Davide Moroni

Finite and Infinite Type Artin Groups:Topological Aspects and Cohomological Computations

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Chapter 1

Introduction

The main topic of this work is the study of the cohomology of Artin groups, with emphasis on a particular class of infinite type groups known as affine Artin groups.

Let (W, S) be a Coxeter system of rank $n = |S|$. Loosely speaking, the Artin group G_W associated to (W, S) is the group obtained by dropping the relations $s^2 = 1$ $(s \in S)$ in the standard presentation for W. We say that an Artin group G_W is of *finite type* when W is finite.

Finite type Artin groups have special features which in general do not have an analogue for infinite type groups. The most important difference for our purposes is the lack of an appropriate classifying space for infinite type Artin groups, whereas the question for finite type Artin groups was settled by Deligne [Del72].
Recall indeed that a finite Coxeter group W may be geometrically represented as a (orthogonal) linear reflection group in $\mathbf{R}^n$. Let $\mathcal{A}^{\mathbf{R}}$ be the real arrangement of hyperplanes given by the mirrors of the reflections in W and let $\mathcal{A}$ its complexification. Note that W acts freely on the complement $Y(W) := \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}} H$ and let $X(W) := Y(W)/W$. It is known that $X(W)$ has the Artin group G_W as fundamental group [Bri71a] and, by a result of Deligne, $X(W)$ is as well a $k(\pi, 1)$ space.
For the infinite type, we can no more represent the Coxeter group as a group of orthogonal reflections; however it is possible to regard it as the group of linear (not-orthogonal) reflections w.r.t. the walls of a polyhedral cone C of maximal dimension in $V = \mathbf{R}^n$ [Vin71]. It can be shown that the union $U = \bigcup_{w \in W} wC$ of W-translates of C is a convex cone and that W acts properly on the interior U^0 of U. We may now rephrase the construction used in the finite case as follows. Let $\mathcal{A}$ the complexified arrangement of the mirrors of the reflections in W and consider $I := \{v \in V \otimes \mathbb{C} \,|\, \Re(v) \in U^0\}$. Then W acts on $Y(W) = I \setminus \bigcup_{H \in \mathcal{A}} H$ and we can form the quotient $X(W) := Y(W)/W$. It was first shown in [vdL83] that G_W is indeed the fundamental group of X, but in general it is only conjectured that the space

$X(W)$ is a $k(\pi,1)$. This conjecture is known to be true for 1) Artin group of large type [Hen85], 2) Artin group satisfying the FC condition [CD95] and 3) for the affine Artin group of type $\tilde{A}_n, \tilde{C}_n$ [Oko79].

For what regards cohomology of Artin groups, the available literature is mainly focused in finite type Artin groups.
Starting with the classical braid groups (i.e. Artin groups of type A_n), the cohomology with trivial coefficients was computed in the seventies by F. Cohen [Coh76], and independently by A. Veinstein ([Vai78], see also [Arn68], [Bri71a], [BS72], [Fuk70]). For Artin groups of type C_n, D_n it was computed in [Gor78], while for exceptional cases in [Sal94] it was given as an abelian module, while the ring structure was computed in [Lan00].
Other cohomologies with twisted coefficients were later considered: an interesting case is over the module of Laurent polynomials $\mathbb{Q}[q^{\pm 1}]$, which gives the $\mathbb{Q}$-cohomology of the Milnor fibre of the naturally associated bundle. The theory of hypergeometric functions (as described in [Gel86]; see also [OT01][Var95]) provides further motivation to the study of twisted coefficients cohomology for Artin groups.
Again for the case of classical braids, many authors provided computations, independently and using different methods ([Fre88], [Mar96], [DPS01]), while for cases C_n, D_n see [DPSS99] (here the authors use the resolution coming from topological considerations discovered in [Sal94],[DS96]; an equivalent resolution was independently discovered by using purely algebraic methods in [Squ94]). Over the integral Laurent polynomials $\mathbb{Z}[q^{\pm 1}]$ not many computations exist: see [CS04] for the exceptional cases and recently [Cal05a] for the case of braid groups, and [DSS97] for the top cohomologies in all cases. By contrast, as regards Artin groups of non-finite type, some computations were provided in [SS97][CD95].

The main goal of this work is to start an analysis of the cohomology of Artin groups associated to Coxeter groups of affine type (also known as affine Weyl groups for their connection with Lie algebras).

Preliminarily, we studied the 'toric' version of an affine reflection arrangement. For affine Artin groups of type $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$ these arrangements (or a quotient of them) may be straightened to ordinary finite hyperplane arrangements.
Analyzing the straightened picture, we obtain embeddings of the affine Artin groups $G_{\tilde{A}_n}$, $G_{\tilde{C}_n}$ into the finite type Artin group $G_{B_{n+1}}$. We obtain as immediate corollary the results by Okonek [Oko79] and we extend a conjecture by Charney and Davis [CD95] remarking that $G_{\tilde{C}_n}$ has trivial center.
We found out that a similar analysis has been presented in [All02] and so these results are at least implicit in this paper.
As regards to the Artin group of type $\tilde{B}_n$, we are instead able to solve the $k(\pi,1)$-problem for it; this seems to be new.

Then, for cohomological computations we used as a main tool the so-called Salvetti complex (as described in [Sal94]; see also chapter 3); the algebraic generalization of this complex by DeConcini-Salvetti [DS96] provides an effective way to determine the cohomology of the orbit space $X(W)$ with values in an arbitrary G_W-module. When $X(W)$ is a $k(\pi,1)$ space, of course, we get the cohomology of the group G_W.

Consider the $G_{\tilde{A}_n}$-module consisting of Laurent polynomials $\mathbb{Q}[q^{\pm 1}]$, where the action of the standard generators of $G_{\tilde{A}_n}$ is by $(-q)$-multiplication. The inclusion $G_{\tilde{A}_{n-1}} < G_{B_n}$ turns out to be very useful to completely determine the cohomology of $G_{\tilde{A}_n}$ over this twisted system. Indeed, by Shapiro's lemma and a degree shift result, we may reduce to the equivalent computation of the cohomology of G_{B_n} over the module $\mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$, where the action is $(-q)-$multiplication for the standard generators associated to the first $n-1$ nodes of the Dynkin diagram, while is $(-t)-$multiplication for the generator associated to the last node.
To compute these latter cohomology groups, we use techniques similar to [DPS01]: a natural filtration of the Salvetti complex and the associated spectral sequence.
As a corollary we derive the trivial $\mathbb{Q}-$cohomology of $G_{\tilde{A}_{n-1}}$.
We remark that using another well-known inclusion (e.g. implicit in [Bri71b]) of G_{B_n} into the classical braid group $\mathrm{Br}_{n+1} := G_{A_n}$, we also find an isomorphism with the cohomology of Br_{n+1} over a certain representation, namely the irreducible $(n+1)-$dimensional representation of Br_{n+1} found in [TYM96], twisted by an abelian representation.
We also describe the cohomology of the braid group over the irreducible representation in [TYM96].

Having shown that the orbit space $X(\tilde{B}_n)$ is a $k(\pi,1)$ space, we consider finally the cohomology of $G_{\tilde{B}_n}$ over the module $\mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$, where the action is $(-t)$-multiplication for the node in the leave of weight 4 and $(-q)$-multiplication for the remaining ones.
In this case the Salvetti complex exhibits a natural involution and we may compute separately the cohomology of the invariant and anti-invariant subcomplexes. The invariant part turns out to have the cohomology of G_{B_n} over the double-weighted module described before, while for the anti-invariant part we resort to the spectral sequence of a natural filtration. We remark that the columns in the E_1-term of the spectral sequence correspond to the cohomology of the anti-invariant part for the Artin group of type D_k $(1 \leq k \leq n)$, which has been completely computed in [DPSS99]. However, in order to cope with the higher order boundary maps in the spectral sequence, we need some cohomology generators that are not easily deduced from the argument in [DPSS99]; thus we preferred to reprove everything from scratch.

As additional topics, that justify the wider-range title of this thesis, we provide further applications of the more general theory exposed in [DS00]. For any Coxeter group W, in particular, a free resolution of the trivial $\mathbb{Z}[W]$-module $\mathbb{Z}$ is provided, that allows the computation of the cohomology of W in an arbitrary coefficients module. As first application, we are then able to determine the cohomology of the Artin group and Coxeter group associated to the complete graph having all edges of weight 3. Actually, with some more effort one can generalize computations to the case where the weight of each edge is some fixed natural number p (≥ 2) (we are going to write this more general result in future). For the Artin group we considered coefficients in the module $\mathbb{Z}[q^{\pm 1}]$, where the standard generators of the group act by $(-q)$-multiplication, while the trivial and sign representations are used as coefficients module when dealing with the Coxeter group. The result for Artin groups was already exposed in [SS97], whereas the treatment of the Coxeter group is new.
Then we present an approach (see [Sal05]), based on the analysis of sheaves over posets, to homology computations for affine Artin groups with coefficient in a particular rank 1 system. With this method we are able to determine the top and top-1 homology of the affine Artin group of type $\tilde{A}_n$. Although we have already obtained using a different strategy the complete list of cohomology groups for $G_{\tilde{A}_n}$, we believe that the presented method retains its interest in view of its combinatorial treatment.
Finally we discuss the genus problem for affine Artin groups. By contrast with the finite type case (treated in [DS00], [DPS04], [Aro05]), here the answer is relatively simple.

In more detail, the material is organized as follows.

In chapter 2 we collect some basic definitions and results regarding Coxeter groups and Artin groups.
In chapter 3 we review the theory developed in [Sal94][DS96][DS00], stating the results for arbitrary (not necessarily finite type) Coxeter groups. We present as applications of the theory the determination of the cohomology of the Artin and Coxeter group associated to the complete graph (section 3.3), an approach (based on the analysis of sheaves over posets) to homology computations for affine Artin groups (section 3.4) and the genus problem for affine Artin groups (section 3.5).
Chapter 4 is devoted to topological constructions that allow to establish inclusions of the affine Artin groups of type $\tilde{A}_n$ and $\tilde{C}_n$ into the finite type Artin group of type B_{n+1} (sections 4.2-4.3). In section 4.4 we solve instead the $k(\pi,1)$-problem for the Artin group of type $\tilde{B}_n$. We then switch to representation theory for Artin group, analyzing representation induced by the previously found inclusions (section 4.5).
In chapter 5 we determine the cohomology of the Artin group G_{B_n} with coefficients in the double weight local system $R_{q,t} = \mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$, where the

action is by $(-q)$-multiplication for the standard generators associated to the first $n-1$ nodes of the Coxeter diagram and $(-t)$-multiplication for the last one. We obtain as main corollary the cohomology of the affine Artin group $G_{\tilde{A}_n}$ with coefficients in the module $\mathbb{Q}[q^{\pm 1}]$, where the action of its standard generators is given by $(-q)$-multiplication (section 5.3.2). We also determine the cohomology of the braid group with coefficients in the Tong-Yang-Ma representation (section 5.3.3).
In chapter 6 we determine the cohomology of $G_{\tilde{B}_n}$ over the double-weighted module $R_{q,t} = \mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$[1].

The results of chapter 5 were announced in [CMSb] and have been submitted as a joint paper with Filippo Callegaro and Mario Salvetti [CMSa].

The results of chapter 6 (also in collaboration with the same authors) will be submitted shortly. Other results, like the computation of the cohomology for the complete graph, the cohomology of sheaves over posets with application to computation of the Schwartz genus are almost ready to be submitted.

Acknowledgement

This work has been prepared in strict collaboration with prof. Mario Salvetti. The results in chapters $5-6$ are obtained in collaboration also with Filippo Callegaro.

[1] At the time of submitting this preliminary version, chapter 6 is still under preparation.

Chapter 2

Coxeter and Artin groups

In this chapter we recall briefly some basic facts about Coxeter groups and Artin groups with the two-fold aim of fixing notation and having at hand facts that will be used frequently throughout the remaining chapters. Almost every cited results for Coxeter groups may be found in standard textbooks such as [Bou68] and [Hum90], to which we refer for details if not otherwise stated.
The only exception is represented by the more specialistic section 2.4.3, in which we determine the expression for a double-weight analogue of ordinary Poincaré series for the Coxeter group of type B_n. This result may be found in [Rei93], but, for convenience, an elementary enumerative proof is provided.

2.1 Definition of Coxeter groups and first properties

A *Coxeter graph* is a finite undirected graph, whose edges are labelled with integers ≥ 3 or with the symbol ∞.

Let S be the vertex set of a Coxeter graph. For every pair of vertices $s, t \in S$ $(s \neq t)$ joined by an edge, define $m(s,t)$ to be the label of the edge joining them. If s, t are not joined by an edge, set by convention $m(s,t) = 2$. Let also $m(s,s) = 1$.

Associated to a Coxeter graph, we have a Coxeter system (W, S), consisting of a group W and a set of generators $S \subset W$, such that W has presentation:

$$W = \langle s \in S \mid (st)^{m(s,t)} = 1 \ \forall s, t \in S \text{ such that } m(s,t) \neq \infty \rangle$$

We will say that (W, S) is irreducible iff its Coxeter graph is connected. Further, the rank of (W, S) is defined as the cardinality $|S|$ of the generating set S.

2.1.1 Length function and parabolic subgroups

Note that every element $w \in W$ may be written as a product $w = s_1 s_2 \cdots s_k$ for some $s_i \in S$ (not necessarily distinct). We define the *length* $\ell(w)$ of w as the minimal length of an expression of w in term of the generators $s \in S$. An expression for w of minimal length will be called *reduced*. Note that $\ell(w) = 0$ iff $w = 1$ and, in general, for $s \in S$, we have $\ell(ws) = \ell(w) \pm 1$.

Given a subset $I \subset S$, let W_I be the subgroup of W generated by the $s \in I$. Such a group will be called a *parabolic subgroup* of W.

Proposition 2.1.1 *Let (W, S) be a Coxeter system and $I \subset S$. Then:*

1. *(W_I, I) is a Coxeter system on its own.*
2. *The length functions with respect W and W_I agree on W_I.*
3. *Let $W^I = \{w \in W \mid \ell(ws) > \ell(w) \forall s \in I\}$. Then for every $w \in W$ there exist unique w^I, w_I such that $w = w^I \cdot w_I$. Moreover $\ell(w) = \ell(w^I) + \ell(w_I)$.*

□

In the following, the elemts of W^I will be called *minimal length coset representatives.*

2.1.2 Matsumoto's theorem

Let $(M, \cdot)$ a monoid. For $s, t \in M$, we define

$$\mathrm{Prod}(s, t; m) = \underbrace{ststs\ldots}_{m-\text{terms}}$$

Let now (W, S) be a Coxeter system. Note that, given the relations $s^2 = 1$ (for $s \in S$), a relation of type $(s, t)^{m(s,t)}$ may be restated as:

$$\mathrm{Prod}(s, t; m(s, t)) = \mathrm{Prod}(t, s; m(s, t))$$

We have the following theorem (see e.g. [GP00], section 1.2):

Theorem 2.1.2 *Let (W, S) be a Coxeter system and $(M, \cdot)$ a monoid. Suppose given a map $f : S \to M$ such that:*

$$\mathrm{Prod}(f(s), f(t); m(s, t)) = \mathrm{Prod}(f(t), f(s); m(s, t))$$

Then there exists a unique extension $\hat{f} : W \to M$ of f, such that $\hat{f}(w) = f(s_1) \cdots f(s_k)$ whenever $w = s_1 \cdots s_k$ is a reduced expression.

□

Roughly speaking, Matsumoto theorem states that $\hat{f}(w)$, as defined in the theorem, does not depend on the choice of a reduced expression for w.

2.2 Geometric representation of W

Any Coxeter system can be represented as a concrete group generated by (not necessary orthogonal) reflections. This concrete representation is important for analyzing the structure of W and it will turn out to be essential for further topological constructions.

Consider the real vector space V generated by the vectors $\{\alpha_s \,|\, s \in S\}$ and define a symmetric bilinear for B on V:

$$B(\alpha_s, \alpha_{s'}) = -\cos\left(\frac{\pi}{m(s,s')}\right)$$

with the convention that $B(\alpha_s, \alpha_{s'}) = -1$ in case $m(s,s') = \infty$. Let H_s be the orthogonal hyperplane to α_s w.r.t. B.

For each $s \in S$ define the reflection ρ_s of V as:

$$\rho_s(\alpha_{s'}) = \alpha_{s'} - 2B(\alpha_s, \alpha_{s'})\alpha_s$$

Note that $\rho_s(\alpha_s) = -\alpha_s$ and ρ_s fixes H_s pointwise.

Theorem 2.2.1 *The homomorphism $\rho : W \ni s \mapsto \rho_s \in \mathrm{GL}(V)$ defines a faithful representation of W. Moreover $\rho(W)$ preserves the bilinear form B.* □

In the following, for $\lambda \in V$, $w \in W$, we will write $w\lambda$ in place of $\rho(w)\lambda$ when no confusion is possible.

2.2.1 Positive roots and interpretation of length function

The *root system* of W is defined as the orbit of the canonical basis vectors of V under the action of W, i.e.:

$$\Phi = \{w(\alpha_s) \,|\, \forall s \in S,\ \ \forall w \in W\}$$

Note that this set consists entirely of unit vectors w.r.t. B, since the form B is W-invariant and $B(\alpha_s, \alpha_s) = 1$. Of course, a root α may be written as a real linear combination $\alpha = \sum c_s \alpha_s$. We will say that α is positive (resp. negative) if $c_s \geq 0$ (resp. ≤ 0) for all $s \in S$. Let Φ^+ (resp.Φ^-) be the set of positive (resp. negative) roots.
The set $\Delta = \{\alpha_s \,|\, s \in S\}$ will be called the set of simple roots. We also define the following partial order on Φ: for $\alpha, \beta \in \Phi$, say

$$\alpha \geq \beta \qquad \text{iff } \alpha - \beta \text{ is a non-negative combination of } \Delta$$

Theorem 2.2.2 *Let $w \in W$ and $s \in S$. Then:*

1. *If $\ell(ws) > \ell(w)$, then $w(\alpha_s) \in \Phi^+$.*

2. *If $\ell(ws) < \ell(w)$, then $w(\alpha_s) \in \Phi^-$.*

In particular $\Phi = \Phi^+ \cup \Phi^-$.
Moreover, for any $w \in W$, $\ell(w)$ equals the number of positive roots sent to negative by w, i.e.:

$$\ell(w) = |w(\Phi^+) \cap \Phi^-|$$

2.2.2 Fundamental domain for W and Tits cone

To get more insight in the geometric representation, it is convenient to switch to the contragredient representation $W \to \mathrm{GL}(V^*)$. For $f \in V^*$, $\alpha \in V$, let $\langle f, \alpha \rangle$ be the natural pairing between V^* and V. Then for $w \in W$ the contragredient representation is defined by:

$$\langle w(f), \alpha \rangle = \langle f, w^{-1}(\alpha) \rangle$$

For each $s \in S$, consider the hyperplane:

$$Z_s = \{f \in V^* \,|\, \langle f, \alpha_s \rangle = 0\}$$

and the two half-spaces:

$$\begin{aligned} A_s =& \{f \in V^* \,|\, \langle f, \alpha_s \rangle > 0\} \\ A'_s =& \{f \in V^* \,|\, \langle f, \alpha_s \rangle < 0\} = s(A_s) \end{aligned}$$

Clearly, s fixes Z_s pointwise.
Let C be the intersection of all the A_s for $s \in S$. Note that C is open and its closure $K =: \bar{C} = \bigcap_{s \in S} \bar{A}_s$ is a convex cone.
We can partition D according to the action of the parabolic subgroups W_Γ ($\Gamma \subset S$). Let

$$C_\Gamma = \big(\bigcap_{s \in \Gamma} Z_s\big) \cap \big(\bigcap_{s \notin \Gamma} A_s\big) \tag{2.1}$$

Note that W_Γ fixes C_Γ pointwise. It will be a consequence of theorem 2.2.3 below that W_Γ is precisely the stabilizer of any point $x \in C_\Gamma$.
Further, the Tits cone of W is defined as $U := \bigcup_{w \in W} K$. This is a W-stable subset of V^* and it will be showed to be also convex.

Theorem 2.2.3 ([Vin71]; see also [Bou68]) *With the previous notation, we have:*

1. *Let $w \in W$ and $\Gamma, \Gamma' \subset S$. If $w(C_\Gamma) \cap C_{\Gamma'} \neq \emptyset$, then $\Gamma = \Gamma'$ and $w \in W_\Gamma$. In particular, W_Γ is exactly the stabilizer of every point $x \in C_\Gamma$ and $\mathcal{C} = \{w(C_\Gamma) \,|\, \Gamma \subset S,\, w \in W\}$ is a partition of U.*

2. *K is a fundamental domain for the action of W on U*

3. *U is a convex cone. Moreover, the segment joining two points of U is covered by a finite number of elements of the family $\mathcal{C}$*

4. *The interior U^0 of U is W-stable and W acts properly on it.*

5. *Let $K^f := \{x \in K \mid |\mathrm{stab}(x)| < \infty\} = \bigcup_{\Gamma : |W_\Gamma| < \infty} C_\Gamma$. Then $K^f = K \cap U^0$.*

□

2.3 Special cases

2.3.1 Finite Coxeter groups

Let (W, S) be a Coxeter system as before and consider its geometric representation defined on V. In particular let B the associated bilinear form and $U \subset V^\star$ the Tits cone.

Theorem 2.3.1 *The following conditions are equivalent:*

1. *W is finite.*

2. *The Tits cone is the whole space, i.e. $U = V^*$.*

3. *The form B is positive definite.*

4. *W is a finite (orthogonal) reflection group.*

□

In particular, the usual classification result for finite reflection groups applies:

Theorem 2.3.2 *Let (W, S) be a finite irreducible Coxeter system. Then its Coxeter graph is isomorphic to precisely one of the graph shown in table 2.1.*

□

We will be mainly concerned with the infinite series of type $A, B = C, D$. For completeness, we briefly realize the associated root system and discuss some features that will be relevant further on. In particular, for future use, we describe root systems of type B and C separately, although they give rise to the same Coxeter group. Let $e_1, \ldots, e_m$ be the standard basis of $\mathbf{R}^m$.

• A_n $(n \geq 1)$

Let $V = \{x = (x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} \mid \sum x_i = 0\}$. Define $\Phi = \{e_i - e_j \,|1 \leq i \neq j \leq n+1\}$ and take Δ to be given by:

$$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3, \ldots, \alpha_n = e_n - e_{n+1}$$

The positive roots are then given by:

$$\Phi^+ = \{e_i - e_j \,|1 \leq i < j \leq n+1\}$$

Therefore $|\Phi^+| = \binom{n+1}{2}$.

Note that the root $\tilde{\alpha} = e_1 - e_n$ satisfies $\tilde{\alpha} \geq \alpha$ for every $\alpha \in \Phi$ and it is

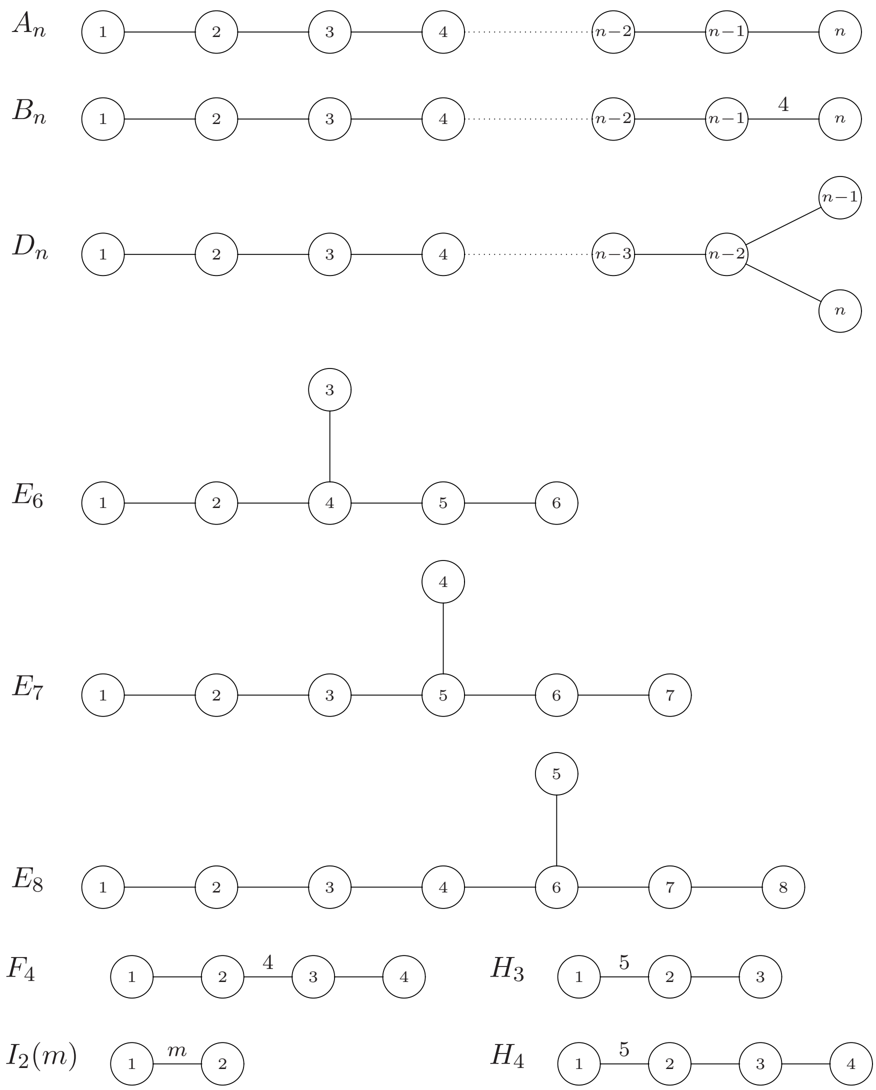


Table 2.1: Graphs of finite Coxeter groups

called the *highest* root.
As it is well-known W is the symmetric group $\mathcal{S}_{n+1}$on $n+1$ letters.
• B_n $(n \geq 2)$
Let $V = \mathbf{R}^n$ and define

$$\Phi = \{\pm e_i \,|\, 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j \,|\, 1 \leq i \neq j \leq n\}$$

So Φ consists of the $2n^2$ vector of length 1 and $\sqrt{2}$ in the standard lattice. The simple roots of Δ are given by:

$$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3, \ldots, \alpha_{n-1} = e_{n-1} - e_n,\ \alpha_n = e_n$$

and the highest root is now given by $\tilde{\alpha} = e_1 + e_2$.
W is the group of *signed* permutations on n letters and, as such, it has a semidirect decomposition $W = \mathcal{S}_n \ltimes (\mathbb{Z}/2)^n$.
• C_n $(n \geq 2)$
In this case Φ may be defined as the inverse root system of type B_n. More precisely, define

$$\Phi = \{\pm 2e_i \,|\, 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j \,|\, 1 \leq i \neq j \leq n\}$$

So Φ consists of the $2n^2$ vector of length 2 and $\sqrt{2}$ in the standard lattice. The simple roots of Δ are given by:

$$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3, \ldots, \alpha_{n-1} = e_{n-1} - e_n,\ \alpha_n = 2e_n$$

and the highest root is now given by $\tilde{\alpha} = 2e_1$.

• D_n $(n \geq 4)$
Let $V = \mathbf{R}^n$ and define

$$\Phi = \{\pm e_i \pm e_j \,|\, 1 \leq i \neq j \leq n\}$$

So Φ consists of the $2n(n-1)$ vector of length $\sqrt{2}$ in the standard lattice. The simple roots of Δ are given by:

$$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3, \ldots, \alpha_{n-1} = e_{n-1} - e_n,\ \alpha_n = e_{n-1} + e_n$$

and the highest root is now given by $\tilde{\alpha} = e_1 + e_2$.
W is the group of signed permutations on n letters that involve an *even* number of sign changes. As such, it has a semidirect decomposition $W = \mathcal{S}_n \ltimes (\mathbb{Z}/2)^{n-1}$.

2.3.2 Affine Coxeter groups

We will now describe a class of infinite Coxeter groups which arise as affine reflection groups in the Euclidean space. First, we will realize these groups as a semidirect product of a Weyl group and a lattice of translations (the coroot lattice); in such a way they are naturally groups of affine reflections. Then we will consider the associated abstract Coxeter group and establish a connection between the two concrete realizations: the affine realization and the geometric representation given by the Tits cone.

Weyl groups Let (W, S) be a finite Coxeter system and $\rho : W \to \mathrm{GL}(V)$ its geometric representation. Let also $(\cdot, \cdot)$ be the scalar product on V induced by the positive definite form B. We say that W is *crystallographic* reflection group if W stabilizes a (maximal rank) lattice in V.

Proposition 2.3.3 *W is crystallographic iff each integer $m(i, j)$ ($i, j \in S$, $i \neq j$) is $2, 3, 4$ or 6.*

□

This proposition can be used together with theorem 2.3.2 to find a list of crystallographic groups. Indeed, the propositions rules out H_3, H_4 and all dihedral groups $I_2(m)$ for $m \neq 3, 4, 6$. Let Φ be the root system associated to (W, S). We say that Φ is crystallographic if:

$$c_{\alpha\beta} = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z} \qquad \forall \alpha, \beta \in \Phi$$

We say in this case that W is a *Weyl group* and call the $c_{\alpha,\beta}$ the Cartan integers.
Note that the crystallographic condition implies that $s_\alpha\beta$ is obtained by β adding an integral multiple of α. This readily implies that W is crystallographic.

From the classification theorem of crystallographic root systems [Bou68], it follows that Weyl groups are precisely the crystallographic reflection groups. Notice, however, that there are two distinct root systems (those of type B_n, C_n), which give rise to the same Coxeter group, as previously observed.

Let Φ a crystallographic root system. For $\alpha \in \Phi$, define $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ to be a *coroot* and let $\Phi^\vee$ the set of all coroots. $\Phi^\vee$ is a crystallographic root system in its own right and it is called the inverse root system of Φ. Note that B_n and C_n are inverse of each other.

Various lattices in V are associated to Φ. We have the root lattice $L(\Phi)$ defined as the $\mathbb{Z}$-span of Φ, the coroot lattice $L(\Phi^\vee)$, the weight lattice

$$\hat{L}(\Phi) = \{v \in V \mid (v, \alpha^\vee) \in \mathbb{Z} \ \forall \alpha \in \Phi\}$$

and the coweight lattice

$$\hat{L}(\Phi^\vee) = \{v \in V \mid (v, \alpha) \in \mathbb{Z} \; \forall \alpha \in \Phi\}$$

Note that $L(\Phi) \subset \hat{L}(\Phi)$ as a subgroup of finite index f, where f is given by the determinant of the Cartan integers matrix. Similarly for the coroots and coweight lattices.

Affine Weyl groups Let W an irreducible Weyl group acting on V as before and be Φ its root system.
For $\alpha \in \Phi$ and $k \in \mathbb{Z}$, we define the affine hyperplane:

$$H_{\alpha,k} = \{v \in V \mid (v, \alpha) = k\}$$

This is the translate of the linear hyperplane orthogonal to α by the translation $t(\frac{k}{2}\alpha^\vee)$. We henceforth define the affine reflection w.r.t $H_{\alpha,k}$ as:

$$s_{\alpha,k}(v) = v - ((v, \alpha) - k)\alpha^\vee$$

Indeed, this transformation fixes $H_{\alpha,k}$ pointwise and sends $0 \mapsto k\alpha^\vee$.

We define the affine Weyl group W_a to be the subgroup of $\mathrm{Aff}(V)$ generated by the $s_{\alpha,k}$ for $\alpha \in \Phi$, $k \in \mathbb{Z}$.

Proposition 2.3.4 *W_a is the semidirect product of W with the translation group corresponding to the coroot lattice, i.e.*

$$W_a = W \ltimes L(\Phi^\vee)$$

Indeed W normalizes the translations by a vector in $L(\Phi^\vee)$ and, thus, we can then form their semidirect product. The equation $s_{\alpha,k} = t(k\alpha^\vee)s_\alpha$ shows that W_a is included in this semidirect product. Viceversa, by the same equation, $L(\Phi^\vee)$ is included in W_a. Since trivially W is contained in W_a, the affine group W_a coincides with $W \ltimes L(\Phi^\vee)$.

Note that W also normalize the coweight lattice $\hat{L}(\Phi^\vee)$, and we can form the *extended affine group* $\hat{W}_a = W \ltimes \hat{L}(\Phi^\vee)$ which contains W_a as a subgroup of finite index; indeed $\hat{W}_a/W_a = \hat{L}(\Phi^\vee)/L(\Phi^\vee)$.

Consider the hyperplane arrangement $\mathcal{A} = \{H_{\alpha,k} \mid \alpha \in \Phi, \, k \in \mathbb{Z}\}$ and its complement $\mathcal{M}(\mathcal{A}) = V \setminus \bigcup_{H \in \mathcal{A}} H$. Let $\mathcal{C}$ be the set of connected components of $\mathcal{M}(\mathcal{A})$. Each element of $\mathcal{C}$ is called an alcove. A particular instance of an alcove is given by:

$$C_0 = \{v \in V \mid 0 < (v, \alpha) < 1 \, \forall \alpha \in \Phi^+\}$$

Indeed this set is contained in $\mathcal{M}(\mathcal{A})$, is convex (and so connected)and, further, any element outside C_0 is separated by at least one hyperplane of type $H_{\alpha,0}$ or $H_{\alpha,1}$.

Lemma 2.3.5 *Let Δ be simple roots and $\tilde{\alpha}$ the highest root of Φ, then*

$$C_0 = \{v \in V \mid 0 < (v, \alpha) \, \forall \alpha \in \Delta, \, (v, \tilde{\alpha}) < 1\}$$

Thus C_0 is a simplex having as *walls* the hyperplanes H_α $(\alpha \in \Delta)$ and $H_{\tilde{\alpha},1}$

Theorem 2.3.6

1. *W_a permutes $\mathcal{C}$ transitively.*
2. *The closure of C_0 is a fundamental domain for the action of W_a on V.*
3. *W_a is generated by the set of reflections S_a w.r.t. the walls of C_0.*
4. *(W_a, S_a) is a Coxeter system.*

□

It is easy to find the Coxeter graph of an affine weyl group W_a. Just start with the corresponding graph for W and add a node corresponding to the further reflection $s_{\tilde{\alpha}}$. Connect this node to the node for s_α by an edge having weight the order of $s_\alpha s_{\tilde{\alpha}}$. In this way we recover the graphs shown in table 2.2, where, as customary, $\tilde{\alpha}$ has been replaced by α_0.

The graphs of table 2.2 are precisely the Coxeter graph for which the associated bilinear form B is positive semidefinite but not definite.

We may represent the abstract Coxeter group corresponding to an irreducible affine Weyl group W_a by means of the geometric representation discussed in section 2.2. Let V' be the real vector space with base α_s, $(s \in S_a)$, and $\sigma : W_a \to \mathrm{GL}(V')$ be the geometric representation, with associated bilinear form B. Let A be the matrix of this bilinear form w.r.t. the basis $\{\alpha_s\}$. Since W_a is irreducible, the matrix A is indecomposable (i.e. it can not be written in block diagonal form after a permutation of the basis).

Lemma 2.3.7 *Let A be a real symmetric matrix of dimension n which is positive semidefinite and indecomposable, satisfying:*

$$a_{ij} \leq 0 \qquad \forall i \neq j$$

Then:

1. $\ker A = \{x \in \mathbf{R}^n \mid x^t A x = 0\}$ *and it has dimension* ≤ 1.
2. *The smallest eigenvalue of A has multiplicity 1 and the relative eigenvector has all strictly positive entries.* □

Then, if A is positive semidefinite but not definite, it has a one dimensional kernel spanned by a vector λ with strictly positive entries. Further λ spans the radical $(V')^\perp$. Let H'_s be the orthogonal complement of α_s with respect B. Note that $(V')^\perp = \bigcap H'_s$ and therefore is fixed pointwise by W_a. It follows that the hyperplane $Z = \{f \in (V')^* \mid \langle f, \lambda \rangle = 0\}$ in the dual space is left fixed by the contragredient action.

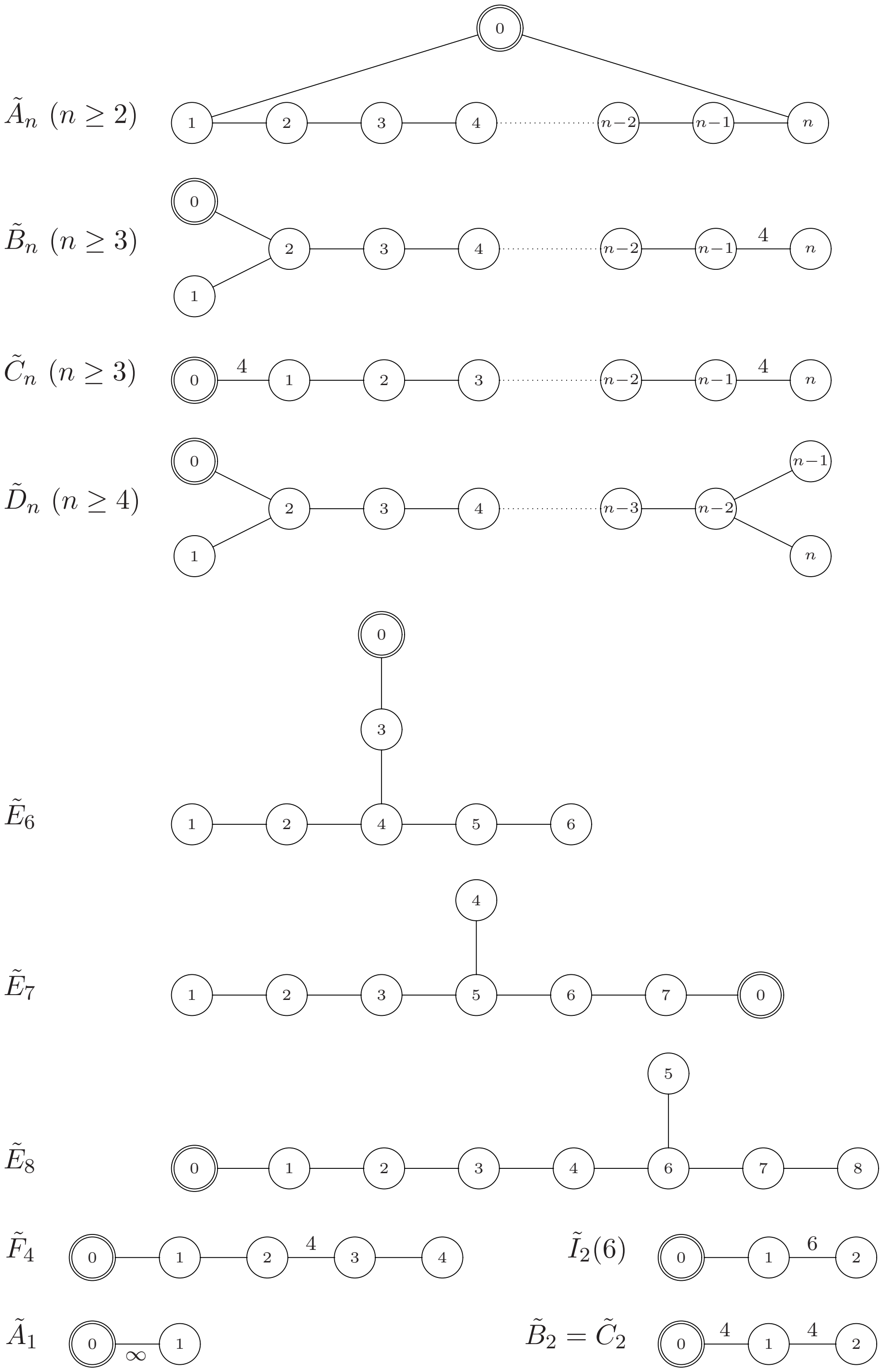


Table 2.2: Coxeter diagrams of affine Weyl groups

Since B is positive definite on the quotient $V'/(V')^{\perp}$ and since Z is dual to $V'/(V')^{\perp}$, the hyperplane Z inherits an Euclidean space structure. Note that W stabilizes as well the affine hyperplane $E = \{f \in (V')^* \mid \langle f, \lambda\rangle = 1\}$ and the Euclidean structure of Z transfers to E.
Consider the hyperplans

$$Z_s = \{f \in (V')^* \mid \langle f, \alpha_s\rangle = 0\}$$

Since λ, being a strictly positive combination of the basis vector, is in particular non proportional to any α_s, we have that $E_s = E \cap Z_s$ is an affine hyperplane in E. The associated action of s turns out to be an orthogonal transformation of E fixing E_s and having order 2. In conclusion W_a is realized as a group of affine reflection in E.

This realization coincides with the previously described one. Indeed the order of ss' characterizes the angle between E_s and $E_{s'}$ and, thus, the configuration of the hyperplanes E_s $(s \in S)$ coincides with the configuration of the walls of the alcove C_0. We further remark a consequence of this discussion:

Corollary 2.3.8 *Let W_a be an affine Weyl group, $\sigma : W_a \to \mathrm{GL}(V')$ its geometric representation and λ a positive vector that spans $(V')^{\perp}$. Then the Tits cone U of W_a is the half space*

$$U = \{f \in (V')^* \mid \langle f, \lambda\rangle \geq 0\}$$

□

2.4 Poincaré series

In this section, after defining some notation regarding q- and (q,t)-analogue of natural numbers, we describe basic results on standard Poincaré series. Finally we discuss a (q,t)-analogue of Poincaré series for B_n, that will be useful for future computations in cohomology.

2.4.1 Notation

We recall some notations. We define the q-analog of the number m by the polynomial

$$[m]_q := 1 + q + \cdots q^{m-1} = \frac{q^m - 1}{q - 1}.$$

It is easy to see that $[m] = \prod_{i|m} \varphi_i(q)$, where $\varphi_i(q)$ is the i-th cyclotomic polynomial in the variable q. Moreover we define the q-factorial $[m]_q!$ as the product

$$[m]_q! := \prod_{i=1}^{m} [i]_q \tag{2.2}$$

and the q-analog of the binomial $\binom{m}{i}$ as the polynomial

$$\left[\begin{array}{c} m \\ i \end{array}\right]_q := \frac{[m]_q!}{[i]_q![m-i]_q!} \tag{2.3}$$

We can also define the (q,t)-analog of an even number

$$[2m]_{q,t} := [m]_q(1+tq^{m-1}) \tag{2.4}$$

and of the double factorial

$$[2m]_{q,t}!! := \prod_{i=1}^{m}[2i]_{q,t} \;=\; [m]_q! \prod_{i=0}^{m-1}(1+tq^i) \tag{2.5}$$

and the polynomial

$$\left[\begin{array}{c} m \\ i \end{array}\right]'_{q,t} := \frac{[2m]_{q,t}!!}{[2i]_{q,t}!![m-i]_q!} \;=\left[\begin{array}{c} m \\ i \end{array}\right]_q \prod_{j=i}^{m-1}(1+tq^j) \tag{2.6}$$

2.4.2 Standard Poincaré series

Let (W,S) be a Coxeter system. For every subset $X \subset W$, we define:

$$X(q) = \sum_{x \in X} q^{\ell(x)}$$

Note that this is well defined, since for every positive integer s, there is only a finite number of elements having length s.
When $X = W$, we obtain, by definition, the *Poincaré series* $W(q)$ of W.

It follows from proposition 2.1.1, that, for every subset $I \subset S$, the series $W_I(q)$ is the Poincaré series of W_I and moreover:

$$W(q) = W_I(q)W^I(q) \tag{2.7}$$

For finite Coxeter groups, Poincaré series (actually polynomial) are well-known and extensively studied. Here we just mention some basics results that will be useful later.

Proposition 2.4.1 *Let W be a finite Coxeter group of rank n. Then its Poincaré polynomial factors as:*

$$W(q) = \prod_{i=1}^{n}[d_i]_q$$

where $d_1, \ldots, d_n$ are called the characteristic degrees *of W*

Group	Degrees
A_n	$2, 3, \ldots, n+1$
B_n	$2, 4, 6, \ldots, 2n$
D_n	$2, 4, 6, \ldots, 2n-2, n$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$
F_4	$2, 6, 8, 12$
H_3	$2, 6, 10$
E_4	$2, 12, 20, 30$
$I_2(m)$	$2, m$

Table 2.3: Characteristic degrees of finite Coxeter groups

□

The characteristic degrees of W are best studied in relation with the algebra of polynomial invariants of the grouup and they are completely known (see table 2.3). In particular we have:

$$\begin{aligned} A_n(q) =& [n+1]_q! \\ B_n(q) =& [2n]_q!! \end{aligned}$$

2.4.3 Double weighted Poincaré polynomial for B_n

For future use in cohomology computations, we are interested in a (q,t)-analogue of the usual Poincaré series for B_n. This result and similar ones are studied in [Rei93].

Consider the Coxeter group W of type B_n with its standard generating reflections $s_1, s_2, \ldots, s_n$.
For $w \in W$, let $n(w)$ be the number of times s_n appears in a reduced expression for w.

Lemma 2.4.2 *The number $n(w)$ is well defined, i.e. it does not depend on the choice of a reduced expression for w.*

Proof. Consider the monoid of natural numbers $(\mathbb{N}, +)$ and define $f : S \to \mathbb{N}$ by $s_i \mapsto 0$ for $1 \leq i < n$ and $s_n \mapsto 1$. Since $m(s_i, s_n)$ is an even number for $i \neq n$, we have

$$\mathrm{Prod}(f(s_i), f(s_n); m(s_i, s_n)) = \mathrm{Prod}(f(s_n), f(s_i); m(s_i, s_n))$$

Thus we may apply Matsumoto's theorem (theorem 2.1.2) to get the desired result.

□

For $X \subset W$, define the two-variate series

$$X(q,t) = \sum_{w \in X} q^{\ell(w)-n(w)} t^{n(w)} \tag{2.8}$$

For $X = W$, this series is called the (q,t)*-weighted Poincaré series* of W. Note that specializing to $t = q$, we recover ordinary Poincaré series.

Since $n(w_1 w_2) = n(w_1) + n(w_2)$ whenever $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$, we have in particular:

$$W(q,t) = W_I(q,t) W^I(q,t)$$

for any $I \subset S$.

Proposition 2.4.3 ([Rei93])

$$W(q,t) = [2n]_{q,t}!!$$

Proof. Consider the parabolic subgroup W_I associated to the subset of reflections $I = \{s_1, \ldots, s_{n-1}\}$. Notice that W_I is isomorphic to the symmetric group on n letters A_{n-1} and that it has index 2^n in B_n. Then, by equation 2.8:

$$\begin{aligned} W(q,t) &= \sum_{w \in W} q^{\ell(w)-n(w)} t^{n(w)} \\ &= \Big(\sum_{w' \in W^I} q^{\ell(w')-n(w')} t^{n(w')} \Big) \cdot \Big(\sum_{w'' \in W_I} q^{\ell(w'')-n(w'')} t^{n(w'')} \Big) \end{aligned} \tag{2.9}$$

Clearly, for elements $w'' \in W_I$, we have $n(w'') = 0$; so the second factor in 2.9 reduces to ordinary Poincaré series for A_{n-1}:

$$\sum_{w'' \in W_I} q^{\ell(w'')-n(w'')} t^{n(w'')} = [n]_q!$$

To deal with the first factor, instead, we explicitly enumerate the elements of W^I. For $i = 1, \ldots, n$ let:

$$p_i := s_i s_{i+1} \cdots s_n \tag{2.10}$$

We have the following lemma:

Lemma 2.4.4 *1. $p_i = s_i s_{i+1} \cdots s_n$ is a reduced expression.*

2. For $i_1 < i_2 < \cdots < i_{r-1} < i_r$, we have

$$\ell(p_{i_r} p_{i_{r-1}} \cdots p_{i_2} p_{i_1}) = \sum_{j=1}^{r} \ell(p_{i_j})$$

Thus a reduced expression for $p_{i_r} p_{i_{r-1}} \cdots p_{i_2} p_{i_1}$ is obtained by concatenation of reduced expressions for the p_is.

3. $W^I = \{p_{i_r}p_{i_{r-1}}\cdots p_{i_2}p_{i_1} \mid i_1 < i_2 < \cdots < i_{r-1} < i_r\}$

Assuming lemma 2.4.4 and noting that

$$\begin{aligned} n(p_{i_r}p_{i_{r-1}}\cdots p_{i_2}p_{i_1}) &= r \\ \ell(p_{i_r}p_{i_{r-1}}\cdots p_{i_2}p_{i_1}) &= \sum_{j=1}^{r}\ell(p_{i_j}) = \sum_{j=1}^{r}(n+1-i_j) \end{aligned}$$

we may rewrite the first factor in formula 2.9 as:

$$\sum_{w'\in W^I} q^{\ell(w')-n(w')}t^{n(w')} = \prod_{i=0}^{n-1}(1+tq^i)$$

Finally,

$$W(q,t) = \Big(\prod_{i=0}^{n-1}(1+tq^i)\Big)[n]_q! = [2n]_{q,t}!!$$

□

Lemma 2.4.4 will be proved elementarily in section 2.4.4 below.

2.4.4 Proof of lemma 2.4.4

Before proving lemma 2.4.4, we will digress a little bit on the study of the length function for the group B_n. In particular we will present a simple method to count the length of an arbitrary element of B_n.

Length of elements of B_n as signed permutations

It is well known that, for the symmetric group, the length of an element w.r.t. standard generator is given by the number of inversions, i.e. given $w \in A_{n-1}$,:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ i_1 & i_2 & i_3 & i_4 & \dots & i_n \end{pmatrix}$$

the length of w is given by the number of couples $i_l > i_m$ with $l < m$. Indeed, this is of course equivalent to the number of positive roots sent to negative ones and we may thus apply theorem 2.2.2. Representing elements of B_n as signed permutations, we may count the length of an element in a similar fashion. Let be given $w \in B_n$,

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ *_1 i_1 & *_2 i_2 & *_3 i_3 & *_4 i_4 & \dots & *_n i_n \end{pmatrix}$$

where each occurrence of $*_i$ should be replaced by ± 1.
Consider $\Phi_k^+ = \{e_k\} \cup \{e_k \pm e_j \mid j > k\}$ and let $\ell_k = |w(\Phi_k^+) \cap \Phi^-|$.

The following lemma is straightforward:

Lemma 2.4.5 *The length of an element $w \in B_n$ can be decomposed as $\ell(w) = \sum_{k=1}^{n} \ell_k(w)$. Further:*

1. *If $*_k = 1$, than ℓ_k equals the number of inversions of type $i_k > i_j$ (disregarding the sign $*_j$) with $j > k$.*

2. *If $*_k = -1$, then ℓ_k equals the number of non-inversions of type $i_k < i_j$ (with $j > k$)* incremented *by $n + 1 - k$.*

$\square$

For example the length of the element

$$w_0 = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ -1 & -2 & -3 & -4 & \dots & -n \end{pmatrix}$$

can be computed as

$$\ell(w_0) = (2n-1) + (2n-3) + \cdots + 1 = n^2$$

Therefore w_0 is the element of maximal length in B_n (recall that the number of positive roots is n^2).
Now, the element $p_i = s_i s_{i+1} \cdots s_n$ of formula 2.10 corresponds to the signed permutation:

$$p_i = \begin{pmatrix} 1 & \dots & i-1 & i & \dots & n-1 & n \\ 1 & \dots & i-1 & i+1 & \dots & n & -i \end{pmatrix}$$

and thus its length is

$$\ell(p_i) = (n-i)\,(\text{inversions}) + 1(\text{sign change in the last position}) = n + 1 - i$$

Therefore the expression we used to define p_i is reduced.

Proof of lemma 2.4.4

We start showing that $p_i \in W^I$. For this is enough to show that $\ell(p_i s) > \ell(p_i)$ for every $s \in I$. Looking to the action of p_i on the root space, we have:

$$p_i(e_k) = \begin{cases} e_k & k < i \\ e_{k+1} & i \leq k \leq n-1 \\ -e_i & k = n \end{cases}$$

Therefore $p_i(e_k - e_{k+1}) > 0$ for $1 \leq k \leq n$. Thus we get $p_i \in W^I$ by theorem 2.2.2.

In the same way, explicit computation shows that for $i_1 < i_2 < \ldots < i_s$ we have:

$$p_{i_s} p_{i_{s-1}} \cdots p_{i_2} p_{i_1}(e_k) = \begin{cases} e_k & k < i_1 \\ e_{k+1} & i_1 \leq k < i_2 - 1 \\ e_{k+2} & i_2 - 1 \leq k < i_3 - 2 \\ \cdots & \\ e_{k+l} & i_l - l + 1 \leq k < i_{l+1} - (l+1) + 1 \\ \cdots & \\ e_{k+s} & i_s - s + 1 \leq k < n - s + 1 \\ -e_{i_{n+1-k}} & n - s + 1 \leq k \leq n \end{cases}$$

It follows readily that $p_{i_s} p_{i_{s-1}} \cdots p_{i_2} p_{i_1}(e_k - e_{k+1}) > 0$ and so, again by theorem 2.2.2, $p_{i_s} p_{i_{s-1}} \cdots p_{i_2} p_{i_1}$ is a minimal coset representative for I. Further the expression for $x = p_{i_s} p_{i_{s-1}} \cdots p_{i_2} p_{i_1}$ obtained by concatenation of the expressions for the p_is is reduced. Indeed computing its length according to lemma 2.4.5, we have:

$$\ell(x) = \sum_{l=1}^{s} (n - s + 1 - i_l + l - 1) + \sum_{l=1}^{s} l = \sum_{l=1}^{s} (n + 1 - i_l) = \sum_{l=1}^{s} \ell(p_{i_l})$$

Finally varying s and $i_1 < \ldots < i_s$, we obtain 2^n distinct coset representatives (no duplicate arises: this is clear, for example, from the explicit action on the roots). So lemma 2.4.4 is completely proved. □

2.5 Artin groups and the $k(\pi, 1)$ problem

2.5.1 Definition of Artin groups and first properties

Let (W, S) be an arbitrary Coxeter system.

Definition 2.5.1 *[BS72] The Artin group G_W is the group given by the following presentation:*

$\langle g_s | s \in S, \mathrm{Prod}(g_s, g_t, m(s,t)) = \mathrm{Prod}(g_t, g_s, m(s,t))$ *for* $s \neq t$, $m(s,t) \neq \infty \rangle$

Loosely speaking, G_W is the group obtained from the presentation for W omitting the relations $s^2 = 1$ $(s \in S)$.

There is indeed a canonical projection $\pi : G_W \to W$ defined by $\pi(g_s) = s$. We may fit π in the exact sequence:

$$1 \longrightarrow PG_W \xrightarrow{i} G_W \xrightarrow{\pi} W \longrightarrow 1$$

By definition PG_W is the *pure Artin group*. By Matsumoto's theorem 2.1.2, we have a set-theoretic section ψ:

$$\begin{tikzcd} 1 \arrow[r] & PG_W \arrow[r, "i"] & G_W \arrow[r, "\pi"] & W \arrow[l, bend left, "\psi"] \arrow[r] & 1 \end{tikzcd} \tag{2.11}$$

defined by $\psi(w) = g_{s_1} \cdots g_{s_k}$ whenever $w = s_1 \cdots s_k$ is a reduced expression.

An Artin group G_W is said to be of *finite* type if the associated Coxeter group W is finite.

2.5.2 Reflection arrangements and the $k(\pi, 1)$ problem

Finite type Artin groups

Recall that for type A_n, the group G_{A_n} may be identified with the braid group $\mathrm{Br}(n+1)$ in $n+1$ strings, which we briefly describe (see e.g. [Bir74] or [PS97]). Consider $n+1$ distinct points $p_1, \ldots, p_{n+1}$ in $\mathbb{C}$. We may then represent an element of $\mathrm{Br}(n+1)$ as a collection of $n+1$ non-intersecting simple paths in $\mathbb{C} \times [0,1]$ joining the points in $\{(p_i, 1) \mid 1 \leq i \leq n+1\}$ to the points in $\{(p_i, 0) \mid 1 \leq i \leq n+1\}$. Two such *braids* represent the same element of the group if they are homotopic. Modulo this identification, braids form a group, where composition is given by path gluing. The inverse of an element is obtained by mirror reflection w.r.t the plane $\mathbb{C} \times \{1/2\}$.

From this description it is clear that the group $G_{A_n}$is the fundamental group of the space $X(A_n)$ of unordered $(n+1)-$*tuples* of distinct points in $\mathbb{C}$. Of course this space is obtained taking the complement $Y(A_n)$ of the hyperplanes $x_i = x_j$ in $\mathbb{C}^{n+1}$ and quotienting it by the action of the symmetric group A_n. In particular note that the involved hyperplanes are precisely the (complexified) mirrors of the reflections in A_n. Further observe that the pure group PG_{A_n} is the fundamental group of $Y(A_n)$.

This construction can be repeated verbatim for any Artin group of finite type. Let W be a finite Coxeter group of rank n and consider its geometric representation. Let $\mathcal{A}$ the real hyperplane arrangement in $\mathbb{R}^n$ consisting in the mirrors of reflections in W. For an hyperplane $H \in \mathcal{A}$, consider its complexification $H_{\mathbb{C}} := H \otimes \mathbb{C}$. Define

$$Y(W) := \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}} H_{\mathbb{C}}$$

Note that we have an induced free and properly discontinuous action of W on $Y(W)$. The quotient space $X(W) := Y(W)/W$ has the following remarkable property:

Theorem 2.5.2 ([Bri71a]) *The Artin group G_W is the fundamental group of $X(W)$.* □

In the case A_n it is easy to show by induction on n that $Y(A_n)$ is a $k(\pi, 1)$ space and, thus, $X(A_n)$ is a classifying space for the group G_{A_n}. Indeed the projection on the first n coordinates is a fibration [FN62], the fiber being $\mathbb{C}$ with n points removed; we may thus apply the long exact sequence of the fibration.

Similar fibration techniques (not always based on linear projections) work for the cases B_n, $I_2(m)$, D_n, F_4 [Bri71b].

The question in general was settled by Deligne [Del72], who showed that $X(W)$ is a $k(\pi, 1)$ space for any finite Coxeter group. Indeed he proved the following stronger result. Recall that a real arrangement $\mathcal{A}$ is said to be *simplicial* if any of its chambers is a simplicial cone (i.e. after a suitable basis change, the chamber consists precisely of the points with all coordinates positive).

Theorem 2.5.3 ([Del72]) *Let $\mathcal{A}$ be a finite real central arrangement and let $Y(\mathcal{A})$ be the complement of its complexification. If $\mathcal{A}$ is simplicial, then $Y(\mathcal{A})$ is a $k(\pi, 1)$ space.* □

Since reflection arrangements are known to be simplicial [Bou68]:

Theorem 2.5.4 *$Y(W)$ and hence $X(W)$ are $k(\pi, 1)$ spaces.* □

Therefore for a finite type Artin group G_W, we can exhibit an explicit finite dimensional classifying space $X(W)$. In the following chapter we will see how to construct a finite dimensional cell complex homotopic to $X(W)$ that will be useful for the computation of cohomology of Artin groups.

Infinite type Artin groups

The previous topological construction may be repeated with minor changes for infinite type Artin groups. Nevertheless, in general, we have no more a result analogous to theorem 2.5.4.

Let (W, S) be a Coxeter system of rank n and consider the geometric representation given by the Tits cone U. Consider the interior U^0 of U and define the complexified cone as:

$$I := \{v \in \mathbb{C}^n \mid \Re(v) \in U^0\}$$

Let $\mathcal{A}$ be the hyperplane arrangement consisting of the mirrors of reflections in W. Then W acts freely and properly discontinuously on

$$Y(W) := I \setminus \bigcup_{H \in \mathcal{A}} H_{\mathbb{C}} \tag{2.12}$$

Let

$$X(W) := Y(W)/W \tag{2.13}$$

be the orbit space. The following is an extension of theorem 2.5.2:

Theorem 2.5.5 ([vdL83]) *The Artin group G_W is the fundamental group of $X(W)$.* □

Remark. In the following chapter, an explicit cellular complex $\mathcal{S}$ homotopic to $X(W)$ will be described [Sal94]. Analyzing the 2-skeleton of $\mathcal{S}$, it is possible to recover theorem 2.5.5.

As announced, however, it is not known wether $X(W)$ is a classifying space for G_W. This has been verified for some classes of Artin groups, namely

- Artin group of large type [Hen85]. This result is based on an extension of the notion of simplicial arrangement.
- Artin group satisfying the FC condition [CD95].
- The affine Artin groups of type $\tilde{A}_n, \tilde{C}_n$ [Oko79] (rediscovered for $\tilde{A}_n$ in [CP03]).

We refer to the original papers for definitions; here we just mention that FC condition does not hold for Artin groups whose Coxeter graph has a subgraph of affine type.

We will discuss somewhat extensively the $k(\pi, 1)$ problem for affine Artin groups in chapter 4.

Chapter 3

Cohomology of Coxeter and Artin groups: General theory and applications

In this chapter we will introduce a family of complexes $(C_*^{(d)}, \partial)$ $(d \in \mathbb{N} \cup \infty)$ associated to a general Coxeter system (W, S); in particular the complex $(C_*^{(\infty)}, \partial)$ provides a free resolution of the trivial $\mathbb{Z}[W]$ module $\mathbb{Z}$, thus allowing, in principle, computations of the cohomology $H^*(W; M)$ in any coefficient system. The complex $(C_*^{(1)}, \partial)$ describes instead the cohomology of the orbit space $X(W)$ defined in 2.13. When $X(W)$ is a $k(\pi, 1)$, the complex is an effective way to determine the cohomology of G_W. We present as well three applications of this theory. First we determine the cohomology of the Coxeter and Artin groups associated to the complete graph, then we present an approach to homology computations for affine Artin groups based on sheaves over posets and finally we discuss the genus problem for Coxeter groups.

3.1 M-Complexification of hyperplane arrangements in real cones

Here and in next section we state results from [Sal94][DS00] (see also [Sal87], [DS96]), extending them to general type Coxeter group. As stated in the original papers, the proofs work with minor changes and will be omitted.

3.1.1 Basic definitions

Let M be a real vector space of dimension $d+1$ and let $b_1, \ldots, b_{d+1}$ be a basis of M. An element $X \in M^n \cong \mathbb{R}^n \otimes M$ will be then written as $X = \sum X_i \otimes b_i$ or, formally, as $X \equiv X_1 + \ldots + X_{d+1}$.

Let I be an open cone in $\mathbb{R}^n$ and consider its M-complexification:

$$I^{(d+1)} := \{X \in M^n \,|\, X_1 \in I\}$$

Let $\mathcal{A}$ be a central hyperplane arrangement in $\mathbb{R}^n$ such that every hyperplane $H \in \mathcal{A}$ cuts I. Consider the M-complexification $H_M := H \otimes M$ of an hyperplane in $\mathbb{R}^n$ and define:

$$Y^{(d+1)} := Y^{(d+1)}(\mathcal{A}) := I^{(d+1)} \setminus \bigcup_{H \in \mathcal{A}} H_M$$

Note that for $d = 1$, when I is the open Tits cone of a Coxeter group, this space coincides with the 'complexification' described in 2.5.2.

The decomposition of $\mathbb{R}^n$ into facets operated by $\mathcal{A}$ induces a stratification Φ (resp. $\tilde{\Phi}$) of $\mathbb{R}^n$ (resp. I). Facets are partially ordered according to reverse inclusion of their closures:

$$F \preceq G \quad \Leftrightarrow \quad \bar{F} \supseteq G$$

In this way, Φ becomes a ranked poset, where $\mathrm{rk}(F) = \mathrm{codim}(F)$; note that $\tilde{\Phi}$ can be identified with an order ideal of Φ.

Given a facet $F \in \Phi$, let $\mathcal{A}_F = \{H \in \mathcal{A} \,|\, A \supseteq F\}$ be the subarrangement of hyperplanes containing F and consider the induced stratifications Φ_F of $\mathbb{R}^n$. We have a projection map $\phi_F : \Phi \to \Phi_F$, which sends a stratum in Φ in the unique stratum of Φ_F containing it. Note that ϕ_F admits a left inverse: for $G \in \Phi_F$, among the elements of $\phi_F^{-1}(G)$ there exists a unique $F' := \psi^F(G) \in \Phi$ satisfying $F' \prec F$. Further if $F \in \tilde{\Phi}$, clearly $\mathrm{Im}(\psi^F \phi_F) \subseteq \tilde{\Phi}$.

3.1.2 Stratifications for M-complexified cones

With these notions at hand, we can now describe a stratification for $I^{(d+1)}$ induced by the subspace arrangement $\{H_M \,|\, H \in \mathcal{A}\}$.
Consider the set of descending chains of length $d+1$ in $\tilde{\Phi}$:

$$\tilde{\Phi}^{(d+1)} = \{\mathcal{F} = (F_1, \ldots, F_{d+1}) \,|\, F_i \in \Phi,\ F_1 \succeq \ldots \succeq F_{d+1}\}$$

A subset $\hat{\mathcal{F}} \subseteq I^{(d+1)}$ is associated to any of such chains:

$$\hat{\mathcal{F}} := \{X \in I^{(d+1)} \,|\, X_1 \in F_1,\ X_k \in \phi_{F_{k-1}}(F_k) \text{ for } 2 \leq k \leq d+1\}$$

It can be shown that each $\hat{\mathcal{F}}$ is homeomorphic to an open ball and that $\{\hat{\mathcal{F}} \,|\, \mathcal{F} \in \tilde{\Phi}^{(d+1)}\}$ is a stratification of $I^{(d+1)}$. Further, let

$$\tilde{\Phi}_0^{(d+1)} := \{\mathcal{F} \in \tilde{\Phi}^{(d+1)} \,|\, \mathrm{rk}(F_{d+1}) = 0\}$$

be the subset of $\tilde{\Phi}^{(d+1)}$ consisting of descending chains that ends up with a chamber. It is then easily seen that $Y^{(d+1)}$ is the union of the strata corresponding to $\tilde{\Phi}_0^{(d+1)}$, i.e.:

$$Y^{(d+1)} = \bigcup_{\mathcal{F} \in \tilde{\Phi}_0^{(d+1)}} \hat{\mathcal{F}}$$

Since the elements of $\tilde{\Phi}^{(d+1)}$ are associated with a stratification, we may turn $\tilde{\Phi}^{(d+1)}$ into a poset by reverse inclusion of closures; in a more combinatorial fashion, the partial order may be described as:

$$\mathcal{F} \preceq \mathcal{G} \quad \Leftrightarrow \quad \begin{cases} F_1 \preceq G_1 \\ \phi_{F_{k-1}}(F_k) \preceq \phi_{F_{k-1}}(G_k) \quad \text{for } 2 \leq k \leq d+1 \end{cases} \tag{3.1}$$

3.1.3 Associated cellular complexes

We will now build a finite cellular complex homotopic to $Y^{(d+1)}$.
We first select a point inside each stratum of $\tilde{\Phi}^{(d+1)}$. Start choosing one point $\nu(F)$ inside each facet $F \in \tilde{\Phi}$. Given $\mathcal{F} = (F_1 \prec \ldots F_{d+1}) \in \tilde{\Phi}^{(d+1)}$, let F_k be the first facet in $\mathcal{F}$ being equal to F_{d+1}; then $\nu(\mathcal{F}) := \nu(F_1)b_1 + \ldots \nu(F_k)b_k \in \hat{\mathcal{F}}$.

The abstract simplicial complex of strictly ascending chains in $\tilde{\Phi}^{(d+1)}$ may then be concretely realized inside $I^{(d+1)}$: for a chain $\mathcal{C} = (\mathcal{F}_0 \prec \ldots \prec \mathcal{F}_r)$, we may indeed consider the affine r-simplex $s(\mathcal{C})$ having vertex $\nu(\mathcal{F}_0), \ldots, \nu(\mathcal{F}_r)$.

For $\mathcal{F} \in \tilde{\Phi}^{(d+1)}$, let

$$e(\mathcal{F}) := \bigcup_{\mathcal{C} \preceq \mathcal{F}} s(\mathcal{C})$$

the union being over all chains bounded by $\mathcal{F}$ from above. Then $e(\mathcal{F})$ is a triangulated cell dual to $\mathcal{F}$ whose boundary is given by:

$$\partial(e(\mathcal{F})) = \bigcup_{\mathcal{F}' \prec \mathcal{F}} e(\mathcal{F}')$$

Let:

$$\mathcal{Q}^{(d+1)} := \bigcup_{\mathcal{F} \in \tilde{\Phi}^{(d+1)}} e(\mathcal{F})$$
$$\mathcal{P}^{(d)} := \bigcup_{\mathcal{F} \in \tilde{\Phi}_0^{(d+1)}} e(\mathcal{F})$$

and note that $\mathcal{P}$ is a regular subcomplex of $\mathcal{Q}$; for if $\mathcal{F}$ ends up with a chamber, so does any $\mathcal{F}' \prec \mathcal{F}$ by eq. 3.1.

Further, consider the projection $\pi : I^{(d+1)} \to I^{(d)}$ onto the first d coordinates. This induces a morphism of posets

$$\begin{array}{ccc} \tilde{\Phi}^{(d+1)} & \xrightarrow{\pi} & \tilde{\Phi}^{(d)} \\ \mathcal{F} = (\mathcal{F}', F_{d+1}) & \longmapsto & \mathcal{F}' \end{array}$$

and of cellular complexes

$$\begin{array}{ccc} \mathcal{Q}^{(d+1)} & \xrightarrow{\pi} & \mathcal{Q}^{(d)} \\ e(\mathcal{F}) = e\big((\mathcal{F}', F_{d+1})\big) & \longmapsto & e(\mathcal{F}') \end{array}$$

which restricts to a surjective map $\pi : \mathcal{P}^{(d)} \to \mathcal{Q}^{(d)}$.
The following theorem may be proved as in [DS00] using essentially the same retractions exhibited in [Sal87]:

Theorem 3.1.1 (Theorem 1.4.7 in [DS00]) *1. $\mathcal{P}^{(d)}$ is a deformation retract of Y^{d+1}.*

2. The map $\pi_{|e(\mathcal{F}',F_{d+1})}$ is an homeomorphism between $e\big((\mathcal{F}', F_{d+1})\big) \subseteq \mathcal{P}^{(d)}$ and $e(\mathcal{F}') \subseteq \mathcal{Q}^{(d)}$. Thus, the number of cells of $\mathcal{P}^{(d)}$ over $e(\mathcal{F}')$ equals the number of chambers K such that $K \preceq F'_d$.

3. For $\mathcal{F}' \in \tilde{\Phi}^{(d)}$ and K a chamber in $\tilde{\Phi}$ such that $K \preceq F'_d$, we have

$$\dim e((\mathcal{F}', K)) = \operatorname{codim}_{I^{(d)}} \mathcal{F}' = \sum_{i=1}^{d} \operatorname{rk}(F'_i) \qquad \square$$

3.1.4 The limit space $\mathcal{P}^{(\infty)}$

Note that the natural inclusion $i_d : I^{(d)} \to I^{(d+1)}$, induces an injective poset homomorphism $i_d : \tilde{\Phi}^{(d)} \to \tilde{\Phi}^{(d+1)}$ defined by:

$$(F_1, \ldots, F_{d-1}, F_d) \mapsto (F_1, \ldots, F_{d-1}, F_d, F_d)$$

and in turn inclusions:

$$\mathcal{Q}^{(d)} \hookrightarrow \mathcal{Q}^{(d+1)}$$
$$\mathcal{P}^{(d-1)} \hookrightarrow \mathcal{P}^{(d)}$$

It is then easy to prove:

Proposition 3.1.2 *1. $\mathcal{P}^{(d)}$ is obtained from $\mathcal{P}^{d-1}$ attaching cells of dimension $\geq d$*

2. The limit space $\mathcal{P}^{(\infty)} := \lim_{d\to\infty} \mathcal{P}^{(d)}$ is contractible.

3. $\mathcal{P}^{(d)}$ is $(d-1)$-connected. $\square$

3.2 Reflection arrangements

We may now apply these constructions to a general Coxeter group (W, S). Let I be the open Tits cone in a geometric realization and $\mathcal{A}$ be the arrangement of mirrors of reflections in W.

The diagonal action of W on $I^{(d+1)}$ induces actions on the stratification

$$\gamma \cdot (F_1, \dots, F_{d+1}) = (\gamma(F_1), \dots, \gamma(F_{d+1})) \qquad \text{for } \gamma \in W$$

and (choosing suitable points $\nu(F)$ for $F \in \tilde{\Phi}$) on the cellular complex $\mathcal{Q}^{(d+1)}$:

$$\gamma \cdot e(\mathcal{F}) = e(\gamma\mathcal{F}) \qquad \text{for } \gamma \in W$$

By theorem 2.2.3, the W-action is free on $Y^{(d+1)}$ and $\mathcal{P}^{(d)}$. Since the retractions used in theorem 3.1.1 can be chosen W-equivariant, we have:

Corollary 3.2.1 *The orbit space $\mathcal{S}^{(d)} := \mathcal{P}^{(d)}/W$ is a deformation retract of $X^{(d+1)} := Y^{(d+1)}/W$.* □

Let C be the fundamental chamber of W and, for $\Gamma \subseteq S$, let C_Γ be as in formula 2.1.

By theorem 2.2.3, we know that the facets in the stratification $\tilde{\Phi}$ are precisely the W-translates of faces C_Γ having finite stabilizer (i.e. satisfying $|W_\Gamma| < \infty$). Then, clearly each facet $\mathcal{F} \in \tilde{\Phi}^{(d+1)}$ is uniquely determined by a couple $(\mathbf{\Gamma}, \gamma)$ where:

$$\mathbf{\Gamma} = (\Gamma_1 \supseteq \Gamma_2 \supseteq \dots \supseteq \Gamma_{d+1})$$

is a flag of subset of S satisfying $|W_{\Gamma_1}| < \infty$ and $\gamma \in W^{\Gamma_{d+1}}$ is a minimal coset representative for the parabolic subgroup $W_{\Gamma_{d+1}}$. Conversely, any such couple $(\mathbf{\Gamma}, \gamma)$ determine a facet of $\tilde{\Phi}^{(d+1)}$. Hereinafter, if $\mathcal{F}$ corresponds to $(\mathbf{\Gamma}, \gamma)$, we will write $e(\mathbf{\Gamma}, \gamma)$ for the cell $e(\mathcal{F}) \in \mathcal{Q}^{(d+1)}$.

The theory now proceeds exactly as in [DS00] [DS96], provided we consider only flags $\mathbf{\Gamma} = (\Gamma_1 \supseteq \Gamma_2 \supseteq \dots \Gamma_{d+1})$ satisfying $|W_{\Gamma_1}| < \infty$. Thus the exposition will be even more sketchy.

The combinatorial characterization of facets is suitable for the analysis of the W-action on $\mathcal{P}^{(d+1)}$. Indeed one can prove:

Proposition 3.2.2 *The orbit space $\mathcal{S}^{(d)}$ can be described as the quotient of $\mathcal{Q}^{(d)}$ under the equivalence relation that identifies two cells $e(\mathbf{\Gamma}, \gamma)$, $e(\mathbf{\Gamma}, \gamma')$ using the homeomorphism of $I^{(d)}$ induced by $\gamma' \cdot \gamma^{-1}$.* □

After giving an orientation to the cells in $\mathcal{Q}^{(d)}$, it is possible to determine the boundary maps. Consider the complex $(C_*^{(d)}, \partial)$ where each $C_k^{(d)}$ is the free $\mathbb{Z}[W]$-module

$$C_k^{(d)} := \bigoplus_{\substack{\mathbf{\Gamma} : \sum_1^d |\Gamma_i| = k \\ |W_{\Gamma_1}| < \infty}} \mathbb{Z}[W] e(\mathbf{\Gamma})$$

Note that the generators of C_* as a $\mathbb{Z}$-module are in one to one correspondence with the cells of $\mathcal{P}^{(d)}$. The expression of the boundary is the following:

$$\partial e(\mathbf{\Gamma}) = \sum_{\substack{1 \le i \le d \\ |\Gamma_i| > |\Gamma_{i+1}|}} \sum_{\tau \in \Gamma_i} \sum_{\substack{\beta \in W_{\Gamma_i}^{\Gamma_i \setminus \{\tau\}} \\ \beta^{-1}\Gamma_{i+1}\beta \subseteq \Gamma_i \setminus \{\tau\}}} (-1)^{\alpha(\mathbf{\Gamma}, i, \tau, \beta)} \beta e(\mathbf{\Gamma}')$$

where

$$\mathbf{\Gamma}' = (\Gamma_1, \ldots, \Gamma_{i-1}, \Gamma_i \setminus \{\tau\}, \beta^{-1}\Gamma_{i+1}\beta, \ldots, \beta^{-1}\Gamma_d\beta)$$

and $(-1)^{\alpha(\mathbf{\Gamma}, i, \tau, \beta)}$ is an incidence index. To get a precise expression for $\alpha(\mathbf{\Gamma}, i, \tau, \beta)$, fix a linear order on S and let

$$\mu(\Gamma_i, \tau) := |j \in \Gamma \text{ s.t. } j \le \tau|$$
$$\sigma(\beta, \Gamma_j) := |(a, b) \in \Gamma_j \times \Gamma_j \text{ s.t. } i < j \text{ and } \beta(i) > \beta(j)|$$

in other words, $\mu(\Gamma_i, \tau)$ is the number of reflections in Γ_i less or equal to τ and $\sigma(\beta, \Gamma_j)$ is the number of inversions operated by β on Γ_j. Then we define:

$$\alpha(\mathbf{\Gamma}, i, \tau, \beta) = i\ell(\beta) + \sum_{j=1}^{i-1} |\Gamma_j| + \mu(\Gamma_i, \tau) + \sum_{j=i+1}^{d} \sigma(\beta, \Gamma_j)$$

where ℓ is the length function in the Coxeter group.

Theorem 3.2.3 (Theorem 3.1.6 in [DS00]) *The complex $(C_*, \mathbb{Z})$ computes the integer homology of $\mathcal{P}^{(d)}$, i.e:*

$$H_*(C_*^{(d)}, \partial) \cong H_*(\mathcal{P}^{(d)}; \mathbb{Z}) \qquad \square$$

Let $C_* =: \lim_{d\to\infty} C_*^{(d)}$. Since $\mathcal{P}^\infty$ is contractible, we also have:

Theorem 3.2.4 (Theorem 3.1.7 in [DS00]) *The complex (C_*, ∂) provides a free resolution of the trivial $\mathbb{Z}[W]$-module $\mathbb{Z}$, i.e.:*

$$0 \longleftarrow \mathbb{Z} \longleftarrow C_0 \longleftarrow C_1 \longleftarrow C_2 \longleftarrow \cdots$$

is exact. $\square$

Note that this is the algebraic counterpart of the fact that $\mathcal{S}^{(\infty)} := \lim_{d\to\infty} \mathcal{S}^{(d)} \cong \mathcal{P}^{(\infty)}/W$ is a $k(\pi, 1)$-space for W.

In view of theorem 3.2.4, the complex (C_*, ∂) allows in principle homology and cohomology computations in any coefficient module. In homology, for example, letting M be any $\mathbb{Z}[W]$-module, the chain complex $(C_* \otimes_{\mathbb{Z}[W]} M, \partial \otimes \mathrm{Id})$ has homology:

$$H_*(C_* \otimes_{\mathbb{Z}[W]} M) \cong H_*(W; M)$$

3.2.1 Cohomology of the orbit space $X(W)$

Recall that for $d = 1$, the orbit space $X^{(d+1)} = Y^{(d+1)}/W$ is precisely the space $X(W)$ defined in 2.13 and, as such, it has G_W as fundamental group. Using similar ideas, it is possible to exhibit a chain complex (D_*, ∂) for the homology of the universal cover $\tilde{X}(W)$ of $X(W)$. Let D_k be the free $\mathbb{Z}[G_W]$-module

$$D_k := \bigoplus_{\substack{\Gamma : |\Gamma| = k \\ |W_\Gamma| < \infty}} \mathbb{Z}[G_W]e(\Gamma) \tag{3.2}$$

and consider as boundary maps:

$$\partial e(\Gamma) = \sum_{\tau \in \Gamma} \sum_{\beta \in W_\Gamma^{\Gamma \setminus \{\tau\}}} (-1)^{\alpha(\Gamma,\tau,\beta)} \psi(\beta) e(\Gamma') \tag{3.3}$$

where $\Gamma' = \Gamma \setminus \{\tau\}$, the function $\psi : W \to G_W$ is the set-theoretic section of formula 2.11, and the incidence index is defined by

$$\alpha(\Gamma, \tau, \beta) = \ell(\beta) + \mu(\Gamma, \tau)$$

We have the following:

Theorem 3.2.5 ([DS96]) *The chain complex (D_*, ∂) computes the integer homology of $\tilde{X}_W$.* □

Remark. Note that we obtain a more combinatorial formulation of the $k(\pi, 1)$ problem for general Coxeter groups. Indeed the orbit space $X(W)$ is a $k(\pi, 1)$ if and only if (D_*, ∂) is acyclic. In this case, similarly to theorem 3.2.4, we have a finite dimensional free resolution of the trivial $\mathbb{Z}[G_W]$-module $\mathbb{Z}$. In particular, for any G_W-module, we get an explicit cochain complex for the cohomology $H^*(G_W; M)$.

3.3 Example: The complete graph

As an example of the theory just presented, we compute the cohomology of the Coxeter and braid group associated to the complete graph with values in various coefficients systems. Let $\Xi = \Xi_n$ be the complete graph on n vertices, with all edges having weight 3. and let $W = W(\Xi)$ the associated Coxeter group, that is the group having $\{s_i \mid 1 \le i \le n\}$ as generators and relations:

$$\begin{aligned} s_i^2 &= 1 \\ s_i s_j s_i &= s_j s_i s_j \qquad \text{for } i \neq j \end{aligned}$$

Note that a parabolic subgroup W_Γ of W is finite iff $|\Gamma| \le 2$.
Let also G_W be the associated Artin group. By a result of Hendriks [Hen85] (see also [SS97]), we have:

Theorem 3.3.1 *The orbit space $X(W)$ is a $k(G_W, 1)$ space, so its cohomology coincides with that of the Artin group G_W.*

□

3.3.1 Cohomology of $G_{W(\Xi_n)}$

Let as usual $R = \mathbb{Z}[q^{\pm 1}]$ and consider the local coefficient system R_q on $X(W)$, defined by sending the standard generators of G_W to (-q)-multiplication. By the theory exposed in 3.2, we have the following 2-dimensional complex computing the $H^*(G_W; R_q)$.

$$\begin{aligned} C^0 =& R \cdot z \\ C^1 =& \bigoplus_{i=1}^{n} R \cdot v_i \\ C^2 =& \bigoplus_{1 \leq i < j \leq n} R \cdot e_{i,j} \end{aligned}$$

$$\begin{aligned} \mathrm{d}z =& (-q-1) \sum_{i=1}^{n} v_i \\ \mathrm{d}v_k =& (q^2+q+1) \big(\sum_{j:j>k} e_{k,j} - \sum_{i:i<k} e_{i,k} \big) \end{aligned}$$

It is not difficult to compute the cohomology of this complex.

Theorem 3.3.2 ([SS97]) *The cohomology of the Artin group $G_{W(\Xi_n)}$ with values in R_q is given by:*

$$H^i(G_{W(\Xi_n)}; R_q) = \begin{cases} 0 & i = 0 \\ R/[2] & i = 1 \\ (R)^{\oplus \binom{n-1}{2}} \oplus (R/[3])^{\oplus (n-1)} & i = 2 \end{cases}$$

where, as usual, we denote with $[n] = [n]_q = \frac{q^n - 1}{q-1}$ the q-analogue of the integer n.

Proof. Note that the 1-cocycles Z^1 are given by $Z^1 = R \sum v_i$. Indeed, let $\alpha = \sum_i c_i v_i \in Z^1$. Given a couple $i < j$, note that v_i, v_j are incident to $e_{i,j}$ with opposite signs and there are no further incidences with $e_{i,j}$. Therefore we must have $c_i = c_j$ for all couples $i < j$ and thus $c_1 = \ldots = c_n$.

To analyze the 2-coboundaries B^2, it is convenient to choose a new basis for C^2. Let $g_k = \sum_{j:j>k} e_{k,j} - \sum_{i:i<k} e_{i,k}$ and consider

$$C_1^2 = \bigoplus_{k=1}^{n-1} Rg_k$$
$$C_2^2 = \bigoplus_{1\leq i<j<n} Re_{i,j}$$

and note that $C^2 = C_1^2 \oplus C_2^2$. Then, it clear that $B^2 = [3]C_1^2$. The claimed result follows now easily. □

By the same token used in the proof of 3.3.2, and substituting $q = \pm 1$ in the coboundary formulas, one can obtain the cohomology with values respectively in the sign and trivial representations $\mathbb{Z}[\mp 1]$.

Corollary 3.3.3 *We have:*

$$H^i(G_{W(\Xi_n)}; \mathbb{Z}[1]) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} & i = 1 \\ (\mathbb{Z})^{\oplus\binom{n-1}{2}} & i = 2 \end{cases}$$
$$H^i(G_{W(\Xi_n)}; \mathbb{Z}[-1]) = \begin{cases} 0 & i = 0 \\ \mathbb{Z}/2 & i = 1 \\ (\mathbb{Z})^{\oplus\binom{n-1}{2}} \oplus (\mathbb{Z}/3)^{\oplus(n-1)} & i = 2 \end{cases}$$
□

3.3.2 Cohomology of $W(\Xi_n)$ with coefficients in the trivial and sign representation

Let $I_n := \{1, 2, \ldots, n\}$ and consider the flags of subset in I_n inducing finite parabolic subgroups: $\mathbf{\Gamma} = (\Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_k \supsetneq \emptyset \,|\, |\Gamma_i| \leq 2, \Gamma_i \subset I_n)$. A complex for the homology of the associated Coxeter group is given by:

$$C_k = \bigoplus_{\substack{r+s=k \\ r\geq s}} C_{r,s}$$

where

$$C_{r,s} = R < e(\mathbf{\Gamma}) | \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_s \supsetneq \Gamma_{s+1} \supseteq \Gamma_r \supsetneq \emptyset >$$

is the complex of flags of length r with exactly s elements of cardinality 2. We will denote a generator of $C_{r,s}$ as

$$e_{r,s}((i_0,i_1),i_k) = e(\overbrace{\underbrace{\{i_0,i_1\} \supseteq \{i_0,i_1\} \supseteq \ldots \supseteq \{i_0,i_1\}}_{s} \supsetneq \{i_k\} \supseteq \ldots \{i_k\}}^{r})$$

with $i_0 < i_1$ and $k = 0, 1$. When $r = s$ or $s = 0$, obvious identifications are understood. For example, for $r = s$ we have $e_{r,s}((i_0, i_1), i_0) = e_{r,s}((i_0, i_1), i_1)$ and thus the last index is redundant.
It is not difficult to compute the boundary maps. We restrict ourself to coefficients in the trivial and sign representations, i.e. we let $R = \mathbb{Z}[\pm 1]$. We write compactly the two boundary maps letting t be the action of a basic reflection on the ring (thus $t = \pm 1$). The boundary map ∂ may be decomposed into $\partial = \partial_1 + \partial_2$, where $\partial_1 : C_{r,s} \to C_{r-1,s}$ and $\partial_2 : C_{r,s} \to C_{r,s-1}$, and

$$\partial_1^2 = \partial_2^2 = \partial_1\partial_2 + \partial_2\partial_1 = 0$$

In such a way we have a double complex structure on $\{C_{r,s}\}_{r,s}$. The explicit expression of the first boundary $\partial - 1$ is given by:

$$\partial_1 e_{r,s}((i_0, i_1), i_k) = (-1)^{s+1}(t + (-1)^r)e_{r-1,s}((i_0, i_1), i_k)$$

For $r > s$, the second boundary ∂_2 is defined by:

$$\partial_2 e_{r,s}((i_0, i_1), i_k) = e_{r,s-1}((i_0, i_1), i_0) - e_{r,s-1}((i_0, i_1), i_1)$$

Note in particular that $e_{r,s}((i_0, i_1), i_0)$ and $e_{r,s}((i_0, i_1), i_1)$ have the same image under ∂_2.
For $r = s$, we have instead:

$$\partial_2 e_{r,r}((i_0, i_1), i_k) = (1 + (-1)^r t + t^2)\big(e_{r,r-1}((i_0, i_1), i_0) - e_{r,r-1}((i_0, i_1), i_1)\big)$$

For the computation of the associated coboundary maps d_1 and d_2, let $f^{r,s}((i_0, i_1), i_k)$ be the dual to $e_{r,s}((i_0, i_1), i_k)$.
Then for $r > s$ we have

$$\mathrm{d}_1 f^{r,s}((i_0, i_1), i_k) = (-1)^{s+1}(t + (-1)^{r+1})f^{r+1,s}((i_0, i_1), i_k) \tag{3.4}$$

For $r = s > 0$, omitting the redundant index for the element in the singleton:

$$\mathrm{d}_1 f^{r,s}((i_0, i_1), \cdot) = (-1)^{s+1}(t+(-1)^{r+1})(f^{r+1,s}((i_0, i_1), i_0)+f^{r+1,s}((i_0, i_1), i_1)) \tag{3.5}$$

Finally, for $r = s = 0$:

$$\mathrm{d}_1 f^{0,0}((\cdot, \cdot), \cdot) = \sum_{1 \le j \le n} (1 - t)(f^{1,0}((\cdot, \cdot), j)) \tag{3.6}$$

This concludes formulas for d_1.

Formulas for d_2 are exhibited distinguishing among 4 cases.
a) $r - s > 1$, $s > 0$

$$\mathrm{d}_2 f^{r,s}((i_0, i_1), i_k) = (-1)^k(f^{r,s+1}((i_0, i_1), i_0) + f^{r,s+1}((i_0, i_1), i_1)) \tag{3.7}$$

b) $(r,s) = (s+1, s)$, $s > 0$

$$\mathrm{d}_2 f^{r,s}((i_0,i_1),i_k) = (-1)^k(2+(-1)^r)f^{r,s+1}((i_0,i_1),\cdot) \tag{3.8}$$

c) $(r,s) = (r,0)$, $r > 1$

$$\begin{aligned}\mathrm{d}_2 f^{r,0}((\cdot,\cdot),i) = &\sum_{j:\,j>i}\big(f^{r,1}((i,j),i)+f^{r,1}((i,j),j)\big)+\\ &-\sum_{l:\,l<i}\big(f^{r,1}((l,i),l)+f^{r,1}((l,i),i)\big)\end{aligned} \tag{3.9}$$

d) $(r,s) = (1,0)$

$$\mathrm{d}_2 f^{1,0}((\cdot,\cdot),i) = (2-t)\Big(\sum_{j:\,j>i} f^{r,1}((i,j),\cdot) - \sum_{l:\,l<i} f^{r,1}((l,i),\cdot)\Big) \tag{3.10}$$

Using these coboundary formulas, we now determine separately the cohomology of $W(\Xi_n)$ with values in the trivial and sign representation.

Trivial representation ($t = 1$)

To start with, we analyze the structure of the double complex.

For $r > s$, we let $D^{r,s}$ be the module generated by the elements $\bar{f}^{r,s}(i_0,i_1) := f^{r,s}((i_0,i_1),i_0) + f^{r,s}((i_0,i_1),i_1)$. Let also $E^{r,s} := \langle f((i_0,i_1),i_0)\rangle$ in such a way that $C^{r,s} = D^{r,s} \oplus E^{r,s}$.

Lemma 3.3.4 *For $2 \le s \le r-1$, we have:*

$$D^{r,s} = \ker(\mathrm{d}_2^{r,s}) = \mathrm{d}_2(C^{r,s-1})$$

In particular the r-th column $C^{r,\cdot}$ of the double complex is exact in dimension s for $2 \le s \le r-1$.

Proof. Formula follows readily from eq. 3.7.

□

The following decomposition of $D^{r,1}$ will be useful. Let:

$$\begin{aligned} D_1^{r,1} &:= \langle g_i^{r,1} := \mathrm{d}_2(f^{r,0}((\cdot,\cdot),i) \,|\, 1 \le i < n\,\rangle \\ D_2^{r,1} &:= \langle \bar{f}(i_0,i_1) \,|\, 1 \le i_0 < i_1 < n\,\rangle \end{aligned}$$

Note that $D_1^{r,1}$ is an $(n-1)$ dimensional free module and that

$$D^{r,s} = D_1^{r,s} \oplus D_2^{r,s}$$

Lemma 3.3.5 *Let $z^r := \sum_{i=1}^n f^{r,0}((\cdot,\cdot),i) \in C^{r,0}$. We have:*

$$\begin{aligned} \ker(\mathrm{d}_2^{r,0}) &= \langle z^r\rangle \\ D_1^{r,1} &= \mathrm{d}_2(C^{r,0}) \\ H^1_{\mathrm{d}_2}(C^{r,\cdot}) &\cong D_2^{r,1} \end{aligned}$$

□

We further define some maps between the modules in the double complex. Let $S^{r-1,r} : C^{r-1,r} \to C^{r,r}$ be defined by

$$f^{r,r-1}((i_0, i_1), i_k) \mapsto (-1)^k f^{r,r}((i_0, i_1), i_k)$$

Note that $S^{r-1,r}$ is surjective and its kernel is exactly $D^{r,r-1}$. Moreover, substituting $t = 1$ in the coboundary formula 3.8, we have $\mathrm{d}_2^{r,r-1} = S^{r-1,r}$ for odd r odd, while $\mathrm{d}_2^{r,r-1} = 3S^{r-1,r}$ otherwise.

Let $P^{r,r} : C^{r,r} \to D^{r+1,r} \subset C^{r+1,r}$ be the symmetrization operator defined by $f^{r,r}((i_0, i_1), \cdot) \mapsto \bar{f}^{r+1,r}(i_0, i_1)$.
It follows from equation 3.5 that $d_1^{r,r} = 2P^{r,s}$ for r odd and zero otherwise. In the same way, for $r > s$, let $Q^{r,s} : C^{r,s} \to C^{r+1,s}$ be the isomorphism defined by $f^{r,s}((i_0, i_1), i_k) \mapsto f^{r+1,s}((i_0, i_1), i_k)$. From equation 3.4, it is then clear that $d_1^{r,s} = 2Q^{r,s}$ for r odd and zero otherwise.

We can summarize the setting in the following diagram, where arrows not shown are understood to be zero.

$$\begin{tikzcd}
 & & & & & & & & & C^{9,9} \arrow[r,"2"] & \\
 & & & & & & & & C^{8,8} & C^{9,8} \arrow[u,"1"] \arrow[r,"2"] & \\
 & & & & & & & C^{7,7} \arrow[r,"2"] & C^{8,7} \arrow[u,"3"] & C^{9,7} \arrow[u] \arrow[r,"2"] & \\
 & & & & & & C^{6,6} & C^{7,6} \arrow[u,"1"] \arrow[r,"2"] & C^{8,6} \arrow[u] & C^{9,6} \arrow[u] \arrow[r,"2"] & \\
 & & & & & C^{5,5} \arrow[r,"2"] & C^{6,5} \arrow[u,"3"] & C^{7,5} \arrow[u] \arrow[r,"2"] & C^{8,5} \arrow[u] & C^{9,5} \arrow[u] \arrow[r,"2"] & \\
 & & & & C^{4,4} & C^{5,4} \arrow[u,"1"] \arrow[r,"2"] & C^{6,4} \arrow[u] & C^{7,4} \arrow[u] \arrow[r,"2"] & C^{8,4} \arrow[u] & C^{9,4} \arrow[u] \arrow[r,"2"] & \\
 & & & C^{3,3} \arrow[r,"2"] & C^{4,3} \arrow[u,"3"] & C^{5,3} \arrow[u] \arrow[r,"2"] & C^{6,3} \arrow[u] & C^{7,3} \arrow[u] \arrow[r,"2"] & C^{8,3} \arrow[u] & C^{9,3} \arrow[u] \arrow[r,"2"] & \\
 & & C^{2,2} & C^{3,2} \arrow[u,"1"] \arrow[r,"2"] & C^{4,2} \arrow[u] & C^{5,2} \arrow[u] \arrow[r,"2"] & C^{6,2} \arrow[u] & C^{7,2} \arrow[u] \arrow[r,"2"] & C^{8,2} \arrow[u] & C^{9,2} \arrow[u] \arrow[r,"2"] & \\
 & C^{1,1} \arrow[r,"2"] & C^{2,1} \arrow[u,"3"] & C^{3,1} \arrow[u] \arrow[r,"2"] & C^{4,1} \arrow[u] & C^{5,1} \arrow[u] \arrow[r,"2"] & C^{6,1} \arrow[u] & C^{7,1} \arrow[u] \arrow[r,"2"] & C^{8,1} \arrow[u] & C^{9,1} \arrow[u] \arrow[r,"2"] & \\
C^{0,0} & C^{1,0} \arrow[u,"1"] \arrow[r,"2"] & C^{2,0} \arrow[u] & C^{3,0} \arrow[u] \arrow[r,"2"] & C^{4,0} \arrow[u] & C^{5,0} \arrow[u] \arrow[r,"2"] & C^{6,0} \arrow[u] & C^{7,0} \arrow[u] \arrow[r,"2"] & C^{8,0} \arrow[u] & C^{9,0} \arrow[u] \arrow[r,"2"] &
\end{tikzcd}$$

Theorem 3.3.6 *The cohomology of the Coxeter group $W(\Xi_n)$ with trivial coefficients is given by:*

$$H^i(W(\Xi_n); \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ (\mathbb{Z}/2)^{\oplus \binom{n-1}{2}} & \text{if } i = 3 \\ 0 & \text{if } i = 2s + 1,\ s > 1 \\ \mathbb{Z}/2 & \text{if } i = 4s - 2,\ s > 0 \\ \mathbb{Z}/2 \oplus (\mathbb{Z}/3)^{\oplus \binom{n}{2}} & \text{if } i = 4s,\ s > 0 \end{cases}$$

Proof. Although the proof can be based on a spectral sequence argument, we prefer to present a direct argument, based on diagram chasing. Let in the following $W = W(\Xi_n)$.

We start computing low dimensional cohomology groups.

a) Since $\mathrm{d}_1^{0,0} = 0$, we have $H^0(W;\mathbb{Z}) = \mathbb{Z}$.

b) $H^1(W;\mathbb{Z}) = 0$.

Indeed $\mathrm{d}_1^{1,0}$ is injective.

c) $H^2(W;\mathbb{Z}) = \mathbb{Z}/2$.

Let $c = c^{1,1} + c^{2,0} \in C^{1,1} \oplus C^{2,0}$ be a cocycle. Since $\mathrm{d}_2(C^{2,0}) = D_1^{2,0}$, we must have $\mathrm{d}_1(c^{1,1}) \in D_1^{2,0}$. But then $c^{1,1} = \mathrm{d}_2 c^{1,1}$ and, modulo coboundaries, we can assume that $c^{2,0} = 0$. Now, as in lemma 3.3.5, we know that $\ker \mathrm{d}_2^{2,0}$ is generated by z^2. Since $\ker \mathrm{d}_2^{1,0}$ is as well generated by z^1 and $\mathrm{d}_1(z^1) = 2z^2$, the claimed result follows.

d) $H^3(W;\mathbb{Z}) = (\mathbb{Z}/2)^{\oplus\binom{n-1}{2}}$.

We are interested in the following portion of the double complex:

$$\begin{tikzcd}
 & C^{2,2} & \\
C^{1,1} \arrow[r, "2"] & \boxed{C^{2,1}} \arrow[u, "3"] & C^{3,1} \\
 & C^{2,0} \arrow[u] & \boxed{C^{3,0}} \arrow[u] \arrow[r, "2"] & C^{4,0}
\end{tikzcd}$$

Let $c = c^{2,1} + c^{3,0}$ be a cocycle. Since $\mathrm{d}_1^{3,0}$ is injective, $c^{3,0} = 0$. Further, we must have $c^{2,1} \in D^{2,1}$. Now $\mathrm{d}_1(C^{1,1}) + \mathrm{d}_2(C^{2,0}) = 2D^{2,1} + D_1^{2,1} = D_1^{2,1} \oplus 2D_2^{2,1}$. Therefore, $H^3(W;\mathbb{Z}) = D^{2,1}/(D_1^{2,1} \oplus 2D_2^{2,1}) \cong D_2^{2,1}/(2D_2^{2,1})$.

e)$H^4(W;\mathbb{Z}) = \mathbb{Z}/2 \oplus (\mathbb{Z}/3)^{\oplus\binom{n}{2}}$.

We are interested in the following portion of the double complex:

$$\begin{tikzcd}
\boxed{C^{2,2}} & C^{3,2} & \\
C^{2,1} \arrow[u, "3"] & \boxed{C^{3,1}} \arrow[u] \arrow[r, "2"] & C^{4,1} \\
 & C^{3,0} \arrow[u] \arrow[r, "2"] & \boxed{C^{4,0}} \arrow[u]
\end{tikzcd}$$

Clearly the computation split into two parts, since the diagram exhibits two blocks with no arrow between them. The block on the left gives the summand $(\mathbb{Z}/3)^{\oplus\binom{n}{2}}$, while the second gives $\mathbb{Z}/2$ by an argument similar to point c).

We are now ready to compute higher cohomology groups. Note that, when considering the cohomology in degree i, the computation splits in many parts, since the portion of the double complex we are interested in exhibits various block. Upper left block can be of four types, according to the remainder $i \bmod (4)$.

If $i = 4k - 1$, the block is:

$$\begin{tikzcd}
& C^{2k,2k} \\
C^{2k-1,2k-1} \arrow[r, "2"] & \boxed{C^{2k,2k-1}} \arrow[u, "3"] \\
& C^{2k,2k-2} \arrow[u]
\end{tikzcd}$$

and, since d_2 is exact there, the contribution of the block to cohomology is zero.

If $i = 4k+1$, the block is:

$$\begin{tikzcd}
C^{2k+1,2k+1} & \\
\boxed{C^{2k+1,2k}} \arrow[u, "1"] \arrow[r, "2"] & C^{2k+2,2k} \\
C^{2k+1,2k-1} \arrow[u] \arrow[r, "2"] & \boxed{C^{2k+2,2k-1}} \arrow[u] \\
& C^{2k+2,2k-2} \arrow[u]
\end{tikzcd} \tag{3.11}$$

Let $c = c^{2k+1,2k} + c^{2k+2,2k-1}$ be a cocycle. Then we must have $\mathrm{d}_2 c^{2k+1,2k} = 0$. Since d_2 is exact in that point, it follows that $c^{2k+1,2k} = \mathrm{d}_2 c^{2k+1,2k-1}$ and, modulo cobundaries, we can assume $c^{2k+1,2k} = 0$. In turn, we must have $\mathrm{d}_2 c^{2k+2,2k-1} = 0$. But since d_2 is exact in dimension $(2k+2, 2k-1)$, $c^{2k+2,2k-1}$ is a coboundary as well and the block does not contribute any summand to cohomology.

If $i = 4k$, the upper block is:

$$\begin{tikzcd}
\boxed{C^{2k,2k}} \\
C^{2k,2k-1} \arrow[u, "3"]
\end{tikzcd}$$

and contribute with the summand $(\mathbb{Z}/3)^{\oplus\binom{n}{2}}$.

Finally, if $i = 4k+2$, the upper block is:

$$\begin{tikzcd}
\boxed{C^{2k+1,2k+1}} \arrow[r, "2"] & C^{2k+2,2k+1} \\
C^{2k+1,2k} \arrow[u, "1"] \arrow[r, "2"] & \boxed{C^{2k+2,2k}} \arrow[u] \\
& C^{2k+2,2k-1} \arrow[u]
\end{tikzcd}$$

which is easily seen to provide no contribution to cohomology.

Middle blocks are all of the form:

$$
\begin{tikzcd}
C^{s,t+1} & \\
\boxed{C^{s,t}} \arrow[u, "1"] \arrow[r, "2"] & C^{s+1,t} \\
C^{s,t-1} \arrow[u] \arrow[r, "2"] & \boxed{C^{s+1,t-1}} \arrow[u] \\
& C^{s+1,t-2} \arrow[u]
\end{tikzcd}
$$

and provide no contribution to cohomology as is seen by the same argument used for diagram 3.11.

Lower blocks are of two types according to the remainder $i \bmod(2)$. If $i = 2k-1$, the lower block is:

$$
\begin{tikzcd}
C^{2k-1,1} & \\
\boxed{C^{2k-1,0}} \arrow[u] \arrow[r, "2"] & C^{2k,0}
\end{tikzcd}
$$

and gives no contribution to cohomology since $\mathrm{d}_1^{2k-1,0}$ is injective.
If $i = 2k$, the lower block is:

$$
\begin{tikzcd}
C^{2k-1,2} & \\
\boxed{C^{2k-1,1}} \arrow[u] \arrow[r, "2"] & C^{2k,1} \\
C^{2k-1,0} \arrow[u] \arrow[r, "2"] & \boxed{C^{2k,0}} \arrow[u]
\end{tikzcd}
$$

and this gives a $\mathbb{Z}/2$ summand by an argument similar to the computation of $H^2(W;\mathbb{Z})$ (case c).
Grouping the summands relative to dimension i, one gets the claimed result.

□

Sign representation ($t = -1$)

Substituting $t = -1$ in the coboundary formulas, the double complex now looks like:

$$
\begin{tikzcd}
 & & & & & & & & & C^{9,9} \\
 & & & & & & & & C^{8,8} \arrow[r, "2"] & C^{9,8} \arrow[u, "3"'] \\
 & & & & & & & C^{7,7} & C^{8,7} \arrow[r, "2"] \arrow[u, "1"'] & C^{9,7} \arrow[u] \\
 & & & & & & C^{6,6} \arrow[r, "2"] & C^{7,6} \arrow[u, "3"'] & C^{8,6} \arrow[r, "2"] \arrow[u] & C^{9,6} \arrow[u] \\
 & & & & & C^{5,5} & C^{6,5} \arrow[r, "2"] \arrow[u, "1"'] & C^{7,5} \arrow[u] & C^{8,5} \arrow[r, "2"] \arrow[u] & C^{9,5} \arrow[u] \\
 & & & & C^{4,4} \arrow[r, "2"] & C^{5,4} \arrow[u, "3"'] & C^{6,4} \arrow[r, "2"] \arrow[u] & C^{7,4} \arrow[u] & C^{8,4} \arrow[r, "2"] \arrow[u] & C^{9,4} \arrow[u] \\
 & & & C^{3,3} & C^{4,3} \arrow[r, "2"] \arrow[u, "1"'] & C^{5,3} \arrow[u] & C^{6,3} \arrow[r, "2"] \arrow[u] & C^{7,3} \arrow[u] & C^{8,3} \arrow[r, "2"] \arrow[u] & C^{9,3} \arrow[u] \\
 & & C^{2,2} \arrow[r, "2"] & C^{3,2} \arrow[u, "3"'] & C^{4,2} \arrow[r, "2"] \arrow[u] & C^{5,2} \arrow[u] & C^{6,2} \arrow[r, "2"] \arrow[u] & C^{7,2} \arrow[u] & C^{8,2} \arrow[r, "2"] \arrow[u] & C^{9,2} \arrow[u] \\
 & C^{1,1} & C^{2,1} \arrow[r, "2"] \arrow[u, "1"'] & C^{3,1} \arrow[u] & C^{4,1} \arrow[r, "2"] \arrow[u] & C^{5,1} \arrow[u] & C^{6,1} \arrow[r, "2"] \arrow[u] & C^{7,1} \arrow[u] & C^{8,1} \arrow[r, "2"] \arrow[u] & C^{9,1} \arrow[u] \\
C^{0,0} \arrow[r, "2"] & C^{1,0} \arrow[u, "3"'] & C^{2,0} \arrow[r, "2"] \arrow[u] & C^{3,0} \arrow[u] & C^{4,0} \arrow[r, "2"] \arrow[u] & C^{5,0} \arrow[u] & C^{6,0} \arrow[r, "2"] \arrow[u] & C^{7,0} \arrow[u] & C^{8,0} \arrow[r, "2"] \arrow[u] & C^{9,0}
\end{tikzcd}
$$

The following is the analogue of theorem 3.3.7 for the sign representation and it can be proved by similar methods.

Theorem 3.3.7 *The cohomology of the Coxeter group $W(\Xi_n)$ with coefficients in the sign representation is given by:*

$$
H^i(W(\Xi_n); \mathbb{Z}[-1]) = \begin{cases} 0 & \text{if } i = 4s,\ s \geq 0 \\ \mathbb{Z}/2 & \text{if } i = 2s-1,\ s > 0 \\ (\mathbb{Z})^{\oplus \binom{n-1}{2}} \oplus (\mathbb{Z}/3)^{\oplus (n-1)} & \text{if } i = 2 \\ (\mathbb{Z}/3)^{\oplus \binom{n}{2}} & \text{if } i = 4s+2,\ s > 0 \end{cases}
$$

□

Remark. In particular for $n = 2$, one recovers the known results for the cohomology of the Coxeter group A_2 (see [DS00], Thm. 5.1).

3.4 Rank 1 local systems on affine Artin groups

In this section we restrict our analysis to the the homology of the orbit space $X(W)$ with coefficients in a simple rank 1 local coefficient system R_q. In this situation, the complex for the homology of $X(W)$, introduced in section 3.2.1, takes a particular pleasant form. Further, when W is an affine group, we are able relate the homology $H_*(X(W); R_q)$ with the cohomology of a weighted sheaf over a lattice. As an example of this relation, we determine the top and top−1)homology of the orbit space $X(\tilde{A}_n)$. We conclude with a brief cohomological reformulation of the presented constructions.

The material of this section was presented by Prof. Salvetti in a talk delivered at the University of Tokyo (see also [Sal05]).

3.4.1 Preliminaries

Let (W, S) be a Coxeter system and G_W be the associated Artin group. Fix a commutative ring A with 1 (usually $A = \mathbb{Z}$, $\mathbb{Q}$)and let $R := A[q^{\pm 1}]$ be the ring of Laurent polynomial over A. We can consider the following representation of G_W:

$$\begin{aligned} \rho : G_W &\to R^* < \mathrm{Aut}(R) \\ g_s &\mapsto -q \end{aligned}$$

obtained sending the standard generators of G_W into $(-q)$-multiplication (the only reason of the minus sign is to simplify subsequent formulas). Denote with R_q the the ring R with the prescribed structure of G_W-module. From formula 3.2 and theorem 3.2.5, a chain complex for the homology $H_*(X(W); R_q)$ is given by:

$$D_k(W) := \bigoplus_{\substack{J \subseteq S \\ |J| = k \\ |W_J| < \infty}} R \cdot e_J \tag{3.12}$$

Let $\psi : W \to G_W$ be the canonical set theoretic function and notice that for an element $\beta \in W$ we have clearly:

$$\rho(\psi\beta) = (-q)^{\ell(\beta)}$$

Thus we may rewrite the boundary map in equation 3.3 as follows:

$$\begin{aligned} \partial e_J &= \sum_{\substack{I \subset J \\ J = I \cup \{\tau\}}} \sum_{\beta \in W_J^I} (-1)^{\alpha(J,\tau,\beta)} \rho(\psi\beta) \cdot e_I = \\ &= \sum_{\substack{I \subset J \\ J = I \cup \{\tau\}}} \sum_{\beta \in W_J^I} (-1)^{\mu(J,\tau)} q^{\ell(\beta)} \cdot e_I = \\ &= \sum_{\substack{I \subset J \\ |J| = |I| + 1}} [I : J] \frac{W_J(q)}{W_I(q)} \cdot e_I \end{aligned} \tag{3.13}$$

where $W_I(q)$ denotes the Poincaré series of the parabolic subgroup W_I (see section 2.4) and $[I : J] := (-1)^{\mu(J, J \setminus I)}$ is an incidence index.

To motivate our next construction, note that we can formally rewrite the boundary map in 3.13 as:

$$\partial(\frac{1}{W_J(q)}e_J) = \sum_{\substack{I \subset J \\ |J|=|I|+1}} [I:J]\frac{1}{W_I(q)} \cdot e_I \tag{3.14}$$

In particular note that the fractions $\frac{e_I}{W_I(q)}$ behave like the cells of a simplicial scheme.

We may therefore consider the complex

$$D^0_k(W) := \bigoplus_{\substack{J \subseteq S \\ |J|=k \\ |W_J|<\infty}} R \cdot e^0_J$$

with boundary:

$$\partial^0(e^0_J) = \sum_{\substack{I \subset J \\ |J|=|I|+1}} [I:J]\, e^0_I \tag{3.15}$$

It is then clear by the previous discussion that the diagonal map:

$$\begin{aligned}\Delta : D_*(W) &\to D^0_*(W)\\ e_J &\mapsto W_J(q)e^0_J\end{aligned}$$

is an injective complex homomorphism. Then there is an exact sequence of complexes:

$$0 \longrightarrow D_*(W) \xrightarrow{\Delta} D^0_*(W) \xrightarrow{\pi} L_*(W) \longrightarrow 0 \tag{3.16}$$

where

$$L_k(W) := \bigoplus_{\substack{J \subseteq S \\ |J|=k \\ |W_J|<\infty}} R/W_J(q) \cdot \bar{e}_J$$

is the quotient complex.

Remark. Recall that for an affine Coxeter system (W,S) of rank $n+1$, a parabolic subgroup W_I is finite if and only if I is a proper subset of S. In particular the poset of finite parabolic subgroups is isomorphic to the poset of proper subsets of $I_{n+1} = \{1,\dots,n+1\}$ (that is the boolean lattice minus its maximum).

Assume hereinafter that (W,S) is an affine Coxeter system of rank $n+1$. Then, by the remark, the homology of $D^0_*(W)$ is the reduced homology of a $n-1$-sphere modulo a degree shift:

$$H_k(D^0_*(W)) \cong \tilde{H}_{k-1}(S^{n-1};R) \cong \begin{cases} 0 & \text{if } k \neq n \\ R & \text{if } k = n \end{cases}$$

Using the long exact sequence associated to the short exact sequence of complexes 3.16:

$$\xrightarrow{\pi_*} H_{k+1}(L_*) \xrightarrow{\delta} H_k(D_*) \xrightarrow{\Delta_*} H_k(D^0_*) \xrightarrow{\pi_*} H_k(L_*) \xrightarrow{\delta} H_{k-1}(D_*) \xrightarrow{\Delta_*} \tag{3.17}$$

we have then isomorphisms for $0 \leq k \leq n-2$:

$$H_k(D_*(W)) \cong H_{k+1}(L_*(W))$$

whereas the top and top-1 homology groups of $D_*(W)$ fit into:

$$0 \longrightarrow H_n(D_*(W)) \longrightarrow R \longrightarrow H_n(L_*(W)) \xrightarrow{\delta} H_{n-1}(D_*(W)) \longrightarrow 0 \tag{3.18}$$

Therefore we may attempt to determine $H_*(X(W);R_q)$ by analyzing first the homology $H_*(L_*(W))$, which is in turn a special case of the homology of a sheaf over a poset as described below.

3.4.2 Weighted Sheaves over posets

Let $(P, \prec)$ be a poset.

Definition 3.4.1 *(see also [Bac75][Yuz87]). Define a* sheaf of rings *over P as a collection*

$$\{A_x,\ x \in P\}$$

of commutative rings and a collection of ring homomorphisms

$$\{\rho_{x,y}:\ A_y \to A_x, \qquad x \preceq y\}$$

satisfying

$$\rho_{x,x} = id_{A_x}\ ;$$

$$x \preceq y \preceq z\ \Rightarrow\ \rho_{x,z} = \rho_{x,y}\ \rho_{y,z}$$

Let $\mathcal{C}_P$ be the small category associated to P, with

$$Ob(\mathcal{C}_P) = P$$

and

$$\begin{array}{rcll} \mathrm{Hom}(x,y) & = & \{(x,y)\} & \text{if } x \preceq y \\ & = & \emptyset & \text{otherwise} \end{array}$$

So, a sheaf of rings is a contravariant functor from $\mathcal{C}_P$ to the category of rings. As such it can be equivalently called a *diagram of rings.*

We will use a particular class of sheaves. From now on, we fix $R = \mathbb{Q}[q^{\pm 1}]$ to be the ring Laurent polynomial over the rational numbers, although all that will be said generalizes to the class of principal ideal domains. Note that the divisibility relation is an ordering in R, which becomes a poset. In particular any poset homomorphism

$$\psi : (P, \prec) \ \rightarrow \ (R, |) \ : \ x \longrightarrow \psi(x) = p_x \ \in \ R$$

defines a sheaf over P by the collections

$$\{ \ R/(p_x) \ , \ x \in P \}$$

and

$$\{ \ i_{x,y} : \ R/(p_y) \ \rightarrow \ R/(p_x) \ \}$$

where $i_{x,y}$ is the map induced by the identity of R.

We call the triple (P, R, ψ) a *w*eighted sheaf over P and the coefficients p_x the *w*eights of the sheaf.

Any simplicial scheme K over a finite set

$$I_{n+1} \ := \ \{1, ..., n+1\}$$

is a poset with partial ordering

$$\sigma \prec \tau \ \Leftrightarrow \sigma \subset \tau.$$

It is convenient to denote by $C^0_*(K)$ the algebraic complex for the reduced simplicial $R-$homology of K, shifted in dimension by one, so that in dimension k one has

$$C^0_k(K) \ = \bigoplus_{\substack{\sigma \in K \\ |\sigma| = k}} R \cdot \ e^0_\sigma$$

where e^0_σ is a generator associated to a given orientation of σ ($C^0_0(K) = \mathbb{Z}\, e^0_\emptyset$); the boundary is given by

$$\partial^0(e^0_\sigma) \ = \sum_{|\tau|=k-1} [\tau : \sigma] \ e^0_\tau$$

where $[\tau : \sigma]$ denotes the incidence number holding ± 1 if $\tau \prec \sigma$ and vanishing otherwise.

We want to define a modified complex for the homology of weighted sheaves.

Definition 3.4.2 *The* weighted complex *associated to the weighted sheaf* (K, R, ψ) *is the algebraic complex*

$$L_* \ := \ L_*(K)$$

defined by

$$L_k \ := \bigoplus_{|\sigma|=k} \frac{R}{(p_\sigma)}\ \bar{e}_\sigma$$

and boundary

$$\partial : \ L_k \ \rightarrow \ L_{k-1}$$

induced by ∂^0 :

$$\partial(a_\sigma \ \bar{e}_\sigma) \ = \ \sum_{\tau \prec \sigma} [\tau : \sigma] \ i_{\tau,\sigma}(a_\sigma) \ \bar{e}_\tau$$

Remark. Let (W, S) be a Coxeter system and consider the simplicial scheme K associated to finite parabolic subgroup of W. Then clearly the complexes $D^0_*(W)$, $L_*(W)$ introduced in the previous section 3.4.1 arise respectively as the complex $D^0_*(K)$ and as the weighted complex $L_*(K)$ associated to the homomorphism $\psi : K \ni I \rightarrow W_I(q) \in R$.

Decomposition and filtration of $L_*(K)$

Let as before $L := L_*(K)$ the algebraic complex associated to (K, R, ψ). It is natural to consider the decomposition of L_* into its φ_d-primary components $L^{(d)}_*$, where as usual we denote with φ_d the d-th cyclotomic polynomial.

More precisely, for $\sigma \in K$, let $p_\sigma = \psi(\sigma)$ be the weight of the cell and let:

$$m(d, \sigma) := \text{max power of } \varphi_d \text{ dividing } p_\sigma$$

Then the φ_d-primary component $L^{(d)}_*$ is defined as:

$$L^{(d)}_k \ := \bigoplus_{|\sigma|=k} \frac{R}{(\varphi_d^{m(d,\sigma)})}\ \bar{e}_\sigma$$

Of course, we may define an increasing filtration for $L^{(d)}$ by the involved powers of φ_d:

$$F^s(L^{(d)}) \ := \bigoplus_{m(d,\sigma)\le s} \frac{R}{(\varphi_d^{m(d,\sigma)})}\ \bar{e}_\sigma$$

In the same manner, we may define an increasing filtration of the simplicial complex K by:

$$K^{(d)}_s \ := \ \{\sigma \in K \mid m(d, \sigma) \le s\}$$

In the following proposition we summarize what we get by standard results in homological algebra.

Proposition 3.4.3 *For a weighted sheaf* (K, R, ψ) *there exists a spectral sequence*

$$E^0_{p,q} \quad \Rightarrow \quad H_*(L^{(d)})$$

that abuts to the homology of the φ_d-primary component of the associated algebraic complex L_*.
Moreover the E^1-term:

$$E^1_{p,q} = H_{p+q}(F^p/F^{p-1}) \cong H_{p,q}(K_p^{(d)}, K_{p-1}^{(d)}; R/\varphi_d^p)$$

is isomorphic to the relative homology with trivial coefficients of the simplicial complexes pair $(K_p^{(d)}, K_{p-1}^{(d)})$. □

3.4.3 Top and Top-1 homology of $G_{\tilde{A}_n}$

As an example of these ideas, we determine the homology groups $H_k(G_{\tilde{A}_n}; R_q)$ for $k = n-1, n$, i.e. in dimension top and top-1.

Let $S =: I_{n+1}$. For any subset $J \subset S$, let Γ be the subgraph of the Coxeter diagram for $\tilde{A}_n$ spanned by the vertices in J. Let also $\Gamma_1, \dots, \Gamma_k$ be the connected components of Γ and set $a_i := |\Gamma_i| + 1$. Since every connected component Γ_i gives rise to a Coxeter graph of type $A_{|\Gamma_i|}$, the Poincaré series for the parabolic subgroup $(\tilde{A}_n)_J$ is $\psi(J) := [a_1]![a_2]! \cdots [a_k]!$.

Let K be the simplicial scheme of finite parabolic subgroup of $\tilde{A}_n$ and consider the weighted sheaf (K, R, ψ). We start analyzing the associated algebraic complex L_*.

Let $d > 1$ be an integer and consider the φ_d-primary component of L_*.

Lemma 3.4.4

$$L^{(d)} = \bigoplus_{J \subsetneq I_{n+1}} \frac{R}{(\varphi_d^{\lfloor a_1/d \rfloor + \ldots + \lfloor a_k/d \rfloor})}$$

where $\lfloor \cdot \rfloor$ is the floor function.

Proof. It is enough to observe that the maximal power of d dividing $a!$ is $\lfloor a/d \rfloor$. □

We may now determine the homology $H_k(G_{\tilde{A}_n}; R_q)$ for $k = n-1, n$. Indeed we have:

Theorem 3.4.5

$$H_n(G_{\tilde{A}_n}; R_q) = R$$

$$H_{n-1}(G_{\tilde{A}_n}; R_q) = \bigoplus_{\substack{d \mid n+1 \\ d \geq 2}} \left(\frac{R}{\varphi_d}\right)^{d-1}$$

Proof. Recall that we have an exact sequence:

$$0 \longrightarrow H_n(G_{\tilde{A}_n}; R_q) \xrightarrow{\Delta_*} H_n(D^0_*) \longrightarrow H_n(L_*) \xrightarrow{\delta} H_{n-1}(G_{\tilde{A}_n}; R_q) \longrightarrow 0 \tag{3.19}$$

and observe that $H_n(D^0_*) \cong R$ is generated by $z^0 = \sum_{i=1}^{n+1}(-1)^i e^0_{S\setminus\{i\}}$. From the exact sequence and the description of the diagonal map Δ, it is then clear that $H_n(G_{\tilde{A}_n}; R_q) \cong R$ with generator $z = \sum_{i=1}^{n+1}(-1)^i e_{S\setminus\{i\}}$. Since the image of z in $H_n(D^0_*)$ is $\Delta_*(z) = [n+1]!z^0$, we are left to analyze the sequence:

$$0 \longrightarrow \frac{R}{[n+1]!} \longrightarrow H_n(L_*) \xrightarrow{\delta} H_{n-1}(G_{\tilde{A}_n}; R_q) \longrightarrow 0$$

We determine $H_n(L_*)$ first using the decomposition in primary components and the related filtration. Fix thus an integer $d \geq 2$ and let $m =: \lfloor n+1/d \rfloor$. By lemma 3.4.4, $F^m L_n^{(d)} = L_n^{(d)}$ while $F^{m-1} L_n^{(d)} = 0$.
Similarly $F^{m-2} L_{n-1}^{(d)} = 0$, whereas:

$$\frac{F^m L_{n-1}^{(d)}}{F^{m-1} L_{n-1}^{(d)}} = \bigoplus_{\substack{1 \leq i < j \leq n+1 \\ \langle j-i \rangle_d \leq \langle n+1 \rangle_d}} \left(\frac{R}{(\varphi_d)^m} \right) \bar{e}_{S\setminus\{i,j\}}$$

$$\frac{F^{m-1} L_{n-1}^{(d)}}{F^{m-2} L_{n-1}^{(d)}} = \bigoplus_{\substack{1 \leq i < j \leq n+1 \\ \langle j-i \rangle_d > \langle n+1 \rangle_d}} \left(\frac{R}{(\varphi_d)^{m-1}} \right) \bar{e}_{S\setminus\{i,j\}}$$

where we denote with $\langle a \rangle_d$ the remainder class of a modulo d with values in $0, 1, \ldots, d-1$.
Looking at the incidence relations between cells, it is easy to see that:

$$H_n\left(\frac{F^m L_*^{(d)}}{F^{m-1} L_*^{(d)}} \right) \cong \begin{cases} \left(\frac{R}{(\varphi_d)^m} \right) \cdot \bar{z} & \text{for } d \nmid n+1 \\ \bigoplus_{\lambda=1}^{d} \left(\frac{R}{(\varphi_d)^m} \right) \cdot \bar{z}_\lambda & \text{for } d \mid n+1 \end{cases}$$

where $\bar{z}$ and $\bar{z}_\lambda$ are defined by:

$$\bar{z} =: \sum_{i=1}^{n+1} (-1)^i \bar{e}_{S\setminus\{i\}}$$

$$\bar{z}_\lambda =: \sum_{i=0}^{m-1} (-1)^{\lambda+id} \bar{e}_{S\setminus\{\lambda+id\}}$$

Note in particular that for a divisor $d \mid n+1$, we have $\bar{z} = \sum_\lambda \bar{z}_\lambda$.

Exploiting the short exact sequence:

$$0 \longrightarrow F^{m-1} \longrightarrow F^m \longrightarrow F^m/F^{m-1} \longrightarrow 0$$

we may identify $H_n(F^m)$ with the kernel of the connecting homomorphism $H_n(F^m/F^{m-1}) \to H_{n-1}(F^{m-1})$. When $d \nmid n+1$, the generator z is obviously in the kernel. When $d \mid n+1$ we should check whether the boundary of a linear combination $\sum a_\lambda z_\lambda$ is zero modulo φ_d^{m-1}. By simple calculations, it turns out that the kernel is generated by $\bar{z} = \sum_\lambda \bar{z}_\lambda$ and by the multiples $(\varphi_d)^{m-1} z_\lambda$ for $\lambda = 1, \ldots, d-1$. Since $H_n(F^m) = H_n(L_*^{(d)})$, the stated result follows now easily from the exact sequence 3.19. □

Remark. Using this strategy, we computed the complete list of homology groups of $G_{\tilde{A}_n}$ up to $n = 6$, however I was unable to generalize the result. The complete determination (for cohomology) has been obtained in chapter 5 using an inclusion of Artin groups presented in chapter 4. It is likely, however, that a more careful combinatorial study of the filtered simplicial complex K could give more insight in the proposed solution and may also convey a method generalizable to other affine Artin groups.

3.4.4 Cohomological version

The constructions carried out so far for homology have of course a cohomological counterpart, obtained using the dual of the diagonal map Δ. We are not going into the details but we just mention a result that can be easily proved with these methods and that will be useful later.

Proposition 3.4.6 *Let W be an affine weyl group of rank $n+1$ and let $M = \mathbb{Z}[q^{\pm 1}]$ or $M = \mathbb{Z}[-1]$. Then $H^n(G_W; M)$ has rank 1 has M-module.*

Proof. Note that the entries of the diagonal map Δ are non-zero in M. Then both results follow straightforwardly from suitable analogue of the exact sequence in equation 3.18. □

3.5 The genus problem for general Coxeter group

As another application, we discuss the genus problem for general Coxeter groups.

3.5.1 Schwarz and homological genera

We start recalling the definition of Schwarz genus and discussing briefly some of its properties (we refer to [Vas92] for details; the original exposition is in [Sch61]).

Definition 3.5.1 *For a locally trivial fibration $f : Y \to X$, the Schwarz genus $g(f)$ is the minimal cardinality of an open cover $\mathcal{U}$ of X such that f admits a section over each set $U \in \mathcal{U}$.*

Remark. The Schwarz genus is the extension to fibrations of the Lusternik-Schnirelmann category of a topological space; indeed the category of a path connected topological space coincides with the Schwarz genus of its Serre fibration.

When X has the homotopy type of a finite dimensional CW complex, we have an upper bound for the genus of whatever fibration:

Theorem 3.5.2 ([Sch61]) *If X has the homotopy type of a CW complex of dimension N, then $g(f) \leq N + 1$.*

□

Let now $f : Y \to X$ be a regular G-covering. Consider the classifying space BG for G and the universal G-bundle $EG \to BG$ (see e.g. [Ste51]). Then there is a unique (up to homotopy) classifying map $a : X \to BG$ such that the covering f is induced as pull-back of the universal G-bundle $EG \to BG$.
Let M be an arbitrary G-module and a^*M be the local system on X induced by the map a.

Definition 3.5.3 *The homological M-genus of $f : Y \to X$ is the smallest integer $h_M(f)$ such that the induced map in cohomology:*

$$a^* : H^j(BG; M) \to H^j(X; a^*M)$$

is zero in degree j for $j \geq h_M(f)$.
The homological genus is defined as the maximum $h(f) = \max_M h_M(f)$ of the homological M-genera.

Homological genus provides a lower bound for Schwarz genus:

Theorem 3.5.4 ([Sch61]) *For any regular covering $f : Y \to X$, we have $g(f) \geq h(f)$.* □

3.5.2 The genus problem and Coxeter groups

Let W be a Coxeter group and consider the regular covering $f_W : Y(W) \to X(W)$. We are interested in the genus $g(f_W)$ of f_W.
When $W = A_n$, this number has an interesting interpretation. Recall that in this case, $X(W)$ is the space of monic polynomials of degree $n+1$ with distinct roots and $Y(W)$ is the space of such roots. Suppose that f admits a section s on an open set $U \subset X(W)$. The section s may be then understood as a function that "computes" the roots of a polynomial $P \in U$. The Schwarz genus corresponds then to the minimal number of such functions needed to compute the roots of any polynomial in $X(W)$.
Smale [Sma87] links this problem to the topological complexity $\tau(n, \epsilon)$ of an algorithm that computes the roots of a polynomial of degree $n+1$ with a given precision ϵ. In particular for small ϵ, one has $\tau(n, \epsilon) \geq g(f_{A_n}) - 1$. Further it is known that the previous inequality becomes an equality when $n + 1$ is a prime power.

For a finite Coxeter group W of rank n, it was shown in [DS00] that the Schwarz genus reaches the upper bound in theorem 3.5.2 (i.e. $g(f_W) = n+1$), except when $W = A_n$ and $n+1$ is not a prime power. In this case, the exact determination of the genus is still open; for the first non prime power $n+1 = 6$, it is known that $g(f_{A_5}) = 5$ (and so the genus does not reach the upper bound of theorem 3.5.2) [DPS04]. Recently, Arone [Aro05] showed by different methods that $g(f_{A_n}) < n+1$ when $n+1 \neq p^k, 2p^k$ for p prime.

Recall that we have an inclusion $i : X(W) \hookrightarrow X^{(\infty)}(W)$ and that $X(W)$ may be identified with the subcomplex of $X^{(\infty)}(W)$ consisting of cells of type $\mathbf{\Gamma} = (\Gamma_1 \supseteq \emptyset \supseteq \emptyset \supseteq \ldots)$.

Let M be a W-module and M' the local coefficient system on X_W induced by M via i. The associated map of cochains

$$i^* : C^*(W; M) \to C^*(X(W); M')$$

may then be easily described as the restriction of $c \in C^*(W; M)$ to the chains for $X(W)$. Let n be the maximal cardinality of a subset $I \subset S$ s.t. $|W_I| < \infty$. Then in degree n we have:

$$\begin{tikzcd}
{} \arrow[r] & C^{n-1}(W;M) \arrow[r] \arrow[d] & C^n(W;M) \arrow[r] \arrow[d] & C^{n+1}(W;M) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & C^{n-1}(X(W);M') \arrow[r] & C^n(X(W);M') \arrow[r] & 0 \arrow[r] & {}
\end{tikzcd} \tag{3.20}$$

The following proposition is useful in the computation of the homological genus.

Proposition 3.5.5 *Let $M = \mathbb{Z}[-1]$ be the sign representation. Then the map $i^* : H^n(W; M) \to H^n(X(W); M')$ is an epimorphism.*

Proof. Let $c \in C^n(W, M)$ satisfying $c(\Gamma) = 0$ for all $\Gamma = (\Gamma_1 \supseteq \Gamma_2 \supseteq \ldots)$ having $|\Gamma_2| > 0$. By diagram 3.20,it is enough to show that every such c is a cocycle. Notice that $\mathrm{d}c(\Gamma_1 \supseteq \Gamma_2 \supseteq \ldots) = 0$ if $|\Gamma_2| > 1$ or $|\Gamma_3| > 0$, so we are left to analyze the behavior on the $\Gamma = (\Gamma_1 \supseteq \Gamma_2 \supseteq \emptyset)$ satisfying $|\Gamma_2| = 1$, for which we just compute. Let $\Gamma_1 = \{i_1 < i_2 < \ldots < i_n\}$, $\Gamma_2 = \{i_k\}$ for some k $(1 \leq k \leq n)$. We have:

$$\begin{aligned}
\mathrm{d}c(\Gamma) =& c(\partial\Gamma) = \sum_{\beta \in W_{\Gamma_2}} (-1)^{\alpha}(\Gamma, 2, i_k, \beta)\rho(\beta)c(\Gamma_1) = \\
=& (-1)^{n+k}(1 + \rho(s_{i_k}))c(\Gamma) \\
=& 0 \qquad \square
\end{aligned}$$

Let now W_a be an affine Weyl group. From proposition 3.4.6, we know that the top-cohomology of $X(W_a)$ with coefficients in the sign representation does not vanish. Using proposition 3.5.5, the homological genus $h(f_{W_a})$ is greater than $n+1$. Since $X(W_a)$ has dimension n, using theorems 3.5.4 and 3.5.2, we get:

Corollary 3.5.6 *Let W_a be an affine Weyl group of rank $n+1$. Then the Schwarz genus of the fibration $Y(W_a) \to X(W_a)$ is precisely $n+1$.* □

Remark. It should be observed that in the previous corollary the machinery of proposition 3.4.6 is only needed to deal with the affine Weyl group of type $\tilde{A}_n$, while in the other cases the result follows readily from the finite type case. Indeed any affine group $W_a \neq \tilde{A}_n$ has a finite parabolic subgroup H of rank n of type $\neq A_n$. Since $H^n(X(W_a); \mathbb{Z}[-1])$ is onto $H^n(X(H); \mathbb{Z}[-1])$, the latter being non trivial (see [DS00]), the top cohomology $H^n(X_{W_a}; \mathbb{Z}[-1])$ is non-trivial as well.

Chapter 4

Inclusions of Artin groups and induced representations

In this chapter we will study several inclusions of Artin groups, which arise from topological constructions. We will start with a well known result (see e.g [Cri99]), which states that the group G_{B_n} can be regarded as the group of braids on the annulus. Then, using this suggestion, we work out using Tietze moves a new presentation of G_{B_n} that exhibits the group as the semidirect product $\mathbb{Z} \ltimes G_{\tilde{A}_{n-1}}$. Thus we have an inclusion $G_{\tilde{A}_{n-1}} < G_{B_n}$. This results appears in [KP02], but we give a different proof. Finally, we give some topological motivation for this inclusion, showing that the space $X(\tilde{A}_{n-1})$ is homotopic to a cover of $X(B_n)$. This implies in particular that $X(\tilde{A}_{n-1})$ is a $K(\pi, 1)$. Further we study an analogous inclusion of $G_{\tilde{C}_{n-1}}$ in G_{B_n}. We exploit this inclusion to observe that $G_{\tilde{C}_{n-1}}$ has trivial center (prop. 4.3.4), answering partially to a conjecture of Charney and Peifer. Indeed in [CP03] it is conjectured that affine Artin groups have trivial center. This is already known for $G_{\tilde{A}_n}$ [JA85]. By contrast, finite type Artin groups are known to have infinite cyclic center.

Finally we discuss the case of $G_{\tilde{B}_n}$, showing that $X(\tilde{B}_n)$ is a $K(\pi, 1)$ space. Up to our knowledge, this seems to be new.

The ubiquitous strategy (suggested by Prof. Salvetti) when dealing with an affine Artin groups W_a is to transform by exponentiation $Y(W_a)$ into a toric arrangement and then try to straighten it to a new simpler hyperplane arrangement.
We found out that this kind of analysis as been performed, although with different goals, by Allcock; in particular the treatment of $\tilde{A}_n$ and $\tilde{C}_n$ is essentially the same of [All02].

We then exploit these inclusions, to find out interesting induced representations. In particular consider the representation of $G_{\tilde{A}_{n-1}}$ on $\mathbb{Z}[q^{\pm 1}]$ defined sending the standard generators to $(-q)$-multiplication. Then this induces a representation of G_{B_n} on $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ where all generators acts again by

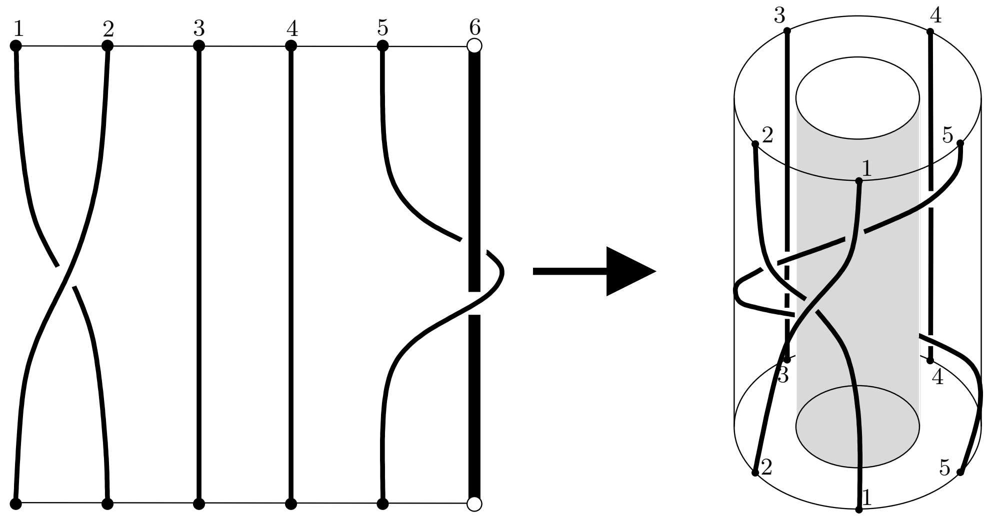


Figure 4.1: A braid in Br_6^6 represented as an annular braid on 5 strands.

$(-q)$-multiplication except the last one which acts by $(-t)$-multiplication. In turn, this representation induces a G_{A_n} representation which is isomorphic to the Tong-Yang-Ma representation [TYM96] tensored by a one dimensional representation. Similar considerations are applied to the inclusion $G_{\tilde{C}_{n-1}} < G_{B_n}$.
These assertions will be useful in chapter 5, where they will allow to relate several cohomology groups by means of Shapiro's lemma.

4.1 G_{B_n} as the annular braid group

Let $\mathrm{Br}_{n+1} = G_{A_n}$ be the braid group on $n+1$-strands and $\mathrm{Br}_{n+1}^{n+1} < \mathrm{Br}_{n+1}$ be the subgroup of braids fixing the $(n+1)$-th strand. The group Br_{n+1}^{n+1} is called the annular braid group, since it can be regarded as the group of braids on n strands on the annulus (see figure 4.1). Indeed, pictorially, we can just thick the $(n+1)$-th strand and use it as the axis of a hollow cylinder.

Let $\epsilon_1, \dots, \epsilon_{n-1}, \bar{\epsilon}_n$ be the standard generators of G_{B_n} and $\sigma_1, \dots, \sigma_n$ those of Br_{n+1} (see table 4.1).

The annular braid group is isomorphic to the Artin braid group G_{B_n} of type B_n (see e.g. [Cri99][Lam94]; it has been noted that this is also implicit in [Bri71b]). Indeed:

A_n	$\sigma_1 — \sigma_2 — \sigma_3 — \sigma_4 - - \quad - - \sigma_{n-2} — \sigma_{n-1} — \sigma_n$
B_n	$\epsilon_1 — \epsilon_2 — \epsilon_3 — \epsilon_4 - - \quad - - \epsilon_{n-2} — \epsilon_{n-1} \overset{4}{—} \bar{\epsilon}_n$

Table 4.1: Graphs of type A_n, B_n; each node is labelled with the corresponding generator in the braid group

Theorem 4.1.1 ([Cri99]) *The map*

$$\begin{aligned} G_{B_n} &\to \mathrm{Br}_{n+1}^{n+1} < \mathrm{Br}_{n+1} \\ \epsilon_i &\mapsto \sigma_i \quad \textit{for } 1 \le i \le n-1 \\ \bar{\epsilon}_n &\mapsto \sigma_n^2 \end{aligned}$$

is an isomorphism. □

Proof. We use the fact that the orbit space of the complex hyperplane complement under the action of the associated Coxeter group has the braid group as fundamental group. Using the same notation of chapter 3, let thus:

$$\begin{aligned} Y(A_n) &= \{x \in \mathbb{C}^{n+1} \,|\, x_i \neq x_j \ \forall i \neq j; \ \sum x_i = 0\} \\ X(A_n) &= Y(A_n)/S_{n+1} \\ Y(B_n) &= \{x \in \mathbb{C}^n \,|\, x_i \neq \pm x_j \ \forall i \neq j; \ x_k \neq 0 \, \forall k\} \\ X(B_n) &= Y(B_n)/B_n \end{aligned}$$

Since $B_n \cong S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, we may consider the intermediate covering $Y(B_n)/(\mathbb{Z}/2\mathbb{Z})^n$ of $X(B_n)$.
Note that the covering $\gamma : Y(B_n) \to Y(B_n)/(\mathbb{Z}/2\mathbb{Z})^n$ may be explicitly realized as follows. Let:

$$Y'(B_n) := \{y \in \mathbb{C}^n \,|\, y_i \neq y_j \ \forall i \neq j; \ y_k \neq 0 \ \forall k\} \tag{4.1}$$

and define $\gamma : Y(B_n) \to Y'(B_n)$ as $\gamma(x_1, \ldots, x_n) = (x_1^2, \ldots, x_n^2)$.

The symmetric group S_n acts on $Y'(B_n)$ by simply permuting coordinates. Consider as well the S_n action on $Y(A_n)$ given by permutation of the first n coordinates. The map:

$$\begin{aligned} \pi : \ Y'(B_n) &\to Y(A_n) \\ (x_1, \ldots, x_n) &\to (x_1 + \eta, \ldots, x_n + \eta, \eta) \\ (x_1 - x_{n+1}, \ldots, x_n - x_{n+1}) &\leftarrow (x_1, \ldots, x_{n+1}) \end{aligned}$$

where

$$\eta = \frac{-1}{n+1} \sum_{i=1}^{n} x_i$$

is easily seen to be an S_n-equivariant isomorphism. Therefore:

$$X(B_n) \cong Y'(B_n)/S_n \cong Y(A_n)/S_n$$

and G_{B_n} is the fundamental group of any of these spaces.
Since $[S_{n+1} : S_n] = n + 1$, it follows that G_{B_n} is a subgroup of index $n + 1$ in Br_{n+1}. This group consists of those closed paths in $X(A_n)$ that lift to closed paths in $Y(A_n)/S_n$. In terms of braids, this means that the endpoint of the $n + 1$ strands coincides with its starting point, so G_{B_n} is identified with Br_{n+1}^{n+1}, as required.
Now we determine an expression of this map in terms of the generators of G_{B_n}. Recall that the standard simple roots of B_n are given by $\alpha_1 = x_1 - x_2, \dots, \alpha_{n-1} = x_{n-1} - x_n$, $\alpha_n = x_n$ and that we can take as generators of the Artin G_{B_n} groups small twists around these hyperplanes. To be more precise, fix $p = (-n, -n + 1, \dots, -1) \in Y(B_n)$ as base point. For $1 \leq i < n$ consider the path $g^{(i)} : [0, 1] \to Y(B_n)$ defined by:

$$g^{(i)}(t) = (p_1, \dots, p_{i-1}, (p_i + p_{i+1} - e^{\pi it})/2, (p_i + p_{i+1} + e^{\pi it})/2, p_{i+2}, \dots, p_n)$$

and finally:

$$g^{(n)}(t) = (p_1, p_2, \dots, p_{n-1}, -e^{\pi it})$$

These paths become closed in $X(B_n)$ and their images are indeed the standard generators $\epsilon_1, \dots, \epsilon_{n-1}, \bar{\epsilon}_n$ of G_{B_n}. Let $q = (q_1, \dots, q_{n+1}) := \pi\gamma(p)$ be the base point of $Y(A_n)$. Note that for $1 \leq i < n$, the image of $g^{(i)}$ via the equivariant homomorphism, is a similar path in $Y(A)$ interchanging q_i and q_{i+1}, while other points are left unbraided. In the quotient $Y(A_n)/A_n$, this is homotopic to the generator $\sigma_i \in \mathrm{Br}_{n+1}$. The last path $g^{(n)}$ is instead sent to a full twist of q_n around q_{n+1}, while other points are left unbraided. In the quotient, this is homotopic to the square σ_n^2 of last generator σ_n of Br_{n+1}.

□

4.2 $G_{\tilde{A}_{n-1}}$ as a subgroup of G_{B_n}

Using the suggestion given by the identification with the annular braid group, a new interesting presentation for G_{B_n} can be worked out.

Let $\tau = \bar{\epsilon}_n \epsilon_{n-1} \cdots \epsilon_2 \epsilon_1$. As annular braid, τ may be represented as in figure 4.2. It is easy to verify that:

$$\tau^{-1} \epsilon_i \tau = \epsilon_{i+1} \quad \text{for } 1 \leq i < n - 1$$

i.e. conjugation by τ shifts forward the first $n - 2$ standard generators. By analogy, let $\epsilon_n = \tau^{-1} \epsilon_{n-1} \tau$.

We have the following:

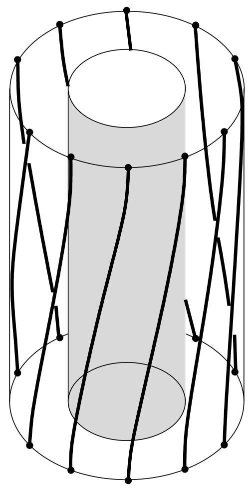


Figure 4.2: As an annular braid, the element τ is obtained turning the bottom annulus by a rotation of $2\pi/n$.

Theorem 4.2.1 ([KP02](see also [tom98])) *The group G_{B_n} has presentation $\langle \mathcal{G}|\mathcal{R}\rangle$ where*

$$\begin{aligned}\mathcal{G} =&\{\tau, \epsilon_1, \epsilon_2, \ldots, \epsilon_n\}\\ \mathcal{R} =&\{\epsilon_i\epsilon_j = \epsilon_j\epsilon_i \quad \textit{for } i \neq j-1, j+1\}\cup\\ &\{\epsilon_i\epsilon_{i+1}\epsilon_i = \epsilon_{i+1}\epsilon_i\epsilon_{i+1}\}\cup\\ &\{\tau^{-1}\epsilon_i\tau = \epsilon_{i+1}\}\end{aligned}$$

where are all indexes have to be taken modulo n. □

The proof of theorem 4.2.1 presented in the original paper is algebraic and based on Tietze transformations. We present a somewhat shorter proof based as well on Tietze transformations.

Let $< G|R >$ be a group presentation where G is the generating set and R, a set of words in G, represents relations among generators. Recall the four Tietze transformations (see e.g. [MKS66]):

T1 If ω is a relation in G, that can be derived from relations in R, then add ω to R

T2 If ω is a relation in R that can be derived from the other relations in R, then remove ω from R

T3 If x is any word in the generators G and t is a symbol not in G, then add t to G and tx^{-1} to R.

T4 If a relation has the form tx^{-1} where t is a symbol in G, then remove t from G and replace any occurrence of t in R by x

Theorem 4.2.2 (Tietze) *Two finite presentations represent the same group iff there is a finite sequence of transformations T1–T4, that transform the first presentation in the second.*

□

Proof of Theorem 4.2.1

Recall that G_{B_n} has presentation:

Generators		ϵ_i $\bar{\epsilon}_n$	$1 \leq i < n$
Relations	A	$\epsilon_i\epsilon_j = \epsilon_j\epsilon_i$	$\|j-i\| \geq 2$
	B	$\epsilon_i\epsilon_{i+1}\epsilon_i = \epsilon_{i+1}\epsilon_i\epsilon_{i+1}$	$1 \leq i < n-1$
	C	$\epsilon_i\bar{\epsilon}_n = \bar{\epsilon}_n\epsilon_i$	$1 \leq i < n-1$
	D	$\epsilon_{n-1}\bar{\epsilon}_n\epsilon_{n-1}\bar{\epsilon}_n = \bar{\epsilon}_n\epsilon_{n-1}\bar{\epsilon}_n\epsilon_{n-1}$	

We will make several applications of the following basic identity:

Lemma 4.2.3 *Let $\delta_{n-1} = \epsilon_{n-1}\epsilon_{n-2}\cdots\epsilon_2\epsilon_1 \in G_{B_n}$, then we have:*

$$\delta_{n-1}^{-1}\epsilon_i\delta_{n-1} = \epsilon_{i+1} \qquad 1 \leq i < n-1$$

Further, this relation follows from the relations of type A,B in the previous table. □

Note that $\tau = \bar{\epsilon}_n\delta_{n-1}$ where δ_{n-1} is defined in lemma 4.2.3.
We operate a Tietze move T3, adding τ to the generating set. Then we use T4 to get rid of generator $\bar{\epsilon}_n$, substituting each occurrence with $\bar{\epsilon}_n = \tau\delta_{n-1}^{-1}$.
Relations are thus given by (for C' we use lemma 4.2.3):

A	$\epsilon_i\epsilon_j = \epsilon_j\epsilon_i$	$\|j-i\| \geq 2$
B	$\epsilon_i\epsilon_{i+1}\epsilon_i = \epsilon_{i+1}\epsilon_i\epsilon_{i+1}$	$1 \leq i < n-1$
C'	$\tau^{-1}\epsilon_i\tau = \epsilon_{i+1}$	$1 \leq i < n-1$
D'	$\epsilon_{n-1}\tau\delta_{n-1}^{-1}\epsilon_{n-1}\tau\delta_{n-1}^{-1} = \tau\delta_{n-1}^{-1}\epsilon_{n-1}\tau\delta_{n-1}^{-1}\epsilon_{n-1}$	

Let, by analogy with relations C', $\epsilon_n = \tau^{-1}\epsilon_{n-1}\tau$. Add ϵ_n to generating set using transformation T3.
Note that the relation:

$$\epsilon_n\epsilon_{n-1}\epsilon_n = \epsilon_{n-1}\epsilon_n\epsilon_{n-1}$$

may be deduced from relations already in the presentation. Indeed:

$$\begin{aligned}\epsilon_n\epsilon_{n-1}\epsilon_n =&(\tau^{-1}\epsilon_{n-1}\tau)\epsilon_{n-1}(\tau^{-1}\epsilon_{n-1}\tau) =\\ =&\tau^{-1}(\epsilon_{n-1}(\tau\epsilon_{n-1}\tau^{-1})\epsilon_{n-1})\tau =\\ =&\tau^{-1}(\epsilon_{n-1}\epsilon_{n-2}\epsilon_{n-1})\tau =\\ =&\tau^{-1}(\epsilon_{n-2}\epsilon_{n-1}\epsilon_{n-2})\tau =\\ =&\epsilon_{n-1}\epsilon_n\epsilon_{n-1}\end{aligned}$$

Therefore we can add this relation (T1 move).
Now relation D' may be written as

$$\epsilon_n \delta_{n-1}^{-1} \tau \epsilon_n \delta_{n-1}^{-1} = \delta_{n-1}^{-1} \tau \epsilon_n \delta_{n-1}^{-1} \epsilon_{n-1} \tag{4.2}$$

Note that from relation C', it follows that $\delta_{n-1}^{-1}\tau = \tau\epsilon_1\delta_{n-1}^{-1}\epsilon_n^{-1}$. So the relation in eq. 4.2is equivalent to:

$$\epsilon_n \tau \epsilon_1 \delta_{n-1}^{-1} \delta_{n-1}^{-1} = \tau \epsilon_1 \delta_{n-1}^{-1} \delta_{n-1}^{-1} \epsilon_{n-1}$$

which is turn equivalent to:

$$\tau^{-1}\epsilon_n\tau = \epsilon_1 \underbrace{(\delta_{n-1}^{-1}(\delta_{n-1}^{-1}\epsilon_{n-1})\delta_{n-1})}\delta_{n-1}\epsilon_1^{-1}$$

Applying lemma 4.2.3to the expression underbraced this is equivalent to:

$$\tau^{-1}\epsilon_n\tau = \epsilon_1(\epsilon_1\delta_{n-1}^{-1})\delta_{n-1}\epsilon_1^{-1}$$

In summary D' is equivalent to

$$\tau^{-1}\epsilon_n\tau = \epsilon_1$$

Finally use T1 to introduce relation

$$\epsilon_1\epsilon_n\epsilon_1 = \epsilon_n\epsilon_1\epsilon_n$$

that is easily seen to hold. □

Letting $\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_n$ be the standard generator of the Artin braid group of type $\tilde{A}_{n-1}$, we have the following immediate corollary of theorem 4.2.1:

Corollary 4.2.4 ([KP02]) *The map*

$$G_{\tilde{A}_{n-1}} \ni \tilde{\sigma}_i \mapsto \epsilon_i \in G_{B_n}$$

gives an isomorphism between $G_{\tilde{A}_{n-1}}$ and the subgroup of G_{B_n} generated by $\epsilon_1, \ldots, \epsilon_n$. Moreover, we have a semidirect product decomposition $G_{B_n} \cong G_{\tilde{A}_{n-1}} \rtimes \langle\tau\rangle$. □

We have thus a 'curious' inclusion of the Artin group of infinite type $\tilde{A}_{n-1}$ into the Artin group of finite type B_n.

4.2.1 A topological proof of inclusion

Consider the complex hyperplane arrangement of type $\tilde{A}_{n-1}$ and let $Y(\tilde{A}_{n-1})$ be its complement:

$$Y(\tilde{A}_{n-1}) = \{x \in \mathbb{C}^n |\, x_i - x_j \notin \mathbb{Z}\ \forall i \neq j;\ \sum_i x_i = 0\} \tag{4.3}$$

The group $\tilde{A}_{n-1}$ acts on this space in the following way. In the standard basis $\{\alpha_i = e_i - e_{i+1}\}$ for the root system A_{n-1}, recall that the highest positive root is given by $\tilde{\alpha} = e_1 - e_n$. From the general theory of affine Weyl groups, the group is generated by the reflections s_i with respect to (central) hyperplane $H_{\alpha_i} = \{x_i - x_{i+1} = 0\}$, plus one extra reflection s_0 w.r.t. the affine hyperplane $H_{-\tilde{\alpha},1} = \{(x, -\tilde{\alpha}) = 1\} = \{-x_1 + x_n = 1\}$. Reflection s_i $(1 \leq i \leq n-1)$ switches as usual the $i, i+1$-th coordinates, whereas

$$s_0(x_1, \dots, x_n) = (x_n - 1, x_2, \dots, x_{n-1}, x_1 + 1)$$

Recall also that $\tilde{A}_{n-1} = L \rtimes A_{n-1}$ where L is the coroot lattice. In the present situation, L is the intersection of the standard lattice $\mathbb{Z}^n \subset \mathbb{C}^n$ with the hyperplane $\{\sum x_i = 0\}$.

Recall also that the orbit space $X(\tilde{A}_{n-1}) \equiv Y(\tilde{A}_{n-1})/\tilde{A}_{n-1}$ has $G_{\tilde{A}_{n-1}}$ as fundamental group.

Theorem 4.2.5 *The space $X(\tilde{A}_{n-1})$ is homotopic to an infinite cyclic covering $X(B_n)$.*

From this theorem, we may recover as a straightforward corollary the following result:

Corollary 4.2.6 *The space $X(\tilde{A}_{n-1})$ is a $k(\pi, 1)$ space for the group $G_{\tilde{A}_{n-1}}$.*

Proof. Since B_n is finite, $X(B_n)$ is a $k(\pi, 1)$ space by Deligne result (theorem 2.5.4). From theorem 4.2.5, it follows that $X(\tilde{A}_{n-1})$ is a $k(\pi, 1)$ space as well. □

The $k(\pi, 1)$-property for $X(\tilde{A}_{n-1})$ was already shown in [Oko79] and rediscovered by different methods in [CP03]. However the proof presented here is very elementary.

Proof of theorem 4.2.5 We first consider the intermediate covering $\pi : Y(\tilde{A}_{n-1}) \to Y(\tilde{A}_{n-1})/L$. This covering can explicitly described as follows. Consider the map $\mathbb{C}^n \ni (x_1, \dots x_n) \mapsto (e^{2\pi \mathrm{i} x_1}, \dots, e^{2\pi \mathrm{i} x_n}) \in (\mathbb{C}^*)^n$. Note that this map induces the desired covering:

$$\pi : Y(\tilde{A}_{n-1}) \to Y'(\tilde{A}_{n-1}) := \{y \in \mathbb{C}^n |\, y_i \neq 0;\ y_i - y_j \neq 0;\ \prod_i y_i = 1\}$$

Note also that $Y'(\tilde{A}_{n-1}) = Y'(B_n) \cap \{\prod_i y_i = 1\}$ where $Y'(B_n) \cong Y(B_n)/(\mathbb{Z}/2\mathbb{Z})^n$ is defined in eq. 4.1.

We have now an infinite cyclic covering:

$$\begin{aligned} \eta :\ & Y'(\tilde{A}_{n-1}) \times \mathbb{C} \to Y'(B_n) \\ & \big((y_1, \dots, y_n), \lambda\big) \mapsto (e^{2\pi \mathrm{i} \lambda} y_1, \dots, e^{2\pi \mathrm{i} \lambda} y_n) \end{aligned}$$

$$\bar{\gamma}_1 \overset{4}{—} \gamma_2 — \gamma_3 — \gamma_4 - - \qquad - - \gamma_{n-2} — \gamma_{n-1} \overset{4}{—} \bar{\gamma}_n$$

Table 4.2: Graph of type $\tilde{C}_{n-1}$; each node is labelled with the corresponding generator in the Artin group

The map η is clearly equivariant for the action of the symmetric group S_n on both spaces. Since $X(\tilde{A}_{n-1}) = Y'(\tilde{A}_{n-1})/S_n$ and $X(B_n) = Y'(B_n)/S_n$, we have therefore a covering:

$$\begin{aligned} X(\tilde{A}_{n-1}) \times \mathbb{C} &\to X(B_n) \\ \big(\{y_1, \dots, y_n\}, \lambda\big) &\mapsto \{e^{2\pi \mathrm{i}\lambda} y_1, \dots, e^{2\pi \mathrm{i}\lambda} y_n\} \end{aligned}$$

□

Remark. One can show that the homomorphism of fundamental groups induced by the covering in theorem 4.2.5 takes the form prescribed in corollary 4.2.4.

4.3 $G_{\tilde{C}_{n-1}}$ as a subgroup of G_{B_n}

Consider the complex hyperplane arrangement of type $\tilde{C}_{n-1}$ and let $Y(\tilde{C}_{n-1})$ be its complement:

$$Y(\tilde{C}_{n-1}) = \{x \in \mathbb{C}^{n-1} |\, x_i \pm x_j \notin \mathbb{Z} \ \forall i \neq j; 2x_i \notin \mathbb{Z} \ \forall i\} \tag{4.4}$$

Recalling that $\tilde{C}_{n-1} = L(\hat{\Phi}) \rtimes B_{n-1}$, the action of $\tilde{C}_{n-1}$ on this space may be described in the following way: B_{n-1} acts as groups of signed permutations on the coordinates, while $L(\hat{\Phi})$ acts as translations of integer vectors.

The orbit space $X(\tilde{C}_{n-1}) = Y(\tilde{C}_{n-1})/\tilde{C}_{n-1}$ has $G_{\tilde{C}_{n-1}}$ as fundamental group. We label the standard generators of this group as in table 4.2.

Theorem 4.3.1 *We have:*

1. *The space $X(\tilde{C}_{n-1})$ is homotopic to a covering of $X(B_n)$.*
2. *The map*

$$\begin{aligned} G_{\tilde{C}_{n-1}} &\to G_{B_n} \\ \bar{\gamma}_1 &\mapsto \epsilon_1^2 \\ \gamma_i &\mapsto \epsilon_i \quad \textit{for } 2 \leq i \leq n-1 \\ \bar{\gamma}_n &\mapsto \bar{\epsilon}_n \end{aligned}$$

is injective. □

As in the previous section, we recover the following result by Okonek [Oko79]:

Corollary 4.3.2 *The space $X(\tilde{C}_{n-1})$ is a $k(\pi, 1)$ space for the group $G_{\tilde{C}_{n-1}}$.* □

Proof of theorem 4.2.5

Consider the function $f : \mathbb{C} \setminus \frac{1}{2}\mathbb{Z} \ni x \mapsto z = \frac{1-e^{2\pi i x}}{1+e^{2\pi i x}} \in \mathbb{C}$. Applying f componentwise to $Y(\tilde{C}_{n-1})$ we obtain a covering:

$$\pi : Y(\tilde{C}_{n-1}) \to Y'(\tilde{C}_{n-1}) := \{z \in \mathbb{C}^{n-1} | \, z_i \neq 0; \; z_i \pm z_j \neq 0; \; z_i \neq \pm 1\}$$

Note that $Y'(\tilde{C}_{n-1}) \cong Y(\tilde{C}_{n-1})/L(\hat{\Phi})$. We may further quotient out for the subgroup $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ of sign changes in $B_{n-1} = (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_{n-1}$. This is done applying componentwise the function $y = z^2$. We thus get:

$$Y''(\tilde{C}_{n-1}) := Y'(\tilde{C}_{n-1})/(\mathbb{Z}/2\mathbb{Z})^{n-1} \cong \{y \in \mathbb{C}^{n-1} | \, y_i \neq 0; \; y_i - y_j \neq 0; \; y_i \neq +1\}$$

Let $Y'(B_n) = \{y = (y_0, \ldots, y_{n-1}) \in \mathbb{C}^n \,|\, y_i \neq y_j \; \forall i \neq j; \; y_i \neq 0 \; \forall i\}$. Apart from a shifting of the indices, this is the same space defined in eq. 4.1 and so $Y'(B_n) \cong Y(B_n)/(\mathbb{Z}/2)^n$. Now notice that $Y''(\tilde{C}_{n-1}) \cong Y'(B_n) \cap \{y_0 = 1\}$.

We have now an infinite cyclic covering:

$$\begin{aligned} &\eta : Y''(\tilde{C}_{n-1}) \times \mathbb{C} \to Y'(B_n) \\ &\big((y_1, \ldots, y_{n-1}), \lambda\big) \mapsto (e^{2\pi i \lambda}, e^{2\pi i \lambda} y_1, \ldots, e^{2\pi i \lambda} y_{n-1}) \end{aligned}$$

Let the symmetric group S_{n-1} acts on $Y'(B_n)$ by permuting the last $(n-1)$ coordinates. Note that η is equivariant for the action of the symmetric group S_{n-1} on these spaces. Since $X(\tilde{C}_{n-1}) = Y''(\tilde{C}_{n-1})/S_{n-1}$, we have a covering $X(\tilde{C}_{n-1}) \times \mathbb{C} \to Y'(B_n)/S_{n-1}$. Composing with the covering $Y'(B_n)/S_{n-1} \to Y'(B_n)/S_n = X(B_n)$, we have the desired covering map. The second statement follows from the first and an explicit description of the covering map at the level of fundamental groups, similar to that presented in the proof of theorem 4.1.1; so the proof will be rather sketchy.
Fix $p = (p_1, \ldots, p_{n-1})$ as base point of $Y(\tilde{C}_{n-1})$ where each p_i is real and satisfies:

$$1/2 > p_1 > p_2 > \ldots > p_{n-1} > 0$$

and take 0 as base point of $\mathbb{C}$. Actually, the presented covering map does not induce directly the stated homomorphism of fundamental groups. Indeed, for the computation of the image of $\bar{\gamma}_1$ (resp. $\bar{\gamma}_n$) we may consider the path g_1 (resp. g_n) in $Y(\tilde{C}_{n-1})$ which is constant in all components except the first (resp. the last). In the distinguished component they are respectively semicircle around $1/2$ and 0 (see figure 4.3). The path around 0 is sent via $f(x)^2$ to a full turn around 0, while the other is sent to a full turn around the point ∞ to infinity. The image of g_n in $Y'(B_n)$ correspond readily to the generator $\bar{\epsilon}_n$, while the image in $Y'(B_n)$ of g_1 winds around

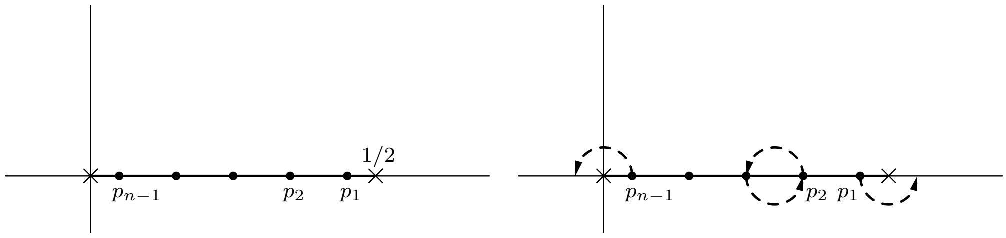


Figure 4.3: Having fixed $p = (p_1, \dots, p_{n-1})$ as basis point for $Y(\tilde{C}_n)$, the generator $\bar{\gamma}_1$ corresponds to a path in $Y(\tilde{C}_n)$ constant in all the coordinates but the first, in which it is the small semicircle around $1/2$ depicted in the figure. Similarly, the generator $\bar{\gamma}_n$ may be represented by a path whose projection on the last coordinate is a small semicircle with center 0. The remaining generators exchange instead couples of adjacent coordinates.

all the remaining coordinates (see plots (a)-(c) in figure 4.4). To correct this situation, we may consider the function $y \mapsto h(y) = \frac{y}{y-1}$. This map fixes 0 and takes ∞ to 1. Applying this function componentwise, we get an automorphism of $Y''(\tilde{C}_{n-1})$(see plot (d) in figure 4.4). Identifying again $h(Y''(\tilde{C}_{n-1})) \cong Y'(B_n) \cap \{y_0 = 1\}$ and proceeding as before, we get the desired homomorphism of fundamental groups. $\square$

Remark. From the identification $Y''(\tilde{C}_{n-1}) \cong Y'(B_n) \cap \{y_0 = 1\}$, we can equivalently conclude that $Y(\tilde{C}_{n-1})$ is a $k(\pi, 1)$ in the following way. Note that the map $Y'(B_n) \to \mathbb{C}^*$ given by the projection on the first coordinate is a locally trivial fibration and that $Y''(\tilde{C}_{n-1})$ is the fiber over 1. Then apply the homotopy long exact sequence of the fibration.

Using the interpretation of G_{B_n} as annular braid group, we can identify also $G_{\tilde{C}_{n-1}}$ as a subgroup of the braid group.

Consider the subgroup $\mathrm{Br}_{n+1}^{(1,n+1)} < \mathrm{Br}_{n+1}$ of $(1, n+1)$-pure braids, i.e. the subgroup of braids for which the start- and end-point of the first and last strands coincide. Note that $\mathrm{Br}_{n+1}^{(1,n+1)}$ is a subgroup of index n of Br_{n+1}^{n+1}. Let $P_2 \cong \mathbb{Z}$ be the group of pure braids on 2 strands. We have a forgetting homomorphism $f : \mathrm{Br}_{n+1}^{(1,n+1)} \to P_2$, obtained deleting the strands from 2 to n in a braid belonging to $\mathrm{Br}_{n+1}^{(1,n+1)}$. Let $Q_{n-1} := \ker f$; pictorially $Q_{n-1}$is the subgroup of braids for which the first and last strands may be straightened. Equivalently, Q_{n-1} can be regarded as the braid group in $n-1$ strands on the surface $\mathbb{C} \setminus \{0, 1\}$. We may also split the exact sequence

$$\begin{tikzcd} 1 \arrow[r] & Q_{n-1} \arrow[r] & \mathrm{Br}_{n+1}^{(1,n+1)} \arrow[r, "s"] & P_2 \cong \mathbb{Z} \arrow[r] \arrow[l, bend left] & 1 \end{tikzcd}$$

defining $s(1) = \tau^n$, where as usual τ is the element of Br_{n+1}^{n+1} represented

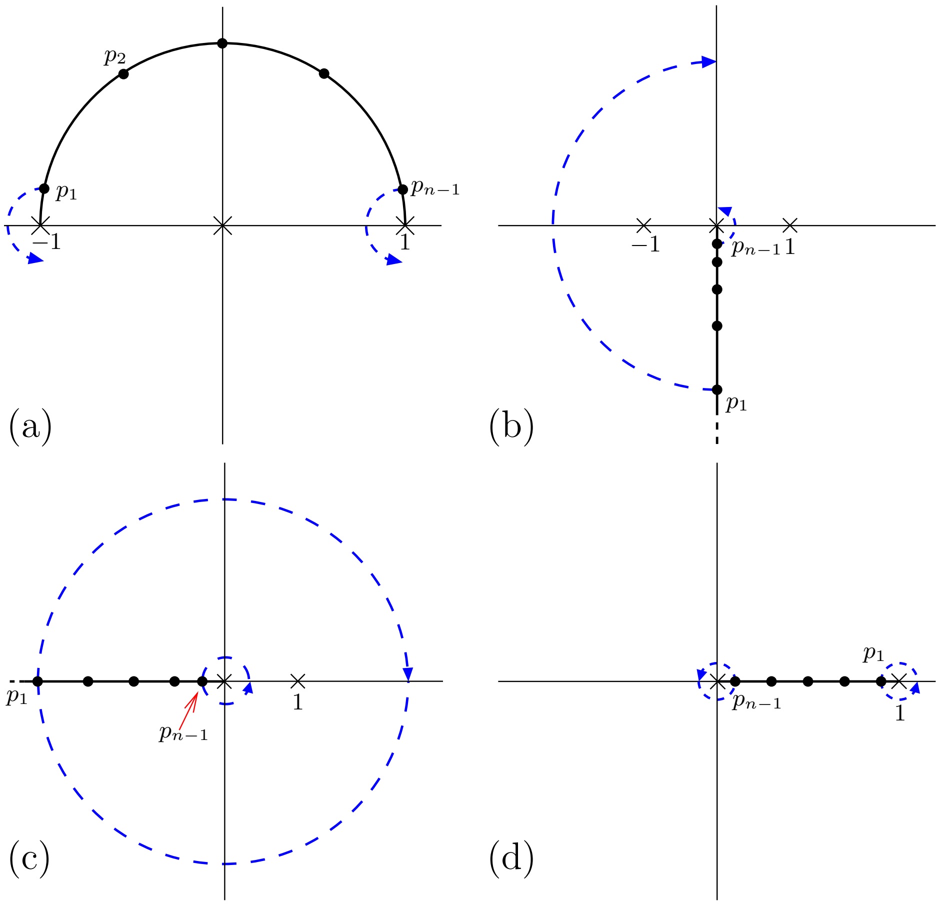


Figure 4.4: The plots in the figure represent step by step how the paths in figure 4.3 are modified by the effect of the functions used in the proof. Plot (a) is obtained form figure 4.3 applying the exponential map; (b) is obtained from (a) applying $z \mapsto 1-z/1+z$; (c) is obtained from (b) applying $z \mapsto z^2$. In summary plot (c) represent the effect of the map $f(x)^2$ used in the text. Finally plot (d) shows the behavior of the corrective function $z \mapsto z/z-1$.

in figure 4.2. Since τ^n is in the center of Br_{n+1}^{n+1}, we have a direct product decomposition $\mathrm{Br}_{n+1}^{(1,n+1)} = Q_{n-1} \times \mathbb{Z}$.

Let $Y'(B_n)$ be the space defined in eq. 4.1 and consider the action of S_{n-1} given by permutation of the last $n-1$ coordinates. As in the proof of theorem 4.1.1, one can show that $\pi_1(Y'(B_n)/S_{n-1}) = \mathrm{Br}_{n+1}^{(1,n+1)}$.
As a corollary of the proof of theorem 4.3.1, we then have:

Corollary 4.3.3 *The group $G_{\tilde{C}_{n-1}}$ is isomorphic to Q_{n-1}.*
In particular, we have a direct product decomposition:

$$\mathrm{Br}_{n+1}^{(1,n+1)} = G_{\tilde{C}_{n-1}} \times \mathbb{Z}$$

Proof. Consider the infinite cyclic covering $X(\tilde{C}_{n-1}) \times \mathbb{C} \to Y'(B_n)/S_{n-1}$ given in the proof of theorem 4.3.1 and note that a path $\big(f_0(t), \{f_1(t), \ldots, f_{n-1}(t)\}\big)$ in $Y'(B_n)/S_{n-1}$ lifts to a closed path in $X(\tilde{C}_{n-1}) \times \mathbb{C}$ if and only if the winding number of $f_0(\cdot)$ around 0 is null. Via the identification $\pi_1(Y'(B_n)/S_{n-1}) = \mathrm{Br}_{n+1}^{(1,n+1)}$, this means exactly that the first strand does not wind around the $(n+1)$-th one.

□

It is known that the group $G_{\tilde{A}_{n-1}}$ has trivial center [JA85]. Mimicking the proof of this fact given in [CP03](prop. 1.3), we can show:

Proposition 4.3.4 *The group $G_{\tilde{C}_{n-1}}$ has trivial center.*

Proof. Recall that for a finite type Artin group of rank n, the maximal rank of an abelian subgroup is precisely n. Let r be the maximal rank of an abelian subgroup of $G_{\tilde{C}_{n-1}}$. Since $G_{B_{n-1}} < G_{\tilde{C}_{n-1}}$ as a parabolic subgroup, we have $r \geq n-1$. Note that if H be an abelian subgroup of $G_{\tilde{C}_{n-1}}$ having rank r, then $H \times \langle \tau^n \rangle$ is a rank $r+1$ abelian subgroup of G_{B_n}. Thus we have also $r+1 \leq n$, which forces $r = n-1$.
Now let $z \in Z = Z(G_{\tilde{C}_{n-1}})$ be a non-trivial central element. Consider for $i = 1, \ldots, n$ the parabolic subgroups $G_{\tilde{C}_{n-1}}(i)$, generated by words that do not contain the i-th generator. We claim that there exists i_0 s.t. $\langle z \rangle \cap G_{\tilde{C}_{n-1}}(i_0) = 1$. Suppose that this does not hold. Then for every i, we have $z^{a_i} \in G_{\tilde{C}_{n-1}}(i)$ for some integer a_i. Let $a = \prod a_i$. Then $z^a \in \bigcap G_{\tilde{C}_{n-1}}(i) = 1$. Since G_{B_n} is torsion-free (just as any other finite type Artin groups), we conclude that $z = 1$, which is absurd, since z is assumed to be non trivial. Thus, we can assume that for some i_0 we have $\langle z \rangle \cap G_{\tilde{C}_{n-1}}(i_0) = 1$.

Since $G_{\tilde{C}_{n-1}}(i_0)$ is a rank $n-1$ finite type Artin group, it admits an abelian subgroup G of rank $n-1$. But then $G \times \langle z \rangle$ is a rank n abelian subgroup of $G_{\tilde{C}_{n-1}}$, which is absurd.

□

4.4 $G_{\tilde{B}_n}$ and the $k(\pi, 1)$ problem

This section is devoted to prove that $X(\tilde{B}_n)$ is a $k(\pi, 1)$ space. Up to the best of our knowledge, this result seems to be new. The proof is based on an "exponentiation" argument with Deligne's theorem 2.5.3 as further ingredient.

Recall that the complement of the complexified affine reflection arrangement of type B_n is given by:

$$Y := Y(\tilde{B}_n) = \{x \in \mathbb{C}^n \mid x_i \pm x_j \notin \mathbb{Z} \text{ for all } i \neq j,\ x_k \notin \mathbb{Z} \text{ for all } k\}$$

The Coxeter group $\tilde{B}_n$ is the semidirect product of the Coxeter group B_n and the coroot lattice L, consisting of integer vectors whose coordinates add up to an even number. The group $\tilde{B}_n$ acts freely on Y: B_n is identified with the group of signed permutations of the coordinates of Y, while L acts simply by translations.

On Y we have as well a free action by translations of the coweight lattice $\hat{L}$, identified with the standard lattice $\mathbb{Z}^n \subset \mathbb{C}^n$.

Theorem 4.4.1 *$Y(\tilde{B}_n)$ and, hence, $X(\tilde{B}_n)$ are $k(\pi, 1)$ spaces.*

Proof. We first explicitly describe the covering $Y \to Y/\hat{L}$ applying the exponential map $y = \exp(2\pi\mathrm{i}x)$ componentwise to Y:

$$\begin{array}{ccc} Y & \xrightarrow{\quad\pi\quad} & Y/\hat{L} \cong \{y \in \mathbb{C}^n \mid y_i \neq y_j^{\pm 1},\ y_k \neq 0, 1\} \\ & & \\ (x_1, \ldots, x_n) & \longmapsto & \big(\exp(2\pi\mathrm{i}x_1), \ldots, \exp(2\pi\mathrm{i}x_n)\big) \end{array}$$

Note now that the function

$$\mathbb{C} \setminus \{0, 1\} \ni y \mapsto g(y) = \frac{1+y}{1-y} \in \mathbb{C} \setminus \{\pm 1\}$$

satisfies $g(y^{-1}) = -g(y)$. Further g is invertible, its inverse being given by $z \mapsto \frac{z-1}{z+1}$. Therefore applying g componentwise to $Y/\hat{L}$, we have:

$$Y/\hat{L} \cong \{z \in \mathbb{C}^n \mid z_i \neq \pm z_j,\ z_k \neq \pm 1\}$$

Consider now the arrangement $\mathcal{A}^{\mathbf{R}}$ in $\mathbf{R}^{n+1}$ consisting of the hyperplanes $x_i = \pm x_j$ for $1 \leq i < j \leq n+1$ and $x_{n+1} = 0$ and let $Y(\mathcal{A})$ be the complement of its complexification.

We have an homeomorphism

$$\eta : Y/\hat{L} \times \mathbb{C}^* \to Y(\mathcal{A})$$

defined by

$$\eta\big((z_1, \ldots, z_n), \lambda\big) = (\lambda z_1, \ldots, \lambda z_n, \lambda)$$

To show that $Y/\hat{L}$ is a $k(\pi,1)$, it is then sufficient to show that $Y(\mathcal{A})$ is a $k(\pi,1)$. We will show in lemma 4.4.2 below that $\mathcal{A}^{\mathbf{R}}$ is simplicial, and therefore the result follows from Deligne's theorem 2.5.3. □

Lemma 4.4.2 *Let $\mathcal{A}$ be the real arrangement in $\mathbf{R}^{n+1}$ consisting of the hyperplanes $x_i = \pm x_j$ for $1 \leq i < j \leq n+1$ and $x_{n+1} = 0$. Then $\mathcal{A}$ is simplicial.*

Proof. Note that $\mathcal{A}$ is the union of the reflection arrangement $\mathcal{A}(D_{n+1})$ of type D_{n+1} and the hyperplane $x_{n+1} = 0$. Hence we study how the chambers of $\mathcal{A}(D_{n+1})$ are cut by the hyperplane $x_{n+1} = 0$. Since the Coxeter group D_{n+1} acts transitively on the collection of chambers, it is enough to consider how the fundamental chamber C_0 of $\mathcal{A}(D_{n+1})$ is cut by the D_{n+1}-translates of the hyperplane $x_{n+1} = 0$. Since D_{n+1} acts on $\mathbf{R}^{n+1}$ as the group of signed permutations of the coordinates involving an even number of sign changes, the D_{n+1}-translates of the hyperplane $x_{n+1} = 0$ are the coordinate hyperplanes $x_k = 0$, for $k = 1, 2, \ldots, n+1$. Let $e_1, \ldots, e_{n+1}$ be the canonical basis of $\mathbf{R}^{n+1}$ and fix a simple system Δ for D_{n+1} consisting of the roots $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, …, $\alpha_n = e_n - e_{n+1}$, $\alpha_{n+1} = e_n + e_{n+1}$. Let also $Z_i = \{x \in \mathbf{R}^{n+1} \mid (x, \alpha_i) > 0\}$.
Then, the fundamental chamber is given by $C_0 = \bigcap_{i=1}^{n+1} Z_i$ and it is of course a simplicial cone, since $\alpha_1, \ldots, \alpha_{n+1}$ form a basis of $\mathbf{R}^{n+1}$. Note that the coordinates of a point $x \in C_0$ satisfy $x_1 > x_2 > \ldots > x_n > x_{n+1} > -x_n$. In particular $x_i > 0$ for $i = 1, 2, \ldots, n$, and thus C_0 all lies on one side of the hyperplane $x_i = 0$, for $i = 1, 2, \ldots, n$. It remains to consider how x_{n+1} cuts C_0. We start noting that $\alpha_1, \ldots, \alpha_n, e_{n+1}$ is a basis of $\mathbf{R}^{n+1}$ and that its positive span contains the root α_{n+1}. It follows that $C_0 \cap \{x_{n+1} > 0\} = \bigcap_{i=1}^{n+1} Z_i \cap \{x_{n+1} > 0\} = \bigcap_{i=1}^{n} Z_i \cap \{x_{n+1} > 0\}$ is a simplicial cone.
Similarly, $\alpha_1, \ldots, \alpha_{n-1}, \alpha_{n+1}, -e_{n+1}$ is a basis of $\mathbf{R}^{n+1}$ and its positive span contains the root α_n. So $C_0 \cap \{-x_{n+1} > 0\} = \bigcap_{i=1}^{n+1} Z_i \cap \{-x_{n+1} > 0\} = \bigcap_{i=1}^{n-1} Z_i \cap Z_{n+1} \cap \{x_{n+1} > 0\}$ is a simplicial cone as well. □

4.5 Representations of Artin groups

The previous inclusions of Artin groups studied in sections 4.1-4.3 allow to construct interesting representations of Artin groups starting with very simple ones by the method of induction/restriction. We will study some instances of this construction that will be useful in relating various cohomology group by means of Shapiro's lemma. Further we recover a representation of the braid group (known as the Tong-Yang-Ma representation) that, despite its simplicity, seems to have been discovered only in 1996 (see [Sys01]).
Since it is not very-well known, we start recalling Tong-Yang-Ma representation.

4.5.1 Tong-Yang-Ma representation

Let R be a ring and $V_R = \bigoplus_{i=1}^{n+1} R[u^{\pm 1}]e_i$ be an $(n+1)$-dimensional free $\mathbb{R}[u^{\pm 1}]$ module.

Definition 4.5.1 *The Tong-Yang-Ma representation [TYM96] is the representation*

$$\rho_R : G_{A_n} \to \mathrm{Gl}_{R[u^{\pm 1}]}(V_R)$$

defined w.r.t. the basis $e_1, \ldots, e_{n+1}$ *by:*

$$\rho_R(\sigma_i) = \begin{pmatrix} I_{i-1} & & & \\ & 0 & 1 & \\ & u & 0 & \\ & & & I_{n-i} \end{pmatrix}$$

where I_j *denote the* j*-dimensional identity matrix and all other entries are zero.*

Note that, specializing to $u = 1$, ρ_R induces a representation of the symmetric group S_{n+1} whose direct summands are one copy of the standard representation and one copy of the trivial one.

We briefly discuss the relevance of Tong-Yang-Ma representation.

Let $\rho_{\mathbb{C}}(\lambda)$ the specialization of the complex Tong-Yang-Ma representation, obtained setting $u = \lambda \in \mathbb{C}^*$.

Proposition 4.5.2 *The complex representation* $\rho_{\mathbb{C}}(\lambda)$ *is irreducible for* $\lambda \neq 1$.

□

Let $\eta : G_{A_n} \to \mathrm{Gl}_r(\mathbb{C})$ be a complex representation. Then the *corank* of η is defined as $\mathrm{rk}(\eta(\sigma_i) - I)$, where I is the identity of $\mathrm{Gl}_r(\mathbb{C})$.
Since the standard generators σ_i $(1 \leq i \leq n)$ are all conjugate, the corank is well defined.

Note that the Tong-Yang-Ma representation has corank 2. The following theorem says that it is indeed an 'universal' example:

Theorem 4.5.3 ([Sys01]) *For* $n \geq 6$*, every irreducible complex representation of* G_{A_n} *of corank* 2 *is equivalent to* $\rho_{\mathbb{C}}(\lambda)$ *for some* $\lambda \in \mathbb{C}^* \setminus \{1\}$.

Corank two representations are of key importance in the classification of irreducible complex representations of braid groups. Indeed we have:

Theorem 4.5.4 ([Sys01]) *For* $n \geq 8$*, every* $n+1$*-dimensional complex irreducible representation of the braid group* G_{A_n} *is equivalent to a tensor product of a one-dimensional representation and an* $n+1$*-dimensional representation of corank* 2.

For small n the question is settled in [FLSV03], where a full classification of complex G_{A_n}-representation of dimension $\leq n+1$ is provided.

4.5.2 The standard representation of G_{B_n}

Motivated by the Tong-Yang-Ma representation, we define a new representation for the Artin group G_{B_n}.

Definition 4.5.5 *The* standard *representation of the Artin group G_{B_n} is the representation*

$$\eta = \eta_R : G_{B_n} \to \mathrm{Gl}_n(R[u^{\pm 1}, v^{\pm 1}])$$

defined w.r.t. the standard generators $\epsilon_1, \ldots, \bar{\epsilon}_n$ of G_{B_n} by:

$$\eta(\epsilon_i) = \left(\begin{array}{c|cc|c} I_{i-1} & & & \\ \hline & 0 & 1 & \\ & u & 0 & \\ \hline & & & I_{n-i-1} \end{array}\right)$$

for $1 \leq i < n$ while

$$\eta(\bar{\epsilon}_n) = \left(\begin{array}{c|c} I_{n-1} & \\ \hline & v \end{array}\right)$$

where I_j denote the j-dimensional identity matrix and all other entries are zero.

Note that the restriction to G_{B_n} of the Tong-Yang-Ma representation for G_{A_n} splits into a one dimensional representation and an n dimensional representation which is equivalent to the standard representation specialized to $v = u$.

Let $\eta_{\mathbb{C}}(\lambda, \mu)$ the specialization of the complex standard representation, obtained setting $u = \lambda \in \mathbb{C}^*$, $v = \mu \in \mathbb{C}^*$. We have:

Proposition 4.5.6 *The complex representation $\rho_{\mathbb{C}}(\lambda, \mu)$ is irreducible except when $\lambda = \mu = 1$.*

$\square$

Thus the standard representation parameterizes a family of complex irreducible representations of G_{B_n}. It would be interesting to see wether (at least for large n) these are the only representations satisfying some constraint on the corank (note that here we should define the corank of the representation as the couple $\big(\mathrm{rk}(\eta(\epsilon_i) - \mathrm{Id}), \mathrm{rk}(\eta(\bar{\epsilon}_n) - \mathrm{Id})\big)$, since the last generator is not conjugate to the first ones).

4.5.3 Induced representations

Let $M := \mathbb{Q}[q^{\pm 1}]$. We indicate by M_q the $G_{\tilde{A}_{n-1}}$-module where the action of the standard generators is $(-q)$-multiplication.
Consider also the inclusion $G_{\tilde{A}_{n-1}} < G_{B_n}$ given in section 4.2.

Proposition 4.5.7 *We have*

$$\mathrm{Ind}_{G_{\tilde{A}_{n-1}}}^{G_{B_n}}(M_q) \cong M[t^{\pm 1}]_{q,t}$$

$$\mathrm{CoInd}_{G_{\tilde{A}_{n-1}}}^{G_{B_n}}(M_q) \cong M[[t^{\pm 1}]]_{q,t}$$

where the action of G_{B_n} *on* $M[t^{\pm 1}]_{q,t}$ *(and on* $M[[t^{\pm 1}]]_{q,t}$*) is given by* $(-q)$*-multiplication for the generators* $\epsilon_1, \ldots, \epsilon_{n-1}$ *and* $(-t)$*-multiplication for the last generator* $\bar{\epsilon}_n$*.*

Proof. We start with the induced representation. By corollary 4.2.4, any element of $\mathrm{Ind}_{G_{\tilde{A}_{n-1}}}^{G_{B_n}} M_q := \mathbb{Z}[G_{B_n}] \otimes_{G_{\tilde{A}_{n-1}}} M_q$ can be represented as a sum of elements of the form $\tau^\alpha \otimes q^m$. Now, we have an isomorphism of $\mathbb{Z}[G_{B_n}]$-modules

$$\mathbb{Z}[G_{B_n}] \otimes_{G_{\tilde{A}_{n-1}}} M_q \to M[t^{\pm 1}]_{q,t}$$

defined by sending $\tau^\alpha \otimes q^m \mapsto (-1)^{n\alpha} t^\alpha q^{(n-1)\alpha+m}$. Since $\tau = \bar{\epsilon}_n \epsilon_{n-1} \cdots \epsilon_1$, the result follows.
As for the coinduced representation, we have by definition:

$$\mathrm{CoInd}_{G_{\tilde{A}_{n-1}}}^{G_{B_n}}(M_q) := \mathrm{Hom}_{G_{\tilde{A}_{n-1}}}(\mathbb{Z}G_{B_n}, M_q)$$

Let $f \in \mathrm{Hom}_{G_{\tilde{A}_{n-1}}}(\mathbb{Z}G_{B_n}, M_q)$ and $Q_\alpha = f(\tau^\alpha) \in M_q$. We may define the following isomorphism of $\mathbb{Z}[G_{B_n}]$-modules

$$\mathrm{Hom}_{G_{\tilde{A}_{n-1}}}(\mathbb{Z}G_{B_n}, M_q) \to M[[t^{\pm 1}]]_{q^{-1},t}$$

obtained sending $f \mapsto \sum_\alpha Q_\alpha (-t)^\alpha (-q)^{-(n-1)\alpha}$. The result follows renaming $q \to q^{-1}$. □

It is interesting to note what happens inducing again via the inclusion $G_{B_n} < G_{A_n}$.
Let $\rho = \rho_{\mathbb{Q}} : G_{A_n} \to \mathrm{Gl}(V)$ be the rational Tong-Yang-Ma representation for G_{A_n}.

Proposition 4.5.8 *The induced module* $\mathrm{Ind}_{G_{B_n}}^{G_{A_n}} M[t^{\pm 1}]_{q,t}$ *is equivalent to the Tong-Tang-Ma representation* V *tensored with a one dimensional representation. More precisely, we have:*

$$\mathrm{Ind}_{G_{B_n}}^{G_{A_n}} M[t^{\pm 1}]_{q,t} \cong M_q \otimes V$$

where the action of G_{A_n} *on* M_q *is defined sending the standard generators to* $(-q)$*-multiplication.*

Proof. We first describe the cosets $G_{A_n}/G_{B_n} \cong \mathrm{Br}_{n+1}/\mathrm{Br}_{n+1}^{n+1}$. For $1 \le i \le n+1$, let:

$$P_i = \{x \in \mathrm{Br}_{n+1} \,|\, \text{the endpoint of the } i\text{-th strand is the } n+1\text{-st base point}\}$$

These sets are obviously disjoint and cover Br_{n+1}. Also for $x \in P_i$, we have $x\mathrm{Br}_{n+1}^{n+1} = P_i$. Thus the P_is are the cosets $\mathrm{Br}_{n+1}/\mathrm{Br}_{n+1}^{n+1}$. We now choose as coset representatives the elements $\alpha_i = (\sigma_i\sigma_{i+1}\cdots\sigma_{n-1})\sigma_n(\sigma_i\sigma_{i+1}\cdots\sigma_{n-1})^{-1}$ for $1 \le i \le n-1$, $\alpha_n = \sigma_n$, $\alpha_{n+1} = e$.
By definition of induced representation, $\mathrm{Ind}_{G_{B_n}}^{G_{A_n}} M[t^{\pm 1}]_{q,t} = \bigoplus_{i=1}^{n+1} M[t^{\pm 1}]e_i$ with the following action. For an element $x \in G_{A_n}$, write $x\alpha_k = \alpha_{k'}x'$ with $x' \in G_{B_n}$. Then x acts on an element $r \cdot e_k \in \bigoplus_{i=1}^{n+1} M[t^{\pm 1}]e_i$ as $x(r \cdot e_k) = (x'r) \cdot e_{k'}$.

We thus figure out the action of Br_{n+1} on the cosets. Note that for every index i we have $\alpha_i^2 \in \mathrm{Br}_{n+1}^{n+1}$. The following lemma then characterizes completely this action.

Lemma 4.5.9 *For $1 \le i \le n-1$, we have:*

$$\begin{aligned} \sigma_i\alpha_j &= \alpha_j\sigma_i \qquad j \ne i, i+1 \\ \sigma_i\alpha_i &= \alpha_{i+1}\alpha_i^2\sigma_i^{-1} \\ \sigma_i\alpha_{i+1} &= \alpha_i\sigma_i \end{aligned}$$

while for the last generator σ_n the action is given by:

$$\begin{aligned} \sigma_n\alpha_i &= \alpha_i(\sigma_{i+1}\cdots\sigma_{n-1})^{-1}\sigma_i(\sigma_{i+1}\cdots\sigma_{n-1}) \qquad i \ne n, n+1 \\ \sigma_n\alpha_n &= \alpha_{n+1}\sigma_n^2 \\ \sigma_n\alpha_{n+1} &= \alpha_n \end{aligned}$$

□

Lemma 4.5.9 is best proven 'pictorially'. However it is not difficult (but lengthy) to prove these formulas by induction using just the braid relations.

Using lemma 4.5.9 and the definitions, one can write the induced representation in the following matrix form:

$$\sigma_i \mapsto \left(\begin{array}{c|cc|c} -qI_{i-1} & & & \\ \hline & 0 & -q & \\ & q^{-1}t & 0 & \\ \hline & & & -qI_{n-i} \end{array}\right)$$

for $1 \le i \le n-1$, whereas

$$\sigma_n \mapsto \left(\begin{array}{c|cc} -qI_{n-1} & & \\ \hline & 0 & 1 \\ & -t & 0 \end{array}\right)$$

Conjugating by $U = \mathrm{Diag}(1, 1, \ldots, 1, -q^{-1})$, one can bring the last 2×2 block for the σ_n-action in the same form of the 2×2 blocks of the other generators:

$$U \left(\begin{array}{c|cc} -qI_{n-1} & & \\ \hline & 0 & 1 \\ & -t & 0 \end{array}\right) U^{-1} = \left(\begin{array}{c|cc} -qI_{n-1} & & \\ \hline & 0 & -q \\ & q^{-1}t & 0 \end{array}\right)$$

Finally, setting $u = -q^{-2}t$, one obtains the desired result. $\square$

We have a similar result for the inclusion $G_{\tilde{C}_{n-1}} < G_{B_n}$. In this case the standard representation of G_{B_n} (definition 4.5.5) arises as induced representation of a one dimensional representation of $G_{\tilde{C}_{n-1}}$.

Let $N := \mathbb{Q}[s^{\pm1}, t^{\pm1}, q^{\pm1}]$. We indicate by $N_{s,q,t}$ the one-dimensional $G_{\tilde{C}_{n-1}}$-module where the action of the standard generators $\bar{\gamma}_1, \gamma_2, \ldots, \gamma_{n-1}, \bar{\gamma}_n$ is as follows: $\bar{\gamma}_1$ acts by $(-s)$-multiplication, $\bar{\gamma}_n$ acts by $(-t)$-multiplication and all other generators act by $(-q)$-multiplication. Consider as well the one dimensional G_{B_n} module $P_{q,t} = \mathbb{Q}[q^{\pm1}, t^{\pm1}]$, where the action is defined sending the first standard generators to $(-q)$-multiplication and the last to $(-t)$-multiplication.

Let also $\eta = \eta_{\mathbb{Q}} : G_{B_n} \to \mathrm{Gl}(U)$ be the rational standard representation for G_{B_n}.

Proposition 4.5.10 *We have*

$$\mathrm{Ind}_{G_{\tilde{C}_{n-1}}}^{G_{B_n}} N_{s,q,t} \cong P_{q,t} \otimes U$$

Proof. We induce in two steps using $\mathrm{Br}_{n+1}^{(1,n+1)}$ as pivot.
Since $\mathrm{Br}_{n+1}^{(1,n+1)} \cong G_{\tilde{C}_{n-1}} \times \langle \tau^n \rangle$,

$$\mathrm{Ind}_{G_{\tilde{C}_{n-1}}}^{\mathrm{Br}_{n+1}^{(1,n+1)}} N_{s,q,t} \cong N_{s,q,t}[r^{\pm1}]$$

where the action of τ^n is given by $(-r)$-multiplication.

Now, we may choose as coset representatives for $G_{B_n}/\mathrm{Br}_{n+1}^{(1,n+1)}$ the elements $\tau^0, \tau^{-1}, \ldots, \tau^{-n+1}$. Using the definition of induced representation and noting that

$$\begin{aligned} \epsilon_1 &= \tau^{-1}(\bar{\epsilon}_n \epsilon_{n-1} \cdots \epsilon_2 \epsilon_1^2) \in \tau^{-1}\mathrm{Br}_{n+1}^{(1,n+1)} \\ \epsilon_n &= \tau^{-n+1}(\tau^n \epsilon_2^{-1} \cdots \bar{\epsilon}_n^{-1}) \in \tau^{-n+1}\mathrm{Br}_{n+1}^{(1,n+1)} \end{aligned}$$

it is easy to write the representation in matrix form:

$$\epsilon_i \mapsto \left(\begin{array}{c|cc|c} -qI_{i-1} & & & \\ \hline & 0 & (-t)^{-1}(-q)^{-(n-2)} & \\ & st(-q)^{n-2} & 0 & \\ \hline & & & -qI_{n-i} \end{array}\right)$$

for $1 \leq i \leq n-1$, whereas

$$\bar{\epsilon}_n \mapsto \left(\begin{array}{c|c} -tI_{n-1} & \\ \hline & (-r)^{-1}\big(s^{-1}t^{-1}(-q)^{-(n-2)}\big)^{n-1} \end{array}\right)$$

Set $\lambda = (-t)^{-1}(-q)^{-n+1}$. Conjugating by $u = \mathrm{diag}(1, \lambda, \lambda^2, \ldots, \lambda^{n-1})$, the matrix form for $\bar{\epsilon}_n$ is unchanged, while

$$\epsilon_i \mapsto \left(\begin{array}{c|cc|c} -qI_{i-1} & & & \\ \hline & 0 & -q & \\ & (-q)(-q^{-2}s) & 0 & \\ \hline & & & -qI_{n-i} \end{array}\right)$$

The result then follows setting $u = -q^{-2}s$, $v = -(-r)^{-1}s^{-n+1}t^{-n}(-q)^{-(n-2)(n-1)}$.
□

4.5.4 Linearity of affine Artin groups

We conclude with a remark connected to representation theory of Artin groups.

Recall that a group is said to be *linear* if it can be faithfully represented as a group of linear transformation of a vector space.

Krammer [Kra00] and Bigelow [Big01] have shown by different methods that a certain representation (called Lawrence-Krammer representation) of the braid group Br_n is faithful. Subsequently, this result has been extended to all Artin group of finite type [CW02]. Since a subgroup of a linear group is itself linear, as a corollary of theorems 4.2.5 and 4.3.1, we have:

Corollary 4.5.11 *The affine Artin groups $G_{\tilde{A}_n}$ and $G_{\tilde{C}_n}$ are linear.*
□

Chapter 5

Cohomology of G_{B_n} and applications

In this chapter we determine the cohomology of the Artin group G_{B_n} with values in two local systems, namely the double-weight local system $R_{q,t} = \mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$ (section 5.1), in which the action is by $(-q)$-multiplication for the standard generators in the first $n-1$ nodes of the Dynkin diagram and $(-t)$-multiplication for the last, and in the local system obtained from $R_{q,t}$ specializing q to -1 (section 5.2). Finally we use a degree shift result to replace in the previuos cohomology group Laurent polynomials in t with Laurent series; using Shapiro lemma and the inclusion of chapter 4, we explore the cohomology of $G_{\tilde{A}_n}$ and derive the cohomology of G_{A_n} with coefficients in the Tong-Yang-Ma representation (section 5.3).

5.1 Cohomology of G_{B_n} with a double-weight local system

Let $R_{q,t}$ be the local system for G_{B_n} over the ring of Laurent polynomias $R = \mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$ where the action is $(-q)-$multiplication for the standard generators associated to the first $n-1$ nodes of the Dynkin diagram, while is $(-t)-$multiplication for the generator associated to the last node.

We write $\varphi_m(q)$ for the m-th cyclotomic polynomial in the variable q. In order to state our result we need to define the R-modules

$$\{m\}_i = R/(\varphi_m(q), q^i t + 1)$$

For $m = 1$ we set:

$$\{1\}_i = R/(q^i t + 1).$$

Note that the modules $\{m\}_i$ and $\{m'\}_{i'}$ are isomorphic as $\mathbb{Q}[q^{\pm 1}]$-modules if and only if $m = m'$ and are isomorphic as $\mathbb{Q}[t^{\pm 1}]$-modules if and only if $m = m'$ and $(m, i) = (m, i')$.

Our main result is the following

Theorem 5.1.1

$$H^i(G_{B_n}, R_{q,t}) = \begin{cases} \bigoplus_{d|n, 0\le k\le d-2}\{d\}_k \oplus \{1\}_{n-1} & \textit{if } i = n \\ \bigoplus_{d|n, 0\le k\le d-2, d\le \frac{n}{j+1}}\{d\}_k & \textit{if } i = n-2j \\ \bigoplus_{d\nmid n, d\le \frac{n}{j+1}}\{d\}_{n-1} & \textit{if } i = n-2j-1 \end{cases}$$

□

To perform our computation we will use the Salvetti complex and the spectral sequence induced by a natural filtration.

Recall that the complex that compute the cohomology of G_{B_n} over $R_{q,t}$ is given as follows (see also chapter 3):

$$C_n^* = \bigoplus_{\Gamma\subseteq I_n} R.\Gamma \tag{5.1}$$

where I_n denotes the set $\{1, \ldots, n\}$ and the graduation is given by $|\Gamma|$.

The set I_n correspond to the set of nodes of the Dynkin diagram of B_n and in particular the last element, n, correspond to the last node.

It is convenient to consider also the complex $\overline{C}_n^*$ for the cohomology of G_{A_n} over the local system $R_{q,t}$. In this case the action associated to a standard generator is always the $(-q)$-multiplication and thus the complex $\overline{C}_n$ and its cohomology are free as t-modules. The complex $\overline{C}_n^*$ is isomorphic to C_n^* as a module.

In both complexes the coboundary map is

$$\delta(q,t)(\Gamma) = \sum_{j\in I_n\setminus\Gamma} (-1)^{\sigma(j,\Gamma)} \frac{W_{\Gamma\cup\{j\}}(q,t)}{W_\Gamma(q,t)}(\Gamma\cup\{j\}) \tag{5.2}$$

where $\sigma(j,\Gamma)$ is the number of elements of Γ that are less than j. In the case A_n, the polynomial $W_\Gamma(q,t)$ is the Poincaré series of the parabolic subgroup $W_\Gamma \subseteq A_n$ generated by the elements in the set Γ, with weight q for each standard generator, while in the case B_n it is the Poincaré polynomial of the parabolic subgroup $W_\Gamma \subseteq B_n$ generated by the elements in the set Γ, with weight q for the first $n-1$ generators and t for the last generator.

Using Proposition 2.4.3 we can give an explicit computation of the coefficients $\frac{W_{\Gamma\cup\{j\}}(q,t)}{W_\Gamma(q,t)}$. We will use as well the notation introduced in section 2.4.1.

Recall that if Γ correspond to a connected component of the Dynkin diagram of B_n without the last element, then

$$W_\Gamma(q,t) = [m+1]_q!,$$

where $m =\mid \Gamma \mid$. If Γ correspond to a connected component containing the last element, then

$$W_\Gamma(q,t) = [m]_{q,t}!!,$$

where $m =\mid \Gamma \mid$.

If Γ is the union of several connected components of the Dynkin diagram, $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_k$, then $W_\Gamma(q,t)$ is the product

$$\prod_{i=1}^{k} W_{\Gamma_i}(q,t)$$

of the factors corresponding to the different components.

If $j \notin \Gamma$ we can write $\Gamma(j)$ for the connected component of $\Gamma \cup \{j\}$ containing j. Suppose that $m =\mid \Gamma(j) \mid$ and i is the number of element in $\Gamma(j)$ greater than j. Then, if $n \in \Gamma(j)$ we have

$$\frac{W_{\Gamma\cup\{j\}}(q,t)}{W_\Gamma(q,t)} = \begin{bmatrix} m \\ i \end{bmatrix}'_{q,t}$$

and

$$\frac{W_{\Gamma\cup\{j\}}(q,t)}{W_\Gamma(q,t)} = \begin{bmatrix} m+1 \\ i+1 \end{bmatrix}_{q}$$

otherwise.

We can represent a generator Γ by a binary string A of n elements, namely we set the i-th element to be 1 if i belongs to Γ and 0 otherwise. For example the set $\Gamma = \{2,3,5,7,8\} \subseteq I_8$ is associated to the string 01101011.

We introduce a decreasing filtration F on the complex (C_n^*, δ); by definition F^sC_n is the subcomplex generated by the strings of type $A1^s$. We have thus the chain of inclusions

$$C_n = F^0C_n \supset F^1C_n \supset \cdots \supset F^nC_n = R.1^n \supset F^{n+1}C_n = 0.$$

Note that we have the following isomorphism of complexes:

$$(F^sC_n/F^{s+1}C_n) \cong \overline{C}_{n-s-1}[s] \tag{5.3}$$

where $\overline{C}_{n-s-1}$ is the complex for $G_{A_{n-s-1}}$ and, as customary, for a complex D^*, we denote with $D^*[s] := D^{*-s}$ the complex with graduation shifted by s. By means of the filtration F we can construct a spectral sequence E_* to compute the cohomology of the complex (C_n^*, δ). The equality 5.3 tells us how the columns of the E_0 term of the spectral sequence looks like. In fact for $0 \leq s \leq n-2$ we have

$$E_0^{s,r} = \frac{F^sC_n^{(s+r)}}{F^{s+1}C_n^{(s+r)}} = \overline{C}_{n-s-1}^{(s+r)}[s] = \overline{C}_{n-s-1}^{(r)} \tag{5.4}$$

So the s-th column computes the cohomology of $G_{A_{n-s-1}}$.
For $s = n-1$ and $s = n$ the only non trivial elements in the spectral sequence are

$$E_0^{n-1,0} = E_0^{n,0} = R. \tag{5.5}$$

It is then clear that the E_1 term of the spectral sequence is given for $0 \leq s \leq n-2$ by

$$E_1^{s,r} = H^r(G_{A_{n-s-1}}, R_{q,t}) = H^r(G_{A_{n-s-1}}, \mathbb{Q}[q^{\pm 1}]_q)[t^{\pm 1}] \tag{5.6}$$

since the t-action is trivial. For $s = n-1$ and $s = n$ the only non trivial elements instead are

$$E_1^{n-1,0} = E_1^{n,0} = R. \tag{5.7}$$

In order to prove Theorem 5.1.1 we need to state the following lemmas.

Lemma 5.1.2 *Let $I(n,k)$ be the ideal generated by the polynomials $\begin{bmatrix} n \\ n-d \end{bmatrix}'_{q,t}$ for $d \mid n$ and $d \leq k$. If $k \mid n$ the map*

$$\alpha_{n,k} : R/(\varphi_k(q)) \to R/I(n,k-1)$$

induced by the multiplication by $\begin{bmatrix} n \\ n-k \end{bmatrix}'_{q,t}$ is well defined and is injective.

Remark. The fact that this map is well defined will follow automatically from the general theory of spectral sequences, as it is clear from the proof of Theorem 5.1.1. However, below we prove it by other means.

Proof. Let d, k be positive integers such that $d \mid n$ and $k \mid n$. We can observe that $\varphi_d(q) \mid \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$ if and only if $d \nmid k$. Moreover each factor φ_d appears in $\begin{bmatrix} n \\ k \end{bmatrix}_q$ at most with exponent 1.

Let $J(n,k)$ be the ideal generated by the polynomials $\begin{bmatrix} n \\ n-d \end{bmatrix}_q$ for $d \mid n$ and $d \leq k$. It is easy to see that we have the following inclusion:

$$\prod_{i=n-k}^{n-1} (1 + tq^i) J(n,k) \subseteq I(n,k).$$

Moreover $J(n,k)$ is a principal ideal and is generated by the product

$$p_{n,k}(q) = \prod_{d \mid n, k<d} \varphi_d(q).$$

It follows that $\begin{bmatrix} n \\ n-k \end{bmatrix}_q \varphi_k(q) \in J(n, k-1)$ and so $\begin{bmatrix} n \\ n-k \end{bmatrix}'_{q,t} \varphi_k(q) \in I(n, k-1)$. This proves that the map $\alpha_{n,k}$ is well defined.

Now we can note that the factor $\varphi_k(q)$ divides each generator of $I(n, k-1)$, but does not divide $\begin{bmatrix} n \\ n-k \end{bmatrix}'_{q,t}$. This imply that $\alpha_{n,k}$ is not the zero map and that every polynomial in the kernel of $\alpha_{n,k}$ must be a multiple of $\varphi_k(q)$, hence the map must be injective. □

Lemma 5.1.3 *Let $I(n)$ be the ideal generated by the polynomials $\begin{bmatrix} n \\ n-d \end{bmatrix}'_{q,t}$ for $d \mid n$. Then $I(n)$ is the direct product of the ideals $I_{i,d} = (\varphi_d(q), q^i t + 1)$ for $d \mid n$ and $0 \leq i \leq d-2$ and of the ideal $I_{n-1} = (q^{n-1}t + 1)$. Moreover the ideals $I_{i,d}$ and I_{n-1} are pairwise coprime.*

Proof: Note that the polynomial $(1 + tq^{n-1})$ divides each generator of the ideal $I(n)$, so we can write

$$I(n) = (1 + tq^{n-1})\widetilde{I}(n)$$

where $\widetilde{I}(n)$ is the ideal generated by the polynomials

$$\widetilde{\begin{bmatrix} n \\ n-d \end{bmatrix}}'_{q,t} := \begin{bmatrix} n \\ n-d \end{bmatrix}'_{q,t} / (1 + tq^{n-1})$$

Let $n = d_1 > \cdots > d_h = 1$ be the list of all the divisors of n in decreasing order. If we set

$$\begin{aligned} P_i :=& \varphi_{d_i}(q) \text{ and} \\ Q_i :=& \prod_{j=d_{i+1}+1}^{d_i} (1 + tq^{n-j}) \end{aligned}$$

we can rewrite our ideal as

$$\begin{aligned} \widetilde{I}(n) = \Bigg(&\begin{bmatrix} n \\ n-d_h \end{bmatrix}, \begin{bmatrix} n \\ n-d_{h-1} \end{bmatrix} Q_{h-1}, \begin{bmatrix} n \\ n-d_{h-2} \end{bmatrix} Q_{h-2}Q_{h-1}, \ldots \\ &\ldots, \begin{bmatrix} n \\ n-d_2 \end{bmatrix} Q_2 \cdots Q_{h-1}, Q_1 \cdots Q_{h-1} \Bigg) \end{aligned} \tag{5.8}$$

We claim that we can reduce to the following set of generators:

$$\begin{aligned} \widetilde{I}(n) = &(P_1 \cdots P_{h-1}, P_1 \cdots P_{h-2}Q_{h-1}, P_1 \cdots P_{h-3}Q_{h-2}Q_{h-1} \cdots \\ &\ldots, P_1 Q_2 \cdots Q_{h-1}, Q_1 \cdots Q_{h-1}) \end{aligned} \tag{5.9}$$

The first generator is the same in both equations and the j-th generator in equation (5.9) divides the corresponding generator in equation (5.8). Now suppose that a factor $\varphi_m(q)$ divides $\left[\begin{array}{c} n \\ n-d_j \end{array}\right]$ but does not divide $P_1 \cdots P_{j-1}$. We may distinguish two cases:

i) Suppose that $m \nmid n$. Then we can get rid of the factor $\varphi_m(q)$ in $\left[\begin{array}{c} n \\ n-d_j \end{array}\right]$ with an opportune combination with the polynomial $P_1 \cdots P_{h-1}$.

ii) Suppose $m \mid n$. Then $m = d_l$ for some $l > j$ and we can get rid of $\varphi_m(q)$ using a suitable combination with the polynomial $P_1 \cdots P_{l-1} Q_l \cdots Q_{h-1}$.

We may now proceed inductively. Supposing we have already reduced the first $j-1$ terms, we can reduce the j-th term of the ideal in equation (5.8) to the corresponding term in equation (5.9).

Now we observe that if J, I_1, I_2 are ideals and $I_1 + I_2 = (1)$, then $(J, I_1 I_2) = (J, I_1)(J, I_2)$. Since the polynomials P_i are all coprime, we can apply $h-2$ times this elementary fact to the ideal $\widetilde{I}(n)$. At the i-th step we set $I_1 = (P_i)$, $I_2 = (P_{i+1} \cdots P_{h-1}, P_{i+1} \cdots P_{h-2} Q_{h-1}, \ldots, Q_{i+1} \cdots Q_{h-1})$, $J = (Q_1 \cdots Q_{h-1})$. So we can factor $\widetilde{I}(n)$ as

$$(P_1, Q_1 \cdots Q_{h-1})(P_2 \cdots P_{h-1}, P_2 \cdots P_{h-2} Q_{h-1}, Q_2 \cdots Q_{h-1}) = \cdots$$

$$\cdots = (P_1, Q_1 \cdots Q_{h-1})(P_2, Q_2 \cdots Q_{h-1}) \cdots (P_{h-1}, Q_{h-1}).$$

Finally we can split $(P_s, Q_s \cdots Q_{h-1})$ as the product

$$(P_s, 1 + tq^{n-d_s}) \cdots (P_s, 1 + tq^{n-d_h-1}).$$

So we have reduced the ideal $I(n)$ in the product stated in the Lemma and it is easy to check that all the ideals of the splitting are coprime. □

Proof of Theorem 5.1.1 We can now prove our Theorem using the spectral sequence described in the equations 5.6 and 5.7. The E_1 term is completely known (see [DPS01]); we introduce the following notation for the generators of the spectral sequence, as in the original paper:

$$\begin{aligned}
w_h &= 01^{h-2}0 \\
z_h &= 1^{h-1}0 + (-1)^h 01^{h-1} \\
b_h &= 01^{h-2} \\
c_h &= 1^{h-1} \\
z_h(i) &= \sum_{j=0}^{i-1} (-1)^{hj} w_h^j z_h w_h^{i-j-1} \\
v_h(i) &= \sum_{j=0}^{i-2} (-1)^{hj} w_h^j z_h w_h^{i-j-2} b_h + (-1)^{h(i-1)} w_h^{i-1} c_h
\end{aligned}$$

We write $\{m\}[t^{\pm 1}]$ for the module $R/(\varphi_m(q))$. The E_1-term of the spectral sequence has a module $\{m\}[t^{\pm 1}]$ in position (s, r) if and only if one of the following condition is satisfied:

a) $m \mid n - s - 1$ and $r = n - s - 2\frac{n-s-1}{m}$;

b) $m \mid n - s$ and $r = n - s + 1 - 2(\frac{n-s}{m})$.

Moreover we have modules R in position $(n-1, 0)$ and $(n, 0)$.
For example in case $n = 6$ the E_1 term is given by:

$$\begin{array}{|ccccccc}
\{6\}[t^{\pm 1}] & & & & & & \\
\{5\}[t^{\pm 1}] & \{5\}[t^{\pm 1}] & & & & & \\
\{3\}[t^{\pm 1}] & \{4\}[t^{\pm 1}] & \{4\}[t^{\pm 1}] & & & & \\
0 & 0 & \{3\}[t^{\pm 1}] & \{3\}[t^{\pm 1}] & & & \\
\{2\}[t^{\pm 1}] & \{2\}[t^{\pm 1}] & \{2\}[t^{\pm 1}] & \{2\}[t^{\pm 1}] & \{2\}[t^{\pm 1}] & & \\
0 & 0 & 0 & 0 & 0 & R & R \\
\hline
\end{array}$$

We now look at the map $d_1 : E_1^{n-1,0} \to E_1^{n,0}$. Recall that $E_1^{n-1,0}$ is generated by the string 01^{n-1} and $E_1^{n,0}$ is generated by the string 1^n and so the map

$$d_1^{n-1,0} : E_1^{n-1,0} \to E_1^{n,0}$$

is given by the multiplication by $\left[\begin{array}{c} n \\ n-1 \end{array} \right]'_{q,t} = [n]_q(1 + tq^{n-1})$ and is injective. It turns out that $E_2^{n-1,0} = 0$ and $E_2^{n,0} = R/([n]_q(1+tq^{n-1}))$. Moreover all the following terms $E_j^{n,0}$ are quotient of $E_2^{n,0}$.

Note that every map between modules of kind $\{m\}[t^{\pm 1}]$ and $\{m'\}[t^{\pm 1}]$ must be zero if $m \neq m'$. So we can study our spectral sequence considering only maps between the same kind of modules.

First let us consider an integer m that does not divide n. We may suppose that $m \mid n + c$ with $1 \leq c < m$ and set $i = \frac{n+c}{m}$. The modules of type $\{m\}[t^{\pm 1}]$ are:

$$\begin{array}{ll}
E_1^{\lambda m - c - 1, n + c - \lambda(m-2) - 2i + 1} & \text{generated by: } z_m(i - \lambda)01^{\lambda m - c - 1} \\
E_1^{\lambda m - c, n + c - \lambda(m-2) - 2i + 1} & \text{generated by: } v_m(i - \lambda)01^{\lambda m - c}
\end{array}$$

for $\lambda = 1, \ldots, i - 1$.

Here is a picture for this case ($\boldsymbol{h} = \{m\}[t^{\pm 1}]$):

$$\begin{array}{llll}
\boldsymbol{h} \xrightarrow{d_1} \boldsymbol{h} & & & \\
& \cdots \xrightarrow{d_1} \cdots & & \\
& & \boldsymbol{h} \xrightarrow{d_1} \boldsymbol{h} & \\
& & & \searrow^{0} \\
& & & \qquad R/I
\end{array}$$

The map

$$d_1 : E_1^{\lambda m-c-1,n+c-\lambda(m-2)-2i+1} \to E_1^{\lambda m-c,n+c-\lambda(m-2)-2i+1}$$

is given by the multiplication by $\begin{bmatrix} \lambda m - c \\ \lambda m - c - 1 \end{bmatrix}'_{q,t} = [\lambda m - c]_q(1+tq^{\lambda m-c-1})$. Since $\varphi_m(q) \nmid [\lambda m - c]_q$ the map is injective and in the E_2-term we have:

$$\begin{aligned} E_2^{\lambda m-c-1,n+c-\lambda(m-2)-2i+1} &= 0 \\ E_2^{\lambda m-c,n+c-\lambda(m-2)-2i+1} &= \{m\}_{\lambda m-c-1} = \{m\}_{m-c-1} \end{aligned}$$

for $\lambda = 1, \ldots, i-1$.

The other map we have to consider is

$$d_m^{n-m,m-1} : E_m^{n-m,m-1} \to E_m^{n,0}.$$

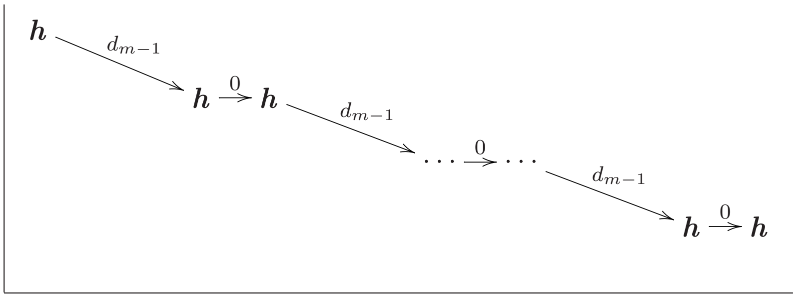
The module $E_m^{n-m,m-1} = \{m\}_{m-c-1}$ is generated by $1^{m-1}01^{n-m}$ and so the map is the multiplication by $\begin{bmatrix} n \\ n-m \end{bmatrix}'_{q,t}$. Since $(1 + tq^{n-1})$ divides the coefficient $\begin{bmatrix} n \\ n-m \end{bmatrix}'_{q,t}$, the image of the map must be contained in $(1 + tq^{n-1})E_m^{n,0}$ that is a quotient of $R/([n]_q)$, but $(\varphi_m(q), [n]_q) = (1)$ and so the map is null.

As a consequence the E_2 part described before collapses to E_∞ and we have a copy of $\{m\}_{m-c-1}$ as a direct summand of $H^{n-2j-1}(C_n)$ for $j = 0, \ldots, i-2$, that is for $m \leq \frac{n}{j+1}$.

Now we consider an integer m that divides n and let $i = \frac{n}{m}$. The modules of type $\{m\}[t^{\pm 1}]$ are:

$$\begin{array}{llll}
E_1^{\lambda m-1,n-\lambda(m-2)-2i+1} & \text{generated by:} & z_m(i-\lambda)01^{\lambda m-1} & \text{for } 1 \leq \lambda \leq i-1 \\
E_1^{\lambda m,n-\lambda(m-2)-2i+1} & \text{generated by:} & v_m(i-\lambda)01^{\lambda m} & \text{for } 0 \leq \lambda \leq i-1
\end{array}$$

The situation is shown in the next diagram ($\boldsymbol{h} = \{m\}[t^{\pm 1}]$):



The map

$$d_1 : E_1^{\lambda m-1, n-\lambda(m-2)-2i+1} \to E_1^{\lambda m, n-\lambda(m-2)-2i+1}$$

is given by the multiplication by $\left[\begin{array}{c} \lambda m \\ \lambda m - 1 \end{array}\right]'_{q,t} = [\lambda m]_q(1+tq^{\lambda m-1})$, but in this case the coefficient is zero in the module $\{m\}[t^{\pm 1}]$ because $\varphi_m(q) \mid [\lambda m]_q$ and so we have that $E_1 = \cdots = E_{m-1}$. So we have to consider the map

$$d_{m-1}^{\lambda m, n-\lambda(m-2)-2i+1} : E_{m-1}^{\lambda m, n-\lambda(m-2)-2i+1} \to E_1^{(\lambda+1)m-1, n-(\lambda+1)(m-2)-2i+1}$$

for $\lambda = 0, \ldots, i-2$.

This map corresponds to the multiplication by

$$\left[\begin{array}{c} (\lambda+1)m - 1 \\ \lambda m \end{array}\right]'_{q,t} = \left[\begin{array}{c} (\lambda+1)m - 1 \\ \lambda m \end{array}\right]_q \prod_{j=\lambda m+1}^{(\lambda+1)m-1} (1+tq^{j-1}).$$

It is easy to see that the polynomial $\left[\begin{array}{c} (\lambda+1)m - 1 \\ \lambda m \end{array}\right]_q$ is prime with the torsion $\varphi_m(q)$ and so the map $d_{m-1}^{\lambda m, n-\lambda(m-2)-2i+1}$ is injective and the cokernel is isomorphic to

$$R/(\varphi_m(q), \prod_{j=\lambda m+1}^{(\lambda+1)m-1} (1+tq^{j-1})) \simeq \bigoplus_{0 \le k \le m-2} \{m\}_k.$$

As a consequence we have that

$$E_m^{\lambda m-1, n-\lambda(m-2)-2i+1} = \bigoplus_{0 \le k \le m-2} \{m\}_k \qquad \text{for } 1 \le \lambda \le i-1$$

$$E_m^{\lambda m, n-\lambda(m-2)-2i+1} = 0 \qquad \text{for } 0 \le \lambda \le i-2.$$

and all these modules collapse to E_∞. This means that we can find $\varphi_m(q)$-torsion only in $H^{n-2j}(C_n)$ and for $j \ge 1$ the summand is given by

$$\bigoplus_{0 \le k \le m-2} \{m\}_k$$

for $d \leq \frac{n}{j+1}$.

We still have to consider all the terms $E_m^{n-m,m-1} = \{m\}[t^{\pm 1}]$ for $m \mid n$. Here the maps we have to look at are the following:

$$d_m^{n-m,m-1} : E_m^{n-m,m-1} \to E_m^{n,0}.$$

These maps correspond to the multiplication by the polynomials $\begin{bmatrix} n \\ n-m \end{bmatrix}'_{q,t}$.

Moreover recall that $E_1^{n,0} = R/(\begin{bmatrix} n \\ n-1 \end{bmatrix}'_{q,t})$. We can now use Lemma 5.1.2 to say that all the maps $d_m^{n-m,m-1}$ are injective and Lemma 5.1.3 to say that

$$E_{n+1}^{n,0} = E_\infty^{n,0} = \bigoplus_{m|n, 0\leq k \leq d-2} \{m\}_k \oplus \{1\}_{n-1}.$$

Since $E_\infty^{n,0} = H^n(C_n)$, this complete the proof of the Theorem. □

5.2 Cohomology of G_{B_n} with a single-weight local system

The cohomology of G_{B_n} over the module $\mathbb{Q}[t^{\pm 1}]$, where the action is trivial for generators corresponding to the first $n-1$ nodes in the Dynkin diagram and $(-t)-$multiplication for the last one, may be easily deduced by similar methods. Indeed it is computed by the complex C_n^* of section 5.1 where we specialize q to -1. We employ the analogous filtration and associated spectral sequence. Since the $\mathbb{Q}$-cohomology of the braid group is of rank 1 in dimension ≤ 1 and vanishes elsewhere, the setting now is much simpler. Using a formula analog to (5.6) we get

$$\begin{aligned} E_1^{s,r} &= \mathbb{Q}[t^{\pm 1}] \quad \text{if} \quad 0 \leq s \leq n,\ r = 0 \quad \text{or} \quad 0 \leq s \leq n-2,\ r = 1 \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{5.10}$$

Next from formula (5.2) it follows

$$d_1^{s,r} = \{[s+1]_q\ (1 + q^s t)\}_{\{q=-1\}},\ r = 0, 1$$

Therefore $d_1^{s,r} = 0$ for s odd while $d_1^{s,r} = 1 + t$ for s even. It follows that in E_2 the odd columns are obtained from the same columns of E_1 quotiented by $1+t$. The even columns vanish everywhere for n odd. If n is even, instead we have non-trivial modules only in position $(n-2, 1)$ and $(n, 0)$; these two non vanishing modules are given by:

$$E_2^{n-2,1} = E_2^{n,0} = \mathbb{Q}[t^{\pm 1}].$$

It is easily seen that the only possible non vanishing boundaries are of the form:

$$d_2^{s,1}: \; E_2^{s,1} \;\to\; E_2^{s+2,0}$$

and these corresponds to multiplication by

$$\left[\begin{array}{c} s+2 \\ s \end{array}\right]'_{[q=-1],t}.$$

Up to an invertible, the latter holds $(1+t)(1-t)$. Then d_2 vanishes except for $d_2^{n-2,1}$ in case n even. Collecting these remarks, we have:

Theorem 5.2.1

$$\begin{array}{rcll} H^k(G_{B_n};\mathbb{Q}[t^{\pm 1}]) & = & \mathbb{Q}[t^{\pm 1}]/(1+t) & 1 \le k \le n-1 \\ H^n(G_{B_n};\mathbb{Q}[t^{\pm 1}]) & = & \mathbb{Q}[t^{\pm 1}]/(1+t) & \textit{for odd } n \\ H^n(G_{B_n};\mathbb{Q}[t^{\pm 1}]) & = & \mathbb{Q}[t^{\pm 1}]/(1-t^2) & \textit{for even } n \end{array}$$

□

5.3 Shapiro's Lemma, degree shift and consequences

The previous inclusions of Artin groups allow to relate the homology of the involved groups by means of Shapiro's lemma (see for instance [Bro82]), of which we explore some consequences.

5.3.1 Shapiro's lemma

Shapiro's lemma allows to relate the homology of a subgroup $H < G$ with some coefficient module M to that of G with coefficients in the induced module $\mathrm{Ind}_H^G(M) := \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M$; more precisely:

$$H_*(H;M) \cong H_*(G;\mathrm{Ind}_H^G(M))$$

In cohomology an analogous statement holds, indeed:

$$H^*(H;M) \cong H^*(G;\mathrm{CoInd}_H^G(M))$$

where $\mathrm{CoInd}_H^G(M) := \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M)$ denotes the coinduced representation.

Recall that the induced and coinduced representations are equivalent whenever H is a *finite* index subgroup of G.

5.3.2 Cohomology of $G_{\tilde{A}_{n-1}}$

We have now the ingredients to determine the cohomology of $G_{\tilde{A}_{n-1}}$ over the local system $M_q := \mathbb{Q}[q^{\pm 1}]$, where the action of the standard generators is $(-q)$-multiplication.

Considering the inclusion $G_{\tilde{A}_{n-1}} < G_{B_n}$ and applying proposition 4.5.7, we have by Shapiro's lemma:

Proposition 5.3.1

$$H^*(G_{\tilde{A}_{n-1}}; M_q) \cong H^*(G_{B_n}; M[[t^{\pm 1}]]_{q,t}) \qquad \square$$

Notice that, in the previous section, we have determined $H^*(G_{B_n}; M[t^{\pm 1}]_{q,t})$, i.e. we have considered (for computational convenience), coefficient in the ring of Laurent polynomials in t; to determine $H^*(G_{\tilde{A}_{n-1}}; M_q)$ we now need instead coefficient in the Laurent series in t. The following proposition is useful in the present situation:

Proposition 5.3.2 (Degree shift)

$$H^*(G_{B_n}; M[[t^{\pm 1}]]_{q,t}) \cong H^{*+1}(G_{B_n}; M[t^{\pm 1}]_{q,t}).$$

Proof. Consider the following decreasing filtration G on the complex (C_n^*, δ) described in eq. 5.1; by definition $G^s C_n$ is the subcomplex generated by the strings of type $1^s A$ (thus this is the 'left' version of the filtration F used in the proof of theorem 5.1.1). We will prove by induction on n that the complex (C_n^*, δ) is well filtered with respect G in the sense of [Cal05b], Definition 1.2. Let, consistently with the previous sections, $R = M[t^{\pm 1}] = \mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$.

The basis of the induction $n = 0$ is trivially verified. So let $n > 0$. We check one by one the conditions i)-iv) of [Cal05b]:

i) Clearly, the filtration is compatible with the boundary maps and $G^0 C_n = C_n^*$, $G^{n+1} C_n = 0$.

ii) $G^n C_n = (G^n C_n)^n \cong G^{n-1} C_n / G^n C_n = (G^{n-1} C_n / G^n C_n)^n \cong R$. This is straightforward too.

iii) Identify, through the isomorphisms in point ii), the induced differential $d_1 : G^{n-1} C_n / G^n C_n \to G^n C_n / G^{n+1} C_n$ with a multiplication map $R \ni x \mapsto p \cdot x \in R$. Write $p = \sum_{a \le i \le b} c_i t^i$, where $c_i \in M$ and a, b are some integer numbers. Then it is required that c_a, c_b are invertible of M. In the present situation, we have by formula 5.2 that $p = \prod_{i=0}^{n-1} (1 + q^i t)$ and so the condition clearly holds.

iv) Finally we should check that for $k \neq n-1, n$, the induced complex $G^k C_n / G^{k+1} C_n$ is also well filtered. But now note that $G^k C_n / G^{k+1} C_n$ is just the complex $C_{n-k-1}^*[k]$, where C_{n-k-1}^* is the complex for $G_{B_{n-k-1}}$. By induction, this is well filtered.

Since (C_n^*, δ) is well filtered, we may then apply theorem 1 in [Cal05b] to get the desired result. □

Collecting theorem 5.1.1, proposition 5.3.1 and 5.3.2, we have:

Theorem 5.3.3

$$H^i(G_{\tilde{A}_n}; M_q) = \begin{cases} \bigoplus_{d|n+1} \left(\frac{\mathbb{Q}[q^{\pm 1}]}{\varphi_d}\right)^{(d-1)} \oplus \mathbb{Q}[q^{\pm 1}] & \textit{if } i = n \\ \bigoplus_{d|n+1,\ d \leq \frac{n+1}{j+1}} \left(\frac{\mathbb{Q}[q^{\pm 1}]}{\varphi_d}\right)^{(d-1)} & \textit{if } i = n-2j \\ \bigoplus_{d \nmid n+1,\ d \leq \frac{n+1}{j+1}} \frac{\mathbb{Q}[q^{\pm 1}]}{\varphi_d} & \textit{if } i = n-2j-1 \end{cases}$$

□

We may rephrase the same considerations for the trivial representation of $G_{\tilde{A}_n}$. Note how proposition 5.3.1 translates in this setting:

Proposition 5.3.4 *We have*

$$H_*(G_{\tilde{A}_{n-1}}; \mathbb{Q}) \cong H_*(G_{B_n}, \mathbb{Q}[t^{\pm 1}])$$

$$H^*(G_{\tilde{A}_{n-1}}; \mathbb{Q}) \cong H^*(G_{B_n}, \mathbb{Q}[[t^{\pm 1}]])$$

where the action of G_{B_n} on $\mathbb{Q}[t^{\pm 1}]$ (and on $R[[t^{\pm 1}]]$) is trivial for the generators $\epsilon_1, \ldots, \epsilon_{n-1}$ and t-multiplication for the last generator $\bar{\epsilon}_n$. □

Further degree shift continues to hold true:

$$H^*(G_{B_n}; \mathbb{Q}[[t^{\pm 1}]]) \cong H^{*+1}(G_{B_n}; \mathbb{Q}[t^{\pm 1}])$$

Indeed, after having specialized q to -1, we may repeat word by word the proof of proposition 5.3.2.

Using now theorem 5.2.1, we have:

Corollary 5.3.5 (Rational cohomology of $G_{\tilde{A}_n}$)

$$\begin{array}{rcll} H^k(G_{\tilde{A}_n}; \mathbb{Q}) & = & \mathbb{Q} & 1 \leq k \leq n-1 \\ H^n(G_{\tilde{A}_n}; \mathbb{Q}[t^{\pm 1}]) & = & \mathbb{Q} & \textit{for even } n \\ H^n(G_{\tilde{A}_n}; \mathbb{Q}[t^{\pm 1}]) & = & \mathbb{Q}^2 & \textit{for odd } n \end{array}$$

□

5.3.3 Cohomology of G_{A_n} with coefficient in the Tong-Yang-Ma representation

Recall that G_{B_n} may be identified with a subgroup of G_{A_n} of index $n+1$. By the use of Shapiro's lemma we may thus relate cohomology of the two groups. Here we just mention the following fact. Provide $\mathbb{Q}[t^{\pm 1}]$ with a G_{B_n}-module structure, where as usual the standard generators corresponding to the first $n-1$ nodes in the Dynkin diagram act trivially whereas the last acts by $(-t)$-multiplication.

Proposition 5.3.6 *We have the following isomorphism of cohomology groups:*

$$H^*(G_{B_n}; \mathbb{Q}[t^{\pm 1}]) \cong H^*(G_{A_n}; V)$$

where V is the Tong-Yang-M representation of G_{A_n} defined in 4.5.1. In particular the cohomology

$$H^*(G_{A_n};\ V)$$

is given as in theorem 5.2.1.

Proof. The proposition follows from proposition 4.5.8 and the fact that coinduced representation is equivalent to the induced one for finite index subgroup. □

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