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UNIVERSITÀ DEGLI STUDI DI ROMA “LA SAPIENZA”

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DILATION THEORY FOR C*-DYNAMICAL SYSTEMS

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Contents

1. Introduction

In the operator framework of quantum mechanics we define a dynamical system by the triple $(\mathfrak{A}, \Phi, \varphi)$, where $\mathfrak{A}$ is a C^*-algebra, Φ is an unital completely positive map and φ is a state on $\mathfrak{A}$. In particular, if this map Φ is a *-automorphism, $(\mathfrak{A}, \Phi, \varphi)$ is said be a conservative dynamical system.
The dilation problem for dynamical system $(\mathfrak{A}, \Phi, \varphi)$ is related with question wheter it is possible to interpret an irrevesible evolution of a physical system as the projection of a unitary reversible evolution of a larger system $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}\right)$ [**9**].
In [**26**] we find a good description of what we intend for dilation of a dynamical system: *The idea of dilation is to understand the dynamics Φ of $\mathfrak{A}$ as projection from the dynamics $\widehat{\Phi}$ of $\widehat{\mathfrak{A}}$. In statistical physic the algebras $\mathfrak{A}$ and $\widehat{\mathfrak{A}}$ may be considered as algebras of quantum mechanical observable so that $\mathfrak{A}$ models the description of a small system embedded into a big one modelled by $\widehat{\mathfrak{A}}$. In the classical example $\mathfrak{A}$ is the algebra of random variables describing a brownian particle moving on a liquid in thermal equilibrium and $\widehat{\mathfrak{A}}$ is the algebra of random variables describing both the molecules of the liquid and particle.*
Many authors in the last years have studied the dilatative problem, we cite the pioneer works of Arveson [**1**], Evans and Lewis [**7**], [**8**], and Vincent-Smith [**31**]. In absence of an invariant faithful state, Arveson, Evans and Lewis have verified that the dilations have been constructed for every completely positive map defined on W^*-algebra, while Vincent-Smith using a particular definition of dilation, shows that every W^*-dynamical system admits a reversible dilation. In our work we will assume the concept of dilation given by Kümmerer and Maassen in [**12**] and [**13**]. It is our opinion that this definition is that that describes better the physical processes.
The statement of the problem is the following:
Given a dynamical system $(\mathfrak{A}, \Phi, \varphi)$, to construct a conservative dynamical system $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}\right)$ containing it in the following sense. there is an injective linear *-multiplicative map $i : \mathfrak{A} \to \widehat{\mathfrak{A}}$ and a projection $\mathcal{E}$ of norm one of $\widehat{\mathfrak{A}}$ onto $i(\mathfrak{A})$ such that the diagram

$$\begin{array}{ccccc}
\widehat{\mathfrak{A}} & & \xrightarrow{\widehat{\Phi}^n} & & \widehat{\mathfrak{A}} \\
& \overset{\widehat{\varphi}}{\searrow} & & \overset{\widehat{\varphi}}{\swarrow} & \\
i \uparrow & & \mathbb{C} & & \downarrow \mathcal{E} \\
& \overset{\varphi}{\nearrow} & & \overset{\varphi}{\nwarrow} & \\
\mathfrak{A} & & \xrightarrow{\Phi^n} & & \widehat{\mathfrak{A}}
\end{array}$$

commutes for each $n \in \mathbb{N}$.
The$\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}, i, \mathcal{E}\right)$ is said to be a reversible dilation of the dynamical system $(\mathfrak{A}, \Phi, \varphi)$, furthermore an dilation is unital if the injective map $i : \mathfrak{A} \to \widehat{\mathfrak{A}}$ is unital.
Kummer in [**12**] estabilishes that the existence of a reversible dilation depends on the existence of adjoint map in this sense:
A completely positive map $\Phi^+ : \mathfrak{A} \to \mathfrak{A}$ is a $\varphi-$adjoint of the completely positive map

Φ if for each a, b belongs to $\mathfrak{A}$ we obtain that $\varphi(b(\Phi(a))) = \varphi(\Phi^{+}(b)a)$.
The principal purpose of our work is to establish under which condition is possible to costruct a reversible dilation that keeps the ergodic and weakly mixing properties of the original dynamical system. An found difficulty has been that to determine the existence of the expectation conditioned as described in the preceding scheme (In fact generally, the exisistence of a conditional expectation between C*-algebras is fairly exceptional [1].) and the presence of an invariant state subsequently complicates the matters.
This thesis is organized as follow.
In chapter 1 we introduce some preliminaries concept and we show the following generalization of the theorem of Stinespring:
Gives an unital completely positive map $\Phi : \mathfrak{A} \to \mathfrak{A}$ on C*-algebra with unit $\mathfrak{A}$, there is a representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$ and an isometry $\mathbf{V}$ on the Hilbert space $\mathcal{H}$ such that $\pi(\Phi(a)) = \mathbf{V}\pi(a)\mathbf{V}^{*}$ for each element a belong to $\mathfrak{A}$. Subsequently we have used results contained in the paper [**20**] to show that all W^{*}-dynamical systems for which the dinamic Φ is a *-homomorphism with $\varphi-$adjoint, admit an unital reversible dilation.
In chapter 2 using the generalized Stinespring theorem and Nagy-Foias dilation theory for the linear contraction on Hilbert space, we proof that every dynamical system $(\mathfrak{A}, \Phi, \varphi)$ has a multiplicative dilations $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}, i, \mathcal{E}\right)$, that is a dilation in which the dynamic $\widehat{\Phi} : \widehat{\mathfrak{A}} \to \widehat{\mathfrak{A}}$ is not a *-automorphism of algebras, but an injective *-homomorphism. This dilation keeps ergodic and weakly mixing properties of the original dynamic system. We also recover a results on the existence of dilation for W^{*}-dynamical systems determined by Muhly-Solel their paper [**16**]. We make to notice that our proof differs for the method and the approach to that of the two preceding authors. For the methodologies applied by the authors, and relative results, the reader can see the further jobs [**15**] and [**17**].
In chapter 3 we apply Hilbert module methods to show the existence of a particular dilations $\left(\widehat{\mathfrak{M}}, \widehat{\Phi}, \widehat{\varphi}, i, \mathcal{E}\right)$ of W*-dynamical system $(\mathfrak{M}, \Phi, \varphi)$ where the dynamic $\widehat{\Phi}$ is a completely positive map such that $\mathfrak{M}$ is included in the multiplicative domains $\mathcal{D}\left(\widehat{\Phi}\right)$ of $\widehat{\Phi}$. Also $\left(\widehat{\mathfrak{M}}, \widehat{\Phi}, \widehat{\varphi}, i, \mathcal{E}\right)$ keeps the ergodic and mixing properties of the C*-dynamical system $(\mathfrak{M}, \Phi, \varphi)$.

[1] For the existence of expectation conditioned the reader can see Takesaki [**29**] .

CHAPTER 1

Dynamical systems and their dilations

In this chapter using the results of Niculescu, Ströh and Zsido contained in their paper [**20**], we have show that a dynamical system with dynamics described by a homomorphism that admits adjoint as defined by Kummerer in [**12**], can be dilated to a minimal reversible dynamical system. Moreover this reversible system take the ergodic property of the original dynamical system. Fundamental ingredient of the proof is the the theory of the dilation of Nagy-Foias for the linear contractions on the Hilbert space

1. Preliminaries

In this first section, we shortly introduce some results on the completely positive maps[1]. For further details on the subject, the reader can see the Paulsen's books cited in the bibliography.
A self-adjoint subspace $\mathcal{S}$ of a C*-algebra $\mathfrak{A}$ that contains the unit of $\mathfrak{A}$ is called operator system of $\mathfrak{A}$, while a linear map $\Phi : \mathcal{S} \to \mathfrak{B}$ between the operator system $\mathcal{S}$ and the C*-algebra $\mathfrak{B}$ is positive if it maps positive elements of $\mathcal{S}$ in positive elements of $\mathfrak{B}$.
The set of all $n \times n$ matrices, with entries from $\mathcal{S}$, is denoted with $\mathbb{M}_n(\mathcal{S})$. We define a new linear map $\Phi_n : \mathbb{M}_n(\mathcal{S}) \to \mathbb{M}_n(\mathfrak{A})$ thus defined:

$$\Phi_n\left(|x_{i,j}|_{i,j}\right) = |\Phi(x_{i,j})|_{i,j}, \qquad x_{i,j} \in \mathcal{S},\ i,j = 1,2...n.$$

The linear map Φ is said be n-positive if the linear map Φ_n is positive and we call Φ completely positive if Φ is n-positive for all $n \in \mathbb{N}$.
We observe that if $\mathfrak{A}$ and $\mathfrak{B}$ are C*-algebra, a linear map $\Phi : \mathfrak{A} \to \mathfrak{B}$ is cp-map if and only if

$$\sum_{i,j} b_i^* \Phi(a_i^* a_j) b_j \geq 0$$

for each $a_1, a_2, ...a_n \in \mathfrak{A}$ and $b_1, b_2....b_n \in \mathfrak{B}$.

Proposition 1.1. *If $\Phi : \mathcal{S} \to \mathfrak{B}$ is a cp-map, then*

$$\|\Phi\| = \|\Phi(\mathbf{1})\|$$

Proof. See [**22**] proposition 3.5. □

If $\Phi : \mathfrak{A} \to \mathfrak{B}$ is an unital cp map between C*-algebras, we have that Φ has norm 1. A fundamental result in the theory of the cp-maps is given by the *extension theorem of Arveson* [**1**]*:*

Proposition 1.2. *Let $\mathcal{S}$ be an operator system of the C*-algebra $\mathfrak{A}$, and $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ a cp-map. Then there is a cp-map, $\Phi_{ar} : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$, extending Φ.*

[1] Briefly cp-map.

PROOF. See [**22**] proposition 6.5. ☐

Let us recall the fundamental definition of conditional expectation.
Let $\mathfrak{B}$ be a Banach algebra (in generally without unit) and let $\mathfrak{A}$ be a subalgebra of Banach of $\mathfrak{B}$. We recal that a projection $\mathcal{P}$ is a continuous linear map from $\mathfrak{B}$ onto $\mathfrak{A}$ satisfying $\mathcal{P}(a) = a$ for each $a \in \mathfrak{A}$, while a quasi-conditional expectation $\mathcal{Q}$ is a projection from $\mathfrak{B}$ onto $\mathfrak{A}$ satisfying $\mathcal{Q}(xby) = x\mathcal{Q}(b)\,y$ for each $x, y \in \mathfrak{A}$, and $b \in \mathfrak{B}$.
An conditional expectation is a quasi-conditional expectation of norm 1.
In the case that $\mathfrak{A}$ and $\mathfrak{B}$ are C*-algebras there is the following result of the 1957 of Tomiyama:

PROPOSITION 1.3. *The linear map $\mathcal{E} : \mathfrak{B} \to \mathfrak{A}$ is a conditional expectation if and only if is a projection of norm 1.*

PROOF. See [**2**], proposition 6.10. ☐

We observe that every conditional expectation is a cp-map.
In fact for each $a_1, a_2, ...a_n \in \mathfrak{A}$ and $b_1, b_2....b_n \in \mathfrak{B}$, we obtain:

$$\sum_{i,j} a_i^* \mathcal{E}\left(b_i^* b_j\right) a_j = \mathcal{E}\left(\sum_{i,j} a_i^* b_i^* b_j a_j\right) \geq 0.$$

The *multiplicative domains* of the cp map $\Phi : \mathfrak{A} \to \mathfrak{B}$ is the set

$$\mathcal{D}(\Phi) = \{a \in \mathfrak{A} : \Phi(a^*)\,\Phi(a) = \Phi(a^* a) \text{ and } \Phi(a)\,\Phi(a^*) = \Phi(aa^*)\}\,, \tag{1}$$

furthermore we have the following relation (cfr.[**22**]):

$$a \in \mathcal{D}(\Phi) \text{ if and only if } \Phi(a)\,\Phi(b) = \Phi(ab)\,,\ \Phi(b)\,\Phi(a) = \Phi(ba) \text{ for all } b \in \mathfrak{A}.$$

2. Stinespring Dilations for the cp map

We examine a concrete C*-algebra $\mathfrak{A}$ of $\mathcal{B}(\mathcal{H})$ with unit and an unital cp-map $\Phi : \mathfrak{A} \to \mathfrak{A}$. By the Stinespring theorem for the cp-map Φ, we can deduce a triple $(\mathbf{V}_\Phi, \sigma_\Phi, \mathcal{L}_\Phi)$ constituted by a Hilbert space $\mathcal{L}_\Phi$, of the reprensentation $\sigma_\Phi : \mathfrak{A} \to \mathcal{B}(\mathcal{L}_\Phi)$ and a linear contraction $\mathbf{V}_\Phi : \mathcal{H} \to \mathcal{L}_\Phi$ such that

$$\Phi(a) = \mathbf{V}_\Phi^* \sigma_\Phi(a)\,\mathbf{V}_\Phi, \qquad a \in \mathfrak{A}. \tag{2}$$

We recall to the reader[2] that the Hilbert space $\mathcal{L}_\Phi$ is the quotient space of $\mathfrak{A} \otimes_\Phi \mathcal{H}$ by the equivalence relation given by the linear space $\{a \otimes_\Phi \Psi : \|a \otimes \Psi\| = 0\}$, where

$$\langle a_1 \otimes_\Phi \Psi_1; a_2 \otimes_\Phi \Psi_2\rangle_{\mathcal{L}_\Phi} = \langle \Psi_1; \Phi(a_1^* a_2)\,\Psi_2\rangle_{\mathcal{H}}$$

and $\sigma_\Phi(a)\,x \otimes_\Phi \Psi = ax \otimes_\Phi \Psi$, for each $x \otimes_\Phi \Psi \in \mathcal{L}_\Phi$ with $\mathbf{V}_\Phi \Psi = \mathbf{1} \otimes_\Phi \Psi$ for each $\Psi \in \mathcal{H}$. Since Φ is unital map the linear operator $\mathbf{V}_\Phi$ is an isometry whit adjoint $\mathbf{V}_\Phi^*$ defined by

$$\mathbf{V}_\Phi^* a \otimes_\Phi \Psi = \Phi(a)\,\Psi,$$

for each $a \in \mathfrak{A}$ and $\Psi \in \mathcal{H}$.

[2]For further details cfr.[**22**] and [**23**].

PROPOSITION 1.4. *The unital cp-map Φ is a multiplicative if and only if $\mathbf{V}_\Phi$ is an unitary.*
Moreover for each $x \in \mathcal{D}(\Phi)$ we have

$$\sigma_\Phi(x)\,\mathbf{V}_\Phi\mathbf{V}_\Phi^* = \mathbf{V}_\Phi\mathbf{V}_\Phi^*\sigma_\Phi(x) = \sigma_\Phi(x)\,.$$

PROOF. For each $\Psi \in \mathcal{H}$ we obtain the follow implication:

$$a \otimes_\Phi \Psi = \mathbf{1} \otimes_\Phi \Phi(a)\,\Psi \quad \Longleftrightarrow \quad \Phi(a^*a) = \Phi(a^*)\,\Phi(a)\,,$$

since

$$\|a \otimes_\Phi \Psi - 1 \otimes_\Phi \Psi(a)\,\Psi\| = \langle \Psi, \Phi(a^*a)\,\Psi\rangle - \langle \Psi, \Phi(a^*)\,\Phi(a)\,\Psi\rangle\,.$$

Furthermore, for each $a \in \mathfrak{A}$ and $\Psi \in \mathcal{H}$ we have $\mathbf{V}_\Phi\mathbf{V}_\Phi^* a \otimes_\Phi \Psi = \mathbf{1} \otimes_\Phi \Phi(a)\,\Psi$. □

Let $\Phi : \mathfrak{A} \to \mathfrak{B}$ an unital cp map between C*-algebra $\mathfrak{A}$ and $\mathfrak{B}$, for each $a \in \mathfrak{A}$ we have:

$$\Phi(a^*a) = \mathbf{V}_\Phi^*\sigma_\Phi(a^*)\,\sigma_\Phi(a^*)\,\mathbf{V}_\Phi \geq \mathbf{V}_\Phi^*\sigma_\Phi(a^*)\,\mathbf{V}_\Phi\mathbf{V}_\Phi^*\sigma_\Phi(a^*)\,\mathbf{V}_\Phi = \Phi(a^*)\,\Phi(a)\,,$$

this shows that the *Kadison inequality:*

$$\Phi(a^*)\,\Phi(a) \leq \Phi(a^*a) \tag{3}$$

is satisfied.
We now need a simple lemma:

LEMMA 1.1. *Let $\mathfrak{M}_i \subset B(\mathcal{H}_i)$ with $i = 1, 2$, are von Neumann algebra and the linear positive map $\Phi : \mathfrak{M}_1 \to \mathfrak{M}_2$ is wo$-$ continuous, then is w^*-continuous.*

PROOF. Let $\{x_\alpha\}$ an increasing net in $\mathfrak{M}_1^+$ with least upper bound x, we have that x_α converges $\sigma-$continuous to x, it follow that x_α converges *wo*-continuous to x and since for hypothesis $\Phi(x_\alpha) \leq \Phi(x)$ in $\mathfrak{M}_2^+$ and $\Phi(x_\alpha) \to \Phi(x)$ in *wo*-continuous, we have $\Phi(x) = \operatorname{lub}\Phi(x_\alpha)$, then Φ is w^*-continuous. □

A simple consequence of the lemma is the following proposition:

PROPOSITION 1.5. *If $\mathfrak{M} \subset B(\mathcal{H})$ is a von Neumann algebra and $\Phi : \mathfrak{M} \to \mathfrak{M}$ is normal cp map, then the Stinespring representation $\sigma_\Phi : \mathfrak{M} \to \mathcal{B}(\mathcal{L}_\Phi)$ is normal.*

PROOF. Let $\{x_\alpha\}$ an increasing net in $\mathfrak{M}^+$ with least upper bound x, for each $a \otimes_\Phi \Psi \in \mathcal{L}_\Phi$ we obtain:
$\langle a \otimes_\Phi \Psi; \sigma_\Phi(x_\alpha)\,a \otimes_\Phi \Psi\rangle = \langle \Psi; \Phi(ax_\alpha a)\,\Psi\rangle \to \langle \Psi; \Phi(axa)\,\Psi\rangle$ and
$\langle \Psi; \Phi(axa)\,\Psi\rangle = \langle a \otimes_\Phi \Psi; \sigma_\Phi(x)\,a \otimes_\Phi \Psi\rangle\,.$
Therefore $\sigma_\Phi(x_\alpha) \to \sigma_\Phi(x)$ in *wo*-topology. □

The Stinespring theorem admit the following extension:

THEOREM 1.1. *Let $\mathfrak{A}$ be a C*-algebra with unit and $\Phi : \mathfrak{A} \to \mathfrak{A}$ an unital cp-map, then there exists a faithful representation $(\pi_\infty, \mathcal{H}_\infty)$ of $\mathfrak{A}$ and an isometry $\mathbf{V}_\infty$ on Hilbert Space $\mathcal{H}_\infty$ such that:*

$$\mathbf{V}_\infty^*\pi_\infty(a)\,\mathbf{V}_\infty = \pi_\infty(\Phi(a)) \qquad a \in \mathfrak{A}, \tag{4}$$

where

$$\sigma_0 = id, \qquad \Phi_n = \sigma_n \circ \Phi$$

and $(\mathbf{V}_n, \sigma_{n+1}, \mathcal{H}_{n+1})$ *is the Stinespring dilation of* Φ_n *for every* $n \geq 0$,

$$\mathcal{H}_\infty = \bigoplus_{j=0}^{\infty} \mathcal{H}_j, \qquad \mathcal{H}_j = \mathfrak{A} \otimes_{\Phi_{j-1}} \mathcal{H}_{j-1}, \quad \textit{for } j \geq 1 \textit{ and } \mathcal{H}_0 = \mathcal{H}; \tag{5}$$

and

$$\mathbf{V}_\infty(\Psi_0, \Psi_1, \Psi_2, ...) = (0, \mathbf{V}_0\Psi_0,\ \mathbf{V}_1\Psi_1, ...)$$

for each $(\Psi_0, \Psi_1, \Psi_2, ...) \in \mathcal{H}_\infty$.
Furthermore the map Φ *is a homomorphism if and only if* $\mathbf{V}_\infty\mathbf{V}_\infty^* \in \pi_\infty(\mathfrak{A})'$.

Proof. By the Stinespring theorem there is triple $(\mathbf{V}_0, \sigma_1, \mathcal{H}_1)$ such that for each $a \in \mathfrak{A}$ we have $\Phi(a) = \mathbf{V}_0^*\sigma_1(a)\mathbf{V}_0$. The application $a \in \mathfrak{A} \to \sigma_1(\Phi(a)) \in \mathcal{B}(\mathcal{H}_1)$ is composition of cp-maps therefore also it is cp map. Set $\Phi_1(a) = \sigma_1(\Phi(a))$. By appling the Stinespring theorem to Φ_1, we have a new triple $(\mathbf{V}_1, \sigma_2, \mathcal{H}_2)$ such that $\Phi_1(a) = \mathbf{V}_1^*\sigma_2(a)\mathbf{V}_1$. By induction for $n \geq 1$ define $\Phi_n(a) = \sigma_n(\Phi(a))$ we have a triple $(\mathbf{V}_n, \sigma_{n+1}, \mathcal{H}_{n+1})$ such that $\mathbf{V}_n : \mathcal{H}_n \to \mathcal{H}_{n+1}$ and $\Phi_n(a) = \mathbf{V}_n^*\sigma_{n+1}(a)\mathbf{V}_n$.
We get the Hilbert space $\mathcal{H}_\infty$ defined in 5 and the injective reppresentation of the C*-algebra $\mathfrak{A}$ on $\mathcal{H}_\infty$:

$$\pi_\infty(a) = \bigoplus_{n\geq 0} \sigma_n(a) \tag{6}$$

with $\sigma_0(a) = a$, for each $a \in \mathfrak{A}$.
Let $\mathbf{V}_\infty : \mathcal{H}_\infty \to \mathcal{H}_\infty$ be the isometry defined by

$$\mathbf{V}_\infty\ (\Psi_0, \Psi_1....\Psi_n...) = (0, \mathbf{V}_0\Psi_0, \mathbf{V}_1\Psi_1....\mathbf{V}_n\Psi_n...), \qquad \Psi_i \in \mathcal{H}_i. \tag{7}$$

The adjoint operator of $\mathbf{V}_\infty$ is

$$\mathbf{V}_\infty^*\ (\Psi_0, \Psi_1....\Psi_n...) = \left(\mathbf{V}_0^*\Psi_1, \mathbf{V}_1^*\Psi_2....\mathbf{V}_{n-1}^*\Psi_n...\right), \qquad \Psi_i \in \mathcal{H}_i, \tag{8}$$

therefore

$$\begin{aligned}\mathbf{V}_\infty^*\pi_\infty(a)\mathbf{V}_\infty \bigoplus_{n\geq 0}\Psi_n &= \bigoplus_{n\geq 0}\mathbf{V}_n^*\sigma_{n+1}(a)\mathbf{V}_n\Psi_n = \bigoplus_{n\geq 0}\Phi_n(a)\Psi_n = \\ &= \bigoplus_{n\geq 0}\sigma_n(\Phi(a))\Psi_n = \pi_\infty(\Phi(a))\bigoplus_{n\geq 0}\Psi_n.\end{aligned}$$

We notice that let $\mathbf{E}_n = \mathbf{V}_n\mathbf{V}_n^*$ be the orthogonal projection of $\mathcal{B}(\mathcal{H}_{n-1})$, we have:

$$\mathbf{E}(\Psi_0, \Psi_1...\Psi_n..) = (0, \mathbf{E}_0\Psi_1, \mathbf{E}_1\Psi_2, ...\mathbf{E}_n\Psi_{n+1}...).$$

Let Φ be a multiplicative map then for each $(\Psi_0, \Psi_1...\Psi_n.....) \in \mathcal{H}_\infty$ we get:

$$\mathbf{V}_\infty\mathbf{V}_\infty^*(\Psi_0, \Psi_1...\Psi_n..) = (0, \Psi_1, \Psi_2, ...\Psi_{n+1}...), \tag{9}$$

then

$$\mathbf{V}_\infty\mathbf{V}_\infty^*\pi_\infty(a) = \pi_\infty(a)\mathbf{V}_\infty\mathbf{V}_\infty^*,$$

while for the vice-versa for each $a, b \in \mathfrak{A}$ we obtain:

$$\begin{aligned}\pi_\infty(\Phi(a))\pi_\infty(\Phi(b)) &= \mathbf{V}_\infty^*\pi_\infty(a)\mathbf{V}_\infty\mathbf{V}_\infty^*\pi_\infty(b)\mathbf{V}_\infty = \mathbf{V}_\infty^*\pi_\infty(a)\pi_\infty(b)\mathbf{V}_\infty = \\ &= \mathbf{V}_\infty^*\pi_\infty(ab)\mathbf{V}_\infty = \pi_\infty(\Phi(ab)).\end{aligned}$$

□

Remark 1.1. *Let $\mathfrak{M}$ be a von Neumann algebra and Φ is normal, then the representation $(\pi_\infty, \mathcal{H}_\infty)$ of $\mathfrak{M}$ on $\mathcal{H}_\infty$ is normal, since the Stinespring representations $(\mathbf{V}_n, \sigma_{n+1}, \mathcal{H}_{n+1})$ of the cp-maps $\Phi_n = \mathfrak{M} \to \mathcal{B}(\mathcal{H}_n)$, are normal representations.*

We observe that $\mathbf{V}_\infty \notin \pi_\infty(\mathfrak{A})$ and $\mathbf{V}_\infty \mathbf{V}_\infty^* \notin \pi_\infty(\mathfrak{A})$.
Indeed if x is an element $x \in \mathfrak{A}$ such that $\pi_\infty(x) = \mathbf{V}_\infty$, we have for definition that for every $(\Psi_0, \Psi_1, ...\Psi_n...) \in \mathcal{H}_\infty$

$$(x\Psi_0, \sigma_1(x)\Psi_1, ...\sigma_n(x)\Psi_n...) = (0, \mathbf{V}_0\Psi_0, \mathbf{V}_1\Psi_1, ...\mathbf{V}_n\Psi_n...),$$

therefore $x = 0$.
If exists $a \in \mathfrak{A}$ such that $\mathbf{V}_\infty \mathbf{V}_\infty^* = \pi_\infty(a)$ then for each $(\Psi_0, \Psi_1...\Psi_n..) \in \mathcal{H}_\infty$ we have

$$\pi_\infty(a)(\Psi_0, \Psi_1, ...\Psi_n...) = (0, \mathbf{V}_0\mathbf{V}_0^*\Psi_0, \mathbf{V}_1\mathbf{V}_1^*\Psi_1, ...\mathbf{V}_n\mathbf{V}_n^*\Psi_n...)$$

it follows that $a = 0$.

Remark 1.2. *If x belong to multiplicative domains $\mathcal{D}(\Phi)$ we have*

$$\pi_\infty(x)\mathbf{V}_\infty\mathbf{V}_\infty^* = \mathbf{V}_\infty\mathbf{V}_\infty^*\pi_\infty(x) = \pi_\infty(x).$$

Moreover let $\mathbf{F} = \mathbf{I} - \mathbf{V}_\infty\mathbf{V}_\infty^*$, we have $\mathbf{F}\pi_\infty(\mathfrak{A})\mathbf{V} = 0$ if and only if the cp map Φ is multplicative. In fact for each $a, b \in \mathfrak{A}$ we get

$$(\mathbf{F}\pi_\infty(a)\mathbf{V})^* \mathbf{F}\pi_\infty(b)\mathbf{V} = \pi_\infty(\Phi(ab) - \Phi(a)\Phi(b)).$$

We study some simple property of the linear contraction $\mathbf{V}_\infty^*$.

Proposition 1.6. *The linear contraction $\mathbf{V}_\infty$ satisfies the relation*

$$\ker(I - \mathbf{V}_\infty) = \ker(I - \mathbf{V}_\infty^*) = 0.$$

Moreover for each $\Psi \in \mathcal{H}_\infty$, we have

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} \mathbf{V}_\infty^k \Psi = \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} \mathbf{V}_\infty^{k^*} \Psi = 0,$$

with

$$\lim_{n\to\infty} \left\langle \Psi, \mathbf{V}_\infty^k \Psi \right\rangle = 0.$$

Moreover for each $A \in \mathcal{B}(\mathcal{H}_\infty)$ we obtain:

$$\lim_{n\to\infty} \mathbf{V}_\infty^k A^* A \mathbf{V}_\infty^{k^*} \Psi = 0.$$

Proof. Let $(\Psi_0, \Psi_1, ...\Psi_n...) \in \mathcal{H}_\infty$ with $\mathbf{V}_\infty(\Psi_0, \Psi_1, ...\Psi_n...) = (\Psi_0, \Psi_1, ...\Psi_n...)$.
For definition

$$(0, V_0\Psi_0, V_1\Psi_1, \quad V_n\Psi_n...) = (\Psi_0, \Psi_1, ...\Psi_n...)$$

it follow that $(\Psi_0, \Psi_1, ...\Psi_n...) = (0, 0, ...0....)$.
It is well known that the relation $\ker(\mathbf{I} - \mathbf{V}_\infty) = \ker(\mathbf{I} - \mathbf{V}_\infty^*)$ is always true for linear contraction on the Hilbert spaces[3].
The relation $\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} \mathbf{V}_\infty^k \Psi = 0$ follow by the mean ergodic theory of von Neumann.
For the second relation we get:
$\mathbf{V}_\infty^k \mathbf{V}_\infty^{k^*} \Psi = \left(0, , 0......0, \mathbf{J}_{k-1,0}\mathbf{J}_{k-1,0}^* \Psi_k, \mathbf{J}_{k,1}\mathbf{J}_{k,1}^* \Psi_{k+1}, \mathbf{J}_{k+1,2}\mathbf{J}_{k+1,2}^* \Psi_{k+2} \cdots\right)$

[3]See [**19**] proposition 1.3.1.

where for each $h,k \in N$ with $h > k$ we set:
$\mathbf{J}_{k,h} = V_h V_{h+1} \circ\circ\circ V_k$.
$\left\|\mathbf{V}_\infty^k \mathbf{V}_\infty^{k*}\Psi\right\|^2 = \sum\limits_{\alpha=k}^{n} \left\|\mathbf{J}_{k-1+\alpha,k-\alpha}\mathbf{J}_{k-1+\alpha,k-\alpha}^*\Psi_\alpha\right\|^2 \leq \sum\limits_{\alpha=k}^{n} \|\Psi_\alpha\|^2$
since $\left\|\mathbf{J}_{k-1+\alpha,k-\alpha}\mathbf{J}_{k-1+\alpha,k-\alpha}^*\right\| \leq 1$
Then $\lim\limits_{n\to\infty} \sum\limits_{\alpha=k}^{n} \|\Psi_\alpha\|^2 = 0$ it follow that $\lim\limits_{n\to\infty} \left\|\mathbf{V}_\infty^k \mathbf{V}_\infty^{k*}\Psi\right\| = 0$.
Furthermore we get:
$\left\langle \Psi, \mathbf{V}_\infty^k A^* A \mathbf{V}_\infty^{k*}\Psi\right\rangle \leq \|A\|^2 \left\langle \Psi, \mathbf{V}_\infty^k \mathbf{V}_\infty^{k*}\Psi\right\rangle$.

Since $\frac{1}{n+1}\sum\limits_{k=0}^{n} \left\langle \Psi, \mathbf{V}_\infty^k\Psi\right\rangle \to 0$ we have $D - \lim\limits_{n\to\infty} \left\langle \Psi, \mathbf{V}_\infty^k\Psi\right\rangle = 0$[4] but we get
$\left|\left\langle \Psi, \mathbf{V}_\infty^k\Psi\right\rangle\right| = \sum\limits_{\alpha=k}^{n} \left|\left\langle \Psi_\alpha, \mathbf{J}_{k-1+\alpha,k-\alpha}\Psi_\alpha\right\rangle\right| \leq \sum\limits_{\alpha=k}^{n} \|\Psi_\alpha\|^2$
then $\lim\limits_{n\to\infty} \left\langle \Psi, \mathbf{V}_\infty^k\Psi\right\rangle = 0$. □

* * *

Proposition 1.1 leads to the following definition:

Definition 1.1. *Let $\Phi : \mathfrak{A} \to \mathfrak{A}$ be a cp-map, a triple $(\pi, \mathcal{H}, \mathbf{V})$ costitued by a faithful representation $\pi : \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ on the Hilbert space $\mathcal{H}$ and by a linear isometry $\mathbf{V}$, such that for each $a \in \mathfrak{A}$ we get:*

$$\pi(\Phi(a)) = \mathbf{V}^*\pi(a)\mathbf{V} \tag{10}$$

is a isometric covariant representation of the cp map Φ.

For our purposes it will be necessary to find an isometric covariant representation of appropriate dimensions, this is possible for the following theorem:

Proposition 1.7. *Let $\Phi : \mathfrak{A} \to \mathfrak{A}$ be cp-map with isometric representation $(\pi, \mathcal{H}, \mathbf{V})$, if Φ isn't an automorphism, for each cardinal number $\mathfrak{c}$ there exist an isometric covariant representation $(\pi_\mathfrak{c}, \mathcal{H}_\mathfrak{c}, \mathbf{V}_\mathfrak{c})$ with the following property:*
Representation π_∞ is an equivalent subrepresentation of $\pi_\mathfrak{c}$ with $\dim \mathcal{H}_\mathfrak{c} \geq \dim(\mathcal{H})$ and $\dim\ker(\mathbf{V}_\mathfrak{c}^*) \geq \mathfrak{c}$;
Moreover there is a cp map $\mathcal{E}_o : \mathcal{B}(\mathcal{H}_\mathfrak{c}) \to \mathcal{B}(\mathcal{H})$ such that for each $a \in \mathfrak{A}$, $T \in \mathcal{B}(\mathcal{H}_\mathfrak{c})$ we have

$$\mathcal{E}_o(\pi_\mathfrak{c}(a)T) = \pi(a)\mathcal{E}_o(T),$$

with

$$\mathcal{E}_o(\mathbf{V}_\mathfrak{c}^* T \mathbf{V}_\mathfrak{c}) = \mathbf{V}^*\mathcal{E}_o(T)\mathbf{V}; \tag{11}$$

Proof. Let $\mathfrak{c}$ be a cardinal number and $\mathcal{L}$ a Hilbert space with $\dim(\mathcal{L}) = \mathfrak{c}$, since Φ isn't automorphism we have $\dim(\ker \mathbf{V}^*) \geq 1$, then there is a vector $\xi_\natural \in \ker \mathbf{V}^*$ of one norm.
We set with $\mathcal{H}_\mathfrak{c}$ the Hilbert space $\mathcal{H}_\mathfrak{c} = \mathcal{H}\overline{\otimes}\mathcal{L}$ and with $\mathbf{V}_\mathfrak{c}$ the linear isometry

$$\mathbf{V}_\mathfrak{c} = \mathbf{V} \otimes \mathbf{I}_\mathcal{L}.$$

[4] Cfr. appendix.

Let $\{e_i\}_{i\in\mathcal{J}}$ be a orthonormal base of the Hilbert space $\mathcal{L}$, we have $card(\mathcal{J}) = \mathfrak{c}$ and

$$\xi_\natural \otimes e_j \in \ker \mathbf{V}_\mathfrak{c}^* \qquad j \in \mathcal{J}.$$

Since for each $j \in \mathcal{J}$ we obtain:

$$\mathbf{V}_\mathfrak{c}^* \left(\xi_\natural \otimes e_j\right) = \left(\mathbf{V}^* \otimes \mathbf{I}_\mathcal{L}\right)\left(\xi_\natural \otimes e_j\right) = \mathbf{V}^*\xi_\natural \otimes e_j = \mathbf{0} \otimes e_j = 0,$$

it follow that $\dim\left(\ker \mathbf{V}_\mathfrak{c}^*\right) \geq \mathfrak{c}$.
The faithfull *-representation $\pi_\mathfrak{c} : \mathfrak{A} \to \mathcal{B}\left(\mathcal{H}_\mathfrak{c}\right)$ defined by

$$\pi_\mathfrak{c}\left(a\right) = \pi\left(a\right) \otimes \mathbf{I}_\mathcal{L}, \qquad a \in \mathfrak{A}$$

satisfies the relation 10.
In fact for each $a \in \mathfrak{A}$ we obtain:

$$\mathbf{V}_\mathfrak{c}^*\pi_\mathfrak{c}\left(a\right)\mathbf{V}_\mathfrak{c} = \left(\mathbf{V}^* \otimes \mathbf{I}_\mathcal{L}\right)\left(\pi\left(a\right) \otimes \mathbf{I}_\mathcal{L}\right)\left(\mathbf{V} \otimes \mathbf{I}_\mathcal{L}\right) = \mathbf{V}^*\pi\left(a\right)\mathbf{V} \otimes \mathbf{I}_\mathcal{L} =$$
$$= \pi\left(\Phi\left(a\right)\right) \otimes \mathbf{I}_\mathcal{L} = \pi_\mathfrak{c}\left(\Phi\left(a\right)\right).$$

Let $l_o \in \mathcal{L}$ vector of one norm and $\Pi_{l_o} : \mathcal{H}_\mathfrak{c} \to \mathcal{H}$ the linear isometry

$$\Pi_{l_o}h = h \otimes l_o, \qquad h \in \mathcal{H},$$

with adjoint

$$\Pi_{l_o}^* h \otimes l = \left\langle l, l_o\right\rangle h, \qquad h \in \mathcal{H},\ l \in \mathcal{L}.$$

The cp map $\mathcal{E}_o : \mathcal{B}\left(\mathcal{H}_\mathfrak{c}\right) \to \mathcal{B}\left(\mathcal{H}\right)$ so defined:

$$\mathcal{E}_o\left(T\right) = \Pi_{l_o}^* T \Pi_{l_o}, \quad T \in \mathcal{B}\left(\mathcal{H}_\mathfrak{c}\right) \tag{12}$$

for each $a \in \mathfrak{A}$, $T \in \mathcal{B}\left(\mathcal{H}_\mathfrak{c}\right)$ enjoys of the following property:

$$\mathcal{E}_o\left(\pi_\mathfrak{c}\left(a\right)T\right) = \pi\left(a\right)\mathcal{E}_o\left(T\right).$$

In fact for each $h_1, h_2 \in \mathcal{H}_\infty$ we obtain
$\left\langle h_2, \mathcal{E}_o\left(\pi_\mathfrak{c}\left(a\right)T\right)h_1\right\rangle = \left\langle \pi_\mathfrak{c}\left(a^*\right)\Pi_{l_o}h_2, T\Pi_{l_o}h_1\right\rangle = \left\langle \pi\left(a^*\right)h_2 \otimes l_o, T\Pi_{l_o}h_1\right\rangle =$
$= \left\langle \pi\left(a^*\right)h_2, \Pi_{l_o}T\Pi_{l_o}h_1\right\rangle = \left\langle \pi\left(a^*\right)h_2, \mathcal{E}_o\left(T\right)h_1\right\rangle = \left\langle h_2, \pi\left(a\right)\mathcal{E}_o\left(T\right)h_1\right\rangle.$
We now verify the relation 11.
For each $h_1, h_2 \in \mathcal{H}$ we have:
$\left\langle h_2, \mathcal{E}_o\left(\mathbf{V}_\mathfrak{c}^* T \mathbf{V}_\mathfrak{c}\right)h_1\right\rangle = \left\langle \mathbf{V}_\mathfrak{c}\Pi_{l_o}h_2, T\mathbf{V}_\mathfrak{c}\Pi_{l_o}h_1\right\rangle = \left\langle \mathbf{V}h_2 \otimes l_o, T\mathbf{V}h_1 \otimes l_o\right\rangle =$
$= \left\langle \Pi_{l_o}\mathbf{V}h_2, T\Pi_{l_o}\mathbf{V}h_1\right\rangle = \left\langle \mathbf{V}h_2, \Pi_{l_o}^* T\Pi_{l_o}\mathbf{V}h_1\right\rangle = \left\langle \mathbf{V}h_2, \mathcal{E}_o\left(T\right)\mathbf{V}h_1\right\rangle =$
$= \left\langle h_2, \mathbf{V}^*\mathcal{E}_o\left(T\right)\mathbf{V}h_1\right\rangle.$ □

Lemma 1.2. *Let $\mathfrak{A}$ be an unit C^*-algebra and $\theta_o : \mathfrak{A} \to \mathfrak{B}\left(\mathcal{H}_o\right)$ representation of $\mathfrak{A}$, then for every infinite cardinal number $\mathfrak{c} \geq \dim\left(\mathcal{H}_o\right)$ there is a representation $\theta : \mathfrak{A} \to \mathfrak{B}\left(\mathcal{H}\right)$ such that*

$$\theta\left(a\right) = \bigoplus_{j\in J}\theta_o\left(a\right)$$

with

$$\mathcal{H} = \bigoplus_{j\in J}\mathcal{H}_o$$

and $card(J) = \mathfrak{c}$.

Proof. Let $\mathcal{H}$ be an any Hilbert space with $\dim(\mathcal{H}) = \mathfrak{c}$ with $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ orthonormal bases of $\mathcal{H}_o$ and of $\mathcal{L}$ respectively. For definition we have that $card\{J\} = \mathfrak{c}$ while $card\{I\} = \dim(\mathcal{H}_o)$.
The cardinal number $\mathfrak{c}$ isn't finte then for the notes rules of the cardinal arithmetic it results that $card\{I \times J\} = card\{J\}$. Then we can write that

$$J = \dot{\cup}\{I \times j : j \in J\} = \dot{\cup}\{I_j : j \in J\}$$

with $card(I_j) = \dim(\mathcal{H}_o)$.
In fact for every $j \in J$ the norm closure of the $span\{f_k : k \in I_j\}$ is isomorphic to the Hilbert space $\mathcal{H}_o$.
We get

$$\mathcal{H} = \bigoplus_{j\in J} \overline{span\{f_k : k \in I_j\}} = \bigoplus_{j\in J} \mathcal{H}_o,$$

and for each $a \in \mathfrak{A}$, $\Psi_j \in \mathcal{H}_o$ we define

$$\theta(a) \bigoplus_{j\in J} \Psi_j = \bigoplus_{j\in J} \theta_o(a) \Psi_j.$$

□

We now have a further generalization of the theorem 1.1:

Corollary 1.1. *Let $\Phi : \mathfrak{A} \to \mathfrak{A}$ be a cp-map. if Φ isn't an automorphism, there exists an isometric covariant representation $(\pi, \mathcal{H}, \mathbf{V})$ and a representation $\theta : \mathfrak{A} \to \mathfrak{B}(\ker(\mathbf{V}^*))$ such that*

$$\theta(a) = \bigoplus_{j\in J} \pi_\infty(a), \qquad a \in \mathfrak{A},$$

where J is a set of cardinalty

$$dim(\mathcal{H}) \geq card(J) \geq \dim(\mathcal{H}_\infty),$$

and $\mathcal{H}_\infty$ is the Hilbert space 5.

Proof. Let $\mathfrak{c}$ be the infinite cardinal number with $\mathfrak{c} \geq \dim \mathcal{H}_\infty$, for the proposition 1.7 there is an isometric covariant representation $(\pi_\mathfrak{c}, \mathcal{H}_\mathfrak{c}, \mathbf{V}_\mathfrak{c})$ subequivalent to π_∞ with $\dim(\ker \mathbf{V}_\mathfrak{c}) \geq \mathfrak{c}$. Then for the preceding lemma there is a *-representation $\theta = \bigoplus_{j\in J} \pi_\infty$ with $card(J) = \dim(\ker \mathbf{V}_\mathfrak{c})$. □

3. Nagy-Foiaş Dilations Theory

Let $\mathbf{T}$ and $\mathbf{S}$ be operators on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. We call $\mathbf{S}$ a dilation of $\mathbf{T}$ if $\mathcal{H}$ is a subspace of $\mathcal{K}$ and the following condition is satisfied for each $n \in \mathbb{N}$:

$$\mathbf{T}^n \Psi = \mathbf{P}_\mathcal{H} \mathbf{S}^n \Psi, \qquad \Psi \in \mathcal{H},$$

where $\mathbf{P}_\mathcal{H}$ denotes the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$.
Given a contraction operator $\mathbf{T}$ on the Hilbert space $\mathcal{H}$, the defect operator $\mathbf{D_T}$ is defined by

$$\mathbf{D_T} = \sqrt[2]{\mathbf{I} - \mathbf{T}^*\mathbf{T}}.$$

Moreover we define the following operator $\widehat{\mathbf{T}}$ on the Hilbert space $\mathcal{K} = \mathcal{H} \oplus l^2\left(\overline{\mathbf{D_T}\mathcal{H}}\right)$[5]:

$$\widehat{\mathbf{T}} = \begin{vmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{C_T} & \mathbf{W} \end{vmatrix}, \tag{13}$$

where the operators $\mathbf{W} : l^2\left(\overline{\mathbf{D_T}\mathcal{H}}\right) \to l^2\left(\overline{\mathbf{D_T}\mathcal{H}}\right)$ and $\mathbf{C_T} : \mathcal{H} \to l^2\left(\overline{\mathbf{D_T}\mathcal{H}}\right)$ are so defined:

$$\mathbf{W}\left(\xi_0, \xi_1 ... \xi_n ..\right) = \left(0, \xi_0, \xi_1 .... \xi_n ..\right), \qquad \xi \in l^2\left(\overline{\mathbf{D_T}\mathcal{H}}\right)$$

and

$$\mathbf{C_T} h = \left(\mathbf{D_T} h, 0, ...0...\right), \quad h \in \mathcal{H},$$

$\mathbf{D_T}$ is the defect operator of $\mathbf{T}$. Moreover for each $(\xi_0, \xi_1, ..\xi_n ...) \in l^2\left(\overline{\mathbf{D_T}\mathcal{H}}\right)$ we have:

$$\mathbf{C_T^*}\left(\xi_0, \xi_1, ...\xi_n ...\right) = \mathbf{D_T}\xi_0,$$

and

$$\mathbf{C_T^*}\mathbf{C_T} = \mathbf{I} - \mathbf{T}^*\mathbf{T}.$$

We observe that for each $\xi \in l^2\left(\overline{\mathbf{D_T}\mathcal{H}}\right)$:

$$\mathbf{W}^*\left(\xi_0, \xi_1 ... \xi_n ..\right) = \left(\xi_1 .... \xi_n ..\right),$$

and

$$\mathbf{D_{W^*}}\left(\xi_0, \xi_1 ... \xi_n ..\right) = \left(\xi_0, 0, 0, ...0..\right)$$

where $\mathbf{D_{W^*}}$ is the defect operator of the contraction $\mathbf{W}^*$, therefore $\mathbf{D_{W^*}}$ is the orthogonal projection of the space $\overline{\mathbf{D_T}\mathcal{H}}$. Obviously $\widehat{\mathbf{T}}$ is a dilation of $\mathbf{T}$ and a simple calculation shows that $\widehat{\mathbf{T}}$ is an isometric, therefore $\widehat{\mathbf{T}}$ is an isometric dilation of $\mathbf{T}$. An isometric dilation $\widehat{\mathbf{T}}$ on $\mathcal{K}$ of $\mathbf{T}$ is minimal if $\mathcal{H}$ is cyclic for $\mathbf{T}$; that is

$$\mathcal{K} = \bigvee_{n \in \mathbb{N}} \widehat{\mathbf{T}}^n \mathcal{H},$$

moreover it is shown that the 13 is the only, up to unitary equivalences, minimal dilation of $\mathbf{T}$.
The dilations $\left(\widehat{\mathbf{T}}_1, \mathcal{K}_1\right)$ and $\left(\widehat{\mathbf{T}}_2, \mathcal{K}_2\right)$ of $\mathbf{T}$ are equivalent if exists an unitary operator $\mathbf{U} : \mathcal{K}_1 \to \mathcal{K}_2$ such that $\mathbf{U}\widehat{\mathbf{T}}_1 = \widehat{\mathbf{T}}_2\mathbf{U}$ and $\mathbf{U}_{|\mathcal{H}} = id$.
We recall the following proposition:

Proposition 1.8. *Every contraction operator* $\mathbf{T}$ *on the Hilbert space* $\mathcal{H}$ *has a unitary dilation* $\widehat{\mathbf{T}}$ *on a Hilbert space* $\mathcal{K}$ *such that (minimal property)*

$$\mathcal{K} = \bigvee_{n \in \mathbb{Z}} \widehat{\mathbf{T}}^n \mathcal{H}.$$

The operator $\widehat{\mathbf{T}}$ *is then determined by* $\mathbf{T}$ *uniquely (up to unitary equivalences).*

Proof. See [**18**] theorem 1.1. □

[5]For further details cfr.[**18**] and [**19**]

4. Dilations Theory for Dynamical Systems

We define a C*-*dynamical systems* a couple $(\mathfrak{A},\Phi)$ constituted by an unital C*-algebra $\mathfrak{A}$ and an unital cp-map $\Phi:\mathfrak{A}\to\mathfrak{A}$.
A state φ on $\mathfrak{A}$ is say be $\Phi-$invariant if for each $a\in\mathfrak{A}$ we have

$$\varphi\left(\Phi\left(a\right)\right)=\varphi\left(a\right). \tag{14}$$

The $C^{*}-$dynamical systems with invariant state φ is a triple $(\mathfrak{A},\Phi,\varphi)$ where φ is a $\Phi-$invariant state on $\mathfrak{A}$.
A $W^{*}-$*dynamical systems* is a couple $(\mathfrak{M},\Phi)$ constituted by a von Neumann Algebra $\mathfrak{M}$ and an unital normal cp-map $\Phi:\mathfrak{M}\to\mathfrak{M}$.
The $W^{*}-$dynamical systems with invariant state φ is a triple $(\mathfrak{M},\Phi,\varphi)$ where φ is a faithful normal $\Phi-$invariant state on $\mathfrak{M}$.
A $C^{*}-$dynamical systems $(\mathfrak{A},\Phi)$ is say be *multiplicative* if Φ is a homomorphism, while is say be *invertible* if the cp-map Φ is invertible. We have a *reversible C^{*}-dynamical systems* $(\mathfrak{A},\Phi)$ if Φ is an automorphism of $C^{*}-$algebras.

REMARK 1.3. *We observe that from the Kadison inequality 3, for every $a\in\mathfrak{A}$ we have:*

$$\varphi\left(\Phi\left(a^{*}\right)\Phi\left(a\right)\right)\leq\varphi\left(a^{*}a\right).$$

Let $(\mathfrak{A},\Phi,\varphi)$ be a C^{*}-dynamical systems with invariant state φ and $(\mathcal{H}_{\varphi},\pi_{\varphi},\Omega_{\varphi})$ its GNS. We define for each $a\in\mathfrak{A}$, the following operator of $\mathcal{B}\left(\mathcal{H}_{\varphi}\right)$:

$$\mathbf{U}_{\varphi}\pi_{\varphi}\left(a\right)\Omega_{\varphi}=\pi_{\varphi}\left(\Phi\left(a\right)\right)\Omega_{\varphi}. \tag{15}$$

For definition, for each $a\in\mathfrak{A}$ we have

$$\left\|\pi_{\varphi}\left(\Phi\left(a\right)\right)\Omega_{\varphi}\right\|^{2}=\varphi\left(\Phi\left(a^{*}\right)\Phi\left(a\right)\right)\leq\varphi\left(a^{*}a\right)=\left\|\pi_{\varphi}\left(a\right)\Omega_{\varphi}\right\|^{2}.$$

Then $\mathbf{U}_{\varphi}:\mathcal{H}_{\varphi}\to\mathcal{H}_{\varphi}$ is linear contraction of Hilbert spaces.

EXAMPLE 1 (Commutative case). *Let $(\mathfrak{M},\varphi,\Phi)$ be a abelian W^{*}- dynamical system, as well known, the commutative algebra $\mathfrak{M}$ can be represented in the form $L^{\infty}\left(X\right)$ for some classic probability space (X,Σ,μ) where $\varphi\left(f\right)=\int f\,d\mu$ for each $f\in L^{\infty}\left(X\right)$. The GNS of φ is costitued by $\left(L^{2}\left(X\right),\pi_{\varphi},\Omega_{\varphi}\right)$ whit $\pi_{\varphi}\left(f\right)\Psi=f\cdot\Psi$ for each $f\in L^{\infty}\left(X\right)$ and $\Psi\in L^{2}\left(X\right)$. Moreover for the linear contraction $\mathbf{U}_{\varphi}$ we get $\mathbf{U}_{\varphi}\Psi=\Phi\left(f\right)\cdot\Psi$ for each $f\in L^{\infty}\left(X\right)$ and $\Psi\in L^{2}\left(X\right)$.*

We have the following result for the ergodic theory:

PROPOSITION 1.9. *Let $(\mathfrak{A},\Phi,\varphi)$ be a dynamical system and $(\mathcal{H}_{\varphi},\pi_{\varphi},\Omega_{\varphi})$ the GNS of the state φ. There exists a unique linear contraction $\mathbf{U}_{\Phi}$ on the $\mathcal{H}_{\varphi}$ where the relation 15 holds and denoting the orthogonal projection on the linear space* $\ker\left(\mathbf{I}-\mathbf{U}_{\varphi}\right)=\ker\left(\mathbf{I}-\mathbf{U}_{\varphi}^{*}\right)$ *by $\mathbf{P}_{\varphi}$, we have*

$$\mathbf{U}_{\varphi}\mathbf{P}_{\varphi}=\mathbf{P}_{\varphi}\mathbf{U}_{\varphi}=\mathbf{P}_{\varphi}\quad\textit{and}\quad\frac{1}{n+1}\sum_{k=0\varphi}^{n}\mathbf{U}_{\varphi}^{k}\to\mathbf{P}_{\varphi}\quad\textit{in so-topology.} \tag{16}$$

If the application Φ is homomorphism, then $\mathbf{U}_{\varphi}$ is an isometry on $\mathcal{H}_{\varphi}$ such that

$$\mathbf{U}_{\varphi}\mathbf{U}_{\varphi}^{*}\in\pi_{\varphi}\left(\Phi\left(\mathfrak{A}\right)\right)^{\prime}\subset\mathcal{B}\left(\mathcal{H}_{\varphi}\right) \tag{17}$$

and

$$\mathbf{U}_{\varphi}\pi_{\varphi}(a) = \pi_{\varphi}(\Phi(a))\,\mathbf{U}_{\varphi}, \qquad a \in \mathfrak{A}. \tag{18}$$

Proof. See [**20**] lemma 2.1. □

4.1. Dilations for Dynamical Systems. We now give the fundamental definition of dilation of a dynamical system.

Definition 1.2. *Let $(\mathfrak{A}, \Phi, \varphi)$ be a C*-dynamical system. The 5-tuple $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}, i, \mathcal{E}\right)$ composed by a C*-dynamical system $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}\right)$ and cp-maps $\mathcal{E} : \widehat{\mathfrak{A}} \to \mathfrak{A}$, $i : \mathfrak{A} \to \widehat{\mathfrak{A}}$, is say be a dilation of $(\mathfrak{A}, \Phi, \varphi)$ if for each $a \in \mathfrak{A}$ and $n \in \mathbb{N}$ we have*

$$\mathcal{E}\left(\widehat{\Phi}^{n}(i(a))\right) = \Phi^{n}((a)),$$

and for each $x \in \widehat{\mathfrak{A}}$

$$\widehat{\varphi}(x) = \varphi(\mathcal{E}(x)).$$

Two dilations $\left(\widehat{\mathfrak{A}}_1, \widehat{\Phi}_1, \widehat{\varphi}_1, i_1, \mathcal{E}_1\right)$ and $\left(\widehat{\mathfrak{A}}_2, \widehat{\Phi}_2, \widehat{\varphi}_2, i_2, \mathcal{E}_2\right)$ of the C^*-dynamical system $(\mathfrak{A}, \Phi, \varphi)$ are *equivalent* if exists an automorphism $\Lambda : \widehat{\mathfrak{A}}_1 \to \widehat{\mathfrak{A}}_2$ such that

$$\Lambda \circ \widehat{\Phi}_1 = \widehat{\Phi}_2 \circ \Lambda, \quad \widehat{\varphi}_2 = \widehat{\varphi}_1 \circ \Lambda \ \text{ and } \ \mathcal{E}_2 \circ \Lambda = \mathcal{E}_1, \quad \Lambda \circ i_1 = i_2. \tag{19}$$

The dilation $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}, i, \mathcal{E}\right)$ of the C*-dynamical system $(\mathfrak{A}, \varphi, \Phi)$, is say be a *reversible [multiplicative]* dilation if $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}\right)$ is a reversible [multiplicative] C*-dynamical system.
The dilation $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}, i, \mathcal{E}\right)$ of the C*-dynamical system $(\mathfrak{A}, \varphi, \Phi)$, is say be a *unital dilation* if the cp-map i is unital, i.e. $i(\mathbf{1}_{\mathfrak{A}}) = \mathbf{1}_{\widehat{\mathfrak{A}}}$.

Remark 1.4. *Let $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}, i, \mathcal{E}\right)$ be a reversible dilation of $(\mathfrak{A}, \varphi, \Phi)$, for definition we have that $\mathcal{E} \circ i = id_{\mathfrak{A}}$ where i is injective map while $\mathcal{E}$ is surjective map.*

We have a first proposition that affirms that the map $\mathcal{E}$ is a conditional expectation.

Proposition 1.10. *Let $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}, i, \mathcal{E}\right)$ be a reversible dilation of $(\mathfrak{A}, \varphi, \Phi)$, for each $a, b \in \mathfrak{A}$, $x \in \widehat{\mathfrak{A}}$ we have:*

$$\mathcal{E}(i(a)\,x\,i(b)) = a\mathcal{E}(x)\,b.$$

Proof. For each $a \in \mathfrak{A}$ we obtain

$$\mathcal{E}(i(a^*)\,i(a)) = a^*a,$$

since $a^*a = \mathcal{E}(i(a^*a)) \geq \mathcal{E}(i(a^*)\,i(a)) \geq \mathcal{E}(i(a^*))\,\mathcal{E}(i(a)) = a^*a$. Then for each $a \in \mathfrak{A}$, the element $i(a)$ is in the multiplicative domains of $\mathcal{E}$, it follow by the relation 1 that $\mathcal{E}(i(a)X) = \mathcal{E}(i(a))\,\mathcal{E}(X)$ and $\mathcal{E}(Xi(a)) = \mathcal{E}(X)\,\mathcal{E}(i(a))$ for each $X \in \widehat{\mathfrak{A}}$. □

We observe that if $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}, i, \mathcal{E}\right)$ be a reversible dilation of $(\mathfrak{A}, \varphi, \Phi)$ we have

$$\mathcal{E}(i(a_1)\,i(a_2)\cdots i(a_n)) = a_1a_2\cdots a_n$$

for each $a_1,\ a_2, ...a_n \in \mathfrak{A}$, since

$$\mathcal{E}\left(i\left(a_1\right)i\left(a_2\right)\cdots i\left(a_n\right)\right) = a_1\mathcal{E}\left(i\left(a_2\right)\cdots i\left(a_n\right)\right).$$

Then

$$\mathcal{E}\left(\left(i\left(a\right)i\left(b\right)-i\left(ab\right)\right)^*\left(i\left(a\right)i\left(b\right)-i\left(ab\right)\right)\right)=0$$

and

$$\widehat{\varphi}\left(\left(i\left(a\right)i\left(b\right)-i\left(ab\right)\right)^*\left(i\left(a\right)i\left(b\right)-i\left(ab\right)\right)\right)=0.$$

From this last relation we have the following remark:

REMARK 1.5. *Let* $\left(\widehat{\mathfrak{M}},\widehat{\varphi},\widehat{\Phi},i,\mathcal{E}\right)$ *be a reversible dilation of the W*-dynamical system* $\left(\mathfrak{M},\Phi,\varphi\right)$ *, then the map* i *is multiplicative (but is not necessarily unital) and* $i\circ\mathcal{E}:\widehat{\mathfrak{M}}\rightarrow\widehat{\mathfrak{M}}$ *is (unique) conditional expectation on von Neumann algebra* $i\left(\mathfrak{M}\right)''$[6].

We have now an important definition:

DEFINITION 1.3. *The reversible dilation* $\left(\widehat{\mathfrak{A}},\widehat{\Phi},\widehat{\varphi},i,\mathcal{E}\right)$ *of the* C^**-dynamical system* $\left(\mathfrak{A},\Phi,\varphi\right)$ *is to said be minimal if*

$$\widehat{\mathfrak{A}} = C^*\left(\bigcup_{k\in\mathbb{Z}}\widehat{\Phi}^k\left(i\left(\mathfrak{A}\right)\right)\right)$$

while is to said be Markov if

$$\widehat{\mathfrak{A}} = C^*\left(\bigcup_{k\in\mathbb{N}}\widehat{\Phi}^k\left(i\left(\mathfrak{A}\right)\right)\right).$$

We study now the relation between the representations GNS of the C*-dynamical system $\left(\mathfrak{A},\Phi,\varphi\right)$ and one its possible dilation $\left(\widehat{\mathfrak{A}},\widehat{\Phi},\widehat{\varphi},i,\mathcal{E}\right)$.
Let $\mathbf{Z}:\mathcal{H}_\varphi\rightarrow\mathcal{H}_{\widehat{\varphi}}$ be the linear operator thus defined:

$$\mathbf{Z}\pi_\varphi\left(a\right)\Omega_\varphi = \pi_{\widehat{\varphi}}\left(i\left(a\right)\right)\Omega_{\widehat{\varphi}},\qquad a\in\mathfrak{A} \tag{20}$$

The operator is an isometry since

$$\left\|\mathbf{Z}\ \pi_\varphi\left(a\right)\Omega_\varphi\right\|^2 = \widehat{\varphi}\left(i\left(a^*\right)i\left(a\right)\right) = \widehat{\varphi}\left(i\left(a^*a\right)\right) = \varphi\left(a^*A\right) = \left\|\pi_\varphi\left(a\right)\Omega_\varphi\right\|^2.$$

Moreover for each $x\in\widehat{\mathfrak{A}}$ we have:

$$\left\langle\mathbf{Z}^*\pi_{\widehat{\varphi}}\left(x\right)\Omega_{\widehat{\varphi}}\pi_\varphi\left(a\right)\Omega_\varphi\right\rangle = \widehat{\varphi}\left(x^*i\left(a\right)\right) = \varphi\left(\mathcal{E}\left(x^*\right)a\right) = \left\langle\pi_\varphi\left(\mathcal{E}\left(x\right)\right)\Omega_{\widehat{\varphi}},\pi_\varphi\left(a\right)\Omega_\varphi\right\rangle.$$

Then

$$\mathbf{Z}^*\pi_{\widehat{\varphi}}\left(x\right)\Omega_{\widehat{\varphi}} = \pi_\varphi\left(\mathcal{E}\left(x\right)\right)\Omega_\varphi, \tag{21}$$

and a simple calculation shows that for each $a\in\mathfrak{A}$ and $x\in\widehat{\mathfrak{A}}$ we obtain:

$$\mathbf{Z}\pi_\varphi\left(a\right) = \pi_{\widehat{\varphi}}\left(i\left(a\right)\right)\mathbf{Z} \tag{22}$$

and

$$\mathbf{Z}^*\pi_{\widehat{\varphi}}\left(x\right)\mathbf{Z} = \pi_\varphi\left(\mathcal{E}\left(x\right)\right). \tag{23}$$

[6] Cfr.[**22**] Proposition 3.5.

We notice that the operator $\mathbf{Q} = \mathbf{Z}\mathbf{Z}^*$ is the ortogonal projection on the Hilbert space generated by the vectors $\left\{\pi_{\widehat{\varphi}}\left(i\left(a\right)\right)\Omega_{\widehat{\varphi}} : a \in \mathfrak{A}\right\}$ with

$$\mathbf{Q}\pi_{\widehat{\varphi}}\left(x\right)\Omega_{\widehat{\varphi}} = \pi_{\widehat{\varphi}}\left(i\left(\mathcal{E}\left(x\right)\right)\right)\Omega_{\widehat{\varphi}}, \qquad x \in \widehat{\mathfrak{A}}. \tag{24}$$

For all $n \in \mathbb{N}$ we have

$$\mathbf{U}_{\varphi}^{n} = \mathbf{Z}^*\mathbf{U}_{\widehat{\varphi}}^{n}\mathbf{Z}, \tag{25}$$

since for each $a \in \mathfrak{A}$:
$\mathbf{Z}^*\mathbf{U}_{\widehat{\varphi}}^{n}\mathbf{Z}\pi_{\varphi}\left(a\right)\Omega_{\varphi} = \mathbf{Z}^*\pi_{\widehat{\varphi}}\left(\widehat{\Phi}^{n}\left(i\left(a\right)\right)\right)\Omega_{\widehat{\varphi}} = \pi_{\widehat{\varphi}}\left(\mathcal{E}\left(\widehat{\Phi}^{n}\left(i\left(a\right)\right)\right)\right)\Omega_{\widehat{\varphi}} =$
$= \pi_{\widehat{\varphi}}\left(\Phi^{n}\left(a\right)\right)\Omega_{\varphi} = \mathbf{U}_{\varphi}^{n}\pi_{\varphi}\left(a\right)\Omega_{\varphi}.$
We study now the relation between the orthogonal projections $\mathbf{P}_{\varphi} = \left[\ker\left(\mathbf{I} - \mathbf{U}_{\varphi}\right)\right]$ and $\mathbf{P}_{\widehat{\varphi}} = \left[\ker\left(\mathbf{I} - \mathbf{U}_{\widehat{\varphi}}\right)\right]$.
From the relation 25 for each $N \in \mathbb{N}$ we have the relation

$$\frac{1}{N+1}\sum_{k=0}^{N}\mathbf{U}_{\varphi}^{k} = \mathbf{Z}^*\left(\frac{1}{N+1}\sum_{k=0}^{N}\mathbf{U}_{\widehat{\varphi}}^{k}\right)\mathbf{Z}$$

it follow that

$$\mathbf{P}_{\varphi} = \mathbf{Z}^*\mathbf{P}_{\widehat{\varphi}}\mathbf{Z}. \tag{26}$$

Proposition 1.11. *Let* $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}, i, \mathcal{E}\right)$ *be a dilation of the C*-dynamical system* $\left(\mathfrak{A}, \varphi, \Phi\right)$ *the unitary operator* $\mathbf{U}_{\widehat{\varphi}}$ *is a dilation of the contraction* $\mathbf{Z}\mathbf{U}_{\varphi}\mathbf{Z}^*$*.*
Moreover to equivalent dilations of the C-dynamical system corresponds equivalent dilations of the linear contraction* $\mathbf{U}_{\varphi}$*.*

Proof. We observe that for each $a \in \mathfrak{A}$ and $n \in \mathbb{N}$ we have:
$\left(\mathbf{Z}\mathbf{U}_{\varphi}\mathbf{Z}^*\right)^{n}\pi_{\widehat{\varphi}}\left(a\right)\Omega_{\widehat{\varphi}} = \mathbf{Q}\mathbf{U}_{\widehat{\varphi}}^{n}\mathbf{Z}\pi_{\widehat{\varphi}}\left(a\right)\Omega_{\widehat{\varphi}} = \mathbf{Q}\pi_{\widehat{\varphi}}\left(\widehat{\Phi}^{n}\left(i\left(a\right)\right)\right)\Omega_{\widehat{\varphi}} =$
$= \pi_{\widehat{\varphi}}\left(i\left(\Phi^{n}\left(a\right)\right)\right)\Omega_{\widehat{\varphi}} = \mathbf{Z}\mathbf{U}_{\varphi}^{n}\pi_{\widehat{\varphi}}\left(a\right)\Omega_{\widehat{\varphi}} = \left(\mathbf{Z}\mathbf{U}_{\varphi}\mathbf{Z}^*\right)^{n}\mathbf{Z}\pi_{\widehat{\varphi}}\left(a\right)\Omega_{\widehat{\varphi}},$
consequently for each $\Psi \in \mathcal{H}_{\varphi}$ we have

$$\mathbf{Q}\mathbf{U}_{\widehat{\varphi}}^{n}\mathbf{Z}h{=}\left(\mathbf{Z}\mathbf{U}_{\mathbf{\Phi}}\mathbf{Z}^*\right)^{n}\Psi.$$

Let $\left(\widehat{\mathfrak{A}}_1, \widehat{\Phi}_1, \widehat{\varphi}_1, i_1, \mathcal{E}_1\right)$ and $\left(\widehat{\mathfrak{A}}_2, \widehat{\Phi}_2, \widehat{\varphi}_2, i_2, \mathcal{E}_2\right)$ are two equivalent dilations of the C*-dynamical system $\left(\mathfrak{A}, \Phi, \varphi\right)$ with automorphism $\Lambda : \widehat{\mathfrak{A}}_1 \to \widehat{\mathfrak{A}}_2$ defined in 19.
We set for each $a \in \mathfrak{A}$

$$\Lambda_{\natural}\pi_{\widehat{\varphi}_1}\left(a\right)\Omega_{\widehat{\varphi}_1} = \pi_{\widehat{\varphi}_2}\left(\Lambda\left(a\right)\right)\Omega_{\widehat{\varphi}_2},$$

we have an unitary operator $\Lambda_{\natural} : \mathcal{H}_{\widehat{\varphi}_1} \to \mathcal{H}_{\widehat{\varphi}_2}$ such that

$$\Lambda_{\natural} \circ \mathbf{U}_{\widehat{\varphi}_1} = \mathbf{U}_{\widehat{\varphi}_2} \circ \Lambda_{\natural}.$$

□

We have the following remark:

Remark 1.6. *If* $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}\right)$ *is a minimal dilation, in general, it is not said that the operator* $\mathbf{U}_{\widehat{\varphi}}$ *is minimal unitary dilation of* $\mathbf{U}_{\varphi}$*.*
In fact the Hilbert space $\mathcal{H}_{\widehat{\varphi}}$ *is the norm closed linear space generate by the set of elements*

$$\left\{\mathbf{U}_{\widehat{\varphi}}^{n_1}\pi_{\widehat{\varphi}}\left(i\left(a_1\right)\right)\cdots\mathbf{U}_{\widehat{\varphi}}^{n_k}\pi_{\widehat{\varphi}}\left(i\left(a_k\right)\right)\Omega_{\widehat{\varphi}} : a_i \in \mathfrak{A}, \quad n_i \in \mathbb{N}\right\}$$

while the space $\bigvee_{n\in\mathbb{N}} \mathbf{U}_{\widehat{\varphi}}^{n}\mathbf{Z}\mathcal{H}_{\varphi}$ *is generate by the set of elements*

$$\left\{\mathbf{U}_{\widehat{\varphi}}^{n}\pi_{\widehat{\varphi}}\left(i\left(a\right)\right)\Omega_{\widehat{\varphi}} : a \in \mathfrak{A}, \quad n \in \mathbb{Z}\right\}.$$

We see now an example of as the Nagy dilation for the contraction on the Hilbert space is applied to the dilation theory of dynamical systems.

Example 2. *Let* $\mathcal{H}$ *be a Hilbert space and* $\mathbf{V}$ *an isometry on* $\mathcal{H}$*, we get the unital cp-map* $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$

$$\Phi\left(\mathbf{A}\right) = \mathbf{V}^{*}\mathbf{A}\mathbf{V}, \qquad \mathbf{A} \in\mathcal{B}\left(\mathcal{H}\right),$$

and φ *is a* Φ*-invariant state of* $\mathcal{B}(\mathcal{H})$*. In this way we get the* C^**-dynamic system* $(\mathcal{B}(\mathcal{H}), \Phi, \varphi)$*.*
Let $\left(\mathcal{K}, \widehat{\mathbf{V}}\right)$ *be the Nagy dilation of the isometry* $\mathbf{V}^*$*:*

$$\widehat{\mathbf{V}} = \begin{vmatrix} \mathbf{V}^{*} & \mathbf{0} \\ \mathbf{C} & \mathbf{W} \end{vmatrix},$$

and Hilbert space $\mathcal{K} = \mathcal{H} \oplus l^{2}(\mathcal{I})$*.*
We have an auntomorphims $\widehat{\Phi} : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$

$$\widehat{\Phi}\left(\mathbf{X}\right) = \widehat{\mathbf{V}}\mathbf{X}\widehat{\mathbf{V}}^{*}, \qquad \mathbf{X} \in\mathcal{B}\left(\mathcal{H}\right),$$

such that for each $\mathbf{A} \in\mathcal{B}(\mathcal{H})$ *we have:*

$$\mathbf{J}^{*}\widehat{\Phi}^{n}\left(\mathbf{J}\mathbf{A}\mathbf{J}^{*}\right)\mathbf{J} = \Phi\left(\mathbf{A}\right).$$

The C^**-dynamical systems* $\left(\mathcal{B}(\mathcal{K}), \widehat{\Phi}, \widehat{\varphi}\right)$ *with*

$$\widehat{\varphi}\left(\mathbf{X}\right) = \varphi\left(\mathbf{J}^{*}\mathbf{X}\mathbf{J}\right), \qquad \mathbf{X} \in\mathcal{B}\left(\mathcal{K}\right)$$

is a reversible dilation of $(\mathcal{B}(\mathcal{H}), \Phi, \varphi)$*, since*

$$\begin{array}{ccc} \mathcal{B}(\mathcal{K}) & \overset{\widehat{\Phi}^{n}}{\longrightarrow} & \mathcal{B}(\mathcal{K}) \\ i\uparrow & & \downarrow\mathcal{E} \\ \mathcal{B}(\mathcal{H}) & \overset{\Phi^{n}}{\longrightarrow} & \mathcal{B}(\mathcal{H}) \end{array}$$

is a commutative diagram, where:
the application $\mathcal{E} : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ *is the unital cp-map*

$$\mathcal{E}\left(\mathbf{X}\right) = \mathbf{J}^{*}\mathbf{X}\mathbf{J}, \qquad \mathbf{X} \in\mathcal{B}\left(\mathcal{K}\right)$$

while $i : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ *is the *-multiplicative map (non unital)*

$$i\left(\mathbf{A}\right) = \mathbf{J}\mathbf{A}\mathbf{J}^{*}, \qquad \mathbf{X} \in\mathcal{B}\left(\mathcal{K}\right).$$

We observe that $\widehat{\varphi}$ *is a* $\widehat{\Phi}-$*invariant state, since*

$$\widehat{\varphi}\left(\widehat{\Phi}\left(\mathbf{X}\right)\right) = \varphi\left(\mathbf{J}^{*}\widehat{\Phi}\left(\mathbf{X}\right)\mathbf{J}\right) = \varphi\left(\mathbf{J}^{*}\widehat{\mathbf{V}}\mathbf{X}\widehat{\mathbf{V}}^{*}\mathbf{J}\right) = \varphi\left(\mathbf{V}^{*}\mathbf{J}^{*}\mathbf{X}\mathbf{J}\mathbf{V}\right) = \varphi\left(\mathbf{J}^{*}\mathbf{X}\mathbf{J}\right) = \widehat{\varphi}\left(\mathbf{X}\right)$$

for all $\mathbf{X} \in\mathcal{B}(\mathcal{K})$*.*

* * *

We now study the problem list that we have with the dilations of composition.
Let $(\mathfrak{A}, \Phi, \varphi)$ be a C*-dynamical system and $(\mathfrak{A}_o, \Phi_o, \varphi_o, \mathcal{E}_o, i_o)$ a its Markov multiplicative dilation.
If the C*-dynamical system $(\mathfrak{A}_o, \Phi_o, \varphi_o)$ admits a minimal reversible dilation $(\mathfrak{A}_\times, \Phi_\times, \varphi_\times, \mathcal{E}_\times, i_\times)$, we have the follow diagram:

$$\begin{array}{ccc} \mathfrak{A}_\times & \xrightarrow{\Phi_{oo}^n} & \mathfrak{A}_\times \\ i_\times \uparrow & & \downarrow \mathcal{E}_\times \\ \mathfrak{A}_o & \xrightarrow{\Phi_o^n} & \mathfrak{A}_o \\ i_o \uparrow & & \downarrow \mathcal{E}_o \\ \mathfrak{A} & \xrightarrow{\Phi^n} & \mathfrak{A} \end{array} \qquad \begin{array}{ll} \mathfrak{A}_o = C^* \left(\bigcup_{k\in\mathbb{N}} \Phi_o^k \left(i_o \left(\mathfrak{A} \right) \right) \right), & \varphi_o = \varphi \circ \mathcal{E}_o \\ \mathfrak{A}_\times = C^* \left(\bigcup_{k\in\mathbb{Z}} \Phi_\times^k \left(i_\times \left(\mathfrak{A} \right) \right) \right), & \varphi_\times = \varphi_o \circ \mathcal{E}_\times \end{array}$$

Then the 5-tuple $(\mathfrak{A}_\times, \widehat{\varphi}, \Phi_\times, \mathcal{E}, i)$ with $\mathcal{E} = \mathcal{E}_o \circ \mathcal{E}_\times$ and $i = i_\times \circ i_o$ with $\widehat{\varphi} = \widehat{\varphi} \circ \mathcal{E}$, is a reversible dilation of the C*-dynamical system $(\mathfrak{A}, \Phi, \varphi)$, but in generally it is not minimal.
We observe that if φ is faithful state on $\mathfrak{A}$ then φ_o is faithful state on $\mathfrak{A}$ if and only if $\mathcal{E}_o$ is a faithful cp-map.

4.2. The $\varphi-$Adjoint of morphism. Let $(\mathfrak{A}, \Phi, \varphi)$ be C*-algebra dynamical system, a cp map $\Phi^+ : \mathfrak{A} \to \mathfrak{A}$ is said to be φ-adjoint of Φ, if for each $a \in a$ we have

$$\varphi \left(\Phi \left(a \right) b \right) = \varphi \left(a \Phi^+ \left(b \right) \right).$$

We observe that $\left(\Phi^+\right)^+ = \Phi$.
Moreover every reversible C*-dynamical system admits a φ-adjoint where $\Phi^+ = \Phi^{-1}$.
If Φ admits a φ-adjoint, for each $a \in \mathfrak{A}$ we have

$$\mathbf{U}_\varphi^* \pi_\varphi \left(a \right) \Omega_\varphi = \pi_\varphi \left(\Phi^+ \left(a \right) \right) \Omega_\varphi,$$

since for each $a, b \in \mathfrak{A}$, we get:

$$\left\langle \mathbf{U}_\varphi^* \pi_\varphi \left(b \right) \Omega_\varphi, \pi_\varphi \left(a \right) \Omega_\varphi \right\rangle = \varphi \left(b^* \Phi \left(a \right) \right) = \varphi \left(\Phi^+ \left(b^* \right) a \right) = \left\langle \pi_\varphi \left(\Phi^+ \left(b \right) \right) \Omega, \pi_\varphi \left(a \right) \Omega_\varphi \right\rangle.$$

We introduce a necessary condition for the existence of a reversible dilation (cfr.[**12**] proposition 2.1.8).

PROPOSITION 1.12. *Let $(\mathfrak{A}, \Phi, \varphi)$ be a C*-dynamical system with a reversible dilation $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}, \mathcal{E}, i\right)$. Then Φ has a $\varphi-$adjoint Φ^+ and $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}^{-1}, \widehat{\varphi}, \mathcal{E}, i\right)$ is a dilation of the C*-dynamical system $(\mathfrak{A}, \Phi^+, \varphi)$.*

PROOF. For $a, b \in \mathfrak{A}$ and $n \in \mathbb{N}$ we have:

$$\begin{aligned} \varphi \left(a \Phi^n \left(b \right) \right) &= \varphi \left(a \mathcal{E} \left(\widehat{\Phi}^n \left(i \left(b \right) \right) \right) \right) = \varphi \left(\mathcal{E} \left(a \widehat{\Phi}^n \left(i \left(b \right) \right) \right) \right) = \widehat{\varphi} \left(i \left(a \right) \widehat{\Phi}^n \left(i \left(b \right) \right) \right) = \\ &= \widehat{\varphi} \left(\widehat{\Phi}^{-n} \left(i \left(a \right) \right) i \left(b \right) \right) = \varphi \left(\mathcal{E} \left(\widehat{\Phi}^{-n} \left(i \left(a \right) \right) \right) b \right). \end{aligned}$$

Then the $\varphi-$adjoint of Φ results to be $\Phi^+ = \mathcal{E} \circ \widehat{\Phi}^{-1} \circ i$. □

REMARK 1.7. *Let $(\mathfrak{A}, \Phi, \varphi)$ be a C*-dynamical system with a φ-adjont Φ^+. If Φ^+ is a multiplivative map we have*

$$\mathbf{U}_\varphi \mathbf{U}_\varphi^* = I.$$

Furthermore, if φ is a faithful state we have

$$\Phi\left(\Phi^{+}(a)\right)=a$$

for each $a\in\mathfrak{A}$.

We have now the follow proposition:

PROPOSITION 1.13. *Let $(\mathfrak{A},\Phi,\varphi)$ be a C^*-dynamical system with a φ-adjont Φ^+, we have*
1-

$$\mathbf{U}_{\varphi}^{*}\pi_{\varphi}(a)\,\mathbf{U}_{\varphi}=\pi_{\varphi}\left(\Phi^{+}(a)\right)$$

if and only if for each $a,b,c\in\mathfrak{A}$:

$$\varphi\left(b\Phi^{+}(a)\,c\right)=\varphi\left(\Phi(b)\,a\Phi(c)\right). \tag{27}$$

2-

$$\mathbf{U}_{\varphi}\pi_{\varphi}(a)\,\mathbf{U}_{\varphi}^{*}=\pi_{\varphi}\left(\Phi(a)\right)$$

if and only if for each $a,b,c\in\mathfrak{A}$:

$$\varphi\left(b\Phi(a)\,c\right)=\varphi\left(\Phi^{+}(b)\,a\Phi^{+}(c)\right) \tag{28}$$

PROOF. We have:
$\langle\pi_{\varphi}(b^*)\,\Omega_{\varphi},\pi_{\varphi}(\Phi^{+}(a))\,\pi_{\varphi}(c)\,\Omega_{\varphi}\rangle=\varphi\left(b\Phi^{+}(a)\,c\right)=\varphi\left(\Phi(b)\,a\Phi(c)\right)=$
$=\langle\mathbf{U}_{\varphi}\pi_{\varphi}(b^*)\,\Omega_{\varphi},\pi_{\varphi}((a))\,\mathbf{U}_{\varphi}\pi_{\varphi}(c)\,\Omega_{\varphi}\rangle=\left\langle\pi_{\varphi}(a^*)\,\Omega_{\varphi},\mathbf{U}_{\varphi}^{*}\pi_{\varphi}((a))\,\mathbf{U}_{\varphi}\pi_{\varphi}(c)\,\Omega_{\varphi}\right\rangle,$
while for the second relation we obtain:
$\left\langle\pi_{\varphi}(b^*)\,\Omega_{\varphi},\mathbf{U}_{\varphi}\pi_{\varphi}(a)\,\mathbf{U}_{\varphi}^{*}\pi_{\varphi}(c)\,\Omega_{\varphi}\right\rangle=\langle\pi_{\varphi}(\Phi^{+}(b^*))\,\Omega_{\varphi},\pi_{\varphi}(a)\,\pi_{\varphi}(\Phi^{+}(c))\,\Omega_{\varphi}\rangle=$
$=\varphi\left(\Phi^{+}(b^*)\,a\Phi^{+}(c)\right)=\varphi\left(b^*\Phi(a)\,c\right)=\langle\pi_{\varphi}(b)\,\Omega_{\varphi},\pi_{\varphi}(\Phi(a))\,\pi_{\varphi}(c)\,\Omega_{\varphi}\rangle.$ □

4.3. The (φ,n)-multiplicative maps. Let φ be a state on a C*-algebra $\mathfrak{A}$ and $\Phi:\mathfrak{A}\to\mathfrak{A}$ Cp- map, if there is a $n\in\mathbb{N}$ such that for each $a_{1,}a_{2}...a_{n}\in\mathfrak{A}$ we get

$$\varphi\left(\prod_{j=0}^{n}\Phi(a_j)\right)=\varphi\left(\Phi\left(\prod_{j=0}^{n}a_j\right)\right), \tag{29}$$

then the Φ is said to be (φ,n)-*multiplicative.*
The next proposition characterizes the $(\varphi,2)$-multiplicative maps:

REMARK 1.8. *Let φ be a faithful state on a C^*-algebra $\mathfrak{A}$, every $(\varphi,2)$-multiplicative map $\Phi:\mathfrak{A}\to\mathfrak{A}$ is a *-homomorphism.*

PROOF. Cfr. [**6**] lemma III-2 □

A simple consequence of the definition is given by the following proposition:

PROPOSITION 1.14. *Let $(\mathfrak{A},\varphi,\Phi)$ a C^*-dynamical system, then the dynamic Φ is $(\varphi,2)$-multiplicative if and only if $\mathbf{U}_{\varphi}$ is isometric.*

PROOF. For definition for each $a,b\in\mathfrak{A}$ we have:
$\langle\mathbf{U}_{\Phi}\pi_{\varphi}(b)\,\Omega_{\varphi},\mathbf{U}_{\Phi}\pi_{\varphi}(a)\,\Omega_{\varphi}\rangle=\langle\pi_{\varphi}(\Phi(b))\,\Omega_{\varphi},\pi_{\varphi}(\Phi(a))\,\Omega_{\varphi}\rangle=$
$=\varphi\left(\Phi(b^*)\,\Phi(a)\right)=\varphi\left(\Phi(b^*a)\right)=\varphi\left(\Phi(b^*a)\right)=\varphi(b^*a)=$
$=\langle\pi_{\varphi}(b)\,\Omega_{\varphi},\pi_{\varphi}(a)\,\Omega_{\varphi}\rangle.$ □

5. Spatial Morphism

Let $(\mathfrak{A}, \Phi, \varphi)$ be a C*-dynamical system and $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$ the GNS of the state φ. We set with $\mathfrak{M} = \pi_\varphi(\mathfrak{A})''$the von Neumann subalgebra of $\mathcal{B}(\mathcal{H}_\varphi)$ and ω the defined state on $\mathfrak{M}$ as

$$\omega(X) = \langle \Omega_\varphi, X\Omega_\varphi \rangle, \qquad X \in \mathfrak{M}.$$

We say that the cp map Φ is *spatial*[7] if there exists an unique normal, unital cp map $\Phi_\sharp : \mathfrak{M} \to \mathfrak{M}$ such that for each $a \in \mathfrak{A}$, we obtain:

$$\Phi_\sharp(\pi_\varphi(a)) = \pi_\varphi(\Phi(a)).$$

We have a W^*-dynamical system $(\mathfrak{M}, \Phi_\sharp, \omega)$ since ω is $\Phi_\sharp-$invariant.
C*-dynamical system $(\mathfrak{A}, \Phi, \varphi)$ is said to be a *separating* if Ω_φ is cyclic for $\pi_\varphi(\mathfrak{A})'$.

PROPOSITION 1.15. *Let $(\mathfrak{A}, \Phi, \varphi)$ be a separating C*-dynamical system. Then Φ is spatial morphism, and for each $X \in \mathfrak{M}$ we have:*

$$\Phi_\sharp(X)\Omega_\varphi = \mathbf{U}_\varphi X\Omega_\varphi.$$

If Φ is omomorphism the $\Phi_\sharp$ is an automorphism of von Neumann algebra.

PROOF. It's a trivial consequence of the proposition 3.1 of [**20**]. □

An important characterization for the dilations of W^*-dynamical systems is given by the following proposition:

PROPOSITION 1.16. *Let $(\mathfrak{A}, \Phi, \varphi)$ be a separating C^*-dynamical systems , the following conditions are equivalent:*
- *Φ commutes with the automorphism modular group σ_t^φ of $(\mathfrak{M}_\varphi, \varphi)$:*

$$\sigma_t^\varphi(\Phi_\sharp(\pi_\varphi(a))) = \Phi_\sharp(\sigma_t^\varphi(\pi_\varphi(a))), \quad t \in \mathbb{R}, \quad a \in \mathfrak{A};$$

- $\mathbf{U}_\varphi \Delta^{it} = \Delta^{it}\mathbf{U}_\varphi$ *for all $t \in \mathbb{R}$, where Δ is the modular operator of φ;*
- $\mathbf{U}_\varphi$ *commutes with modular coniugation* $\mathbf{J}_\varphi$ *of* φ;
- *There exists an unique cp-map $\Phi^+ : \mathfrak{M}_\varphi \to \mathfrak{M}_\varphi$ such that for each $a \in \mathfrak{M}$ we have*

$$\pi_\varphi\left(\Phi^+(a)\right)\Omega_\varphi = \mathbf{U}_\varphi^* \pi_\varphi(a)\Omega_\varphi.$$

PROOF. It's a consequence of the proposition 3.3 of [**20**]. □

We obtain a necessary condition for the existence of dilations of W^*-dynamical systems (see [**12**] and [**14**]):

REMARK 1.9. *The morphism Φ commutes with the automorphism modular group σ_t^φ of $(\mathfrak{M}, \varphi)$ if and only if the Φ admit φ-adjoint.*

* * *

Let $(\mathbf{V}, \mathcal{H})$ be isometry on the Hilbert space $\mathcal{H}$, we set with $\left(\widehat{\mathbf{V}}, \widehat{\mathcal{H}}\right)$ the minimal unitary dilation of $(\mathbf{V}, \mathcal{H})$ and $\mathbf{Z} : \mathcal{H} \to \widehat{\mathcal{H}}$ isometry operator such that

$$\mathbf{ZV} = \widehat{\mathbf{V}}\mathbf{Z}.$$

[7] Cfr. [**3**] par.4.

Let $\mathfrak{F}$ the set of the operator net $\{\mathbf{T}_j\}_{j\in\mathbb{N}}$ of $\mathfrak{B}(\mathcal{H})$ with the follow property:
▶ - $\sup\{\|\mathbf{T}_j\| : j\in\mathbb{N}\}\leq\infty$
▶ - $\mathbf{VT}_0=\mathbf{T}_1\mathbf{V}$
▶ - $\mathbf{VV}^*\mathbf{T}_j=\mathbf{T}_j\mathbf{VV}^*\ j\geq 1$
For every net $\mathbf{t}=\{\mathbf{T}_j\}_{j\in\mathbb{N}}$ belong to $\mathfrak{F}$ we define

$$\mathbf{S}_n(\mathbf{t})=\mathbf{ZT}_0\mathbf{Z}^*+\sum_{j=1}^{n}\widehat{\mathbf{V}}^{-j}\mathbf{ZT}_j\mathbf{FZ}^*\widehat{\mathbf{V}}^j$$

where $\mathbf{F}=\mathbf{I}-\mathbf{VV}^*$ is orthogonal projection on the space $\ker(\mathbf{V}^*)$.
We have another fundamental proposition:

Proposition 1.17. *For every element $\mathbf{t}=\{\mathbf{T}_j\}_{j\in\mathbb{N}}$ belong to $\mathfrak{F}$, the net $\{\mathbf{S}_n(\mathbf{t})\}_{n\in\mathbb{N}}$ converges respect to the strong operator topology and*

$$\mathbf{S}(\mathbf{t})=So-\lim_{n\to\infty}\sum_{j=2}^{n}\left[\widehat{\mathbf{V}}^{-(j-1)}\mathbf{Z}\left(\mathbf{T}_{j-1}-\mathbf{V}^*\mathbf{T}_j\mathbf{V}\right)\mathbf{Z}^*\widehat{\mathbf{V}}^{(j-1)}+\widehat{\mathbf{V}}^{-n}\mathbf{ZT}_n\mathbf{Z}^*\widehat{\mathbf{V}}^n\right]$$

Moreover for each $\mathbf{t}=\{\mathbf{T}_j\}_{j\in\mathbb{N}}$ and $\mathbf{r}=\{\mathbf{R}_j\}_{j\in\mathbb{N}}$ belongs to $\mathfrak{F}$ we have

$$\mathbf{S}(\mathbf{t})\,\mathbf{S}(\mathbf{r})=\mathbf{S}(\mathbf{t}\cdot\mathbf{r})$$

where $\mathbf{t}\cdot\mathbf{r}=\{\mathbf{T}_j\circ\mathbf{R}_j\}_{j\in\mathbb{N}}$.

Proof. Cfr [**20**] section 6. □

A simple consequence of the preceding proposition is the following theorem, it is a first important result in the dilation theory of the dynamic systems:

Proposition 1.18. *Let $(\mathfrak{A},\Phi,\varphi)$ be a multiplicative C^*-dynamical system, we set with $\left(\widehat{\mathbf{U}}_\varphi,\widehat{\mathcal{H}}_\varphi,\mathbf{Z}_\varphi\right)$ the minimal unitary dilation of the linear isometry $\mathbf{U}_\varphi$ defined in 15:*

$$\mathbf{U}_\varphi\pi_\varphi(a)\,\Omega_\varphi=\pi_\varphi(\Phi(a))\,\Omega_\varphi.$$

Let $\mathbf{Z}_\varphi:\mathcal{H}_\varphi\to\widehat{\mathcal{H}}_\varphi$ be the linear isometry satisfying $\mathbf{Z}_\varphi\mathbf{U}_\varphi=\widehat{\mathbf{U}}_\varphi\mathbf{Z}_\varphi$.
Then exist a representation $\widehat{\pi}:\mathfrak{A}\to\mathcal{B}\left(\widehat{\mathcal{H}}_\varphi\right)$ such that for each $a\in\mathfrak{A}$ we have

$$\widehat{\pi}(a)\,\mathbf{Z}_\varphi=\mathbf{Z}_\varphi\pi_\varphi(a)\tag{30}$$

and

$$\widehat{\pi}(\Phi(a))=\widehat{\mathbf{U}}_\varphi\widehat{\pi}(a)\,\widehat{\mathbf{U}}_\varphi^*,\tag{31}$$

with

$$\widehat{\pi}(a)=\mathbf{Z}_\varphi\pi_\varphi(a)\,\mathbf{Z}_\varphi^*+\sum_{k=1}^{\infty}\widehat{\mathbf{U}}_\varphi^{-k}\mathbf{Z}_\varphi\pi_\varphi\left(\Phi^k(a)\right)\mathbf{FZ}_\varphi^*\widehat{\mathbf{U}}_\varphi^k=\tag{32}$$

$$=So-\lim_{n\to\infty}\left[\widehat{\mathbf{U}}_\varphi^{-n}\mathbf{Z}_\varphi\pi_\varphi(\Phi^n(a))\,\mathbf{Z}_\varphi^*\widehat{\mathbf{U}}_\varphi^n\right],\tag{33}$$

where $\mathbf{F}$ is the projection $\mathbf{I}-\mathbf{U}_\varphi\mathbf{U}_\varphi^*\in\pi_\varphi(\Phi(\mathfrak{A}))'$ and the series converges respect to the strong operator topology of $\mathcal{B}\left(\widehat{\mathcal{H}}_\varphi\right)$.

*Furthemore, the so-topology closure of the * subalgebra generate by the set:*

$$\mathfrak{B} = \bigcup_{k\in\mathbb{Z}} \widehat{\mathbf{U}}_{\varphi}^{k}\widehat{\pi}\left(\mathfrak{A}\right)\widehat{\mathbf{U}}_{\varphi}^{-k} = \bigcup_{k\in\mathbb{N}} \widehat{\mathbf{U}}_{\varphi}^{-k}\widehat{\pi}\left(\mathfrak{A}\right)\widehat{\mathbf{U}}_{\varphi}^{k} \tag{34}$$

of $\mathcal{B}\left(\widehat{\mathcal{H}}_{\varphi}\right)$ is a von Neumann algebra $\mathfrak{M}$ and $\widehat{\Omega} = \mathbf{Z}_{\varphi}\Omega_{\varphi}$ is a cyclic vector for $\mathfrak{M}$ satisfying $\widehat{\mathbf{U}}_{\varphi}\widehat{\Omega} = \widehat{\Omega}$ and for each $a \in \mathfrak{A}$ we have:

$$\varphi\left(a\right) = \left\langle \widehat{\Omega}, \widehat{\pi}\left(a\right)\widehat{\Omega}\right\rangle.$$

PROOF. See [**20**] proposition 6.1. □

Next proposition certifies that for the *multiplicative* C*-dynamical system the φ-adjunction is a sufficient condition for the existence of a reversible dilation.

THEOREM 1.2. *Let $(\mathfrak{A}, \Phi, \varphi)$ be multiplicative C*-dynamical system with φ faithful state. If Φ admit a φ-adjoint Φ^{+} then there exists a minimal reversible dilation $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}, i, \mathcal{E}\right)$ where:*

- *The C*-algebra $\widehat{\mathfrak{A}}$ is the norm closed of the algebra $\mathfrak{B}$ defined in 34;*
- *The cp map i is the representation $\widehat{\pi}$ defined in 30[8];*
- *The automorphism $\widehat{\Phi} : \widehat{\mathfrak{A}} \to \widehat{\mathfrak{A}}$ is thus defined:*

$$\widehat{\Phi}\left(X\right) = \widehat{\mathbf{U}}_{\varphi}X\widehat{\mathbf{U}}_{\varphi}^{*}; \qquad X \in \widehat{\mathfrak{A}}; \tag{35}$$

- *The conditional expectation $\mathcal{E} : \widehat{\mathfrak{A}} \to \mathfrak{A}$ is defined through the expression:*

$$\mathcal{E}\left(\widehat{\mathbf{U}}_{\varphi}^{-k}\widehat{\pi}\left(a\right)\widehat{\mathbf{U}}_{\varphi}^{k}\right) = \pi_{\varphi}\left(\Phi^{+k}\left(a\right)\right), \qquad a \in \mathfrak{A}, .k \in \mathbb{N}, \tag{36}$$

while for the state we have

$$\widehat{\varphi}\left(X\right) = \varphi\left(\mathcal{E}\left(X\right)\right) \qquad X \in \widehat{\mathfrak{A}}.$$

PROOF. We get the following inclusions for each $n \geq 0$:

$$\pi_T\left(\mathfrak{A}\right) \subset \widehat{\mathbf{U}}_{\varphi}^{*}\pi_T\left(\mathfrak{A}\right)\widehat{\mathbf{U}}_{\varphi} \subset \widehat{\mathbf{U}}_{\varphi}^{-2}\pi_T\left(\mathfrak{A}\right)\widehat{\mathbf{U}}_{\varphi}^{2} \subset \cdots \subset \widehat{\mathbf{U}}_{\varphi}^{-n}\pi_T\left(\mathfrak{A}\right)\widehat{\mathbf{U}}_{\varphi}^{n} \subset \cdots$$

since we have

$$\widehat{\mathbf{U}}_{\varphi}\widehat{\pi}\left(\mathfrak{A}\right)\widehat{\mathbf{U}}_{\varphi}^{*} = \widehat{\pi}\left(\Phi\left(\mathfrak{A}\right)\right) \subset \widehat{\pi}\left(\mathfrak{A}\right),$$

then

$$\widehat{\pi}\left(\mathfrak{A}\right) \subset \widehat{\mathbf{U}}_{\varphi}^{-1}\widehat{\pi}\left(\mathfrak{A}\right)\widehat{\mathbf{U}}_{\Phi}.$$

We observe that every element X belong to algebra $\mathfrak{B}$ defined in 34 has this writing:

$$X = \widehat{\mathbf{U}}_{\varphi}^{-n}\pi_T\left(x\right)\widehat{\mathbf{U}}_{\varphi}^{n}$$

for some $x \in \mathfrak{A}$ and $n \in \mathbb{N}$.
We define the application $\mathcal{E} : \mathfrak{B} \to \pi_{\varphi}\left(\mathfrak{A}\right)$ in the following way:

$$\mathcal{E}\left(\widehat{\mathbf{U}}_{\varphi}^{-k}\widehat{\pi}\left(a\right)\widehat{\mathbf{U}}_{\varphi}^{k}\right) = \pi_{\varphi}\left(\Phi^{+k}\left(a\right)\right), \quad a \in \mathfrak{A}. \tag{37}$$

[8]Then i is a unital homomorphism.

We now verify that the application $\mathcal{E}$ is well defined.
Let

$$\widehat{\mathbf{U}}_{\varphi}^{-k}\widehat{\pi}\left(a\right)\widehat{\mathbf{U}}_{\varphi}^{k}=\widehat{\mathbf{U}}_{\varphi}^{-h}\widehat{\pi}\left(b\right)\widehat{\mathbf{U}}_{\varphi}^{h},$$

we obtain for each $c\in\mathfrak{A}$ the following equalities:

$$\left\langle \pi_{\varphi}\left(c\right)\Omega_{\varphi},\pi_{\varphi}\left(\Phi^{+k}\left(a\right)\right)\Omega_{\varphi}\right\rangle =\varphi\left(c^{*}\Phi^{+k}\left(a\right)\right)=\varphi\left(\Phi^{k}\left(c^{*}\right)a\right)=$$

$$=\left\langle \widehat{\pi}\left(\Phi^{k}\left(c\right)\right)\Omega_{\widehat{\varphi}},\widehat{\pi}\left(a\right)\Omega_{\widehat{\varphi}}\right\rangle =\left\langle \widehat{\mathbf{U}}_{\varphi}^{k}\widehat{\pi}\left(c\right)\Omega_{\widehat{\varphi}},\widehat{\pi}\left(a\right)\Omega_{\widehat{\varphi}}\right\rangle =$$

$$=\left\langle \widehat{\pi}\left(c\right)\Omega_{\widehat{\varphi}},\widehat{\mathbf{U}}_{\varphi}^{-k}\widehat{\pi}\left(a\right)\widehat{\mathbf{U}}_{\varphi}^{k}\Omega_{\widehat{\varphi}}\right\rangle =\left\langle \widehat{\pi}\left(c\right)\Omega_{\widehat{\varphi}},\widehat{\mathbf{U}}_{\Phi}^{-h}\widehat{\pi}\left(b\right)\widehat{\mathbf{U}}_{\Phi}^{h}\Omega_{\widehat{\varphi}}\right\rangle =$$

$$=\varphi\left(\Phi^{h}\left(c^{*}\right)b\right)=\varphi\left(\Phi^{h}\left(c^{*}\right)b\right)=\left\langle \pi_{\varphi}\left(c\right)\Omega_{\varphi},\pi_{\varphi}\left(\Phi^{+h}\left(b\right)\right)\Omega_{\varphi}\right\rangle .$$

Then

$$\pi_{\varphi}\left(\Phi^{+k}\left(a\right)\right)\Omega_{\varphi}=\pi_{\varphi}\left(\Phi^{+h}\left(b\right)\right)\Omega_{\varphi}$$

and since the vector Ω_{φ} is separating, for $\pi\left(\mathfrak{A}\right)$ we have $\pi_{\varphi}\left(\Phi^{+k}\left(a\right)\right)=\pi_{\varphi}\left(\Phi^{+h}\left(b\right)\right).$
The linear application $\mathcal{E}:\mathfrak{B}\rightarrow\pi_{\varphi}\left(\mathfrak{A}\right)$ is a positive continuous map, since for each $a\in\mathfrak{A}$ we have

$$\left\Vert \mathcal{E}\left(\widehat{\mathbf{U}}_{\varphi}^{-k}\widehat{\pi}\left(a\right)\widehat{\mathbf{U}}_{\varphi}^{k}\right)\right\Vert =\left\Vert \pi_{\varphi}\left(\Phi^{+k}\left(a\right)\right)\right\Vert \leq\left\Vert a\right\Vert =\left\Vert \widehat{\mathbf{U}}_{\varphi}^{-k}\widehat{\pi}\left(a\right)\widehat{\mathbf{U}}_{\varphi}^{k}\right\Vert ,$$

and

$$\mathcal{E}\left(\left(\widehat{\mathbf{U}}_{\varphi}^{-k}\widehat{\pi}\left(a\right)\widehat{\mathbf{U}}_{\varphi}^{k}\right)^{*}\left(\widehat{\mathbf{U}}_{\varphi}^{-k}\widehat{\pi}\left(a\right)\widehat{\mathbf{U}}_{\varphi}^{k}\right)\right)=\pi_{\varphi}\left(\Phi^{+k}\left(a^{*}a\right)\right)\geq0,$$

moreover for each $a\in\mathfrak{A}$ and $X\in\mathfrak{B}$ we have

$$\mathcal{E}\left(\widehat{\pi}\left(a\right)X\right)=\pi_{\varphi}\left(a\right)\mathcal{E}\left(X\right). \tag{38}$$

In fact, if $X=\widehat{\mathbf{U}}_{\varphi}^{-k}\widehat{\pi}\left(x\right)\widehat{\mathbf{U}}_{\varphi}^{k}$ and $\widehat{\pi}\left(a\right)=\widehat{\mathbf{U}}_{\varphi}^{-k}\widehat{\pi}\left(y\right)\widehat{\mathbf{U}}_{\varphi}^{k}$ with $x,y\in\mathfrak{A}$, we have for each $b\in\mathfrak{A}$ that

$$\left\langle \pi_{\varphi}\left(b\right)\Omega_{\varphi},\pi_{\varphi}\left(a\right)\mathcal{E}\left(X\right)\Omega_{\varphi}\right\rangle =\left\langle \pi_{\varphi}\left(b\right)\Omega_{\varphi},\pi_{\varphi}\left(a\right)\pi_{\varphi}\left(\Phi^{+k}\left(x\right)\right)\Omega_{\varphi}\right\rangle =$$

$$=\varphi\left(b^{*}a\Phi^{+k}\left(x\right)\right)=\varphi\left(\Phi^{k}\left(b^{*}a\right)x\right)=\left\langle \widehat{\pi}\left(\Phi^{k}\left(a^{*}b\right)\right)\Omega_{\widehat{\varphi}},\widehat{\pi}\left(x\right)\Omega_{\widehat{\varphi}}\right\rangle =$$

$$=\left\langle \widehat{\pi}\left(b\right)\Omega_{\widehat{\varphi}},\widehat{\pi}\left(a\right)\widehat{\mathbf{U}}_{\varphi}^{k*}\widehat{\pi}\left(x\right)\widehat{\mathbf{U}}_{\varphi}^{k}\Omega_{\widehat{\varphi}}\right\rangle =\left\langle \widehat{\pi}\left(b\right)\Omega_{\widehat{\varphi}},\widehat{\mathbf{U}}_{\varphi}^{-k}\widehat{\pi}\left(yx\right)\widehat{\mathbf{U}}_{\varphi}^{k}\Omega_{\widehat{\varphi}}\right\rangle =$$

$$=\varphi\left(\Phi^{k}\left(b^{*}\right)yx\right)=\varphi\left(b^{*}\Phi^{+k}\left(yx\right)\right)=\left\langle \pi_{\varphi}\left(b\right)\Omega_{\varphi},\pi_{\varphi}\left(\Phi^{+k}\left(yx\right)\right)\Omega_{\varphi}\right\rangle .$$

It follow that

$$\pi_{\varphi}\left(a\right)\mathcal{E}\left(X\right)\Omega_{\varphi}=\mathcal{E}\left(\widehat{\pi}\left(a\right)X\right)\Omega_{\varphi}=\pi_{\varphi}\left(\Phi^{+k}\left(yx\right)\right)\Omega_{\varphi},$$

again, the vector Ω_{φ} is separating for $\pi\left(\mathfrak{A}\right)$ then the relation 38 it's hold.
Then for each $a,b\in\mathfrak{A}$ and $X\in\mathfrak{B}$ we have:

$$\mathcal{E}\left(\widehat{\pi}\left(a\right)X\widehat{\pi}\left(b\right)\right)=\pi_{\varphi}\left(a\right)\mathcal{E}\left(X\right)\pi_{\varphi}\left(b\right),$$

moreover for each $a_{i}\in\mathfrak{A}$ and $X_{i}\in\mathfrak{B},\ \ i=1,2,..m$, we obtain:

$$\sum_{i,j}\pi_{\varphi}\left(a_{i}^{*}\right)\mathcal{E}\left(X_{i}^{*}X_{j}\right)\pi_{\varphi}\left(a_{j}\right)=\sum_{i,j}\mathcal{E}\left(\widehat{\pi}\left(a_{i}^{*}\right)X_{i}^{*}X_{j}\widehat{\pi}\left(a_{j}\right)\right)\geq0,$$

it follow that the map $\mathcal{E} : \mathfrak{B} \to \pi_{\varphi}(\mathfrak{A})$ is a cp-map and it is extended for continuity to all the C*-algebra $\widehat{\mathfrak{A}}$.
We define the following state $\widehat{\varphi}$ on the C*-algebra $\widehat{\mathfrak{A}}$

$$\widehat{\varphi}(X) = \varphi(\mathcal{E}(x)).$$

In conclusion, we have the following commutative diagram:

$$\begin{array}{ccc} \widehat{\mathfrak{A}} & \xrightarrow{\widehat{\Phi}^n} & \widehat{\mathfrak{A}} \\ \pi_{\varphi} \uparrow & & \downarrow \mathcal{E} \\ \mathfrak{A} & \xrightarrow{\Phi^n} & \mathfrak{A} \end{array}$$

with

$$\widehat{\varphi}\left(\widehat{\Phi}(X)\right) = \widehat{\varphi}(X),$$

for each $X \in \widehat{\mathfrak{A}}$, since:

$$\widehat{\varphi}\left(\widehat{\Phi}(X)\right) = \widehat{\varphi}\left(\widehat{\mathbf{U}}_{\varphi}^{-k+1}\widehat{\pi}(x)\widehat{\mathbf{U}}_{\varphi}^{k+1}\right) = \varphi\left(\Phi^{+(k+1)}(x)\right) = \varphi\left(\Phi^{+k}(x)\right) = \widehat{\varphi}(\mathcal{E}(X)).$$

□

We analyze the ergodic properties of the dilation determined by the preceding theorem.

THEOREM 1.3. *If the state φ of $(\mathfrak{A}, \Phi, \varphi)$ is ergodic [weakly mixing] then the state $\widehat{\varphi}$ of the dilation $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}, i, \mathcal{E}\right)$ is ergodic [weakly mixing].*

PROOF. Let $X, Y \in \widehat{\mathfrak{A}}$ with $X = \widehat{\mathbf{U}}_{\varphi}^{-n}\widehat{\pi}(x)\widehat{\mathbf{U}}_{\varphi}^{n}$ and $Y = \widehat{\mathbf{U}}_{\varphi}^{-m}\widehat{\pi}(y)\widehat{\mathbf{U}}_{\varphi}^{m}$. We determine the following limit:

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\left[\widehat{\varphi}\left(X\widehat{\Phi}^k(Y)\right) - \widehat{\varphi}(X)\widehat{\varphi}(Y)\right].$$

.For each $k \geq m$ we have:

$$\begin{aligned} \widehat{\varphi}\left(X\widehat{\Phi}^k(Y)\right) &= \widehat{\varphi}\left(\widehat{\mathbf{U}}_{\varphi}^{-n}\widehat{\pi}(x)\widehat{\mathbf{U}}_{\varphi}^{n}\widehat{\mathbf{U}}_{\varphi}^{(-m+k)}\widehat{\pi}(y)\widehat{\mathbf{U}}_{\varphi}^{(m-k)}\right) = \\ &= \widehat{\varphi}\left(\widehat{\mathbf{U}}_{\varphi}^{-n}\widehat{\pi}(x)\widehat{\mathbf{U}}_{\varphi}^{n}\widehat{\pi}\left(\Phi^{(k-m)}(y)\right)\right) = \\ &= \varphi\left(\mathcal{E}\left(\widehat{\mathbf{U}}_{\varphi}^{-n}\widehat{\pi}(x)\widehat{\mathbf{U}}_{\varphi}^{n}\widehat{\pi}\left(\Phi^{(k-m)}(y)\right)\right)\right) = \\ &= \varphi\left(\Phi^{+n}(x)\left(\Phi^{k-m}(y)\right)\right) = \varphi\left(x\left(\Phi^{k-m+n}(y)\right)\right). \end{aligned}$$

Then

$$\begin{aligned} \widehat{\varphi}\left(X\widehat{\Phi}^k(Y)\right) - \widehat{\varphi}(X)\widehat{\varphi}(Y) &= \varphi\left(x\left(\Phi^{(k-m+n)}(y)\right)\right) - \varphi\left(\Phi^{+n}(x)\right)\varphi\left(\Phi^{+m}(y)\right) = \\ &= \varphi\left(x\left(\Phi^{(k-m+n)}(y)\right)\right) - \varphi(x)\varphi(y) \end{aligned}$$

It follows that

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\left[\widehat{\varphi}\left(X\widehat{\Phi}^k(Y)\right) - \widehat{\varphi}(X)\widehat{\varphi}(Y)\right] =$$

$$= \lim_{N\to\infty} \frac{1}{N+1} \sum_{k=m}^{N} \left[\varphi\left(x \left(\Phi^{(k-m+n)}(y) \right) \right) - \varphi(x)\varphi(y) \right] =$$

$$= \lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \left[\varphi\left(x \left(\Phi^{k}(y) \right) \right) - \varphi(x)\varphi(y) \right] = 0.$$

The proof of the weakly mixing is performed in the same way. □

We conclude this section with the following remark

REMARK 1.10. *Let* $(\mathfrak{A}, \Phi, \varphi)$ *be* C^**-dynamical system with faithful state* φ*. If the dynamic* Φ *admit a multiplicative* φ*-adjoint* Φ^+ *the operator* $\mathbf{U}_\varphi^*$ *is isometric. Then exchanging the roles, in the precedent theorem, of* Φ *with* Φ^+ *and of* $\mathbf{U}_\varphi$ *with* $\mathbf{U}_\varphi^*$*, it is easy to verify that also in this case the dynamic system* $(\mathfrak{A}, \Phi, \varphi)$ *admits a revesible dilation with "good" ergodic properties.*

CHAPTER 2

Towards the reversible dilations

We will use the generalization of the Stinespring theorem of the precedent chapter to establish the existence of a Markov multiplicative dilation for a generic C*-dynamical system. The proof founds it on the property of particular operator system associated to our system. In this section we also recover a results on the existence of dilation for W^*-dynamical systems determined by Muhly and Solel in [**16**].

1. Multiplicative dilation

Let $(\mathfrak{A}, \Phi, \varphi)$ be a C*-dynamical system with $\mathfrak{A}$ a C*-subalgebra of $\mathcal{B}(\mathcal{H})$ and $(\pi_\infty, \mathcal{H}_\infty, \mathbf{V}_\infty)$ its Stinespring representation of theorem 4.
Let $\mathbf{U}$ be the Nagy Foiaş dilation of the the linear contraction $\mathbf{V}_\infty^*$:

$$\mathbf{U} = \begin{vmatrix} \mathbf{V}_\infty^* & \mathbf{0} \\ \mathbf{C}_1 & \mathbf{W} \end{vmatrix}, \tag{39}$$

it is the minimal isometric dilation of $\mathbf{V}_\infty^*$.
The defectes operator $\mathbf{D}_{\mathbf{V}^*} = \sqrt[2]{\mathbf{I} - \mathbf{V}_\infty \mathbf{V}_\infty^*}$ of $\mathbf{V}_\infty^*$ coincides with the orthogonal projection $\mathbf{F} = \mathbf{I} - \mathbf{V}_\infty \mathbf{V}_\infty^*$ on $\ker \mathbf{V}_\infty^*$, therefore

$$\mathcal{K} = \mathcal{H} \oplus l^2 (\ker \mathbf{V}_\infty^*)$$

and for each $h \in \mathcal{H}$ we have

$$\mathbf{C}_1 h = (\mathbf{F}h, 0, ...0...) .$$

Moreover for each $(\xi_0, \xi_1, \xi_2......) \in l^2 (\ker \mathbf{V}_\infty^*)$ we get:

$$(\mathbf{C}_1 \mathbf{C}_1^* + \mathbf{W}\mathbf{W}^*) (\xi_0, \xi_1, \xi_2...0...) = (\xi_0, 0, ...0...) + (0, \xi_1, \xi_2...0...) = (\xi_0, \xi_1, \xi_2...0...)$$

then $\mathbf{C}_1 \mathbf{C}_1^* = \mathbf{I} - \mathbf{W}\mathbf{W}^*$ it follows that the operator $\mathbf{U}$ is an unitary.

Remark 2.1. *The operator* $\mathbf{U}^*$ *is the minimal unitary dilation of the isometry* $\mathbf{V}_\infty$.

We observe that for each $n \in \mathbb{N}$ the operator $\mathbf{U}^n$ is of the type

$$\mathbf{U}^n = \begin{vmatrix} \mathbf{V}_\infty^{*^n} & \mathbf{0} \\ \mathbf{C}_n & \mathbf{W}^n \end{vmatrix}, \tag{40}$$

while for the operator $\mathbf{C}_n : \mathcal{H}_\infty \to l^2 (\ker \mathbf{V}_\infty^*)$ we obtain

$$\mathbf{C}_n = \sum_{j=0}^{n-1} \mathbf{W}^{(n-1)-j} \mathbf{C}_1 \mathbf{V}_\infty^{j^*} \tag{41}$$

with $\mathbf{C}_0 = 0$.
In fact we give

$$\mathbf{U}^n\mathbf{U} = \begin{vmatrix} \mathbf{V}_\infty^{(n+1)^*} & \mathbf{0} \\ \mathbf{C}_n\mathbf{V}^* + \mathbf{W}^n\mathbf{C}_1 & \mathbf{W}^{n+1} \end{vmatrix} = \begin{vmatrix} \mathbf{V}_\infty^{(n+1)^*} & \mathbf{0} \\ \mathbf{C}_{n+1} & \mathbf{W}^{n+1} \end{vmatrix} \mathbf{U}^{n+1}$$

and for induction follow that
$\mathbf{C}_{n+1} = \mathbf{C}_n\mathbf{V}_\infty^* + \mathbf{W}^n\mathbf{C}_1 = \left(\sum_{j=0}^{n-1} \mathbf{W}^{(n-1)-j}\mathbf{C}_1\mathbf{V}_\infty^{j^*}\right)\mathbf{V}_\infty^* + \mathbf{W}^n\mathbf{C}_1 =$
$= \sum_{j=0}^{n} \mathbf{W}^{(n-1)-j}\mathbf{C}_1\mathbf{V}_\infty^{j^*}$.
For each $\Psi \in \mathcal{H}$ and $n > 0$ we obtain:

$$\mathbf{C}_n\Psi = \left(\mathbf{F}\mathbf{V}_\infty^{(n-1)^*}\Psi, \mathbf{F}\mathbf{V}_\infty^{(n-2)^*}h, \ldots \overset{(n-1)\ step}{\mathbf{F}\Psi}, \mathbf{0}, ..\mathbf{0}..\right). \tag{42}$$

while for each $\bigoplus_{j=0}^{\infty}\xi_j \in l^2(\ker \mathbf{V}_\infty^*)$ we have:

$$\mathbf{C}_n^* \bigoplus_{j=0}^{\infty}\xi_j = \sum_{j=1}^{n} \mathbf{V}_\infty^{(n-j)}\mathbf{F}\xi_{j-1}. \tag{43}$$

In fact we have
$\mathbf{C}_n^* \bigoplus_{i=0}^{\infty}\xi_i = \sum_{j=0}^{n-1} \mathbf{V}^j\mathbf{C}_1^*\mathbf{W}^{(n-1)-j^*} \bigoplus_{i=0}^{\infty}\xi_i = \sum_{j=0}^{n-1} \mathbf{V}_\infty^j\mathbf{C}_1^*(\xi_{n-1-j}, \xi_{n-j}, \xi_{n+1-j}, \ldots) =$
$= \sum_{j=0}^{n-1} \mathbf{V}_\infty^j\mathbf{F}\xi_{n-1-j} = \sum_{j=1}^{n} \mathbf{V}_\infty^{(n-j)}\mathbf{F}\xi_{j-1}$.
By the unitary property of the operator $\mathbf{U}$, we have the following relations:

$$\mathbf{C}_m^*\mathbf{C}_n = \left[\mathbf{V}_\infty^{n^*}; \mathbf{V}_\infty^m\right] = \mathbf{V}_\infty^{n^*}\mathbf{V}_\infty^m - \mathbf{V}_\infty^m\mathbf{V}_\infty^{n^*} \tag{44}$$

while

$$\mathbf{C}_m\mathbf{C}_n^* = \left[\mathbf{W}^{n^*}; \mathbf{W}^m\right] = \mathbf{W}^{n^*}\mathbf{W}^m - \mathbf{W}^m\mathbf{W}^{n^*}. \tag{45}$$

Furthermore

$$\mathbf{C}_n\mathbf{V}_\infty^m = \begin{cases} \mathbf{C}_{n-m} & n > m \\ \mathbf{0} & n \le m \end{cases}; \quad \text{and} \quad \mathbf{C}_m^*\mathbf{W}^n = \begin{cases} \mathbf{0} & n \ge m \\ \mathbf{C}_{n-m}^* & n < m \end{cases};$$

We observe that for $n \in \mathbb{N}$ we have: $\mathbf{C}_n\mathbf{V}^n = \mathbf{W}^{n^*}\mathbf{C}_n = 0$.
For unitary operator $\mathbf{U}$ we have the follow property:

PROPOSITION 2.1. *The unitary operator* $\mathbf{U}$ *satisfies the relation*

$$\ker(I - \mathbf{U}) = \ker(I - \mathbf{U}^*) = 0.$$

Furthermore for each $\Psi \in \mathcal{K}$, *we have*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \mathbf{U}^k\Psi = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \mathbf{U}^{k^*}\Psi = 0$$

and

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \langle \xi, C_k\Psi \rangle = 0,$$

for each $\xi \in l^2(\ker \mathbf{V}_\infty^*)$ $\Upsilon \in \mathcal{H}_\infty$.

PROOF. Let $\Psi = \Upsilon \oplus \xi \in \mathcal{H}_\infty \oplus l^2 \left(\ker \mathbf{V}_\infty^*\right)$ with $\mathbf{U}\Upsilon \oplus \xi = \Upsilon \oplus \xi$.
For definition

$$\left|\begin{array}{cc} \mathbf{V}_\infty^* & \mathbf{0} \\ \mathbf{C}_1 & \mathbf{W} \end{array}\right| \left|\begin{array}{c} \Upsilon \\ \xi \end{array}\right| = \left|\begin{array}{c} \mathbf{V}_\infty^*\Upsilon \\ \mathbf{C}_1\Upsilon + \mathbf{W}\xi \end{array}\right| = \left|\begin{array}{c} \Upsilon \\ \xi \end{array}\right|$$

and $\ker\left(\mathbf{I} - \mathbf{V}_\infty^*\right) = \{0\}$ it follow that $\Upsilon = 0$ and $\mathbf{W}\xi = \xi$ then $\xi = 0$ since

$$(0, \xi_0, \xi_1, \quad \xi_n ...) = (\xi_0, \xi_1, ...\xi_n ...) .$$

The relation $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \mathbf{U}^k \Psi = 0$ follow by the mean ergodic theory of von Neumann.
We observe that $D\text{-}\lim_{k\to\infty} \left\langle \Psi, \mathbf{U}^k\Psi \right\rangle = 0$[1].

For the second relation for each $\Psi = \Upsilon \oplus \xi \in \mathcal{K}$ we get:

$$\left\langle \Psi, \mathbf{U}^k \Psi \right\rangle = \left\langle \Upsilon, \mathbf{V}_\infty^{k^*}\Upsilon \right\rangle + \left\langle \xi, \mathbf{C}_k \Upsilon \right\rangle + \left\langle \xi, \mathbf{W}^k \xi \right\rangle ,$$

where $\lim_{k\to\infty} \left|\left\langle \xi, \mathbf{W}^k \xi \right\rangle\right|^2 = \lim_{k\to\infty} \sum_{j=0}^{k-1} \|\xi_j\|^2 = 0$ and $\lim_{k\to\infty} \left\langle \Upsilon, \mathbf{V}_\infty^{k^*}\Upsilon \right\rangle = 0$ by the proposition 1.6.
Then $D\text{-}\lim_{k\to\infty} \left\langle \Psi, \mathbf{U}^k \Psi \right\rangle = D\text{-}\lim_{k\to\infty} \left\langle \xi, \mathbf{C}_k \Upsilon \right\rangle = 0$ it follow that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left\langle \xi, \mathbf{C}_k \Upsilon \right\rangle = 0.$$

□

We have a simple proposition:

PROPOSITION 2.2. *Let $(\mathfrak{A}, \Phi, \varphi)$ be a C^*-dynamical system with $\mathfrak{A} \subset \mathfrak{B}(\mathcal{H})$. There exist an injective representation $(\mathcal{K}, \widehat{\pi})$ of the C^*algebra $\mathfrak{A}$ and a isometry $\mathbf{J} : \mathcal{H} \to \mathcal{K}$ such that for each $a \in \mathfrak{A}$ and natural number $n \geq 0$, we have:*

$$\mathbf{J}^* \left(\mathbf{U}^n \pi(a) \mathbf{U}^{n*}\right) \mathbf{J} = \pi\left(\Phi^n(a)\right).$$

PROOF. From the corollary 1.1 there exists an isometric covariant representation $(\pi, \mathcal{H}, \mathbf{V})$ of Φ and an unital homomorphism $\theta : \mathfrak{A} \to \mathfrak{B}\left(\ker\left(\mathbf{V}^*\right)\right)$.
For each $a \in \mathfrak{A}$ we define the representation

$$\widehat{\pi}(a) = \left|\begin{array}{cc} \pi(a) & \mathbf{0} \\ \mathbf{0} & \mathbf{\Theta}(a) \end{array}\right|, \tag{46}$$

where for each $\xi_j \in \ker \mathbf{V}^*$ with $j \in \mathbb{N}$:

$$\mathbf{\Theta}(a) \bigoplus_{j=0}^{\infty} \xi_j = \bigoplus_{j=0}^{\infty} \theta(a) \xi_j ,$$

The representation $\widehat{\pi}$ is injective map and for each natural number $n \geq 0$ we have:

$$\mathbf{U}^n \widehat{\pi}(a) \mathbf{U}^{n*} = \left|\begin{array}{cc} \pi\left(\Phi^n(a)\right); & \mathbf{V}^{n^*} \pi(a) \mathbf{C}_n^* \\ \mathbf{C}_n \pi(a) \mathbf{V}^n; & \mathbf{C}_n \pi(a) \mathbf{C}_n^* + \mathbf{W}^n \mathbf{\Theta}(a) \mathbf{W}^{n*} \end{array}\right|. \tag{47}$$

If $\mathbf{J}$ is defined by $\mathbf{J}h = h \oplus 0$ for every $h \in \mathcal{H}$, we have the thesis. □

[1] Cfr. appendix.

For each $X \in \mathcal{B}(\mathcal{K})$ we define

$$\mathcal{E}_{1,1}(X) = \mathbf{J}^* X \mathbf{J}. \tag{48}$$

The map $\mathcal{E}_{1,1} : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ is a normal cp-map and for each $X \in \mathcal{B}(\mathcal{K})$, and $a, b \in \mathfrak{A}$ we obtain

$$\mathcal{E}_{1,1}(\widehat{\pi}(a) X \widehat{\pi}(b)) = \pi(a) \mathcal{E}_{1,1}(X) \pi(b).$$

Since if $X = |X_{i,j}|_{i,j=1,2}$ we have:

$$\begin{vmatrix} \pi(a) & \mathbf{0} \\ \mathbf{0} & \mathbf{\Theta}(a) \end{vmatrix} \begin{vmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{vmatrix} \begin{vmatrix} \pi(b) & \mathbf{0} \\ \mathbf{0} & \mathbf{\Theta}(b) \end{vmatrix} = \begin{vmatrix} \pi(a) X_{1,1} \pi(b) & * \\ * & * \end{vmatrix}.$$

* * *

Theorem 2.1. *Let $(\mathfrak{A}, \Phi, \varphi)$ be a C^*-dynamical system with $\mathfrak{A} \subset \mathfrak{B}(\mathcal{H})$. There is a C^*-dynamical system $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}\right)$, where $\widehat{\mathfrak{A}}$ is the C^*-subalgebra of $\mathfrak{B}(\mathcal{K})$ thus definied:*

$$\widehat{\mathfrak{A}} = C^* \left(\bigcup_{n \geq 0} \mathbf{U}^n \widehat{\pi}(\mathfrak{A}) \mathbf{U}^{n^*} \right); \tag{49}$$

*while the injective *-homorphism $\widehat{\Phi} : \widehat{\mathfrak{A}} \to \widehat{\mathfrak{A}}$ is defined by:*

$$\widehat{\Phi}(X) = \mathbf{U} X \mathbf{U}^*, \qquad X \in \widehat{\mathfrak{A}}; \tag{50}$$

and the state $\widehat{\varphi}$ on $\widehat{\mathfrak{A}}$ is

$$\widehat{\varphi}(X) = \varphi_\sharp(\mathcal{E}X), \qquad X \in \widehat{\mathfrak{A}},$$

where $\varphi_\sharp$ is a state on $\mathcal{B}(\mathcal{H})$ that extends φ;
such that for each $n \in \mathbb{N}$

$$\begin{array}{ccc} \widehat{\mathfrak{A}} & \stackrel{\widehat{\Phi}^n}{\longrightarrow} & \widehat{\mathfrak{A}} \\ \widehat{\pi} \uparrow & & \downarrow \mathcal{E}_{1,1} \\ \mathfrak{A} & \stackrel{\Phi^n}{\longrightarrow} & \mathcal{B}(\mathcal{H}) \end{array}$$

is a commutative diagram:

$$\mathcal{E}_{1,1}\left(\widehat{\Phi}^n(\widehat{\pi}(a))\right) = \Phi^n(a), \qquad a \in \mathfrak{A};$$

where the cp map $\widehat{\pi} : \mathfrak{A} \to \widehat{\mathfrak{A}}$ is the representation defined in 46 while $\mathcal{E} : \widehat{\mathfrak{A}} \to \mathcal{B}(\mathcal{H})$ is the unital cp-map defined by the relation 48;

Proof. We have for each $a \in \mathfrak{A}$:

$$\mathcal{E}_{1,1}\left(\widehat{\Phi}^n(\widehat{\pi}(A))\right) = \mathbf{J}^* \widehat{\Phi}^n(\widehat{\pi}(a)) \mathbf{J} = \mathbf{J}^* \left(\begin{vmatrix} \mathbf{V}^{n^*} \pi(a) \mathbf{V}^n & * \\ * & * \end{vmatrix} \right) \mathbf{J} = \Phi^n(a).$$

Let $\Phi_o : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ the unital cp-map defined by

$$\Phi_o(A) = \mathbf{V}^{n^*} A \mathbf{V}^n, \qquad A \in \mathcal{B}(\mathcal{H})$$

and φ_o Hahn-Banach extension of φ on $\mathcal{B}(\mathcal{H})$.
We set

$$\varphi_n = \frac{1}{n+1} \sum_{k \geq 0 \omega}^{n} \varphi_o \circ \Phi_o^k$$

the set $\{\varphi_n\}_{n\in\mathbb{N}}$ is a net of the unital ball $\mathcal{B}(\mathcal{H})_1^*$ of the fuctional on $\mathcal{B}(\mathcal{H})$. It is well known that the set $\mathcal{B}(\mathcal{H})_1^*$ is w^*-compact. Then our net admits at least a point limit $\varphi_\sharp$ that belong to $\mathcal{B}(\mathcal{H})_1^*$:

$$\varphi_\sharp = w^* - \lim_i \varphi_{n_i} \tag{51}$$

Moreover $\varphi_\sharp$ is Φ_o-invariant and for each $a \in \mathfrak{A}$ we have that $\varphi_\sharp(a) = \varphi(a)$.
Since for each $N \in \mathbb{N}$ we obtain:

$$\varphi_N(a) = \frac{1}{n+1}\sum_{k\geq 0\omega}^{n} \varphi_o\left(\Phi_o^k(a)\right)\frac{1}{n+1}\sum_{k\geq 0\omega}^{n}\varphi\left(\Phi^k(a)\right) = \varphi(a).$$

The state $\widehat{\varphi}$ is a $\widehat{\Phi}$-invariant since for definition, for each $X \in \widehat{\mathfrak{A}}$, we get

$$\widehat{\varphi}\left(\widehat{\Phi}(X)\right) = \varphi_\sharp\left(\mathcal{E}\widehat{\Phi}(X)\right) = \varphi_\sharp(\mathbf{V}^* X_{1,1}\mathbf{V}) = \varphi_\sharp(X_{1,1}) = \widehat{\varphi}(X),$$

in fact

$$\widehat{\Phi}(X) = \mathbf{U}\begin{vmatrix} X_{1,1}; & X_{1,2} \\ X_{2,1}; & X_{2,2}\end{vmatrix}\mathbf{U}^* = \begin{vmatrix} \mathbf{V}^*\pi(a)\mathbf{V} & * \\ * & * \end{vmatrix}. \tag{52}$$

□

The preceding theorem leads to a result that it approaches of very to our definition of Markov dilation for a C*-dynamic system. To get a dilation in our sense, we have to determine a good algebra $\mathfrak{B}$ of $\mathcal{B}(\mathcal{K})$ with the following property:

$$\widehat{\pi}(\mathfrak{A}) \subset \mathfrak{B} \text{ with } \mathcal{E}_{1,1}(\mathfrak{B}) \subset \mathfrak{A} \text{ and } \mathbf{U}\mathfrak{B}\mathbf{U}^* \subset \mathfrak{B}.$$

In this way we get that the cp-map $\mathcal{E}_{1,1} : \widehat{\mathfrak{A}} \to \mathcal{B}(\mathcal{H})$ is a conditional expectation between $\widehat{\mathfrak{A}}$ and $\widehat{\pi}(\mathfrak{A})$.
This will be the purpose of the next paragraph.

1.1. The construction of multiplicative dilations. Let $\Phi : \mathfrak{A} \to \mathfrak{A}$ be a cp map with $\mathfrak{A} \subset \mathfrak{B}(\mathcal{H})$, the triple $(\pi_\infty, \mathcal{H}_\infty, \mathbf{V}_\infty)$ is the isometric covariant representation 4 of Φ and $\mathbf{U}$ be Nagy isometry dilation of $\mathbf{V}^*$ on the Hilbert space $\mathcal{K} = \mathcal{H} \oplus l^2(\ker \mathbf{V}^*)$.
Let $\Gamma : l^2(\ker \mathbf{V}^*) \to \mathcal{H}$ the linear operator so defined:

$$\Gamma = \sum_{k\geq 0}^{n} \mathbf{V}^{(k+1)^*}\pi_\infty(a_k)\,\mathbf{C}_1^*\mathbf{W}^{k^*}, \qquad a_k \in \mathfrak{A}, \quad k = 1, 2...n. \tag{53}$$

The operator Γ is say be a ($\mathbf{U}$,Φ)-associated operator.
With a simple calculus for each $\xi_i \in \ker \mathbf{V}^*$, we obtain that

$$\Gamma \bigoplus_{i=0}^{\infty}\xi_i = \sum_{k=0}^{n}\mathbf{V}^{(k+1)^*}\pi_\infty(a_k)\,\mathbf{F}\xi_k \tag{54}$$

while for each $h \in \mathcal{H}$:

$$\Gamma^* h = \left(\mathbf{F}\pi_\infty(a_0^*)\,\mathbf{V}h, \mathbf{F}\pi_\infty(a_1^*)\,\mathbf{V}^2 h, ...\mathbf{F}\pi_\infty(a_n^*)\,\mathbf{V}^{n+1}h, 0....\right). \tag{55}$$

REMARK 2.2. *If the elements a_k belong to the multiplicative domains of Φ, we get that*

$$\Gamma = \sum_{k\geq 0}^{n} \mathbf{V}^{(k+1)^*} \pi_\infty (a_k) \mathbf{C}_1^* \mathbf{W}^{k^*} = 0.$$

In fact for each $k = 1, 2..n$ we obtain:

$$\pi_\infty (a_k) \mathbf{C}_1^* \bigoplus_{i=0}^{\infty} \xi_i = \pi_\infty (a_k) \mathbf{F} l_0 = \mathbf{F} \pi_\infty (a_k) \xi_0.$$

Therefore in the multiplicative case the only $(\mathbf{U},\Phi)$*-associated operator are the void operators.*

We have a first fundamental proposition:

PROPOSITION 2.3. *For every* $(\mathbf{U},\Phi)$*-associated operators* Γ_1 *and* Γ_2*, we have the following result:*

$$\Gamma_1 \Gamma_2^* \in \pi_\infty (\mathfrak{A})$$

in particulary if $\Gamma_i = \sum_{k\geq 1}^{n_i} \mathbf{V}^{k^*} \pi_\infty (a_{i,k}) \mathbf{C}_1^* \mathbf{W}^{(k-1)^*}$ $i = 1, .2$ *we have:*

$$\Gamma_1 \Gamma_2^* = \pi_\infty \left(\Phi^{k-1} \left(\sum_{k\geq 1}^{n} \left[\Phi \left(a_{1,k} a_{2,k}^* \right) - \Phi (a_{1,k}) \Phi \left(a_{2,k}^* \right) \right] \right) \right).$$

PROOF. We have:

$$\Gamma_1 \Gamma_2^* = \sum_{k\geq 0}^{n} \mathbf{V}^{(k+1)^*} \pi_\infty (a_{1,k}) \mathbf{C}_1^* \mathbf{W}^{k^*} \cdot \sum_{j\geq 0}^{n} \mathbf{W}^j \mathbf{C}_1 \pi_\infty \left(a_{2,k}^* \right) \mathbf{V}^{j+1} =$$

$$= \sum_{k,j\geq 0}^{n} \mathbf{V}^{(k+1)^*} \pi_\infty (a_{1,k}) \mathbf{C}_1^* \mathbf{W}^{k^*} \mathbf{W}^j \mathbf{C}_1 \pi_\infty \left(a_{2,k}^* \right) \mathbf{V}^{j+1};$$

and for the relations 45 we obtain:

$$\mathbf{C}_1^* \mathbf{W}^{(k-1)^*} \mathbf{W}^{j-1} \mathbf{C}_1 = \mathbf{C}_1^* \mathbf{C}_1 \delta_{i,j} \quad \text{where} \quad \delta_{i,j} = \begin{cases} \mathbf{I} & k = j \\ 0 & k \neq j \end{cases}$$

It follow that:

$$\Gamma_1 \Gamma_2^* = \sum_{k\geq 1}^{n} \mathbf{V}^{k^*} \pi_\infty (a_{1,k}) \mathbf{C}_1^* \mathbf{C}_1 \pi_\infty \left(a_{2,k}^* \right) \mathbf{V}^k =$$

$$= \sum_{k\geq 1}^{n} \mathbf{V}^{k^*} \pi_\infty (a_{1,k}) (\mathbf{I} - \mathbf{V}\mathbf{V}^*) \pi_\infty \left(a_{2,k}^* \right) \mathbf{V}^k =$$

$$= \pi_\infty \left(\sum_{k\geq 1}^{n} \left[\Phi^k \left(a_{1,k} a_{2,k}^* \right) - \Phi^{k-1} \left(\Phi (a_{1,k}) \Phi \left(a_{2,k}^* \right) \right) \right] \right). \qquad \square$$

We have a new operator systems $\mathcal{S}_o$ of $\mathcal{B}\left(l^2 (\ker \mathbf{V}^*)\right)$ thus defined:

$$\mathcal{S}_o = \left\{ \mathbf{T} \in \mathcal{B}\left(l^2 (\ker \mathbf{V}^*)\right) : \Gamma_1 \mathbf{T} \Gamma_2^* \in \pi_\infty (\mathfrak{A}) \text{ for every } (\mathbf{U},\Phi)\text{-ass. op. } \Gamma_1, \Gamma_2 \right\}.$$

By the preceding proposition, we have that $\mathbf{I}\in\mathcal{S}_o$.
If $\Pi_k : l^2(\ker\mathbf{V}^*)\to\ker\mathbf{V}^*$ is the linear operator defined for each $\bigoplus_{i=0}^{\infty}\xi_i\in l^2(\ker\mathbf{V}^*)$ by

$$\Pi_j\bigoplus_{i=0}^{\infty}\xi_i=\xi_j,\quad j\in\mathbb{N},$$

and we set for every $\mathbf{T}\in\mathcal{B}\left(l^2(\ker\mathbf{V}^*)\right)$

$$\mathbf{T}=\left|\begin{array}{cccccc} T_{0,0} & T_{0,1} & \cdot & \cdot & T_{0,n} & \cdot\\ T_{1,0} & T_{1,1} & \cdot & \cdot & T_{1,n} & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ T_{m,0} & T_{m,1} & \cdot & \cdot & T_{m,n} & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right|,$$

where $T_{i,j}=\Pi_i\mathbf{T}\Pi_j^*$ for all $i,j\in\mathbb{N}$:

$$T\bigoplus_{i=0}^{\infty}\xi_i=\bigoplus_{i=0}^{\infty}\sum_{j=0}^{\infty}T_{i,j}\xi_j.$$

We study some simple property of the operator systems $\mathcal{S}_o$.

Proposition 2.4. *If* $\mathbf{T}\in\mathcal{S}_o$ *for each* Γ_1 *and* Γ_2 $(\mathbf{U},\Phi)$*-associated operators we have:*

$$\Gamma_1\mathbf{T}\Gamma_2^*=\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\mathbf{V}^{*k+1}\pi_\infty(a_{1,i})\,\mathbf{F}T_{i,j}\mathbf{F}\pi_\infty\left(a_{2,j}^*\right)\mathbf{V}^{j+1},$$

whit

$$\Gamma_i=\sum_{k\geq1}^{n_i}\mathbf{V}^{k^*}\pi_\infty(a_{i,k})\,\mathbf{C}_1^*\mathbf{W}^{(k-1)^*}\quad \textit{for}\quad i=1,2.$$

Then the linear operator T *of* $\mathcal{B}\left(l^2(\ker\mathbf{V}^*)\right)$ *belong to* S_o *if and only if for each* $a,b\in\mathfrak{A}$ *and* $i,j\in N$, *we get*

$$\mathbf{V}^{(i+1)^*}\pi_\infty(a)\,\mathbf{F}T_{i,j}\mathbf{F}\pi_\infty(b)\,\mathbf{V}^{(j+1)}\in\pi_\infty(\mathfrak{A}).$$

Proof. From the relations 54 and 55, for each $h\in\mathcal{H}$ we have:

$$\Gamma_1\mathbf{T}\Gamma_2^*h=\Gamma_1T\bigoplus_{i\in\mathbb{N}}\mathbf{F}\pi_\infty\left(a_{2,i}^*\right)\mathbf{V}^{(i+1)}h=\Gamma_1\bigoplus_{i\in\mathbb{N}}\sum_{j=0}^{n_2}T_{i,j}\mathbf{F}\pi_\infty\left(a_{2,i}^*\right)\mathbf{V}^{(j+1)}h=$$
$$=\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\mathbf{V}^{(i+1)^*}\pi_\infty(a_{1,i})\,\mathbf{F}T_{i,j}\mathbf{F}\pi_\infty\left(a_{2,i}^*\right)\mathbf{V}^{(j+1)}h.$$

□

We now analyze the existing relations between operator system $\mathcal{S}_o$ and unitary operator $\mathbf{U}$ of $\mathfrak{B}\left(\mathcal{H}\oplus l^2(\ker\mathbf{V}^*)\right)$.

Lemma 2.1. *For every* $a\in\mathfrak{A}$ *and* $(\mathbf{U},\Phi)$*-associated operator* Γ, *we have*

$$\mathbf{C}_1\pi_\infty(a)\,\mathbf{C}_1^*\in\mathcal{S}_o,\qquad \mathbf{C}_1\Gamma\mathbf{W}^*\in\mathcal{S}_o.$$

Proof. Let $\Gamma_i = \sum_{k\geq 0}^{n_i} \mathbf{V}^{(k+1)^*}\pi_\infty(a_{i,k})\,\mathbf{C}_1^*\mathbf{W}^{k^*}$ $i = 1, 2$, the ($\mathbf{U}$,Φ)-associated operators, since for each $n > 0$ we have $\mathbf{W}^{n^*}\mathbf{C}_1 = 0$ we obtain:

$$\Gamma_1\mathbf{C}_1 = \mathbf{V}^*\pi_\infty(a_{1,1})\,\mathbf{C}_1^*\mathbf{C}_1$$

then:

$\Gamma_1\mathbf{C}_1\pi_\infty(a)\,\mathbf{C}_1^*\Gamma_2^* = \mathbf{V}^*\pi_\infty(a_{1,1})\,\mathbf{C}_1^*\mathbf{C}_1^*\pi_\infty(a)\,\mathbf{C}_1^*\mathbf{C}_1\pi_\infty\left(a_{2,1}^*\right)\mathbf{V} =$
$= \mathbf{V}^*\pi_\infty(a_{1,1})\,\mathbf{F}\mathbf{F}_1\pi_\infty\left(a_{2,1}^*\right)\mathbf{V} =$
$= \pi_\infty\left(\Phi(a_{1,1}aa_{2,1}) - \Phi(a_{1,1}a)\,\Phi(a_{2,1}) - \Phi(a_{1,1})\,\Phi(aa_{2,1}) - \Phi(a_{1,1})\,\Phi(a)\,\Phi(a_{2,1})\right)$

For the second relation we have:

$\Gamma_1\mathbf{C}_1\Gamma\mathbf{W}^*\Gamma_2^* = \mathbf{V}^*\pi_\infty(a_{1,1})\,\mathbf{C}_1^*\mathbf{C}_1\Gamma\mathbf{W}^*\Gamma_2^+$

and if $\Gamma = \sum_{j\geq 0}^{n} \mathbf{V}^{*j+1}\pi_\infty(a_j)\,\mathbf{C}_1^*\mathbf{W}^{*j}$ we get:

$$\Gamma\mathbf{W}^*\Gamma_2^* = \sum_{k\geq 0}^{n} \mathbf{V}^{(k+1)^*}\pi_\infty(a_k)\,\mathbf{C}_1^*\mathbf{C}_1\pi_\infty(a_{2,k+1})\,\mathbf{V}^{(k+2)}$$

therefore

$$\Gamma_1(\mathbf{C}_1\Gamma\mathbf{W}^*)\,\Gamma_2^* = \sum_{k\geq 0}^{n} \mathbf{V}^*\pi_\infty(a_{1,1})\,\mathbf{C}_1^*\mathbf{C}_1\mathbf{V}^{(k+1)^*}\pi_\infty(a_k)\,\mathbf{C}_1^*\mathbf{C}_1\pi_\infty(a_{2,k+1})\,\mathbf{V}^{(k+2)},$$

and with a simple algebric calculus we get:

$\mathbf{V}^*\pi_\infty(a_{1,1})\,\mathbf{C}_1^*\mathbf{C}_1\mathbf{V}^{(k+1)^*}\pi_\infty(a_k)\,\mathbf{C}_1^*\mathbf{C}_1\pi_\infty(a_{2,k+1})\,\mathbf{V}^{(k+2)} =$
$= \mathbf{V}^*\pi_\infty(a_{1,1})\,(\mathbf{I} - \mathbf{V}\mathbf{V}^*)\,\mathbf{V}^{(k+1)^*}\pi_\infty(a_k)\,(\mathbf{I} - \mathbf{V}\mathbf{V}^*)\,\pi_\infty(a_{2,k+1})\,\mathbf{V}^{(k+2)} =$
$= \Phi\left(a_{1,1}\cdot\Phi^{(k+1)}(aa_{2,k+1})\right) - \Phi\left(a_{1,1}\cdot\Phi^k(\Phi(a_k)\cdot\Phi(a_{2,k+1}))\right) -$
$-\Phi(a_{1,1})\cdot\Phi^{(k+2)}(a_ka_{2,k+1}) + \Phi(a_{1,1})\left(\Phi^k(a_k)\cdot\Phi(a_{2,k+1})\right).$ □

The set

$$\mathcal{S} = \left\{\begin{vmatrix} \pi_\infty(a) & \Gamma \\ \Gamma^* & T \end{vmatrix} : a \in \mathfrak{A},\ \Gamma \text{ is a } (\mathbf{U},\Phi)\text{-ass. op. and } T \in \mathcal{S}_o\right\} \tag{56}$$

is a operator systems of $\mathcal{B}(\mathcal{K})$ with the following properties:

Proposition 2.5. *The operator system $\mathcal{S}$ is a $\mathbf{U}$-invariant set:*

$$\mathbf{U}\mathcal{S}\mathbf{U}^* \subset \mathcal{S}.$$

Proof. If $\mathbf{S} = \begin{vmatrix} \pi_\infty(a) & \Gamma \\ \Gamma^* & T \end{vmatrix}$ is an element of $\mathcal{S}$, we obtain

$$\mathbf{U}\mathbf{S}\mathbf{U}^* = \begin{vmatrix} \mathbf{V}^*\pi_\infty(a)\,\mathbf{V}; & \mathbf{V}^*\pi_\infty(a)\,\mathbf{C}_1^* + \mathbf{V}^*\Gamma\mathbf{W}^* \\ \mathbf{C}_1\pi_\infty(a)\,\mathbf{V} + \mathbf{W}\Gamma^*\mathbf{V}; & \mathbf{C}_1\pi_\infty(a)\,\mathbf{C}_1^* + \mathbf{W}\Gamma^*\mathbf{C}_1^* + \mathbf{C}_1\Gamma\mathbf{W}^* + \mathbf{W}T\mathbf{W}^* \end{vmatrix},$$

where $\mathbf{V}^*\Gamma\mathbf{W}^*$ and $\mathbf{V}^*\pi_\infty(a)\,\mathbf{C}_1^*$ are ($\mathbf{U}$,Φ)-associated operators.
For the lemma 2.1 we have $\mathbf{C}_1\pi_\infty(a)\,\mathbf{C}_1^*$, $\mathbf{W}\Gamma^*\mathbf{C}_1^* \in \mathcal{S}_o$.
Moreover $\mathbf{W}T\mathbf{W}^* \in S_o$ since we have

$$\Gamma_i\mathbf{W} = \sum_{k\geq 0}^{n_i} \mathbf{V}^{(k+1)^*}\pi_\infty(a_{i,k})\,\mathbf{C}_1^*\mathbf{W}^{k^*}\mathbf{W} = \mathbf{V}^*\sum_{k\geq 0}^{n_i-1} \mathbf{V}^{(k+1)^*}\pi_\infty(a_{i,k})\,\mathbf{C}_1^*\mathbf{W}^{k^*} = \mathbf{V}^*\widetilde{\Gamma}_i$$

where $\mathbf{C}_1^*\mathbf{W} = 0$ and $\widetilde{\Gamma}$ is the ($\mathbf{U}$,Φ)-associated operator

$$\widetilde{\Gamma} = \sum_{k\geq 0}^{n_i-1} \mathbf{V}^{(k+1)^*}\pi_\infty\left(a_{i,k}\right)\mathbf{C}_1^*\mathbf{W}^{k^*}.$$

It follow that

$$\Gamma_1\left(\mathbf{WTW}^*\right)\Gamma_2^* = \mathbf{V}^*\left(\widetilde{\Gamma}_i\mathbf{T}\widetilde{\Gamma}_i^*\right)\mathbf{V},$$

and for hyphothesis $\widetilde{\Gamma}_i\mathbf{T}\widetilde{\Gamma}_i^* \in \pi_\infty\left(\mathfrak{A}\right)$. □

The next proposition is fundamental to establish the existence of a conditional expectation between the C*-subalgebra $C^*\left(\mathcal{S}\right)$ of $\mathcal{B}\left(\mathcal{K}\right)$ generated by the operator system $\mathcal{S}$ and C*-algebra $\widehat{\pi}\left(\mathfrak{A}\right)$.
Let $X = \begin{vmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{vmatrix} \in \mathcal{B}\left(\mathcal{K}\right)$ we have the *-linear map $\mathcal{E}_{i,j}$ thus defined:

$$\mathcal{E}_{i,j}\left(X\right) = X_{i,j}. \tag{57}$$

We have a first result:

Lemma 2.2. *For each* $S_1, S_2, ...S_n \in \mathcal{S}$ *we have:*

$$\mathcal{E}_{1,1}\left(\prod_{i=1}^n S_i\right) \in \mathfrak{A}$$

Proof. We have these simple properties:

$$\mathcal{E}_{1,1}\left(\prod_{i=1}^n S_i\right) = \mathcal{E}_{1,1}\left(\prod_{i=1}^{n-1} S_i\right)\mathcal{E}_{1,1}\left(S_n\right) + \mathcal{E}_{1,2}\left(\prod_{i=1}^{n-1} S_i\right)\mathcal{E}_{2,1}\left(S_n\right);$$

$$\mathcal{E}_{1,2}\left(\prod_{i=1}^n S_i\right) = \mathcal{E}_{1,1}\left(\prod_{i=1}^{n-1} S_i\right)\mathcal{E}_{1,2}\left(S_n\right) + \mathcal{E}_{1,2}\left(\prod_{i=1}^{n-1} S_i\right)\mathcal{E}_{2,2}\left(S_n\right).$$

and for induction on the length n of the elements $\prod_{i=1}^n S_i$ we have the thesis.
In fact if $\mathcal{S}_{oo}$ is the set of operator

$$\mathcal{S}_{oo} = \left\{\pi_\infty\left(a\right)\Gamma T : a \in \mathfrak{A},\ \Gamma \text{ is a } \left(\mathbf{U},\Phi\right)\text{-associated operator and } T \in \mathcal{S}_o\right\},$$

we have:
For $n = 1$ we obtain that $\mathcal{E}_{1,1}\left(S_1\right) \in \mathfrak{A}$ and $\mathcal{E}_{1,2}\left(S_1\right) \in \mathcal{S}_{oo}$;
For $n-1$ we assumed that $\mathcal{E}_{1,1}\left(\prod_{i=1}^{n-1} S_i\right) \in \mathfrak{A}$ and $\mathcal{E}_{1,2}\left(\prod_{i=1}^{n-1} S_i\right) \in \mathcal{S}_{oo}$;
from the relations written above, we get that the assertion is true for each $n \in \mathbb{N}$. □

Proposition 2.6. *There exists a cp map* $\mathcal{E} : C^*\left(\mathcal{S}\right) \to \mathfrak{A}$ *such that:*

$$\mathcal{E}\left(X\right) = \mathcal{E}_{1,1}\left(X\right), \qquad X \in C^*\left(\mathcal{S}\right) \tag{58}$$

and for each $a_i \in \mathfrak{A}$, $T_i \in \mathcal{S}_o$, $i = 1, 2$ *and* $X \in C^*\left(\mathcal{S}\right)$, *we have:*

$$\mathcal{E}\left(\left(\pi_\infty\left(a_1\right) \oplus T_1\right)X\left(\pi_\infty\left(a_2\right) \oplus T_2\right)\right) = \pi_\infty\left(a_1\right)\mathcal{E}\left(X\right)\pi_\infty\left(a_2\right)$$

PROOF. Let $\mathcal{E}_{1,1}:\mathcal{B}(\mathcal{K})\to\mathcal{B}(\mathcal{H})$ be the unital cp map 57, for each $X\in C^*(\mathcal{S})$ we obtain $\mathcal{E}_{1,1}(X)\in\mathfrak{A}$, since the elements X of $C^*(\mathcal{S})$ are sum of elements of the type $\prod_{i=1}^{n} S_i$ with $S_i\in\mathcal{S}$ for all $i=1,2...n$, from the preceding lemma the thesis follows.
With a simple calculation, for each $X\in C^*(\mathcal{S})$ and $a_1,a_2\in\mathfrak{A}$, $\mathbf{T}_1,\mathbf{T}_2\in\mathcal{S}_o$, we have

$$\left|\begin{array}{cc}\pi_\infty(a_1) & 0\\ 0 & \mathbf{T}_1\end{array}\right|\left|\begin{array}{cc}X_{1,1} & X_{1,2}\\ X_{2,1} & X_{2,2}\end{array}\right|\left|\begin{array}{cc}\pi_\infty(a_2) & 0\\ 0 & \mathbf{T}_2\end{array}\right|=\left|\begin{array}{cc}\pi_\infty(a_1)X_{1,1}\pi_\infty(a_2) & *\\ * & *\end{array}\right|$$

and

$$\mathcal{E}\left(\left|\begin{array}{cc}\pi_\infty(a_1)X_{1,1}\pi_\infty(a_2) & *\\ * & *\end{array}\right|\right)=\pi_\infty(a_1)\mathcal{E}(X)\pi_\infty(a_2).$$

□

The next proposition establishes the existence of multiplicative dilations for C*-dynamical systems.

THEOREM 2.2. *Let $(\mathfrak{A},\Phi,\varphi)$ be a C*-dynamical systems with $\mathfrak{A}\subset\mathfrak{B}(\mathcal{H})$ and let $(\pi_\infty,\mathcal{H}_\infty,\mathbf{V}_\infty)$ be the isometric covariant representation defined in 4. If there exists a *-multiplicative linear map*

$$\Theta:\mathfrak{A}\to\mathfrak{B}\left(l^2(\ker\mathbf{V}^*)\right)\tag{59}$$

such that for each $a\in\mathfrak{A}$ we get

$$\pi_\infty(a)\oplus\Theta(a)\in C^*(\mathcal{S}).$$

Then $(\mathfrak{A},\Phi,\varphi)$ admit a Markov multiplicative dilation $\left(\widehat{\mathfrak{A}},\widehat{\Phi},\widehat{\varphi},\mathcal{E},\widehat{\pi}\right)$ where:

(1) *The cp map $\widehat{\pi}:\mathfrak{A}\to C^*(\mathcal{S})$ is thus defined:*
$$\widehat{\pi}(a)=\pi_\infty(a)\oplus\Theta(a),\qquad a\in\mathfrak{A};\tag{60}$$
(2) *$\widehat{\mathfrak{A}}$ is a subalgebra with unit of $C^*(\mathcal{S})$:*
$$\widehat{\mathfrak{A}}=C^*\left(\bigcup_{n\geq 0}\mathbf{U}^n\widehat{\pi}(\mathfrak{A})\mathbf{U}^{n*};\mathbf{I}\right);\tag{61}$$
(3) *The injective *-homorphism $\widehat{\Phi}:\widehat{\mathfrak{A}}\to\widehat{\mathfrak{A}}$ is defined by:*
$$\widehat{\Phi}(X)=\mathbf{U}X\mathbf{U}^*,\qquad X\in\widehat{\mathfrak{A}};\tag{62}$$
(4) *The conditional expectation $\mathcal{E}:C^*(\mathcal{S})\to\mathfrak{A}$ is defined by the relation 58 and the state $\widehat{\varphi}$ on $\widehat{\mathfrak{A}}$ is thus defined*
$$\widehat{\varphi}(X)=\varphi(\mathcal{E}X),\qquad X\in\widehat{\mathfrak{A}}.$$

PROOF. Since $\widehat{\pi}(\mathfrak{A})\subset\mathcal{S}$ and $\mathbf{U}\mathcal{S}\mathbf{U}^*\subset\mathcal{S}$ it follows that

$$\mathbf{U}\widehat{\pi}(\mathfrak{A})\mathbf{U}^*\subset\mathbf{U}C^*(\mathcal{S})\mathbf{U}^*\subset C^*(\mathcal{S}).$$

Then $\widehat{\mathfrak{A}}\subset C^*(\mathcal{S})$ and the injective *-homorphism 62 is well defined.
For definition, the map $\widehat{\pi}:\mathfrak{A}\to\widehat{\mathfrak{A}}$ is injective *-multiplicative linear map:

$$\widehat{\pi}(a)=\left|\begin{array}{cc}\pi_\infty(a) & 0\\ 0 & \Theta(a)\end{array}\right|$$

and for each $n \in \mathbb{N}$

$$\widehat{\Phi}^n\left(\widehat{\pi}\left(a\right)\right) = \begin{vmatrix} \mathbf{V}_\infty^{n^*}\pi_\infty\left(a\right)\mathbf{V}_\infty^n; & \mathbf{V}_\infty^{n^*}\pi_\infty\left(a\right)\mathbf{C}_n^* \\ \mathbf{C}_n\pi_\infty\left(a\right)\mathbf{V}_\infty^{n^*}; & \mathbf{C}_nA\mathbf{C}_n^* + \mathbf{W}^n\mathbf{\Theta}\left(a\right)\mathbf{W}^{n*} \end{vmatrix}.$$

For each $a, b \in \mathfrak{A}$ and $X \in \widehat{\mathfrak{A}}$ we obtain

$$\mathcal{E}\left(\widehat{\pi}\left(a\right)X\widehat{\pi}\left(b\right)\right) = \pi_\infty\left(a\right)\mathcal{E}\left(X\right)\pi_\infty\left(b\right),$$

moreover

$$\mathcal{E}\left(\widehat{\Phi}^n\left(\widehat{\pi}\left(A\right)\right)\right) = \mathcal{E}\left(\begin{vmatrix} \mathbf{V}_\infty^{n^*}\pi_\infty\left(a\right)\mathbf{V}_\infty^n & * \\ * & * \end{vmatrix}\right) = \Phi^n\left(a\right).$$

For each $X \in \widehat{\mathfrak{A}}$ we have [2]:

$$\widehat{\varphi}\left(\widehat{\Phi}\left(X\right)\right) = \varphi\left(\mathcal{E}\left(\mathbf{U}^*X\mathbf{U}\right)\right) = \varphi\left(\mathbf{V}_\infty^*X_{1,1}\mathbf{V}_\infty\right) = \varphi\left(\Phi\left(\mathcal{E}X\right)\right) = \varphi\left(\mathcal{E}X\right) = \widehat{\varphi}\left(X\right).$$

□

The theorem is easily adaptable to W*-dynamical systems[3]:

THEOREM 2.3. *Let* $(\mathfrak{M}, \Phi, \varphi)$ *be a W*-dynamical systems with* $\mathfrak{M} \subset \mathfrak{B}(\mathcal{H})$ *and let* $(\pi_\infty, \mathcal{H}_\infty, \mathbf{V}_\infty)$ *be the normal isometric covariant representation defined in 4. If there exists a normal *-multiplicative linear map*

$$\Theta : \mathfrak{A} \to \mathfrak{B}\left(l^2\left(\ker \mathbf{V}^*\right)\right) \tag{63}$$

such that for each $a \in \mathfrak{A}$ *we get*

$$\pi_\infty\left(a\right) \oplus \Theta\left(a\right) \in \mathcal{S}''.$$

Then $(\mathfrak{M}, \Phi, \varphi)$ *admit a Markov multiplicative dilation* $\left(\widehat{\mathfrak{M}}, \widehat{\Phi}, \widehat{\varphi}, \mathcal{E}, \widehat{\pi}\right)$ *where:*

(1) *The cp map* $\widehat{\pi} : \mathfrak{M} \to \mathcal{S}''$ *is thus defined:*

$$\widehat{\pi}\left(a\right) = \pi_\infty\left(a\right) \oplus \Theta\left(a\right), \qquad a \in \mathfrak{M}; \tag{64}$$

(2) $\widehat{\mathfrak{M}}$ *is a von Neumann algebra:*

$$\widehat{\mathfrak{M}} = \left(\bigcup_{n \geq 0} \mathbf{U}^n\widehat{\pi}\left(\mathfrak{A}\right)\mathbf{U}^{n*}\right)''; \tag{65}$$

(3) *The injective *-homorphism* $\widehat{\Phi} : \widehat{\mathfrak{M}} \to \widehat{\mathfrak{M}}$ *is defined by:*

$$\widehat{\Phi}\left(X\right) = \mathbf{U}X\mathbf{U}^*, \qquad X \in \widehat{\mathfrak{M}}; \tag{66}$$

---

[2]In fact if $X = \left|X_{i,j}\right|_{i,j} \in \widehat{\mathfrak{A}}$, the explicit calculation is the following:

$$\widehat{\Phi}^n\left(X\right) = \begin{vmatrix} \mathbf{V}_\infty^{n^*}X_{1,1}\mathbf{V}_\infty^n; & \mathbf{V}_\infty^{n^*}X_{1,1}\mathbf{C}_n^* + \mathbf{V}_\infty^{n^*}X_{1,2}\mathbf{W}^{n*} \\ \mathbf{C}_nX_{1,1}\mathbf{V}_\infty^n + \mathbf{W}^nX_{2,1}\mathbf{V}_\infty^n; & \begin{matrix}\left(\mathbf{C}_nX_{1,1} + \mathbf{W}^nX_{2,1}\right)\mathbf{C}_n^* + \\ + \left(\mathbf{C}_nX_{1,2} + \mathbf{W}^nX_{2,2}\right)\mathbf{W}^{n*}\end{matrix} \end{vmatrix}.$$

[3]Cfr. Theorem 2.24 of [**16**].

(4) *The normal conditional expectation* $\mathcal{E} : \mathcal{S}'' \to \mathfrak{M}$ *is defined by the relation 58 while normal state* $\widehat{\varphi}$ *on* $\widehat{\mathfrak{M}}$ *is defined by*

$$\widehat{\varphi}(X) = \varphi(\mathcal{E}X), \qquad X \in \widehat{\mathfrak{M}}.$$

PROOF. It is a simple variation of the proof of the preceding theorem. □

1.2. On the existence of the multiplicative dilations. Let $(\mathfrak{A}, \Phi, \varphi)$ be a C*-dynamical systems with $\mathfrak{A} \subset \mathfrak{B}(\mathcal{H})$.
We study some property of the operator systems $\mathcal{S}_o$ of $\mathcal{B}\left(l^2(\ker \mathbf{V}^*)\right)$ associated to our dynamical systems.

PROPOSITION 2.7. *Let* Γ_1 *and* Γ_2 *are* $(\mathbf{U},\Phi)$*-associated operators, we have*

$$\Gamma_1^+ \pi_\infty(a)\Gamma_2 \in \mathcal{S}_o$$

for each $a \in \mathfrak{A}$.
Moreover the linear space $\mathfrak{A}_\sharp$ *generated by the elements*

$$\bigoplus_{k\in\mathbb{N}} \mathbf{F}\pi_\infty(\mathfrak{A})\mathbf{V}^{(k+1)}\pi_\infty(\mathfrak{A})\mathbf{V}^{(k+1)^*}\pi_\infty(\mathfrak{A})\mathbf{F}$$

*is a *-subalgebra (without unit) of* $\mathcal{B}(\ker \mathbf{V}^*)$ *with* $\mathcal{S}_o \subset \mathfrak{A}_\sharp$.
If $\mathfrak{A}_{\mathbf{V}}$ *is the C*-subalgebra of* $\mathcal{B}(\mathcal{H}_\infty)$ *generated by the elements*

$$\{\mathbf{F}\pi_\infty(a)\mathbf{V}\pi_\infty(b)\mathbf{V}^*\pi_\infty(c)\mathbf{F} : a, b, c \in \mathfrak{A}\} \cup \{\mathbf{F}\}$$

we obtain

$$\bigoplus_{k\in\mathbb{N}} \mathfrak{A}_{\mathbf{V}} \subset C^*(\mathcal{S}_o)$$

where $C^*(\mathcal{S}_o)$ *is the C*-algebra (with unit* $\mathbf{I}$*) generated by the set* $\mathcal{S}_o$:

$$C^*(\mathcal{S}_o) \subset \mathcal{B}\left(l^2(\ker \mathbf{V}^*)\right).$$

PROOF. The operator $\Gamma_1^+\pi_\infty(\mathfrak{A})\Gamma_2$ belong to $\mathcal{S}_o$ since

$$\Gamma_3\left(\Gamma_1^+\pi_\infty(\mathfrak{A})\Gamma_2\right)\Gamma_4^+ \in \mathcal{S}_o.$$

For every $(\mathbf{U},\Phi)$-associated operators Γ_3 and Γ_4.
Then for each $a_{m,n}, b_{m,n}, c_{m,n} \in \mathfrak{A}$ with $m, n \in \mathbb{N}$ we get

$$T_{m,n} = \mathbf{F}\pi_\infty(a_{m,n})\mathbf{V}^{(m+1)}\pi_\infty(b_{m,n})\mathbf{V}^{(n+1)^*}\pi_\infty(c_{m,n})\mathbf{F} \in \mathcal{B}(\ker \mathbf{V}^*)$$

and let T be operator of $\mathcal{B}\left(l^2(\ker \mathbf{V}^*)\right)$ thus defined $T = |T_{m,n}|_{m,n\in\mathbb{N}}$, we have that $T \in \mathcal{S}_o$, in particular we get:

$$\bigoplus_{k\in\mathbb{N}} \mathbf{F}\pi_\infty(\mathfrak{A})\mathbf{V}^{(k+1)}\pi_\infty(\mathfrak{A})\mathbf{V}^{(k+1)^*}\pi_\infty(\mathfrak{A})\mathbf{F} \subset \mathcal{S}_o.$$

For each $a_i, b_i, c_i \in \mathfrak{A}$ with $i = 1, 2$ we obtain:

$$\mathbf{F}\pi_\infty(a_1)\mathbf{V}^{(k+1)}\pi_\infty(b_1)\mathbf{V}^{(k+1)^*}\pi_\infty(c_1)\mathbf{F} \cdot \mathbf{F}\pi_\infty(a_2)\mathbf{V}^{(k+1)}\pi_\infty(b_2)\mathbf{V}^{(k+1)^*}\pi_\infty(c_2)\mathbf{F} =$$

$$= \mathbf{F}\pi_\infty(a_1)\mathbf{V}^{(k+1)}\pi_\infty\left(b_1\Phi^k\left[\Phi(c_1a_2) - \Phi(c_1)\Phi(a_2)\right]b_2\right)\mathbf{V}^{(k+1)^*}\pi_\infty(c_2)\mathbf{F} \in \mathfrak{A}_\sharp.$$

The last affirmation is of easy proof now. □

REMARK 2.3. *We observe that if exists* $(\mathbf{U},\Phi)$*-associated operators* Γ_1 *and* Γ_2 *such that*

$$\Gamma_1\Gamma_2^* = \mathbf{1}$$

the operator system $\mathcal{S}_o$ *is a* **-subalgebra with unit of* $\mathfrak{B}\left(l^2\left(\ker \mathbf{V}^*\right)\right)$.

We study the relation between the C*-algebra generated of the elements $\mathbf{F}\pi_\infty(\mathfrak{A})\mathbf{F}$ of $\mathcal{B}(\ker \mathbf{V}^*)$ and the C*-algebra generated of the operator system $\mathcal{S}_o$ of $\mathcal{B}\left(l^2\left(\ker \mathbf{V}^*\right)\right)$. For each n-pla $\underline{A} = (a_1, a_2, \quad a_n)$ of operator A_k of $\mathfrak{A}$, we define the follow operator of $\mathcal{B}\left(l^2\left(\ker \mathbf{V}^*\right)\right)$:

$$\mathbf{T}_{\underline{A}} = \bigoplus_{k\in\mathbb{N}} \mathbf{F}\pi_\infty(a_k)\mathbf{F}. \tag{67}$$

PROPOSITION 2.8. *We have that* $T_{\underline{A}} \in \mathcal{S}_o$ *for each n-pla* $\underline{A} = (a_1, a_2, \quad a_n)$ *of elements of* $\mathfrak{A}$. *It follow that:*

$$\bigoplus_{i\in\mathbb{N}} C^*\left(\mathbf{F}\pi_\infty(\mathfrak{A})\mathbf{F}\right) \subset C^*\left(\mathcal{S}_o\right).$$

PROOF. For each $b_1, b_2 \in \mathfrak{A}$ and $k \in \mathbb{N}$, we have

$$\mathbf{V}^{(k+1)^*}\pi_\infty(b_1)\mathbf{F}\pi_\infty(a_k)\mathbf{F}\pi_\infty(b_2)\mathbf{V}^{(k+1)} \in \mathfrak{A},$$

since

$$\begin{aligned}
&\mathbf{V}^{(k+1)^*}\pi_\infty(b_1)\mathbf{F}\pi_\infty(a_k)\mathbf{F}\pi_\infty(b_2)\mathbf{V}^{(k+1)} = \\
&= \mathbf{V}^{(k+1)^*}\pi_\infty(b_1)(\mathbf{I}-\mathbf{V}\mathbf{V}^*)\pi_\infty(a_k)(\mathbf{I}-\mathbf{V}\mathbf{V}^*)\pi_\infty(b_2)\mathbf{V}^{(k+1)} = \\
&= \mathbf{V}^{(k+1)^*}\pi_\infty(b_1)\pi_\infty(a_k)\pi_\infty(b_2)\mathbf{V}^{(k+1)} - \mathbf{V}^{(k+1)^*}\pi_\infty(b_1)\pi_\infty(a_k)\mathbf{V}\mathbf{V}^*\pi_\infty(b_2)\mathbf{V}^{(k+1)} - \\
&- \mathbf{V}^{(k+1)^*}\pi_\infty(b_1)\mathbf{V}\mathbf{V}^*\pi_\infty(a_k)\pi_\infty(b_2)\mathbf{V}^{(k+1)} + \\
&+ \mathbf{V}^{(k+1)^*}\pi_\infty(b_1)\mathbf{V}\mathbf{V}^*\pi_\infty(a_k)\mathbf{V}\mathbf{V}^*\pi_\infty(b_2)\mathbf{V}^{(k+1)} = \\
&= \pi_\infty\left(\Phi^{k+1}(b_1a_kb_2)\right) - \pi_\infty\left(\Phi^k\left(\Phi(b_1a_k)\Phi(b_2)\right)\right) - \pi_\infty\left(\Phi^k\left(\Phi(b_1)\Phi(a_kb_2)\right)\right) + \\
&+ \pi_\infty\left(\Phi^k\left(\Phi(b_1)\Phi(a_k)\Phi(b_2)\right)\right) \in \pi_\infty(\mathfrak{A}).
\end{aligned}$$

□

We have another claim:

$$\pi_\infty(\mathfrak{A}) \oplus C^*\left(\mathcal{S}_o\right) \in C^*\left(\mathcal{S}\right).$$

Indeed, if $a \in \mathfrak{A}$ and $S_k \in \mathcal{S}_o$ we get

$$\begin{vmatrix} \pi_\infty(a) & 0 \\ 0 & \prod_{k=1}^n S_k \end{vmatrix} = \begin{vmatrix} \pi_\infty(a) & 0 \\ 0 & S_1 \end{vmatrix}\begin{vmatrix} I & 0 \\ 0 & S_2 \end{vmatrix}\cdots\cdot\begin{vmatrix} I & 0 \\ 0 & S_n \end{vmatrix},$$

and for each $k = 2, 3...n$

$$\begin{vmatrix} \pi_\infty(a) & 0 \\ 0 & S_1 \end{vmatrix}, \begin{vmatrix} I & 0 \\ 0 & S_k \end{vmatrix} \in \mathcal{S},$$

then

$$\begin{vmatrix} \pi_\infty(a) & 0 \\ 0 & S_1 \end{vmatrix}\begin{vmatrix} I & 0 \\ 0 & S_2 \end{vmatrix}\cdots\cdot\cdot\begin{vmatrix} I & 0 \\ 0 & S_n \end{vmatrix} \in C^*\left(\mathcal{S}\right).$$

From theorem 2.2, the existence of a dilations for the dynamical system is conditioned to the existence of *-linear multiplicatve maps $\Theta : \mathfrak{A} \to C^*\left(\mathcal{S}_o\right)$. We denote with $H\left(\mathfrak{A}, \mathcal{S}_o\right)$ this set of applications.
Then for every $\theta \in H\left(\mathfrak{A}, \mathcal{S}_o\right)$ we get a multiplicative dilation for $(\mathfrak{A}, \Phi, \varphi)$.

For zero $\theta = 0$ we get the *basic* dilation of the our to dynamical system, in this case the representation $\widehat{\pi} : \mathfrak{A} \to \widehat{\mathfrak{A}}$ is given from:

$$\widehat{\pi}(a) = \begin{vmatrix} \pi_\infty(a) & 0 \\ 0 & 0 \end{vmatrix}, \qquad a \in \mathfrak{A}.$$

An example of *-multiplicative map that belong to $H(\mathfrak{A}, \mathcal{S}_o)$ is thus defined:

$$\theta(a)(h_0, h_1 .... h_n ...) = (ah_0, 0, ..0...)$$

for each $a \in \mathfrak{A}$ and $(h_0, h_1 .... h_n ...) \in \mathcal{H}_\infty$.
We observe that for each $a, b, c \in \mathfrak{A}$ we have $\Theta(b) \in \mathcal{S}_o$ since by the proposition 2.3 we have

$$\mathbf{V}^{m^*} \pi_\infty(a) \mathbf{F} \vartheta(b) \mathbf{F} \pi_\infty(c) \mathbf{V}^m = 0,$$

for all $m > 0$.
Furthermore, if Θ is unital map we obtain an unital multiplicative dilation.
For abelian dymanical systems this last case is always possible:

Remark 2.4. *If the characters space $\Omega(\mathfrak{A})$ of the algebra $\mathfrak{A}$ is not void (as in the abelian case), we can take as representation $\theta : \mathfrak{A} \to \mathcal{B}(\ker \mathbf{V}^*)$ the map*

$$\theta(a) = \phi(a) \mathbf{I}, \qquad a \in \mathfrak{A},$$

where ϕ is an any element of $\Omega(\mathfrak{A})$.

A trivial consequence of the preceding propositions is the follow remark:

Remark 2.5. *If there is a *-homomorphism $\theta : \mathfrak{A} \to C^*(\mathbf{F}\pi_\infty(\mathfrak{A})\mathbf{F})$ the C^*-dynamical-system $(\mathfrak{A}, \Phi, \varphi)$ admits a unital multiplicative dilation.*

We give a method to determine the elements of $H(\mathfrak{A}, \mathcal{S}_o)$.

Proposition 2.9. *Let $x_o \in \mathfrak{A}$ and $\mathbf{L} : \pi_\infty(\mathfrak{A}) \to \pi_\infty(\mathfrak{A})$ be a cp-map such that for each $a, b \in \mathfrak{A}$ we have:*

$$\mathbf{L}(a, b) = \mathbf{L}(a) \pi_\infty(\Phi(x_o^* x_o) - \Phi(x_o^*) \Phi(x_o)) \mathbf{L}(b).$$

Then the application

$$\Theta(a) = \bigoplus_{n \in \mathbb{N}} \theta(a),$$

where

$$\theta(a) = \mathbf{F} \pi_\infty(x_o) \mathbf{V} \mathbf{L}(a) \mathbf{V}^* \pi_\infty(x_o^*) \mathbf{F},$$

is an elemen that belong to $H(\mathfrak{A}, \mathcal{S}_o)$.

Proof. The map Θ belong to $H(\mathfrak{A}, \mathcal{S}_o)$ since

$$\mathbf{F} \pi_\infty(x_o) \mathbf{V} \mathbf{L}(a) \mathbf{V}^* \pi_\infty(x_o^*) \mathbf{F} \in \mathfrak{A}_\mathbf{V}$$

where $\mathfrak{A}_\mathbf{V}$ is a C*-algebra defined in the preceding proposition.
The map θ is *-linear and for every $a, b \in \mathfrak{A}$ we have:

$$\theta(a)\theta(b) = \mathbf{F}\pi_\infty(x_o)\mathbf{V}\mathbf{L}(a)\mathbf{V}^*\pi_\infty(x_o^*)\mathbf{F}\pi_\infty(x_o)\mathbf{V}\mathbf{L}(b)\mathbf{V}^*\pi_\infty(x_o^*)\mathbf{F} = \\ = \mathbf{F}\pi_\infty(x_o)\mathbf{V}\mathbf{L}(ab)\mathbf{V}^*\pi_\infty(x_o^*)\mathbf{F} = \theta(ab),$$

since

$$\mathbf{V}^* \pi_\infty(x_o^*) \mathbf{F} \pi_\infty(x_o) \mathbf{V} = \pi_\infty(\Phi(x_o^* x_o) - \Phi(x_o^*) \Phi(x_o)).$$

□

A method to determine the applications described in the precedent proposition is the following:
Let x_o, y_o are elements belongs to $\mathfrak{A}$ such that

$$y_o^* \left[\Phi\left(x_o^* x_o\right) - \Phi\left(x_o^*\right)\Phi\left(x_o\right)\right] y_o = I,$$

the *-linear map $L_{y_o} : \mathfrak{A} \to \mathfrak{A}$

$$L_{y_o}(a) = y_o a y_o^*, \qquad a \in \mathfrak{A}$$

satisfies the relation:

$$L_{y_o}(a)\left[\Phi\left(x_o^* x_o\right) - \Phi\left(x_o^*\right)\Phi\left(x_o\right)\right] L_{y_o}(b) = y_o a y_o^* \left[\Phi\left(x_o^* x_o\right) - \Phi\left(x_o^*\right)\Phi\left(x_o\right)\right] y_o b y_o^* = \\ = y_o a b y_o^* = L_{y_o}(ab).$$

for each $a, b \in \mathfrak{A}$.

EXAMPLE 3. *We consider the matrix algebra* $\mathbf{M}_2(\mathbb{C})$ *and unital cp map* $\Phi : \mathbf{M}_2(\mathbb{C}) \to \mathbf{M}_2(\mathbb{C})$ *thus definied:*

$$\Phi(\mathbf{A}) = \frac{1}{2}\sum_{i,j=1}^{2} \mathbf{E}_{i,j}^* \mathbf{A} \mathbf{E}_{i,j},$$

where $\mathbf{E}_{i,j}$ *are the matrixs:*

$$\mathbf{E}_{1,1} = \begin{vmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{vmatrix}; \quad \mathbf{E}_{1,2} = \begin{vmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & 0 \end{vmatrix}; \quad \mathbf{E}_{2,1} = \begin{vmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{vmatrix}; \quad \mathbf{E}_{2,2} = \begin{vmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{vmatrix}.$$

Then for each $\mathbf{A} \in \mathbf{M}_2(\mathbb{C})$ *we have:*

$$\Phi(\mathbf{A}) = \frac{1}{2}\begin{vmatrix} a_{1,1} + a_{2,2} & \mathbf{0} \\ \mathbf{0} & a_{1,1} + a_{2,2} \end{vmatrix}.$$

Let

$$\mathbf{X}_{\pm} = \begin{vmatrix} \mathbf{r} & \mathbf{0} \\ \mathbf{0} & \mathbf{r} \pm \mathbf{2} \end{vmatrix} \in \mathbf{M}_2(\mathbb{C}), \qquad \mathbf{r} \in \mathbb{R}.$$

we get

$$\Phi\left(\mathbf{X}_{\pm}^2\right) - \Phi\left(\mathbf{X}_{\pm}\right)^2 = \mathbf{I},$$

since

$$\Phi(\mathbf{X}_{\pm}) = \begin{vmatrix} \mathbf{r} \pm \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{r} \pm \mathbf{1} \end{vmatrix}, \quad \Phi\left(\mathbf{X}_{\pm}^2\right) = \begin{vmatrix} \mathbf{r}^2 + 2 \pm 2\mathbf{r} & \mathbf{0} \\ \mathbf{0} & \mathbf{r}^2 + 2 \pm 2\mathbf{r} \end{vmatrix}.$$

It follow that the map

$$\theta(\mathbf{A}) = \mathbf{F}\pi_\infty(X)\,\mathbf{V}\pi_\infty(\mathbf{A})\,\mathbf{V}^*\pi_\infty(X^*)\,\mathbf{F}$$

*is a *-linear multiplicative map (non unital) such that* $\bigoplus_{n\in\mathbb{N}} \theta : \mathbf{M}_2(\mathbb{C}) \to C^*(\mathcal{S}_o)$.

1.3. Faithful dilation. Let $(\mathfrak{A}, \Phi, \varphi)$ be C*-dynamical system with dilation $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}, \mathcal{E}, \widehat{\pi}\right)$ of the theorem 2.2. We define a new C*-algebra with unit

$$\widehat{\mathfrak{A}}_{2,2} = \left\{ X \in \widehat{\mathfrak{A}} : X = \left| \begin{array}{cc} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & X_{2,2} \end{array} \right| \right\},$$

that results to be $\mathbf{U}$-invariant:

$$\mathbf{U}\widehat{\mathfrak{A}}_{2,2}\mathbf{U}^* \subset \widehat{\mathfrak{A}}_{2,2}.$$

DEFINITION 2.1. *The dilation* $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}, \mathcal{E}, \widehat{\pi}\right)$ *is faithful if* $\widehat{\mathfrak{A}}_{2,2} = \{\mathbf{0}\}$.

We observe that if $\widehat{\varphi}$ is faithful state[4] the dilation $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}, \mathcal{E}, \widehat{\pi}\right)$ is faithful.
In fact let $X \in \widehat{\mathfrak{A}}_{2,2}$, for definition we get

$$\widehat{\varphi}\left(X^*X\right)) = \varphi\left(\mathcal{E}_{1,1}\left(X^*X\right)\right) = 0.$$

It follow that $X = 0$.
If we examine the basic dilation[5] the C*-algebra $\widehat{\mathfrak{A}}_{2,2}$ is not zero since

$$\mathbf{I} - \widehat{\pi}\left(\mathbf{1}\right) \in \widehat{\mathfrak{A}}_{2,2}.$$

Then the basic dilation is never faithful.

REMARK 2.6. *If* φ *is faithful state with the property*

$$\varphi\left(a^*a\right) = \varphi\left(aa^*\right), \qquad a \in \mathfrak{A},$$

and the dilation $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}, \mathcal{E}, \widehat{\pi}\right)$ *is faithful the* $\widehat{\varphi}$ *state is faithful.*

2. Ergodic property of the dilation

We study now the ergodic properties of the multiplicative dilation $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}, i, \mathcal{E}\right)$ of theorem 2.2 of the C*-dynamical system $(\mathfrak{A}, \Phi, \varphi)$.
Let $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$ the GNS of φ and $\mathbf{U}_\varphi$ the linear contraction 15 associated with C*-dynamical systems.
We defined the set of the $\Phi-$invariant element of $\mathfrak{A}$:

$$\mathfrak{A}^{\Phi} = \left\{a \in \mathfrak{A} : \Phi\left(a\right) = a\right\},$$

since for each $a \in \mathfrak{A}$ we have $\Phi\left(a^*\right)\Phi\left(a\right) \leq \Phi\left(a^*a\right) \leq a^*a$, the set $\mathfrak{A}^{\Phi}$ is included in the multiplicative domains $\mathcal{D}\left(\Phi\right)$ of Φ and it is a C*-subalgebra with unit of $\mathfrak{A}$.
We have the following implication:

$$X \in \widehat{\mathfrak{A}}^{\widehat{\Phi}} \quad \Longrightarrow \quad \mathcal{E}_{1,1}\left(X\right) \in \mathfrak{A}^{\Phi},$$

and if φ is a *faithful* state we obtain

$$\mathfrak{A}^{\Phi} = \mathbb{C}\mathbf{I} \quad \Longleftrightarrow \quad \dim\ker\left(\mathbf{I} - \mathbf{U}_\varphi\right) = 1.$$

---

[4]Then φ is faithful state since

$$\widehat{\varphi}\left(\widehat{\pi}\left(a^*\right)\widehat{\pi}\left(a\right)\right)) = \varphi\left(a^*a\right)$$

for all $a \in \mathfrak{A}$.

[5]That is when $\theta = 0$.

We have a fundamental lemma for the study of the ergodic property of the dilation.

LEMMA 2.3. *We have the following implication:*

$$\mathfrak{A}^{\Phi} = \mathbb{C}\mathbf{I} \implies \widehat{\mathfrak{A}}^{\widehat{\Phi}} = \mathbb{C}\mathbf{I}.$$

PROOF. We set $\mathcal{E}_{1,1}(X) = \lambda\mathbf{I}$ with λ complex number:

$$X = \begin{vmatrix} \lambda\mathbf{I} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{vmatrix}$$

For hypothesis, for each $n \in \mathbb{N}$ we have $\widehat{\Phi}^n(X) = X$ then[6]:

$$\begin{cases} X_{1,2} = \mathbf{V}_{\infty}^{n^*} X_{1,2} \mathbf{W}^{n^*}; \\ X_{2,1} = \mathbf{W}^n X_{2,1} \mathbf{V}_{\infty}^n; \\ X_{2,2} = (\lambda\mathbf{C}_n + \mathbf{W}^n X_{2,1})\,\mathbf{C}_n^* + (\mathbf{C}_n X_{1,2} + \mathbf{W}^n X_{2,2})\,\mathbf{W}^{n^*}; \end{cases}$$

Let $\xi = (\xi_0, \xi_1, ...\xi_n...) \in l^2(\ker \mathbf{V}_{\infty}^*)$ we have

$$X_{1,2}\xi = \sum_{j=0}^{\infty} L_j \xi_j$$

with $L_j : \ker \mathbf{V}_{\infty}^* \to \mathcal{H}_{\infty}$ linear operators, from the first relation we have:

$$\sum_{j=0}^{\infty} L_j \xi_j = \sum_{j=0}^{\infty} \mathbf{V}_{\infty}^{n^*} L_j \xi_{j+n}.$$

Then if $\xi = (0, 0, ...\xi_p, 0...)$ with $p < n$, we have $L_p\xi_p = 0$, it follow that $X_{1,2} = 0$.
In the same way it verify that the operator $X_{2,1} = 0$.
the third relation becomes now:

$$X_{2,2} = \lambda\mathbf{C}_n\mathbf{C}_n^* + \mathbf{W}^n X_{2,2} \mathbf{W}^{n^*} = \lambda\left(\mathbf{I} - \mathbf{W}^n\mathbf{W}^{n^*}\right) + \mathbf{W}^n X_{2,2} \mathbf{W}^{n^*}.$$

Let $X_{2,2} = |\mathbf{T}_{i,j}|_{i,j\in\mathbb{N}}$ where $\mathbf{T}_{i,j} : \ker \mathbf{V}_{\infty}^* \to \ker \mathbf{V}_{\infty}^*$ are linear operators, we have:

$$\left(\lambda\xi_0, ..\lambda\xi_{n-1}, \sum_{j=0}^{\infty} \mathbf{T}_{0,j}\xi_{j+n}, \sum_{j=0}^{\infty} \mathbf{T}_{1,j}\xi_{j+n}, ..\right) = \left(\sum_{j=0}^{\infty} \mathbf{T}_{0,j}\xi_j, \sum_{j=0}^{\infty} \mathbf{T}_{1,j}\xi_j, ...\right)$$

and if $\xi = (0, 0, ...\xi_{n-1}, 0...)$ we get:

$$(0, ..., \lambda\mathbf{F}\xi_{n-1}, 0, ...0, ...) = (\mathbf{T}_{0,n-1}\xi_{n-1}, ...\mathbf{T}_{n-1,n-1}\xi_{n-1}, ...).$$

Then

$$\mathbf{T}_{i,n-1} = \begin{cases} 0 & i \neq n-1 \\ \lambda\mathbf{F} & i = n-1 \end{cases},$$

follow that $X_{2,2} = \lambda\mathbf{I}$.
We have verified that if $\mathcal{E}_{1,1}(X) \in \mathbb{C}\mathbf{I}$ we obtain $X = \lambda\mathbf{I}$. □

PROPOSITION 2.10. *If φ is a ergodic faithful state we have*

$$\widehat{\mathfrak{A}}^{\widehat{\Phi}} = \mathbb{C}\mathbf{I}.$$

PROOF. It's trivial. □

[6] We have

$$\mathbf{V}_{\infty}^{n^*} X_{1,1} \mathbf{C}_n^* = \lambda \mathbf{V}_{\infty}^{n^*} \mathbf{C}_n^* = 0.$$

2.1. The $\mathcal{Z}_{k,p}$ Operators and ergodic properties. For the study of the ergodic property of the dilations of dynamical systems we have to determine the value of the followings limits:

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\left[\widehat{\varphi}\left(X\widehat{\Phi}^{k}\left(Y\right)\right)-\widehat{\varphi}\left(X\right)\widehat{\varphi}\left(Y\right)\right]$$

and

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\left|\widehat{\varphi}\left(X\widehat{\Phi}^{k}\left(Y\right)\right)-\widehat{\varphi}\left(X\right)\widehat{\varphi}\left(Y\right)\right|$$

for all $X,Y\in\widehat{\mathfrak{A}}$.
We recall that for definition that

$$\widehat{\varphi}\left(X\widehat{\Phi}^{k}\left(Y\right)\right)=\varphi\left(\mathcal{E}_{1,1}\left(X\widehat{\Phi}^{k}\left(Y\right)\right)\right).$$

and

$$\mathcal{E}_{1,1}\left(X\widehat{\Phi}^{k}\left(Y\right)\right)=\mathcal{E}_{1,1}\left(X\right)\mathcal{E}_{1,1}\left(\widehat{\Phi}^{k}\left(Y\right)\right)+\mathcal{E}_{1,2}\left(X\right)\mathcal{E}_{2,1}\left(\widehat{\Phi}^{k}\left(Y\right)\right).$$

For the study of the ergodic property of the our dilations, we can consider only to the elements of the $\widehat{\mathfrak{A}}$ of the type $\prod_{j=1}^{p}\widehat{\Phi}^{n_j}\left(\widehat{\pi}\left(a_j\right)\right)$ with $a_j\in\mathfrak{A}$.

For definition of $\widehat{\mathfrak{A}}$, let $X\in\widehat{\mathfrak{A}}$ for each $\varepsilon>0$ there is $P_\varepsilon=\sum_i\prod_{j=1}^{p_i}\widehat{\Phi}^{n_{i,j}}\left(\widehat{\pi}\left(a_{i,j}\right)\right)\in\widehat{\mathfrak{A}}$ such that

$$\left\|X-P_\varepsilon\right\|<\varepsilon.$$

We have

$$\left|\frac{1}{N+1}\sum_{k=0}^{N}\left[\widehat{\varphi}\left(X\widehat{\Phi}^{k}\left(Y\right)\right)-\widehat{\varphi}\left(P_\varepsilon\widehat{\Phi}^{k}\left(Y\right)\right)\right]\right|\leq\frac{1}{N+1}\sum_{k=0}^{N}\left|\widehat{\varphi}\left(\left[X-P_\varepsilon\right]\widehat{\Phi}^{k}\left(Y\right)\right)\right|\leq\varepsilon\left\|Y\right\|$$

for all $Y\in\widehat{\mathfrak{A}}$.
Moreover for the von Neumann algebras we have, from the bicommutant theorem, that let $X\in\widehat{\mathfrak{M}}$, for each $\varepsilon>0$ there is $P_\varepsilon=\sum_i\prod_{j=1}^{p_i}\widehat{\Phi}^{n_{i,j}}\left(\widehat{\pi}\left(a_{i,j}\right)\right)\in\widehat{\mathfrak{A}}$ such that

$$\widehat{\varphi}\left(\left(X-P_\varepsilon\right)^{*}\left(X-P_\varepsilon\right)\right)<\varepsilon.$$

Then

$$\left|\frac{1}{N+1}\sum_{k=0}^{N}\left[\widehat{\varphi}\left(X\widehat{\Phi}^{k}\left(Y\right)\right)-\widehat{\varphi}\left(P_\varepsilon\widehat{\Phi}^{k}\left(Y\right)\right)\right]\right|\leq\frac{1}{N+1}\sum_{k=0}^{N}\left|\widehat{\varphi}\left(\left[X-P_\varepsilon\right]\widehat{\Phi}^{k}\left(Y\right)\right)\right|\leq$$

$$\leq\frac{1}{N+1}\sum_{k=0}^{N}\left|\widehat{\varphi}\left(X-P_\varepsilon\right)\right|^{2}\left|\widehat{\varphi}\left(\widehat{\Phi}^{k}\left(Y\right)\right)\right|^{2}\leq\varepsilon\left\|Y\right\|.$$

It follow that

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\widehat{\varphi}\left(X\widehat{\Phi}^{k}\left(Y\right)\right)=\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\widehat{\varphi}\left(P_\varepsilon\widehat{\Phi}^{k}\left(Y\right)\right).$$

* * *

We have a fundamental lemma for the ergodic property for our dilation.

LEMMA 2.4. *If the multiplicative map* Θ *of the theorem 2.2 is of the shape*

$$\Theta = \bigoplus_{n\in\mathbb{N}} \vartheta,$$

for each $Y \in \widehat{\mathfrak{A}}$ *and* $a \in \mathfrak{A}$ *there exists* $n_o \in \mathbb{N}$ *such that for each* $k > n_o$ *we obtain:*

$$\mathcal{E}_{1,2}(X)\,\mathbf{W}^k\Theta(a)\,Y_{2,1}\mathbf{V}^k = 0.$$

PROOF. If $X_1 = \widehat{\Phi}^n(\widehat{\pi}(x))$ with $x \in \mathfrak{A}$, we have

$$\mathcal{E}_{1,2}\left(\widehat{\Phi}^n(\widehat{\pi}(x))\right)\mathbf{W}^k\Theta(a)\,Y_{2,1}\mathbf{V}^k = \mathbf{V}^{n^*}\pi_\infty(x)\,\mathbf{C}_n^*\mathbf{W}^k\Theta(a)\,Y_{2,1}\mathbf{V}^k$$

and for $k > n$ we obtain that the operator $\mathbf{W}^{k^*}\mathbf{C}_n = 0$.
For induction on the length p of the string of X :

$$X_p = \prod_{k=1}^{p}\widehat{\Phi}^{n_k}(\widehat{\pi}(x_k)), \qquad x_1, x_2, ...x_p \in \mathfrak{A},$$

we assume true the relation for $p-1$ step, then there is a n_o such that for each $k > n_o$ we have:

$$\mathcal{E}_{1,2}\left(\prod_{k=1}^{p-1}\widehat{\Phi}^{n_k}(\widehat{\pi}(x_k))\right)\mathbf{W}^k\Theta(a)\,Y_{2,1}\mathbf{V}^k = 0.$$

For p step we have

$$\mathcal{E}_{1,2}\left(\prod_{k=1}^{p}\widehat{\Phi}^{n_k}(\widehat{\pi}(x_k))\right) = \mathcal{E}_{1,1}\left(\prod_{k=1}^{p-1}\widehat{\Phi}^{n_k}(\widehat{\pi}(x_k))\right)\mathcal{E}_{1,2}\left(\widehat{\Phi}^{n_p}(\widehat{\pi}(x_p))\right) + \\ + \mathcal{E}_{1,2}\left(\prod_{k=1}^{p-1}\widehat{\Phi}^{n_k}(\widehat{\pi}(x_k))\right)\mathcal{E}_{2,2}\left(\widehat{\Phi}^{n_p}(\widehat{\pi}(x_p))\right),$$

it follow that

$$\mathcal{E}_{1,2}(X_p)\,\mathbf{W}^k\Theta(a)\,Y_{2,1}\mathbf{V}^k = \mathcal{E}_{1,1}(X_{p-1})\,\mathcal{E}_{1,2}\left(\widehat{\Phi}^{n_p}(\widehat{\pi}(x_p))\right)\mathbf{W}^k\Theta(a)\,Y_{2,1}\mathbf{V}^k + \\ + \mathcal{E}_{1,2}(X_{p-1})\,\mathcal{E}_{2,2}\left(\widehat{\Phi}^{n_p}(\widehat{\pi}(x_p))\right)\mathbf{W}^k\Theta(a)\,Y_{2,1}\mathbf{V}^k$$

where for $k > m_1$

$$\mathcal{E}_{1,2}\left(\widehat{\Phi}^{n_p}(\widehat{\pi}(x_p))\right)\mathbf{W}^k\Theta(a)\,Y_{2,1}\mathbf{V}^k = 0.$$

then

$$\mathcal{E}_{1,2}(X_p)\,\mathbf{W}^k\Theta(a)\,Y_{2,1}\mathbf{V}^k = \mathcal{E}_{1,2}(X_{p-1})\,\mathcal{E}_{2,2}\left(\widehat{\Phi}^{n_p}(\widehat{\pi}(x_p))\right)\mathbf{W}^k\Theta(a)\,Y_{2,1}\mathbf{V}^k$$

and

$$\mathcal{E}_{2,2}\left(\widehat{\Phi}^{n_p}(\widehat{\pi}(x_p))\right) = \mathbf{C}_{n_p}\pi_\infty(x_p)\,\mathbf{C}_{n_p}^* + \mathbf{W}^{n_p}\Theta(x_p)\,\mathbf{W}^{n_p^*}.$$

For $k > n_p$ we have $\mathbf{C}_{n_p}^*\mathbf{W}^k = 0$ and we obtain that

$$\mathcal{E}_{2,2}\left(\widehat{\Phi}^{n_p}(\widehat{\pi}(x_p))\right)\mathbf{W}^k\Theta(a)\,Y_{2,1}\mathbf{V}^k\Psi = \mathbf{W}^{n_p}\Theta(x_p)\,\mathbf{W}^{k-n_p}\Theta(a)\,Y_{2,1}\mathbf{V}^k.$$

Since $\Theta(a)$ commute $\mathbf{W}^k$ it follow:

$$\mathcal{E}_{2,2}\left(\widehat{\Phi}^{n_p}(\widehat{\pi}(x_p))\right)\mathbf{W}^k\Theta(a)\,Y_{2,1}\mathbf{V}^k\Psi = \mathbf{W}^k\Theta(x_p)\,\Theta(a)\,Y_{2,1}\mathbf{V}^k$$

then

$$\mathcal{E}_{1,2}(X_p)\,\mathbf{W}^k\Theta(a)\,Y_{2,1}\mathbf{V}^k = \mathcal{E}_{1,2}(X_{p-1})\,\mathbf{W}^k\Theta(x_p a)\,Y_{2,1}\mathbf{V}^k$$

and for inductive hypothesis there existst a natural number n_o such that for each $k > n_o$ we get:

$$\mathcal{E}_{1,2}\left(\prod_{k=1}^{p-1}\widehat{\Phi}^{n_k}\left(\widehat{\pi}\left(x_k\right)\right)\right)\mathbf{W}^k\Theta(x_p a)\,Y_{2,1}\mathbf{V}^k = 0.$$

Let $X \in \widehat{\mathfrak{A}}$ for each $\varepsilon > 0$ there is $P_\varepsilon = \sum_i \prod_{j=1}^{p_i}\widehat{\Phi}^{n_{i,j}}\left(\widehat{\pi}\left(a_{i,j}\right)\right) \in \widehat{\mathfrak{A}}$ such that

$$\|X - P_\varepsilon\| < \varepsilon.$$

For the continuity of the application $\mathcal{E}_{1,2}$ we have

$$\left\|\mathcal{E}_{1,2}(X)\,\mathbf{W}^k\Theta(a)\,Y_{2,1}\mathbf{V}^k\right\| \leq \varepsilon + \left\|\mathcal{E}_{1,2}(P_\varepsilon)\,\mathbf{W}^k\Theta(a)\,Y_{2,1}\mathbf{V}^k\right\|$$

with

$$\mathcal{E}_{1,2}(P_\varepsilon)\,\mathbf{W}^k\Theta(a)\,Y_{2,1}\mathbf{V}^k = 0$$

for $k > m_2$. □

Remark 2.7. *In the case that the multiplicative linear map $\Theta : \mathfrak{A} \to \mathcal{B}(\ker \mathbf{V}^*)$ is not unital, we can easily verify, through the preceding lemma, that for each $X \in \widehat{\mathfrak{A}}$, there exists a natural number n_o such that for each $k > n_o$ we have:*

$$\mathcal{E}_{1,2}(X)\,\mathbf{W}^k Y_{2,1}\mathbf{V}^k = 0.$$

* * *

To simplify our calculations we introduce new symbol.
If $X = \prod_{j=1}^{p}\widehat{\Phi}^{n_j}\left(\widehat{\pi}\left(a_j\right)\right)$ with $a_1, a_2..a_p \in \mathfrak{A}$, we set

$$\mathcal{Z}_{k,p}\left(\prod_{j=1}^{p}\widehat{\Phi}^{n_j}\left(\widehat{\pi}\left(a_j\right)\right)\right) = \mathcal{E}_{1,2}\left(\prod_{j=1}^{p}\widehat{\Phi}^{n_j}\left(\widehat{\pi}\left(a_j\right)\right)\right)\mathcal{E}_{2,1}\left(\widehat{\Phi}^k(Y)\right) \in \pi_\infty(\mathfrak{A})$$

and

$$\mathcal{R}_{k,p}\left(\prod_{j=1}^{p}\widehat{\Phi}^{n_j}\left(\widehat{\pi}\left(a_j\right)\right)\right) = \mathcal{E}_{2,2}\left(\prod_{j=1}^{p}\widehat{\Phi}^{n_j}\left(\widehat{\pi}\left(a_j\right)\right)\right)\mathcal{E}_{2,1}\left(\widehat{\Phi}^k(Y)\right)$$

Then

$$\mathcal{E}_{1,1}\left(\prod_{j=1}^{p}\widehat{\Phi}^{n_j}\left(\widehat{\pi}\left(a_j\right)\right)\widehat{\Phi}^k(Y)\right) = \mathcal{E}_{1,1}\left(\prod_{j=1}^{p}\widehat{\Phi}^{n_j}\left(\widehat{\pi}\left(a_j\right)\right)\right)\pi_\infty\left(\Phi^k\left(Y_{1,1}\right)\right) + $$
$$+\,\mathcal{Z}_{k,p}\left(\prod_{j=1}^{p}\widehat{\Phi}^{n_j}\left(\widehat{\pi}\left(a_j\right)\right)\right).$$

It follow that

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varphi\left(\mathcal{Z}_{k,p}(X)\right) = \lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\left[\widehat{\varphi}\left(X\widehat{\Phi}^k(Y)\right) - \varphi\left(X_{1,1}\widehat{\Phi}^k\left(Y_{1,1}\right)\right)\right].$$

REMARK 2.8. *The $\widehat{\varphi}$ is ergodic state if and only if*

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varphi\left(\mathcal{Z}_{k,p}\left(X\right)\right)=0.$$

While $\widehat{\varphi}$ is weakly mixing state if and only if

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\left|\varphi\left(\mathcal{Z}_{k,p}\left(X\right)\right)\right|=0.$$

We have the following relation for the $\mathcal{Z}_{p,k}$ operators [7]:

$$\mathcal{Z}_{k,p}\left(\prod_{j=1}^{p}\widehat{\Phi}^{n_j}\left(\widehat{\pi}\left(a_j\right)\right)\right)=\mathcal{E}_{1,2}\left(\prod_{j=1}^{p}\widehat{\Phi}^{n_j}\left(\widehat{\pi}\left(a_j\right)\right)\right)\mathbf{C}_kY_{1,1}\mathbf{V}^k$$

and

$$\mathcal{Z}_{k,p}\left(\prod_{j=1}^{p}\widehat{\Phi}^{n_j}\left(\widehat{\pi}\left(a_j\right)\right)\right)=\pi_\infty\left(a_1\right)\mathcal{Z}_{k,p-1}\left(\prod_{j=2}^{p}\widehat{\Phi}^{n_j}\left(\widehat{\pi}\left(a_j\right)\right)\right)+$$
$$+\mathcal{E}_{1,2}\left(\widehat{\Phi}^{n_1}\left(\widehat{\pi}\left(a_1\right)\right)\right)\mathcal{R}_{k,p-1}\left(\prod_{j=2}^{p}\widehat{\Phi}^{n_j}\left(\widehat{\pi}\left(a_j\right)\right)\right).$$

PROPOSITION 2.11. *Let $X=\prod_{j=1}^{p}\widehat{\Phi}^{n_j}\left(\widehat{\pi}\left(a_j\right)\right)$ with $a_1,a_2..a_p\in\mathfrak{A}$, and*

$$n_q=\min\left\{n_j:j=1,2..p\right\}\geq 0.$$

If φ is a ergodic state we have:

$$\lim_{N\to\infty}\tfrac{1}{N+1}\sum_{k=0}^{N}\varphi\left(\mathcal{Z}_{k,p}\left(\prod_{j=1}^{p}\widehat{\Phi}^{n_j}\left(\widehat{\pi}\left(a_j\right)\right)\right)=\lim_{N\to\infty}\tfrac{1}{N+1}\sum_{k=0}^{N}\mathcal{Z}_{k,p}\left(\prod_{j=1}^{p}\widehat{\Phi}^{(n_j-n_q)}\left(\widehat{\pi}\left(a_j\right)\right)\right)\right).$$

Moreover let φ be a weakly meaxing state, if

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\left|\varphi\left(\mathcal{Z}_{k,p}\left(\prod_{j=1}^{p}\widehat{\Phi}^{(n_j-n_q)}\left(\widehat{\pi}\left(a_j\right)\right)\right)\right)\right|=0$$

we have that $\widehat{\varphi}$ is weakly meaxing.

PROOF. We set $\widetilde{X}=\prod_{j=1}^{p}\widehat{\Phi}^{(n_j-n_q)}\left(\widehat{\pi}\left(a_j\right)\right)$, we have:

$$\lim_{N\to\infty}\tfrac{1}{N+1}\sum_{k=0}^{N}\varphi\left(\mathcal{Z}_{k,p}\left(X\right)\right)=\lim_{N\to\infty}\tfrac{1}{N+1}\sum_{k=0}^{N}\left[\widehat{\varphi}\left(X\widehat{\Phi}^k\left(Y\right)\right)-\varphi\left(X_{1,1}\widehat{\Phi}^k\left(Y_{1,1}\right)\right)\right]=$$
$$=\lim_{N\to\infty}\tfrac{1}{N+1}\sum_{k=0}^{N}\left[\widehat{\varphi}\left(\widehat{\Phi}^{n_q}\left(\widetilde{X}\widehat{\Phi}^{k-n_q}\left(Y\right)\right)\right)-\varphi\left(X_{1,1}\right)\varphi\left(Y_{1,1}\right)\right]=$$
$$=\lim_{N\to\infty}\tfrac{1}{N+1}\sum_{k=0}^{N}\left[\widehat{\varphi}\left(\widetilde{X}\widehat{\Phi}^{k-n_q}\left(Y\right)\right)-\varphi\left(X_{1,1}\right)\varphi\left(Y_{1,1}\right)\right]=$$

[7] We recal that by the lemma 2.4, for all $X\in\widehat{\mathfrak{A}}$ we have

$$X_{1,2}\mathcal{E}_{2,1}\left(\widehat{\Phi}^k\left(Y\right)\right)=X_{1,2}\mathbf{C}_kY_{1,1}\mathbf{V}^k.$$

$= \lim\limits_{N\to\infty} \frac{1}{N+1} \sum\limits_{k=0}^{N} \left[\widehat{\varphi}\left(\widetilde{X} \widehat{\Phi}^{k-n_q}(Y) \right) - \varphi\left(\widetilde{X}_{1,1} \right) \varphi(Y_{1,1}) \right] =$

$= \lim\limits_{N\to\infty} \frac{1}{N+1} \sum\limits_{k=0}^{N} \left[\widehat{\varphi}\left(\widetilde{X} \widehat{\Phi}^{k}(Y) \right) - \varphi\left(\widetilde{X}_{1,1} \widehat{\Phi}^{k}(Y_{1,1}) \right) \right] = \lim\limits_{N\to\infty} \frac{1}{N+1} \sum\limits_{k=0}^{N} \varphi\left(\mathcal{Z}_{k,p}\left(\widetilde{X} \right) \right).$

For the second assertion we get:

$\lim\limits_{N\to\infty} \frac{1}{N+1} \sum\limits_{k=0}^{N} \left| \widehat{\varphi}\left(\prod\limits_{j=1}^{p} \widehat{\Phi}^{n_j}(\widehat{\pi}(a_j)) \widehat{\Phi}^{k}(Y) \right) - \varphi(X_{1,1}) \varphi(Y_{1,1}) \right| =$

$= \lim\limits_{N\to\infty} \frac{1}{N+1} \sum\limits_{k=0}^{N} \left| \widehat{\varphi}\left(\prod\limits_{j=1}^{p} \widehat{\Phi}^{(n_j-n_q)}(\widehat{\pi}(a_j)) \widehat{\Phi}^{k}(Y) \right) - \varphi\left(\widetilde{X}_{1,1} \right) \varphi(Y_{1,1}) \right| \leq$

$\leq \lim\limits_{N\to\infty} \frac{1}{N+1} \sum\limits_{k=0}^{N} \left| \varphi\left(\widetilde{X}_{1,1} \widehat{\Phi}^{k}(Y_{1,1}) \right) - \varphi\left(\widetilde{X}_{1,1} \right) \varphi(Y_{1,1}) + \varphi\left(\mathcal{Z}_{k,p}\left(\widetilde{X} \right) \right) \right| \leq$

$\leq \lim\limits_{N\to\infty} \frac{1}{N+1} \sum\limits_{k=0}^{N} \left| \varphi\left(\mathcal{Z}_{k,p}\left(\widetilde{X} \right) \right) \right| + \lim\limits_{N\to\infty} \frac{1}{N+1} \sum\limits_{k=0}^{N} \left| \varphi\left(\widetilde{X}_{1,1} \widehat{\Phi}^{k}(Y_{1,1}) \right) - \varphi\left(\widetilde{X}_{1,1} \right) \varphi(Y_{1,1}) \right|.$

□

REMARK 2.9. *We have:*

$\mathcal{Z}_{k,p}\left(\widetilde{X} \right) = \mathcal{E}_{1,1}\left(\prod\limits_{j=1}^{q-1} \widehat{\Phi}^{(n_j-n_q)}(\widehat{\pi}(a_j)) \right) \mathcal{Z}_{k,p-q}\left(\prod\limits_{j=q}^{p} \widehat{\Phi}^{(n_j-n_q)}(\widehat{\pi}(a_j)) \right) +$

$+\mathcal{E}_{1,2}\left(\prod\limits_{j=1}^{q-1} \widehat{\Phi}^{(n_j-n_q)}(\widehat{\pi}(a_j)) \right) \Theta(a_q) \mathcal{R}_{k,p-q-1}\left(\prod\limits_{j=q+1}^{p} \widehat{\Phi}^{(n_j-n_q)}(\widehat{\pi}(a_j)) \right).$

We see that form they take the $\mathcal{Z}_{k,p}$ operators for $p = 1$.
We observe that when $k > m$ we obtain:

$$\mathcal{Z}_{k,1}\left(\widehat{\Phi}^{m}(\widehat{\pi}(a)) \right) = \pi_\infty\left(\Phi^{m}(a) \Phi^{k-m}(Y_{1,1}) \right) - \pi_\infty\left(\Phi^{m}(a) \Phi^{k}(Y_{1,1}) \right),$$

since

$$\mathcal{Z}_{k,1}\left(\widehat{\Phi}^{m}(\widehat{\pi}(a)) \right) = \mathcal{E}_{1,2}\left(\widehat{\Phi}^{m}(\widehat{\pi}(a)) \right) \mathbf{C}_k Y_{1,1} \mathbf{V}^k = \mathbf{V}^{m^*} \pi_\infty(a) \mathbf{C}_m^* \mathbf{C}_k Y_{1,1} \mathbf{V}^k.$$

We have a simple lemma:

LEMMA 2.5. *If φ is ergodic state for each $a, d \in \mathfrak{A}$, $Y \in \widehat{\mathfrak{A}}$ and $m \in \mathbb{N}$ we have:*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi\left(\pi_\infty(d) \mathcal{Z}_{k,1}\left(\widehat{\Phi}^{m}(\widehat{\pi}(a)) \right) \right) = 0,$$

while if φ is weakly mixing state we obtain:

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \left| \varphi\left(\pi_\infty(d) \mathcal{Z}_{k,1}\left(\widehat{\Phi}^{m}(\widehat{\pi}(a)) \right) \right) \right| = 0.$$

PROOF. We have

$\pi_\infty(d) \mathcal{Z}_{k,1}\left(\widehat{\Phi}^{m}(\widehat{\pi}(a)) \widehat{\Phi}^{k}(Y) \right) = \pi_\infty\left(d\Phi^{m}(a) \Phi^{k-m}(Y_{1,1}) \right) - \pi_\infty\left(d\Phi^{m}(a) \Phi^{k}(Y_{1,1}) \right).$

Moreover

$\lim\limits_{N\to\infty} \frac{1}{N+1} \sum\limits_{k=0}^{N} \left[\varphi\left(d\Phi^{m}(a) \Phi^{k-m}(Y_{1,1}) \right) - \varphi\left(d\Phi^{m}(a) \Phi^{k}(Y_{1,1}) \right) \right] =$

$= \varphi\left(d\Phi^{m}(a) \varphi(Y_{1,1}) \right) - \varphi\left(d\Phi^{m}(a) \varphi(Y_{1,1}) \right) = 0,$

while in the weakly mixing case we get:

$\frac{1}{N+1}\sum_{k=0}^{N}\left|\varphi\left(\pi_{\infty}(d)\,\mathcal{Z}_{k,1}\left(\widehat{\Phi}^{m}(\widehat{\pi}(a))\right)\right)\right| =$

$= \frac{1}{N+1}\sum_{k=0}^{N}\left|\varphi\left(d\Phi^{m}(a)\,\Phi^{k-m}(Y_{1,1})\right) - \varphi\left(d\Phi^{m}(a)\,\Phi^{k}(Y_{1,1})\right) \pm \varphi\left(d\Phi^{m}(a)\right)\varphi(Y_{1,1})\right| \leq$

$\leq \frac{1}{N+1}\sum_{k=0}^{N}\left|\varphi\left(d\Phi^{m}(a)\,\Phi^{k-m}(Y_{1,1})\right) - \varphi\left(d\Phi^{m}(a)\right)\varphi(Y_{1,1})\right| +$

$+\frac{1}{N+1}\sum_{k=0}^{N}\left|\varphi\left(d\Phi^{m}(a)\,\Phi^{k}(Y_{1,1})\right) - \varphi\left(d\Phi^{m}(a)\right)\varphi(Y_{1,1})\right|.$ □

2.2. Ergodic properties for the basic dilation. We study now the ergodic properties of the multiplicative dilation $\left(\widehat{\mathfrak{A}}, \widehat{\Phi}, \widehat{\varphi}, i, \mathcal{E}\right)$ of theorem 2.2 in the case that the multplicative linear map Θ is zero.

THEOREM 2.4. *Let φ be ergodic state, if the cp map Φ admit a φ-adjoin for each $X = \prod_{j=1}^{p}\widehat{\Phi}^{n_j}(\widehat{\pi}(a_j)) \in \widehat{\mathfrak{A}}$, and $b \in \mathfrak{A}$, $Y \in \widehat{\mathfrak{A}}$ we have:*

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varphi\left(\pi_{\infty}(b)\,\mathcal{Z}_{k,p}\left(\prod_{j=1}^{p}\widehat{\Phi}^{n_j}(\widehat{\pi}(a_j))\right)\right) = 0.$$

Then $\widehat{\varphi}$ is an ergodic state.
If φ is weakly mixing state we obtain

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\left|\varphi\left(\mathcal{Z}_{k,p}\left(\prod_{j=1}^{p}\widehat{\Phi}^{n_j}(\widehat{\pi}(a_j))\right)\right)\right| = 0.$$

Then $\widehat{\varphi}$ is weakly mixing state.

PROOF. We show the affirmation for induction on the p lengt of the product.
► For $p = 1$ the affirmation it's true for lemma 2.5.
Let $X = \prod_{j=1}^{p}\widehat{\Phi}^{n_j}(\widehat{\pi}(a_j))$, with $a_j \in \mathfrak{A}$, and $n_q = \min\{n_j : j = 1, 2..p\} \geq 0$.
We have:
$\mathcal{Z}_{k,p}(\pi_{\infty}(b)\,X) = b\mathcal{E}_{1,2}\left(\widehat{\Phi}^{n_q}\left(\widetilde{X}\right)\right)\mathcal{E}_{2,1}\left(\widehat{\Phi}^{k}(Y)\right),$
where
$\widetilde{X} = \prod_{j=1}^{p}\widehat{\Phi}^{n_j - n_q}(\widehat{\pi}(a_q)).$
Therefore
$\mathcal{E}_{1,2}\left(\widehat{\Phi}^{n_q}\left(\widetilde{X}\right)\right)\mathcal{E}_{2,1}\left(\widehat{\Phi}^{k}(Y)\right) = \left[\mathbf{V}^{n_q^*}\widetilde{X}_{1,1}\mathbf{C}^*_{n_q} + \mathbf{V}^{n_q^*}\widetilde{X}_{1,2}\mathbf{W}^{n_q^*}\right]\mathbf{C}_k Y_{1,1}\mathbf{V}^k =$
$= \mathbf{V}^{n_q^*}\widetilde{X}_{1,1}\mathbf{C}^*_{n_q}\mathbf{C}_k Y_{1,1}\mathbf{V}^k + \mathbf{V}^{n_q^*}\widetilde{X}_{1,2}\mathbf{C}^*_{k-n_q}Y_{1,1}\mathbf{V}^{(k-n_q)}\mathbf{V}^{n_q},$
since $\mathbf{C}^*_k\mathbf{W}^{n_q} = \mathbf{C}_{k-n_q}$.
it follow that.
$\mathcal{E}_{1,2}\left(\widehat{\Phi}^{n_q}\left(\widetilde{X}\right)\right)\mathcal{E}_{2,1}\left(\widehat{\Phi}^{k}(Y)\right) = \mathbf{V}^{n_q^*}\widetilde{X}_{1,1}\mathbf{C}^*_{n_q}\mathbf{C}_k Y_{1,1}\mathbf{V}^k + \widehat{\Phi}^{n_q}\left(\widetilde{X}_{1,2}\mathcal{E}_{2,1}\left(\widehat{\Phi}^{(k-n_q)}(Y)\right)\right),$
since $\widetilde{X}_{1,2}\mathcal{E}_{2,1}\left(\widehat{\Phi}^{(k-n_q)}(Y)\right) \in \pi_{\infty}(\mathfrak{A})$. Then

$\varphi\left(\mathcal{Z}_{k,p}\left(\pi_{\infty}\left(b\right)X\right)\right)=$
$=\varphi\left(b\mathbf{V}^{n_q^*}\widetilde{X}_{1,1}\mathbf{C}_{n_q}^{*}\mathbf{C}_kY_{1,1}\mathbf{V}^k\right)+\varphi\left(\widehat{\Phi}^{n_q^+}\left(b\right)\widetilde{X}_{1,2}\mathcal{E}_{2,1}\left(\widehat{\Phi}^{(k-n_q)}\left(Y\right)\right)\right).$
Now we get
$\mathbf{V}^{n_q^*}\widetilde{X}_{1,1}\mathbf{C}_{n_q}^{*}=\mathcal{E}_{1,2}\left(\widehat{\Phi}^{n_q}\left(\widehat{\pi}\left(\widetilde{X}_{1,1}\right)\right)\right)$
while we can write that
$\widetilde{X}_{1,2}=\mathcal{E}_{1,1}\left(\prod_{j=1}^{q}\widehat{\Phi}^{n_j-n_q}\left(\widehat{\pi}\left(a_q\right)\right)\right)\mathcal{E}_{1,2}\left(\prod_{j=q+1}^{p}\widehat{\Phi}^{n_j-n_q}\left(\widehat{\pi}\left(a_q\right)\right)\right).$
It follow that
$\varphi\left(\pi_{\infty}\left(b\right)\mathcal{Z}_{k,p}\left(X\right)\right)=$
$=\varphi\left(b\mathcal{Z}_1\left(\widehat{\Phi}^{n_q}\left(\widehat{\pi}\left(\widetilde{X}_{1,1}\right)\right)\right)\right)+\varphi\left(d\mathcal{Z}_{k-n_q,p-q}\left(\prod_{j=q+1}^{p}\widehat{\Phi}^{n_j-n_q}\left(\widehat{\pi}\left(a_q\right)\right)\right)\right),$
where we set:
$\pi_{\infty}\left(d\right)=\widehat{\Phi}^{n_q^+}\left(b\right)\mathcal{E}_{1,1}\left(\prod_{j=1}^{q}\widehat{\Phi}^{n_j-n_q}\left(\widehat{\pi}\left(a_q\right)\right)\right).$
We can finally write
$\frac{1}{N+1}\sum_{k=0}^{N}\varphi\left(\pi_{\infty}\left(b\right)\mathcal{Z}_{k,p}\left(\prod_{j=1}^{p}\widehat{\Phi}^{n_j}\left(\widehat{\pi}\left(a_j\right)\right)\right)\right)=$
$=\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varphi\left(\pi_{\infty}\left(b\right)\mathcal{Z}_1\left(\widehat{\Phi}^{n_q}\left(\widehat{\pi}\left(\widetilde{X}_{1,1}\right)\right)\right)\right)+$
$+\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varphi\left(d\mathcal{Z}_{k,p-q}\left(\prod_{j=q+1}^{p}\widehat{\Phi}^{n_j-n_q}\left(\widehat{\pi}\left(a_q\right)\right)\right)\right).$
For the lemma 2.5 we obtain
$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varphi\left(\pi_{\infty}\left(b\right)\mathcal{Z}_{k,1}\left(\widehat{\Phi}^{n_q}\left(\widehat{\pi}\left(\widetilde{X}_{1,1}\right)\right)\right)\right)=0,$
while for the inductive hypothesis
$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varphi\left(d\mathcal{Z}_{k,p-q}\left(\prod_{j=q+1}^{p}\widehat{\Phi}^{n_j-n_q}\left(\widehat{\pi}\left(a_q\right)\right)\right)\right)=0.$
► For weakly mixing we obtain that
$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\left|\varphi\left(\pi_{\infty}\left(b\right)\mathcal{Z}_{k,p}\left(\prod_{j=1}^{p}\widehat{\Phi}^{n_j}\left(\widehat{\pi}\left(a_j\right)\right)\right)\right)\right|=$
$\leq\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\left|\varphi\left(b\mathcal{Z}_1\left(\widehat{\Phi}^{n_q}\left(\widehat{\pi}\left(\widetilde{X}_{1,1}\right)\right)\right)\right)\right|+$
$+\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\left|\varphi\left(d\mathcal{Z}_{k-n_q,p-q}\left(\prod_{j=q+1}^{p}\widehat{\Phi}^{n_j-n_q}\left(\widehat{\pi}\left(a_q\right)\right)\right)\right)\right|.$
Again for the lemma 2.5 we have
$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\left|\varphi\left(b\mathcal{Z}_1\left(\widehat{\Phi}^{n_q}\left(\widehat{\pi}\left(\widetilde{X}_{1,1}\right)\right)\right)\right)\right|=0$
and for the inductive hypothesis
$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\left|\varphi\left(d\mathcal{Z}_{k-n_q,p-q}\left(\prod_{j=q+1}^{p}\widehat{\Phi}^{n_j-n_q}\left(\widehat{\pi}\left(a_q\right)\right)\right)\right)\right|=0.$ □

CHAPTER 3

C*-Hilbert module and dilations

In this section we apply Hilbert module methods to show the existence of a particular dilations that include in its multiplicative domains, the C*-algebra of the observables of the original dynamical system. The ergodic properties and the weakly mixing property they have remained.

1. Definitions and notations

We shortly introduce some results on the C*-Hilbert module. For further details on the subject, the reader can see the references [**21**] and [**32**].

DEFINITION 3.1. *Let $\mathfrak{A}$ be a C^*-algebra. A pre-Hilbert $\mathfrak{A}$ -module is a complex vector space $\mathcal{X}$ which is also a right $\mathfrak{A}$ -module, compatible with the complex algebra structure, equipped with an $\mathfrak{A}$-valued inner product*

$$\langle \cdot ; \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathfrak{A}$$

such that for each $X, Y, Z \in \mathcal{X}$, $\alpha, \beta \in \mathbb{C}$ and $a \in \mathfrak{A}$ satisfies the following relations:
$\langle X; \alpha Y + \beta Z \rangle = \alpha \langle X; Y \rangle + \beta \langle X; Z \rangle;$
$\langle X; Y \cdot a \rangle = \langle Y; X \rangle \cdot a;$
$\langle X; Y \rangle^* = \langle Y; X \rangle;$
$\langle X; X \rangle \geq 0;$ *if* $\langle X; X \rangle = 0$ *then* $X = 0$.
We say that $\mathcal{X}$ is a Hilbert $\mathfrak{A}$-module if $\mathcal{X}$ is complete with respect to the topology determined by the norm $\|\cdot\|$ given by

$$\|X\| = \sqrt{\|\langle X; X \rangle\|}.$$

If $\mathcal{X}$ is a Hilbert $\mathfrak{A}$-module, we make the following notations:
Let $\mathcal{B}(\mathcal{X})$ be the Banach space of all bounded linear operators $\mathbf{T} : \mathcal{X} \to \mathcal{X}$, while $\mathfrak{L}(\mathcal{X})$ is the set of all maps $\mathbf{T} \in \mathcal{B}(\mathcal{X})$ for which there is a map $\mathbf{T}^* \in \mathcal{B}(\mathcal{X})$ such that

$$\langle \mathbf{T}X; Y \rangle = \langle X; \mathbf{T}^* Y \rangle$$

for each $X, Y \in \mathcal{X}$.
Let $\mathcal{B}_{\mathfrak{A}}(\mathcal{X})$ be the Banach space of all bounded module homomorphisms $\mathbf{T} : \mathcal{X} \to \mathcal{X}$ that is:

$$\mathbf{T}(X \cdot a) = \mathbf{T}(X) \cdot a$$

for each $X \in \mathcal{X}$ and $a, \in \mathfrak{A}$.
Moreover we have the following inclusion:

$$\mathfrak{L}(\mathcal{X}) \subset \mathcal{B}_{\mathfrak{A}}(\mathcal{X})$$

and the set $\mathfrak{L}(\mathcal{X})$ is also a C*-algebra with unit.
In general, $\mathcal{B}_{\mathfrak{A}}(\mathcal{X})$ is different by $\mathfrak{L}(\mathcal{X})$ and so the theory of Hilbert C*-modules and

the theory of Hilbert spaces are different.
The set $\mathcal{X}^{\#}$ is the Banach space of all bounded module homomorphisms from $\mathcal{X}$ to $\mathfrak{A}$ which becomes a right $\mathfrak{A}$-module, where the action of $\mathfrak{A}$ on $\mathcal{X}^{\#}$ is defined by

$$(a \cdot \mathbf{\Psi}))(X) = a^{*}\mathbf{\Psi}(X),$$

for each $a \in \mathfrak{A}$, $\mathbf{\Psi} \in \mathcal{X}^{\#}$.
We say that $\mathcal{X}$ is *self-dual* if $\mathcal{X} = \mathcal{X}^{\#}$ as right $\mathfrak{A}$-module.
Then if $\mathbf{\Psi} : \mathcal{X} \to \mathfrak{A}$ is an element of $\mathcal{X}^{\#}$ there exisist a unique vector $X_o \in \mathcal{X}^{\#}$ such that

$$\mathbf{\Psi}(X) = \langle X; X_o \rangle$$

for $X \in \mathcal{X}$.

PROPOSITION 3.1. *If $\mathcal{X}$ is self-dual, then*

$$\mathcal{B}_{\mathfrak{A}}(\mathcal{X}) = \mathfrak{L}(\mathcal{X}).$$

PROOF. See [**21**] Proposition 3.4]. □

We have another fundamental proposition:

PROPOSITION 3.2. *If $\mathfrak{A}$ is a W^{*}-algebra, $\mathcal{X}^{\#}$ becomes a self-dual Hilbert $\mathfrak{A}$-module.*

PROOF. See [**21**] Proposition 3.2]. □

A *-representation of a C*-algebra $\mathfrak{B}$ on the Hilbert $\mathfrak{A}$-modulo $\mathcal{X}$ is a *-homorphism $\pi : \mathfrak{B} \to \mathfrak{L}(\mathcal{X})$.
The representation π is *non-degenerate* if is $\pi(\mathfrak{B})\mathcal{X}$ dense in $\mathcal{X}$.
We recall that one rank operator $|X\rangle\langle Y|$ on the Hilbert $\mathfrak{A}$- module $\mathcal{X}$ are thus definied:

$$|X\rangle\langle Y| Z = X \cdot \langle Y, Z \rangle_{\mathcal{X}}$$

for each $X, Y, Z \in \mathcal{X}$.
The set of compact adjointable operators on $\mathcal{X}$ is the closed subspace of $\mathfrak{L}(\mathcal{X})$ generated by the maps $|\cdot\rangle\langle\cdot|$:

$$\mathcal{K}(\mathcal{X}) = \overline{span\{|X\rangle\langle Y| : X, Y \in \mathcal{X}\}}.$$

* * *

We see the existing relations between Cp-map between C*-algebras and Hilbert modules over C*-algebras[1].
Let $\Phi : \mathfrak{A} \to \mathfrak{B}$ unital cp map between C*-algebras with unit $\mathfrak{A}$ and $\mathfrak{B}$.
The set $\mathfrak{X}_{\Phi} = \mathfrak{A}\overline{\otimes}_{\Phi}\mathfrak{B}$ with the $\mathfrak{B}$-valued inner product:

$$\langle A_1 \otimes_{\Phi} B_1; A_2 \otimes_{\Phi} B_2 \rangle = B_1^{*}\Phi(A_1^{*}A_2) B_2,$$

where $A_1, A_2 \in \mathfrak{A}$ and $B_1, B_2 \in \mathfrak{B}$, is a Hilbert $\mathfrak{A} - \mathfrak{B}$-module that is:

$$A_2 \cdot (A_1 \otimes_{\Phi} B_1) \cdot B_2 = (A_2 A_1) \otimes_{\Phi} (B_1 B_2).$$

We have the representation $\pi_{\Phi} : \mathfrak{A} \to \mathfrak{L}(\mathfrak{X}_{\Phi})$ in the following way:

$$\pi_{\Phi}(C) A \otimes_{\Phi} B = CA \otimes_{\Phi} B,$$

[1]For furthermore information cfr.[**21**] section 5].

for each $A,,C \in \mathfrak{A}$ $B \in \mathfrak{B}$.
If $\Omega_\Phi$is the vector $\Omega_\Phi = \mathbf{1} \otimes_\Phi \mathbf{1}$ for each $A \in \mathfrak{M}$ we obtain

$$\Phi(A) = \langle \Omega_\Phi; \pi_\Phi(A)\Omega_\Phi \rangle .$$

The triple $(\mathfrak{L}(\mathfrak{X}_\Phi); \pi_\Phi; \Omega_\Phi)$ is say to be the *GNS* of a cp-map $\Phi : \mathfrak{A} \to \mathfrak{B}$.

REMARK 3.1. *We observe that if $\Phi : \mathfrak{M} \to \mathfrak{M}$ is a cp-map between von Neumann algebra the set $\mathfrak{X}_\Phi = \mathfrak{M}\overline{\otimes}_\Phi\mathfrak{M}$ is a Hilbert $\mathfrak{M}$-module and for the precedent proposition it is self dual.*

PROPOSITION 3.3. *Let $\mathfrak{M}$ be a von Neumann algebra and $\Phi : \mathfrak{M} \to \mathfrak{M}$ be a cp-map. If for each $A_1, A_2 \in \mathfrak{M}$*

$$\Phi(A_1 A A_2) = 0,$$

we have

$$\pi_\Phi(A) = 0.$$

PROOF. For each $A_1 \otimes_\Phi B_1$, $A_2 \otimes_\Phi B_2 \in \mathfrak{X}_\Phi$ we have
$\langle A_1 \otimes_\Phi B_1; \pi_\Phi(A) A_2 \otimes_\Phi B_2 \rangle = B_1^* \cdot \langle A_1 \otimes_\Phi \mathbf{1}; AA_2 \otimes_\Phi \mathbf{1} \rangle \cdot B_2 =$
$= B_1^* \Phi(A_1^* A A_2^*) B_2 = 0,$
then $\langle A_1 \otimes_\Phi B_1; \pi_\Phi(A) A_2 \otimes_\Phi B_2 \rangle = 0.$
since $\mathfrak{X}_\Phi$ is self-dual we obtain $\pi_\Phi(A) = 0.$ □

2. Dilations constructed by using Hilbert modules

Let $\Phi : \mathfrak{A} \to \mathfrak{B}$ be an unital cp-map between C*-algebra $\mathfrak{A}$ and $\mathfrak{B}$, We have the follow applications:
► The Stinespring representation $\pi_\Phi : \mathfrak{A} \to \mathfrak{L}(\mathfrak{X}_\Phi)$, where $\mathfrak{X}_\Phi$ is Hilbert $\mathfrak{A}$-$\mathfrak{B}$ module $\mathfrak{X}_\Phi = \mathfrak{A}\overline{\otimes}_\Phi\mathfrak{B}$;
► The application $\mathcal{E}_\Phi : \mathfrak{L}(\mathfrak{X}_\Phi) \to \mathfrak{A}$ thus defined:

$$\mathcal{E}_\Phi(\mathbf{T}) = \langle \Omega_\Phi; \mathbf{T}\Omega_\Phi \rangle_{\mathfrak{X}_\Phi}, \qquad \mathbf{T} \in \mathfrak{L}(\mathfrak{X}_\Phi)$$

► The application $\mathcal{T}_\Phi : \mathfrak{B} \to \mathfrak{L}(\mathfrak{X}_\Phi)$ defined by[2]:

$$\mathcal{T}_\Phi(b)\, x \otimes_\Phi y = \mathbf{1} \otimes_\Phi b\Phi(x)\, y,$$

for each $x \otimes_\Phi y \in \mathfrak{X}_\Phi$ and $b \in \mathfrak{B}$.

$$\mathcal{T}_\Phi(b^*)\, x \otimes_\Phi y = \mathbf{1} \otimes_\Phi b^*\Phi(x)\, y.$$

We have a first proposition:

PROPOSITION 3.4. *The application $\mathcal{T}_\Phi : \mathfrak{B} \to \mathfrak{L}(\mathfrak{X}_\Phi)$ is an injective *-homomorphism.*

[2]The operator $\mathcal{T}_\Phi(b)$ is one rank operator:

$$|\mathbf{1} \otimes_\Phi b\rangle \langle \Omega_\Phi| ,$$

and

$$\mathcal{T}_\Phi(\mathbf{1}) = |\Omega_\Phi\rangle \langle \Omega_\Phi| .$$

Furthermore we have

$$\mathcal{T}_\Phi(\mathbf{1}) = \mathbf{1} \quad \Longleftrightarrow \quad \Phi \text{ is a multiplicative.}$$

PROOF. For each $x_1 \otimes_\Phi y_1, x_2 \otimes_\Phi y_2 \in \mathfrak{X}_\Phi$ and $b \in \mathfrak{B}$ we have:
$\langle \mathcal{T}_\Phi(b) x_1 \otimes_\Phi y_1; x_2 \otimes_\Phi y_2 \rangle_{\mathfrak{X}_\Phi} = \langle 1 \otimes_\Phi b\Phi(x_1) y_1; x_2 \otimes_\Phi y_2 \rangle_{\mathfrak{X}_\Phi} = (b\Phi(x_1) y_1)^* \Phi(x_2) y_2 =$
$= y_1^* \Phi(x_1^*) b^* \Phi(x_2) y_2 = \langle x_1 \otimes_\Phi y_1; \mathcal{T}_\Phi(b^*) x_2 \otimes_\Phi y_2 \rangle_{\mathfrak{X}_\Phi}.$
While
$\mathcal{T}_\Phi(b_1) \mathcal{T}_\Phi(b_2) x \otimes_\Phi y = \mathcal{T}_\Phi(b_1) \mathbf{1} \otimes_\Phi b_2 \Phi(x) y = \mathbf{1} \otimes_\Phi b_1 b_2 \Phi(x) y = \mathcal{T}_\Phi(b_1 b_2) x \otimes_\Phi y.$ □

We have the property of conditional expectation for the map $\mathcal{E}_\Phi$:

PROPOSITION 3.5. *The application $\mathcal{E}_\Phi : \mathfrak{L}(\mathfrak{X}_\Phi) \to \mathfrak{A}$ is unital cp map such that*

$$\mathcal{E}_\Phi(\mathcal{T}_\Phi(b_1) \mathbf{T} \mathcal{T}_\Phi(b_2)) = b_1 \mathcal{E}_\Phi(\mathbf{T}) b_2$$

for each $b_1, b_2 \in \mathfrak{B}$ and $\mathbf{T} \in \mathfrak{L}(\mathfrak{X}_\Phi)$.

PROOF. For each $b_j \in \mathfrak{B}$ and $\mathbf{T}_j \in \mathfrak{L}(\mathfrak{X}_\Phi)$ with $j = 1, 2...n$ we have

$$\sum_{i,j=1}^{n} b_j^* \mathcal{E}_\Phi\left(\mathbf{T}_j^* \mathbf{T}_i\right) b_i = \sum_{i,j=1}^{n} b_j^* \cdot \left\langle \Omega_\Phi; \mathbf{T}_j^* \mathbf{T}_i \Omega_\Phi \right\rangle_{\mathfrak{X}_\Phi} \cdot b_i =$$

$$= \sum_{i,j=1}^{n} \left\langle \mathbf{1} \otimes_\Phi b_j; \mathbf{T}_j^* \mathbf{T}_i \mathbf{1} \otimes_\Phi b_i \right\rangle_{\mathfrak{X}_\Phi} = \sum_{i,j=1}^{n} \left\langle \sum_{j=1}^{n} \mathbf{T}_j \mathbf{1} \otimes_\Phi b_j; \sum_{i=1}^{n} \mathbf{T}_i \mathbf{1} \otimes_\Phi b_i \right\rangle_{\mathfrak{X}_\Phi} \geq 0.$$

while

$$\mathcal{E}_\Phi(\mathcal{T}_\Phi(b_1) \mathbf{T} \mathcal{T}_\Phi(b_2)) = \langle \Omega_\Phi; \mathcal{T}_\Phi(b_1) \mathbf{T} \mathcal{T}_\Phi(b_2) \Omega_\Phi \rangle_{\mathfrak{X}_\Phi} =$$

$$= \langle \mathbf{1} \otimes_\Phi b_1^*; \mathbf{T} \mathbf{1} \otimes_\Phi b_2 \rangle_{\mathfrak{X}_\Phi} = b_1 \cdot \langle \Omega_\Phi; \mathbf{T} \Omega_\Phi \rangle_{\mathfrak{X}_\Phi} \cdot b_2 = b_1 \mathcal{E}_\Phi(\mathbf{T}) b_2.$$

□

We observe that

$$\Phi = \mathcal{E}_\Phi \circ \pi_\Phi \quad \text{and} \quad id = \mathcal{E}_\Phi \circ \mathcal{T}_\Phi.$$

In fact for each $b \in \mathfrak{B}$ we have

$$b = \langle \Omega_\Phi; \mathcal{T}_\Phi(b) \Omega_\Phi \rangle_{\mathfrak{X}_\Phi}.$$

Let $\Phi : \mathfrak{A} \to \mathfrak{A}$ be unital cp-map, we can define an unital cp-map $\widetilde{\Phi} : \mathfrak{L}(\mathfrak{X}_\Phi) \to \mathfrak{L}(\mathfrak{X}_\Phi)$ by

$$\widetilde{\Phi} = \pi_\Phi \circ \mathcal{E}_\Phi,$$

such that for each $x, y \in \mathfrak{A}$ we obtain

$$\widetilde{\Phi}(\mathcal{T}_\Phi(x) \mathcal{T}_\Phi(y)) = \widetilde{\Phi}(\mathcal{T}_\Phi(x)) \widetilde{\Phi}(\mathcal{T}_\Phi(y)).$$

Indeed for each $a \in \mathfrak{A}$ we have

$$\widetilde{\Phi}(\mathcal{T}_\Phi(a)) = \pi_\Phi(a).$$

Moreover

$$\begin{array}{ccc} \mathfrak{L}(\mathfrak{X}_\Phi) & \xrightarrow{\widetilde{\Phi}^n} & \mathfrak{L}(\mathfrak{X}_\Phi) \\ \mathcal{T}_\Phi \uparrow & & \downarrow \mathcal{E}_\Phi \\ \mathfrak{A} & \xrightarrow{\Phi^n} & \mathfrak{A} \end{array}$$

is a commutative diagram:

$$\mathcal{E}_\Phi\left(\widetilde{\Phi}^n(\mathcal{T}_\Phi(a))\right) = \Phi^n(a)$$

for each $n \in \mathbb{N}$ and $a \in \mathfrak{A}$.
We defined the $\widetilde{\varphi}$ state on $\mathfrak{L}(\mathfrak{X}_\Phi)$ by:

$$\widetilde{\varphi}(T) = \varphi((\mathcal{E}_\Phi T))$$

for each $T \in \mathfrak{L}(\mathfrak{X}_\Phi)$.
W have verified the following theorem of existence

THEOREM 3.1. *The C^*-dynamical system $\left(\mathfrak{L}(\mathfrak{X}_\Phi), \widetilde{\Phi}, \widetilde{\varphi},\right)$ is a non unital dilation of $(\mathfrak{A}, \Phi, \varphi)$ such that*

$$\mathfrak{A} \subset \mathfrak{D}\left(\widetilde{\Phi}\right)$$

where $\mathfrak{D}\left(\widetilde{\Phi}\right)$ is multiplicative domains of the cp-map $\widetilde{\Phi}$.

3. Ergodic property

For the study of the ergodic property of the dilations of dynamical systems we have to determine the value of the followings limits:

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \left[\widetilde{\varphi}\left(\mathbf{X}\widetilde{\Phi}^k(\mathbf{Y})\right) - \widehat{\varphi}(X)\widehat{\varphi}(Y)\right];$$
$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \left|\widehat{\varphi}\left(X\widehat{\Phi}^k(Y)\right) - \widehat{\varphi}(X)\widehat{\varphi}(Y)\right|$$

for all $\mathbf{X}, \mathbf{Y} \in \mathfrak{L}(\mathfrak{X}_\Phi)$.
We observe that for each $k \geq 1$ we get

$$\widetilde{\Phi}^k(\mathbf{Y}) = \pi_\Phi\left(\Phi^{k-1}(\mathcal{E}_\Phi(\mathbf{Y}))\right)$$

since for each $k \geq 1$ we obtain:

$$\widetilde{\Phi}^k(\mathbf{Y}) = \pi_\Phi\left(\Phi\left(\mathcal{E}_\Phi\left(\widetilde{\Phi}^{k-2}(\mathbf{Y})\right)\right)\right).$$

Consequently we have

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \widetilde{\varphi}\left(\mathbf{X}\widetilde{\Phi}^k(\mathbf{Y})\right) = \lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \widetilde{\varphi}\left(\mathbf{X}\pi_\Phi\left(\Phi^k(\mathcal{E}_\Phi(\mathbf{Y}))\right)\right).$$

Also in this circumstance the property of φ-adjoin it is fundamental for ergodicity:

THEOREM 3.2. *Let φ be ergodic state if Φ admit a φ-adjoin $\widetilde{\varphi}$ is an ergodic state. While if φ is a weakly mixing state, $\widetilde{\varphi}$ is a weakly mixing state.*

PROOF. We have

$$\widetilde{\varphi}\left(\mathbf{X}\pi_\Phi\left(\Phi^k(\mathcal{E}_\Phi(\mathbf{Y}))\right)\right) = \varphi\left(\mathcal{E}_\Phi\left(\mathbf{X}\pi_\Phi\left(\Phi^k(\mathcal{E}_\Phi(\mathbf{Y}))\right)\right)\right)$$

and
$\mathcal{E}_\Phi\left(\mathbf{X}\pi_\Phi\left(\Phi^k(\mathcal{E}_\Phi(\mathbf{Y}))\right)\right) = \left\langle \mathbf{X}^*\Omega_\Phi; \pi_\Phi\left(\Phi^k(\mathcal{E}_\Phi(\mathbf{Y}))\right)\Omega_\Phi\right\rangle_{\mathfrak{X}_\Phi} =$
$= \left\langle \mathbf{X}^*\Omega_\Phi; \Phi^k(\mathcal{E}_\Phi(\mathbf{Y})) \otimes_\Phi \mathbf{1}\right\rangle_{\mathfrak{X}_\Phi}.$

For definition of the Hilbert module $\mathfrak{X}_\Phi$, for each $\varepsilon > 0$ there is an element polynomial $p_\varepsilon = \sum_{i,j} a_i \otimes_\Phi b_j$ such that

$$\|\mathbf{X}^*\Omega_\Phi - p_\varepsilon\|_{\mathfrak{X}_\Phi} < \varepsilon$$

therefore

$$\left\langle p_\varepsilon ; \Phi^k\left(\mathcal{E}_\Phi\left(\mathbf{Y}\right)\right) \otimes_\Phi \mathbf{1}\right\rangle_{\mathfrak{X}_\Phi} = \left\langle \sum_{i,j} a_i \otimes_\Phi b_j ; \Phi^k\left(\mathcal{E}_\Phi\left(\mathbf{Y}\right)\right) \otimes_\Phi \mathbf{1}\right\rangle_{\mathfrak{X}_\Phi} =$$

$$= \sum_{i,j} b_j \Phi\left(a_i^* \Phi^k\left(\mathcal{E}_\Phi\left(\mathbf{Y}\right)\right)\right),$$

and

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi\left(\left\langle p_\varepsilon ; \pi_\Phi\left(\Phi^k\left(\mathcal{E}_\Phi\left(\mathbf{Y}\right)\right)\right)\Omega_\Phi\right\rangle_{\mathfrak{X}_\Phi}\right) =$$

$$= \lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi\left(\sum_{i,j} b_j \Phi\left(a_i^* \Phi^k\left(\mathcal{E}_\Phi\left(\mathbf{Y}\right)\right)\right)\right) = \sum_{i,j} \lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi\left(b_j \Phi\left(a_i^* \Phi^k\left(\mathcal{E}_\Phi\left(\mathbf{Y}\right)\right)\right)\right),$$

and for hypothesis

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi\left(\Phi^+\left(b_j\right) a_i^* \Phi^k\left(\mathcal{E}_\Phi\left(\mathbf{Y}\right)\right)\right) = \varphi\left(\Phi^+\left(b_j\right) a_i^*\right) \varphi\left(\mathcal{E}_\Phi\left(\mathbf{Y}\right)\right),$$

then

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi\left(\sum_{i,j} b_j \Phi\left(a_i^* \Phi^k\left(\mathcal{E}_\Phi\left(\mathbf{Y}\right)\right)\right)\right) = \varphi\left(\sum_{i,j} b_j \Phi\left(a_i^*\right)\right) \widetilde{\varphi}\left(\mathbf{Y}\right).$$

We observe

$$\varphi\left(\sum_{i,j} b_j \Phi\left(a_i^*\right)\right) \widetilde{\varphi}\left(\mathbf{Y}\right) = \varphi\left(\left\langle p_\varepsilon ; \Omega_\Phi\right\rangle_{\mathfrak{X}_\Phi}\right) \widetilde{\varphi}\left(\mathbf{Y}\right),$$

since

$$\sum_{i,j} b_j \Phi\left(a_i^*\right) = \left\langle p_\varepsilon ; \Omega_\Phi\right\rangle_{\mathfrak{X}_\Phi}.$$

Therefore

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \widetilde{\varphi}\left(\mathbf{X}\widetilde{\Phi}^k\left(\mathbf{Y}\right)\right) = \lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi\left(\left\langle \mathbf{X}^*\Omega_\Phi ; \Phi^k\left(\mathcal{E}_\Phi\left(\mathbf{Y}\right)\right) \otimes_\Phi \mathbf{1}\right\rangle\right) =$$

$$= \varphi\left(\left\langle p_\varepsilon ; \Omega_\Phi\right\rangle_{\mathfrak{X}_\Phi}\right) \widetilde{\varphi}\left(\mathbf{Y}\right),$$

Furthermore we have:

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \left[\widetilde{\varphi}\left(\mathbf{X}\widetilde{\Phi}^k\left(\mathbf{Y}\right)\right) - \widetilde{\varphi}\left(\mathbf{X}\right)\widetilde{\varphi}\left(\mathbf{Y}\right)\right] =$$

$$= \lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \left[\widetilde{\varphi}\left(\mathbf{X}\widetilde{\Phi}^k\left(\mathbf{Y}\right)\right) - \varphi\left(\left\langle p_\varepsilon ; \Omega_\Phi\right\rangle_{\mathfrak{X}_\Phi}\right) \widetilde{\varphi}\left(\mathbf{Y}\right)\right] +$$

$$+ \left[\widetilde{\varphi}\left(\mathbf{X}\right)\widetilde{\varphi}\left(\mathbf{Y}\right) - \varphi\left(\left\langle p_\varepsilon ; \Omega_\Phi\right\rangle_{\mathfrak{X}_\Phi}\right) \widetilde{\varphi}\left(\mathbf{Y}\right)\right].$$

Since

$$\left|\widetilde{\varphi}\left(\mathbf{X}\right) - \varphi\left\langle p_\varepsilon ; \Omega_\Phi\right\rangle_{\mathfrak{X}_\Phi}\right| = \varphi\left(\left\langle \mathbf{X}^*\Omega_\Phi - p_\varepsilon ; \Omega_\Phi\right\rangle_{\mathfrak{X}_\Phi}\right) \leq \|\mathbf{X}^*\Omega_\Phi - p_\varepsilon ;\|_{\mathfrak{X}_\Phi} < \varepsilon.$$

we obtain

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \left[\widetilde{\varphi}\left(\mathbf{X}\widetilde{\Phi}^k\left(\mathbf{Y}\right)\right) - \widetilde{\varphi}\left(\mathbf{X}\right)\widetilde{\varphi}\left(\mathbf{Y}\right)\right] =$$

$$= \lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \left[\widetilde{\varphi}\left(\mathbf{X}\widetilde{\Phi}^k\left(\mathbf{Y}\right)\right) - \varphi\left(\left\langle p_\varepsilon ; \Omega_\Phi\right\rangle_{\mathfrak{X}_\Phi}\right) \widetilde{\varphi}\left(\mathbf{Y}\right)\right] = 0.$$

For the weakly mixing property we can write

$$\frac{1}{N+1}\sum_{k=0}^{N}\left|\widetilde{\varphi}\left(\mathbf{X}\Phi^{k}\left(\mathcal{E}_{\Phi}\left(\mathbf{Y}\right)\right)\right)-\widetilde{\varphi}\left(\mathbf{X}\right)\widetilde{\varphi}\left(\mathbf{Y}\right)\right|=$$

$$=\frac{1}{N+1}\sum_{k=0}^{N}\left|\widetilde{\varphi}\left(\mathbf{X}\pi_{\Phi}\left(\Phi^{k}\left(\mathcal{E}_{\Phi}\left(\mathbf{Y}\right)\right)\right)\right)-\widetilde{\varphi}\left(\mathbf{X}\right)\widetilde{\varphi}\left(\mathbf{Y}\right)\right|.$$

Therefore we have

$$\frac{1}{N+1}\sum_{k=0}^{N}\left|\widetilde{\varphi}\left(\mathbf{X}\Phi^{k}\left(\mathcal{E}_{\Phi}\left(\mathbf{Y}\right)\right)\right)-\widetilde{\varphi}\left(\mathbf{X}\right)\widetilde{\varphi}\left(\mathbf{Y}\right)\right|\leq$$

$$\leq\frac{1}{N+1}\sum_{k=0}^{N}\left|\varphi\left(\left\langle p_{\varepsilon};\pi_{\Phi}\left(\Phi^{k}\left(\mathcal{E}_{\Phi}\left(\mathbf{Y}\right)\right)\right)\Omega_{\Phi}\right\rangle_{\mathfrak{X}_{\Phi}}\right)-\varphi\left(\left\langle p_{\varepsilon};\Omega_{\Phi}\right\rangle_{\mathfrak{X}_{\Phi}}\right)\widetilde{\varphi}\left(\mathbf{Y}\right)\right|+$$

$$+\left|\varphi\left(\left\langle p_{\varepsilon};\Omega_{\Phi}\right\rangle_{\mathfrak{X}_{\Phi}}\right)\widetilde{\varphi}\left(\mathbf{Y}\right)-\widetilde{\varphi}\left(\mathbf{X}\right)\widetilde{\varphi}\left(\mathbf{Y}\right)\right|+$$

$$+\frac{1}{N+1}\sum_{k=0}^{N}\left|\widetilde{\varphi}\left(\mathbf{X}\pi_{\Phi}\left(\Phi^{k}\left(\mathcal{E}_{\Phi}\left(\mathbf{Y}\right)\right)\right)\right)-\varphi\left(\left\langle p_{\varepsilon};\pi_{\Phi}\left(\Phi^{k}\left(\mathcal{E}_{\Phi}\left(\mathbf{Y}\right)\right)\right)\Omega_{\Phi}\right\rangle_{\mathfrak{X}_{\Phi}}\right)\right|.$$

Moreover

$$\left|\widetilde{\varphi}\left(\mathbf{X}\pi_{\Phi}\left(\Phi^{k}\left(\mathcal{E}_{\Phi}\left(\mathbf{Y}\right)\right)\right)\right)-\varphi\left(\left\langle p_{\varepsilon};\pi_{\Phi}\left(\Phi^{k}\left(\mathcal{E}_{\Phi}\left(\mathbf{Y}\right)\right)\right)\Omega_{\Phi}\right\rangle_{\mathfrak{X}_{\Phi}}\right)\right|=$$

$$=\left|\widetilde{\varphi}\left(\mathbf{X}\pi_{\Phi}\left(\Phi^{k}\left(\mathcal{E}_{\Phi}\left(\mathbf{Y}\right)\right)\right)\right)-\varphi\left(\left\langle p_{\varepsilon};\pi_{\Phi}\left(\Phi^{k}\left(\mathcal{E}_{\Phi}\left(\mathbf{Y}\right)\right)\right)\Omega_{\Phi}\right\rangle_{\mathfrak{X}_{\Phi}}\right)\right|=$$

$$=\left|\varphi\left(\left\langle\mathbf{X}\Omega_{\Phi}-p_{\varepsilon};\widetilde{\Phi}^{k}\left(\mathcal{E}_{\Phi}\left(\mathbf{Y}\right)\right)\Omega_{\Phi}\right\rangle_{\mathfrak{X}_{\Phi}}\right)\right|\leq\left\|\mathbf{X}\Omega_{\Phi}-p_{\varepsilon}\right\|_{\mathfrak{X}_{\Phi}}\left\|\mathbf{Y}\right\|.$$

It follow that

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\left|\widetilde{\varphi}\left(\mathbf{X}\Phi^{k}\left(\mathcal{E}_{\Phi}\left(\mathbf{Y}\right)\right)\right)-\widetilde{\varphi}\left(\mathbf{X}\right)\widetilde{\varphi}\left(\mathbf{Y}\right)\right|=$$

$$=\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\left|\varphi\left(\left\langle p_{\varepsilon};\pi_{\Phi}\left(\Phi^{k}\left(\mathcal{E}_{\Phi}\left(\mathbf{Y}\right)\right)\right)\Omega_{\Phi}\right\rangle_{\mathfrak{X}_{\Phi}}\right)-\varphi\left(\left\langle p_{\varepsilon};\Omega_{\Phi}\right\rangle_{\mathfrak{X}_{\Phi}}\right)\widetilde{\varphi}\left(\mathbf{Y}\right)\right|.$$

$$\frac{1}{N+1}\sum_{k=0}^{N}\left|\varphi\left(\left\langle p_{\varepsilon};\pi_{\Phi}\left(\Phi^{k}\left(\mathcal{E}_{\Phi}\left(\mathbf{Y}\right)\right)\right)\Omega_{\Phi}\right\rangle_{\mathfrak{X}_{\Phi}}\right)-\varphi\left(\left\langle p_{\varepsilon};\Omega_{\Phi}\right\rangle_{\mathfrak{X}_{\Phi}}\right)\widetilde{\varphi}\left(\mathbf{Y}\right)\right|=$$

$$=\frac{1}{N+1}\sum_{k=0}^{N}\left|\varphi\left(\sum_{i,j}\Phi^{+}\left(b_{j}\right)a_{i}^{*}\Phi^{k}\left(\mathcal{E}_{\Phi}\left(\mathbf{Y}\right)\right)\right)-\varphi\left(\sum_{i,j}b_{j}\Phi\left(a_{i}^{*}\right)\right)\widetilde{\varphi}\left(\mathbf{Y}\right)\right|=$$

$$\leq\sum_{i,j}\frac{1}{N+1}\sum_{k=0}^{N}\left|\varphi\left(\Phi^{+}\left(b_{j}\right)a_{i}^{*}\Phi^{k}\left(\mathcal{E}_{\Phi}\left(\mathbf{Y}\right)\right)\right)-\varphi\left(\Phi^{+}\left(b_{j}\right)a_{i}^{*}\right)\widetilde{\varphi}\left(\mathbf{Y}\right)\right|=0.$$

□

APPENDIX A

Algebraic formalism in ergodic theory

In this appendix we shortly give some fundamental definitions of the non-commutative ergodic theory. For further details on the subject, the reader can see the traditional works [**4**] and [**10**] of Doplicher and Kastler and books cited in bibliography.

* * *

The classical dynamic system is constituted by a space of probability (X, Σ, μ) and *measure-preserving transormation* $T : X \to X$ of the probability space (X, Σ, μ) , i.e.

$$\mu\left(T^{-1}(\Delta)\right) = \mu(\Delta)$$

for each $\Delta \in \Sigma$ (cfr.[**11**] section 1.1).
We recall the following definitions (cfr.[**24**] section 2.5):
The transformation T (or, more properly, the system (X, Σ, μ, T)) is called *ergodic* if and only if

- ▶ $\lim\limits_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \mu\left(T^{-k}\Delta \cap \Delta_o\right) = \mu(\Delta)\,\mu(\Delta_o)$ for each $\Delta, \Delta_o \in \Sigma$;
- ▶ We say that T is *weakly mixing* if
  $\lim\limits_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \left|\mu\left(T^{-k}\Delta \cap \Delta_o\right) - \mu(\Delta)\,\mu(\Delta_o)\right|$ for each $\Delta, \Delta_o \in \Sigma$.

In algebraic formalism the dynamic system (X, Σ, μ, T) is corresponds to the W^*-dynamical sistem $(L^\infty(X), \Phi, \varphi)$ where$L^\infty(X)$ is space of the bounded measurable function on (X, Σ, μ) ,the state φ is defined

$$\varphi(f) = \int_X f\, d\mu, \qquad f \in L^\infty(X)$$

while the dynamic Φ:$L^\infty(X) \to L^\infty(X)$ is

$$\Phi(f) = f \circ T, \qquad f \in L^\infty(X).$$

Then in the operator framework of quantum mechanics this definition picks up the following form:

Let $(\mathfrak{A}, \Phi)$ be a C^*-*dynamical systems*, a Φ-invariant state φ on $\mathfrak{A}$ is *ergodic* if and only if

- ▶ $\lim\limits_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi\left(b\Phi^k(a)\right) = \varphi(b)\,\varphi(a)$ for each $a, b \in \mathfrak{A}$;
- ▶ We say that Φ is *weakly mixing* if
  $\lim\limits_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \left|\varphi\left(b\Phi^k(a)\right) - \varphi(b)\,\varphi(a)\right| = 0$ for each $a, b \in \mathfrak{A}$.

Let $(\mathfrak{A}, \Phi, \varphi)$ be a C^*-dynamical systems with invariant state φ and $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$ its GNS. We can define for each $a \in \mathfrak{A}$, the following operator of $\mathcal{B}(\mathcal{H}_\varphi)$:

$$\mathbf{U}_\varphi \pi_\varphi (a) \Omega_\varphi = \pi_\varphi (\Phi (a)) \Omega_\varphi,$$

Then $\mathbf{U}_\varphi : \mathcal{H}_\varphi \to \mathcal{H}_\varphi$ is linear contraction of Hilbert spaces.
A fundamental result for the linear contraction of Hilbert space is the *Mean Ergodic Theory of von Neumann*:

Theorem A.1. *Let $\mathbf{V} : \mathcal{H} \to \mathcal{H}$ is a linear contraction of the Hilbert space $\mathcal{H}$ we have that*

$$\frac{1}{n+1}\sum_{k=0}^{n} \mathbf{V}^k \longrightarrow \mathbf{P} \qquad \textit{in so-topology,}$$

where $\mathbf{P}$ is a orthogonal projection on the linear space $\ker(\mathbf{I} - \mathbf{V}) = \ker(\mathbf{I} - \mathbf{V}^*)$.

Proof. See [**24**] theorem 2.1.1. □

We have the following result for the ergodic theory:

Proposition A.1. *Let $(\mathfrak{A}, \Phi, \varphi)$ be C^*-dynamical systems with invariant state, φ is ergodic state if and only if*

$$\dim(\ker(\mathbf{I} - \mathbf{U}_\varphi)) = \mathbf{1}.$$

Proof. See [**20**] lemma 5.2. □

Another important definition in ergodic theory is that of s*et of zero density* (cfr. [**20**]):
A subset Δ of $\mathbb{N}$ is say to have zero density if

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} 1_\Delta (k) = \lim_{n\to\infty} \frac{card\{[0,n] \cap \Delta\}}{n+1} = 0.$$

An sequence $\{x_n\}_{n\in\mathbb{N}}$ in a topological space X is said to convergence in density to an element $x \in X$ if there exists a subset $\Delta \subset \mathbb{N}$ of density zero such that

$$\lim_{n\to\infty} x'_n = x$$

where $x'_n = x_n$ for each $n \notin \Delta$.
We will also write

$$D - \lim_{n\to\infty} x_n = x.$$

We recall the fundamental lemma of *Koopman-von Neumann:*

Lemma A.1. *If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of positive real numbers, we have*

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} x_k = 0 \qquad \Longleftrightarrow \qquad D - \lim_{n\to\infty} x_n = 0.$$

Proof. See [**24**] lemma 6.2 pag 65. □

For the property of the $D - limit$, we postpone the reader to the reference [**34**]

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