.6.3 for further explanations.

Concerning the economic interpretation of the methodological results listed above we underline the following.

- We have at hand a power series expansion of the investment and capital path where the dependence of the coefficients on the initial investment path is explicit. This means that the short run fluctuations of investment and capital and of their growth rates (which are driven by replacement echoes) can be analyzed in terms of the deviation of the investment's history from the "natural" balanced growth path (see Subsection 7.6.1.1). Moreover the presence of explicit formulae opens the door to a more precise empirical testing of the model.
- We provide a comparison of the model with the standard AK model with depreciation rate of capital equal to 0. First we see that when the lifetime $R$ of machines goes to infinity the vintage AK model reduces to such standard AK model. Moreover we show that in the vintage AK model the "equivalent capital" (see Subsection 7.6.1.2 for a definition) has a constant growth rate.

  This may explain two qualitative characteristic of the model: first the consumption path has a constant growth rate since the decision of the agent is to consume a constant share at the "equivalent capital" which is the key variable of the system (see the closed loop relation (7.50)); second the agent adjusts the investments to keep constant the growth rate of the "equivalent capital" (compare (7.50) and (7.51)) and this gives rise to the fluctuations in the investment path (due to replacement echoes). In this regard this is not a model of business cycle, as already pointed out in [BLPdR05].

- In this setting, differently from the standard AK model with zero depreciation rate of capital, a positive investment rate is compatible with a negative long run growth rate. This enlarge the scenarios where the deviation between growth and investment rates can arise (see e.g. the discussion on this given in [BLPdR05]).

We organize the chapter as follows.

In Section 7.2 we will briefly describe the model of [BLPdR05] and its relationship with the literature on vintage capital models. Moreover, in Subsection 7.2.1 we will describe our approach to the problem: rewrite the model in an infinite dimensional space and apply the Dynamic Programming approach.

Then we come to the technical part of the chapter in Sections 3, 4, 5.

In Section 7.3 we give some preliminary results about the solution of the state equation, the existence of optimal controls, the properties of the value function.

The mathematical core of the work is Section 7.4. Here we give, with complete proofs: the precise formulation of the problem in infinite dimension (Subsection 7.4.1); the formulation of the HJB equation and its explicit solution (Subsection 7.4.2); the closed loop formula for the optimal strategies in explicit form (Subsection 7.4.3).

In Section 7.5 we come back to the original problem proving, as corollaries of the results of Section 4, our results about the explicit form of the value function (Subsection 7.5.1), the explicit closed loop strategies (Subsection 7.5.2) and the asymptotic behavior (long run equilibrium, costate dynamics, transversality conditions, balanced growth paths) of the optimal trajectories (Subsection 7.5.3).

In Section 7.6 we discuss the implications of our results and make a comparison with the previous ones. We divide it in three subsections, referring to the methodological points (I)-(II)-(III) raised above.

Appendix 7.A is devoted to a quick development of the Dynamic Programming approach to the standard AK model with zero depreciation rate of capital. It is given here partly because we did not find it in the literature (even if it is standard), partly for the commodity of the reader to have a sketch of the Dynamic Programming approach in an easy case and to make more clear the comparison with the present model (done in Subsection 7.6.1.2) and the related comments.

## 7.2. The AK model with vintage capital

We deal with the vintage capital model presented in [BLPdR05] as a representative continuous time endogenous growth model with vintage capital. Vintage capital is a well known topic in the growth theory literature of last ten years (see for instance [Par94], [AH94], [JR97], [GGR99], [GJ98] [Ben91], [BdlCL04], [Iac02]). Even in a simple setting like the one of AK models the introduction of vintage capital involves the presence of oscillations in the short-run[2] and this is one of the main features that make the model interesting. Indeed the optimal paths in the model of [BLPdR05] converge asymptotically to a steady state but the transition

[2]As we remind in Appendix 7.A, the optimal trajectories of the standard AK models are simply exponential without transition towards steady state and this is one of main limits of such models.

is complex and involve nontrivial dynamics. So this model can be used to study the contribution of the vintage structure of the capital in the transition and the behavior of the system after economic shocks.

For an in-dept explanation of the model and its background see the Introduction of [BLPdR05] or [BdlCL04]. We report here only its main features. First of all we clarify that the model presented in [BLPdR05] is a vintage version of standard AK model with CRRA (Constant Relative Risk Aversion) utility function (which is recalled in Appendix 7.A in the case of zero depreciation rate of capital).

Obsolescence and deterioration of physical capital are simply modeled assuming that all machines have the same technology and that they have a fixed lifetime $R$ (a constant "scrapping time").

The time is continuous and $t = 0$ is the initial point (the horizon is infinite as usual in growth models). However, since we want to introduce a delay effect in the model due to the vintage capital structure, we assume that the economy exists at least at time $-R$ and that its behavior between $t = -R$ and $t = 0$ is known. So all variables of the model will be defined on $[-R, +\infty)$. Of course their paths between $t = -R$ and $t = 0$ will be considered data of the problem so we will define equations and constraints for $t \geq 0$.

We denote by $k(t)$ the stock of capital at time $t$; $i(t)$ and $c(t)$ are the investment and the consumption at time $t$. All of them are nonnegative. So

$$k(t) \geq 0, \ i(t) \geq 0, \ c(t) \geq 0; \ \forall t \geq 0.$$

The $AK$ technology is the following: the aggregate production at time $t$ is denoted by $y(t)$ and it satisfies, for $t \geq 0$

$$y(t) = a \int_{t-R}^{t} i(s)\mathrm{d}s \qquad a > 0. \tag{7.1}$$

Interpreting the integral in the right hand side as the capital we then have, for $t \geq 0$

$$y(t) = ak(t).$$

We have the following accounting relation, for $t \geq 0$

$$ak(t) = y(t) = i(t) + c(t) \tag{7.2}$$

so the non-negativity of all variables is equivalent to ask that, for $t \geq 0$

$$i(t), c(t) \in [0, y(t)] = [0, ak(t)]. \tag{7.3}$$

If the investment function $i(\cdot)$ is assumed to be sufficiently regular (e.g. continuous), then the above relation (7.1) can be rewritten as a Delay Differential Equation for the capital stock

$$\dot{k}(t) = i(t) - i(t - R) \tag{7.4}$$

with initial datum $k(0)$ given as function of the past investments by

$$k(0) = \int_{-R}^{0} i(s)\mathrm{d}s. \tag{7.5}$$

The equilibrium is the solution of the problem of maximizing, over all investment-consumption strategies that satisfy the above constraints (7.1), (7.2), (7.3), the functional of CRRA (Constant Relative Risk Aversion) type

$$\int_{0}^{+\infty} e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} \mathrm{d}t \tag{7.6}$$

where $\rho > 0$, $\sigma > 0$ (and $\sigma \neq 1$). Also more general set of parameters $\rho$ and $\sigma$ (e.g. $\sigma = 1$ or some cases when $\rho \leq 0$) can be treated without big effort, but we avoid this for simplicity.

From the mathematical point of view this model is an optimal control problem. The state variable is the capital $k$, the control variables are the consumption $c$ and the investment $i$, the state equation is the Delay Differential Equation (7.4) with the initial condition (7.5) (which is somehow unusual, see the following discussion and Notation 7.1 for more explanations); the objective functional is (7.6). A control strategy $c(\cdot), i(\cdot)$ defined for $t \geq 0$ is admissible if it satisfies for every such $t$ the constraints (7.2) and (7.3).

Since the two control functions $i(\cdot)$ and $c(\cdot)$ are connected by the relation (7.2) then we can eliminate the consumption $c(\cdot)$ from the mathematical formulation of the problem. Then the only control function is $i(\cdot)$ giving the present investment (as said above its 'history' in the interval $[-R, 0)$ is the initial datum $\bar{\iota}(\cdot)$). We assume that $i(\cdot) \in L^2_{loc}([0, +\infty); \mathbb{R}^+)$ Given an initial datum $\bar{\iota}(\cdot)$ and an investment strategy $i(\cdot)$ we denote by $k_{\bar{\iota},i}(\cdot)$ the associated solution (see Section 7.3 for its explicit form) of the state equation (7.4). The strategy $i(\cdot)$ will be called admissible if it satisfies the state-control constraints (coming from (7.3)):

$$0 \leq i(t) \leq ak_{\bar{\iota},i}(t) \qquad \forall t \geq 0. \tag{7.7}$$

Now, using (7.2), we write the associated inter-temporal utility from consumption as

$$J(\bar{\iota}(\cdot); i(\cdot)) \stackrel{def}{=} \int_0^\infty e^{-\rho s} \frac{(ak_{\bar{\iota},i}(t) - i(t))^{1-\sigma}}{(1-\sigma)} \mathrm{d}s$$

(we have explicitly written in the functional the dependence on the initial datum $\bar{\iota}(\cdot)$).

Given the above our problem is now the one of maximizing the functional $J(\bar{\iota}(\cdot); i(\cdot))$ over all admissible investment strategies $i(\cdot)$.

It must be noted that the model reduces to the standard AK model with zero depreciation rate of capital (described in the Appendix 7.A) when the delay $R$ (i.e. the "scrapping time") is $+\infty$[3].

**7.2.1. The Dynamic Programming Approach.** In this subsection we see how the general four steps of dynamic programming that we have described in the Introduction, become in this applied case.

**(i)** First of all, given an initial datum $\bar{\iota}(\cdot) \in L^2((-R, 0); \mathbb{R}^+)$ we define the set of admissible strategies given $\bar{\iota}(\cdot)$ as

$$\mathcal{I}_{\bar{\iota}} = \left\{ i(\cdot) \in L^2_{loc}([0, +\infty); \mathbb{R}^+) : \ i(t) \in [0, ak_{\bar{\iota},i}(t)], \ a.e. \right\}$$

and then the value function as

$$V(\bar{\iota}(\cdot)) = \sup_{i(\cdot) \in \mathcal{I}_{\bar{\iota}}} \left\{ \int_0^\infty e^{-\rho s} \frac{(ak_{\bar{\iota},i}(t) - i(t))^{1-\sigma}}{(1-\sigma)} \mathrm{d}s \right\}$$

The first step of DP approach recommends to write the DP Principle We cannot apply DP using state equation in delay form so we write the problem in $M^2$ (as describe in Section 1.3) and we write the HJB equation in $M^2$. We will use, for technical reasons, a setting where the initial data will be both $\bar{\iota}(\cdot)$ and $k(0)$ ignoring the relation $k(0) = \int_{-R}^0 \bar{\iota}(s)\mathrm{d}s$ that connects them. So in Section 7.4 we will consider an artificial value function depending on $\bar{\iota}(\cdot)$ and $k(0)$ and write and solve

[3] Indeed in such case

$$k(t) = \int_{-\infty}^t i(s)\mathrm{d}s = k(0) + \int_0^t i(s)\mathrm{d}s$$

and so the Delay Differential Equation (7.4) becomes the Ordinary Differential Equation of the standard AK model with zero depreciation rate of capital.

the HJB equation for it (see Subsection 7.4.1). After this, in Section 7.5 we will go back to the value function defined here.

**(ii)** The second step of DP approach is now to solve the HJB equation. We will find explicitly a solution of the HJB equation and prove that it is the value function (see Propositions 7.24 and 7.32). The only other examples of explicit solution of the HJB equation in infinite dimension involve, for what we know, linear state equations and quadratic functionals (see Section 1.3 for references).

This HJB equation cannot be treated with the results of the existing literature. This is due, as previously said, to the presence of the state/control constraint, to the unboundedness of the control operator and the non-analyticity of the semigroup given by the solution operator of the state equation (see Remark 7.21 for more details).

**(iii)** The third step will be then to write the closed loop (feedback) formula. This means to write a formula that gives the present value of the optimal control as function only of the present value of the state. In this case the state is infinite dimensional and it is composed, for each $t \geq 0$, by the present value of the capital $k(t)$ and by the past (at time $t$) of the investment strategy $\{i(t+s), s \in [-R,0)\}$. So the closed loop formula will give the present value of the investment $i(t)$ as a function of the present value of the state and of the past of the investment itself (see equation (7.39) for the feedback in infinite dimension and equation (7.49) for its Delay Differential Equation version). This formula will be given in term of the value function and so, using its explicit expression found in step (ii), also the closed loop formula will be given in explicit form. For details see Theorem 7.29 for the result in infinite dimensions and Proposition 7.33 for the Delay Differential Equation version.

**(iv)** The closed loop formula will be then substituted into the state equation (7.9) to get an equation for the optimal state trajectory (the so-called Closed Loop Equation). Such equation will be a Delay Differential Equation, as recalled at point (iii) above, and explicit solutions cannot be given in general. However it allows to study the behavior of the optimal paths and to perform numerical simulations. For details see Theorem 7.35 and Subsections 7.5.3, 7.6.1.1.

## 7.3. Preliminary results on the control problem

We first introduce a notation.

Notation 7.1. *We have to distinguish $\bar{\imath}\colon [-R,0) \to \mathbb{R}^+$ that is part of initial data, $i\colon [0,+\infty) \to \mathbb{R}^+$ that is the control strategy and $\tilde{\imath}\colon [-R,+\infty) \to \mathbb{R}^+$ that is piecewise defined as*

$$\tilde{\imath}(s) = \begin{cases} \bar{\imath}(s) & s \in [-R,0) \\ i(s) & s \in [0,+\infty). \end{cases} \tag{7.8}$$

*$\tilde{\imath}$ is useful to rewrite more formally the state equation as in (7.9) below.*

The state equation is now written as the Delay Differential Equation (on $\mathbb{R}^+$)

$$\begin{cases} \dot{k}(t) = \tilde{\imath}(t) - \tilde{\imath}(t-R), \ \forall t \geq 0, \\ \tilde{\imath}(s) = \bar{\imath}(s), \ \forall s \in [-R,0), \\ k(0) = \int_{-R}^{0} \bar{\imath}(s)\mathrm{d}s, \end{cases} \tag{7.9}$$

where $R \in \mathbb{R}$ is a positive constant, $\bar{\imath}(\cdot)$ and $k(0)$ are the initial conditions. We will assume $\bar{\imath}(\cdot) \geq 0$ and $\bar{\imath}(\cdot) \not\equiv 0$. Moreover $\bar{\imath}(\cdot) \in L^2((-R,0);\mathbb{R}^+)$. For every $i(\cdot) \in L^2_{loc}([0,+\infty);\mathbb{R}^+)$ and every $\bar{\imath}(\cdot) \in L^2((-R,0);\mathbb{R}^+)$ the Delay Differential Equation (7.9) admits a unique locally absolutely continuous solution given by

$$k_{\bar{\imath},i}(t) \stackrel{def}{=} \int_{(t-R)\vee 0}^{t} i(s)\mathrm{d}s \; + \int_{(t-R)\wedge 0}^{0} \bar{\imath}(s)\mathrm{d}s = \int_{(t-R)}^{t} \tilde{\imath}(s)\mathrm{d}s \tag{7.10}$$

The functional to maximize is

$$J(\bar{\imath}(\cdot);i(\cdot)) \stackrel{def}{=} \int_0^\infty e^{-\rho s}\frac{(ak_{\bar{\imath},i}(t)-i(t))^{1-\sigma}}{(1-\sigma)}\mathrm{d}s$$

over the set

$$\mathcal{I}_{\bar{\imath}} \stackrel{def}{=} \{i(\cdot) \in L^2_{loc}([0,+\infty);\mathbb{R}^+) \; : \; i(t) \in [0, ak_{\bar{\imath},i}(t)] \text{ for a.e. } t \in \mathbb{R}^+\}.$$

Here $a$ and $\sigma$ are strictly positive constants with $\sigma \neq 1$. The choice of $\mathcal{I}_{\bar{\imath}}$ implies $k_{\bar{\imath},i}(\cdot) \in W^{1,2}_{\mathrm{loc}}(0,+\infty;\mathbb{R}^+)$ for every $i(\cdot) \in \mathcal{I}_{\bar{\imath}}$.

REMARK 7.2 (**On the irreversibility constraint**). In the definition of $\mathcal{I}_{\bar{\imath}}$ we have imposed two control constraints for each $t \geq 0$: the first is of course $(ak_{\bar{\imath},i}(t) - i(t)) \geq 0$ that means exactly that the consumption cannot be negative; the second is $i(t) \geq 0$, i.e. irreversibility of investments. In the standard AK growth model context (see Appendix 7.A) this assumption is usually replaced by the constraint $k(t) \geq 0$ (or some weaker "no Ponzi game" condition). There are some arguments to believe that $i(t) \geq 0$ is a more natural choice in our delay setting. First of all in the vintage model $i(t)$ is the investment in new capital and so the irreversibility assumption is natural from the economic point of view. Moreover we can observe that, unlike the non-delay case, see (7.69), $i(t) \geq 0$ does not imply a growth of the capital (see Subsection 7.6.3 on this). Finally if this constraint hold on the datum $\bar{\imath}(\cdot)$ (as we assume) the set of admissible strategies is always nonempty. If we take only the constraints $k(t) \geq 0$ and $(ak_{\bar{\imath},i}(t) - i(t)) \geq 0$ then there are examples of initial data $\bar{\imath}(\cdot)$ (not always positive) with $k(0) \geq 0$ such that the set of admissible trajectories is empty (for instance $\bar{\imath}(s) = 2\chi_{[-R,-R/2]}(s) - 2\chi_{(-R/2,0)}(s)$ for $s \in [-R, 0)$). ■

We will name *Problem (**P**)* the problem of finding an optimal control strategy i.e. to find an $i^*(\cdot) \in \mathcal{I}_{\bar{\imath}}$ such that:

$$J(\bar{\imath}(\cdot);i^*(\cdot)) = V(\bar{\imath}(\cdot)) \stackrel{def}{=} \sup_{i(\cdot)\in\mathcal{I}_{\bar{\imath}}} \left\{ \int_0^\infty e^{-\rho s}\frac{(ak_{\bar{\imath},i}(t)-i(t))^{1-\sigma}}{(1-\sigma)}\mathrm{d}s \right\}. \tag{7.11}$$

We now give a preliminary study of the problem concerning the asymptotic behavior of admissible trajectories, the finiteness of the value function, the existence of optimal strategies and the positivity of optimal trajectories.

**7.3.1. Asymptotic behavior of admissible trajectories.** To find conditions ensuring the finiteness of the value function we need first to study the asymptotic behavior of the admissible trajectories, in particular to determine which is the maximum asymptotic growth rate of the capital.

PROPOSITION 7.3. *Given an initial datum $\bar{\imath}(\cdot) \in L^2((-R,0);\mathbb{R}^+)$ and a control $i(\cdot) \in L^2_{loc}([0,+\infty);\mathbb{R}^+)$, we have that the solution $k_{\bar{\imath},i}(\cdot)$ of (7.9) is dominated at any time $t \geq 0$ by the solution $k^M(\cdot)$ obtained taking the same initial datum $\bar{\imath}(\cdot)$ and the admissible control defined by the feedback relation $i^M(t) = ak^M(t)$ for all $t \geq 0$ (that is the maximum of the range of admissibility).*

Proof. The statement follows from the integral form of the Delay Differential Equation (equation (7.10)). Indeed by the admissibility constraints (7.7) we have, for $t \in [0, R]$,

$$k_{\bar{\imath},i}(t) = \int_{t-R}^{0} \bar{\imath}(s)\mathrm{d}s + \int_0^t i(s)\mathrm{d}s \le \int_{t-R}^{0} \bar{\imath}(s)\mathrm{d}s + \int_0^t ak_{\bar{\imath},i}(s)\mathrm{d}s$$

while the function $k^M(\cdot)$ satisfies, for $t \in [0, R]$,

$$k^M(t) = \int_{t-R}^{0} \bar{\imath}(s)\mathrm{d}s + \int_0^t ak^M(s)\mathrm{d}s$$

Given these the inequality $k_{\bar{\imath},i}(t) \le k^M(t)$ for $t \in [0, R]$, follows from a straightforward application of the Gronwall inequality (see e.g. [Hen81] page 6).
For $t \in (R, 2R]$ we have, arguing as above

$$k_{\bar{\imath},i}(t) = \int_{t-R}^{R} i(s)\mathrm{d}s + \int_R^t i(s)\mathrm{d}s \le \int_{t-R}^{R} ak_{\bar{\imath},i}(s)\mathrm{d}s + \int_R^t ak_{\bar{\imath},i}(s)\mathrm{d}s$$

while the function $k^M(\cdot)$ satisfies, for $t \in (R, 2R]$,

$$k^M(t) = \int_{t-R}^{R} ak^M(s)\mathrm{d}s + \int_R^t ak^M(s)\mathrm{d}s$$

Since from the first step we know that $k_{\bar{\imath},i}(t) \le k^M(t)$ for $t \in [0, R]$ then we have, calling $g(t) = \int_{t-R}^{R} ak^M(s)\mathrm{d}s$ for $t \in (R, 2R]$:

$$k_{\bar{\imath},i}(t) \le g(t) + \int_R^t ak_{\bar{\imath},i}(s)\mathrm{d}s$$

$$k^M(t) = g(t) + \int_R^t ak^M(s)\mathrm{d}s$$

and then the Gronwall inequality gives the claim for $t \in (R, 2R]$. The claim for every $t \ge 0$ follows by an induction argument on the same line of the above steps. $\square$

Observe now that, by its definition, $k^M(\cdot)$ is the unique solution of

$$\begin{cases} \dot{k^M}(t) = \tilde{\imath}^M(t) - \tilde{\imath}^M(t-R) \\ \tilde{\imath}^M(s) = \bar{\imath}(s) \qquad for\ s \in [-R, 0) \\ k^M(0) = \int_{-R}^{0} \bar{\imath}(r)\mathrm{d}r > 0 \end{cases} \tag{7.12}$$

and then for $t \ge R$, $k^M(t) = h(t)$ where $h(\cdot)$ the unique solution of

$$\begin{cases} \dot{h}(t) = a(h(t) - h(t-R))\ for\ t \ge R \\ h(s) = k^M(s)\ for\ s \in [0, R) \end{cases} \tag{7.13}$$

For equation (7.13) we can apply standard statements on Delay Differential Equations as follows.

We define the *characteristic equation* of the Delay Differential Equation (7.13) as

$$z = a(1 - e^{-zR}), \qquad z \in \mathbb{C} \tag{7.14}$$

The characteristic equation is defined for general linear Delay Differential Equations as described in [DVGVLW95] (page 27). In our case, by a convexity argument, we can easily prove the following result.

Proposition 7.4. *There exists exactly one strictly positive root of (7.14) if and only if* $aR > 1$. *Such root* $\xi$ *belongs to* $(0, a)$. *If* $aR \le 1$ *then the only root with non negative real part is* $z = 0$.

Since, as we will see in Proposition 7.6 and Remark 7.7, the maximum characteristic root give the maximum rate of growth of the solution, to rule out the cases where growth cannot occur it is natural to require the following.

HYPOTHESIS 7.5. $aR > 1$.

We will assume from now on that Hypothesis 7.5 holds. Assuming Hypothesis 7.5 we have

$$\begin{array}{lcl} g \in (0,\xi) & \Longrightarrow & g < a(1-e^{-gR}) \\ g \in (-\infty,0)\cup(\xi,+\infty) & \Longrightarrow & g > a(1-e^{-gR}). \end{array} \tag{7.15}$$

PROPOSITION 7.6. *Let Hypothesis 7.5 hold true. Given an initial datum* $\bar\iota(\cdot) \in L^2((-R,0);\mathbb{R}^+)$ *with* $\bar\iota(\cdot) \not\equiv 0$ *and a control* $i(\cdot) \in L^2_{loc}([0,+\infty);\mathbb{R}^+)$, *we have that for every* $\varepsilon > 0$

$$\lim_{t\to+\infty} \frac{k^M(t)}{e^{(\xi+\varepsilon)t}} = 0.$$

PROOF. First we observe that being $\bar\iota(\cdot) \not\equiv 0$, $k^M(t)$ is strictly positive for each $t \geq 0$. To prove this it is enough to observe that, for $t \geq 0$,

$$k^M(t) = \int_{(t-R)\wedge 0}^{0} \bar\iota(s)\mathrm{d}s + \int_{(t-R)\vee 0}^{t} ak^M(s)\mathrm{d}s$$

and to argue by contradiction. Moreover, as we said above, for $t \geq R$, $k^M(t) = h(t)$ where $h(\cdot)$ the unique solution of (7.13). Now the solution $h(t)$ of (7.13) is continuous on $[R,+\infty)$ (see [BDPDM92] page 207). Moreover (see [DVGVLW95] page 34) there exist at most $N < +\infty$ (complex) roots $\{\lambda_j\}_{j=1}^N$ of the characteristic equation with real part exceeding $\xi$ and there exist $\{p_j\}_{j=1}^N$ $\mathbb{C}$-valued polynomial such that

$$h(t) = o(e^{(\xi+\varepsilon)t}) + \sum_{j=1}^{N} p_j(t)e^{\lambda_j t} \qquad \textit{for } t \to +\infty \tag{7.16}$$

for every $\varepsilon > 0$. Since $k^M(t)$ and so $h(t)$ remain strictly positive for all $t \geq R$, then all the $p_j$ vanish. So we have proved the claim. □

REMARK 7.7 (**On the Hypothesis 7.5**). Hypothesis 7.5 has a clear economic meaning: if there are no strictly positive root we can see, as in Proposition 7.3, that the maximal growth of the capital stock[4] is not positive since the the stock of capital always goes to zero. So positive growth would be excluded from the beginning. Moreover Hypothesis 7.5 is verified when we take the limit of the model as $R$ goes to $+\infty$ which is "substantially" the standard AK model with zero depreciation rate of capital. In this case we will have $\xi \to a$. ■

The above Proposition 7.6 is what we need to analyze the finiteness of the value function. Before to proceed with it we give a refinement of Proposition 7.4 that give a more detailed analysis of the solutions of characteristic equation (7.14) and so of the solution of equation (7.13) that will be useful later, see the proof of Proposition 7.36 and Subsection 7.6.1.1.

PROPOSITION 7.8. *Assuming Hypothesis 7.5 we can state that:*

(a) *The characteristic equation (7.14) has only simple roots.*
(b) *There are exactly 2 real roots of (7.14), i.e.* $\xi$ *and* 0.

[4]That occurs re-investing all capital.

(c) *There is a sequence* $\{\lambda_k,\ k=1,2,...\}\subset\mathbb{C}$ *such that* $\{\lambda_k,\overline{\lambda}_k\ k=1,2,...\}$ *are the only complex and non real roots of (7.14).*

*For each* $k$ *we have* $R\cdot Im\lambda_k\in(2k\pi,(2k+1)\pi)$.

*The real sequence* $\{Re\lambda_k,\ k=1,2,...\}$, *is strictly negative and strictly decreasing to* $-\infty$. *Finally*

$$Re\lambda_1<\xi-a. \tag{7.17}$$

PROOF. First of all we observe that $z$ is a root of (7.14) if and only if $w=zR$ is a root of

$$w=aR-aRe^{-w}. \tag{7.18}$$

Then it is enough to apply Theorem 3.2 p. 312 and Theorem 3.12 p.315 of [DVGVLW95]. The only statements which are not contained there are the fact that $Re\lambda_k\to-\infty$ as $k\to+\infty$ and the inequality (7.17). To see the first observe that, from (7.18) it follows, calling $\mu_k=R\cdot Re\lambda_k$ and $\nu_k=R\cdot Im\lambda_k$

$$aRe^{-\mu_k}\sin\nu_k=\nu_k\quad\Longrightarrow\quad e^{-\mu_k}>\frac{\nu_k}{aR}$$

and the claim is proved since $\nu_k\to+\infty$ as $k\to+\infty$. The proof of inequality (7.17) uses elementary arguments but it is a bit long so we give only a sketch of it. First of all by (7.14) we get that, when $aR>1$

$$\xi>a\left(1-\frac{1}{(aR)^2}\right) \tag{7.19}$$

while, for $aR>5$

$$\xi>a\left(1-\frac{1}{(aR)^3}\right). \tag{7.20}$$

Moreover using (7.18) we get that

$$e^{-2\mu_1}-(a-\mu_1)^2=\nu_1^2>4\pi^2.$$

Now the function $h(\mu)=e^{-2\mu}-(a-\mu)^2$ is strictly decreasing on $(-\infty,0)$ and using (7.19) and (7.20) we get that $h(\xi R-aR)<4\pi^2$. This gives $\mu_1<\xi R-aR$ and so the claim. □

**7.3.2. Finiteness of the value function.** We now introduce the following assumption that, given Hypothesis 7.5 will be a sufficient condition for the finiteness of the value function for every initial datum[5].

HYPOTHESIS 7.9. $\rho>\xi(1-\sigma)$.

From now on we will assume that Hypotheses 7.5 and 7.9 hold. Now, thanks to Proposition 7.3 and Hypothesis 7.9 we can exclude two opposite cases: on one hand, when $\sigma<1$, the existence of some $\bar\iota(\cdot)$ in which $V(\bar\iota(\cdot))=+\infty$ (Corollary 7.10), on the other hand, when $\sigma>1$, the existence of some $\bar\iota(\cdot)$ in which $V(\bar\iota(\cdot))=-\infty$ (Corollary 7.12).

COROLLARY 7.10. $V(\bar\iota(\cdot))<+\infty$ *for all* $\bar\iota(\cdot)$ *in* $L^2((-R,0);\mathbb{R}^+)$.

PROOF. For $\sigma>1$ it is true since $J(\bar\iota(\cdot);i(\cdot))\le0$ always. For $\sigma\in(0,1)$ we observe that for every $i(\cdot)\in L^2_{loc}([0,+\infty);\mathbb{R}^+)$,

$$J(\bar\iota(\cdot);i(\cdot))\le\frac{1}{1-\sigma}\int_0^{+\infty}e^{-\rho t}(ak_{\bar\iota,i}(t))^{1-\sigma}\mathrm{d}t\le\frac{1}{1-\sigma}\int_0^{+\infty}e^{-\rho t}(ak^M(t))^{1-\sigma}\mathrm{d}t.$$

[5]Indeed in the standard AK model with zero depreciation rate of capital such condition with $\xi=a$ is also necessary, see e.g. [FGS06]. In our case a similar result can be proved but we avoid it for simplicity.

so from the definition of the value function, Proposition 7.3 and Hypothesis 7.9, the claim follows. □

LEMMA 7.11. *Given any initial datum $\bar{\imath}(\cdot) \in L^2((-R,0);\mathbb{R}^+)$, $\bar{\imath}(\cdot) \not\equiv 0$ there exists an $\varepsilon > 0$ and an admissible control strategy $i(\cdot)$ such that $i(t) = \varepsilon$ for all $t \geq R$. Moreover there exists a $\delta > 0$ such that the control defined by the feedback formula $i_\delta(t) = ak_\delta(t) - \delta$ for all $t \geq 0$ is admissible and $i_\delta(t) \geq \delta > 0$ for all $t \geq 0$.*

PROOF. **The idea:** We give a constructive proof in four steps. We first find a small $\alpha > 0$ and a $\beta < R$ such that the (constant) control $i(t) = \varepsilon_1 \stackrel{def}{=} \frac{a\alpha\beta}{4}$ is admissible in the interval $(0, \frac{\beta}{4})$; then we see that such control can be lengthened defining, on the interval $[\frac{\beta}{4}, R - \frac{\beta}{4})$, $i(t) = \varepsilon_2 \stackrel{def}{=} \min\{\frac{a^2\alpha\beta^2}{32}, \varepsilon_1\}$. Furthermore we prove that we can extend such control on $[R - \frac{\beta}{4}, R)$ putting $i(t) = \varepsilon \stackrel{def}{=} \min\{\frac{a\varepsilon_2}{2}\frac{R}{4}, \varepsilon_2\}$. Observe that in view of the "minima" in the definitions of $\varepsilon_1, \varepsilon_2, \varepsilon$, the control is decreasing on $[0, R)$. Eventually (fourth step) we see that on the interval $[R, +\infty)$ we can put our control constantly $\varepsilon$. The statement for $\delta$ follows from this construction.

**The proof:**

first step: In view of the fact that $\bar{\imath}(\cdot) \not\equiv 0$ we can choose a positive number $\alpha$ such that

$$\beta \stackrel{def}{=} m(\{s \in (-R, 0) \ \ s.t. \ \ \bar{\imath}(s) \geq \alpha\} > 0$$

where $m$ is the Lebesgue measure. So

$$\int_{t-R}^{0} \bar{\imath}(s)\mathrm{d}s \geq \frac{\alpha\beta}{2}$$

for all $t \in (0, \frac{\beta}{2})$ and in particular it is true for $t \in (0, \frac{\beta}{4})$. Now for $t \in [0, \frac{\beta}{4})$ we can put $i(t) = \varepsilon_1 \stackrel{def}{=} \frac{a\alpha\beta}{4} > 0$ obtaining that

$$a\int_{t-R}^{t} \tilde{\imath}(s)\mathrm{d}s \geq a\int_{-R+\frac{\beta}{4}}^{0} \bar{\imath}(s)\mathrm{d}s \geq a\alpha\frac{\beta}{2} > a\alpha\frac{\beta}{4} = i(t)$$

so the strategy is admissible on $[0, \frac{\beta}{4})$. For such choice of $i(t)$ we have $ak(t) - i(t) \geq \frac{a\alpha\beta}{4}$ for $t \in [0, \frac{\beta}{4})$.

second step: Choosing $i(\cdot)$ in the interval $[0, \frac{\beta}{4})$ as in the first step, and for $t \in [\frac{\beta}{4}, R - \frac{\beta}{4})$, $i(t) = \varepsilon_2 \stackrel{def}{=} \min\{\frac{a^2\alpha\beta^2}{32}, \varepsilon_1\} > 0$ (in view of the previous integral such constant is in the range of admissible control for all $t$ in the interval) we have that for all $t \in [\frac{\beta}{4}, R - \frac{\beta}{4})$

$$a\int_{t-R}^{t} \tilde{\imath}(s)\mathrm{d}s \geq a\int_{0}^{t} i(s)\mathrm{d}s \geq a\frac{\beta}{4}\frac{a\alpha\beta}{4} = \frac{a^2\alpha\beta^2}{16} > a^2\alpha\frac{\beta^2}{32} \geq i(t)$$

so the strategy is admissible on $[\frac{\beta}{4}, R - \frac{\beta}{4})$. For such choice of $i(t)$ we have $ak(t) - i(t) \geq \frac{a\alpha\beta^2}{32}$ for $t \in [\frac{\beta}{4}, R - \frac{\beta}{4})$.

third step: In particular we have put $i(t) = \varepsilon_2 > 0$ for $t \in (R/2, R - \frac{\beta}{4})$ and so, with a step similar to the previous one, we can put $i(t) = \varepsilon \stackrel{def}{=} \min\{\frac{a\varepsilon_2}{2}\frac{R}{4}, \varepsilon_2\} > 0$ for $t \in [R - \frac{\beta}{4}, 0)$ and we have on such interval $ak(t) - i(t) \geq \frac{\varepsilon_2}{2}\frac{R}{4}$.

fourth step: In view of the "minima" in the definition of $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon$ we have that $\varepsilon \leq \varepsilon_2 \leq \varepsilon_1$ and that $i(t) \geq \varepsilon$ in the interval $[0, R)$. So, choosing $i(t) = \varepsilon$ for all $t \geq R$, we get an admissible control, indeed $\varepsilon > 0$ and

$$a\int_{t-R}^{t} i(s)\mathrm{d}s \geq aR\varepsilon > \varepsilon$$

(the last follows by (H1)), for all $t \geq R$. We have that, on $[R, \infty)$, $ak(t) - i(t) \geq \frac{aR-1}{2}\varepsilon$.

The second statement, related to the $\delta$ constant, follows from the previous proof and from the observation we have done during the proof with respect to the term $ak(t) - i(t)$. If we consider the strategy of the previous proof we have that

$$ak(t) - i(t) \geq \delta \stackrel{def}{=} \min\{\varepsilon, \frac{aR-1}{2}\varepsilon\}$$

and $i(t) \geq \delta$ for all $t \geq 0$. Now if we consider such $\delta$ the strategy given by the feedback formula $i_\delta(t) = ak_\delta(t) - \delta$ satisfies the inequality $i_\delta(t) > i(t)$ (where $i(\cdot)$ is the strategy defined in first, second and third steps) for all $t \geq 0$ arguing as in the proof of the first statement of Proposition 7.3. Then we get that $i_\delta(t) \geq i(t) \geq \delta$ for all $t \geq 0$ so it is admissible and the claim is proved. □

Corollary 7.12. *If $\bar{\iota}(\cdot) \in L^2((-R, 0); \mathbb{R}^+)$ and $\bar{\iota}(\cdot) \not\equiv 0$ there exists a control $\theta(\cdot) \in \mathcal{I}_{\bar{\iota}}$ such that $J(\bar{\iota}(\cdot); \theta(\cdot)) > -\infty$.*

Proof. It is sufficient to take the control $i_\delta(t)$ s.t. $ak(t) - i_\delta(t) = \delta > 0$ found in previous lemma. □

**7.3.3. Existence of optimal strategies.** We now state and prove the existence of optimal paths.

Proposition 7.13. *An optimal control exists in $\mathcal{I}_{\bar{\iota}}$, i.e. we can find in $\mathcal{I}_{\bar{\iota}}$ an admissible strategy $i^*(\cdot)$ such that $V(\bar{\iota}(\cdot)) = J(\bar{\iota}(\cdot); i^*(\cdot))$.*

Proof. The proof is a simple application of a direct method (see also [BLPdR05]. We will adapt the scheme of Askenazy and Le Van in [ALV99] to our formulation.

We will indicate with $\mu$ the measure on $\mathbb{R}^+$ given by $\mathrm{d}\mu(t) = e^{(-\varepsilon-\xi)t}\mathrm{d}t$ where $\mathrm{d}t$ is the Lebesgue measure and $\varepsilon > 0$ is fixed. By $L^1(0, +\infty; \mathbb{R}; \mu)$, or simply $L^1(\mu)$ we will denote the space of all Lebesgue measurable functions that are integrable with respect to $\mu$.

We consider $\mathcal{I}_{\bar{\iota}}$ as subset of $L^1(\mu)$.

We know that on a space of finite measure $\mu$ a subset $G$ of $L^1(\mu)$ is relatively (sequentially) compact for the weak topology if and only if: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every set $I$ with $\mu(I) < \delta$ and for all $f \in G$ we have $\int_I f(x)\mathrm{d}\mu(x) < \varepsilon$ (this property is also known as *Dunford - Pettis criterion* see for example [DS66] page 294 Corollary 11). In our case constraint (7.7), Proposition 7.3 and Proposition 7.6 guarantee such property for $\mathcal{I}_{\bar{\iota}}$.
We choose now a maximizing sequence $i_n(\cdot) \in \mathcal{I}_{\bar{\iota}}$; thanks to the Dunford-Pettis criterion we can we can find a subsequence $i_{n_m}(\cdot) \in L^1(\mu)$ and $i^*(\cdot) \in L^1(\mu)$ such that $i_{n_m}(\cdot) \rightharpoonup i^*(\cdot) \in L^1(\mu)$. The functional

$$J(\bar{\iota}(\cdot); \cdot) : L^1(\mu) \supseteq \mathcal{I}_{\bar{\iota}} \to \overline{\mathbb{R}},$$

that brings any $i(\cdot) \in L^1(\mu)$ to $J(\bar{\iota}(\cdot); i(\cdot))$, is concave and so it is weakly upper semicontinuous on $L^1(\mu)$ and so $J(\bar{\iota}(\cdot); i^*(\cdot)) \geq \limsup_{m\to\infty} J(\bar{\iota}(\cdot); i_{n_m}(\cdot))$.
It remain to show that $i^*(\cdot) \in \mathcal{I}_{\bar{\iota}}$, i.e. $i^*(\cdot)$ satisfies the constraints (7.7). For the positivity constraint $i_{n_m}(\cdot) \rightharpoonup i^*(\cdot)$ and $i_{n_m}(\cdot) \geq 0$ imply $i^*(\cdot) \geq 0$ since non-negativity constraints are preserved under weak convergence. Concerning the other constraint we observe that, thanks to (7.10) we know that $k_{\bar{\iota}, i_{n_m}}(\cdot) \to k_{\bar{\iota}, i^*}(\cdot)$ uniformly on the compact sets and so $ak_{\bar{\iota}, i^*}(t) \geq i^*(t)$ almost everywhere. This also implies that $i^*(\cdot) \in L^2_{loc}([0, +\infty); \mathbb{R}^+)$. □

**7.3.4. Strict positivity of optimal trajectories.** We can now prove the strict positivity of optimal trajectories that we will use in Section 7.5. We have already proved the strict positivity of the capital path $k^M$ in the proof of Proposition 7.6.

LEMMA 7.14. *Let $\bar\imath(\cdot)$ be in $L^2((-R,0);\mathbb{R}^+)$ and $\bar\imath(\cdot)\not\equiv 0$ and let $i^*(\cdot)\in\mathcal{I}_{\bar\imath}$ be an optimal strategy then $k_{\bar\imath,i^*}(t)>0$ for all $t\in[0,+\infty)$.*

PROOF. For simplicity we will drop the $*$ writing $i(\cdot)$ instead of $i^*(\cdot)$ along this proof. If there exist $\bar t\in(0,+\infty)$ such that $k_{\bar\imath,i}(\bar t)=0$ then by (7.10) (and a simple Gronwall-type argument) $k_{\bar\imath,i}(t)=0$ for all $t\geq\bar t$.
So if $\sigma>1$ the statement is a consequence of Corollary 7.12.

Then suppose that ($\sigma<1$) and that there exist a first $\bar t>0$ such that $k_{\bar\imath,i}(\bar t)=0$. We assume that such $\bar t$ is greater than $R/2$ but this imposition can be easily overcome (indeed noting that $\bar t>0$ we can choose $n\in\mathbb{N}$ such that $\bar t>R/n$ and proceed in a similar way).

We note that $k_{\bar\imath,i}(\bar t)=0$ implies $i=0$ in the set $[\bar t-R,\bar t]$.

Thanks to the fact that $k_{\bar\imath,i}(t)$ is positive and continuous until $\bar t$ and that $i=0$ (or $\tilde\imath=0$) in the set $[\bar t-R,\bar t]$ we can say that exist $\varepsilon>0$ such that the measure of the set

$$\Theta^\varepsilon\overset{def}{=}\{t\in[\bar t-R/2,\bar t]\ :\ ak_{\bar\imath,i}-i(t)>\varepsilon\}$$

is strictly positive (for the Lebesgue measure $m$): let be $h=m(\Theta^\epsilon)>0$. We choose $\varrho<\varepsilon$ and define the new strategy $i_\varrho(\cdot)$:

$$i_\varrho(t)=\begin{cases} i(t)+\varrho & =\varrho \quad for\ t\in\Theta^\varepsilon\\ i(t) & \quad\quad otherwise\end{cases}$$

From the choice of $\Theta^\varepsilon$ and $\varrho$ we obtain that $i_{\varepsilon,\varrho}(\cdot)$ is in $\mathcal{I}_{\bar\imath}$. The following estimate is valid:

$$J(\bar\imath(\cdot);i_\varrho(\cdot))=I_1+I_2+I_3+I_4\overset{def}{=}\int_0^{\bar t-R/2}e^{-\rho t}\frac{(ak_{\bar\imath,i}(t)-i(t))^{1-\sigma}}{1-\sigma}\mathrm{d}t+$$

$$+\int_{([\bar t-R/2,\bar t]-\Theta^\varepsilon)}e^{-\rho t}\frac{(ak_{\bar\imath,i_\varrho}(t)-i(t))^{1-\sigma}}{1-\sigma}\mathrm{d}t+$$

$$+\int_{\Theta^\varepsilon}e^{-\rho t}\frac{(ak_{\bar\imath,i_\varrho}(t)-i_\varrho(t))^{1-\sigma}}{1-\sigma}\mathrm{d}t+\int_{\bar t}^{\bar t+R}e^{-\rho t}\frac{(ak_{\bar\imath,i_\varrho}(t)-i_\varrho(t))^{1-\sigma}}{1-\sigma}\mathrm{d}t.$$

Moreover we have the following estimates (we use that $i=0$ on the set $[\bar t-R,\bar t]$):

$$I_2\geq I_2'\overset{def}{=}\int_{([\bar t-R/2,\bar t]-\Theta^\varepsilon)}e^{-\rho t}\frac{(ak_{\bar\imath,i}(t))^{1-\sigma}}{1-\sigma}\mathrm{d}t$$

$$I_3\geq\int_{\Theta^\varepsilon}e^{-\rho t}\frac{(ak_{\bar\imath,i}(t)-\varrho)^{1-\sigma}}{1-\sigma}\mathrm{d}t\geq\ (linearizing)$$

$$\geq I_3^1-I_3^2\overset{def}{=}\int_{\Theta^\varepsilon}e^{-\rho t}\frac{(ak_{\bar\imath,i}(t))^{1-\sigma}}{1-\sigma}-\int_{\Theta^\varepsilon}e^{-\rho t}\varepsilon^{-\sigma}\varrho\mathrm{d}t+o(\varrho).$$

Furthermore:

$$I_4\geq\int_{\bar t}^{\bar t+R/2}e^{-\rho t}\frac{(ak_{\bar\imath,i_\varrho}(t)-i_\varrho(t))^{1-\sigma}}{1-\sigma}\mathrm{d}t\geq\int_{\bar t}^{\bar t+R/2}e^{-\rho t}\frac{(ah\varrho)^{1-\sigma}}{1-\sigma}\mathrm{d}t$$

So $I_3^2=a_1\varrho$ and $I_4\geq a_2\varrho^{1-\sigma}$ where $a_1$ and $a_2$ are positive constants independent by $\varrho$. Summarizing:

$$J(\bar\imath(\cdot);i_{\varepsilon,\varrho}(\cdot))\geq(I_1+I_2'+I_3^1)+(-I_3^2+I_4)\geq J(\bar\imath(\cdot);i(\cdot))+(-a_1\varrho+a_2\varrho^{1-\sigma})+o(\varrho)$$

so for $\varrho$ small enough we have $J(\bar{\imath}(\cdot); i_{\varrho}(\cdot)) > J(\bar{\imath}(\cdot); i(\cdot))$ and this is a contradiction. □

## 7.4. Writing and solving the infinite dimensional problem

**7.4.1. Rewriting *Problem (P)* in infinite dimensions.** Some of the definitions and results presented in this subsection was already introduced in Section 1.3. We choose to rewrite them for reader convenience.

Given any $t \geq 0$ we indicate the " history" of investments at time $t$ with $\tilde{\imath}_t$ which is defined as follows:

$$\begin{cases} \tilde{\imath}_t \colon [-R, 0] \to \mathbb{R} \\ \tilde{\imath}_t(s) = \tilde{\imath}(t+s) \end{cases} \tag{7.21}$$

The capital stock can then be rewritten as

$$k(t) = \int_{-R}^{0} \tilde{\imath}_t(s) \mathrm{d}s$$

and so the Delay Differential Equation (7.9) can be rewritten as

$$\begin{cases} \dot{k}(t) = B(\tilde{\imath}_t) \\ (k(0), \tilde{\imath}_0) = (\int_{-R}^{0} \bar{\imath}(s)\mathrm{d}s,\ \bar{\imath}) \end{cases} \tag{7.22}$$

where B is the continuous linear map

$$\begin{cases} B \colon C([-R, 0]) \to \mathbb{R} \\ B(f) = f(0) - f(-R) \end{cases}$$

Equation (7.22) has a pointwise meaning only if the control is continuous but always has an integral sense (as in (7.10)).
The link between the initial condition for $k(t)$ and $\tilde{\imath}_t$ (that is $k(0) = \int_{-R}^{0} \tilde{\imath}_0(s)\mathrm{d}s$) has a clear economic meaning but is, so to speak, nonstandard from a mathematical point of view. We " suspend" it in this section and will reintroduce it in section 7.5 when we will find the optimal feedback for *Problem (**P**)*. So we consider now initial data given by $(k_0, \bar{\imath})$ where $k_0$ and $\bar{\imath}$ have no relationship. Our problem becomes a bit more general:

$$\begin{cases} \dot{k}(t) = B(\tilde{\imath}_t) \\ (k(0), \tilde{\imath}_0) = (k_0,\ \bar{\imath}) \end{cases} \tag{7.23}$$

Its solution is

$$k_{k_0, \bar{\imath}, i}(t) = k_0 - \int_{-R}^{0} \bar{\imath}(s)\mathrm{d}s + \int_{t-R}^{t} \tilde{\imath}(s)\mathrm{d}s \tag{7.24}$$

Clearly for every $t \geq 0$, $k_{\int_{-R}^{0} \bar{\imath}(s)\mathrm{d}s, \bar{\imath}, i}(t) = k_{\bar{\imath}, i}(t)$ as defined in equation (7.10). Now we introduce the infinite dimensional space in which we re-formulate the problem, it is:

$$M^2 \stackrel{def}{=} \mathbb{R} \times L^2(-R, 0)$$

A generic element $x$ of $M^2$ will be denoted as a couple $(x^0, x^1)$. The scalar product on $M^2$ will be the one on a product of Hilbert spaces i.e.:

$$\langle (x^0, x^1), (z^0, z^1) \rangle_{M^2} \stackrel{def}{=} x^0 z^0 + \langle x^1, z^1 \rangle_{L^2}$$

for every $(x^0, x^1), (z^0, z^1) \in M^2$. Now we introduce the operator $A^*$ on $M^2$:

$$\begin{cases} D(A^*) \stackrel{def}{=} \{(\psi^0, \psi^1) \in M^2 \ : \ \psi^1 \in H^1(-R, 0),\ \psi^0 = \psi^1(0)\} \\ A^* \colon D(A^*) \to M^2 \\ A^*(\psi^0, \psi^1) \stackrel{def}{=} (0, \frac{\mathrm{d}}{\mathrm{d}s}\psi^1) \end{cases}$$

Abusing of notation it is also possible to confuse, on $D(A^*)$, $\psi^1(0)$ with $\psi^0$ and redefine

$$\begin{cases} B\colon D(A^*) \to \mathbb{R} \\ B(\psi(0),\psi) = B\psi = \psi(0) - \psi(-R) \in \mathbb{R} \end{cases}$$

NOTATION 7.15. *We will indicate with $F$ the application*

$$F\colon L^2(-R,0) \to L^2(-R,0)$$
$$z \mapsto F(z)$$

*where*

$$F(z)(s) \stackrel{def}{=} -z(-R-s) \tag{7.25}$$

*and with $S$ the application*

$$S\colon L^2(-R,0) \to \mathbb{R}$$
$$S\colon z \mapsto \int_{-R}^{0} z(s)\mathrm{d}s$$

DEFINITION 7.16 (**Structural state**). *Given initial data $(k_0, \bar{\iota})$ we set for simplicity $y = (k_0, F(\bar{\iota})) \in M^2$ (that will be the initial datum in the Hilbert setting).*

*Given $\bar{\iota} \in L^2((-R,0))$, $i \in L^2_{loc}[0,+\infty)$, $k_0 \in \mathbb{R}$ and $k_{k_0,\bar{\iota},i}(t)$ as in (7.24) we define the* structural state *of the system the couple $x_{y,i}(t) = (x^0_{y,i}(t), x^1_{y,i}(t)) \stackrel{def}{=} (k_{k_0,\bar{\iota},i}(t), F(\tilde{\iota}_t))$. In view of what we have said $x^0_{y,i}(t) \in \mathbb{R}$ and $x^1_{y,i}(t) \in L^2((-R,0);\mathbb{R})$ and so $x_{y,i}(t) \in M^2$*

THEOREM 7.17. *Assume that $\bar{\iota} \in L^2((-R,0)$, $i \in L^2_{loc}[0,+\infty)$, $k_0 \in \mathbb{R}$ $y = (k_0, F(\bar{\iota}))$, then, for every $T > 0$, the structural state $x_{y,i}(t) = (x^0_{y,i}(t), x^1_{y,i}(t)) = (k_{k_0,\bar{\iota},i}(t), F(\tilde{\iota}_t))$ is the unique solution in*

$$\Pi \stackrel{def}{=} \left\{ f \in C([0,T];M^2) \ : \ \frac{d}{\mathrm{d}t} j^* f \in L^2((0,T);D(A^*)') \right\} \tag{7.26}$$

*to the equation:*

$$\begin{cases} \frac{d}{\mathrm{d}t}x(t) = Ax(t) + B^*i(t), \quad t > 0 \\ x(0) = y = (k_0, F(\bar{\iota})) \end{cases} \tag{7.27}$$

*where $j^*$, $A$ and $B^*$ are the dual maps of the continuous linear operators*

$$j\colon D(A^*) \hookrightarrow M^2,$$
$$A^*\colon D(A^*) \to M^2,$$
$$B\colon D(A^*) \to \mathbb{R}.$$

*Here $j$ is simply the embedding, $D(A^*)$ is equipped with the graph norm and $D(A^*)'$ is the topological dual of $D(A^*)$.*

PROOF. We have already see this theorem in the general case in Theorem 1.33. The proof can be found in [BDPDM92] Theorem **5.1** page 258. □

REMARK 7.18 (**On the adjoint of the operators $A^*$ and $B$**). $A$ is the adjoint of the linear operator $A^*$ and so it is linear and continuous from $M^2$ to $D(A^*)' = \mathcal{L}(D(A^*),\mathbb{R})$. The explicit expression of $A(\psi^0,\psi^1)$ for the the couples in which $\psi^1$ is differentiable is

$$A(\psi^0,\psi^1)[(\varphi^0,\varphi^1)] = \psi^1(0)\varphi^1(0) - \psi^1(-R)\varphi^1(-R) -$$
$$- \int_{-R}^{0} \frac{d}{\mathrm{d}s}\psi^1(s)\varphi^1(s)\mathrm{d}s \quad \forall(\varphi^0,\varphi^1) \in D(A^*)$$

Endowing $D(A^*)$ with the graph norm we get that $A$ is continuous and can be extended on all $M^2$ by density.
The expression for $B^*$ is simpler and it is

$$\begin{cases} B^* : \mathbb{R} \to D(A^*)' \\ B^* i = i(\delta_0 - \delta_{-R}) \end{cases}$$

Here $\delta_0$ and $\delta_{-R}$ are the Dirac deltas in 0 and $-R$ respectively and they are elements of $D(A^*)'$. ■

We want to formulate an optimal control problem in infinite dimensions that, thanks to results of the previous section, "contains" the *Problem (**P**)*. To do this we need first the following result that extends the existence and uniqueness results of the previous Theorem 7.17.

THEOREM 7.19. *The equation*

$$\begin{cases} \frac{d}{dt} x(t) = Ax(t) + B^* i(t), \quad t > 0 \\ x(0) = y \end{cases}$$

*for $y \in M^2$, $i \in L^2_{loc}[0, +\infty)$ has a unique solution in $\Pi$ (defined in (7.26)).*

PROOF. We have already seen this theorem in the general case in Theorem 1.33. The proof can be found in [BDPDM92] Theorem **5.1** page 258. □

Now we can formulate our optimal control problem in infinite dimensions. The state space is $M^2$, the control space is $\mathbb{R}$, the time is continuous. The state equation in $M^2$ is given by

$$\begin{cases} \frac{d}{dt} x(t) = Ax(t) + B^* i(t), \quad t > 0 \\ x(0) = y \end{cases} \tag{7.28}$$

for $y \in M^2$, $i \in L^2_{loc}[0, +\infty)$. Thanks to Theorem 7.19 it has a unique solution $x_{y,i}(t)$ in $\Pi$, so $t \mapsto x^0_{y,i}(t)$ is continuous and it makes sense to consider the set of controls

$$\mathcal{I}_y \stackrel{def}{=} \{ i \in L^2_{loc}([0, +\infty); \mathbb{R}^+) \ : \ i(t) \in [0, a x^0_{y,i}(t)] \ \text{ for a.e. } t \in \mathbb{R}^+ \}$$

We define the objective functional as

$$J_0(y; i) \stackrel{def}{=} \int_0^\infty e^{-\rho s} \frac{(a x^0_{y,i}(t) - i(t))^{1-\sigma}}{(1-\sigma)} \mathrm{d}s$$

The value function is then:

$$\begin{cases} V_0(y) \stackrel{def}{=} \sup_{i \in \mathcal{I}_y} \left\{ \int_0^\infty e^{-\rho s} \dfrac{(a x^0_{y,i}(t) - i(t))^{1-\sigma}}{(1-\sigma)} \mathrm{d}s \right\}, & if \ \mathcal{I}_y \neq \emptyset \\ V_0(y) \stackrel{def}{=} -\infty, & if \ \mathcal{I}_y = \emptyset \end{cases}$$

REMARK 7.20 (**Connection with the starting problem**). If we have for some $\bar{\imath}(\cdot) \in L^2((-R, 0); \mathbb{R}^+)$

$$y = \big( S(\bar{\imath}), F(\bar{\imath}) \big)$$

we find $\mathcal{I}_y = \mathcal{I}_{\bar{\imath}}$, $J_0(y; i) = J(\bar{\imath}; i)$ and $V_0(y) = V(\bar{\imath})$ and the solution of the differential equation (7.28) is given by Definition 7.16 as given in Theorem 7.17. ■

**7.4.2. The *HJB* equation and its explicit solution.** We now describe the Hamiltonians of the system. First of all we introduce the *current value* Hamiltonian: it will be defined on a subset $E$ of $M^2 \times M^2 \times \mathbb{R}$ (the product of state space, co-state space and control space) given by[6]

$$E \stackrel{def}{=} \{(x, P, i) \in M^2 \times M^2 \times \mathbb{R} \ : x^0 > 0,\ i \in [0, ax^0],\ P \in D(A^*)\}$$

and its form is the following: ($\langle i, BP\rangle_{\mathbb{R}}$ is simply the product on $\mathbb{R}$):

$$H_{CV}(x, P, i) \stackrel{def}{=} \langle x, A^* P\rangle_{M^2} + \langle i, BP\rangle_{\mathbb{R}} + \frac{(ax^0 - i)^{1-\sigma}}{(1-\sigma)}$$

When $\sigma > 1$ the above is not defined in the points in which $ax^0 = i$. In such points we set then $H_{CV} = -\infty$. In this way we take $H_{CV}$ with values in $\overline{\mathbb{R}}$.

We can now define the maximum value Hamiltonian (that we will simply call Hamiltonian) of the system: we name $G$ the subset of $M^2 \times M^2$ (the product of state space and co-state space) given by:

$$G \stackrel{def}{=} \{(x, P) \in M^2 \times M^2 \ : \ x^0 > 0,\ P \in D(A^*)\}$$

The Hamiltonian is given by:

$$\begin{cases} H : G \to \overline{\mathbb{R}} \\ H : (x, P) \mapsto \sup_{i \in [0, ax^0]} H_{CV}(x, P, i) \end{cases}$$

The *HJB* equation is

$$\rho w(x) \ - H(x, Dw(x)) = 0$$

i.e.

$$\rho w(x) \ - \sup_{i \in [0, ax^0]} \left\{ \langle x, A^* Dw(x)\rangle_{M^2} + \langle i, BDw(x)\rangle_{\mathbb{R}} + \frac{(ax^0 - i)^{1-\sigma}}{(1-\sigma)} \right\} = 0 \tag{7.29}$$

Now we give the definition of solution of the HJB equation.

Remark 7.21 (**On the definition of solution of the HJB equation**). As we have already noted the HJB equation (7.29) cannot be treated with the results of the existing literature. This is due to the presence of the state/control constraint (i.e. the investments that are possible at time $t \geq 0$ depend on $k$ at the same time $t$: $i(t) \in [0, ak(t)]$), to the unboundedness of the control operator (i.e. the term $BDV(x^0, x^1)$) and the non-analyticity of the semigroup generated by the operator $A$. To overcome these difficulties we have to give a suitable definition of solution. We will require the following facts:

(i) the solution of the HJB equation (7.29) is defined on a open set $\Omega$ of $M^2$ and is $C^1$ on such set;

(ii) on a subset $\Omega_1 \subseteq \Omega$, closed in $\Omega$ where the trajectories interesting from the economic point of view must remain, the solution has differential in $D(A^*)$ (on $D(A^*)$ also the Dirac $\delta$ and so $B$ make sense);

(iii) the solution satisfies (7.29) on $\Omega_1$. ∎

Definition 7.22 (**Regular solution**). *Let $\Omega$ be an open set of $M^2$ and $\Omega_1 \subseteq \Omega$ a subset closed in $\Omega$. An application $w \in C^1(\Omega; \mathbb{R})$ is a solution of the HJB equation (7.29) on $\Omega_1$ if $\forall x \in \Omega_1$*

$$\begin{cases} (x, Dw(x)) \in G \\ \\ \rho w(x) - H\big(x, Dw(x)\big) = 0 \end{cases}$$

[6] Recall that an element $x \in M^2$ is done by two components: $x^0$ and $x^1$, so $x = (x^0, x^1)$.

REMARK 7.23 (**On the form of the Hamiltonian**). If $P \in D(A^*)$ and $(BP)^{-1/\sigma} \in (0, ax^0]$, by elementary arguments, the function

$$H_{CV}(x, P, \cdot) \colon [0, ax^0] \to \mathbb{R}$$

admits a unique maximum point given by

$$i^{MAX} = ax^0 - (BP)^{-1/\sigma} \in [0, ax^0)$$

and then we can write the Hamiltonian in a simplified form:

$$H((x^0, x^1), P) = \langle (x^0, x^1), A^* P \rangle_{M^2} + ax^0 BP + \frac{\sigma}{1-\sigma} (BP)^{\frac{\sigma-1}{\sigma}} \tag{7.30}$$

The expression for $i^{MAX}$ will be used to write the solution of the *Problem (**P**)* in closed-loop form. ■

We can now give an explicit solution of the HJB equation. For an explanation of how we guess it (that comes from the economic interpretation of the term $\Gamma_0$ defined in the following formula (7.31)) see Remark 7.25 and Subsection 7.6.1.2. First define, for $x \in M^2$ the quantity

$$\Gamma_0(x) \stackrel{def}{=} x^0 + \int_{-R}^{0} e^{\xi s} x^1(s) \mathrm{d}s \tag{7.31}$$

and then define the set $\Omega \subset M^2$ (which will be the $\Omega$ of the Definition 7.22) as

$$\Omega \stackrel{def}{=} \left\{ x = (x^0, x^1) \in M^2 \ : \ x^0 > 0, \ \Gamma_0(x) > 0 \right\}$$

Finally $\left( \text{calling } \alpha = \dfrac{\rho - \xi(1-\sigma)}{\sigma \xi} \right)$ we define the set $Y \subseteq \Omega$ (which will be the $\Omega_1$ of the Definition 7.22) as

$$Y \stackrel{def}{=} \left\{ x = (x^0, x^1) \in \Omega \ : \ \Gamma_0(x) \le \frac{1}{\alpha} x^0 \right\} \tag{7.32}$$

It is easy to see that $\Omega$ is an open subset of $M^2$ while $Y$ is closed in $\Omega$. We are now ready to present an explicit solution of the HJB equation (7.29) which, in next subsection, will be proved to be the value function under an additional assumption[7].

PROPOSITION 7.24. *If Hypotheses 7.5 and 7.9 hold the function*

$$v \colon \Omega \to \mathbb{R}$$

$$v(x) \stackrel{def}{=} \nu [\Gamma_0(x)]^{1-\sigma} \tag{7.33}$$

*with*

$$\nu = \left( \frac{\rho - \xi(1-\sigma)}{\sigma} \cdot \frac{a}{\xi} \right)^{-\sigma} \frac{1}{(1-\sigma)} \cdot \frac{a}{\xi}$$

*is differentiable in all $x \in \Omega$ and is a solution of the HJB equation (7.29) on $Y$ in the sense of Definition 7.22.*

PROOF. The function $v$ is of course continuous and differentiable in every point of $\Omega$ and its differential in $x = (x^0, x^1)$ is

$$Dv(x) = (\nu(1-\sigma)[\Gamma_0(x)]^{-\sigma}, \{ s \mapsto \nu(1-\sigma)[\Gamma_0(x)]^{-\sigma} e^{\xi s} \})$$

[7]The same procedure is used for the standard AK model, see Appendix 7.A in the case of zero depreciation rate of capital.

So $Dv(x) \in D(A^*)$ for every $x \in \Omega$.
We can also calculate explicitly $A^*Dv$ and $BDv$ getting:

$$A^*Dv(x) = (0, \{s \mapsto \nu(1-\sigma)[\Gamma_0(x)]^{-\sigma}\xi e^{\xi s}\}) \tag{7.34}$$

$$BDv(x) = \nu(1-\sigma)[\Gamma_0(x)]^{-\sigma}(1-e^{-\xi R}) > 0 \tag{7.35}$$

so, using the characteristic equation (7.14)

$$[BDv(x)]^{-1/\sigma} = \left(\frac{\rho - \xi(1-\sigma)}{\sigma} \cdot \frac{a}{\xi}\right)\Gamma_0(x) \tag{7.36}$$

Form the definition of $\Omega$ we have $[BDv(x)]^{-1/\sigma} > 0$ for $x \in \Omega$. Moreover if $x \in Y$ then

$$\Gamma_0(x) \leq \frac{1}{\alpha}x^0 \tag{7.37}$$

and then $[BDv(x)]^{-1/\sigma} \leq ax^0$. So we can use Remark 7.23 and write the Hamiltonian in the form of equation (7.30). Substituting (7.34) and (7.35) in (7.30) we find, by straightforward calculations, the relation:

$$\rho v(x) - \langle x, A^*Dv(x)\rangle_{M^2} - -ax^0 BDv(x) - \frac{\sigma}{1-\sigma}(BDv(x))^{\frac{\sigma-1}{\sigma}} = 0. \tag{7.38}$$

The claim is proved. □

REMARK 7.25 (**On the solution of the HJB equation**). The reason why the function $v$ solves the HJB equation comes from the meaning of the quantity $\Gamma_0(x)$ which we call “equivalent capital” (see Subsection 7.6.1.2). Indeed from this interpretation it is reasonable to expect the value function to be $(1-\sigma)$-homogeneous with respect to $\Gamma_0(x)$ where there are no corner solutions.

Moreover the choice of $Y$ comes from the need of avoiding corner solutions. Indeed we know that in the standard AK model, in presence of corner solutions, the value function is different (see Appendix 7.A for the case of zero depreciation rate of capital). The same would happen here. To prove that $v$ is the value function in next subsection we will need to prove that the closed loop strategy coming from $v$ are admissible and this will be true assuming another restriction on the parameters of the model. This is a key point to solve the theoretical problem of [BLPdR05] mentioned at point (II) of the Introduction and in Subsection 7.6.2. ■

### 7.4.3. Closed loop in infinite dimensions.

We begin with some definitions.

DEFINITION 7.26 ($\mathbf{AFS_y}$). *Given $y \in M^2$ we will call $\Phi \in C(M^2)$ an* admissible feedback (closed loop) strategy related to the initial point $y$ *if the equation.*

$$\begin{cases} \frac{d}{dt}x(t) = Ax(t) + B^*(\Phi(x(t))), & t > 0 \\ x(0) = y \end{cases}$$

*has an unique solution $x_\Phi(t)$ in $\Pi$ and $\Phi(x_\Phi(\cdot)) \in \mathcal{I}_y$. We will indicate the set of admissible feedback (closed loop) strategies related to $y$ with $AFS_y$.*

DEFINITION 7.27 ($\mathbf{OFS_y}$). *Given $y \in M^2$ we will call $\Phi$ an* optimal feedback (closed loop) strategy related to $y$ *if it is in $AFS_y$ and*

$$V_0(y) = \int_0^{+\infty} e^{-\rho t}\frac{\left(ax^0_\Phi(t) - \Phi(x_\Phi(t))\right)^{1-\sigma}}{(1-\sigma)}\mathrm{d}t.$$

*We will indicate the set of optimal feedback (closed loop) strategies related to $y$ with $OFS_y$.*

We have a solution of the HJB equation (7.29) only in a part of the state space (that is $Y$). So we can prove that closed loop optimal strategies are optimal only if they remain in $Y$. This means that we have to impose another condition on the parameters of the problem. As we will remark in Section 7.6.1 such hypothesis is reasonable from an economic point of view as it requires to rule out corner solutions.

HYPOTHESIS 7.28. $\dfrac{\rho - \xi(1-\sigma)}{\sigma} \leq a.$

From now on we will assume that Hypotheses 7.5, 7.9, 7.28 hold true.

THEOREM 7.29. *Given $\bar{\iota} \in L^2((-R,0);\mathbb{R}^+)$ with and $\bar{\iota} \not\equiv 0$, if we call $y = (S(\bar{\iota}), F(\bar{\iota}))$, then the application*

$$\Phi \colon M^2 \to \mathbb{R}$$

$$\Phi(x) \stackrel{def}{=} ax^0 - \left( \frac{\rho - \xi(1-\sigma)}{\sigma} \cdot \frac{a}{\xi} \right) \Gamma_0(x) \tag{7.39}$$

*is in $OFS_y$.*

PROOF. **Part 1.** We prove that $\Phi \in AFS_y$.
We claim that

$$\begin{cases} \frac{d}{\mathrm{d}t} x_\Phi(t) = Ax_\Phi(t) + B^*(\Phi(x_\Phi(t))), & t > 0 \\ x_\Phi(0) = y = (S(\bar{\iota}), F(\bar{\iota})) \end{cases} \tag{7.40}$$

has a unique solution in $\Pi$ (defined in 7.26). We consider first the following integral equation (with unknown $i$: along this proof we drop the "tilde" sign to avoid heavy notation).

$$\begin{cases} i(t) = & \left( a - \dfrac{\rho - \xi(1-\sigma)}{\sigma\xi/a} \right) \left( \displaystyle\int_{t-R}^{t} i(s)\mathrm{d}s \right) - \\ & \qquad - \dfrac{\rho - \xi(1-\sigma)}{\sigma\xi/a} \displaystyle\int_{-R}^{0} e^{\xi s} F(i_t)(s)\mathrm{d}s, \qquad t \geq 0 \\ i(s) = & \bar{\iota}(s), \qquad s \in [-R, 0). \end{cases} \tag{7.41}$$

Such equation has a solution $i$ which is absolutely continuous solution on $[0, +\infty)$ (see for example [BDPDM92] page 287 for a proof). We now claim that $i(t) > 0$ for all $t \geq 0$. First we prove that $i(0) > 0$. Indeed

$$i(0) = \int_{-R}^{0} \left[ a - \frac{\rho - \xi(1-\sigma)}{\sigma\xi/a} \left( 1 - e^{\xi(-R-s)} \right) \right] \bar{\iota}(s)\mathrm{d}s.$$

Since for every $s \in (-R, 0)$, $1 - e^{\xi(-R-s)} < \frac{\xi}{a}$ (in view of the fact that $\xi$ is a positive solution of equation (7.14)) then we get by Hypothesis 7.9

$$i(0) > \int_{-R}^{0} \left[ a - \frac{\rho - \xi(1-\sigma)}{\sigma} \right] \bar{\iota}(s)\mathrm{d}s,$$

so, using Hypothesis 7.28 we obtain $i(0) > 0$. Now, if there exists a first point $\bar{t}$ in which the solution is zero then we have:

$$0 = i(\bar{t}) = \int_{-R}^{0} \left[ a - \frac{\rho - \xi(1-\sigma)}{\sigma\xi/a} \left( 1 - e^{\xi(-R-s)} \right) \right] i_{\bar{t}}(s)\mathrm{d}s$$

but, arguing as for $t = 0$, we can see that the right side is $> 0$ so we have a contradiction.

Now we consider the equation

$$\begin{cases} \frac{d}{\mathrm{d}t} x(t) = Ax(t) + B^*(i(t)), & t > 0; \\ x(0) = y = (S(\bar{\iota}), F(\bar{\iota})). \end{cases} \tag{7.42}$$

We know, thanks to Theorem 7.17, that the only solution in $\Pi$ of such equation is $x(t) = (\eta(t), F(i_t))$ where $\eta(t)$ is the solution of

$$\left\{ \begin{array}{l} \dot{z}(t) = B(i_t) \\ (z(0), i_0) = (S(\bar{\iota}),\ \bar{\iota}) \end{array} \right. \qquad \Big( that\ is \quad \eta(t) = \textstyle\int_{t-R}^{t} i(s)\mathrm{d}s \Big) \tag{7.43}$$

We claim that $x(t)$ is a solution of (7.40). Indeed

$$\Phi(x(t)) = a\eta(t) - \left( \frac{\rho - \xi(1-\sigma)}{\sigma\xi/a} \right) \left( \int_{-R}^{0} e^{\xi s} F(i_t)(s)\mathrm{d}s + \eta(t) \right) \tag{7.44}$$

and so (by (7.41):

$$\begin{aligned} \Phi(x(t)) = \eta(t) \left( a - \frac{\rho - \xi(1-\sigma)}{\sigma\xi/a} \right) + i(t) - \\ - \left( a - \frac{\rho - \xi(1-\sigma)}{\sigma\xi/a} \right) \left( \int_{(t-R)}^{t} i(s)\mathrm{d}s \right) \end{aligned}$$

and by (7.43) we conclude that

$$\Phi(x(t)) = i(t)$$

and so $x(t) = x_\Phi(t)$ is a solution of (7.40) and is in $\Pi$. Moreover thanks to the linearity of $\Phi$ we obtain that $x_\Phi(t)$ is the only solution in $\Pi$. We have now to show that $i(\cdot) = \Phi(x_\Phi(\cdot)) \in \mathcal{I}_y$. The previous steps of the proof gives

$$x_\Phi(t) = (x^0_\Phi(t), x^1_\Phi(t)(\cdot)) = \Big( S(i_t), F(i_t) \Big)$$

where $i$ is absolutely continuous and so in $L^2_{loc}[0, +\infty)$. We claim that $\Phi(x_\Phi(t)) = i(t) \in (0, ax^0_\Phi(t))$. In view of the fact that $i(t) > 0$ for all $t \geq 0$ it is enough to prove that $i(t) < ax^0_\Phi(t)$. Indeed by (7.41)

$$\begin{aligned} ax^0_\Phi(t) - i(t) = \left( \frac{\rho - \xi(1-\sigma)}{\sigma\xi/a} \right) \left( S(i_t) + \int_{-R}^{0} e^{\xi s} F(i_t)(s)\mathrm{d}s \right) \geq \\ \geq \left( \frac{\rho - \xi(1-\sigma)}{\sigma\xi/a} \right) \left( \int_{-R}^{0} i_t(s)(1 - e^{\xi(-R-s)})\mathrm{d} \right) s > 0. \end{aligned}$$

The last inequality is strict due to Hypothesis 7.9 and to the fact that $i(t) > 0$ for all $t > 0$. So $i(t) < ax^0_\Phi(t)$ and we know that $\Phi$ is an admissible feedback strategy related to $y = (S(\bar{\iota}), F(\bar{\iota}))$.

**Part 2.** We prove now that $\Phi \in OFS_y$.

We consider $v$ as defined in Proposition 7.24. It is easy to see from the first part of the proof that $x_\Phi(t)$ remain in $Y$ as defined in (7.32) and so the Hamiltonian (as in the proof of Proposition 7.24) can be expressed in the simplified form of equation (7.30).

We introduce the function:

$$\begin{aligned} &v_0(t,x) \colon \mathbb{R} \times \Omega \to \mathbb{R} \\ &v_0(t,x) \stackrel{def}{=} e^{-\rho t} v(x) \qquad (v\ is\ defined\ in\ (7.33)) \end{aligned}$$

Using that $(Dv(x_\Phi(t))) \in D(A^*)$ and that the application $x \mapsto Dv(x)$ is continuous with respect to the norm of $D(A^*)$, we find:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} v_0(t, x_{\Phi(t)}) &= -\rho v_0(t, x_\Phi(t)) + \\ &\quad + \langle D_x v_0(t, x_\Phi(t)) | Ax_\Phi(t) + B^* i(t) \rangle_{D(A^*) \times D(A^*)'} = \\ &= -\rho e^{-\rho t} v(x_\Phi(t)) + \\ &\quad + e^{-\rho t} \Big( \langle A^* Dv(x_\Phi(t)), x_\Phi(t) \rangle_{M^2} + \langle BDv(x_\Phi(t)), i(t) \rangle_{\mathbb{R}} \Big) \end{aligned} \tag{7.45}$$

By definition (note that $J_0(y;i) = J_0(y;\Phi(x_\Phi)))$:

$$v(y) - J_0(y;i) = v(x_\Phi(0)) - \int_0^\infty e^{-\rho t}\frac{(ax_\Phi^0(t) - \Phi(x_\Phi)(t))^{1-\sigma}}{(1-\sigma)}\mathrm{d}t =$$

Then, using (7.45) (we use Proposition 7.3 to guarantee that the integral is finite), we obtain

$$= \int_0^\infty e^{-\rho t}\Big(\rho v(x_\Phi(t)) - \langle A^* Dv(x_\Phi(t)), x_\Phi(t)\rangle_{M^2} - \langle BDv(x_\Phi(t)), i(t)\rangle_{\mathbb{R}}\Big)\mathrm{d}t-$$

$$- \int_0^\infty e^{-\rho t}\left(\frac{(ax_\Phi^0(t) - i(t))^{1-\sigma}}{(1-\sigma)}\right)\mathrm{d}t =$$

$$= \int_0^\infty e^{-\rho t}\Big(\rho v(x_\Phi(t)) - \langle A^* Dv(x_\Phi(t)), x_\Phi(t)\rangle_{M^2} -$$

$$- \langle BDv(x_\Phi(t)), i(t)\rangle_{\mathbb{R}} - \frac{(ax_\Phi^0(t) - i(t))^{1-\sigma}}{(1-\sigma)}\Big)\mathrm{d}t =$$

$$= \int_0^\infty e^{-\rho t}\Big(H(x_\Phi(t), Dv(x_\Phi(t))) - H_{CV}(x_\Phi(t), Dv(x_\Phi(t)), i(t))\Big)\mathrm{d}t \tag{7.46}$$

The conclusion follows from Remark 7.20 and by the three observations listed below.

(1) Noting that $H(x_\Phi(t), Dv(x_\Phi(t))) \geq H_{CV}(x_\Phi(t), Dv(x_\Phi(t)), i(t))$ the (7.46) implies that, for every admissible control $i$, $v(y) - J_0(y;i) \geq 0$ and then $v(y) \geq V_0(y)$.

(2) The original maximization problem is equivalent to the problem of finding a control $i$ that minimizes $v(y) - J_0(y;i)$

(3) The feedback strategy $\Phi$ achieves $v(y) - J_0(y;i) = 0$ that is the minimum in view of point 1.

□

From the above proof we get an explicit expression for the value function $V_0$ that we state in the following corollary.

COROLLARY 7.30. *Given any $\bar{\iota} \in L^2((-R,0);\mathbb{R}^+)$ and setting $y = (S(\bar{\iota}), F(\bar{\iota}))$ we have that*

$$V(\bar{\iota}) = V_0(y) = v(y).$$

REMARK 7.31 (**Further regularity of the optimal strategies**). From Theorem 7.29 it follows that the optimal control $i^* \colon \mathbb{R}^+ \to \mathbb{R}$ is in $H^1_{loc}[0,+\infty)$. Moreover for every $\theta \in \mathbb{N}$ we have $i^*|_{(\theta R,+\infty)}(t) \in H^\theta_{loc}[\theta R,+\infty)$. ■

## 7.5. Back to *Problem (P)*

We now use the results we found in the infinite dimensional setting to solve the original optimal control *Problem (**P**)*.

**7.5.1. The explicit form of the value function.** First of all observe that, given any initial datum $\bar{\iota}(\cdot) \in L^2((-R,0);\mathbb{R}^+)$ and writing $y = (S(\bar{\iota}), F(\bar{\iota}))$, the quantity $\Gamma_0(y)$ defined in (7.31) becomes

$$\begin{aligned}\Gamma(\bar{\iota}(\cdot)) &\overset{def}{=} \Gamma_0\left(S(\bar{\iota}), F(\bar{\iota})\right) = \int_{-R}^0 \left(1 - e^{-\xi(R+s)}\right)\ \bar{\iota}(s)\mathrm{d}s \\ &= k(0) - \int_{-R}^0 e^{-\xi(R+s)}\ \bar{\iota}(s)\mathrm{d}s\end{aligned} \tag{7.47}$$

A comment on the meaning of such quantity is given in Section 7.6.1.2. Now, as a consequence of Corollary 7.30 we have:

PROPOSITION 7.32. *If Hypotheses 7.5, 7.9 and 7.28 hold, the explicit expression for the value function $V$ related to* Problem (**P**) *is*

$$V(\bar{\imath}(\cdot)) = \nu[\Gamma(\bar{\imath}(\cdot))]^{1-\sigma} = \nu\left(k(0) - \int_{-R}^{0} e^{-\xi(R+s)}\ \bar{\imath}(s)\mathrm{d}s\right)^{1-\sigma}$$

*where*

$$\nu = \left(\frac{\rho - \xi(1-\sigma)}{\sigma} \cdot \frac{a}{\xi}\right)^{-\sigma} \frac{1}{(1-\sigma)} \cdot \frac{a}{\xi}$$

**7.5.2. Closed loop optimal strategies for *Problem (P)*.** We now use the closed loop in infinite dimension to write explicitly the closed loop formula and the closed loop equation for *Problem (**P**)*. First of all we recall that, given $t \geq 0$, $\bar{\imath}(\cdot) \in L^2((-R,0);\mathbb{R}^+)$ and $i(\cdot) \in \mathcal{I}_{\bar{\imath}}$ the "history" $\tilde{\imath}_t(\cdot) \in L^2((-R,0);\mathbb{R}^+)$ is defined as in (7.21) and we can write

$$\Gamma(\tilde{\imath}_t(\cdot)) = \int_{-R}^{0} \left(1 - e^{-\xi(R+s)}\right)\ \tilde{\imath}_t(s)\mathrm{d}s = \int_{t-R}^{t} \left(1 - e^{-\xi(R+s)}\right)\ \tilde{\imath}(s)\mathrm{d}s. \tag{7.48}$$

We use the $^*$ for the optimal investment (and capital) so $\tilde{\imath}^*_t(\cdot) \in L^2((-R,0);\mathbb{R}^+)$ is the history of the optimal investment.

Next we apply Theorem 7.29 (in particular (7.39) and (7.41)) and (7.10) getting the following result whose proof is immediate.

PROPOSITION 7.33. *Let Hypotheses 7.5, 7.9 and 7.28 be satisfied, given an initial datum $\bar{\imath}(\cdot) \in L^2((-R,0);\mathbb{R}^+)$ in equation (7.9) the optimal investment strategy $i^*(\cdot)$ and the related capital stock trajectory $k^*(\cdot)$ satisfy for all $t \geq 0$:*

$$i^*(t) \quad = \quad ak^*(t) - \left(\frac{\rho - \xi(1-\sigma)}{\sigma} \cdot \frac{a}{\xi}\right)\Gamma(\tilde{\imath}^*_t(\cdot)). \tag{7.49}$$

*so calling $c^*(t) = ak^*(t) - i^*(t)$ we have*

$$c^*(t) \quad = \quad \left(\frac{\rho - \xi(1-\sigma)}{\sigma} \cdot \frac{a}{\xi}\right)\Gamma(\tilde{\imath}^*_t(\cdot)). \tag{7.50}$$

We now want to find a more useful closed loop formula. We start by the following lemma.

LEMMA 7.34. *Let Hypotheses 7.5, 7.9 and 7.28 be satisfied, given an initial datum $\bar{\imath}(\cdot) \in L^2((-R,0);\mathbb{R}^+)$ in equation (7.9), there exist constants $\Lambda = \Lambda(\bar{\imath}(\cdot)) > 0$ , $g \in \mathbb{R}$ ($g$ independent of $\bar{\imath}(\cdot)$) such that the optimal investment strategy $i^*(\cdot)$ for* Problem (**P**) *and the related capital stock trajectory $k^*(\cdot)$ satisfy for all $t \geq 0$:*

$$ak^*(t) - i^*(t) = \Lambda e^{gt} \tag{7.51}$$

*(i.e. the optimal consumption path is of exponential type). Moreover*

$$g = \frac{\xi - \rho}{\sigma} \in [\xi - a, \xi) \tag{7.52}$$

*and*

$$\Lambda = \left(\frac{\rho - \xi(1-\sigma)}{\sigma} \cdot \frac{a}{\xi}\right)\Gamma(\bar{\imath}(\cdot)) \tag{7.53}$$

PROOF. By Proposition 7.33, equation (7.49), along the optimal trajectories we have, for $t \geq 0$:

$$ak^*(t) - i^*(t) = \left(\frac{\rho - \xi(1-\sigma)}{\sigma} \cdot \frac{a}{\xi}\right)\Gamma(\tilde{\imath}^*_t(\cdot)).$$

Now let us note that

$$\Gamma(\tilde{\imath}^*_t(\cdot)) = \int_{-R}^{0} e^{\xi s} F(\tilde{\imath}^*_t(\cdot))(s)\mathrm{d}s + k^*(t) = \langle \psi, x(t) \rangle$$

where $\psi = (\psi^0, \psi^1) \in M^2$ with $\psi^0 = 1$, $\psi^1(s) = e^{\xi s}$ and $x(t)$ is the structural state as in Definition 7.16 . We calculate now the the derivative of such expression: it is easy to see that $\psi \in D(A^*)$. So we have (by Theorem 7.17)

$$\frac{d}{dt}\left(\int_{-R}^{0} e^{\xi s} F(\tilde{\imath}_t^*(\cdot))(s)\mathrm{d}s + k^*(t)\right) = \frac{d}{dt}\left\langle \psi, x(t)\right\rangle_{M^2} =$$

(by equation (7.27) and by the definitions of $A^*$ and $B$)

$$= \langle A^*\psi, x(t)\rangle_{M^2} + \langle B\psi, i^*(t)\rangle_{\mathbb{R}} = \big\langle (0, \xi\psi^1(s)), x(t)\big\rangle_{M^2} + \big\langle (1-e^{-\xi R}), i^*(t)\big\rangle_{\mathbb{R}} =$$

(finding $x(t)$, the scalar products and using the (7.49))

$$= \left[\xi\left(\int_{-R}^{0} e^{\xi s} F(\tilde{\imath}_t^*(\cdot))(s)\mathrm{d}s\right)\right] +$$

$$\left[(1-e^{-\xi R})\left(ak^*(t) - \left(\frac{\rho-\xi(1-\sigma)}{\sigma\xi/a}\right)\left(\int_{-R}^{0} e^{\xi s} F\left(\tilde{\imath}_t^*(\cdot)\right)(s)\mathrm{d}s + k^*\right)\right)\right] =$$

(by simple calculations)

$$= \left(\xi - \frac{\rho-\xi(1-\sigma)}{\sigma}\right)\left(\int_{-R}^{0} e^{\xi s} F(\tilde{\imath}_t^*(\cdot))(s)\mathrm{d}s + k^*(t)\right)$$

$$= g\left(\int_{-R}^{0} e^{\xi s} F(\tilde{\imath}_t^*(\cdot))(s)\mathrm{d}s + k^*(t)\right)$$

and so we have the claim. The bounds for $g$ simply follows by Hypotheses 7.9 and 7.28. Finally, since, from (7.51) $\Lambda = ak^*(0) - i^*(0)$ from (7.49) for $t = 0$ we find (7.53) observing that $\tilde{\imath}_0^*(\cdot) = \bar{\imath}(\cdot)$. □

Using the above Lemma 7.34 we can now write a more useful closed loop formula with the associated closed loop equation.

THEOREM 7.35. *Let Hypotheses 7.5, 7.9 and 7.28 be satisfied, given an initial datum* $\bar{\imath}(\cdot) \in L^2((-R,0);\mathbb{R}^+)$ *in equation (7.9), the optimal investment strategy for* Problem (**P**) $i^*(\cdot)$ *is connected with the related state trajectory* $k^*(\cdot)$ *by the following closed loop formula for all* $t \geq 0$*:*

$$i^*(t) = ak^*(t) - \Lambda e^{gt} \tag{7.54}$$

*where* $\Lambda = \Lambda(\bar{\imath}(\cdot))$ *is given in (7.53).*

*Moreover the optimal investment strategy* $i^*(\cdot)$ *is the unique solution in* $H1_{loc}([0,+\infty);\mathbb{R})$ *of the following integral equation:*

$$\begin{cases} \tilde{\imath}^*(t) = & a\displaystyle\int_{t-R}^{t} \tilde{\imath}^*(s)\mathrm{d}s - \Lambda e^{gt} \qquad\qquad t \geq 0 \\ \tilde{\imath}^*(s) = & \bar{\imath}(s), \qquad s \in [-R, 0). \end{cases} \tag{7.55}$$

*Finally the optimal capital stock trajectory* $k^*(\cdot)$ *is the only solution in* $H^1_{loc}([0,+\infty);\mathbb{R}^+)$ *of the following integral equation:*

$$k^*(t) = \int_{(t-R)\wedge 0}^{0} \bar{\imath}(s)ds + \int_{(t-R)\vee 0}^{t} \left[ak(s) - \Lambda e^{gs}\right] ds, \qquad t \geq 0. \tag{7.56}$$

**7.5.3. Growth rates and asymptotic behavior.** We have seen that along the optimal path the consumption is exponential. Nevertheless the optimal investment and the capital stock have a more irregular behavior that depends on initial data. We can anyway describe the asymptotic behavior of them. Calling $c^*(t) = ak^*(t) - i^*(t) = \Lambda e^{gt}$ the optimal consumption path we have the following.

PROPOSITION 7.36. *Let Hypotheses 7.5, 7.9 and 7.28 be satisfied, given an initial datum $\bar{\iota}(\cdot) \in L^2((-R,0);\mathbb{R}^+)$ in equation (7.9), defining, for $t \geq 0$, the optimal detrended paths as:*

$$\begin{aligned} k_g(t) &\stackrel{def}{=} e^{-gt}k^*(t) \\ i_g(t) &\stackrel{def}{=} e^{-gt}i^*(t) \\ c_g(t) &\stackrel{def}{=} e^{-gt}c^*(t) \end{aligned}$$

*we have that the optimal detrended consumption path $c_g(t) = \big(ak_g(t) - i_g(t)\big)$ is constant and equal to $\Lambda$. Moreover there exist positive constants $i_B$ and $k_B$ such that*

$$\lim_{t\to+\infty} i_g(t) = i_B \qquad \text{and} \qquad \lim_{t\to+\infty} k_g(t) = k_B.$$

*We have, when $g \neq 0$*

$$i_B = \frac{\Lambda}{\frac{a}{g}(1-e^{-gR}) - 1} > 0 \qquad \text{and} \qquad k_B = \frac{1-e^{-gR}}{g} \cdot i_B = \frac{\Lambda}{a - \frac{g}{1-e^{-gR}}} > 0.$$

*while, when $g = 0$,*

$$i_B = \frac{\Lambda}{aR-1} > 0 \qquad \text{and} \qquad k_B = R \cdot i_B = \frac{\Lambda R}{aR-1} > 0.$$

PROOF. The proof of existence of the limits is proved also in [BLPdR05] using the transversality conditions. Here we use the integral equation (7.55) and the explicit form of $\Lambda$ given in (7.53).

From(7.55) we can easily find that $i(t)$ satisfies, for $t \geq 0$ the following Delay Differential Equation

$$\left\{ \begin{array}{l} i'(t) = a(i(t) - i(t-R)) - \Lambda g e^{gt}, \;\; \forall t \geq 0, \\ i(s) = \bar{\iota}(s), \;\; \forall s \in [-R,0), \\ i(0) = a\int_{-R}^{0} \bar{\iota}(s)\mathrm{d}s - \Lambda, \end{array} \right. \tag{7.57}$$

The solution of this linear non homogeneous Delay Differential Equation is the sum of the solution of the associated linear homogeneous Delay Differential Equation plus a convolution term (see [HVL93] page 23). In our case it means that the solution of equation (7.57) can be written as:

$$i(t) = \int_0^t -\gamma(t-s)\Lambda g e^{gs}\mathrm{d}s + \gamma(t)i(0) - a\int_{-R}^{0}\gamma(t-R-s)\bar{\iota}(s)\mathrm{d}s \tag{7.58}$$

where $\gamma(t)$ is the solution of the following Delay Differential Equation:

$$\left\{ \begin{array}{l} \gamma'(t) = a(\gamma(t) - \gamma(t-R)) \;\forall t \geq 0, \\ \gamma(s) = 0, \;\; \forall s \in [-R,0), \\ \gamma(0) = 1 \end{array} \right. \tag{7.59}$$

We observe that equation (7.59) is similar to the Delay Differential Equation we have seen in equation (7.13). In particular the characteristic equation is the same and it is (like in (7.14))

$$a(1-e^{-zR}) - z = 0 \tag{7.60}$$

From Proposition 7.8 such characteristic equation has only simple roots: the only real roots are $\xi$ and 0 and all other complex roots have real part in $(-\infty, \xi - a)$

so, even if $g<0$ we have $Re\lambda_j<g$ for each $j=1,2,\ldots$. Applying Corollary 6.4 of [DVGVLW95], page 168 we see that the solution of (7.59) can be written as

$$\gamma(t)=\alpha_\xi e^{\xi t}+\alpha_0+\sum_{j=1}^{+\infty}\left[\alpha_j e^{\lambda_j t}+\overline{\alpha}_j e^{\overline{\lambda}_j t}\right]$$

where the series converges uniformly on compact subsets of $(0,+\infty)$, $\alpha_\xi,\alpha_0$ are real numbers and $\alpha_j$ are complex numbers.

We have now only to substitute such expression in (7.58). In view of the linearity of (7.58) with respect to $\gamma$ we can analyze the contribution of three parts of $\gamma$ in three steps: first we estimate the term due to $\alpha_\xi e^{\xi t}$, second we consider the term $\alpha_0$ and then the series. We start with $\alpha_\xi e^{\xi t}$: its contribution to $i(t)$ is (in view of (7.58) is (using (7.47) and (7.53))

$$\int_0^t -\alpha_\xi e^{\xi(t-s)}\Lambda g e^{gs}\mathrm{d}s+a\int_{-R}^0 \overline{\iota}(s)\mathrm{d}s\alpha_\xi e^{\xi t}-\Lambda\alpha_\xi e^{\xi t}-a\int_{-R}^0\alpha_\xi e^{\xi(t-R-s)}\overline{\iota}(s)\mathrm{d}s=$$

$$=e^{\xi t}\left(\frac{\alpha_\xi\Lambda g}{g-\xi}+\alpha_\xi a\int_{-R}^0\overline{\iota}(s)\mathrm{d}s-\alpha_\xi a\int_{-R}^0\overline{\iota}(s)e^{-\xi(R+s)}\mathrm{d}s-\alpha_\xi\Lambda\right)+e^{gt}\left(-\frac{\alpha_\xi\Lambda g}{g-\xi}\right)=$$

$$=e^{\xi t}\alpha_\xi\left(\frac{\Lambda g}{g-\xi}+a\Gamma(\overline{\iota})-\Lambda\right)+e^{gt}\alpha_\xi\left(-\frac{\Lambda g}{g-\xi}\right)=$$

$$=e^{\xi t}\alpha_\xi\left(\frac{\Lambda g}{g-\xi}+a\Lambda\frac{\xi}{a}\frac{\sigma}{\rho-\xi(1-\sigma)}-\Lambda\right)+e^{gt}\alpha_\xi\left(-\frac{\Lambda g}{g-\xi}\right)=$$

$$=e^{\xi t}\alpha_\xi\Lambda\left(\frac{\frac{\xi-\rho}{\sigma}}{\frac{\xi-\rho}{\sigma}-\xi}+\frac{\xi\sigma}{\rho-\xi(1-\sigma)}-1\right)+e^{gt}\alpha_\xi\left(-\frac{\Lambda g}{g-\xi}\right)=0+e^{gt}\alpha_\xi\left(-\frac{\Lambda g}{g-\xi}\right)$$

Then the part $\alpha_\xi e^{\xi t}$ gives in $i(t)$ a contribution of $e^{gt}\alpha_\xi\left(-\frac{\Lambda g}{g-\xi}\right)$.

The contribution of the term $\alpha_0$ is:

$$-\int_0^t\alpha_0\Lambda g e^{gs}\mathrm{d}s+a\alpha_0\int_{-R}^0\overline{\iota}(s)\mathrm{d}s-\Lambda\alpha_0-a\alpha_0\int_{-R}^0\overline{\iota}(s)\mathrm{d}s=-\alpha_0\Lambda e^{gt}.$$

Now to analyze the contribution of the series we use the dominated convergence theorem that allows to exchange the series and the integral. Then for each term $\alpha_j e^{\lambda_j t}$ we can develop the integrals as above obtaining the sum of two terms

$$-\frac{\alpha_j\Lambda g}{g-\lambda_j}e^{gt}+\left[\frac{\alpha_j\Lambda g}{g-\lambda_j}+\alpha_j a(\Gamma_j-\Lambda)\right]e^{\lambda_j t}$$

where

$$\Gamma_j:=\int_{-R}^0(1-e^{-\lambda_j(R+s)})\overline{\iota}(s)\mathrm{d}s.$$

Since $Re\lambda_j<g$ for each $j$, then the second term is of order smaller or equal to the first. The same can be done for the terms $\overline{\alpha}_j e^{\overline{\lambda}_j t}$.

So the solution can be written in the form

$$Ce^{gt}+o(e^{gt})$$

where

$$C=-\Lambda g\left[\frac{\alpha_\xi}{g-\xi}+\frac{\alpha_0}{g}+\sum_{j=1}^{+\infty}Re\left(\frac{\alpha_j}{g-\lambda_j}\right)\right].$$

This proves the first statement of the Proposition: there exist positive constants $i_B$ and $k_B$ such that

$$\lim_{t\to+\infty}i_g(t)=i_B\qquad \textit{and}\qquad \lim_{t\to+\infty}k_g(t)=k_B.$$

We now calculate such $i_B$ and $k_B$.

From(7.55) we find that $i_g(t)$ satisfies, for $t \geq 0$ the following integral equation:

$$i_g(t) = a \int_{-R}^{0} e^{gs} i_g(t+s)\mathrm{d}s - \Lambda \tag{7.61}$$

and then $i_B$ has to satisfy

$$i_B = a i_B \int_{-R}^{0} e^{gs}\mathrm{d}s - \Lambda$$

so we use the (7.61) to find the value of $i_B$ and $k_B$.

The fact that $i_B > 0$ follows, for $g \neq 0$ from the fact that $g < \xi$ (Hypothesis 7.9) and from (7.15); for $g = 0$ from Hypothesis 7.5. □

REMARK 7.37 (**On the costate variable in our setting**). In the Dynamic Programming approach the costate is (under suitable assumptions) the gradient of the value function along optimal trajectories. In our work we treat the problem in an infinite dimensional setting so the costate is a function of $t$ with infinite dimensional values, more precisely its value at each time $t$ is an element of $M^2$ that we call $\lambda_0(t)$. It has two parts: $\lambda_0^0(t)$ which is a real number and $\lambda_0^1(t)$ which is a function for each $t$: the history of $\lambda_0^0(t)$ as introduced in Subsection 7.4.1.

Which is the relation between such costate and the "standard" costate introduced in [BLPdR05], equations (13) and (14)) that we call it simply $\lambda$ and is a real valued function?

From the definition given in [BLPdR05] and from the results we have proved it can be seen (see also Proposition 11 and equation (27) of [BLPdR05]) that along optimal trajectories

$$\lambda(t) = e^{-\xi t} \cdot \frac{a c_g(t)^{-\sigma}}{\sigma g + \rho}$$

One other side

$$\lambda_0(t) = \nabla V_0(S(i_t^*), F(i_t^*))$$

and from the explicit form of $V_0$ given in equation (7.33) we find that (see the proof of Proposition 7.24)

$$\lambda_0^0(t) = \lambda(t)$$

so $\lambda_0^1$ is the history of $\lambda$. ■

REMARK 7.38 (**On the transversality condition**). In the necessary and sufficient conditions proved in [BLPdR05] the following transversality condition arises

$$\lim_{t\to\infty} \lambda(t)k(t) = 0.$$

We can observe that *ex post* such condition is verified along optimal trajectories we have found. Indeed, as observed in Proposition 7.36 and Remark 7.38 $\lambda(t) = O(e^{-\xi t})$ and $k(t) = O(e^{gt})$. It may also be possible to prove *ex ante* that such property holds using the concavity of the value function on the line of [Ben82]. ■

We now look at the existence of Balanced Growth Paths (BGP).

DEFINITION 7.39 (**BGP**). *We will say that an optimal pair for* Problem (**P**) $(k^*, i^*)$ *is a* Balanced Growth Path *(BGP) if there exist* $a_0, b_0 > 0$*, and real numbers* $a_1, b_1$ *such that*

$$\tilde{\imath}^*(s) = a_0 e^{a_1 s} \quad for\ s \in [-R, +\infty) \qquad k^*(s) = b_0 e^{b_1 s} for\ s \in [0, +\infty).$$

PROPOSITION 7.40. *Let Hypotheses 7.5, 7.9 and 7.28 be satisfied, the only BGPs of the model are the trajectories of the form*

$$\tilde{\imath}^*(s) = a_0 e^{gs}, \quad \textit{for } s \in [-R, +\infty); \qquad k^*(s) = b_0 e^{gs}, \textit{for } s \in [0, +\infty); \tag{7.62}$$

*where* $b_0 = k(0)$ *and* $a_0$ *and* $b_0$ *are connected by the relation:*

$$b_0 = a_0 \int_{-R}^{0} e^{gs} \mathrm{d}s = \frac{a_0}{g} \left(1 - e^{-gR}\right) \tag{7.63}$$

PROOF. We give only a sketch of the proof avoiding standard calculations.

We know that the optimal discounted investment follows the Delay Differential Equation (7.61). If we substitute inside such relation the generic solution $a_0 e^{(a_1 - g)s}$ we find that $a_1 = g$. So the only possible BGPs are the ones described in (7.62).

We substitute then the solution $a_0 e^{gt}$ in (7.55) and we find that the solution of the form (7.62) are optimal. □

In [BLPdR05], Sections 4.2, 4.3 it is proved that detrended consumption is constant over time and that balanced growth path are of the form given in Proposition 7.40. In particular equation (7.63) is the analogous of equation (19) in [BLPdR05]. Apart from other theoretical points, the main progress made here is the fact that we calculate explicitly the constant $\Lambda$.

## 7.6. Discussion and comparison with the previous results

We now discuss the results of Sections 3-4-5 comparing them with the ones of [BLPdR05], emphasizing the novelties and their economic implications.

We proceed by discussing in detail the three methodological points $(I)-(II)-(III)$ raised in the Introduction. We devote a subsection to each one of them.

**7.6.1. The explicit form of the value function and its consequences in the study of the optimal paths.** In [BLPdR05] it is shown that the detrended co-state path $\hat{\lambda}(t) := \lambda(t)e^{\xi t}$ and the optimal detrended consumption path $c_g(\cdot) := e^{-gt}c^*(t)$ are both constant (depending only on the initial data) but none is said about the explicit expression of the constants. Moreover the value function and its relation with the costate are not considered.

Here the value function is explicitly given (Proposition 7.32) and using its closed form, we explicitly calculate such constants[8] i.e.

$$\hat{\lambda}(t) \equiv \frac{a}{\xi}\Lambda^{-\sigma} \qquad \text{and} \qquad c_g(t) \equiv \Lambda$$

where $\Lambda$ is given by (7.53).

Moreover in [BLPdR05] it is shown that the optimal detrended investment path $i_g(t) = e^{-gt}i^*(t)$ and the optimal detrended capital path $k_g(t) = e^{-gt}k^*(t)$ converge asymptotically to a constant (respectively $i_B$ and $k_B$) but nothing is said about their value.

Here, using (7.53) and the closed loop equations (7.55)-(7.56) for the optimal investment and capital trajectories, we determine the explicit form of the constants $i_B$ and $k_B$, given in Proposition 7.36. This way the dependence of the long run equilibrium on the initial datum is explicitly calculated and a comparative statics can be easily performed.

In the following two subsections we discuss some implications of such explicit formulae.

[8]To calculate the co-state $\hat{\lambda}(t)$ one has to observe that it is the gradient of the value function as in Remark 7.38.

7.6.1.1. *The study of short run fluctuations.* The closed loop equations (7.55)-(7.56) for the optimal investment and capital cannot be explicitly solved (apart from very special cases) but they turn out to be useful in studying the qualitative properties of $i_g(\cdot)$ and $k_g(\cdot)$ and of their short run growth rates such as the presence of oscillations and of short run deviations between saving rates and growth rates (see [BLPdR05], Section 5.1).

To see this we first make some remarks on the integral equation (7.55). From Proposition 7.36 and its proof we know that the optimal investment $i^*(\cdot)$ (that solves the Delay Differential Equation (7.55)) can be written as

$$i^*(t) = i_B e^{gt} + \sum_{j=1}^{+\infty} e^{Re\lambda_j t} \left[ i_j^1 \cos(Im\lambda_j t) + i_j^2 \sin(Im\lambda_j t) \right]$$

where the $\lambda_j$ is the sequence described in Proposition 7.8 - (c) giving the complex and non real roots of the characteristic equation ordered with decreasing real part. We have $Re(\lambda_j) < \xi - a \leq g$ for each $j$ and all $\lambda_j$'s are simple roots. Moreover for each compact interval $I$ the number of $\lambda_j$'s with real part in $I$ is finite. Finally $i_B$ is known from Proposition 7.36 and, with the notation used in the proof of Proposition 7.36, for $j \in \mathbb{N}$

$$i_j^1 = 2Re\left( \frac{\alpha_j \Lambda g}{g - \lambda_j} + \alpha_j a(\Gamma_j - \Lambda) \right), \qquad i_j^2 = -2Im\left( \frac{\alpha_j \Lambda g}{g - \lambda_j} + \alpha_j a(\Gamma_j - \Lambda) \right),$$

can be calculated from the initial datum $\bar{\iota}$, the characteristic roots $\{\lambda_j\}_{j\in\mathbb{N}}$ and the coefficients $\{\alpha_j\}_{j\in\mathbb{N}}$. The $\lambda_j$'s and the $\alpha_j$'s can be calculated at least in numerical way (see for example [DVGVLW95], chapters IV and VI).

So we have a *main part* given by $i_B e^{gt}$, that determines the asymptotic behavior, and a *rest*, that gives the short run fluctuations, given by the series. To get a *first order approximation* of the fluctuations in the long run it is enough to take only the term with $Re(\lambda_1)$.

When the initial datum $\bar{\iota}$ is on the steady state $\bar{\iota}_0$ no fluctuation arise so $i_j^1 = i_j^2 = 0$ for each $j$. Otherwise the size of the coefficients $i_j^1, i_j^2$ will depend on the deviation from the steady state, $\bar{\iota} - \bar{\iota}_0$, through the terms $\Lambda$ and $\Gamma_j$.

Using equation (7.56) (or (7.10)) we can moreover approximate the short run fluctuations of the optimal capital

$$k^*(t) = k_B e^{gt} + \sum_{j=1}^{+\infty} e^{Re\lambda_j t} \left[ k_j^1 \cos(Im\lambda_j t) + k_j^2 \sin(Im\lambda_j t) \right].$$

The term $k_B$ is known while $k_j^1, k_j^2$ (as $i_j^1, i_j^2$) can be calculated from $\bar{\iota}$, $\lambda_j$ and $\alpha_j$.

Using the above formulae we can also study the behavior of the output and investment rate ($\frac{y^*(\cdot)'}{y^*(\cdot)} = \frac{k^*(\cdot)'}{k^*(\cdot)}$ and $\frac{i^*(\cdot)'}{i^*(\cdot)}$) in particular through the study of its first order approximation.

Finally the above formulae can be a good basis for an empirical testing of the model.

7.6.1.2. *The "equivalent capital" and the convergence to the standard AK model.* We compare the model treated in this work with the standard one dimensional AK model with zero depreciation rate of capital (which is described in Appendix 7.A, so we send the reader to it for all formulae).

The value function is given by the formula

$$V(\bar{\iota}(\cdot)) = \nu[\Gamma(\bar{\iota}(\cdot))]^{1-\sigma} \tag{7.64}$$

where

$$\Gamma(\bar{\iota}(\cdot)) \stackrel{def}{=} \int_{-R}^{0} \left(1 - e^{-\xi(R+s)}\right) \bar{\iota}(s)\mathrm{d}s = k(0) - \int_{-R}^{0} e^{-\xi(R+s)} \bar{\iota}(s)\mathrm{d}s \tag{7.65}$$

and

$$\nu = \left( \frac{\rho - \xi(1-\sigma)}{\sigma} \cdot \frac{a}{\xi} \right)^{-\sigma} \frac{1}{(1-\sigma)} \cdot \frac{a}{\xi} \tag{7.66}$$

The quantity $\Gamma(\bar{\imath}(\cdot))$ has a clear economic interpretation: it is the initial amount of capital minus the value of scrapped investments discounted at rate $-\xi$ and may be interpreted, in this model, as the initial equivalent amount of infinitely durable capital, since the term $\int_{-R}^{0} e^{-\xi(R+s)}\ \bar{\imath}(s)\mathrm{d}s$ is exactly what is lost for production due to the fact that machines are scrapped after a finite time $R$[9]. For $R < +\infty$ such quantity is strictly less than the capital (except for the degenerate case $\bar{\imath} \equiv 0$). When $R \to +\infty$, i.e. such amount tends to the initial capital $k(0)$ since the discounted integral term disappear.

If we take $t > 0$, the quantity $\Gamma(\tilde{\imath}_t(\cdot))$ (recall that $\tilde{\imath}_t(\cdot)$ is the history of investments at time $t$, see (7.21)) is the "equivalent capital" at time $t$. The feedback formula (7.50) shows that the consumption is chosen by taking a constant share of $\Gamma(\tilde{\imath}_t(\cdot))$. Moreover Lemma 7.34, together with formula (7.50) shows that $\Gamma(\tilde{\imath}_t(\cdot))$ grows at constant rate $g$.

In view of this we may say that the key variable of the model is the "equivalent capital" which has a constant growth rate $g$ due to the AK nature of the model. The consumption path is simply a constant share of the "equivalent capital" while the investment fluctuates to keep it growing at such constant rate. So when $R < +\infty$ the "equivalent capital" plays the role of the capital in the standard AK model.

The standard one dimensional AK model with zero depreciation rate of capital (see Appendix 7.A) can be seen as the limit case of the model treated here when $R = +\infty$. Indeed in such standard AK model the value function $V_0$ depends on $k(0)$ and is

$$V_0(k(0)) = \nu_0 [k(0)]^{1-\sigma} \tag{7.67}$$

where

$$\nu_0 = \left( \frac{\rho - a(1-\sigma)}{\sigma} \right)^{-\sigma} \frac{1}{(1-\sigma)} \tag{7.68}$$

Since $\xi \to a$ as $R \to +\infty$ then we clearly have, for every initial datum $\bar{\imath}(\cdot)$:

$$\lim_{R \to +\infty} V(\bar{\imath}(\cdot)) = V_0(k(0)).$$

Similarly, as $R \to +\infty$ we have (calling $g_{AK}$ the growth rate of the optimal paths in the standard AK model with zero depreciation rate of capital),

$$\Lambda(\bar{\imath}(\cdot)) \to \frac{\rho - a(1-\sigma)}{\sigma} k(0), \qquad g = \frac{\xi - \rho}{\sigma} \to \frac{a - \rho}{\sigma} = g_{AK},$$

so the optimal consumption path converges uniformly on compact sets to the one of the standard AK model described in Appendix 7.A. Consequently the closed loop formula (7.54) converges and passing to the limit in equations (7.55)-(7.56) we get the same convergence for the optimal investment and capital paths.

It is worth to remark that we are comparing the model treated here with an AK model with zero depreciation rate of capital because is not easy task to re-conduct a vintage capital model to a model with constant and positive depreciation rate of capital.

[9] To understand better this fact we can think in a discrete time setting as follows: given the initial distribution of investments, $\bar{\imath}(\cdot)$, the machines bought in the period $[t_0, t_0+1)$, with $t_0 \in [-R, 0) \cap \mathbb{Z}$ are $\bar{\imath}(t_0)$ and are scrapped at time $t_0 + R$. If they would have been infinitely durable then their value at the scrapping time should be $e^{-\xi(R+t_0)}\ \bar{\imath}(t_0)$. The discount rate is $\xi$ i.e. the maximum rate of reproduction of capital.

**7.6.2. The problem of admissibility of the candidate optimal paths.** In [BLPdR05] it is not proved that the candidate optimal trajectory of capital and investment is admissible (see the discussion in Section 4.3, p. 60 of [BLPdR05]) leaving an unsolved gap in the analysis of the model.

Here we can prove that such candidate optimal trajectory is admissible. Indeed, using the closed loop form given by (7.49) and the Hypothesis 7.28 (i.e. $\frac{(\rho-\xi(1-\sigma))}{\sigma} \leq a$) we see, in the proof of Theorem 7.29, that the optimal investment $i^*(t)$ remains in the interval $(0, ak(t))$ for all $t \geq 0$.

The emergence of this theoretical problem comes from the strategy used in [BLPdR05] (and in much of the literature on continuous time endogenous growth models) to attack the problem: first we consider the problem without taking account of the "difficult" state-control constraint (7.3) focusing on interior solutions ([BLPdR05], p.54) and then check afterwards if the optimal paths for the simplified problem also satisfy (7.3). Of course this may not be true, or, even if it is true as in this case, it may be very hard to check.

In our approach we always take account of (7.3) and then it cannot happen that we find a non-admissible candidate optimal trajectory. We also focus on interior solutions but we provide an if and only if condition on parameters (Hypothesis 7.28) for the existence of interior solutions. This can be done explicitly since we know the explicit form of the value function.

To understand better this point one can consider the standard AK model with zero depreciation rate of capital where one adds the constraint $i(t) \geq 0$ for $t \geq 0$ (see Appendix 7.A.2). In this case interior solutions arise if and only if $g_{AK} = \sigma^{-1}(a-\rho) > 0$ (i.e. the economy grows at a strictly positive rate on the optimal paths). If this is not the case then the optimal investment path is constantly 0, so also the capital and the consumption are constant.

In the model of this work interior solutions arise for every nonzero initial datum $\bar{\iota}(\cdot)$ if and only if $g \geq \xi - a$[10] which is exactly (Hypothesis 7.28) and reduces to $g_{AK} \geq 0$[11] when $R \to +\infty$. Differently from the standard AK model here when Hypothesis 7.28 does not hold we do not have constant optimal paths: this depends on the shape of the initial investments profile $\bar{\iota}$.

**7.6.3. The assumptions on the parameters.** We work under more general and sharper assumptions on the parameters that include cases which are interesting from the economic point of view. Indeed the Hypotheses in [BLPdR05] are:

$$(H1) \quad aR > 1.$$
$$(H2) \quad \rho > (1-\sigma)a.$$
$$(H3) \quad \frac{\rho-\xi}{\sigma} < 0.$$

The first (H1) is the same of Hypothesis 7.5.

The second (H2) is strictly stronger than Hypothesis 7.9 because $\xi < a$. This means that we can prove the existence and characterize the form of the optimal trajectories in a more general case. Moreover, in the standard AK model with zero depreciation rate of capital, (H2) is an if and only if conditions for the finiteness of the value function and the existence of optimal paths (see Appendix 7.A, formula (7.71)). Our Hypothesis 7.9 has "substantially" the same meaning for the AK vintage model. Indeed the maximum rate of growth of capital is $a$ in the standard AK model and $\xi$ in the vintage one and it may be proved that the value function is

[10] One may expect that interior solutions arise when a strict inequality is satisfied. This is not the case here, as it comes from the proof of Theorem 7.29.

[11] This is not the same of $g_{AK} > 0$ that guarantees interior solutions. This comes from the passage to the limit as $R \to +\infty$.

somewhere infinite when Hypothesis 7.9 is not satisfied. Note also that in the limit for $R \to +\infty$ the Hypothesis 7.9 tends to (7.71).

Concerning assumption (H3) we also see that it is strictly stronger than Hypothesis 7.28: we can re-write (H3) as $g > 0$ while Hypothesis 7.28 is $g \geq \xi - a$ so our results also cover cases where negative growth rates arise. Since investments always remain positive the occurrence of strictly negative long run growth rates in the AK vintage capital model increases the number of cases where deviation between growth and investment rates can arise (see [BLPdR05] for a discussion on this).

It must also be noted that the assumptions (H2) and (H3) are not compatible for certain values of $\sigma$. Indeed (H2) means $\rho > (1-\sigma)a$ while (H3) means $\rho < \xi$. So, when $\xi \leq a(1-\sigma)$, i.e. when $\sigma \leq e^{-\xi R}$, (H2) and (H3) are not satisfied together. This means that the results of [BLPdR05] do not cover cases with small $\sigma$.

## 7.A. The standard AK-model with zero depreciation rate of capital

In this appendix we briefly recall the setup of the classical linear growth model (named *AK-model* with Rebelo [Reb91]) with CRRA utility function and zero depreciation rate of capital. We show how to find the optimal paths with the Dynamic Programming approach. This way the comparison with the AK vintage capital model can be more clear for the reader. Another reason to write this appendix is the fact that, in the classical literature, see e.g. the Barro and Sala-i-Martin's book [BSiM95] this model is treated with the maximum principle.

We call $y(t)$ the output level at time $t$, which is a linear function of the stock of capital $k(t)$: $y(t) = ak(t)$ for some positive constant $a$. $c(t)$ and $i(t)$ are the consumption and the investment at time $t$ and the system is subject to an accounting equation of the form

$$y(t) = i(t) + c(t).$$

The capital stock follows the state equation (here we use the consumption as control variable, before we have chosen the investment, it is the same in view of the above relation)

$$\begin{cases} \dot{k}(t) = ak(t) - c(t), \\ k(0) = k_0 > 0. \end{cases} \tag{7.69}$$

We want to maximize (over the set of locally integrable consumption paths) the intertemporal utility function given by

$$\int_0^{\infty} e^{-\rho s} \frac{c(t)^{1-\sigma}}{1-\sigma} \mathrm{d}t \tag{7.70}$$

under the constraints $c(t), k(t) \geq 0$ for all $t \geq 0$. We assume

$$\rho - a(1-\sigma) > 0 \tag{7.71}$$

Hypothesis (7.71) is not only sufficient but also necessary to guarantee that the finiteness of the value function and the existence of optimal strategies (see e.g. on this [FGS06]).

In order to compare in a proper way this standard AK model with the one treated in the paper we analyze separately the case where investments can be negative and the case where we impose positivity of them.

**7.A.1. The DP approach for possibly negative investments.** Now we see how to perform the steps (i),..., (iv) of the Dynamic Programming approach

described in Subsection 7.2.1 in this one dimensional case.
Step (i): we write the HJB equation of the problem. It appears as

$$\rho v(k) - \sup_{c \geq 0} \left( v'(k)\,(ak - c) + \frac{c^{1-\sigma}}{1-\sigma} \right) = 0.$$

Step (ii): we solve the HJB equation. It is easy to check that the function[12]

$$v(k) = \nu k^{1-\sigma} \tag{7.72}$$

with $\nu = \frac{1}{1-\sigma} \left( \frac{\rho - a(1-\sigma)}{\sigma} \right)^{-\sigma}$ is a solution of the HJB equation and it is also the value function of the problem.
Step (iii): we use the value function to solve the optimal control problem in closed loop form. We consider the closed loop relation given by

$$\begin{cases} \Phi\colon \mathbb{R}^+ \to \mathbb{R}^+ \\ \Phi(k) \overset{def}{=} \arg\max_{c \in \mathbb{R}^+} \left( v'(k)\,(ak - c) + \dfrac{c^{1-\sigma}}{1-\sigma} \right) = \\ \qquad = (v'(k))^{-1/\sigma} = \dfrac{\rho - a(1-\sigma)}{\sigma} \cdot k. \end{cases}$$

Using a verification theorem it can be proved that the strategy given by such relation is optimal.
Step (iv): We substitute $c = \Phi(k)$ in the state equation:

$$\begin{cases} \dot{k}^*(t) = ak^*(t) - \Phi(k^*(t)) = \left( \dfrac{a - \rho}{\sigma} \right) k^*(t), \\ k^*(0) = k_0. \end{cases} \tag{7.73}$$

So, calling $g_{AK} = \frac{a-\rho}{\sigma}$ the optimal capital and consumption path are:

$$\begin{cases} k^*(t) = e^{g_{AK} t} k_0; \\ c^*(t) = \Phi(k^*(t)) = \left( \dfrac{\rho - a(1-\sigma)}{\sigma} \right) e^{g_{AK} t} k_0. \end{cases}$$

and the investment is

$$i^*(t) = ak^*(t) - c^*(t) = g_{AK} e^{g_{AK} t} k_0.$$

We have positive growth rate $g_{AK}$ if and only if $a \geq \rho$. Moreover the optimal investment has always the same sign of the growth rate $g_{AK}$.

**7.A.2. The DP approach for positive investments.** We call this case the constrained case while the previous is the unconstrained one. When $a \geq \rho$ the optimal path for the unconstrained case is admissible for the constrained case too, as $i^*(\cdot)$ is always positive. This does not happen when $a < \rho$. In such case the solution of the HJB equation is different as the sup is done over $c \in [0, ak]$ instead over $c \geq 0$:

$$\rho v(k) - \sup_{c \in [0, ak]} \left( v'(k)\,(ak - c) + \frac{c^{1-\sigma}}{1-\sigma} \right) = 0.$$

Arguing as in the unconstrained case one finds that the value function is $v(k) = \frac{a}{\rho}(ak)^{-\sigma}$. We perform the step (iii) and (iv). The optimal feedback map is $\Phi(k) =$

[12] It is clear from the structure of the problem that the value function must be $(1-\sigma)$-homogeneous; then the constant $\nu$ is calculated substituting into the HJB equation.

$ak$ so the optimal paths are constant, i.e. for every $t \geq 0$

$$\begin{cases} k^*(t) = k_0 \\ c^*(t) = ak_0 \\ i^*(t) = 0 \end{cases}$$

REMARK 7.41. We briefly point out the relations between the assumptions on the AK vintage model and the standard one, recalling that the second is the limit of the first for $R \to +\infty$ (see Subsections 7.6.2 and 7.6.3 for comments).

The Hypothesis 7.5 (i.e. $aR > 1$) means that strictly positive growth is possible and in the one dimensional case reduces to ask $a > 0$.

The Hypothesis 7.9 (i.e. $\rho > (1-\sigma)\xi$) is substantially an if and only if condition for existence and is the analogous of (7.71) (indeed for $R \to \infty$ they are the same).

The Hypothesis 7.28 (i.e. $\frac{\rho-\xi(1-\sigma)}{\sigma} \leq a$) guarantees that the optimal investment strategy is not a corner solution. The analogous assumption in the standard AK model is $a > \rho$. ■

CHAPTER 8

# An AK vintage model for obsolescence and depreciation

In this Chapter we present a contribution based a model presented in [BdRM]. Here we use the explicit solution of a first order HJB equation in $M^2$ is used to study a model for obsolescence and physical depreciation.

The main ideas are similar to that we have seen in Chapter 7 but the different explicit form of the HJB equation and of the the state equation needs to adapt the proofs for the new case. When the proofs are "too similar" we refer to results of Chapter 7. In particular we will skip the proofs of some preliminary results (Subsection 8.4.1) that exploit the same ideas used in Section 7.3. The formal analogy with respect to the problem treated in Chapter 7 is very significant in the Hilbert space formulation so that we write the two HJB equations in the same way (but the operators involved are different and the variables are not the same). See [Fab06] for a family of problems that can be treated using the same techniques.

As in Chapter 7 we have chosen to emphasize with the notation the economic meaning of the variable: $y$ for the (net) production - that is the state variable, $k$ for the capital, $i$ for the investment - that is the control... So the notations can be different from that we used in first two parts of the thesis. Moreover we have chosen to maintain the Chapter (almost) self-contained so it is possible the reader find some concepts he already found in other parts of the thesis.

We begin presenting in Section 8.1 the model of [BdRM]. In Section 8.3 we will use the instruments presented in Section 1.3 to restate the problem in $M^2$. In Section 8.4 we will use the instruments of the dynamic programming to solve in closed loop form the problem. In last section (Section 8.5) we will come back to the original problem and using the results obtained in Hilbert formulation we will obtain results on original (delay) optimal control problem. We will find the value function of original problem and its solution in closed form. We will find a "constant of the motion" along optimal trajectory and we will study the Balanced Growth Paths and asymptotically behavior.

## 8.1. The model

The production function is $AK$[1]:

$$Y(t) = ak(t),\ a > 0$$

where $k(t)$ is the stock of capital at time $t$, which is given by

$$k(t) = \int_{t-R(t)}^{t} n(z,t)\, i(z)\, dz \tag{8.1}$$

where $i(z)$ is investment at time $z$, $R(t)$ is the age of capital scrapped at time $t$, $n(z,t)$ is the average productivity of vintage $z$ at time $t$ relative to the average productivity of capital.

[1]Here the constant is called $a$ instead of $A$ not to be confused with the generator of the semigroup.

We assume that the relative average productivity of each vintage depends on the resources devoted to its maintenance and repair. The unit maintenance and repair cost of vintage $z$ at time $t$ is assumed to be an increasing function of its age, $t-z$, and its relative average productivity,

$$\omega\left(t-z, n\left(z,t\right)\right)=\beta e^{\gamma(t-z)}n\left(z,t\right)^{\mu}+\eta,\ \beta>0\ ,\ \eta>0,\ \mu>1.$$

Production net of the maintenance and repair costs is

$$y\left(t\right)=\int_{t-R(t)}^{t}Q\left(t-z,n\left(z,t\right)\right)i\left(z\right)dz,$$

where

$$Q\left(t-z,n\left(z,t\right)\right)=an\left(z,t\right)-\omega\left(t-z,n\left(z,t\right)\right) \tag{8.2}$$

is the average profitability of vintage $z$ at time $t$.

The relative average productivity of each vintage at time $t$ is chosen in order to maximize the average profitability, $Q\left(t-z,n\left(z,t\right)\right)$. The first order condition of this maximization problem implies that the relative average productivity of vintage $z$ at time $t$ is a decreasing function of its age and it is given by

$$n\left(z,t\right)=n\left(t-z\right)=n_0e^{-\frac{\gamma}{\mu-1}(t-z)}. \tag{8.3}$$

where $n_0=\left(\frac{a}{\beta\mu}\right)^{\frac{1}{\mu-1}}$.

Substitution from $n\left(t-z\right)$ into (8.2) yields:

$$Q\left(t-z\right)\equiv Q\left(t-z,n\left(t-z\right)\right)=\Omega e^{-\frac{\gamma}{\mu-1}(t-z)}-\eta \tag{8.4}$$

which is the average profitability of vintage $z$ at time $t$ and where $\Omega=\left(\mu-1\right)\beta\left(\frac{a}{\beta\mu}\right)^{\frac{\mu}{\mu-1}}$.

A vintage will be used until that its profitability is zero, which implies that the lifetime of a vintage, $R$, is given by

$$Q\left(R\right)=\Omega e^{-\frac{\gamma}{\mu-1}R}-\eta=0. \tag{8.5}$$

From previous equation it follows that the lifetime of a vintage is constant. We assume that $\Omega>\eta$ which is needed to guarantee that there is a strictly positive lifetime of capital.

The stock of capital at time $t$ can vary due to (i) gross investment, (ii) the change of the relative average productivity of capital, which is physical depreciation, and (iii) the scrapping of unprofitable vintages, which is called obsolescence.

Differentiating equation (8.1), and using that $n\left(t-z\right)$ is given by (8.3), yields the following evolution law of capital:

$$\dot{k}\left(t\right)=n_0i\left(t\right)-\left(l\left(t\right)+\delta\right)k\left(t\right) \tag{8.6}$$

where

$$l\left(t\right)=n_0e^{-\frac{\gamma}{\mu-1}R}\frac{i(t-R)}{k(t)} \tag{8.7}$$

is the fraction of scrapped capital at time $t$ because it becomes unprofitable and consequently it is called the obsolescence rate, and

$$\delta=\frac{\gamma}{\mu-1} \tag{8.8}$$

is the decline rate of the average relative productivity of each vintage and consequently it is called the physical depreciation rate.

So the model allows to disentangle obsolescence and physical depreciation, indeed there is a big difference between the behavior of the obsolescence rate and the physical depreciation rate: while the physical depreciation rate $\delta$ is constant, the obsolescence rate $l(t)$ is not so because it depends on the scrapped investment-capital ratio.

We assume that the planner have to solve the following problem:

$$\text{Max} \int_0^\infty \left(\frac{y(t)-i(t)}{1-\sigma}\right)^{1-\sigma} \mathrm{e}^{-\rho t}\,\mathrm{d}t \tag{8.9}$$

s.t.

$$y\,(t) = \int_{t-R}^{t} Q\,(t-z)\,i\,(z)\,dz \tag{1}$$

where $Q\,(t-z)$ is given by (8.4), and given $i(t) = \bar{\imath}(t) > 0$ for all $t \in [-R, 0)$. Parameters $\rho > 0$, $\sigma > 0$, and $\sigma \neq 1$.

so

$$y(t) = \int_{t-R}^{t} (\Omega e^{-\delta(t-s)} - \eta) i(s) \mathrm{d}s \tag{8.10}$$

where $i(t)$ is the investment at time $t$.

$$\eta = \Omega e^{-\delta R} \tag{8.11}$$

Differentiating equation (8.10) and using equation (8.11) we find that $y$ satisfies the following delay differential equation:

$$\dot{y}(t) = (\Omega - \eta) i(t) - \delta\Omega \int_{-R}^{0} e^{\delta r} i(r+t) \mathrm{d}r$$

## 8.2. The optimal control problem

We use the notation introduced in Notation 7.1 to distinguish $\bar{\imath}\colon [-R, 0) \to \mathbb{R}$ that is part of initial data and $i\colon [0, +\infty) \to \mathbb{R}$ that is the control

$$\tilde{\imath}(s) = \begin{cases} \bar{\imath}(s) & s \in [-R, 0) \\ i(s) & s \in [0, +\infty) \end{cases} \tag{8.12}$$

Now we give a precise formulation of the optimal control problem related to the model. We consider the state equation

$$\begin{cases} \dot{y}(t) = (\Omega - \Omega e^{-\delta R})\tilde{\imath}(t) - \delta\Omega \int_{-R}^{0} e^{\delta r} \tilde{\imath}(r+t) \mathrm{d}r \\ \tilde{\imath}(s) = \bar{\imath}(s)\ \forall s \in [-R, 0) \\ y(0) = \int_{-R}^{0} \bar{\imath}(s)(\Omega e^{\delta s} - \Omega e^{-\delta R}) \mathrm{d}s \end{cases} \tag{8.13}$$

where $\bar{\imath}(s) \in L^2((-R, 0); \mathbb{R}^+)$ with $\bar{\imath} \not\equiv 0$ and $y(0) \in \mathbb{R}$ are the initial conditions, $R, \Omega, \delta \in \mathbb{R}$ are positive constants.

For every $i\colon \mathbb{R}^+ \to \mathbb{R}$ locally integrable and every $\bar{\imath} \in L^2((-R, 0); \mathbb{R}^+)$ the (8.13) admits a unique locally absolutely continuous solution given by:

$$y_{\bar{\imath},i}(t) = \int_{(t-R)}^{t} \tilde{\imath}(s)(\Omega e^{-\delta(t-s)} - \Omega e^{-\delta R}) \mathrm{d}s \tag{8.14}$$

We want to maximize the functional

$$J(\bar{\imath}(\cdot); i(\cdot)) \stackrel{def}{=} \int_0^\infty e^{-\rho s} \frac{(y_{\bar{\imath},i}(t) - i(t))^{1-\sigma}}{(1-\sigma)} \mathrm{d}s$$

over the set

$$\mathcal{I}_{\bar{\imath}} \stackrel{def}{=} \{i(\cdot) \in L^2_{loc}([0, +\infty); \mathbb{R}^+)\ :\ i(t) \in [0, y_{\bar{\imath},i}(t)] \text{ for a.e. } t \in \mathbb{R}^+\}.$$

Here $\sigma$ is strictly positive constants with $\sigma \neq 1$. The choice of $\mathcal{I}_{\bar{\iota}}$ implies $y_{\bar{\iota},i}(\cdot) \in W^{1,2}_{\mathrm{loc}}((0,+\infty);\mathbb{R}^+)$ for every $i(\cdot) \in \mathcal{I}_{\bar{\iota}}$.

We will name *Problem* ***(P)*** the problem of finding an optimal control strategy i.e. to find an $i^*(\cdot) \in \mathcal{I}_{\bar{\iota}}$ such that:

$$J(\bar{\iota}(\cdot); i^*(\cdot)) = V(\bar{\iota}(\cdot)) \stackrel{def}{=} \sup_{i(\cdot)\in\mathcal{I}_{\bar{\iota}}} \left\{ \int_0^\infty e^{-\rho s} \frac{(y_{\bar{\iota},i}(t) - i(t))^{1-\sigma}}{(1-\sigma)} \mathrm{d}s \right\}. \tag{8.15}$$

We'll call $V$ *value function.*

## 8.3. The infinite dimensional setting

### 8.3.1. Rewriting *Problem (P)* in infinite dimensions.

NOTATION 8.1. *We use the notation $\tilde{\imath}$ (and in general $z_t$ for some variable $z$) that we introduced in (1.24). we consider the continuous linear application*

$$B\colon C[-R,0] \to \mathbb{R}$$

$$B\colon \kappa \mapsto (\Omega - \Omega e^{-\delta R})\kappa(0) - \delta\Omega \int_{-R}^0 e^{\delta r}\kappa(r)\mathrm{d}r$$

*and the continuous linear application*

$$S\colon L^2(-R,0),\mathbb{R}) \to \mathbb{R}$$

$$S\colon \bar{\iota} \mapsto \int_{-R}^0 \bar{\iota}(s)(\Omega e^{\delta s} - \Omega e^{-\delta R})\mathrm{d}s$$

*so we can re-write the state equation as*

$$\begin{cases} \dot{y}(t) = B(\tilde{\imath}_t) \\ \tilde{\imath}(s) = \bar{\iota}(s)\ \forall s \in [-R,0) \\ y(0) = S(\bar{\iota}) \end{cases} \tag{8.16}$$

*Also equation (8.16) has not a pointwise meaning but has anyway an integral sense. Eventually we'll put $\nu_B$ the finite measure associate to $B$.*

We can rewrite the (8.16) as:

$$\begin{cases} \dot{y}(t) = B(\tilde{\imath}_t) \\ (y(0), \tilde{\imath}_0) = (S(\bar{\iota}),\ \bar{\iota}) \end{cases} \tag{8.17}$$

Similarly to what we did in Chapter 7 we embed the problem in the family:

$$\begin{cases} \dot{y}(t) = B(\tilde{\imath}_t) \\ (y(0), \tilde{\imath}_0) = (y_0,\ \bar{\iota}) \end{cases} \tag{8.18}$$

which has solution

$$y_{y_0,\bar{\iota},i}(t) = y_0 - S(\bar{\iota}) + \int_{(t-R)\vee 0}^t i(s)(\Omega e^{-\delta(t-s)} - \Omega e^{-\delta R})\mathrm{d}s + \tag{8.19}$$

$$+ \int_{(t-R)\wedge 0}^0 \bar{\iota}(s)(\Omega e^{-\delta(t-s)} - \Omega e^{-\delta R})\mathrm{d}s$$

Note that $y_{S(\bar{\iota}),\bar{\iota},i}(t) = y_{\bar{\iota},i}(t)$. Similarly to Chapter 7 the problem can be reformulated in $M^2$ using the generator of a $C_0$-semigroup defined in (1.35) and setting (abusing notation)

$$\begin{cases} B\colon D(A^*) \to \mathbb{R} \\ B(\psi(0),\psi) = B\psi \in \mathbb{R} \end{cases}$$

NOTATION 8.2. *We'll indicate with $F$ the application*

$$F\colon L^2(-R,0) \to L^2(-R,0))$$
$$i \mapsto F(i)$$

*where*

$$F(i)(s) \overset{def}{=} \int_{-R}^{s} i(-s+r)\mathrm{d}\nu_B(r) = \int_{-R}^{s} -\delta\Omega i(-s+r)e^{\delta r}\mathrm{d}r \tag{8.20}$$

DEFINITION 8.3 (**Structural state**). *Taken $\bar{\imath} \in L^2(-R,0)$, $i \in L^2_{loc}[0,+\infty)$, $y_0 \in \mathbb{R}$ and $y_{y_0,\bar{\imath},i}(t)$ we define the* structural state *of the system the couple $x_{y,i}(t) = (x^0_{y,i}(t), x^1_{y,i}(t)) = (y_{y_0,\bar{\imath},i}(t), F(\tilde{\imath}_t))$ where $y \overset{def}{=} (y_0, F(\bar{\imath}))$*

THEOREM 8.4. *Assume that $\bar{\imath} \in L^2(-R,0)$, $i \in L^2_{loc}[0,+\infty)$, $y_0 \in \mathbb{R}$ and that $y_{y_0,\bar{\imath},i}(t)$ is defined by (8.19), then, for each $T > 0$, $x_{p,i}(t) = (x^0_{p,i}(t), x^1_{p,i}(t)) = (y_{y_0,\bar{\imath},i}(t), F(\tilde{\imath}_t))$ is the unique solution in*

$$\Pi \overset{def}{=} \left\{ f \in C([0,T];M^2) \ : \ \frac{d}{\mathrm{d}t} f \in L^2((0,T);D(A^*)') \right\} \tag{8.21}$$

*to the equation:*

$$\begin{cases} \frac{d}{\mathrm{d}t}x(t) = Ax(t) + B^* i(t), & t > 0 \\ x(0) = (y_0, F(\bar{\imath})) \end{cases} \tag{8.22}$$

*where $j^*$, $A$ and $B^*$ are the dual maps of the continuous linear operators*

$$\begin{aligned} &j\colon D(A^*) \hookrightarrow M^2 \\ &A^*\colon D(A^*) \to M^2 \\ &B\colon D(A^*) \to \mathbb{R} \end{aligned}$$

*and $D(A^*)$ is equipped with the graph norm.*

PROOF. We have already see this theorem in the general case in Theorem 1.33. The proof can be found in [BDPDM92] Theorem **5.1** page 258. ☐

**8.3.2. The optimal control problem in infinite dimensions.** Now we formulate an optimal control problem in infinite dimensions that, thanks to results of the previous section, "contains" the *Problem (**P**)*. We'll need first of all a

THEOREM 8.5. *The equation*

$$\begin{cases} \frac{d}{\mathrm{d}t}x(t) = Ax(t) + B^* i(t), & t > 0 \\ x(0) = p \end{cases}$$

*for $p \in M^2$, $i \in L^2_{loc}[0,+\infty)$ has an unique solution in $\Pi$ (defined in (8.21))*

PROOF. We have already see this theorem in the general case in Theorem 1.33. The proof can be found in [BDPDM92] Theorem **5.1** page 258. ☐

After this long preamble we can methodically formulate the optimal control problem in infinite dimensions: We consider the state equation in $M^2$ given by

$$\begin{cases} \frac{d}{\mathrm{d}t}x(t) = Ax(t) + B^* i(t), & t > 0 \\ x(0) = p \end{cases}$$

for $p \in M^2$, $i \in L^2_{loc}[0,+\infty)$. Thanks to Theorem 8.5 it has a unique solution $x_{p,i}(t)$ in $\Pi$, so $t \mapsto x^0_{p,i}(t)$ is continuous and it makes sense to consider the set of controls

$$\hat{\mathcal{I}}_p \overset{def}{=} \{ i \in L^2_{loc}[0,+\infty) \ : \ i(t) \in [0, x^0_{p,i}(t)] \ \ \tilde{\forall}\, t \in \mathbb{R}^+ \}$$

The objective functional is

$$\hat{J}(p,i) \stackrel{def}{=} \int_0^\infty e^{-\rho s} \frac{(x^0_{p,i}(t) - i(t))^{1-\sigma}}{(1-\sigma)} \mathrm{d}s.$$

The value function is then:

$$\begin{cases} \hat{V}(p) \stackrel{def}{=} \sup_{i \in \hat{\mathcal{I}}_p} \left\{ \int_0^\infty e^{-\rho s} \frac{(x^0_{p,i}(t)-i(t))^{1-\sigma}}{(1-\sigma)} \mathrm{d}s \right\} & if\ \mathcal{I}_p \neq \emptyset \\ \hat{V}(p) = -\infty & if\ \mathcal{I}_p = \emptyset \end{cases}$$

Remark 8.6. *If we set*

$$p = \big(S(\bar{\iota}), F(\bar{\iota})\big)$$

*we find* $\hat{\mathcal{I}}_p = \mathcal{I}_{\bar{\iota}}$, $\hat{J}(p,i) = J(\bar{\iota}, i)$ *and* $\hat{V}(p) = V(\bar{\iota})$ *and the solution of differential equation of Theorem 8.5 is given by 8.3.*

## 8.4. The dynamic programming approach

**8.4.1. Preliminary results.** It is natural, as in the problem without delay, to introduce a first restriction that ensure the finiteness of the value function $V$ at every point. To check the finiteness of value function is useful to observe that the maximal growth of $y(t)$ is obtained when the investment is maximal (that is $i(t) = y(t)$, the proof can be done as in Proposition 7.3). The delay differential equation related to such strategy is

$$\dot{y}(t) = (\Omega - \Omega e^{-\delta R})(y(t)) - \delta\Omega \int_{-R+t}^{t} e^{\delta r} y(r+t)\mathrm{d}r \tag{8.23}$$

The *characteristic equation* of (8.23) is

$$z = ((\Omega - \Omega e^{-\delta R}) - \frac{(\delta\Omega)}{\delta + z}(1 - e^{-(\delta+z)R})) \tag{8.24}$$

Lemma 8.7. *A strictly positive root of (8.24) exists if and only if*

$$(H1) \quad \Omega > \frac{\delta}{1 - \delta e^{-\delta R}(\frac{1}{\delta} + R)} > 0$$

*If a strictly positive root $\xi$ exists it is unique.*

Proof. **First step: If (H1) is satisfied a strictly positive root exists**. If we name $\psi$ the continuous function

$$\psi\colon [0,+\infty) \to \mathbb{R}$$

$$\xi \mapsto ((\Omega - \Omega e^{-\delta R}) - \frac{(\delta\Omega)}{\delta+\xi}(1 - e^{-(\delta+\xi)R}))$$

we see that $\psi(0) = 0$,

$$\lim_{\xi\to\infty} \psi(\xi) = (\Omega - \Omega e^{-\delta R})$$

and $\psi'(0) = \Omega(\frac{1}{\delta} - \frac{1}{\delta}e^{-\delta R} - Re^{-\delta R}) > 1$.

**Second step: the root is unique and the condition (H1) is necessary**

It is enough to observe that $\psi$ is a strictly concave function. We have to calculate two derivatives:

$$\psi'(\xi) = \frac{\delta\Omega}{(\delta+\xi)^2}(1 - e^{-(\delta+\xi)R}) - \frac{\delta\Omega}{(\delta+\xi)}(Re^{-(\delta+\xi)R})$$

and

$$\psi''(\xi) = (\delta\Omega)\Big(\frac{-2}{(\delta+\xi)^3}\big(1 - e^{-(\delta+\xi)R}\big) + \frac{1}{(\delta+\xi)^2}\big(Re^{-(\delta+\xi)R}\big) + \\ + \frac{1}{(\delta+\xi)^2}\big(Re^{-(\delta+\xi)R}\big) - \frac{1}{(\delta+\xi)}\big(-R^2 e^{-(\delta+\xi)R}\big)\Big)$$

So the second derivative has the form:

$$-g(x) \stackrel{def}{=} -\Big(2 - 2e^{-x} - 2xe^{-x} - x^2e^{-x}\Big)$$

and

$$g'(x) = x^2e^{-x} > 0 \quad when\ x > 0$$

moreover $g(0) = 0$ than $-g(x)$ is always negative (except in 0) and so $\psi''(\xi) < 0$ for all $\xi \in [0, +\infty)$ and $\psi$ is strictly concave and we have the claim. □

REMARK 8.8. *We could go on for a while without Hypothesis (H1): all results we see in this section don't use substantially such hypothesis, but (H1) has a strong economic meaning: if there aren't positive root we can see, as in Proposition 8.9 that, also re-investing all $y(t)$, it goes to zero and then it is not possible have positive growth.*

We will assume from now the Hypothesis (H1). We will call $\xi$ the strictly positive root of characteristic equation.

Observing that for, all admissible control and every $t \geq 0$, $y(t) \leq f(t)$ where $f(\cdot)$ solves the differential equation

$$\begin{cases} \dot{f}(t) = (\Omega - \Omega e^{-\delta R})f(t) \\ f(0) = y(0) \end{cases},$$

we have

PROPOSITION 8.9. *The solution of (8.23) is continuous on $\mathbb{R}^+$ and satisfies for every $\epsilon > 0$*

$$\frac{y(t)}{e^{(\Omega - \Omega e^{-\delta R} + \epsilon)t}} \to 0 \quad when\ \ t \to +\infty$$

Now, thanks to Proposition 8.9 we can introduce an hypothesis that will allow us to exclude the existence of some $\bar{\iota}$ in which $V(\bar{\iota}) = +\infty$. The assumption is

$$(H2) \quad \rho > (\Omega - \Omega e^{-\delta R})(1 - \sigma)$$

and we'll always assume it in the sequel.

COROLLARY 8.10. *$V(\bar{\iota}) < +\infty$ for all $\bar{\iota}$ in $L^2_{loc}([0, +\infty); \mathbb{R}^+)$.*

PROOF. It is a consequence of Proposition 8.9 □

Now we state two proposition that can be proved using the same techniques of Chapter 7.

PROPOSITION 8.11. *An optimal control exist in $\mathcal{I}_{\bar{\iota}}$, that is: we can find in $\mathcal{I}_{\bar{\iota}}$ an admissible strategy $i^*(t)$ such that $V(\bar{\iota}) = J(\bar{\iota}, i^*)$.*

PROOF. The proof can be done similarly to Proposition 7.13. □

We can now give a result that we'll use in the section 8.5:

LEMMA 8.12. *Let $\bar{\iota}$ be in $L^2((-R, 0); \mathbb{R}^+)$ and let $i^* \in \mathcal{I}_{\bar{\iota}}$ be an optimal strategy then $y_{\bar{\iota}, i^*}(t) \neq 0$ for all $t \in [0, +\infty)$.*

PROOF. The proof can be done similarly to Lemma 7.14. □

**8.4.2. HJB equation and dynamic programming in infinite dimensions.** We now describe the Hamiltonians of the system. First of all we introduce the *current value* Hamiltonian: it will be defined on a subset $E$ of $M^2 \times M^2 \times \mathbb{R}$ (the product of state space, co-state space and control space) given by[2]

$$E \stackrel{def}{=} \{(x, P, i) \in M^2 \times M^2 \times \mathbb{R} \ : x^0 > 0, \ i \in [0, ax^0], \ P \in D(A^*)\}$$

and its form is the following:

$$H_{CV}(x, P, i) \stackrel{def}{=} \langle x, A^* P\rangle_{M^2} + \langle i, BP\rangle_{\mathbb{R}} + \frac{(x^0 - i)^{1-\sigma}}{(1-\sigma)}$$

When $\sigma > 1$ the above is not defined in the points in which $ax^0 = i$. In such points we set then $H_{CV} = -\infty$. In this way we take $H_{CV}$ with values in $\overline{\mathbb{R}}$.

We can now define the maximum value Hamiltonian (that we will simply call Hamiltonian) of the system: we name $G$ the subset of $M^2 \times M^2$ (the product of state space and co-state space) given by:

$$G \stackrel{def}{=} \{(x, P) \in M^2 \times M^2 \ : \ x^0 > 0, \ P \in D(A^*)\}$$

The Hamiltonian is given by:

$$\begin{cases} H \colon G \to \overline{\mathbb{R}} \\ H \colon (x, P) \mapsto \sup_{i \in [0, x^0]} H_{CV}(x, P, i) \end{cases}$$

The *HJB* equation is

$$\rho w(x) \ - H(x, Dw(x)) = 0$$

i.e.

$$\rho w(x) \ - \sup_{i \in [0, ax^0]} \left\{ \langle x, A^* Dw(x)\rangle_{M^2} + \langle i, BDw(x)\rangle_{\mathbb{R}} + \frac{(ax^0 - i)^{1-\sigma}}{(1-\sigma)} \right\} = 0 \tag{8.25}$$

DEFINITION 8.13 (**Regular solution**). *Let $\Omega$ be an open set of $M^2$ and $Y \subseteq \Omega$ a closed subset. An application $w \in C^1(\Omega; \mathbb{R})$ satisfies the HJB equation on $Y$ if $\forall (p^0, p^1) \in Y$*

$$\begin{cases} Dw(p^0, p^1) \in D(A^*) \\ \rho w(p^0, p^1) - H\big((p^0, p^1), Dw(p^0, p^1)\big) = 0 \end{cases}$$

REMARK 8.14. *If $P \in D(A^*)$ and $(BP)^{-1/\sigma} \in (0, x^0]$, by elementary arguments, the function*

$$H_{CV}(x, P, \cdot) \colon [0, x^0] \to \mathbb{R}$$

*admits exactly a maximum in the point*

$$i^{MAX} = x^0 - (BP)^{-1/\sigma} \ \in [0, x^0)$$

*and then we can write the Hamiltonian in a simplified form:*

$$H((x^0, x^1), P) = \langle (x^0, x^1), A^* P\rangle_{M^2} + x^0 BP + \frac{\sigma}{1-\sigma} (BP)^{\frac{\sigma-1}{\sigma}} \tag{8.26}$$

*The expression for $i^{MAX}$ will be used to write the solution of the* problem (**P**) *in closed-loop form.*

We define

$$X \stackrel{def}{=} \left\{ (x^0, x^1) \in M^2 \ : \ x^0 > 0, \ \left( x^0 + \int_{-R}^{0} e^{\xi s} x^1(s) \mathrm{d}s \right) > 0 \right\}$$

[2]Recall that an element $x \in M^2$ is done by two components: $x^0$ and $x^1$, so $x = (x^0, x^1)$.

and $\left(\text{naming } \omega = \frac{\rho-\xi(1-\sigma)}{\sigma\xi}\right)$

$$Y \stackrel{def}{=} \left\{(x^0, x^1) \in X \ : \ \int_{-R}^{0} e^{\xi s} x^1(s)\mathrm{d}s \leq x^0 \frac{1-\omega}{\omega}\right\} \tag{8.27}$$

It is easy to see that $X$ is an open set of $M^2$ and $Y \subseteq X$ is closed in $X$. We define, for $x \in M^2$ the quantity

$$\Gamma_0(x) \stackrel{def}{=} x^0 + \int_{-R}^{0} e^{\xi s} x^1(s)\mathrm{d}s \tag{8.28}$$

PROPOSITION 8.15. *Let Hypotheses* $(H1)$ *and* $(H2)$ *be satisfied, then*

$$v \colon X \to \mathbb{R}$$

$$v(x) = v(x^0, x^1) \stackrel{def}{=} \nu\left(\int_{-R}^{0} e^{\xi s} x^1(s)\mathrm{d}s + x^0\right)^{1-\sigma} = \nu\Gamma_0(x)^{1-\sigma}$$

*with*

$$\nu = \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi}\right)^{-\sigma} \frac{1}{(1-\sigma)\xi}$$

*is differentiable in all* $(x^0, x^1) \in X$ *and is solution of the HJB equation in all the points of* $Y$ *in the sense of definition 8.13*

PROOF. $v$ is continuous and differentiable in every point of $X$ and its differential in $(x^0, x^1)$ is

$$Dv(x^0, x^1) = (\nu(1-\sigma)\Gamma_0(x)^{-\sigma}, \{s \mapsto (1-\sigma)\nu\Gamma_0(x)^{-\sigma} e^{\xi s}\})$$

So $Dv(x^0, x^1) \in D(A^*)$ everywhere in $X$.
We can also calculate explicitly $A^*Dv$ and $BDv$, we have:

$$A^*Dv(x^0, x^1) = (0, \{s \mapsto (1-\sigma)\nu\Gamma_0(x)^{-\sigma}\xi e^{\xi s}\}) \tag{8.29}$$

$$BDv(x^0, x^1) = (1-\sigma)\nu\Gamma_0(x)^{-\sigma}((\Omega - \Omega e^{-\delta R}) - (\delta\Omega)\int_{-R}^{0} e^{(\delta+\xi)s}\mathrm{d}s) = \tag{8.30}$$

$$= (1-\sigma)\nu\Gamma_0(x)^{-\sigma}\xi > 0 \quad \textit{on } X \tag{8.31}$$

so

$$(BDv)^{-1/\sigma} = \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi}\left(\int_{-R}^{0} e^{\xi s} x^1(s)\mathrm{d}s + x^0\right)\right) \tag{8.32}$$

$(BDv)^{-1/\sigma} \geq 0$.
If $(x^0, x^1) \in Y$ then

$$(\int_{-R}^{0} e^{\xi s} x^1(s)\mathrm{d}s + x^0) \leq \frac{1}{\omega} x^0 \tag{8.33}$$

and then $(BDv)^{-1/\sigma} \leq x^0$. So we can use Remark 8.14 and use the Hamiltonian in the form of equation (8.26).
Now it is sufficient substitute (8.29) and (8.30) in (8.26) and verify by easy calculations, the relation:

$$\rho v(x^0, x^1) - \langle (x^0, x^1), A^*Dv(x^0, x^1)\rangle_{M^2} -$$
$$-x^0 BDv((x^0, x^1) - \frac{\sigma}{1-\sigma}(BDv((x^0, x^1))^{\frac{\sigma-1}{\sigma}} = 0$$

□

REMARK 8.16. *Note the strong formal analogies with the problem treated in Chapter 7. The HJB equation of the problem appears indeed as (8.25) but the meaning of the objects that appear are different. Indeed the operator* $B$ *in not the same (moreover* $\xi$ *has a different meaning and the state represent another DDE).*

**8.4.3. Closed loop in infinite dimensions.** We begin with some definitions:

DEFINITION 8.17 ($\mathbf{AFS_p}$). *Taken* $p \in M^2$ *we'll call* $\Phi \in C(M^2)$ *an* the admissible feedback strategy related to $p$ *if the equation*

$$\begin{cases} \frac{d}{dt} x_\Phi(t) = A x_\Phi(t) + B^*(\Phi(x_\Phi(t))), & t > 0 \\ x_\Phi(0) = p \end{cases}$$

*has an unique solution* $x_\Phi(t)$ *in* $\Pi$ *and* $\Phi(x_\Phi(\cdot)) \in \hat{\mathcal{I}}_p$ *We'll indicate the set of admissible feedback strategies related to p with* $AFS_p$

DEFINITION 8.18 ($\mathbf{OFS_p}$). *Taken* $p \in M^2$ *we'll call* $\Phi$ *an* optimal feedback strategy related to $p$ *if it is in* $AFS_p$ *and*

$$\hat{V}(p) = \int_0^{+\infty} e^{-\rho t} \frac{\left(x_\Phi^0(t) - \Phi(x_\Phi(t))\right)^{1-\sigma}}{(1-\sigma)} \mathrm{d}t$$

*We'll indicate the set of optimal feedback strategies related to p with* $OFS_p$

We have a solution of the HJB equation only in a part of the state space (just $Y$). So we can prove a feedback result (and so the optimality of the feedback) only if the admissible trajectories remain in $Y$. So we have to make some other hypothesis. A sufficient condition is:

$$(H3) \quad \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi}\right)\left(1 - \frac{\delta}{\delta+\xi} - e^{-(\delta+\xi)R}\right) \leq 1 \tag{8.34}$$

we'll assume it in the sequel.

REMARK 8.19. *We'll see in the Theorem 8.24 the optimal consumption growth as* $\Lambda e^{gt}$ *where* $g = \frac{\xi-\rho}{\sigma}$ *(the optimal growth of other variables is more complex). Then the condition to have a positive growth (of the consumption) is* $g > 0$ *that is* $\frac{\rho-\xi(1-\sigma)}{\sigma\xi} < 1$. *The condition* $g > 0$ *implies (H3) (indeed* $\left(1 - \frac{\delta}{\delta+\xi} - e^{-(\delta+\xi)R}\right)$ *is in* $(0,1)$*).*
*So if we impose the condition of positive growth we automatically have (H3).*

THEOREM 8.20. *Taken* $p = (S(\bar{\iota}), F(\bar{\iota}))$ *for some* $\bar{\iota} \in L^2((-R,0);\mathbb{R}^+)$, *the application*

$$\Phi\colon M^2 \to \mathbb{R}$$
$$\Phi(x) \overset{def}{=} x^0 - \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi}\right)\Gamma_0(x)$$

*is in* $OFS_p$

PROOF. First of all we have to observe that $\Phi \in AFS_p$. We claim that

$$\begin{cases} \frac{d}{dt} x_\Phi(t) = A x_\Phi(t) + B^*(\Phi(x_\Phi(t))), & t > 0 \\ x_\Phi(0) = p = (S(\bar{\iota}), F(\bar{\iota})) \end{cases} \tag{8.35}$$

has a solution in $\Pi$:
We consider the solution $i$ of the following delay-differential equation

$$\begin{cases} i(t) = & \left(1 - \frac{\rho-\xi(1-\sigma)}{\sigma\xi}\right)\left(\int_{(t-R)}^t i(s)(\Omega e^{-\delta(t-s)} - \Omega e^{-\delta R})\mathrm{d}s\right) - \\ & - \frac{\rho-\xi(1-\sigma)}{\sigma\xi} \int_{-R}^0 e^{\xi s} F(i_t)(s)\mathrm{d}s \\ i(s) = & \bar{\iota} \quad \forall\, s \in [-R, 0) \end{cases} \tag{8.36}$$

that has an absolute continuous solution $i$ on $[0,+\infty)$ (see for example [BDPDM92] page 287 for a proof).
Then we consider the equation

$$\begin{cases} \frac{d}{dt} x = Ax + B^*(i(t)), & t > 0 \\ x(0) = p = (S(\bar{\iota}), F(\bar{\iota})) \end{cases} \tag{8.37}$$

We know, thanks to Theorem 8.4, that the only solution in $\Pi$ of such equation is $(y(t), F(i_t))$ where $y(t)$ is the solution of

$$\begin{cases} \dot{y}(t) = B(i_t) \\ (y(0), i_0) = (S(\bar{\iota}),\ \bar{\iota}) \end{cases}$$

We claim that $x(t)$ is solution of (8.35) indeed

$$\Phi(x(t)) = y(t) - \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi}\right)\left(\int_{-R}^{0} e^{\xi s} F(i_t)(s)\mathrm{d}s + y(t)\right) \tag{8.38}$$

and so (by (8.36):

$$\Phi(x(t)) = y(t)\left(1 - \frac{\rho - \xi(1-\sigma)}{\sigma\xi}\right) + i(t) -$$
$$-\left(1 - \frac{\rho - \xi(1-\sigma)}{\sigma\xi}\right)\left(\int_{(t-R)}^{t} i(s)(\Omega e^{-\delta(t-s)} - \Omega e^{-\delta R})\mathrm{d}s\right)$$

and by (8.37) we conclude that

$$\Phi(x(t)) = i(t)$$

and so $x(t) = x_\Phi(t)$ is a solution of (8.35) and is in $\Pi$. Moreover thanks to the linearity of $\Phi$ we can observe that $x_\Phi(t)$ is the only solution in $\Pi$. We have now to observe that $\Phi(x_\Phi(\cdot)) \in \hat{\mathcal{I}}_p$, but the previous steps of the proof show that

$$x_\Phi(t) = \Big(S(i_t), F(i_t)\Big)$$

where $i_{|[0,+\infty)}$ is absolute continuous and so $L^2_{loc}$. We claim that $\Phi(x_\Phi(t)) = i(t) \in [0, x^0_\Phi(t))$ that is (by (8.38))

$$\left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi}\right)\left(\int_{-R}^{0} e^{\xi s} F(i_t)(s)\mathrm{d}s + y(t)\right) \in \Big(0, S(i_t)\Big]$$

First we prove that $i(t) \geq 0$:

$$\left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi}\right)\left(\int_{-R}^{0} e^{\xi s} x^1_\Phi(s)\mathrm{d}s + x^0_\Phi\right) =$$
$$= \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi}\right)\left(\int_{-R}^{0} \Omega i_t(r)\left(e^{\delta r} - e^{-\delta R} -\right.\right.$$
$$\left.\left. - \frac{\delta}{\delta+\xi} e^{\delta r} e^{(\delta+\xi)r} + \frac{\delta}{\delta+\xi} e^{\delta r} e^{-(\delta+\xi)R}\right)\mathrm{d}r\right) \leq$$
$$\leq \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi}\right)\left(\int_{-R}^{0} \Omega i_t(r)\Big((e^{\delta r} - e^{-\delta R})\Big(1 - \frac{\delta}{\delta+\xi} e^{-R(\delta+\xi)}\Big)\Big)\mathrm{d}r\right) =$$
$$= \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi}\right)\Big(1 - \frac{\delta}{\delta+\xi} e^{-(\delta+\xi)R}\Big) S(i_t) \leq$$
$$\leq S(i_t) = x^0_\Phi(t)$$

where the last inequality follows by Hypothesis (H3) and so $i(t) \geq 0$. We prove now that $i(t) < x^0_\Phi(t)$: by (8.36)

$$x^0_\Phi(t) - i(t) = \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi}\right)\left(S(i_t) + \int_{-R}^{0} e^{\xi s} F(i_t)(s)\mathrm{d}s\right)$$

and

$$\left(S(i_t) + \int_{-R}^{0} e^{\xi s} F(i_t)(s)\mathrm{d}s\right) = \int_{-R}^{0} (\Omega e^{\delta r} - \Omega e^{-\delta R}) i(r+t)\mathrm{d}r + \int_{-R}^{0} e^{\xi s} \int_{-R}^{s} i(r-s+t)\mathrm{d}\nu_B(r)\mathrm{d}s =$$
$$= \int_{-R}^{0} \Omega i(r+t)\left(e^{\delta r} - e^{-\delta R} - \frac{\delta}{\delta+\xi} e^{\delta r} e^{(\delta+\xi)r} + \frac{\delta}{\delta+\xi} e^{\delta r} e^{-(\delta+\xi)R}\right)\mathrm{d}r$$

We consider now the application $\Psi\colon [-R,0] \to \mathbb{R}$ given by

$$\Psi\colon r \mapsto e^{\delta r} - e^{-\delta R} - \frac{\delta}{\delta+\xi} e^{\delta r} e^{(\delta+\xi)r} + \frac{\delta}{\delta+\xi} e^{\delta r} e^{-(\delta+\xi)R}$$

We note that:

$$\begin{array}{ll} (i) & \Psi(-R) = 0 \\ (ii) & \Psi(0) = \Omega \frac{1-e^{-\delta R} - \frac{\delta}{\delta+\xi}(1-e^{-(\xi+\delta)R})}{\Omega} = \frac{\xi}{\Omega} > 0 \\ (iii) & \Psi'(r) = e^{\delta r}\left( \frac{(\delta^2+\delta\xi)(1-e^{(\delta+\xi)r}) - \delta^2 (e^{(\delta+\xi)r} - e^{-(\delta+\xi)R})}{\delta+\xi} \right) \end{array}$$

So $\Psi'(r)$ between $-R$ and 0 is in a first part $> 0$ and then $\leq 0$ But we know that $\Psi(-R) = 0$ and $\Psi(0) > 0$ so $\Psi(r) > 0$ for all $r$ in $(-R, 0]$. Eventually $i(t) \geq 0$ for all $t > -R$ and $i_{|[0,+\infty)} \not\equiv 0$ implies

$$x^0_\Phi(t) - i(t) = \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi}\right)\left(S(i_t) + \int_{-R}^0 e^{\xi s} F(i_t)(s) \mathrm{d}s\right) > 0$$

and so $i(t) < x^0_\Phi(t)$.
Now we know that $\Phi$ is an admissible strategy related to $p = (S(\bar\iota), F(\bar\iota))$. It is easy to see from what we have said in this first part of the proof that $x_\Phi(t)$ remain in $Y$ as defined in (8.27) and so the Hamiltonian can be expressed in the simplified form of equation (8.26) and $v$ is a solution of the HJB equation on such points.
We see now that $\Phi \in OFS_p$:
We introduce:

$$\begin{aligned} &\tilde v(t,x)\colon \mathbb{R} \times X \to \mathbb{R} \\ &\tilde v(t,x) \stackrel{def}{=} e^{-\rho t} v(x) \\ &\tilde H\colon \mathbb{R} \times F \to \mathbb{R} \\ &\tilde H(t,x,P) \stackrel{def}{=} e^{-\rho t} H(x,P) = \\ &\quad = \sup_{i\in[0,cx^0]} \left\{ e^{-\rho t}\left( \langle (x^0,x^1), A^*P\rangle_{M^2} + \langle i, BP\rangle_{\mathbb{R}} + \frac{(x^0-i)^{1-\sigma}}{(1-\sigma)} \right)\right\} \end{aligned}$$

Using that $(Dv(x_\Phi(t))) \in D(A^*)$ and that the application $x \mapsto D_x v(x)$ is continuous with respect to the norm of $D(A^*)$, we find:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\tilde v(t, x_{\Phi(t)}) &= -\rho \tilde v(t, x_\Phi(t)) + \\ &\quad + \langle D_x \tilde v(t, x_\Phi(t)) | A x_\Phi(t) + B^* i(t)\rangle_{D(A^*)\times D(A^*)'} = \\ &= -\rho e^{-\rho t} v(x_\Phi(t)) + \\ &\quad + e^{-\rho t}\Big( \langle A^* Dv(x_\Phi(t)), x_\Phi(t)\rangle_{M^2} + BDv(x_\Phi(t)) i(t)\Big) \end{aligned} \tag{8.39}$$

By definition (note that $\hat J(p,i) = \hat J(p, \Phi(x_\Phi))$):

$$v(p) - \hat J(p,i) = v(x_\Phi(0)) - \int_0^\infty e^{-\rho t} \frac{(x^0_\Phi(t) - \Phi(x_\Phi)(t))^{1-\sigma}}{(1-\sigma)} \mathrm{d}t =$$

using (8.39) and Proposition 8.9

$$\begin{aligned} &= \int_0^\infty e^{-\rho t}\Bigg( \rho v(x_\Phi(t)) - \langle A^* Dv(x_\Phi(t)), x_\Phi(t)\rangle_{M^2} - \\ &\qquad - \langle BDv(x_\Phi(t)), i(t)\rangle_{\mathbb{R}} - \frac{(x^0_\Phi(t) - i(t))^{1-\sigma}}{(1-\sigma)} \Bigg) \mathrm{d}t = \end{aligned}$$

$$= \int_0^\infty e^{-\rho t}\Big( H(x_\Phi(t), Dv(x_\Phi(t))) - H_{CV}(x_\Phi(t), Dv(x_\Phi(t)), i(t)) \Big) \mathrm{d}t \tag{8.40}$$

The conclusion follows by three simple observation:

(1) The relation (8.40) implies that $v(p) - \hat{J}(p,i) \geq 0$ (this implies $v(p) \geq \hat{V}(p)$).

(2) The original maximization problem is equivalent to the problem of find a control $i$ that minimize $v(p) - \hat{J}(p,i)$

(3) The feedback strategy $\Phi$ achieves $v(p) - \hat{J}(p,i) = 0$.

and from remark 8.6. □

COROLLARY (OF THE PROOF) 8.21. *Taken $p = (S(\bar{\iota}), F(\bar{\iota}))$ for some $\bar{\iota} \in L^2((-R,0);\mathbb{R}^+)$ we have that*

$$V(\bar{\iota}) = \hat{V}(p) = v(p)$$

*that is: on such point we have an explicit expression for the value function $\hat{V}$ given by $v$.*

## 8.5. Back to *Problem (**P**)*

Now we can use the result we found in infinite dimensional setting to give some results for original optimal control problem regulated by the delay differential equation: *Problem (**P**)*.

First of all observe that, given any $f(\cdot) \in L^2((-R,0);\mathbb{R}^+)$ and writing $y = (S(f), F(f))$, the quantity $\Gamma_0(y)$ defined in (8.28) becomes

$$\begin{aligned}(8.41)\quad \Gamma(f(\cdot)) &\stackrel{def}{=} \Gamma_0\left(S(f), F(f)\right) = \\ &= \int_{-R}^0 \left(\Omega e^{\delta s} - \Omega e^{-\delta R}\right) f(s)\mathrm{d}s + \int_{-R}^0 e^{\xi s} \int_{-R}^s -\delta\Omega f(-s+r)e^{\delta r}\mathrm{d}r\mathrm{d}s = \\ &= y(0) - \int_{-R}^0 e^{\xi s} \int_{-R}^s \delta\Omega f(-s+r)e^{\delta r}\mathrm{d}r\mathrm{d}s\end{aligned}$$

We will use such notation both for initial datum ($f = \bar{\iota}$) and the generic $f = \tilde{\imath}_t$ for some $t \geq 0$.

From Corollary 8.21 we can say that

PROPOSITION 8.22. *Let Hypotheses (H1), (H2) and (H3) be satisfied, then the explicit expression for the value function $V$ related to* Problem (**P**)*, for some $\bar{\iota} \in L^2((-R,0);\mathbb{R}^+)$, is*

$$(8.42)\quad V(\bar{\iota}) = a(\Gamma(\bar{\iota}(\cdot)))^{1-\sigma} = a\left(y(0) - \int_{-R}^0 e^{\xi s} \int_{-R}^s \delta\Omega\bar{\iota}(-s+r)e^{\delta r}\mathrm{d}r\mathrm{d}s\right)^{1-\sigma}$$

*where*

$$a = \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi}\right)^{-\sigma} \frac{1}{(1-\sigma)\xi}$$

Moreover, as corollary of Proposition 8.20 we can give a first solution in closed form for *Problem (**P**)*:

PROPOSITION 8.23. *Let Hypotheses (H1), (H2) and (H3) be satisfied. Taken, for some $\bar{\iota} \in L^2((-R,0);\mathbb{R}^+)$, an initial data $(y(0), \tilde{\imath}_0) = (S(\bar{\iota}), \bar{\iota})$ in equation (8.13) the optimal control for* Problem (**P**) *$i^*$ and the related state trajectory $y^*$ satisfy for all $t \geq 0$:*

$$i^*(t) = y^*(t) - \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi}\right)\Gamma(\tilde{\imath}^*_t(\cdot))$$

THEOREM 8.24. *Let Hypotheses (H1), (H2) and (H3) be satisfied. Taken, for some $\bar\imath \in L^2((-R,0);\mathbb{R}^+)$, an initial data $(y(0),\tilde\imath_0) = (S(\bar\imath),\bar\imath)$ in equation (8.13), there exists a $\Lambda$ such that along the optimal trajectory the optimal control for* Problem $(\mathbf{P})$ *$i^*$ and the related state trajectory $y^*$ satisfy for all $t \geq 0$:*

$$C^*(t) = y^*(t) - i^*(t) = \Lambda e^{gt} \tag{8.43}$$

*where $g = \frac{\xi-\rho}{\sigma}$*
*Moreover the explicit value of $\Lambda$ can be computed and it is*

$$\Lambda = \left(\frac{\rho-\xi(1-\sigma)}{\sigma\xi}\right)\Gamma(\bar\imath(\cdot)) = \tag{8.44}$$
$$= \left(\frac{\rho-\xi(1-\sigma)}{\sigma\xi}\right)\left(y(0) - \int_{-R}^{0} e^{\xi s}\int_{-R}^{s}\delta\Omega\bar\imath(-s+r)e^{\delta r}\mathrm{d}r\mathrm{d}s\right)$$

PROOF. Along optimal trajectory we have:

$$y_{y_0,\bar\imath,i}(t) - i(t) = \left(\frac{\rho-\xi(1-\sigma)}{\sigma\xi}\right)\left(\int_{-R}^{0} e^{\xi s}F(\tilde\imath_t)(s)\mathrm{d}s + y_{y_0,\bar\imath,i}(t)\right)$$

We calculate now the the derivative of such expression: We take $\psi = (1, s\mapsto e^{\xi s}) \in M^2$. It is easy to see that $\psi \in D(A^*)$. So we have (by Theorem 8.4)

$$\frac{d}{dt}\left(\int_{-R}^{0} e^{\xi s}F(\tilde\imath_t)(s)\mathrm{d}s + y_{y_0,\bar\imath,i}(t)\right) = \frac{d}{dt}\langle\psi, x(t)\rangle_{M^2} =$$

(by equation (8.22) and that $\xi$ is a root of (8.24))

$$= \langle A^*\psi, x(t)\rangle_{M^2} + \langle B(\psi), i(t)\rangle_{\mathbb{R}} = \left\langle(0, s\mapsto \xi e^{\xi s}), x(t)\right\rangle_{M^2} + \langle \xi, i(t)\rangle_{\mathbb{R}} =$$

(using the relation given by Proposition (8.23), that give as $i(t)$ on optimal trajectory, and the explicit expression of $x(t)$ (see definition 8.3))

$$= \left[\xi\left(\int_{-R}^{0} e^{\xi s}F(\tilde\imath_t)(s)\mathrm{d}s\right)\right] +$$
$$\left[\xi\left(y_{y_0,\bar\imath,i}(t) - \left(\frac{\rho-\xi(1-\sigma)}{\sigma\xi}\right)\left(\int_{-R}^{0} e^{\xi s}F(\tilde\imath_t)(s)\mathrm{d}s + y_{y_0,\bar\imath,i}\right)\right)\right] =$$

by simple calculations

$$= \left(\xi - \frac{\rho-\xi(1-\sigma)}{\sigma}\right)\left(\int_{-R}^{0} e^{\xi s}F(\tilde\imath_t)(s)\mathrm{d}s + y_{y_0,\bar\imath,i}(t)\right)$$
$$= g\left(\int_{-R}^{0} e^{\xi s}F(\tilde\imath_t)(s)\mathrm{d}s + y_{y_0,\bar\imath,i}(t)\right)$$

and so we have that there exists a $\Lambda$ such that

$$y^*(t) - i^*(t) = \Lambda e^{gt}$$

To find the value of $\Lambda$ we use Proposition 8.23 that gives us the value of the optimal control in 0:

$$\Lambda = y_{\bar\imath,i}(0) - i(0) = \left(\frac{\rho-\xi(1-\sigma)}{\sigma\xi}\right)\left(y_{\bar\imath,i}(0) + \int_{-R}^{0} e^{\xi s}F(i_0)(s)\mathrm{d}s\right)$$

□

The following is a consequence of Theorem 8.24 but we want to emphasizes such result:

COROLLARY 8.25. *Let Hypotheses (H1), (H2) and (H3) be satisfied. Taken an initial data* $(y(0), \tilde{\imath}_0) = (S(\bar{\imath}), \bar{\imath})$ *in equation (8.13)* $i^*$ *is connected with the related state equation trajectory* $y^*$ *by the following optimal feedback strategy:*

$$i^*(t) = y^*(t) - \Lambda e^{gt} \tag{8.45}$$

*for all* $t \geq 0$

COROLLARY 8.26. *Re-scaling the variables:*

$$\bar{y}(t) \stackrel{def}{=} e^{-gt} y^*(t)$$

$$\bar{\imath}(t) \stackrel{def}{=} e^{-gt} i^*(t)$$

*we have that* $c^*(t) = \big(\bar{y}(t) - \bar{\imath}(t)\big)$*, that is the discounted consumption, is constant on optimal trajectories.*

From what we have seen we can state that:

PROPOSITION 8.27. *Let Hypotheses (H1), (H2) and (H3) be satisfied. Taken, for some* $\bar{\imath} \in L^2((-R, 0); \mathbb{R}^+)$*, an initial data* $(y(0), \tilde{\imath}_0) = (S(\bar{\imath}), \bar{\imath})$ *in equation (8.13) the optimal control for* Problem (**P**) $\tilde{\imath}^*(t)$ *is the unique solution in* $H^1_{loc}([0, +\infty); \mathbb{R}^+)$ *of the following delay differential equation:*

$$\begin{cases} \tilde{\imath}^*(t) = \int_{(t-R)}^t \tilde{\imath}^*(s)(\Omega e^{-\delta(t-s)} - \Omega e^{-\delta R}) \mathrm{d}s - \Lambda e^{gt} \\ \tilde{\imath}^*_0 = \bar{\imath} \end{cases} \tag{8.46}$$

*Moreover the optimal* $y(\cdot)$ *is the unique solution in* $H^1_{loc}([0, +\infty); \mathbb{R}^+)$ *of the following integral equation*

$$\begin{aligned} y^*(t) = &\int_{(t-R)\wedge 0}^0 (\Omega e^{\delta s} - \Omega e^{-\delta R}) \bar{\imath}(s) \mathrm{d}s + \\ &+ \int_{(t-R)\vee 0}^t (\Omega e^{-\delta(t-s)} - \Omega e^{-\delta R}) \left[y^*(s) - \Lambda e^{gs}\right] \mathrm{d}s, \qquad t \geq 0. \end{aligned} \tag{8.47}$$

DEFINITION 8.28 (**BGP**). *We'll say that the optimal control for* Problem (**P**) $i^*$ *is a* Balanced Growth Path *(BGP) if there exists* $a_0, b_0 > 0$*, and real numbers* $a_1, b_1$ *such that*

$$\tilde{\imath}^*(s) = a_0 e^{a_1 s} \ \ for\ s \in [-R, +\infty \ \ y^*(s) = b_0 e^{b_1 s} for\ s \in [0, +\infty)$$

We can try to understand if there are BGP in our model.

PROPOSITION 8.29. *If* $g = \frac{\xi - \rho}{\sigma} > 0$ *then there is a non trivial Balanced Growth Path with* $a_1 = b_1 = g$ *and*

$$b_0 = \int_{-R}^0 \Omega(e^{\delta s} - e^{-\delta R}) a_0 e^{gs} \mathrm{d}s$$

PROOF. It is enough to try the solution $a_0 e^{a_1 s}$ in equation (8.46).
We find:

$$\begin{aligned} a_0 e^{a_1 s} = &\left(\int_{-R}^0 \Omega e^{(\delta + a_1)s} \mathrm{d}s - \int_{-R}^0 \Omega e^{a_1 s} e^{-\delta R} \mathrm{d}s\right) a_0 e^{a_1 s} \left(\frac{\xi - \rho}{\sigma \xi}\right) + \\ &+ a_0 e^{a_1 s} \left(\frac{\rho - \xi(1-\sigma)}{\sigma \xi}\right) \int_{-R}^0 e^{\xi s} \int_{-R}^s e^{-a_1 s} e^{(a_1 + \delta) r} \mathrm{d}r \mathrm{d}s \end{aligned}$$

then

$$1 = \Sigma(a_1) \stackrel{def}{=} \left(\Omega \frac{1}{\delta + a_1}(1 - e^{-(\delta + a_1)R}) - \Omega e^{-\delta R} \frac{1}{a_1}(1 - e^{-(+a_1)R})\right) \left(\frac{\xi - \rho}{\sigma \xi}\right) +$$

$$+\left(\frac{\rho-\xi(1-\sigma)}{\sigma\xi}\right)\int_{-R}^{0} e^{(\xi-a_1)s}\delta\Omega\frac{1}{\delta+a_1}(e^{(a_1+\delta)s}-e^{-(a_1+\delta)R})\mathrm{d}s=$$
$$\left(\Omega\frac{1}{\delta+a_1}(1-e^{-(\delta+a_1)R})-\Omega e^{-\delta R}\frac{1}{a_1}(1-e^{-(+a_1)R})\right)\left(\frac{\xi-\rho}{\sigma\xi}\right)+$$
$$+\left(\frac{\rho-\xi(1-\sigma)}{\sigma\xi}\right)\frac{\delta\Omega}{\delta+a_1}\left(\frac{1}{\xi+\delta}(1-e^{-(\xi+\delta)R})-e^{-(a_1+\delta)R}\frac{1}{\xi-a_1}(1-e^{-(\xi-a_1)R})\right)$$

so we can observe that, when $g>0$:

$$\lim_{a_1\to-\infty}\Sigma(a_1)=+\infty \qquad \lim_{a_1\to+\infty}\Sigma(a_1)=0$$

then there exists a $a_1$ such that $\Sigma(a_1)=1$ and than such that $a_0e^{a_1\cdot}$ is the a BGP for all positive $a_0$. But from Theorem 8.24 we can deduce that the only possible choice for a BGP is that $a_1=b_1=g$, then $\Sigma(g)=1$ and $a_0e^{g\cdot}$ is a BGP for all positive $a_0$. The related state evolution will be $y(s)=b_0e^{gs}$ where

$$b_0=\int_{-R}^{0}\Omega(e^{\delta s}-e^{-\delta R})a_0e^{gs}\mathrm{d}s$$

$\square$

PROPOSITION 8.30. *There exist the limits*

$$y_l \overset{def}{=} \lim_{t\to+\infty} e^{-gt}y^*(t) \qquad i_l \overset{def}{=} \lim_{t\to+\infty} e^{-gt}i^*(t)$$

*moreover*

$$\begin{cases} i_l=\frac{\Lambda}{\left(\int_{-R}^{0}\Omega(e^{\delta s}-e^{-\delta R})e^{gs}\mathrm{d}s\right)-1}\\ y_l=\Lambda\left(1+\frac{1}{\left(\int_{-R}^{0}\Omega(e^{\delta s}-e^{-\delta R})e^{gs}\mathrm{d}s\right)-1}\right)\end{cases} \tag{8.48}$$

PROOF. In view of equation (8.46) and equation (8.47) we can develop $i^*(\cdot)$ and $y^*(\cdot)$ as

$$i^*(t)=\sum p_j(t)e^{\theta_j t} \qquad y^*(t)=\sum q_j(t)e^{\sigma_j t}$$

Since $i^*$ and $k^*$ remain $\geq 0$ we can state that the root $\theta_j$ with greater real part (we call it $\bar\theta$) is real and that the $\sigma_j$ with greater real part ($\bar\sigma$) is real. Moreover the fact that $y^*(t)-i^*(t)=\Lambda e^{gt}$ implies that $\bar\theta=\bar\sigma\geq g$. The existence of the limit is proved if we prove that $\bar\theta=\bar\sigma=g$. We suppose now, by contradiction, that $\nu\overset{def}{=}\bar\theta=\bar\sigma>g$. So we can find a $\bar t>R$ such that

$$i^*(t)>ce^{gt} \qquad \textit{for all } t\geq \bar t-R$$

where $c$ is such that

$$\Lambda=\left(\frac{\rho-\xi(1-\sigma)}{\sigma\xi}\right)\left(\int_{-R}^{0}ce^{gs}(\Omega e^{\delta s}-\Omega e^{-\delta R})\mathrm{d}s+c\int_{-R}^{0}e^{\xi s}F(r\mapsto e^{gr})(s)\mathrm{d}s\right)$$

Note that the control given by

$$i_s=\begin{cases} i^*(t) & \textit{for } t\leq\bar t\\ ce^{gt} & \textit{for } t>\bar t\end{cases}$$

is admissible and the related consumption is

$$\begin{cases} \Lambda e^{gt} & \textit{for } t\leq\bar t\\ \Lambda e^{gt} & \textit{for } t\geq\bar t+R\\ >\Lambda e^{gt} & \textit{for } t\in(\bar t,\bar t+R)\end{cases}$$

so $J(i_s) > J(i^*)$ but $i^*(\cdot)$ is optimal and then we have a contradiction. So $\nu = g$ and then the limits does exist. Now we compute the limits: we have that

$$\begin{cases} y_l - i_l = \Lambda \\ y_l = \left(\int_{-R}^{0} \Omega(e^{\delta s} - e^{-\delta R})e^{gs}\mathrm{d}s\right) i_l \end{cases}$$

so

$$\begin{cases} i_l = \frac{\Lambda}{\left(\int_{-R}^{0} \Omega(e^{\delta s} - e^{-\delta R})e^{gs}\mathrm{d}s\right) - 1} \\ y_l = \Lambda \left(1 + \frac{1}{\left(\int_{-R}^{0} \Omega(e^{\delta s} - e^{-\delta R})e^{gs}\mathrm{d}s\right) - 1}\right) \end{cases}$$

$\square$

APPENDIX A

# A verification theorem for the stochastic case in the framework of viscosity solutions

In this appendix, as announced in Chapter 5, we present a verification theorem for the stochastic case, in which a second order HJB is involved. A similar result is available in the finite dimensional case, it appeared first in [Zho93, ZYL97] and then in corrected form in [GŚZ05].

## A.1. Notation, definitions and background

Here we study the verification theorem in stochastic setting. We try to use as much common notation with the previous section as possible. $X$ is a separable Hilbert space, $A$ the generator of a $C_0$ semigroup of contraction, $U$ a compact metric space[1]. We call $\mathcal{S}(X)$ the set of bounded and selfadjoint operators on $X$.

We consider an optimal control problem in relaxed form: we look for optimal control on the set of all admissible stochastic bases . A stochastic base is a five-tuple of the form:

$$\nu = (\Omega^\nu, (\mathcal{F}_t^\nu), \mathbb{P}^\nu, W^\nu(\cdot), u^\nu(\cdot))$$

where $(\Omega^\nu, (\mathcal{F}_t^\nu), \mathbb{P}^\nu)$ is a filtered probability space, $W^\nu(\cdot)$ is a $X$-valued Wiener process with covariance operator $\mathcal{R}$ for a fixed nuclear operator $\mathcal{R}$; $u^\nu(\cdot)$ is a progressively measurable process (with respect to $\mathcal{F}_t^\nu$) which takes values in $U$. The expectation $\mathbb{E} = \mathbb{E}^\nu$ is calculated with respect to the probability $\mathbb{P}^\nu$). The set of all admissible stochastic bases will be called $\Theta$. In the following we will avoid the index $\nu$ but all the elements of the stochastic base will be implicit in the choice of an admissible control $u(\cdot)$.

We consider a state equation of the form[2]

$$\begin{cases} \mathrm{d}x(s) = (Ax(s) + f(x(s), u(s)))\,\mathrm{d}s + \sigma(x(s), u(s))\mathrm{d}W(s) \\ x(t) = x \end{cases} \tag{A.1}$$

We assume:

HYPOTHESIS A.1. *$A$ is the generator of a semigroup of contractions that is*

$$\|e^{sA}\| \le 1 \quad \textit{for all } s \ge 0$$

HYPOTHESIS A.2. *$f\colon X \times U \to X$ and $\sigma\colon X \times U \to \mathscr{L}(X)$ are continuous. Moreover there exists $M > 0$ s.t.*

$$|f(x,u) - f(y,u)| \le M|x-y| \quad \forall\, x, y \in X,\ u \in U$$

$$|\sigma(x,u) - \sigma(y,u)| \le M|x-y| \quad \forall\, x, y \in X,\ u \in U$$

The solution of such stochastic differential equation can be given in mild form as in [DPZ02].

---

[1]The compactness assumption was not assumed in the deterministic case.

[2]In this section we use $\sigma$, as usual, for the drift term. We will use $\omega(\cdot)$ for the moduli of continuity.

We take a cost functional of the form:

$$J(t,x,\nu) = \mathbb{E}^\nu \left( \int_t^T L(r, x(r), u(r)) \mathrm{d}r + h(x(T)) \right)$$

We assume that

Hypothesis A.3. *$L\colon [0,T] \times X \times U \to \mathbb{R}$ and $h\colon X \to \mathbb{R}$ are continuous. Moreover there exists a local modulus $\omega(\cdot,\cdot)$, a positive natural number $m$ and a constant $M > 0$ such that*

$$|L(t,x,u) - L(s,y,u)| \le \omega(|x-y| + |s-t|, |x| \vee |y|) \;\; \forall\, x,y \in X, \; u \in U, \; s,t \in [0,T]$$

$$|L(s,x,u)|, |h(x)| \le M(1 + |x|^m) \;\; \forall\, x \in X, \; u \in U, \; s \in [0,T]$$

.

We define the value function as

$$V(t,x) = \inf_{\nu \in \Theta} J(t,x,\nu) \tag{A.2}$$

and the HJB equation of the problem as

$$\begin{cases} -v_t(t,x) - \langle Ax, \nabla v(t,x) \rangle - H(t,x,\nabla v(t,x), D^2 v(t,x)) = 0 \\ v(T,x) = h(x) \end{cases} \tag{A.3}$$

where the Hamiltonian is

$$H(t,x,p,Q) = \inf_{u \in U} \left( \langle p, f(x,u) \rangle + L(t,x,u) + \frac{1}{2} tr(S(x,u)Q) \right)$$

in which $S(x,u) = \sigma(x,u)\mathcal{R}\sigma^*(x,u)$.

As in deterministic case we want to proceed giving the definition of sub/super solution of the HJB equation but we have to begin with some definitions and observations: **The operator $P$ and $P$-continuity** We use, as in Section 2.2, Chapters 3 and Chapter 5 an operator $P$ that is strictly positive, selfadjoint, bounded and such that $A^*P$ is bounded and[3]

$$A^*P \le P.$$

We can now describe the sets of test functions:

$$\begin{aligned} test1_S = \{\varphi \in C^{1,2}((0,T) \times X) \; : \;\; & \varphi \text{ is P-continuous} \\ & \nabla\varphi\colon X \to D(A^*) \textit{ is continuous}, \\ & \varphi, \partial_t\varphi, \nabla\varphi, D^2\varphi, A^*\nabla\varphi \textit{ are} \\ & \text{locally uniformly continuous and} \\ & \text{have polynomial growth } \} \end{aligned}$$

$$\begin{aligned} test2_S = \{g \in C^{1,2}([0,T] \times X) \; : \;\; & g(t,x) = \eta(t)\psi(x) \textit{ where} \\ & \psi \text{ is radial, non-decreasing,} \\ & \psi, \nabla\psi, D^2\psi \text{ are loc. unif. continuous} \\ & \text{and have polynomial growth.} \\ & \eta \in C^1([0,T]) \text{ is positive } \} \end{aligned}$$

We pass to the definition of sub/super solution

Definition A.4 (**Viscosity subsolution**). *$v \in C([0,T] \times X)$, $v$ $P$-upper semicontinuous, is a (viscosity) subsolution of the HJB equation (A.3) if*

$$v(T,x) \le h(x) \quad \textit{for all } x \in X$$

[3]We have already seen in Remark 2.24 we can always find an operator with such features.

*and for all* $\varphi \in test1_S$ *and* $g \in test2_S$ *and every* $(t,x) \in [0,T) \times X$ *global maximum of* $v - \varphi - g$ *we have*

(A.4)

$$-\varphi_t(t,x) - g_t(t,x) - \langle A^* \nabla\varphi(t,x), x\rangle - H(t,x,\nabla(\varphi+g)(t,x), D^2(\varphi+g)(t,x)) \le 0$$

DEFINITION A.5 (**Viscosity supersolution**). $v \in C([0,T] \times X)$, *$v$ $P$-lower semicontinuous, is a (viscosity)* supersolution *of the HJB equation (A.3) if*

$$v(T,x) \ge h(x) \quad \text{for all } x \in X$$

*and for all* $\varphi \in test1$ *and* $g \in test2$ *and every* $(t,x) \in [0,T) \times X$ *global minimum of* $v + \varphi + g$ *we have*

$$\varphi_t(t,x) + g_t(t,x) + \langle A^* \nabla\varphi(t,x), x\rangle - H(t,x,-\nabla(\varphi+g)(t,x), -D^2(\varphi+g)(t,x)) \ge 0$$

DEFINITION A.6 (**Viscosity solution**). $v \in C([0,T] \times X)$ *is a (viscosity)* solution *of the HJB equation (A.3) if it is at the same time a subsolution and a supersolution*

With previous assumptions the following results can be proved (see [Kel02] chapter 3 for a proof):

LEMMA A.7. *Let Hypotheses A.1, A.2 and A.3 hold. Let* $x(\cdot)$ *be a solution of the (A.1) related to the choice of an admissible base* $\nu \in \Theta$. *Let* $\varphi$ *be in* $test1_S$ *and* $g \in test2_S$, *then one has:*

$$\begin{aligned}(\text{A.5}) \quad \mathbb{E}[\varphi(s,x(s))] - \varphi(t,x) &= \\ = \mathbb{E}\Big[\int_t^T &\partial_t\varphi(r,x(r)) + \langle x(r), A^*\nabla\varphi(r,x(r))\rangle + \\ &+ \langle \nabla\varphi(r,x(r)), f(x(r),u(r))\rangle + \\ &+ \frac{1}{2} tr\left(S(x(r),u(r)) D^2\varphi(r,x(r))\right) \mathrm{d}r\Big]\end{aligned}$$

*and*

$$\begin{aligned}(\text{A.6}) \quad \mathbb{E}[g(s,x(s))] - g(t,x) &\le \\ \le \mathbb{E}\Big[\int_t^T &\partial_t g(r,x(r)) + \langle \nabla g(r,x(r)), f(x(r),u(r))\rangle + \\ &+ \frac{1}{2} tr\left(S(x(r),u(r)) D^2 g(r,x(r))\right) \mathrm{d}r\Big]\end{aligned}$$

THEOREM A.8. *Let Hypotheses A.1, A.2, A.3 hold. Then the value function $V$ defined in equation (A.2) is a solution of the HJB equation.*

## A.2. The verification theorem

We need, as in the proof of deterministic verification theorem, another assumption that is very similar to Hypothesis 5.11:

HYPOTHESIS A.9. *There exists a* $\mathcal{G} \subseteq \{f \in C([0,T] \times X) : f \text{ } P\text{-continuous}\}$ *such that:*

(i) *The value function $V$ is in* $\mathcal{G}$
(ii) *If* $v_1, v_2 \in \mathcal{G}$, $v_1$ *is a subsolution of the HJB equation and* $v_2$ *is a supersolution of the HJB equation then* $v_1 \le v_2$

From $(i)$ and $(ii)$ we know that $V$ is the only solution of the HJB equation in $\mathcal{G}$.

Now we need a definition similar to Definition 5.13:

DEFINITION A.10 ($\mathbf{E^{1,2,+}v(t,x)}$). *Given $v \in C([0,T] \times X)$ and $(t,x) \in [0,T] \times X$ we define $E^{1,2,+}v(t,x)$, or simply $E^s v(t,x)$ as*

$$
\begin{aligned}
&E^s v(t,x) = \\
&\quad = \{(q,p_1,p_2,Q) \in \qquad\qquad : \quad \exists \varphi \in test1,\ g \in test2 \ s.t. \\
&\qquad \in \mathbb{R} \times D(A^*) \times X \times \mathcal{S}(X) \qquad v - \varphi - g \text{ attains a global max. in } (t,x), \\
&\qquad\qquad (\partial_t)(\varphi + g)(t,x) = q, \\
&\qquad\qquad \nabla(\varphi)(t,x) = p_1, \quad \nabla(g)(t,x) = p_2, \\
&\qquad\qquad v(t,x) = \varphi(t,x) + g(t,x) \text{ and} \\
&\qquad\qquad D^2(\varphi + g)(t,x) = Q\}
\end{aligned}
$$

REMARK A.11. *As in the deterministic case if we call*

$$E_1^s v(t,x) = \{(q,p,Q) \in \mathbb{R} \times X \times \mathcal{S}(X) \ : \ p = p_1 + p_2 \text{ with } (q,p_1,p_2,Q) \in Ev(t,x)\}$$

*then $E_1^s v(t,x) \subseteq D^{1,2,+}v(t,x)$ and in the finite dimensional case we have $E_1^s v(t,x) = D^{1,2,+}v(t,x)$. We use $p_1$ and $p_2$ instead of $p$ to underline the different role of $g$ and $\varphi$. We will need this fact in the proofs in the sequel.*

In the proof of verification theorem we will need the following technical lemma:

LEMMA A.12. *Let Hypotheses A.1, A.2 and A.3 hold. Let $(x(\cdot), u(\cdot))$ be an admissible pair at $(t,x)$. Define the processes*

$$z_1(r) = f(x(r), u(r)) \quad z_2(r) = \sigma(x(r), u(r))\sigma^*(x(r), u(r))$$

*Then*

$$\lim_{h \to 0^+} \mathbb{E}\left[\frac{1}{h}\int_s^{s+h} |z_i(r) - z_i(t)| \mathrm{d}r\right] = 0 \ \text{ a.e. } s \in [t,T],\ i = 1,2 \tag{A.7}$$

PROOF. See [GŚZ05] Proposition 3.7 page 2014. □

THEOREM A.13. *Let Hypotheses A.1, A.2, A.3 and A.9 hold. Let $(x(\cdot), u(\cdot))$ be an admissible pair at $(t,x)$, let $v \in \mathcal{G}$ be a subsolution of the HJB equation (see Definition A.4) such that*

$$v(T,x) = h(x) \quad \text{for all } x \text{ in } X \tag{A.8}$$

*Let $q(\cdot) \in L^2_{\mathcal{F}_s}((t,T);\mathbb{R})$, $p_1(\cdot) \in L^2_{\mathcal{F}_s}((t,T);D(A^*))$, $p_2(\cdot) \in L^2_{\mathcal{F}_s}((t,T);X)$ and $Q(\cdot) \in L^2_{\mathcal{F}_s}((t,T);\mathscr{L}(X))$ be such that for almost every $s \in (t,T)$*

$$(q(s), p_1(s), p_2(s), Q(s)) \in E^s v(s, x(s)) \ \ \mathbb{P} - a.s. \tag{A.9}$$

*Moreover we assume that*

$$
\begin{aligned}
\mathbb{E}\Bigg[\int_t^T & \langle p_1(s) + p_2(s), f(s,x(s),u(s))\rangle + q(s) + \langle A^* p_1(s), x(s)\rangle + \\
&+ \frac{1}{2} tr(S(x(s),u(s))Q(s)) + L(s,x(s),u(s)) \mathrm{d}s\Bigg] \leq 0
\end{aligned} \tag{A.10}
$$

*Then*

(a) $v(t,x) \leq V(t,x) \leq J(t,x,\nu)$ $\forall (t,x) \in [0,T] \times X$ *for every stochastic base $\nu$.*
(b) $(x(\cdot), u(\cdot))$ *is an optimal pair at $(t,x)$.*

PROOF. We will follows the scheme of the proof for the finite dimensional case (see [GŚZ05]) but we will have to confront with some difficulties that are typical of the infinite dimensional case. To be more precise: the presence of the unbounded term $A^*\nabla\varphi$ (step 2), the (almost) radial test function $g$ and the related terms (steps 1,4,6), the nuclear covariance of $W$ (steps 5 and 6).

We fix $t_0 \in [t, T]$ s.t. (A.9) and (A.7) hold at $t_0$. We fix a $\omega_0 \in \Omega$ such that the (A.9) holds and the regular conditional probability given $\mathcal{F}^s_{t_0}$, denoted by

$$\mathbb{P}(\cdot|\mathcal{F}^s_{t_0})(\omega_0)$$

is well defined (see [KS99] page 84-85 for a proof that for $\mathbb{P}$-a.e. $\omega_0 \in \Omega$ we can define the regular conditional probability).

We consider $x(t_0)$, $p_1(t_0)$, $p_2(t_0)$, $q(t_0)$, $Q(t_0)$. They are $\mathbb{P}(\cdot|\mathcal{F}^t_{t_0})(\omega_0)$-a.s. constant (see [YZ99] Proposition 2.13 Chapter 1) and they are respectively equal to: $x(t_0)(\omega_0)$, $p_1(t_0)(\omega_0)$, $p_2(t_0)(\omega_0)$, $q(t_0)(\omega_0)$, $Q(t_0)(\omega_0)$. We call $x_0 := x(t_0)(\omega_0)$.

The $X$-valued Wiener process $W$ is still a Brownian motion with fixed covariance operator $\mathcal{R}$ although now $W(t_0) = W(t_0)(\omega_0)$ $\mathbb{P}(\cdot|\mathcal{F}^t_{t_0})(\omega_0)$-a.s.. The space is now equipped with a new filtration:

$$\left(\mathcal{F}^{t_0}_r\right)_{r\in[t_0,T]}$$

and the control process $u(\cdot)$ is adapted to this filtration. Moreover for $\mathbb{P}(\cdot|\mathcal{F}^t_{t_0})(\omega_0)$-a.e. $\omega_0 \in \Omega$ $x(\cdot)$ is a solution of (A.1) on $(t_0, T)$ in the probability space

$$\left(\Omega, \mathcal{F}, \mathbb{P}(\cdot|\mathcal{F}^t_{t_0})(\omega_0)\right)$$

with initial condition $x(t_0) = x_0$.

We will write $\mathbb{E}_{\omega_0}$ to denote the expectation with respect to the measure $\mathbb{P}(\cdot|\mathcal{F}^t_{t_0})(\omega_0)$. We now take two test functions $\varphi$ and $g$ such that:

$$\text{(A.11)}\qquad \begin{array}{ll} \mathbf{a} & \varphi + g > v \ \ \forall (t,x) \neq (t_0, x_0) \\ \mathbf{b} & (\varphi + g)(t_0, x_0) = v(t_0, x_0) \\ \mathbf{c} & \partial_t(\varphi + g)(t_0, x_0) = q(t_0)(\omega_0) \\ \mathbf{d} & \nabla(\varphi)(t_0, x_0) = p_1(t_0)(\omega_0) \ \text{ and } \ \nabla(g)(t_0, x_0) = p_2(t_0)(\omega_0) \\ \mathbf{e} & D^2(\varphi + g)(t_0, x_0) = Q(t_0)(\omega_0) \end{array}$$

So

$$\text{(A.12)}\quad \mathbb{E}_{\omega_0}\left[\frac{v(t_0+h, x(t_0+h)) - v(t_0,x_0)}{h}\right] \leq \\ \leq \mathbb{E}_{\omega_0}\left[\frac{(\varphi+g)(t_0+h, x(t_0+h)) - (\varphi+g)(t_0,x_0)}{h}\right] \leq$$

using Lemma A.7

$$\text{(A.13)}\quad \leq \mathbb{E}_{\omega_0}\Bigg[\frac{1}{h}\int_{t_0}^{t_0+h} \varphi_t(r, x(r)) + \langle x(r), A^*\nabla\varphi(r, x(r))\rangle + \\ + \langle \nabla\varphi(r, x(r)), f(x(r), u(r))\rangle + \frac{1}{2} tr\left(S(x(r), u(r)) D^2\varphi(r, x(r))\right) \mathrm{d}r\Bigg] + \\ + \mathbb{E}_{\omega_0}\Bigg[\frac{1}{h}\int_{t_0}^{t_0+h} \eta'(r)\psi(r, x(r)) + \\ + \eta(r)\langle \nabla\psi(x(r)), f(x(r), u(r))\rangle + \frac{1}{2} tr\left(S(x(r), u(r))\eta(r) D^2\psi(x(r))\right) \mathrm{d}r\Bigg]$$

We pass now to verify the convergences of the different parts of the last expression: we use the fact that a function $f(\cdot)\colon \mathbb{R}^+ \to F$ (where $F$ is a Banach space) admit the limit

$$\lim_{h\to 0^+} f(h) = \alpha$$

if and only if for each sequence of positive numbers $h_n \to 0$ there exists a subsequence $h_{n_j}$ such that

$$\lim_{j\to\infty} f(h_{n_j}) = \alpha$$

As usual we will abuse of the notation using the expression $h_n$ also for the subsequence. Fix now a sequence of positive numbers $\{h_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} h_n = 0$.

**step 1**: Thanks to the continuity of $x(\cdot)$ (see, for instance [DPZ02] Theorem 7.4 page 174), the continuity of $\varphi_t$ and the continuity of $\eta'\psi$ we have that $\mathbb{P}(\cdot|\mathcal{F}^t_{t_0})(\omega_0)$-a.e.

$$\int_{t_0}^{t_0+h_n} \varphi_t(r,x(r)) + \eta'(r)\psi(r,x(r))\mathrm{d}r \xrightarrow[\mathbb{R}]{n\to\infty} \longrightarrow \varphi_t(t_0,x_0) + \eta'(t_0)\psi(t_0,x_0) \tag{A.14}$$

Moreover, under Hypotheses A.2 and A.1 we have (see [DPZ02] equation (7.16)) that there exists a $K_T$ such that for every $l \geq 0$:

$$\mathbb{E}_{\omega_0} \sup_{r\in[t_0,T]} |x(r)|^l \leq K_T(1+|x_0|^l) \tag{A.15}$$

So, using the polynomial growth of $\varphi_t$ and of $\eta'\psi$ (that we have required in the definition of test functions) we can dominate the convergence and find that:

$$\lim_{n\to\infty} \mathbb{E}_{\omega_0}\left[\int_{t_0}^{t_0+h_n} \varphi_t(r,x(r)) + \eta'(r)\psi(r,x(r))\mathrm{d}r\right] = 0$$

**step 2**: We have

$$\begin{aligned}\left|\frac{1}{h_n}\mathbb{E}_{\omega_0}\left[\int_{t_0}^{t_0+h_n} \langle x(r), A^*\nabla\varphi(r,x(r))\rangle - \langle x_0, A^*\nabla\varphi(t_0,x_0)\rangle\,\mathrm{d}r\right]\right| \leq \\ \left|\frac{1}{h_n}\mathbb{E}_{\omega_0}\left[\int_{t_0}^{t_0+h_n} \langle x(r)-x_0, A^*\nabla\varphi(r,x(r))\rangle\,\mathrm{d}r\right]\right| + \\ + \left|\frac{1}{h_n}\mathbb{E}_{\omega_0}\left[\int_{t_0}^{t_0+h_n} \langle x_0, A^*\nabla\varphi(r,x(r)) - A^*\nabla\varphi(t_0,x_0)\rangle\,\mathrm{d}r\right]\right|\end{aligned} \tag{A.16}$$

and (as in the previous step) by the continuity of $x(t)$ and of $A^*\nabla\varphi$ we obtain that both the parts of the last expression go to zero $\mathbb{P}(\cdot|\mathcal{F}^t_{t_0})(\omega_0)$-a.s. and, as before, using the polynomial estimates (A.15) we can dominate the expression and use the Lebesgue theorem.

**step 3**: This step follows, without many variations, the proof of [GŚZ05]. The term

$$\mathbb{E}_{\omega_0}\left[\int_{t_0}^{t_0+h_n} \frac{\langle\nabla\varphi(r,x(r)), f(x(r),u(r))\rangle - \langle\nabla\varphi(t_0,x_0), f(x_0,u(t_0))\rangle}{h_n}\mathrm{d}r\right]$$

can be divided into two parts: it becomes:

$$\begin{aligned}\frac{1}{h_n}\mathbb{E}_{\omega_0}\left[\int_{t_0}^{t_0+h_n} \langle\nabla\varphi(r,x(r)) - \nabla\varphi(t_0,x_0), f(x(r),u(r))\rangle\,\mathrm{d}r\right] + \\ + \frac{1}{h_n}\mathbb{E}_{\omega_0}\left[\int_{t_0}^{t_0+h_n} \langle\nabla\varphi(t_0,x_0), f(x(r),u(r)) - f(x_0,u(t_0))\rangle\,\mathrm{d}r\right]\end{aligned} \tag{A.17}$$

**step 3a**: The first part can be estimate as follows

$$\left|\frac{1}{h_n}\mathbb{E}_{\omega_0}\left[\int_{t_0}^{t_0+h_n}\langle\nabla\varphi(r,x(r))-\nabla\varphi(t_0,x_0),f(x(r),u(r))\rangle\,\mathrm{d}r\right]\right|\leq \tag{A.18}$$
$$\frac{1}{h_n}\mathbb{E}_{\omega_0}\left[\int_{t_0}^{t_0+h_n}|\nabla\varphi(r,x(r))-\nabla\varphi(t_0,x_0)|\,|f(x(r),u(r))|\,\mathrm{d}r\right]\leq$$
$$\mathbb{E}_{\omega_0}\left[\left(\frac{1}{h_n}\int_{t_0}^{t_0+h_n}|\nabla\varphi(r,x(r))-\nabla\varphi(t_0,x_0)|^2\,\mathrm{d}r\right)^{\frac{1}{2}}\times\right.$$
$$\left.\times\left(\frac{1}{h_n}\int_{t_0}^{t_0+h_n}|f(x(r),u(r))|^2\,\mathrm{d}r\right)^{\frac{1}{2}}\right]\leq$$
$$\left(\mathbb{E}_{\omega_0}\left[\frac{1}{h_n}\int_{t_0}^{t_0+h_n}|\nabla\varphi(r,x(r))-\nabla\varphi(t_0,x_0)|^2\,\mathrm{d}r\right]\right)^{\frac{1}{2}}\times$$
$$\times\left(\mathbb{E}_{\omega_0}\left[\frac{1}{h_n}\int_{t_0}^{t_0+h_n}|f(x(r),u(r))|^2\,\mathrm{d}r\right]\right)^{\frac{1}{2}}$$

Arguing as in step 1 we get

$$\left(\mathbb{E}_{\omega_0}\left[\frac{1}{h_n}\int_{t_0}^{t_0+h_n}|\nabla\varphi(r,x(r))-\nabla\varphi(t_0,x_0)|^2\,\mathrm{d}r\right]\right)^{\frac{1}{2}}]\to 0$$

and the other term is bounded indeed, from Hypothesis A.2, A.1 and (A.15) we have

$$\left(\mathbb{E}_{\omega_0}\left[\frac{1}{h_n}\int_{t_0}^{t_0+h_n}|f(x(r),u(r))|^2\,\mathrm{d}r\right]\right)^{\frac{1}{2}}\leq 2M^2\left(1+K_T(1+|x_0|^2)\right)$$

**step 3b**: The second part of (A.17) can be estimated as follows:

$$\left|\frac{1}{h_n}\mathbb{E}_{\omega_0}\left[\int_{t_0}^{t_0+h_n}\langle\nabla\varphi(t_0,x_0),f(x(r),u(r))-f(x_0,u(t_0))\rangle\,\mathrm{d}r\right]\right|\leq \tag{A.19}$$
$$|\nabla\varphi(t_0,x_0)|\,\mathbb{E}_{\omega_0}\left[\frac{1}{h_n}\int_{t_0}^{t_0+h_n}|f(x(r),u(r))-f(x_0,u(t_0))|\right]$$

But in view of the choice of $t_0$ the (A.7) holds and then

$$0=\lim_{n\to\infty}=\mathbb{E}\left[\frac{1}{h_n}\int_t^{t+h_n}|f(x(r),u(r))-f(x_0,u(t_0)|\mathrm{d}r\right]= \tag{A.20}$$
$$\lim_{n\to\infty}=\mathbb{E}\left[\frac{1}{h_n}\mathbb{E}\left[\int_t^{t+h_n}|f(x(r),u(r))-f(x_0,u(t_0)|\mathrm{d}r\,\middle|\,\mathcal{F}_{t_0}^s\right]\right]=$$
$$\lim_{n\to\infty}=\mathbb{E}\left[\mathbb{E}_{\omega_0}\left[\frac{1}{h_n}\int_t^{t+h_n}|f(x(r),u(r))-f(x_0,u(t_0)|\mathrm{d}r\right]\right]$$

This means that

$$\mathbb{E}_{\omega_0}\left[\frac{1}{h_n}\int_t^{t+h_n}|f(x(r),u(r))-f(x_0,u(t_0)|\mathrm{d}r\right]\xrightarrow[L^1(\Omega;\mathbb{R})]{n\to\infty}0$$

So, up to consider a subsequence of $\{h_n\}_{n\in\mathbb{N}}$, we have that for $\mathbb{P}$-a.e. $\omega_0$

$$\mathbb{E}_{\omega_0}\left[\frac{1}{h_n}\int_t^{t+h_n} |f(x(r),u(r)) - f(x_0,u(t_0)|\mathrm{d}r\right] \xrightarrow{n\to\infty} 0 \tag{A.21}$$

and this prove the convergence for the $\omega_0$s that satisfy the (A.21). We proceed assuming that $\omega_0$ is such that the last expression holds.

**step 4**: The term

$$\mathbb{E}_{\omega_0}\left[\frac{1}{h_n}\int_t^{t+h_n} \eta(r)\left\langle \nabla\psi(x(r)), f(x(r),u(r))\right\rangle \mathrm{d}r\right]$$

can be treated with arguments similar to step 3. We obtain that, up to consider a subsequence of $\{h_n\}_{n\in\mathbb{N}}$, for $\mathbb{P}$-a.e. $\omega_0$

$$\begin{aligned}\mathbb{E}_{\omega_0}\Bigg[\frac{1}{h_n}\int_t^{t+h_n} &\eta(r)\left\langle \nabla\psi(x(r)), f(x(r),u(r))\right\rangle - \\ &\eta(t_0)\left\langle \nabla\psi(x_0), f(x_0,u(t_0))\right\rangle \mathrm{d}r\Bigg] \xrightarrow{n\to\infty} 0\end{aligned} \tag{A.22}$$

so we can proceed assuming that $\omega_0$ is such that the last expression holds.

**step 5**: Note that (see [DPZ02] page 416) if $A$ is nuclear operator and $B \in \mathscr{L}(X)$ then $AB$ and $BA$ are nuclear and $tr(AB) = tr(BA)$ moreover if $A_1$ and $A_2$ are nuclear operators than $A_1 + A_2$ is a nuclear operator and $tr(A_1) + tr(A_2) = tr(A_1 + A_2)$. We pass now to estimate the term:

$$\mathbb{E}_{\omega_0}\left[\frac{1}{h_n}\int_{t_0}^{t_0+h_n} \frac{1}{2}tr\left(S(x(r),u(r))D^2\varphi(r,x(r))\right)\mathrm{d}r\right]$$

We argue like in step 4:

$$\begin{aligned}\mathbb{E}_{\omega_0}\Bigg[\frac{1}{h_n}\int_{t_0}^{t_0+h_n} &\frac{1}{2}tr\left(S(x(r),u(r))D^2\varphi(r,x(r))\right) - \\ &\frac{1}{2}tr\left(S(x_0,u(t_0))D^2\varphi(t_0,x_0)\right)\mathrm{d}r\Bigg] = \\ \mathbb{E}_{\omega_0}\Bigg[\frac{1}{h_n}\int_{t_0}^{t_0+h_n} &\frac{1}{2}tr\left(S(x(r),u(r))\left(D^2\varphi(r,x(r)) - D^2\varphi(t_0,x_0)\right)\right) + \\ \mathbb{E}_{\omega_0}\Bigg[\frac{1}{h_n}\int_{t_0}^{t_0+h_n} &\frac{1}{2}tr\left(\left(S(x(r),u(r)) - S(x_0,u(t_0))\right)D^2\varphi(t_0,x_0)\right)\mathrm{d}r\Bigg] =\end{aligned} \tag{A.23}$$

And then, arguing like in step 4, we can conclude that, up to consider a subsequence of $\{h_n\}_{n\in\mathbb{N}}$ and up to exclude another set of measure zero in the choice of $\omega_0$, we have

$$\begin{aligned}\mathbb{E}_{\omega_0}\Bigg[\frac{1}{h_n}\int_{t_0}^{t_0+h_n} &\frac{1}{2}tr\left(S(x(r),u(r))D^2\varphi(r,x(r))\right) - \\ &-\frac{1}{2}tr\left(S(x_0,u(t_0))D^2\varphi(t_0,x_0)\right)\mathrm{d}r\Bigg]\end{aligned} \tag{A.24}$$

**step 6**: The term

$$\mathbb{E}_{\omega_0}\left[\frac{1}{h_n}\int_{t_0}^{t_0+h_n} \frac{1}{2}tr\left(S(x(r),u(r))\eta(r)D^2\psi(x(r))\right)\mathrm{d}r\right]$$

can be treated with arguments similar to step 5 and so, up to consider a subsequence and up to exclude another set of measure zero in the choice of $\omega_0$, we have that

$$\text{(A.25)} \quad \mathbb{E}_{\omega_0}\left[\frac{1}{h_n}\int_{t_0}^{t_0+h_n} \frac{1}{2}tr\left(S(x(r),u(r))\eta(r)D^2\psi(x(r))\right) - \frac{1}{2}tr\left(S(x_0,u(t_0))\eta(t_0)D^2\psi(x_0)\right)\mathrm{d}r\right] \xrightarrow{n\to\infty} 0$$

Now we can summarize what we have seen in the six steps: There exists a set $\Omega' \subseteq \Omega$ of full measure (with respect to the measure $\mathbb{P} = \mathbb{P}^\nu$) such that, for every $\omega_0 \in \Omega'$ and for every sequence of positive numbers $h_n \to 0$, up to consider a subsequence, the convergences of steps 1..6 hold. Then, for $\omega_0$ that varies on a full measure set, we have that the (A.13) admits a limit for $h \to 0$ and it is

$$\text{(A.26)} \quad \varphi_t(t_0,x_0) + \eta'(t_0)\psi(x_0) + \langle x_0, A^*\nabla\varphi(t_0,x_0)\rangle + \langle \nabla\varphi(t_0,x_0), f(x_0,u(t_0))\rangle + \eta(t_0)\langle\nabla\psi(x_0), f(x_0,u(t_0))\rangle + \frac{1}{2}tr\left(S(x_0,u(t_0))D^2\varphi(t_0,x_0)\right) + \frac{1}{2}tr\left(S(x_0,u(t_0))\eta(t_0)D^2\psi(x_0)\right)$$

So, using (A.12) and (A.13) and using the relations (A.11) we have that, for $\omega_0$ in a $\mathbb{P}$-full measure set

$$\text{(A.27)} \quad \limsup_{h\to 0^+}\mathbb{E}_{\omega_0}\left[\frac{v(t_0+h,x(t_0+h)) - v(t_0,x_0)}{h}\right] \le \le q(t_0)(\omega_0) + \langle x_0, A^*p_1(t_0)(\omega_0)\rangle + + \langle p_1(t_0)(\omega_0) + p_2(t_0)(\omega_0), f(x_0,u(t_0))\rangle + \frac{1}{2}tr\left(S(x_0,u(t_0))Q(t_0)(\omega_0)\right)$$

Then

$$\text{(A.28)} \quad \limsup_{h\to 0^+}\mathbb{E}\left[\frac{v(t_0+h,x(t_0+h)) - v(t_0,x_0)}{h}\right] = = \limsup_{h\to 0^+}\mathbb{E}\left[\mathbb{E}_{\omega_0}\left[\frac{v(t_0+h,x(t_0+h)) - v(t_0,x_0)}{h}\right]\right] \le$$

for the Fatou's lemma

$$\le \mathbb{E}\left[\limsup_{h\to 0^+}\mathbb{E}_{\omega_0}\left[\frac{v(t_0+h,x(t_0+h)) - v(t_0,x_0)}{h}\right]\right] \le$$

for equation (A.27)

$$\text{(A.29)} \quad \le \mathbb{E}\Big[q(t_0)(\omega_0) + \langle x_0, A^*p_1(t_0)(\omega_0)\rangle + + \langle p_1(t_0)(\omega_0) + p_2(t_0)(\omega_0), f(x_0,u(t_0))\rangle + \frac{1}{2}tr\left(S(x_0,u(t_0))Q(t_0)(\omega_0)\right)\Big]$$

The end of the proof is similar to the deterministic case one: we use the Lemma 5.12 with $g(s) = \mathbb{E}v(s,x(s))$ and (A.10) and (A.29). Then

$$\mathbb{E}\left[v(s,x(s))\right] - v(t,x) \le \mathbb{E}\left[\int_t^T -L(r,x(r),u(r))\mathrm{d}r\right]$$

But, in view of the fact that $v$ is a subsolution and $v \in \mathcal{G}$ we know that $v \le V$ and using the (A.8) we know that $v(T,x(T)) = V(T,x(T)) = h(x(T))$, then

$$\mathbb{E}\left[\int_t^T L(r,x(r),u(r))\mathrm{d}r + h(x(T))\right] \le V(t,x)$$

and this prove that the pair $x(\cdot), u(\cdot)$ is optimal. $\square$

# Abbreviations

| | |
|---|---|
| CRRA | Constant relative risk aversion |
| DDE | Delay differential equation |
| DP | Dynamic programming |
| DPA | Dynamic programming approach |
| HJ equation | Hamilton-Jacobi equation |
| HJB equation | Hamilton-Jacobi-Bellman equation |
| HJI equation | Hamilton-Jacobi-Isaacs equation |
| LQ | Linear quadratic |
| ODE | Ordinary differential equation |
| PDE | Partial differential equation |

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