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Projectively normal complete symmetric varieties and Fano complete symmetric varieties

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In this thesis we establish some properties of complete symmetric varieties. Let $\overline{G}$ be an adjoint semisimple group over $\mathbf{C}$ and let θ be an involution of $\overline{G}$. We define $\overline{H}$ as the subgroup of the elements fixed by θ and we will say that $\overline{G}/\overline{H}$ is a homogeneous symmetric variety. De Concini and Procesi [CSV I] have defined a wonderful completion of $\overline{G}/\overline{H}$ and this is the unique wonderful completion of $\overline{G}/\overline{H}$. They [CSV II] have also classified the complete symmetric varieties, i.e. the $\overline{G}$-varieties with a dense open orbit isomorphic to $\overline{G}/\overline{H}$ and a $\overline{G}$-equivariant map $Y \to X$ extending the identity of $\overline{G}/\overline{H}$. Indeed they showed that there is an equivalence of categories between the category of complete symmetric varieties and the category of toric varieties over an affine space $\mathbf{A}^l$ considered as a $(\mathbf{C}^*)^l$ variety in the obvious way, where l is the rank of G/H. Moreover there is a one-to-one correspondence between the completion Y of $\overline{G}/\overline{H}$ which lie over X and the elements of a special class of complete toric varieties. One can show that the complete toric variety Z^c corresponding to a complete symmetric variety Y is a subvariety of Y and the open toric variety Z corresponding to Y is open subvariety of Z^c.

In this thesis, unless explicitly stated, we shall always assume that the complete symmetric variety Y is smooth. Recall that by [CSV II] it then follows that: 1) any orbit closure in Y is also smooth; 2) the associated toric varieties Z and Z^c are both smooth. Our first result is a classification of the projective complete symmetric varieties. In particular we will prove that a complete symmetric variety is projective if and only if the corresponding complete toric variety is projective. Therefore we can use results for the classification of the projective toric varieties.

Next we will study the projective normality of the complete symmetric varieties. Chirivì and Maffei [CM II] have proved that, given any two line bundles, say L_1 and L_2, generated by global sections on the wonderful complete symmetric variety X, the product of sections

$$H^0(X, L_1) \otimes H^0(X, L_1) \longrightarrow H^0(X, L_1 \otimes L_2)$$

is surjective. This result implies easily the projective normality of X with respect to any projective embedding by a complete linear system. We will try to generalize this results to any complete symmetric variety. First we will prove that the surjectivity of the product of sections of two ample line bundles on a complete symmetric variety is equivalent to the surjectivity of the product of sections of the restrictions of the line bundles to the corresponding complete toric variety. Thus we will have reduced the problem to a problem on toric varieties. But it is very difficult to verify the surjectivity of the product of sections of any two ample line bundles on a generic complete toric variety. However, we can simplify the problem for the special class of complete toric varieties which we are considering. Indeed we will prove that the surjectivity of the product of sections of two ample line bundle on Z^c, say L_1 and L_2, is equivalent to the surjectivity of the product of sections of the restrictions of the line bundles to

Z. This problem is much simpler, because $H^0(Z, L_1|Z)$ and $H^0(Z, L_2|Z)$ are infinite dimensional vector spaces and it is sufficient to prove that the a suitable finite dimensional subspace of $H^0(Z, (L_1 \otimes L_2)|Z)$ is contained in the image of the product of sections. Indeed we will prove that, given any ample line bundle L on Z^c, $H^0(Z, L|Z)$ is generated by $H^0(Z^c, L|Z^c)$ as an $\mathcal{O}_Z(Z)$-module. Next we will find a infinite number of varieties of every dimension such that, for any ample line bundle L on a such variety Z, the product of sections of L

$$H^0(Z, L) \otimes H^0(Z, L) \longrightarrow H^0(Z, L \otimes L)$$

is surjective. In particular we will prove that all the smooth toric varieties proper over $\mathbf{A}^2$ have this property.

In the last part of this work we will study the Fano complete symmetric varieties. A variety is called a Fano variety if its anticanonical bundle is ample. It easy to show that the anticanonical bundle of the wonderful symmetric variety is always generated by global sections. We will classify the homogeneous symmetric varieties $\overline{G}/\overline{H}$ whose wonderful completion is a Fano variety. In particular we will show that the wonderful completion of $\overline{G}/\overline{H}$ is usually Fano, for example if the involution on the root system is different from $-id$, but there are cases for which the wonderful completion of $\overline{G}/\overline{H}$ is not Fano, for example the homogeneous symmetric varieties associated to the involutions of type CI.

More generally, we want to know which complete symmetric varieties are Fano varieties. Bifet [Bi] has shown that there is a deep relation between the line bundles on a complete symmetric variety and the line bundles on the corresponding toric variety. It is known that there are only a finite number of complete toric Fano varieties of every fixed dimension (see [VK]), up to isomorphisms, and they are classified in low dimension. Thus we can expect that the same facts are true for the complete symmetric varieties. We will prove that there are only a finite number of Fano complete symmetric varieties for every $\overline{G}/\overline{H}$. We will classify them for every $\overline{G}/\overline{H}$ whose rank is 2. More generally we will classify, for every $\overline{G}/\overline{H}$, the Fano complete varieties obtainable through a sequence of blow-ups along closed orbits from the wonderful variety. We will show that there are at most two Fano complete symmetric varieties with such property. Unluckily, if the rank of $\overline{G}/\overline{H}$ is strictly greater than 2, this condition is very restrictive. If the rank of $\overline{G}/\overline{H}$ is 3 we can say a bit more. In this case we will classify the Fano complete symmetric varieties obtainable from the wonderful variety through a sequence of blow-ups along $\overline{G}$ stable subvarieties. This condition is not much restrictive. Indeed it is easy to construct example of varieties that does not satisfy these hypothesis, but usually they are not projective, so a fortiori they are not Fano varieties. We will prove that there are at most eleven Fano complete symmetric varieties with such property. The most important part of the proof of the previous classifications will be the classifications of the corresponding open toric varieties with ample anticanonical bundle.

We will also classify the complete symmetric varieties of rank 2 whose anticanonical bundle is generated by global sections. We will show that there is only a finite number of complete symmetric varieties with such property for every $\overline{G}/\overline{H}$ of rank 2, but this number is arbitrarily large.

Part I
Introduction

1 Homogeneous symmetric varieties

First of all, we will describe some preliminary results and we will fix the notations. In this section we want to describe some properties of the homogeneous symmetric varieties. For details on the homogeneous symmetric varieties see [He], [A], [Bu], [K] or [W]. Let G be a connected and simply-connected semi-simple algebraic group over $\mathbf{C}$ and let θ be an involution of G, we define H as the normalizer of the subgroup of invariants G^θ.

Definition 1.1 *We will say that G/H is a (homogeneous) symmetric variety.*

Sometimes we will say that G/H is the symmetric variety associated to (G, θ). We shall denote by $\overline{G}$ the adjoint group associated to G, i.e. the quotient of G by the center $Z(G)$. One can show that there is an one-to-one correspondence between the involutions of G and the involutions of $\overline{G}$. Moreover an involution of G and the corresponding involution of $\overline{G}$ induce the same involution of the Lie algebra $\mathfrak{g}$ of G (and $\overline{G}$). One can show that G/H is isomorphic to the quotient of $\overline{G}$ by the subgroup $\overline{G}^\theta$ of invariants with respect to the involution associated to θ. Observe that there is an one-to-one correspondence between the involutions of G and the involutions of $\mathfrak{g}$ because G is connected and simply-connected. By abuse of notation, we call θ also the involution on $\mathfrak{g}$ associated to θ.

As example of a symmetric variety we can consider any adjoint group $\overline{G}$ considered as a $G \times G$ homogeneous space. Here the involution is $(G \times G, \theta)$ with $\theta((x, y)) = (y, x)$ for each $x, y \in G$.

Definition 1.2 *We will say that G/H is simple if either G is a simple semi-simple group or G/H is a simple adjoint group.*

If G/H is not simple then there are two connected and simply-connected semi-simple group G_1 and G_2 such that $G = G_1 \times G_2$. Moreover there are an involution θ_1 on G_1 and an involution θ_2 on G_2 such that $\theta((x, y)) = (\theta(x), \theta(y))$, so $G/H = G_1/H_1 \times G_2/H_2$. In this case we will write $(G, \theta) = (G_1, \theta_1) \times (G_2, \theta_2)$.

θ acts diagonally on $\mathfrak{g}$ and it has two eigenvalues, namely 1 and -1. The 1-eigenspace $\mathfrak{h}$ is the Lie algebra of H. Observe that $\mathfrak{h}$ is also the Lie algebra of the subgroup of θ fix-points. Moreover H is the largest subgroup of G whose Lie algebra is $\mathfrak{h}$. Notice that the (-1)-eigenspace is isomorphic to the tangent space of the symmetric variety G/H at H. We want to describe it explicitly, but first we have to choose a suitable maximal torus of G and a suitable Borel subgroup of G. We will say that a stable torus is split if $\theta(t) = t^{-1}$ for each element t of the torus.

Definition 1.3 *Let $T^1 \subset G$ be a split torus of maximal dimension l. We will say that l is the rank of the symmetric variety G/H.*

Choose any maximal torus T which contains T^1, one can show that T is θ-stable.

Notation 1 *Let T^0 be the identity component of the subgroup $T \cap G^\theta$ of invariants of T, we will define S as the quotient $T^1/(T^1 \cap T^0)$ of T^1 by the intersection of T^1 with T^0.*

Observe that $T^1 \cap T^0$ consists of elements of order two, namely elements t such that $t = t^{-1}$. The Lie algebra $\mathfrak{t}$ of T is θ-stable, so we can write $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1$ where $\mathfrak{t}_0$ is the 1-eigenspace and $\mathfrak{t}_1$ is the (-1)-eigenspace. $\mathfrak{t}_0$ is the Lie algebra of T^0 and $\mathfrak{t}_1$ is the Lie algebra of T^1. Since $\mathfrak{t}$ is θ stable, θ induces an involution on $\mathfrak{t}^*$ that we call again θ. Moreover this involution of $\mathfrak{t}^*$ stabilizes the root system ϕ of G and it preserves the Killing form. Observe that we can identify $\mathfrak{t}^*$ with the complexification $\chi^*(T) \otimes_{\mathbf{Z}} \mathbf{C}$ of the group of characters $\chi^*(T)$ of T. Moreover $\chi^*(T)$ is the lattice of integral weights of the root system of $\mathfrak{g}$ and it is stabilized by θ. We have the maps

$$\begin{tikzcd} T \\ T^1 \arrow[u, hook] \arrow[r, two heads] & S, \end{tikzcd}$$

where the vertical map is the inclusion and the horizontal one is the quotient map. These maps induce maps on the corresponding groups of characters.

$$\begin{tikzcd} & \chi^*(T) \arrow[d] \\ \chi^*(S) \arrow[r, hook] & \chi^*(T^1). \end{tikzcd}$$

If we extend by linearity these maps to maps of real vector spaces, then the map $\chi^*(S) \hookrightarrow \chi^*(T^1)$ becomes an isomorphism, so we have a surjective map $\chi^*(T)_{\mathbf{R}} \twoheadrightarrow \chi^*(S)_{\mathbf{R}}$. Moreover the restriction of this map to the (-1)-eigenspace is an isomorphism, so we can identify $\chi^*(S)$ with a lattice M contained in $\chi^*(T)_{\mathbf{R}}$. (Given an abelian group A we set $A_{\mathbf{R}} := A \otimes \mathbf{R}$).

Definition 1.4 *Let N be the dual $Hom(M, \boldsymbol{Z})$ of M, where M is identified to the group of characters of S. Thus N is the group of 1-parameter subgroups of S.*

Notice that $S = N \otimes_{\mathbf{Z}} \mathbf{C}^*$.

Notation 2 *We call ϕ_0 the subset $\{\alpha \in \phi \,|\, \theta(\alpha) = \alpha\}$ of ϕ formed by the roots fixed by θ. Moreover we set $\phi_1 = \phi - \phi_0$.*

One can show that the maximality of $\dim(T^1)$ is equivalent to the fact that $\theta|\mathfrak{g}_\alpha = id_{\mathfrak{g}_\alpha}$ for each $\alpha \in \phi_0$ (here $\mathfrak{g}_\alpha$ is the root space corresponding to α). We can choose a Borel subgroup such that the associated set ϕ^+ of positive roots has the following property: if we set $\phi_1^+ = \phi_1 \cap \phi^+$ and $\phi_0^+ = \phi_0 \cap \phi^+$ then $\theta(\phi_1^+) = -\phi_1^+$, namely the image of any positive root α by θ is either α itself or is a negative root. Finally we can give an explicitly description of $\mathfrak{h}$.

Proposition 1.1 $\mathfrak{h} = \mathfrak{t}_0 \oplus \bigoplus_{\alpha \in \phi_0} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \phi_1^+} \boldsymbol{C}(x_\alpha + \theta(x_\alpha))$*, where x_α is any fixed not zero element of $\mathfrak{g}_\alpha$.*

This proposition implies the Iwasawa decomposition: the (-1)-eigenspace of $\mathfrak{g}$ is isomorphic to $\mathfrak{t}_1 \oplus \bigoplus_{\alpha \in \phi_1^+} \mathfrak{g}_\alpha$. Indeed this space is isomorphically projected onto the tangent space of G/H at H. In particular the dimension of G/H is $\dim \mathfrak{t}_1 + 1/2|\phi_1|$. Observe that $BH \subset G$ is dense in G because Lie$(B) \supset \mathfrak{t}_1 \oplus \bigoplus_{\alpha \in \phi_1^+} \mathbf{C}x_\alpha$. Thus G/H has a dense B orbit, namely G/H is a spherical variety.

We can associate a possibly non reduced root system to the involution θ. This root system is usually called the restricted root system of (G, θ) (or the relative root system of (G, θ)). ϕ is sometimes called the absolute root system of (G, θ). Let Γ be the set of simple roots of ϕ, we set $\Gamma_0 = \Gamma \cap \phi_0$ and $\Gamma_1 = \Gamma \cap \phi_1$. For any root α we set $\alpha^s = \alpha - \theta(\alpha)$. If α^s is not zero, we say that it is a restricted root. Observe that α^s is not zero if and only if α belongs to ϕ_1.

Proposition 1.2 *The set $\widetilde{\phi} = \{\alpha^s | \alpha \in \phi_1\}$ is a possibly not reduced root system of rank l in $M_{\boldsymbol{R}}$. We call it the restricted root system.*

Sometimes the restricted root system is defined as $\{\alpha^s/2 \,|\, \alpha \in \phi_1\}$, because $(\alpha^s/2)(t) = (\alpha)(t)$ for each $t \in \mathfrak{t}_1$. Moreover $(\alpha^s/2)(t) = (\alpha^s)(t) = 0$ for each $t \in \mathfrak{t}_0$. A basis of the restricted root system is the set $\widetilde{\Gamma} = \{\alpha^s \,|\, \alpha \in \Gamma_1\}$ (notice that $\Gamma_1 \subset \phi_1$ so $\alpha^s \neq 0$ for each $\alpha \in \Gamma_1$). Moreover we can choose an order of the simple roots such that $\Gamma_1 = \{\alpha_1, ..., \alpha_l, \alpha_{l+1}, ..., \alpha_r\}$ and the α_i^s are distinct for $i = 1, ..., l$, so $\widetilde{\Gamma} = \{\alpha_1^s, ..., \alpha_l^s\}$.

Now we want to describe the Weyl group of $\widetilde{\phi}$ and the lattice of integral weights of $\widetilde{\phi}$.

Proposition 1.3 (See proposition 1.1.3.3 in [W]) *One can identify the Weyl group W^1 of the restricted root system with the group $\{w \in W : w \cdot \mathfrak{t}_1 \subset \mathfrak{t}_1\}/W_0$, where W is the Weyl group of ϕ and W_0 is the Weyl group of the root system ϕ_0 (in $\chi^*(T^0)_{\boldsymbol{R}}$).*

Notation 3 *By Λ we denote the lattice of integral weights of ϕ and by Λ^+ the set of dominant weights.*

Let ω_α be the fundamental weight corresponding to the simple root α and let $<\ ,\ >$ be the scalar product of $\Lambda_{\mathbf{R}}$. Observe that $<\ ,\ >$ induces the scalar product of $M_{\mathbf{R}}$, so we denote this last scalar product again by $<\ ,\ >$. Given a dominant weight λ, let V_λ be the irreducible representation of G of highest weight λ.

Definition 1.5 *We will say that a dominant weight λ is a spherical weight if there is a not zero vector $k \in V_\lambda$ fixed by $\mathfrak{h}$, namely $h \cdot k = 0$ for each $h \in \mathfrak{h}$.*

If λ is a spherical weight then k is unique up to a scalar and we call it k_λ.

Definition 1.6 *We will say that a dominant weight λ is special if $\theta(\lambda) = -\lambda$.*

Observe that if λ is a special weight then it belongs to $M_{\mathbf{R}}$. One can show that the spherical weights are special. Viceversa given a special weight λ then 2λ is spherical. One can show:

Proposition 1.4 (Lemma 2.1 in [CM I]) *Let Ω^+ be the set of spherical weights and let Ω be the lattice generated by the spherical weights, then $\Omega \cap \Lambda^+ = \Omega^+$.*

Notice that Ω contains M. We want to describe more explicitly the relation between spherical weights and special weights. The involution θ induce an involution $\overline{\theta}$ of the set Γ_1 of the simple roots not fixed by θ. Indeed, given any $\alpha \in \Gamma_1$ there is an (unique) $\overline{\theta}(\alpha)$ in Γ_1 such that $\theta(\alpha) = -\overline{\theta}(\alpha) - \beta_\alpha$, where β_α is a positive linear combination of simple roots fixed by θ. Moreover $\theta(\omega_\alpha) = -\omega_{\overline{\theta}(\alpha)}$ for each α in Γ_1. Observe that, given a weight λ, $\theta(\lambda) = -\lambda$ if and only if $\lambda = \sum_{\alpha \in \Gamma_1} n_\alpha \omega_\alpha$ with $n_{\overline{\theta}(\alpha)} = n_\alpha$ for each α in Γ_1. Let $\widetilde{\omega}_i = \omega_{\alpha_i}$ if $\overline{\theta}(\alpha_i) = \alpha_i$ and let $\widetilde{\omega}_i = \omega_{\alpha_i} + \omega_{\overline{\theta}(\alpha_i)}$ if $\overline{\theta}(\alpha_i) \neq \alpha_i$. Thus a dominant weight λ is special if and only $\lambda = \sum_{i=1}^{l} n_i \widetilde{\omega}_i$ for suitable positive integers $n_1, ..., n_l$. Moreover $\widetilde{\omega}_1, ..., \widetilde{\omega}_l$ are free generators of the lattice generated by the special weights, namely $\{\lambda \in \Lambda : \theta(\lambda) = -\lambda\}$. We will say that a special weight λ is regular if $n_i > 0$ for each $i = 1, ..., l$. Now we can describe Ω explicitly.

Proposition 1.5 (Theorem 2.3 in [CM I]) $\Omega = \bigoplus_{i=1}^{l} \boldsymbol{Z} a_i \widetilde{\omega}_i$ *where $a_i \in \{1, 2\}$ for each i. a_i is equal to 2 if $\theta(\alpha_i) = -\alpha_i$, while it is equal to 1 if $\theta(\alpha_i) \neq -\alpha_i$. In particular $a_i = 1$ if $\overline{\theta}(\alpha_i) \neq \alpha_i$. Moreover, for each i and j we have $< a_i \widetilde{\omega}_i, (\alpha_j^s)^\vee >= b_i \delta_{i,j}$ where $(\alpha_j^s)^\vee$ is the coroot associated to α_j^s and $b_i \in \{1, 2\}$. $b_j = 2$ if and only if $2\alpha_j^s \in \widetilde{\phi}$. In particular, if $\widetilde{\phi}$ is reduced then $a_1 \widetilde{\omega}_1, ..., a_l \widetilde{\omega}_l$ are the fundamental weights dual to $(\alpha_1^s)^\vee, ..., (\alpha_l^s)^\vee$.*

Given a weight λ we define the Ω-support of λ as the set $supp_\Omega(\lambda) = \{\alpha^s \in \widetilde{\Gamma} \mid (\lambda, \alpha^s) \neq 0\}$. Observe that a special weight λ is regular if and only if $supp_\Omega(\lambda) = \widetilde{\Gamma}$.

Notation 4 *Let C^+ be the positive Weyl chamber of the restricted root system.*

Observe that $C^+ = \Lambda_{\mathbf{R}}^+ \cap M_{\mathbf{R}}$.

2 The wonderful symmetric variety

Now we want to describe the wonderful compactification of G/H. Let λ be a spherical weight such that $supp_\Omega(\lambda) = \widetilde{\Gamma}$ and let V be a finite dimensional representation of $\mathfrak{g}$ such that $V = V_\lambda \oplus V'$ for a suitable representation V'. Let $k_{V'} \in V'$ be a vector fixed by $\mathfrak{h}$, so also $k = k_\lambda + k_{V'}$ is fixed by $\mathfrak{h}$. Suppose that each weight of $k_{V'}$ has the form $\lambda - \sum_{i=1}^{l} n_i \alpha_i^s$ where the n_i are positive integers and they are not all zero. Let $[k]$ be the class of k in $\mathbf{P}(V)$, we define X as the closure of $G[k]$ in $\mathbf{P}(V)$. The maps $g \to g[k]$ induce an embedding $G/H \hookrightarrow X$ that is called the "minimal compactification" of G/H. Moreover this construction is independent from the choice of the weight λ and of the representation V'.

We can give another description of the minimal compactification of X. Let $\lambda_1, ..., \lambda_m$ be spherical weights whose Ω-supports are disjoint and such that $supp_\Omega(\lambda_1) \cup ... \cup supp_\Omega(\lambda_m) = \widetilde{\Gamma}$. If we define x_0 as the point

$$([h_{\lambda_1}], ..., [h_{\lambda_m}]) \in \mathbf{P}(V_{\lambda_1}) \times ... \times \mathbf{P}(V_{\lambda_m}),$$

then we can extend the map $G/H \ni gH \to gx_0 \in \mathbf{P}(V_{\lambda_1}) \times ... \times \mathbf{P}(V_{\lambda_m})$ to an isomorphism $X \to \overline{Gx_0}$.

We will need a local description of X. Let V be as before and choose a basis of weight vectors. We define $\widetilde{A}$ as the affine open set of $\mathbf{P}(V)$ where the coordinate corresponding to the highest weight v_λ is not zero. Let $A = \widetilde{A} \cap X$ and observe that A is U^- stable, where U^- is the unipotent group associated to $-\phi_1^+$, namely $U^- = \prod_{-\alpha \in \phi_1^+} U_\alpha$ as a variety. One can show that the closure of $T[k]$ in $\widetilde{A}$ is an affine space $\mathbf{A}^l$ with coordinates $-\alpha_1^s, ..., -\alpha_l^s$. Moreover the map $\varphi : U^- \times \mathbf{A}^l \to A$ given by $\varphi(g, v) = g \cdot v$ is an isomorphism. For each i, let $X_{-\alpha_i^s}$ be the divisor of X whose intersection with $U^- \times \mathbf{A}^l$ is the locus of zeroes of $-\alpha_i^s$. Notice that there is an unique closed orbit in $\mathbf{P}(V)$ and it is contained in X. This implies that X is covered by the G-translates open sets of A. Let P be the parabolic subgroup of G associated to Γ_0, namely the parabolic subgroup whose Lie algebra is $\mathfrak{t} \oplus \bigoplus_{\alpha \in \phi_{\Gamma_0} \cup \phi^+} \mathfrak{g}_\alpha$, where ϕ_{Γ_0} is the root system generated by Γ_0. The previous observations allow ourselves to prove the following theorem.

Theorem 2.1 (Theorem 3.1 in [CSV I]) *Let X be the minimal compactification G/H, then:*

1. *X is a smooth projective G-variety;*
2. *$X \backslash (G \cdot [k])$ is a divisor with normal crossings. It has irreducible components $X_{-\alpha_1^s}, ..., X_{-\alpha_l^s}$ and they are smooth subvarieties of X.*
3. *the G-orbits of X correspond to the subset of $\{1, 2, ..., l\}$ so that the orbit closures are the intersections $X_{-\alpha_{i_1}^s} \cap ... \cap X_{-\alpha_{i_k}^s}$.*
4. *there is an unique closed orbit $\bigcap_{i=1}^{l} X_{-\alpha_i^s}$ and it is isomorphic to G/P.*

This proposition shows that X is a wonderful variety according to the definition of Luna [L]. Moreover it is the unique wonderful compactification of G/H, so we will often call it the wonderful symmetric variety.

3 Line bundles on the wonderful symmetric variety

We want to study the Picard group of X. First of all we consider some properties which are valid on a much more general class of varieties.

Proposition 3.1 *Let G be a connected and simply-connected semi-simple algebraic group and let V be a smooth complete G-variety. Suppose that G acts trivially on $Pic(V)$. Then, given any line bundle L on V, there is a (canonical) linearization of L and $H^i(V,L)$ is a G representation for each i. Thus $Pic(V)$ is isomorphic to the group $Pic_G(V)$ of the G-linearized line bundles.*

It easy to see that if V is a spherical G-variety then our assumption are satisfied and we can say more.

Proposition 3.2 *Let $\widetilde{G}$ a connected reductive group and let V be a spherical $\widetilde{G}$-variety, namely a (smooth) $\widetilde{G}$-variety with a dense orbit with respect to a fixed Borel subgroup of $\widetilde{G}$. If L is any linearized line bundle on V then $H^0(V,L)$ is a multiplicity-free $\widetilde{G}$-representation, namely every $\widetilde{G}$ irreducible representation appears in $H^0(V,L)$ with multiplicity at most 1.*

The following proposition implies that we can identify $Pic(X)$ with a sublattice Λ_X of the lattice of weights.

Proposition 3.3 (Proposition 8.1 in [CSV I]) *The map $Pic(X) \to Pic(G/P)$ induced by the canonical inclusion is injective.*

Remember that we can identify $Pic(G/P)$ with a sublattice of the lattice of weights. Indeed $Pic(G/P) \equiv Pic_G(G/P)$ because G/P is a spherical variety and a linearized line bundle $L \in Pic_G(G/P)$ corresponds to the opposite λ of the character $-\lambda$ with which T acts on the fibre over $P \in G/P$. Explicitly we can define L as follows. We can extent λ to a one-dimensional representation V of P. We define L as the quotient of of $G \times V$ by the action of P defined as follows: $p(g,v) = (gp^{-1}, \lambda(p)v)$ for each $p \in P$ and $(g,v) \in G \times V$. The projection $L \to G/P$ is induced by the projection $G \times V \to G \to G/P$ and G acts on L by $h[(g,v)] = [(hg,v)]$ for each $h,g \in G$ and $v \in V$.

Because of the previous proposition, we denote a line bundle on X with L_λ if its image is the weight λ. Let λ be a dominant weight such that $\mathbf{P}(V_\lambda)$ contains a line r fixed by H, for example $\lambda \in \Omega^+$. One can show that the map $G/H \ni gH \to gr$ can be extended to a morphism $\psi_\lambda : X \to \mathbf{P}(V_\lambda)$. The line bundle $\psi_\lambda^* O(1)$ is L_λ. Indeed if we restrict $\psi_\lambda^* O(1)$ to G/P we obtain the line bundle that correspond to λ (in the previous correspondence between $Pic(G/P)$

and a sublattice of Λ). If L_μ is a line bundle on X such that μ is dominant, then there is a sub-representation of $H^0(X, L_\mu)$ isomorphic to V_μ^*, obtained by pullback of $H^0(\mathbf{P}(V_\mu), \mathcal{O}(1))$ to X. Moreover this representation is unique because $H^0(X, L_\mu)$ is multiplicity free. So we can call it V_μ^* without ambiguity. Moreover

Lemma 3.1 (Lemma 4.6 in [CS]) *$Pic(X)$ is the lattice generated by the dominant weights λ such that $\boldsymbol{P}(V_\lambda)^H$ is not trivial. Moreover if $\boldsymbol{P}(V_\lambda)^H$ is not trivial then it is a point.*

We want give a more explicit description of $Pic(X)$. Remember that there is an involution $\overline{\theta}$ of Γ_1.

Definition 3.1 *We will say that a root $\alpha \in \Gamma_1$ is an exceptional root if $\overline{\theta}(\alpha) \neq \alpha$ and $< \alpha, \theta(\alpha) > \neq 0$. Moreover we will say that G/H is exceptional if there is an exceptional root. We will say that a compactification of G/H is exceptional if G/H is exceptional.*

Observe that $\overline{\theta}(\alpha)$ is exceptional if and only if α is. Moreover one can show that, if G/H is exceptional, then the restricted root system $\widetilde{\phi}$ is not reduced.

Theorem 3.1 (Theorem 4.8 in [CS]) *$Pic(X)$ is generated by the spherical weights and by the fundamental weights corresponding to the exceptional roots.*

Notice that, given an exceptional root $\alpha \in \Gamma_1$, $\theta(\omega_\alpha) = -\omega_{\overline{\theta}(\alpha)}$ and $\omega_\alpha + \omega_{\overline{\theta}(\alpha)}$ is a spherical weight. We will need the following lemma on the line bundles corresponding to the opposite of the simple restricted roots.

Proposition 3.4 (Corollary 8.2 in [CSV I]) *There is a G-invariant section $s_{-\alpha_i^s} \in H^0(X, O(X_{-\alpha_i^s}))$ whose divisor is $X_{-\alpha_i^s}$. Moreover this section is unique up to a non zero scalar.*

In the first part of this work we want to generalize the following theorem.

Theorem 3.2 (Theorem A in [CM II]) *Let L_λ and L_μ be two line bundles generated by global sections on X. Then the product of sections*

$$M_{\lambda,\mu} : H^0(X, L_\lambda) \times H^0(X, L_\mu) \to H^0(X, L_{\lambda+\mu})$$

is surjective.

In [CM II] the previous theorem is stated with the hypothesis that λ and μ are dominants. But we will prove that a line bundle L_λ is generated by global sections if and only if λ is dominant, so our assumptions are equivalent to those ones in [CM II]. As a consequence of the previous theorem we have (see for example [Ha] Exercise II.5.14):

Corollary 3.1 *Let L be a line bundle on X generated by global sections and consider the map $X \to \boldsymbol{P}(H^0(X, L)^*)$ defined by L. Then the cone over the image of X is normal.*

4 Toric varieties

In this section we want to collect some results about toric varieties. For details on toric varieties see [F], [O] or [Da]. See [G] for more results about polytopes and see [R] for more results about convex functions.

It is known that there is an equivalence of categories between the category of embeddings of G/H over X and the the category of embeddings of S over $\mathbf{A}^l$. This suggests to describe the toric varieties before describing the embeddings of G/H over X. Moreover this description will be useful to understand the combinatorial constructions that we will do on the embeddings of G/H over X. Indeed these constructions are very similar to the ones used in the theory of toric varieties, but are more difficult to describe geometrically. For these reasons we will be very detailed in this section.

4.1 First definitions

Let S be the torus $N \otimes_{\mathbf{Z}} \mathbf{C}^* = Spec(\mathbf{C}[M])$ where M is a finitely generated free abelian group and N is the dual $Hom(M, \mathbf{Z})$ of M. We can identify M with the character group of S and N with group of 1-parameter subgroups of S. Given $m \in M$ we call χ^m the associated function on S, so $\chi^{m+m'} = \chi^m \cdot \chi^{m'}$ for all $m, m' \in M$. The χ^m form a basis of semi-invariant vectors for the S-representation $\mathbf{C}[M]$. We want to remark that we define the action of the torus on his ring of coordinate as follows: $(t \cdot f)(t') = f(t^{-1} \cdot t)$ for each $f \in \mathbf{C}[M]$ and $t, t' \in T$. Thus χ^m is a seminvariant function with weight $-m$. Usually the action is defined as follows: $(t \cdot f)(t') = f(t \cdot t)$; so that χ^m is a seminvariant function with weight m. This is possible because T is an abelian group, but we will need to study torus which are subgroup of not abelian group, so we do not use the second definition. We want to describe the S-toric varieties. These are the normal S-varieties which contain an open orbit isomorphic to S. Every toric varieties is associated to a fan in N, so we have to define fans. First of all we introduce the notion of a convex rational polyhedral cone.

Definition 4.1 *σ is a convex rational polyhedral cone in $N_{\boldsymbol{R}}$ if there are vectors $v_1, ..., v_n$ in N such that σ is the cone generated by $v_1, ..., v_n$, namely $\sigma = \sum_{i=1}^n \boldsymbol{R}^+ v_i$. We will denote σ by $\sigma(v_1, ..., v_n)$. σ is a strongly convex rational polyhedral cone if, moreover, it contains no line.*

In what follows we are going to tacitly assume that all cones contained in $N_{\mathbf{R}}$ are strongly convex rational polyhedral cones.

Definition 4.2 *The cone $\sigma^\vee = \{x \in M_{\boldsymbol{R}} \mid x(y) \geq 0 \ \forall y \in \sigma\}$ in $M_{\boldsymbol{R}}$ is called the dual cone of σ. Let $\sigma^\perp = \{x \in M_{\boldsymbol{R}} \,|\, x(y) = 0 \ \forall y \in \sigma\}$ be the subspace of $M_{\boldsymbol{R}}$ of vectors vanishing on σ.*

Observe that $\sigma^\vee$ is a convex rational polyhedral cone and that $\sigma^\vee + (-\sigma^\vee) = M_{\mathbf{R}}$, but it may be not strongly convex. Indeed $\sigma^\perp$ is the largest vector space contained in $\sigma^\vee$. The dimension of σ is the dimension of smallest subspace of

$N_{\mathbf{R}}$ containing σ, namely $\sigma + (-\sigma)$. A not empty subset τ of σ is a face of σ if there is a $m \in \sigma^\vee$ such that $\tau = \sigma \cap \{m\}^\perp = \{y \in \sigma \,|\, m(y) = 0\}$. This means that there is semi-space $V = \{x \in N_{\mathbf{R}} : m(x) \geq 0\}$ such that σ is contained in V and τ is the intersection of σ and of the border $\{x \in N_{\mathbf{R}} : m(x) = 0\}$ of V. Notice that $\{0\}$ is a face of every cone, so we usually do not mention it. Indeed $\{0\}$ corresponds to any vector m in $\sigma^\vee \backslash \sigma^\perp$ (this is true because we have assumed that σ is strongly convex). If $\sigma = \sigma(v_1, ..., v_n)$ then its faces are the cones $\sigma(v_{i_1}, ..., v_{i_k})$. A face τ of σ is a strongly convex rational polyhedral cone. Moreover, given v and v' in σ, $v + v'$ belongs to τ if and only if both v and v' belong to τ.

Definition 4.3 *A fan Δ in N is a set of (strongly convex rational polyhedral) cones with the following two properties:*

- *1) if $\sigma \in \Delta$ and τ is a face of σ then $\tau \in \Delta$;*
- *2) if $\sigma, \sigma' \in \Delta$ then the intersection $\sigma \cap \sigma'$ is a face both of σ and σ'.*

The union $|\Delta| = \bigcup_{\sigma \in \Delta} \sigma$ is called the support of Δ. Let $\Delta(i)$ be the subset of Δ formed by the cones of dimension i.

Usually we will not mention the cone 0 that belongs to each fan. Observe that a fan is uniquely determined by its maximal elements. Now we will describe the toric variety Z associated to a fan Δ. Z has an open cover $\{U_\sigma\}_{\sigma \in \Delta}$ formed by open sets stabilized by the action of S. The open set U_σ is isomorphic to $Spec\, \mathbf{C}[M \cap \sigma^\vee]$ and the intersection of two of these open sets, say U_σ and $U_{\sigma'}$, is the open set $U_{\sigma \cap \sigma'}$ associated to the intersection of the associated cones. In particular, if $\sigma' \subset \sigma$ then $U_{\sigma'} \subset U_\sigma$. Notice that S corresponds to the cone $\{0\}$. For example $\mathbf{C}^n$ is the toric variety associated to the fan formed by the faces of $\sigma(v_1, ..., v_l)$, where $\{v_1, ..., v_l\}$ is a basis of N. We can identify S with $(\mathbf{C}^*)^n$ and it acts on $\mathbf{C}^n$ by $(t_1, ..., t_l) \cdot (x_1, ..., x_l) = (t_1 x_1, ..., t_l x_1)$ for each $(t_1, ..., t_l) \in (\mathbf{C}^*)^n$ and $(x_1, ..., x_l) \in \mathbf{C}^n$. The stable open sets of $\mathbf{C}^n$ are the sets $\{(x_1, ..., x_l) : x_{i_1} \neq 0, ..., x_{i_r} \neq 0\}$ for each subset of $\{1, ..., l\}$. Let $\{m_1, ..., m_l\}$ be the basis of M dual to $\{v_1, ..., v_l\}$. We have $\mathbf{C}[M] = \mathbf{C}[\chi^{m_1}, \chi^{-m_1}, ..., \chi^{m_l}, \chi^{-m_l}]$ and $\mathcal{O}_{\mathbf{C}^n}(\mathbf{C}^n) = \mathbf{C}[\chi^{m_1}, ..., \chi^{m_l}]$. Observe that $\sigma(v_1, ..., v_l)^\vee = \sigma(m_1, ..., m_l)$. The ring of coordinates of $\{(x_1, ..., x_l) : x_{i_1} \neq 0, ..., x_{i_r} \neq 0\}$ is $\mathbf{C}[\chi^{m_1}, ..., \chi^{m_l}, \chi^{-m_{i_1}}, ..., \chi^{-m_{i_r}}]$, so this is the open set associated to $\sigma(v_1, ..., \widehat{v}_{i_1}, ..., \widehat{v}_{i_r}, ..., v_l)$. The action of S on $\mathbf{C}[M]$ is such that $t \cdot \chi^m = -m(t)\chi^m$ for each $t \in S$ and $m \in M$, so $\mathbf{C}[M \cap \sigma^\vee]$ is a subrepresentation. Thus S acts on U_σ. Moreover the U_σ are the only S-stable open sets of Z. Observe that the dimension of the variety Z is equal to the rank of N.

Now we give some example of geometric properties of a toric variety and of the equivalent conditions on the associated fan: 1) a toric variety Z is affine if and only if its fan consists of all the faces of a single cone; 2) a toric variety Z is complete if and only if the support of the associated fan is the whole of space, namely $|\Delta| = N_{\mathbf{R}}$; 3) Z is smooth if and only if for each $\sigma \in \Delta$ there is a subset

$\{v_1, ..., v_r\}$ of a basis of N such that $\sigma = \sigma(v_1, ..., v_r)$. These facts imply that the only affine smooth toric variety associated to a cone of maximal dimension is the affine space.

Let $S_1 = N_1 \otimes_{\mathbf{Z}} \mathbf{C}^*$ and $S_2 = N_2 \otimes_{\mathbf{Z}} \mathbf{C}^*$ be two tori, then every map $S_1 \to S_2$ corresponds to a map $\varphi : N_1 \to N_2$. We call φ also the extension of φ by linearity to a map $N_1 \otimes_{\mathbf{Z}} \mathbf{R} \to N_2 \otimes_{\mathbf{Z}} \mathbf{R}$. Let Z_i be a S_i-toric variety for each i and let Δ_i be the fan of Z_i for each i. There is at most one map $Z_1 \to Z_2$ extending φ. It exists if and only if for each $\sigma \in \Delta_1$ there is a cone $\sigma' \in \Delta_2$ such that $\varphi(\sigma) \subset \sigma'$. Suppose that there is a such map, then it is proper if and only if for each $\sigma' \in \Delta_2$ we have $\varphi^{-1}(\sigma') = \bigcup_{\sigma \in \Delta_1 : \varphi(\sigma) \subset \sigma'} \sigma$. In particular if $S_1 = S_2$ and φ is the identity then $Z_1 \to Z_2$ is proper if and only if $|\Delta_1| = |\Delta_2|$.

Let Z be a S-toric variety and let Δ be its fan. We want to describe the bijective correspondence between the orbits of Z and the cones in Δ. Before we need to describe the quotients of S. Observe that $S = Hom_{\mathbf{Z}}(M, \mathbf{C}^*)$ and $U_\sigma = Hom_{sg}(M \cap \sigma^\vee, \mathbf{C}^*)$ for each $\sigma \in \Delta$ (here $Hom_{sg}(\ ,\ )$ means morphisms of semigroups). Let σ be a cone in $N_{\mathbf{R}}$, then the torus $S' = Hom_{\mathbf{Z}}(M \cap \sigma^\perp, \mathbf{C}^*)$ is a quotient of S, where the quotient map

$$S = Hom_{\mathbf{Z}}(M, \mathbf{C}^*) \twoheadrightarrow S' = Hom_{sg}(M \cap \sigma^\perp, \mathbf{C}^*)$$

is given by restriction and it is associated to the inclusion $\mathbf{C}[M \cap \sigma^\perp] \hookrightarrow \mathbf{C}[M]$. Let N_σ be the sublattice of N generated by σ (as a group) and let $N(\sigma) = N/N_\sigma$, then $S' = N(\sigma) \otimes_{\mathbf{Z}} \mathbf{C}^*$ and the quotient map $S \twoheadrightarrow S'$ is obtained tensoring the quotient map $N \twoheadrightarrow N(\sigma)$ by $\mathbf{C}^*$.

Proposition 4.1 (See proposition 1.6 in [O] or page 54 in [F]) *For each $\sigma \in \Delta$ we can regard the quotient algebraic torus $o_\sigma := Hom_{\mathbf{Z}}(M \cap \sigma^\perp, \mathbf{C}^*)$ of S as a S-orbit in Z. Every S-orbit is of this form and, in this way, Δ is in one-to-one correspondence with the set of S-orbits in Z. Moreover the following holds:*

1. *$o_{\{0\}} = U_{\{0\}} = S$.*
2. *for each $\sigma \in \Delta$, the dimension of o_σ is equal to the codimension $l - dim(\sigma)$ of σ in $N_{\mathbf{R}}$.*
3. *For $\sigma, \tau \in \Delta$, τ is a face of σ if and only if o_σ is contained in the closure of o_τ.*
4. *For $\sigma \in \Delta$, o_σ is the unique closed S-orbit in U_σ and we have $U_\sigma = \bigcup_{\tau \subset \sigma} o_\tau$ (observe that o_σ may not be closed in Z).*
5. *There is a one-to-one correspondence between Δ and the closed subvarieties of Z stabilized by the action of S: every $\sigma \in \Delta$ correspond to closure Z_σ of o_σ. Moreover $Z_\sigma = \bigcup_{\tau \supset \sigma} o_\tau$.*
6. *Let $n \in N$ and let $\sigma \in \Delta$. Then we have $n \in \sigma$ if and only if the one-parameter subgroup γ_n corresponding to n has the property that $lim_{t \to 0} \gamma_n(t)$*

exists in U_σ. In this case, the limit coincides with the identity element of o_τ regarded as an algebraic torus, where τ is the face of σ which contains n in its relative interior.

For example if $Z = \mathbf{C}^l$ then the orbit $o_{\sigma(v_{i_1},...,v_{i_r})}$ associated to $\sigma(v_{i_1},...,v_{i_r})$ is the set $\{(x_1,...,x_l) : x_i = 0 \text{ if and only if } i \in \{i_1,...,i_r\}\}$. Its closure is the set $\{(x_1,...,x_l) : x_i = 0 \text{ if } i \in \{i_1,...,i_r\}\}$. The ring of coordinates of $o_{\sigma(v_{i_1},...,v_{i_r})}$ is $\mathbf{C}[\chi^{m_i}, \chi^{-m_i}]_{i \notin \{i_1,...,i_r\}}$, while the ring of coordinates of its closure is $\mathbf{C}[\chi^{m_i}]_{i \notin \{i_1,...,i_r\}}$. If σ has dimension equal to $\dim N$, then o_σ is a S stable point which we call x_σ.

The inclusion $o_\sigma = Hom_{\mathbf{Z}}(M \cap \sigma^\perp, \mathbf{C}^*) \hookrightarrow U_\sigma = Hom_{\mathbf{Z}}(M \cap \sigma^\vee, \mathbf{C}^*)$ is given by extension by zero. The extension by zero of a group homomorphism is a semigroup homomorphism because $\sigma^\perp$ is a face of $\sigma^\vee$, so given $u, u' \in \sigma^\vee$, $u + u'$ belongs to $\sigma^\perp$ if and only if both u and u' belong to $\sigma^\perp$.

We want to describe the stable closed subvarieties of Z more closely. For each $\tau \in \Delta$, Z_τ is a toric variety with respect to the torus o_τ. The fan of Z_τ in $N(\tau)$ is $\{\sigma + N_\tau \otimes_{\mathbf{Z}} \mathbf{R}/N_\tau \otimes_{\mathbf{Z}} \mathbf{R} : \sigma \in \Delta \text{ and } \tau \subset \sigma\}$. Observe that Z_τ intersects U_σ if and only if $\sigma \supset \tau$. In this case $Z_\tau \cap U_\sigma$ is isomorphic to $Spec\, \mathbf{C}[M \cap \tau^\perp \cap \sigma^\vee] = Hom_{sg}(M \cap \tau^\perp \cap \sigma^\vee, \mathbf{C})$. We want to describe the closed immersion $Z_\tau \hookrightarrow Z$. Z_τ is covered by the U_σ with $\sigma \supset \tau$. The closed immersion $Z_\tau \cap U_\sigma = Hom_{sg}(M \cap \tau^\perp \cap \sigma^\vee, \mathbf{C}) \hookrightarrow U_\sigma = Hom_{sg}(M \cap \sigma^\vee, \mathbf{C})$ is given by extension by zero. The extension by zero of a semigroup homomorphism is a semigroup homomorphism because $\tau^\perp \cap \sigma^\vee$ is a face of $\sigma^\vee$. The closed immersion $Z_\tau \cap U_\sigma = Hom_{sg}(M \cap \tau^\perp \cap \sigma^\vee, \mathbf{C}) \hookrightarrow U_\sigma = Hom_{sg}(M \cap \sigma^\vee, \mathbf{C})$ corresponds to the projection $\mathbf{C}[M \cap \sigma^\vee] \twoheadrightarrow \mathbf{C}[M \cap \tau^\perp \cap \sigma^\vee]$ that takes m to m if $m \in M \cap \tau^\perp \cap \sigma^\vee$ and it takes m to 0 otherwise. This projection is a ring homomorphism because $\tau^\perp \cap \sigma^\vee$ is a face of $\sigma^\vee$. These maps are compatible, namely if $\tau \subset \sigma \subset \sigma'$, then the following diagram commutes

$$\begin{tikzcd}
Z_\tau \cap U_\sigma \arrow[r, hook] \arrow[d, hook] & Z_\tau \cap U_{\sigma'} \arrow[d, hook] \\
U_\sigma \arrow[r, hook] & U_{\sigma'}.
\end{tikzcd}$$

Indeed we can rewrite this diagram as

$$\begin{tikzcd}
Hom_{sg}(M \cap \tau^\perp \cap \sigma^\vee, \mathbf{C}) \arrow[r, hook] \arrow[d, hook] & Hom_{sg}(M \cap \tau^\perp \cap (\sigma')^\vee, \mathbf{C}) \arrow[d, hook] \\
Hom_{sg}(M \cap \sigma^\vee, \mathbf{C}) \arrow[r, hook] & Hom_{sg}(M \cap (\sigma')^\vee, \mathbf{C})
\end{tikzcd}$$

where the vertical maps are extensions by zero and the horizontal ones are restrictions. Notice that $\sigma \subset \sigma'$ implies $\sigma^\vee \supset (\sigma')^\vee$.

4.2 Line bundles

In the following we will consider only smooth toric varieties such that the maximal cones of the associated fan have all dimension equal to dim S. Moreover we call l the dimension of S. In this case Z is covered by the open sets associated to the l-dimensional cones, namely $Z = \bigcup_{\sigma\in\Delta(l)} U_\sigma$. Moreover U_σ is isomorphic to $\mathbf{C}^n$ for each $\sigma \in \Delta(l)$. Observe that this hypotheses are satisfied by every smooth complete toric variety.

Now we want to study the line bundles on Z, but it is easier to describe the equivariant line bundles. We want to observe that, without the previous hypothesis on the maximal cones of Δ, most of the following facts are false.

Definition 4.4 *A real valued function $h : |\Delta| \to \boldsymbol{R}$ on the support of Δ is called a (Δ, M)-linear function if it is $\boldsymbol{Z}$-valued on $N \cap |\Delta|$ and it is linear on each $\sigma \in \Delta$. Let $SF(\Delta, M)$ be the additive group of the (Δ, M)-linear functions.*

Remark. We can think h as function $h : N_{\mathbf{R}} \to \mathbf{R} \cup \{-\infty\}$ such that $h(x)$ is finite if and only if $x \in |\Delta|$.

Definition 4.5 *Let h be a (Δ, M)-linear function and let σ be a cone in $\Delta(l)$. We set $h|\sigma$ as the unique linear function which coincides with h on σ.*

Notice that $h|\sigma \in M$ for each h and σ.

Definition 4.6 *Let $h : N_{\boldsymbol{R}} \to \boldsymbol{R} \cup \{-\infty\}$. We say that h is M-piecewise linear if there is a fan Δ for which h is (Δ, M)-linear.*

We have a natural map $M \hookrightarrow SF(\Delta, M)$ that takes $m \in M$ to the restriction of m to $|\Delta|$. This map is injective, so we can think M as a subset of $SF(\Delta, M)$. An equivariant line bundle on Z is a line bundle $\pi : L \to V$ with an algebraic action of S on L such that π is equivariant (namely $\pi(tz) = t\pi(z)$ for each $t \in S$ and $z \in L$) and the action of each $t \in S$ on L induces a linear map from $\pi^{-1}(x)$ to $\pi^{-1}(tx)$ for each $x \in V$. Let $Pic_S(V)$ be the set of isomorphism classes of equivariant line bundles on V. Let $Div_S(V)$ be the subgroup of $Div(V)$ generated by the S-stable divisors. By proposition 4.1 we have $Div_S(V) = \bigoplus_{\tau\in\Delta(1)} \mathbf{Z} Z_\tau$. The following theorem relates the previous groups. We will say that a vector $v \in M$ is primitive if there is no $v' \in M$ such that $v = av'$ for a suitable integer $a > 1$. Given a cone $\tau \in \Delta(1)$ there is an unique primitive vector $\varrho(\tau)$ contained in τ. Moreover, given any cone $\sigma \in \Delta$, $\sigma = \sum_{\tau\in\Delta(1),\tau\subset\sigma} \mathbf{R}^+ \varrho(\tau)$.

Theorem 4.1 ***(See proposition 2.1 and proposition 2.4 in [O] or pages 63ff in [F])***

1. *We have an isomorphism $SF(\Delta, M) \xrightarrow{\cong} Pic_S(Z)$ which associates an equivariant line bundle L_h to each Δ-linear function h.*

2. *Suppose that $h \in SF(\Delta, M)$. If $m \in M$ satisfies*

$$m(n) \geq h(n) \text{ for all } n \in |\Delta|,$$

then we have a semi-invariant section $\varphi : Z \to L_h$ of L_h of weight $-m$, namely $\varphi(tx) = m(t)(t\varphi(x))$ for each $x \in Z$.

3. *We have an isomorphism $SF(\Delta, M) \xrightarrow{\cong} Div_S(Z)$ which takes h to the divisor*

$$D_h := \sum_{\tau \in \Delta(1)} -h(\varrho(\tau))Z_\tau.$$

In particular D_m is the principal divisor associated to the rational function χ^{-m} on Z.

4. *For $h \in SF(\Delta, M)$ the sheaf of germs of sections of L_h coincides with the invertible sheaf $\mathcal{O}_Z(Z_h)$ associated to the S-invariant divisor Z_h. This sheaf has an action of S and it can be regarded naturally as a S-stable $\mathcal{O}_Z$-submodule of the direct image $j_*\mathcal{O}_S$ with respect to the embedding $j : S \to Z$.*

5. *We have the short exact sequence:*

$$0 \to M \to Pic_S(Z) \to Pic(Z) \to 0$$

Moreover $Pic(Z)$ is free abelian.

The map $M \to Pic_S(Z)$ is the composition of the injection $M \to SF(\Delta, M)$ and the isomorphism $SF(\Delta, M) \to Pic_S(Z)$. We want to give some ideas of the proof. Let σ and γ be two cones in $\Delta(l)$. Observe that $(h|\sigma)(n) = (h|\gamma)(n) = h(n)$ for each $n \in \sigma \cap \gamma$, so $h|\sigma - h|\gamma$ is contained in $M \cap (\sigma \cap \gamma)^\perp \subset M \cap (\sigma \cap \gamma)^\vee$. Thus $h|\sigma - h|\gamma$ and $h|\gamma - h|\sigma$ are regular functions on $U_{\sigma\cap\gamma}$. Remember that Z is covered by the open sets U_σ associated to the maximal cones $\sigma \in \Delta(l)$. Hence we can define a line bundle $L_h = \bigcup_{\sigma\in\Delta(l)}(U_\sigma \times \mathbf{C})$ over Z by gluing $U_\sigma \times \mathbf{C}$ and $U_\gamma \times \mathbf{C}$ along $U_{\sigma\cap\gamma} \times \mathbf{C}$ by the isomorphism $\varphi_{\gamma,\sigma} : U_\sigma \times \mathbf{C} \supset U_{\sigma\cap\gamma} \times \mathbf{C} \xrightarrow{\cong} U_{\sigma\cap\gamma} \times \mathbf{C} \subset U_\gamma \times \mathbf{C}$ defined by $\varphi_{\gamma,\sigma}(x, c) = (x, \chi^{h|\sigma - h|\gamma}(x)c)$ for $(x, c) \in U_{\sigma\cap\gamma} \times \mathbf{C}$. The projections to the first factors glue themselves together to give a map $L_h \to Z$. S acts on L_h by $t(x, c) = (tx, \chi^{-h|\sigma}(t)c)$ for each $t \in S$ and each $(x, c) \in U_\sigma \times \mathbf{C}$.

If h is linear and equal to m then obviously L_m is the trivial bundle $Z \times \mathbf{C}$, because $m|\sigma = m|\gamma$ for each σ and γ in $\Delta(l)$. In this case S acts on L_m by $t(x, c) = (tx, \chi^{-m}(t)c)$.

Let $m \in M$ be such that $m(n) \geq h(n)$ for all $n \in |\Delta|$, then $m - h|\sigma \in M \cap \sigma^\vee$ for each $\sigma \in \Delta(l)$ and $\chi^{m-h|\sigma}$ is a regular function on U_σ. Hence there is a section $\varphi : Z \to L_h$ whose restriction $\varphi|U_\sigma : U_\sigma \to U_\sigma \times \mathbf{C}$ is defined by $(\varphi|U_\sigma)(x) = (x, \chi^{m-h|\sigma}(x))$. This section is obviously semi-invariant with weight $-m$.

For the third point observe that h is determined by his values on the primitive vectors $\varrho(\tau)$ with $\tau \in \Delta(1)$, because $\sigma = \sum_{\tau\in\Delta(1),\varrho\subset\sigma} \mathbf{R}^+\varrho(\tau)$ for each $\sigma \in \Delta$. Notice that, for each $\sigma \in \Delta(l)$, the restriction of $\mathcal{O}_Z(D_h)$ to the open set U_σ is $\mathcal{O}_\sigma \cdot \chi^{h|\sigma}$ and $-h|\sigma$ is the character with which S acts on the fibre over the S-stable point x_σ associated to σ. It is easily seen that $\mathcal{O}_Z(D_h)$ is the sheaf of germs of sections of L_h and that it is a S-stable $\mathcal{O}_Z$-submodule of $j_*\mathcal{O}_S$. Notice that L_h is uniquely determined by the characters $h|\sigma$. This fact will be true also for the complete symmetric varieties.

We now want to describe the canonical bundle of a smooth toric variety.

Proposition 4.2 (see page 70 in [O]) *Let Z be any smooth toric variety with fan Δ and let k be the (Δ, M) linear function such that $k(\varrho(\tau) = 1$ for each $\tau \in \Delta(1)$. Then D_h is a canonical divisor.*

Now we want to describe the space of the sections of L_h as an S-module.

Proposition 4.3 (See lemma 2.3 in [O] or page 66 in [F]) *For each $h \in SF(\Delta, M)$,*

$$Q_h = \{m \in M_{\boldsymbol{R}} : m(n) \geq h(n) \ \forall \ n \in |\Delta|\}$$

is a (possibly empty) convex polyhedron. Moreover

$$H^0(Z; L_h) = \bigoplus_{m\in Q_h\cap M} \boldsymbol{C}\chi^m,$$

where χ^m is a semi-invariant section of weight $-m$.

Proof. Observe that $H^0(U_\sigma,\ j_*\mathcal{O}_S) = H^0(S, \mathcal{O}_S) = \mathbf{C}[M]$. Moreover $H^0(U_\sigma, L_h) = H^0(U_\sigma, \mathcal{O}_Z(D_h))$ is a subspace of $H^0(U_\sigma, j_*\mathcal{O}_S)$ for each $\sigma \in \Delta(l)$ and $\{\chi^m : m \in (h|\sigma) + M \cap \sigma^\vee\}$ is a basis of semi-invariant sections because $\mathcal{O}_Z(D_h)|U_\sigma = O_\sigma \cdot \chi^{h|\sigma}$. Moreover $H^0(Z, L_h) = \bigcap_{\sigma\in\Delta(l)} H^0(U_\sigma, L_h)$. The proposition follows because $(h|\sigma) + M \cap \sigma^\vee = \{m \in M : m(n) \geq (h|\sigma)(n) = h(n) \ \forall \ n \in \sigma\}$. □

We will see that we can recover L_h from Q_h if L_h is generated by global sections. We can consider also higher cohomology groups. In this case we suppose that Z is complete for simplicity.

Proposition 4.4 (See theorem 2.6 in [O] or lemma on page 75 in [F]) *Let Z be a complete variety. For each $h \in SF(\Delta, M)$ and each positive integer q, S acts on the cohomology group $H^q(Z, L_h)$. For each $m \in M$, the eigenspace $H^q(Z, L_h)_m$ with respect to the character m is $H^q(N_{\boldsymbol{R}}, N_{\boldsymbol{R}}\backslash Z(h,m), \boldsymbol{C})$ where $Z(h,m) = \{n \in N_{\boldsymbol{R}} : m(n) \geq h(n)\}$. Thus we have a direct sum decomposition*

$$H^q(Z, L_h) = \bigoplus_{m\in M} H^q(N_{\boldsymbol{R}}, N_{\boldsymbol{R}}\backslash Z(h,m), \boldsymbol{C})\chi^m.$$

Now we want to describe the line bundles generated by global sections, respectively the ample line bundles. To do this we need the definition of a convex function.

Definition 4.7 *Let h be a M-piecewise linear function. We will say that h is (upper) convex if $h(n) + h(n') \leq h(n + n')$ for all $n, n' \in N_{\boldsymbol{R}}$.*

If h is Δ-linear, then it is convex if and only if $h(n)+h(n') \leq h(n+n')$ for all $n, n' \in |\Delta|$ (this definition is the reason why we have chosen $-\infty$ instead of ∞). The convexity of h means that the graph of h lies under the graph of $h|\sigma$ for each $\sigma \in \Delta(l)$. Sometimes the function identically equal to $-\infty$ is considered a convex function, but we prefer to exclude it because it does not correspond to any line bundle on a toric variety.

Definition 4.8 *Given a convex $h \in SF(\Delta, M)$ we will say that h is strictly convex on Δ if $h|\sigma \neq h|\gamma$ for each $\sigma \in \Delta(l)$ and $\gamma \in \Delta(l)$ distinct.*

This condition means that, for each $\sigma \in \Delta(l)$, the graph of h on the complement of σ lies strictly under the graph of $h|\sigma$. Observe that this condition depends on the fan Δ, while the convexity is a condition that depends only on h. We will use the fact that these definitions can be stated without assuming that h has integral values on $N \cap |\Delta|$.

Proposition 4.5 (See theorem 2.7 in [O] or lemma on page 68 in [F]) *Let $h \in SF(\Delta, M)$. L_h is generated by global sections if and only if h is convex.*

The necessity of the condition is easy to show. Let $\sigma \in \Delta(l)$ then U_σ is an affine space and x_σ is the unique S-stable point in U_σ. We have $O(L_h)|U_\sigma = \mathbf{C}[M \cap \sigma^\vee]\chi^{h|\sigma}$, so $\chi^{h|\sigma}$ is the unique section, up to a not zero scalar, which does not vanish on x_σ. Thus, if L_h is generated by global sections, then $h|\sigma \geq h$ for each $\sigma \in \Delta(l)$. This means that h is convex.

Proposition 4.6 (See corollary 2.4 in [O] or pages 70ff in [F]) *Let Z be a (possibly singular) complete toric variety and let $h \in SF(\Delta, M)$. Then L_h is ample if and only if h is strictly convex on Δ.*

Proposition 4.7 (Demazure) ***(See corollary 2.5 in [O] or [De])*** *Let Z be a smooth complete toric variety and let $h \in SF(\Delta, M)$. Then L_h is ample if and only if it is very ample. In particular L_h is very ample if and only if h is strictly convex on Δ.*

We will extend the last two theorems to the case of toric varieties proper on the affine space.

Proposition 4.8 *Suppose that Z is a (possibly singular) toric variety proper over $\boldsymbol{A}^l$ and let $h \in SF(\Delta, M)$. Then L_h is ample if and only if h is strictly convex on Δ. If Z is smooth, then L_h is ample if and only if it is very ample.*

Let L_h be a very ample line bundle on a smooth toric variety Z. Given $\sigma \in \Delta(l)$, the description following the theorem 4.1 implies that the only seminvariant section which does not vanish on x_σ is the seminvariant section with weight $h|\sigma$. Thus there are not cones $\sigma, \sigma' \in \Delta(l)$ such that $\sigma \neq \sigma'$ and $h|\sigma = h|\sigma'$, otherwise x_σ and $x_{\sigma'}$ would have the same image through any immersion $\varphi : Z \to \mathbf{P}(V)$ such that $L_h = \varphi^*(\mathcal{O}(1))$. This is in particular true if Z is as in proposition 4.8. The fact the L_h is ample if h is strictly convex on Δ will be a consequence of a more precise statement. More precisely we will define a complete toric variety Z^c and an ample line bundle L on Z^c such that Z is an open subvariety of Z^c and L_h is the restriction of L to Z.

We now mention some properties of convex functions and convex sets. Remember that a set Q is convex if, for each $p, p' \in Q$, Q contains the segment with endpoints p and p'. Moreover Q is a polyhedron, or polyhedral convex set, if it is the intersection of a finite number of semi-spaces (in general a convex set is the intersection of an infinite number of semi-spaces). If a polyhedron is compact then it is the convex hull of a finite number of points and we will say that it is a polytope. We will say that a polyhedron is rational if all its vertices belong to M.

Theorem 4.2 (See theorem 13.2 in [R] or theorem A.18 in [O]) *Let $C(M_{\boldsymbol{R}})$ be the set of not-empty convex sets in $M_{\boldsymbol{R}}$ and let $SF(N_{\boldsymbol{R}})$ be the set of functions $h : M_{\boldsymbol{R}} \to \boldsymbol{R} \cup \{-\infty\}$ which are positively homogeneous and upper convex, namely $h(av) = ah(v)$ and $h(v + v') \geq h(v) + h(v')$ for each $a \in \boldsymbol{R}^+$ and $v, v' \in N_{\boldsymbol{R}}$.*

1. *We have mutually inverse maps $C(M_{\boldsymbol{R}}) \to SF(N_{\boldsymbol{R}})$ and $SF(N_{\boldsymbol{R}}) \to C(M_{\boldsymbol{R}})$, which respectively send Q to h_Q and h to Q_h, defined as follows:*

$$h_Q(v) = inf\{m(v);\ m \in Q\} \quad for\ v \in N_{\boldsymbol{R}}$$

$$Q_h = \{m \in M_{\boldsymbol{R}} : m(v) \geq h(v),\ \forall\ v \in N_{\boldsymbol{R}}.\}$$

2. *Under the map above, the sum $Q + Q'$ and a positive multiple aQ correspond respectively to the sum function $h + h'$ and the positive multiple ah.*

3. *Q is compact if and only if h_Q has finite value everywhere.*

This theorem implies that, given a line bundle L_h generated by global sections, we can recover h from the polyhedron Q_h, so we can recover L_h from Q_h. h_K is called the support function for K.

Proposition 4.9 (See theorem A.18 in [O]) *The following conditions are equivalent:*

- *$h \in SF(N_{\boldsymbol{R}})$ is the support function for a convex polyhedral set under the correspondence of the theorem 4.2;*

- *there exists a finite decomposition of $h^{-1}(\boldsymbol{R})$ into a union of convex polyhedral cones, such that the restriction of h to each convex polyhedral cone in the decomposition is a linear function. These cones do not intersect in their relative interiors.*

Moreover for any given polyhedron Q, there exists the coarsest such decomposition Δ, which satisfies the following properties:

- *Define*
$$P^\dagger = \{v \in N_{\boldsymbol{R}} : m(v) = h(v)\ \forall v \in P\}$$
for each nonempty face P of Q. Then the map sending P to $P^\dagger$ gives rise to a bijection $\{$nonempty faces of $Q\} \to \Delta$

- *$dim P + dim P^\dagger = dim\, N_{\boldsymbol{R}}$ for each nonempty face P of Q*

- *if $P_1 \supset P_2$ for nonempty faces P_1 and P_2, then $P_1^\dagger \subset P_2^\dagger$*

- *If $\gamma \in \Delta$ and $\gamma = P^\dagger$, then*
$$P = \{m \in Q : m(v) = h(v)\ \forall\ m \in \gamma\} \in \{\textit{nonempty faces of } Q\}$$

Notice that h is a M-piecewise linear function if and only if Q_h is a rational polyhedron. Indeed the vertices of Q_h are the $h|\sigma$ with σ in $\Delta(l)$. We want to remark that the cones in the previous proposition may be not strongly convex. Indeed, they are all strongly convex if and only if Q_h has dimension equal to the dimension of $N_{\mathbf{R}}$. In this case the set of the previous cones and their faces is a fan Δ in N. If moreover $|\Delta| = N_{\mathbf{R}}$, then h is the piecewise linear function associated to an ample line bundle on the complete toric variety corresponding to Δ.

This proposition implies easily the following corollaries.

Corollary 4.1 *Let Z be a complete toric variety and let L_h be a line bundle on Z generated by global sections. Suppose that Q_h has dimension equal to the rank of N, then there is a complete (possibly singular) toric variety Z' dominated by Z and an ample line bundle L' on Z' such that L_h is the pullback of L'.*

Corollary 4.2 *Let Z be a toric variety proper over $\boldsymbol{A}^l$ and let L_h be a line bundle on Z generated by global sections. Then there is a (possibly singular) toric variety Z' dominated by Z and an ample line bundle L' on Z' such that L_h is the pullback of L'. Moreover Z' is proper over $\boldsymbol{A}^l$.*

Now we want to describe the projective toric varieties. Before we need to define the polar convex set and the gauge function of a convex set containing 0.

Definition 4.9 *Let Q be a convex set in $M_{\boldsymbol{R}}$ containing 0, then the set $Q^\circ := \{n \in N_{\boldsymbol{R}} :\ m(n) \geq -1\ \forall m \in Q\}$ is called the polar convex set of Q. Let $h : M_{\boldsymbol{R}} \to \boldsymbol{R} \cup \{-\infty\}$ be the function such that $h(0) = 0$ and $h(u) = -inf\{r \in \boldsymbol{R}^+ :\ u \in rQ\}$ if $u \neq 0$. h is called the gauge function of Q.*

Proposition 4.10 (see p. 28, 125, 174 on [R]) *Let Q be a convex set in $M_{\boldsymbol{R}}$ containing 0, then Q° is a convex set. Moreover:*

- *Q° contains 0 and $(Q^\circ)^\circ = Q$;*
- *Q is a polyhedral convex set if and only if Q° is a polyhedral convex set;*
- *Q is limited if and only if 0 is contained in the interior of Q°. Dually, 0 is contained in the interior of Q if and only if Q° is limited;*
- *the gauge function h of Q is the support function of Q° and $Q = \{m \in M_{\boldsymbol{R}} : \ h(m) \geq -1\}$.*

One can show the following proposition using the proposition 4.6, the theorem 4.2, the proposition 4.9 and the proposition 4.10.

Proposition 4.11 *Let Δ be the fan of a complete toric variety and let h be a (Δ, M) linear function. The following conditions are equivalent:*

- *h is strictly convex on Δ;*
- *Q_h is a rational polytope in $M_{\boldsymbol{R}}$ with vertices $\{h|\sigma : \ \sigma \in \Delta(l)\}$. Moreover $h|\sigma \neq h|\sigma'$ for each $\sigma, \sigma' \in \Delta(l)$ different, i.e. the number of vertices of Q_h is equal to the cardinality of $\Delta(l)$;*
- *Q_h° is a rational polytope in $N_{\boldsymbol{R}}$ with vertices $\{-\frac{1}{h(\varrho(\tau))}\rho(\tau) : \ \tau \in \Delta(1)\}$.*

If one of this condition is verified then the cones of Δ are generated by the faces of Q_h°.

Corollary 4.3 *Let Δ be the fan of a complete toric variety Z. Then Z is projective if and only if there is rational polytope P in $N_{\boldsymbol{R}}$ containing 0 as an internal point and such that the faces of Δ are generated by the faces of P.*

We have similar properties for the smooth toric varieties Z proper over $\mathbf{A}^l$.

Proposition 4.12 *Let Z a smooth toric variety proper over $\boldsymbol{A}^l$, let Δ be the fan of Z and let L_h be a linearized line bundle on Z. L_h is ample if and only if Q_h° is a rational polytope in $N_{\boldsymbol{R}}$ with vertices $\{-\frac{1}{h(\varrho(\tau))}\rho(\tau) : \ \tau \in \Delta(1)\} \cup \{0\}$. In this case the cones of Δ are generated by the faces of Q_h° not containing 0.*

Recall that there is a basis $\{v_1, ...v_l\}$ of N such that the fan of $\mathbf{A}^l$ is formed by the faces of $\sigma(v_1, ...v_l)$.

Corollary 4.4 *Z is quasiprojective if and only if there is a rational polytope Q in $N_{\boldsymbol{R}}$ with the following properties: 1) Q is contained in $\sigma(v_1, ...v_l)$; 2) there are positive constants $a_1, ..., a_l$ such that $0, a_1v_1, ..., a_lv_l$ are vertices of Q and 3) the cones of Δ are generated by the faces of Q_h° not containing 0.*

Observe that Q may have other vertices besides $0, a_1v_1, ..., a_lv_l$.

4.3 S-toric varieties and étale coverings of S

Let S' be a torus and suppose that it is an étale covering of S, i.e. there is a morphism of algebraic group $\pi : S' \to S$ with finite kernel and such that S is the quotient group of S' by $ker\,\pi$. If Z is a S-toric variety then we have a canonical action of S' on Z. Indeed, for each $t \in S'$ and $z \in Z$ we can set $t \cdot z := \pi(t) \cdot z$. We want to study the S' linearized line bundles on Z. In the following we recollect some results which are easily implied by the results of the previous section about the S linearized line bundles. Recall that we consider only smooth toric varieties such that all the maximal cones of the associated fan have dimension equal to dim S.

Let L be a line bundle on Z. First of all, a S-linearization of L induces canonically a S'-linearization of L. This action is defined as follows: $t \cdot x := \pi(t) \cdot x$ for each $t \in S'$ and $x \in L$. Notice that there are S'-linearized line bundles whose linearization is not induced by a S-linearization. We now define some examples. Let M' be the character group of S', we have an injective map $\pi^* : M \hookrightarrow M'$. For each $m \in M'$ we can define a S'-linearization of the trivial bundle $Z \times \mathbf{C}$ as follows: $t \cdot (z, c) = (t \cdot z, -m(t)c)$ for each $t \in S'$ and $(z, c) \in Z \times \mathbf{C}$. We call L_m the S'-linearized line bundle given by the trivial bundle with the previous S' linearization. Observe that if m does not belong to M then the S' linearization of L_m is not induced by a S linearization.

The following proposition is implied by the proposition 4.1

Proposition 4.13 *We have the following commutative diagram with exact rows and injective columns*

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] & Pic_{S'}(Z) \arrow[r] & Pic(Z) \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[u, hook, "\pi^*"] & Pic_S(Z) \arrow[r] \arrow[u, hook] & Pic(Z) \arrow[r] \arrow[u, "id"'] & 0.
\end{tikzcd}$$

In particular $Pic_{S'}(Z) = Pic_S(Z) + M'$.

We want to remark that one can prove that $ker(Pic_{S'}(Z) \longrightarrow Pic(Z))$ is isomorphic to M' using the fact that the character group of S' is discrete. We want to define a group similar to $SF(\Delta, M)$. Notice that M has finite index in M', so we can think M' as a lattice in $M_{\mathbf{R}}$ containing M. Moreover given any lattice with such properties, say $\tilde{M}$, there is a étale covering of S, namely Spec$[\tilde{M}]$, whose character group is the given lattice.

Definition 4.10 *A real valued function $h : |\Delta| \to \boldsymbol{R}$ on the support of Δ is called a (Δ, M')-linear function if h is linear on each $\sigma \in \Delta$. Let $h|\sigma$ be the unique linear function which coincide with h on σ. We request moreover that $h|\sigma$ belongs to M' and that $h|\sigma_1 - h|\sigma_2$ belongs to M for each σ, σ_1 and σ_2 in $\Delta(l)$. Let $SF(\Delta, M')$ be the additive group of the (M', Δ)-linear functions.*

Sometimes we say that an element h of $SF(\Delta, M')$ is a Δ linear function. We can again think a Δ-linear function as a function $h : N_{\mathbf{R}} \to \mathbf{R} \cup \{-\infty\}$. As before we say that a (Δ, M') linear function h is convex if $h(v+v') \geq h(v) + h(v')$ for each $v, v' \in |\Delta|$. We say that h is strictly convex on Δ if moreover $h|\sigma \neq h|\sigma'$ for each $\sigma, \sigma' \in \Delta(l)$. Observe that, given any $h \in SF(\Delta, M')$, there is a positive integer n such that nh is (Δ, M)-linear function.

Definition 4.11 *Let $h : N_{\boldsymbol{R}} \to \boldsymbol{R} \cup \{-\infty\}$. We say that h is piecewise linear if there is a fan Δ and a lattice M' for which h is a (Δ, M')-linear function.*

We can associate a S'-linearized line bundle L_h on Z to each $h \in SF(\Delta, M')$ in a similar way to the S-linearized line bundles associated to an elements of $SF(\Delta, M)$ (see theorem 4.1). The line bundle L_h is associated to the Cartier divisor $\{U_\sigma, \chi^{-h|\sigma}\}_{\sigma \in \Delta(l)}$ and the S'-linearization is defined as follows: $t \cdot (x, c) = (t \cdot x, \chi^{-h|\sigma}(t)c)$ for each $t \in S'$ and $(x, c) \in U_\sigma \times \mathbf{C}$.

Notation 5 *Suppose that we have fixed a lattice M' and a fan Δ associated to a S-toric variety Z. Let σ be an arbitrarily fixed cone in $\Delta(l)$. For each (Δ, M') linear function h we denote with v_h the linear function $h|\sigma$.*

Given two (Δ, M')-piecewise linear function, say h and k, we have $v_{h+k} = v_h + v_k$. The definitions immediately imply that, given any (Δ, M') linear function h, the function $h' = h - v_h$ is a (Δ, M) linear function. Moreover h is strictly convex on Δ (respectively convex) if and only h is. Given a (Δ, M') linear function h, the S'-linearized line bundle L_h on Z is the product of L_{h-v_h} and of L_{v_h}. Observe that the S'-linearization of L_{h-v_h} is induced by a S-linearization and that L_{v_h} is trivial as a line bundle. The following proposition is immediately implied by the theorem 4.1 and by the propositions 4.5, 4.7, 4.8 and 4.13.

Proposition 4.14 *The map $SF(\Delta, M') \to Pic_{S'}(Z)$ the takes h in L_h is an isomorphism. Moreover:*

- *L_h is generated by global sections if and only if h is convex;*
- *Suppose that $|\Delta|$ is $N_{\boldsymbol{R}}$ or $\sigma(v_1, ..., v_l)$. Then L_h is ample if and only if h is strictly convex on Δ. Moreover L_h is very ample if and only if it is ample.*

Given any $h \in SF(\Delta, M')$, $H^0(Z, L_h)$ is a S' representation. Moreover this representation has a basis of seminvariant sections because S' is a torus. The set of the weights of these sections is obviously contained in M'. If we change the linearization of L_h, i.e. if we multiply L_h by a linearized line bundle L_m, then this set of weights is translated with respect to the vector m. Because of the proposition 4.3 there is an one-to-one correspondence between a basis of seminvariant sections of $H^0(Z, L_h)$ and the set of the rational points of Q_{h-v_h}. Hence, we have the following proposition:

Proposition 4.15 *For each $h \in SF(\Delta, M')$,*

$$H^0(Z; L_h) = \bigoplus_{m \in Q_h \cap (M+v_h)} \boldsymbol{C}\chi^m,$$

where χ^m is a semi-invariant section of weight $-m$.

Notice that any piecewise linear function h corresponds to a translate of a suitable rational polyhedron under the correspondence of the theorem 4.2. Indeed $Q_h - v_h$ is the rational polyhedron Q_{h-v_h}.

5 Complete symmetric varieties

A G/H embedding is a complete irreducible G-variety with a G-equivariant open embedding $\varphi : G/H \to Y$. We say that Y is a smooth G/H-embedding if Y is a smooth variety.

Definition 5.1 *Let Y be a G/H-embedding. We will say that Y is a complete symmetric variety if there is a commutative diagram*

$$\begin{tikzcd} G/H \arrow[r, hook] \arrow[d, "id"] & Y \arrow[d, "\pi"] \\ G/H \arrow[r, hook] & X, \end{tikzcd}$$

where $\pi : Y \to X$ is a G-equivariant proper map.

First of all, we want to describe the relation between the complete symmetric varieties and the toric varieties proper over $\mathbf{A}^l$. Let P be the S-principal fibre bundle on X associated to the vector bundle $\bigoplus_{i=1}^l O(X_{-\alpha_i^s})$. Remember that the $X_{-\alpha_i^s}$ are the stable divisors of X and that for each i there is a G-invariant section $s_i \in H^0(X, O(X_{-\alpha_i^s}))$ with divisor $X_{-\alpha_i^s}$. Moreover s_i is unique up to a not zero scalar. The section $\bigoplus_{i=1}^l s_i$ of $\bigoplus_{i=1}^l O(X_{-\alpha_i^s})$ defines an embedding $X \to P \times_S \mathbf{A}^l$ because $P \times_S \mathbf{A}^l$ is isomorphic to $\bigoplus_{i=1}^l O(X_{-\alpha_i^s})$. If $Z \to \mathbf{A}^l$ is a toric variety over $\mathbf{A}^l$, we define $Y = X_Z$ as the fibre product of

$$\begin{tikzcd} & P \times_S Z \arrow[d] \\ X \arrow[r, hook] & P \times_S \mathbf{A}^l. \end{tikzcd}$$

Theorem 5.1 (Proposition 5.1, theorem 5.2 and theorem 5.3 in [CSV II]) *Let Z be any toric variety over $\boldsymbol{A}^l$, and let X_Z be as before.*

1. *G acts on Y and the projection $\pi : Y \to X$ is equivariant.*

2. *If $Z_1 \to Z_2$ is an S-equivariant map then the induced map $X_{Z_1} \to X_{Z_2}$ is G-equivariant.*

3. *The G orbits of Y are in one-to-one correspondence with the S-orbits of Z. Moreover the codimension of an orbit O in Y is equal to the codimension of corresponding orbit o in Z, so it is equal to the dimension of the cone associated to o.*

4. *$\pi^{-1}(\boldsymbol{A}^l)$ is the closure in $\pi^{-1}(U^- \times \boldsymbol{A}^l)$ of the open S-orbit in $\pi^{-1}(\boldsymbol{A}^l)$.*

5. *$\pi^{-1}(U^- \times \boldsymbol{A}^l) \cong U^- \times \pi^{-1}(\boldsymbol{A}^l)$ in a $U^- \times T$ equivariant way.*

6. *The map $Y \to \pi^{-1}(\boldsymbol{A}^l)$ is an equivalence between the category of complete symmetric varieties and the category of toric varieties proper over $\boldsymbol{A}^l$. Moreover Y is smooth if and only if $\pi^{-1}(\boldsymbol{A}^l)$ is smooth.*

7. *The closure of S in X is the toric variety Z_0^c associated to the fan formed by the Weyl chambers and their faces.*

8. *There is an one to one correspondence between complete symmetric varieties and complete toric S-varieties over Z_0^c whose fan is W^1 invariant.*

In this thesis, unless explicitly stated, we shall always assume that the complete symmetric variety Y is smooth. In this case it follows that: 1) any orbit closure in Y is also smooth; 2) the complete toric variety corresponding to Y is smooth. We now introduce some notations that we will often use.

Notation 6 *Let X be the wonderful complete variety and let Y be the complete symmetric variety over X associated to a toric variety Z over $\boldsymbol{A}^l$. We will denote by Z^c the closure of Z in Y. Observe that Z^c is the closure of S in Y. We will call Δ the fan of Z and Δ^c the fan of Z^c. We shall denote the fan of $Z_0 := \boldsymbol{A}^l$ by Δ_0 and the fan of Z_0^c by Δ_0^c. Remember that o_γ is the S-orbit of Z associated to $\gamma \in \Delta$. We will call O_τ the G-orbit of Y corresponding to o_τ. We shall denote by Z_γ the stable subvariety of Z associated to $\gamma \in \Delta$, by Z_γ^c the stable subvariety of Z^c associated to $\gamma \in \Delta^c$ and by Y_γ the stable subvariety of Y associated to $\gamma \in \Delta$.*

Observe that, given $\gamma \in \Delta$, Z_γ may be properly included in Z_γ^c.

Definition 5.2 *We define $\{f_1, ..., f_l\}$ as the basis of M such that $f_i = -\alpha_i^s$ for each i. Moreover we define $\{e_1, ..., e_l\}$ as the basis of N dual to the basis $\{f_1, ..., f_l\}$.*

Observe that e_i is a negative multiple of $\widetilde{w}_i$, so it is a negative multiple of the i-th fundamental weight of the restricted root system.

6 Line bundles on a complete symmetric variety

Now we want to describe the Picard group of Y following [Bi]. Remember that the closed orbits O_σ of Y are in one-to-one correspondence with the maximal cones of Δ. Moreover they are all isomorphic to the unique closed orbit of X through the restriction of the projection, so we can identify $Pic(O_\sigma)$ with $Pic(G/P)$ for each $\sigma \in \Delta(l)$. Remember that we can identify $Pic(X)$ with a sublattice Λ_X of the lattice Λ of integral weights. One can easily show that $Cl(Z)$ is freely generated by the divisors Z_τ associated to the cones $\tau \in \Delta(1)\backslash\Delta_0(1)$. Remember that $Cl(Z)$ is the divisor class group of Z, i.e. the quotient of the divisor group of Z by the group of principal divisors. Notice that $Cl(Z)$ is isomorphic to $Pic(Z)$. The following theorem gives a complete description of $Pic(Y)$.

Theorem 6.1 (Theorem 2.4 in [Bi]) *Let $Y = X_Z$ be a complete symmetric variety. Then*

1. *The maps* $Z \overset{i}{\hookrightarrow} Y \xrightarrow{\pi} X$ *induce the split exact sequence*

$$0 \longrightarrow Pic(X) \xrightarrow{\pi^*} Pic(Y) \xrightarrow{i^*} Pic(Z) \longrightarrow 0,$$

   so $Pic(Y)$ is (not canonically) isomorphic to $Pic(X) \oplus Pic(Z)$.

2. *A section $Cl(Z) \to Cl(Y)$ of the split short exact sequence*

$$0 \longrightarrow Cl(X) \longrightarrow Cl(Y) \longrightarrow Cl(Z) \longrightarrow 0$$

   is given by sending the free generators $[Z_\tau]$, with $\tau \in \Delta(1)\backslash\Delta_0(1)$, to $[Y_\tau]$. Thus

$$Cl(Y) = \pi^* Cl(X) \oplus \bigoplus_{\tau \in \Delta(1)\backslash\Delta_0(1)} \boldsymbol{Z}[Y_\tau].$$

3. *The morphism given by the restriction to the closed orbits*

$$c_1^G : Pic(Y) \to \prod_{\sigma \in \Delta(l)} Pic(O_\sigma)$$

   is injective and its image can be identified with the lattice

$$\Lambda_Y = \{h = (h|\sigma) \in \prod_{\sigma \in \Delta(l)} \Lambda_X \subset \prod_{\sigma \in \Delta(l)} \Lambda : \; h|\sigma - h|\sigma' \in M \cap (\sigma \cap \sigma')^\perp$$
$$\forall\ \sigma,\ \sigma' \in \Delta(l).\}$$

We will indicate with L_h the line bundle whose image is h. Using the proposition 3.1, we get that $Pic(Y)$ is isomorphic to the group of equivariant line bundles $Pic_G(Y)$. Given a line bundle L_h, $-h_\sigma$ is the character of the action of T on the fibre over the T-stable point $O_\sigma \cap Z$.

Definition 6.1 *Let h be in Λ_Y. We will say that h is almost spherical if $h|\sigma \in \Omega$ for each $\sigma \in \Delta(l)$. Moreover we will say that h is spherical if $h|\sigma$ is a spherical weight for each $\sigma \in \Delta(l)$.*

Remember that $h|\sigma$ is spherical if and only if it is dominant and it belongs to Ω.

We define also h^c as the set $(h|\sigma)$ where σ varies in $\Delta^c(l)$ and $-h|\sigma$ is the character of the action of T on the fibre over the T-stable point $x_\sigma \in Z^c$. We say that h^c is (almost) spherical if h is. Notice that h^c is almost spherical if and only if $h|\sigma \in \Omega$ for each $\sigma \in \Delta^c(l)$. The proof of the following proposition is trivial.

Proposition 6.1 *If h is almost spherical, then we can think h as a (Δ, Λ_X) linear function and h^c as a (Δ^c, Λ_X) linear function. Moreover h^c is W^1 invariant, thus, if $w \in W^1$ and $v \in |\Delta|$, then $h^c(w \cdot v) = h(v)$.*

Now we want to do some remarks on the h that are not almost spherical. Let $l+s$ be the rank of $Pic(X)$, we can order the simple roots of ϕ so that $\alpha_1, ..., \alpha_s$ are exceptional roots and $Pic(X)$ is generated by the spherical weights and by the fundamental weights $\omega_{\alpha_1}, ..., \omega_{\alpha_s}$ corresponding respectively to $\alpha_1, ..., \alpha_s$. Thus $Pic(Y) \cong Pic(Z) \oplus \Omega \oplus \bigoplus_{i=1}^{s} \mathbf{Z}\omega_{\alpha_i}$. Therefore, given any dominant weight μ in Λ_X, there are integers a_i and a spherical weight μ' such that $\mu = \mu' + \sum a_i \omega_{\alpha_i}$. Observe that $\omega_{\alpha_i} - \theta(\omega_{\alpha_i}) = \omega_{\alpha_i} + \omega_{\overline{\theta}(\alpha_i)}$ is a spherical weight. Thus we can suppose that the a_i are positive up to exchange some α_i with $\overline{\theta}(\alpha_i)$. *We will say that μ is regular if μ' is regular or, equivalently, if the restriction of the line bundle L_μ to the closed orbit of X is ample.* Given any $h \in \Lambda_Y$ there are integers a_i and an almost spherical Δ-linear function h' such that $h = h' + \sum a_i \omega_{\alpha_i}$ (given two maximal cones σ and σ', $h|\sigma - h|\sigma'$ is a integral combination of restricted simple roots). If $h|\sigma$ is dominant (respectively regular) for each $\sigma \in \Delta(l)$ then we can assume that the $h'|\sigma$ are dominant (respectively regular). Moreover we can assume that the a_i are positive up to exchange some α_i with $\overline{\theta}(\alpha_i)$. We will say that h is convex, respectively strictly convex on Δ, if h' is. Moreover, given any $h \in \Lambda_Y$ and $\sigma \in \Delta(l)$, $h - h|\sigma$ is a (Δ, Λ_X) linear function and $h^c - h|\sigma$ is a (Δ^c, Λ_X) linear function. *We will say by abuse of notation that h is a (Δ, Λ_X)-linear function and h^c is a (Δ^c, Λ_X)-linear function.*

Let L be any line bundle on Y. We want to describe the space of sections of L on Y. We will describe also the space of sections of the restriction of L to Z, respectively to Z^c. Remember that $H^0(Y, L)$ is a multiplicity free representation of G, because Y is a spherical variety (see proposition 3.2). In the same way $H^0(Z, L|Z)$ and $H^0(Z^c, L|Z^c)$ are multiplicity-free T-representations because Z and Z^c have a dense T-orbit. We need a proposition that generalizes proposition 3.4. Remember that Y_τ is the divisor associated to $\tau \in \Delta(1)$.

Lemma 6.1 (Lemma 2.7 in [Bi]) *Let d^τ be the Δ-linear function $c_1^G(Y_\tau)$ associated to Y_τ. Then $d^\tau(\varrho(\tau')) = -\delta_{\tau,\tau'}$, in particular d^τ is $\boldsymbol{Z}$-valued on $|\Delta| \cap N$. Moreover there is a unique, up to scalar, G-invariant section s_τ in $H(Y, L_{d^\tau})$ whose divisor is Y_τ.*

Now we want to define sets in bijective correspondence with bases respectively of $H^0(Y, L)$, $H^0(Z, L|Z)$ and $H^0(Z^c, L|Z^c)$.

Definition 6.2 *Given $h \in \Lambda_Y$ let*

$$\Pi(Z, h) = \{\mu \in \bigcap_{\sigma \in \Delta(l)} (h|\sigma + (M \cap \check{\sigma}))\},$$

$$\Pi(Z^c, h) = \{\mu \in \bigcap_{\sigma \in \Delta^c(l)} (h|\sigma + (M \cap \check{\sigma}))$$

and

$$\Pi(Y, h) = \Pi(Z, h) \cap \Lambda^+.$$

Before we describe the sections of L_h, we want to rewrite the conditions for a weight to belongs to $\Pi(Z, h)$, respectively to $\Pi(Z^c, h)$.

Lemma 6.2 *Let λ be a weight in Λ_X and let h be in Λ_Y. Then the following conditions are equivalent:*

1. $\lambda \in \Pi(Z, h)$
2. $\lambda \geq h$ *as functions on* $|\Delta|$
3. $h = \lambda + \sum_{\tau \in \Delta(1)} a_\tau d^\tau$ *where a_τ is a positive integer for each $\tau \in \Delta(1)$.*

Lemma 6.3 *Let λ be a weight in Λ_X and let h be in Γ_Y. Then the following conditions are equivalent:*

1. $\lambda \in \Pi(Z^c, h)$
2. $\lambda \geq h^c$ *as functions on* $N_{\boldsymbol{R}}$

Notice that $\Pi(Y, h)$ is contained in Λ_X.

Theorem 6.2 (Theorem 3.4 in [Bi]) *Let L_h be a line bundle on Y. Then*

$$H^0(Y, L_h) = \bigoplus_{\mu \in \Pi(Y,h)} V_\mu^*.$$

In particular $H^0(Y, L_h) \neq 0$ if and only if $\Pi(Y, h)$ is not empty.

We want to give an idea of one build of $H^0(Y, L_h)$. Let $\lambda \in \Pi(Y, h)$. Remember that $H^0(X, L_\lambda)$ contains V_λ^*. Moreover $H^0(X, L_\lambda) \subset H^0(Y, \pi^* L_\lambda)$, so $H^0(Y, \pi^* L_\lambda)$ contains a lowest weight vector $v_{-\lambda}$ of weight $-\lambda$. There are positive constants a_τ such that $h - \lambda = \sum_{\tau \in \Delta(1)} a_\tau d^\tau$ because of the lemma 6.2. Thus $v_{-\lambda} \cdot \prod s_\tau^{a_\tau}$ is a not-zero section of $H^0(Y, L_h)$ with weight $-\lambda$ because the sections s_τ are G-invariant. Moreover $v_{-\lambda} \cdot \prod s_\tau^{a_\tau}$ is invariant by the unipotent part of the opposite B^- of the fixed Borel group of G. Thus $H^0(Y, L_h) \supseteq \bigoplus_{\mu \in \Pi(Y,h)} V_\mu^*$. Because of the previous theorem we give the following definition:

Definition 6.3 *Given h in Λ_Y and λ in $\Pi(Y, h)$, we write $h = \lambda + \sum_{\tau \in \Delta(1)} a_\tau d^\tau$ for suitable $a_\tau \in \boldsymbol{Z}^+$. We define $s^{h-\lambda}$ as the section $\prod s_\tau^{a_\tau}$ of $H^0(Y, L_{h-\lambda})$.*

More generally, let L_h and $L_{h'}$ be two line bundles on Y such that $h' \geq h$, namely $h - h' = \sum_{\tau \in \Delta(1)} a_\tau d^\tau$ for positive a_τ. Then the product by the G-invariant section $\prod s_\tau^{a_\tau}$ of $L_{h-h'}$ defines an injective G-equivariant linear map from $H^0(Y, L_{h'})$ to $H^0(Y, L_h)$.

The following proposition is immediately implied by the proposition 4.15. Indeed the T linearization of L_h induces a T^1 linearization of L_h.

Proposition 6.2 *Let L_h be a line bundle on Y. Then*

- $$H^0(Z, L_h|Z) = \bigoplus_{\mu \in \Pi(Z,h)} \boldsymbol{C}\chi^\mu,$$

  where χ^μ is a T-seminvariant section of weight $-\mu$. In particular $H^0(Z, L_h|Z) \neq 0$ if and only if $\Pi(Z, h)$ is not empty.

- $$H^0(Z^c, L_h|Z^c) = \bigoplus_{\mu \in \Pi(Z^c,h)} \boldsymbol{C}\chi^\mu.$$

  In particular $H^0(Z^c, L_h|Z^c) \neq 0$ if and only if $\Pi(Z^c, h)$ is not empty.

Remark. Let $\pi : Y \to Y'$ be an G-equivariant morphism between two complete symmetric varieties and let L_h be a line bundle on Y'. Then the pullback $\pi^*(L_h)$ is the line bundle on Y associated to h because of the last point of the theorem 6.1. Moreover $H^0(Y, \pi^*(L_h)) = H^0(Y', L_h)$ because of the lemma 6.2.

Now we want to explain some relations between the previous sets.

Corollary 6.1 (Corollary 4.1 in [Bi]) *Given $h \in \Gamma_Y$, we have the equality*

$$\Pi(Y, h) = \Pi(Z^c, h) \cap \Lambda^+.$$

In particular the sections of L_h over Z^c (or over Z) determine the sections of L_h over Y.

Proof. Remember that $\Pi(Y,h) = \Pi(Z,h) \cap \Lambda^+$, so it is sufficient to prove that $\Pi(Y,h) \subset \Pi(Z^c,h)$. Let $\mu \in \Pi(Y,h)$, then there is a lowest weight section $s_{-\mu} \in H^0(Y, L_h)$ of weight $-\mu$. It is U^- invariant, so it cannot vanish on Z^c, for otherwise it would vanish on the dense open set $U^- \times Z^c$, thus on Y. $\square$

If h is almost spherical, we can say more.

Proposition 6.3 (Proposition 4.2 and theorem 4.2 in [Bi]) *If $h \in \Lambda_Y$ is almost spherical, then*

$$\Pi(Z^c, h) = \bigcup_{w \in W^1} w \cdot \Pi(Y, h).$$

Moreover the restriction map $H^0(Y, L_h) \to H^0(Z^c, L_h|Z^c)$ is surjective.

We prove only the second part of the proposition. Let $w \cdot \mu \in \Pi(Z^c, h)$ with $\mu \in \Pi(Y,h)$. Let $s \in V_\mu^* \subset H^0(Y, L_h)$ be a section of weight $-w \cdot \mu$, then we can choose another basis of the root system such that s is a lowest weight vector. Observe that we have already proved that a such section cannot vanish on Z^c.

Remember that there is a polyhedron associated to every linearized line bundle on a toric variety. We want to do the same with the line bundles on a complete symmetric variety.

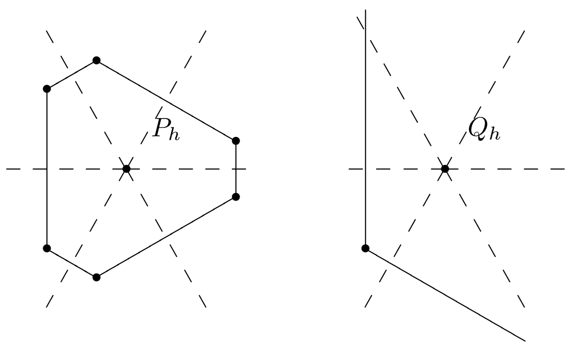
Definition 6.4 *Let Y be a complete symmetric variety and let L_h be a line bundle on Y such that h is almost spherical i.e. $h|\sigma \in \Omega$ for each $\sigma \in \Delta(l)$. We define the polytope associated to h (and L_h) as the polytope*

$$P_h = \{m \in M_{\boldsymbol{R}} : m(v) \geq h^c(v) \ \forall v \in |\Delta^c|\}.$$

Moreover we define the polyhedron associated to h as the polyhedral convex set

$$Q_h = \{m \in M_{\boldsymbol{R}} : m(v) \geq h(v) \ \forall v \in |\Delta|\}.$$

Observe that $P_h = \{m \in M_{\mathbf{R}} : m(\varrho(\tau)) \geq h^c(\varrho(\tau)) \ \forall \ \tau \in \Delta^c(1)\}$ and $Q_h = \{m \in M_{\mathbf{R}} : m(\varrho(\tau)) \geq h(\varrho(\tau)) \ \forall \tau \in \Delta(1)\}$. P_h is the polytope associated to h^c in the correspondence of the theorem 4.2. Under such correspondence, Q_h is associated to h (here we think h as a function $h : N_{\mathbf{R}} \to \mathbf{R} \cup \{-\infty\}$ such that it has finite value exactly on $|\Delta|$). Notice that $\Pi(Z^c, h) = P_h \cap (M + v_h)$, $\Pi(Z, h) = Q_h \cap (M + v_h)$ and $\Pi(Y, h) = P_h \cap C^+ \cap (M + v_h)$. (In the figure we draw an example for the wonderful variety corresponding to an involution such that the restricted root system has type A_2).



We now study an example. Take the wonderful symmetric variety X and let L_λ be a line bundle on X. In this case $\mu \in \Pi(X,\lambda)$ if and only if λ is greater than μ in the dominant order, i.e. $\lambda - \mu$ is a linear combination of the roots with positive integral coefficients (theorem 8.3 in [CSV I]). Remember that $\mu \in \Pi(X,\lambda)$ if and only if $\mu \geq \lambda$ as functions on $\sigma(e_1,...,e_l)$ (lemma 6.2). Observe that in this case Q_h is the polyhedron $\{m \in M_{\mathbf{R}} : m(e_i) \geq \lambda(e_i)\}$. If we write $m = \sum m_i \alpha_i^s$ and $\lambda = \sum \lambda_i \alpha_i^s$ then the inequalities of Q_h are $m_i \leq \lambda_i$. The proof of the following lemma is trivial.

Lemma 6.4 *Let λ and μ be two weights in Ω. Then the following conditions are equivalent.*

1. *$\mu \geq \lambda$ as functions $\sigma(e_1,...,e_l) \to \boldsymbol{R}$;*
2. *$\lambda \succcurlyeq \mu$ in the dominant order, i.e. $\lambda - \mu = \sum a_i \alpha_i^s$ where a_i is a positive integer for each i.*

Thus we have shown that for the wonderful variety X the theorem 6.2 is a restatement of the theorem 8.3 in [CSV I]. *Given two weights λ and μ (in Ω) we will say that $\lambda \geq \mu$ if $\lambda - \mu$ has positive values everywhere on $\sigma(e_1,...,e_l)$.*

Part II
Multiplication of sections

7 Ample line bundles and line bundles generated by global sections

Brion [Br] has found a characterization of the ample line bundles (respectively the line bundles generated by global sections) on a spherical variety. Now we want to find different conditions for a line bundle on a complete symmetric variety to be generated by global sections, respectively to be ample.

Proposition 7.1 *Let L_h be a line bundle on Y. Then*

1. L_h *is generated by global sections if and only if* h *is convex and* $h|\sigma$ *is dominant for each* $\sigma \in \Delta(l)$.

2. L_h *is very ample if and only if* h *is strictly convex on* Δ *and* $h|\sigma$ *is a regular weight for each* $\sigma \in \Delta(l)$.

3. L_h *is ample if and only if it is very ample.*

Proof. The necessity of the conditions in the first two points is easy to show. Indeed, if L_h is generated by global sections, then also the restriction of L_h to Z is generated by global sections, so h is convex. Moreover, $L_h|O_\sigma$ is generated by global sections for each closed orbit O_σ, so $h|\sigma$ is dominant for each $\sigma \in \Delta(l)$. One can show the necessity of the condition in the second point in the same way.

We want prove the sufficiency of the condition in the first point, but before we will prove a lemma.

Lemma 7.1 *If* h *is convex and* $h|\sigma$ *is dominant then the restriction map to the closed orbit* O_σ

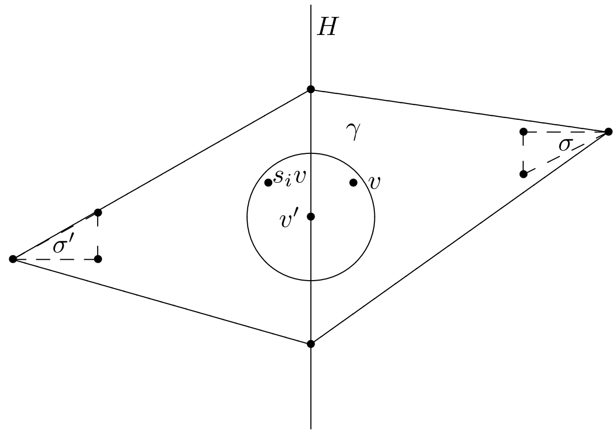
$$H^0(Y, L_h) \to H^0(O_\sigma, L_h|O_\sigma)$$

is surjective.

Proof. Since $h|\sigma$ is dominant, $L_h|O_\sigma$ is generated by global sections and $H^0(O_\sigma, L_h|O_\sigma)$ is the irreducible G-representation $V^*_{h|\sigma}$. Moreover $h|\sigma$ belongs to $\Pi(Y, h)$ because of the convexity of h. Thus there is a lowest weight vector $\varphi \in H^0(Y, L_h)$ of weight $-h|\sigma$. Hence, because of the reductivity of G, it is sufficient to prove that the restriction of φ to O_σ is not zero. Observe that $\varphi = \varphi' \cdot \prod s_{\tau_i}^{a_i}$ where φ' is a lowest weight vector of $V^*_{\lambda_\sigma} \subset H^0(Y, L_{h|\sigma})$ and $a_i > 0$ only if τ_i is not contained in σ. Remember that the line bundle on Y associated to $h|\sigma$ is the pull-back of the line bundle on X associated to $h|\sigma$ and that $H^0(Y, L_{h|\sigma}) = H^0(X, L_{h|\sigma})$. Moreover $\varphi'|O_\sigma \neq 0$ because of the observations following the proposition 3.3. Hence $\varphi|O_\sigma \neq 0$ because s_{τ_i} vanishes exactly on Y_{τ_i} for each τ_i. $\square$

Now we can prove the sufficiency of the condition in the first point. Observe that the locus of base points is closed and stable for the action of G. So, either it is empty or it contains a closed orbit O_σ. Since $h|\sigma$ is dominant, $L_h|O_\sigma$ is generated by global sections. Hence, given any $y \in O_\sigma$ there is a section $\widetilde{s} \in H^0(O_\sigma, L_h|O_\sigma)$ such that $\widetilde{s}(y) \neq 0$. Thus, because of the previous lemma, there is a section $s \in H^0(Y, L_h)$ such that its restriction to O_σ is $\widetilde{s}$, so $s(y) \neq 0$. This is a contradiction.

Now we want to show the sufficiency of the condition in the second point if $h|\sigma$ is a spherical weight for each $\sigma \in \Delta(l)$. First of all we want to show that $L_h|Z$ is very ample. We will prove a stronger result, namely that $L_h|Z^c$ is very ample. This fact is equivalent to the strictly convexity of h^c on Δ^c. h^c is convex because of the first point of the proposition. Suppose by contradiction that there are two distinct maximal cones σ and σ' such that $h|\sigma = h|\sigma'$. We can suppose that σ belongs to Δ because of the symmetry of Δ^c with respect to the Weyl group. Thus σ' cannot belong to Δ because h is strictly convex on Δ. Let γ be the locus of the points v such that $h(v) = (h|\sigma)(v)$. We know that γ is a convex cone by the corollary 4.9. By hypothesis there is an hyperplane H such that it is secant to γ and its intersection with $\sigma(e_1, ..., e_l)$ is a face of $\sigma(e_1, ..., e_l)$. There is an unique i such that e_i does not belong to this hyperplane. For each j let s_j be the orthogonal reflection with respect to the hyperplane generated by $e_1, ..., \hat{e}_j, ..., e_l$. There is a vector $v \in \gamma \cap |\Delta|$ such that also $s_i v$ belongs to γ and we can suppose that v belongs to the interior of $|\Delta|$. Indeed there is a vector v' that belongs to $\gamma \cap |\Delta| \cap H$ because of the convexity of γ (there is a vector of γ in the interior of σ, a fortiori in the interior of $|\Delta|$ and there is another vector of γ in the interior of σ' and a fortiori outside of $|\Delta|$). So we can choose v in a suitable neighborhood of v'.



By hypothesis we know that $h^c(v) = h(v) = (h|\sigma)(v)$ and that $h^c(s_i v) = (h|\sigma)(s_i v) = (s_i \cdot (h|\sigma))(v)$, so $(h|\sigma)(v) = (s_i(h|\sigma))(v)$ because of the invariance of h^c by W^1. Observe that $h|\sigma - s_i \cdot h|\sigma$ is a multiple of α_i^s because of the definition of s_i. It is a strictly positive multiple of α_i^s because $h|\sigma$ is a regular weight (this implies that $h|\sigma$ is a strongly dominant weight with respect to the restricted root system). Thus $((h|\sigma) - s_i(h|\sigma))(v)$ is strictly positive because v is in the interior of $|\Delta|$; this is a contradiction. Observe that we have proved a more general statement. Let Z be a possibly singular toric variety proper over $\mathbf{A}^l$ and let L_{h^c} be a line bundle on Z^c such that h^c is invariant for the action of W^1. If $h^c|\sigma$ is a regular weight for each $\sigma \in \Delta(l)$, h^c is a convex function and h is strictly convex on the fan of Z, then L_{h^c} is ample. Moreover one can easily prove that h^c is convex if its restriction to $\sigma(e_1, ..., e_l)$ is convex and $h^c|\sigma$ is a dominant weight for each $\sigma \in \Delta(l)$. Indeed h^c defines a line bundle on the completion $(Z^{'})^c$ of a resolution of singularities Z' of Z and this line bundle is generated by global sections by the first part of the proposition. We will use this facts to prove the proposition 4.8.

Since L_h is generated by global sections, we have an equivariant morphism $\varphi : Y \to \mathbf{P}(V)$ with $V = H^0(Y, L)^*$. Let U be the locus where φ is not an immersion. We could try to prove this point like the previous one, namely using the fact that the restriction of L_h to Z, respectively to any closed orbit is very ample. Instead we will use the stronger fact that $L_h|Z^c$ is ample and the proposition 6.3, namely the surjectivity of the restriction of the sections from Y to Z^c. Observe that the restriction of the sections to Z is clearly not surjective.

Now we want to show that U is stable and closed in the Euclidean topology. U is the union of two loci: the locus U_1 of the points where the differential of φ is not injective and the locus U_2 of the points where φ is not injective. U_1 and U_2 are G-stable because φ is equivariant. U_1 is closed because it is the locus of the zeroes of the jacobian of φ. Now we want to prove that the closure of U_2 is contained in U. Let $\{x_n\}$ be any sequence in U_2 and suppose that it converges to $x \in Y$. We have to show that x belongs to U. By hypothesis there is a sequence $\{y_n\}$ in U_2 such that $x_n \neq y_n$ and $\varphi(x_n) = \varphi(y_n)$ for each n. Since Y is compact, we can suppose, up to take sub-sequences, that $\{y_n\}$ has limit y in Y. Moreover we have $\varphi(x) = lim_{n\to\infty}(\varphi(x_n)) = lim_{n\to\infty}(\varphi(y_n)) = \varphi(y)$ because of the continuity of φ. If $x \neq y$ then $x \in U_2$. Hence we can suppose that $x = y$. Suppose by contradiction that x does not belong to U, so it does not belong to U_1. Because of the Dini theorem there is a open neighborhood W of x such that $\varphi|W$ is a diffeomorphism onto the image $\varphi(W)$. This is a contradiction because there is an integer n_0 such that x_n and y_n belong to W for each $n > n_0$. Observe that if U_1 is empty then x must be different from y.

Suppose that U is not empty. First suppose that U_1 is not empty, so it contains a closed orbit O_σ. Let x_σ be the intersection of Z and O_σ, so x_σ is a point fixed by T. The map $H^0(Z^c, L_h|Z^c)^* \longrightarrow H^0(Y, L_h)^*$ dual to the restriction map is injective because of the proposition 6.3. Thus we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\quad\varphi\quad} & \mathbf{P}(H^0(Y, L_h)^*) \\ \uparrow & & \uparrow \\ Z^c & \xrightarrow{\quad\varphi'\quad} & \mathbf{P}(H^0(Z^c, L_h|Z^c)^*) \end{array} .$$

φ' is an immersion because h^c is strictly convex on Δ^c, so $\varphi(Z^c)$ is isomorphic to Z^c. Let $[h]$ be the image $\varphi(H)$ of $H \in G/H$ and let $[v_{h|\sigma}]$ be the image of x_σ. Observe that $[v_{h|\sigma}]$ is the class of a highest weight vector of $V_{h|\sigma} \subset H^0(Y, L_h)^*$. We can write $H^0(Y, L_h)^* = V_{h|\sigma} \oplus V'$ for a suitable representation V'. We can choose $h = v_{h|\sigma} + \sum v_i$ where the v_i are weight vectors with weights contained in $h|\sigma + M$. Indeed the weights of the highest weight vectors of $H^0(Y, L_h)^*$ are contained in $h|\sigma + M$ because of the theorem 6.2. The other weight are contained in $h|\sigma + M$ because they are contained in $\Pi(Y, h) + \bigoplus_{\alpha\in\phi} \mathbf{Z}^+(-\alpha)$ and they are special ($t \cdot v_i = v_i$ for each $t \in T^0$). Let $\widetilde{A}$ be the affine open set of $\mathbf{P}(H^0(Y, L_h)^*)$ where a lowest weight vector $s \in H^0(Y, L_h)$ of weight $-h|\sigma$ is not zero, i.e. $\widetilde{A} = v_{h|\sigma} + V'_{h|\sigma} \oplus V'$ where $V_{h|\sigma} = \mathbf{C}v_{h|\sigma} \oplus V'_{h|\sigma}$ in a T-equivariant way. The intersection A of $\widetilde{A}$ and $\varphi(Y)$ is B^- stable. Moreover the intersection of $\varphi(Z^c)$ and A is $\varphi(U_\sigma)$, in particular $\varphi(x_\sigma)$ belongs to A. Indeed the set of points $\{x \in Z^c : \ s(x) = 0\}$ is the union of the divisor Z^c_τ for $\tau \nsubseteq \sigma$.

We want to study the restriction of φ to $U^- \cdot U_\sigma$, where U^- is the unipotent group whose Lie algebra is $\bigoplus_{\alpha \in -\phi_1^+} \mathfrak{g}_\alpha$. Every irreducible component of $H^0(Y, L_h)^*$ is isomorphic to its dual in a θ linear way (see lemma 1.6 [CSV I]), so $H^0(Y, L_h)^*$ is isomorphic to its dual in a θ linear way. Thus there is a not degenerate bilinear form $(\,,)$ on $H^0(Y, L_h)^*$ with following properties: 1) given any two distinct irreducible component V_1 and V_2 of $H^0(Y, L_h)^*$, they are orthogonal; 2) $(gu, v) = (u, \theta(g^{-1})v)$ for each $g \in G$, $u, v \in H^0(Y, L_h)^*$; 3) $(xu, v) = (u, -\theta(x)v)$ for each $x \in \mathfrak{g}$, $u, v \in H^0(Y, L_h)^*$. Let Υ' be the tangent space in $v_{h|\sigma}$ to the orbit $U^- \cdot v_{h|\sigma}$ and let Υ the space generated by Υ' and $v_{h|\sigma}$. One can show that the restriction of $(\ ,\ )$ to Υ is non-degenerate, Υ is stable under P and the orthogonal $\Upsilon^\perp$ is stable under $\theta(P)$ (see lemma 2.4 [CSV I]). A fundamental part of the proof is the following lemma:

Lemma 7.2 (see lemma 2.5 in [CSV I]) *$U_\sigma \subset v_{h|\sigma} + \Upsilon^\perp$.*

Proof. Remember that $h = v_{h|\sigma} + \sum v_i$ where the weight of v_i belongs to $h|\sigma + M$ for each i. We have to show that each v_i belongs to $\Upsilon^\perp$. Given weight vectors $v_1, v_2 \in H^0(Y, L_h)^*$ with weights respectively λ_1 and λ_2, we have $\lambda_1(t)(v_1, v_2) = (tv_1, v_2) = (v_1, \theta(t^{-1})v_2) = -\theta(\lambda_2)(t)(v_1, v_2)$. Thus $\lambda_1 = -\theta(\lambda_2)$ if $(v_1, v_2) \neq 0$. So it is sufficient to study the v_i contained in $V_{h|\sigma}$ and with weight $h|\sigma - \sum n_i \alpha_i^s$ equal to $h|\sigma - \alpha$ for a suitable $a \in \phi_1^+$. Notice that in this case the n_i are all positive. Let h' be the orthogonal projection of h to $V_{h|\sigma}$ and let v_{i_0} a such vector, we have $(x_\alpha + \theta(x_\alpha))h' = 0$ because h is H stable. $(x_\alpha + \theta(x_\alpha))h' - x_\alpha v_{i_0}$ is a sum of weight vectors with weights different from $h|\sigma$, so $x_\alpha v_{i_0} = 0$. Given weight vectors $v_1, v_2 \in V_{h|\sigma}$ with weights respectively λ_1 and λ_2, we have $\lambda_1(t)(v_1, v_2) = -\theta(\lambda_2)(t)(v_1, v_2)$. Thus $\lambda_1 = -\theta(\lambda_2)$ if $(v_1, v_2) \neq 0$, so the only possibly non zero scalar product between v_{i_0} and a vector of the basis of Υ is the one with $x_{\theta(\alpha)} v_{h|\sigma}$. In this case we have $(x_{\theta(\alpha)} v_{h|\sigma}, v_{i_0}) = -(v_{h|\sigma}, \theta(x_{\theta(\alpha)}) v_{i_0}) = 0$. Indeed $\theta(x_{\theta(\alpha)})$ is a multiple of x_α. □

Let π be the projection of $H^0(Z^c, L_h|Z^c)^*$ onto $H^0(Z^c, L_h|Z^c)^*/\Upsilon^\perp$. $U^- \subset \theta(P)$, so U^- acts on $H^0(Z^c, L_h|Z^c)^*/\Upsilon^\perp$ and the projection is equivariant. The affine hyperplane $\pi(\widetilde{A})$ in $H^0(Z^c, L_h|Z^c)^*/\Upsilon^\perp$ is stable by the action of U^-. We have the following lemma.

Lemma 7.3 (see lemma 2.6 in [CSV I]) *The map $j : U^- \to \pi(\widetilde{A})$ defined by $j(u) = \pi(uv_{h|\sigma})$ is an U^- equivariant isomorphism.*

Proof. Υ' is the tangent space of $U^- v_{h|\sigma}$ at $v_{h|\sigma}$, so j is smooth at the identity. Thus j is everywhere smooth because it is U^- equivariant. j is an open immersion because $\dim U^- = \dim \pi(\widetilde{A})$ and U^- has not finite subgroups. Now it is sufficient to observe that an open immersion between two affine space of the same dimension is an isomorphism. □

Thus the tangent space to U_σ at $v_{h|\sigma}$ is orthogonal to Υ and the differential of φ is injective in x_σ. Hence we have proved that $U_1 = \emptyset$, so U_2 is equal to U and it is closed.

Now suppose that U_2 is not empty, so it contains a closed orbit O_σ. Given $x \in O_\sigma$, there is $y \neq x$ such that $\varphi(x) = \varphi(y)$. We want to show that we can choose x and y such that also y belongs to a closed orbit $O_{\sigma'}$. First of all, we can suppose that y belongs to Z. Indeed there is $g \in G$ such that gy belongs to Z. Moreover $gx \in O_\sigma$, $gx \neq gy$ and $\varphi(gx) = \varphi(gy) = g\varphi(x)$. Now observe that there is an one parameter subgroup γ of T such that $y_2 = lim_{t\to 0}\gamma(t)gy$ is a point of Z fixed by T, so y_2 belongs to a closed orbit (see the last point of the proposition 4.1). Moreover $x_2 = lim_{t\to 0}\gamma(t)gx$ belongs to O_σ and $\varphi(x_2) = \varphi(y_2)$. By the previous part of the proof x_2 is different by y_2.

The closed orbits O_σ and $O_{\sigma'}$ are different because the restriction of L to O_σ is very ample. We known that $H^0(O_\sigma, L|O_\sigma) = V^*_{h|\sigma}$. Because of the lemma 7.1 there is a global section s, lowest weight vector of weight $-h|\sigma$, which does not vanish on O_σ. Up to a translation we can suppose that $s(x) \neq 0$. Because h is strictly convex on Δ, s vanishes on the divisor Z_τ of Z associated to a cone τ contained in σ' ($h|\sigma' - h|\sigma \in \check{\sigma} - \sigma^\perp$). Therefore s vanishes on the divisor Y_τ of Y because s is U^- invariant. In particular s vanishes on $O_{\sigma'}$, so $\varphi(x) \neq \varphi(y)$, a contradiction.

Finally we can consider the exceptional case. First of all we want to recall some facts. Let Y be a complete exceptional symmetric variety and let X be the corresponding wonderful variety. We have chosen an order of the simple roots of ϕ such that $\alpha_1, ..., \alpha_s$ are exceptional roots with the following property: $Pic(X)$ is generated by the spherical weights and by the fundamental weights $\omega_{\alpha_1}, ..., \omega_{\alpha_s}$ corresponding respectively to $\alpha_1, ..., \alpha_s$. Moreover, given a piecewise function h in Λ_Y such that $h|\sigma$ is dominant for each $\sigma \in \Delta(l)$, there are integers a_i and a spherical piecewise linear function h' such that $h = h' + \sum a_i\omega_{\alpha_i}$. We can suppose that a_i is positive up to exchange α_i with $\overline{\theta}(\alpha_i)$.

If L_h satisfies the hypotheses of the second point, then $L_{h'}$ is very ample because h' is spherical. Moreover, $L_{h-h'}$ is generated by global sections because $h - h' = \sum a_i\omega_{\alpha_i}$. Thus L_h is the product of a very ample bundle $L_{h'}$ and a bundle L_{h-h_1} generated by global sections, so it is very ample.

The third point is obvious. □

Now, we can prove that a S-linearizated line bundle L_h on Z is ample if and only if h is strictly convex on Δ.

Proposition 4.8 *Suppose that Z is a (possibly singular) toric variety proper over $\boldsymbol{A}^l$ and let $h \in SF(\Delta, M)$. Then L_h is ample if and only if h is strictly convex on Δ. If Z is smooth then L_h is ample if and only if it is very ample.*

Proof. Suppose that L_h is ample, then there is an integer n such that L_{nh} is very ample, so L_{nh} is generated by global sections and nh is convex. Hence L_{nh} is the pullback of a line bundle generated by global sections on a variety Z' dominated by Z with the property that the fan Δ' of Z' has the same support of the fan of Z and nh is strictly convex on Δ'. (Notice that if a cone is contained in $\sigma(e_1, ..., e_l)$ then it contains no line). Let $\varphi : Z \to \mathbf{P}(V)$ be an immersion such that $L_{nh} = \varphi^*\mathcal{O}(1)$, then φ factorizes through Z' because $H^0(Z', L_h) = H^0(Z, L_h)$. Since φ is an immersion, Z' must be Z and nh is strictly convex on Δ, so h is strictly convex on Δ. To prove the viceversa will be sufficient to prove the following lemma.

Lemma 7.4 *Let Z be is a (possibly singular) toric variety proper over $\boldsymbol{A}^l$ and let L be a line bundle on Z generated by global sections. Given any homogeneous symmetric variety G/H of rank l, let Z^c be the complete toric variety associated to Z. Then there is a linearized line bundle L_{h^c} on Z^c generated by global sections and such that the restriction of L_{h^c} to Z is L as line bundle. Moreover we can suppose that: 1) h^c is invariant by W^1; 2) $h^c|\sigma$ is a regular weight for each $\sigma \in \Delta(l)$.*

We can suppose that Z is smooth. Otherwise there is a resolution of singularities Z' of Z. Moreover if L'_{h^c} is a linearized line bundle on $(Z')^c$ such that h^c is W^1 invariant and $L'_{h^c}|Z'$ is the pullback of L, then L'_{h^c} is the pullback of the line bundle L_{h^c} on Z^c. Now it is sufficient to prove the following lemma.

Lemma 7.5 *Let Z be any smooth toric variety proper over $\boldsymbol{A}^l$ and let L be any line bundle on Z generated by global sections (respectively any ample line bundle on Z). Given any homogeneous symmetric variety G/H of rank l, let Y be the complete symmetric variety associated to Z. Then there is a line bundle L' on Y generated by global sections (respectively an ample line bundle L' on Y) whose restriction to Z is L. Moreover we can suppose that the restriction of L' to any closed orbit O_σ of Y is ample even if L is not ample.*

Proof. By theorem 6.1 there is a line bundle L_h on Y whose restriction to Z is L, but it may have base points. Moreover, we can suppose that h is almost spherical. Let λ be a regular spherical weight. Observe that the restriction of L_λ to Z is trivial. Moreover there is a positive integer n such that $(h + n\lambda)|\sigma$ is a regular weight for each $\sigma \in \Delta(l)$, so $L' = L_{h+n\lambda}$ satisfies ours requests. $\square$.

Observe that the line bundle L' is not unique unless Z is a point. We now can conclude the proof the proposition 4.8. Let h be a strictly convex function on Δ, we can suppose, up to exchange the linearization of L_h, that L_{h^c} is ample on Z^c. Indeed we can suppose by the lemma 7.4 that h^c is W^1 invariant and that $h^c|\sigma$ is a regular weight for each $\sigma \in \Delta(l)$. Thus h^c is strictly convex on Δ^c by the proof of the proposition 7.1. The last point of the proposition is implied by the Demazure theorem. $\square$

Remark. We have proved that the line bundle L_{h^c} of the lemma 7.4 is ample if L is ample.

Theorem 7.1 *Let Y be a complete symmetric variety. The following conditions are equivalent:*

1. *Y is projective;*
2. *Z^c is projective;*
3. *Z is quasiprojective.*

Proof. It is sufficient to prove that the third condition implies the first one. If Z is quasiprojective then there is ample line bundle L on Z. Moreover L is the restriction of an ample line bundle on Y because of the lemma 7.5, so Y is projective. $\square$

Now we want to reformulate the proposition 7.1 using h^c instead of h. It is immediately implied from proposition 1.5 that a spherical weight is regular if and only if it is a strongly dominant weight for the restricted root system. Given a spherical Δ-linear function h we can easily show that h is convex if and only if h^c is convex. Indeed if h is convex then the corresponding line bundle L_h on Y is generated by global sections. In particular its restriction to Z^c is generated by global sections, thus h^c is convex. The viceversa is trivial. We have already proved that, given h such that $h|\sigma$ is a regular spherical weight for each $\sigma \in \Delta(l)$, h is strictly convex on Δ if and only if h^c is strictly convex on Δ^c. We want to show that if h^c is an almost spherical convex Δ^c linear function then h is a spherical Δ-linear function. If h^c is also strictly convex on Δ^c, then $h|\sigma$ is regular for each $\sigma \in \Delta(l)$. Given $\sigma \in \Delta(l)$, there is an element $w \in W^1$ such that $w \cdot h|\sigma$ is a dominant weight. Observe that $h|\sigma - w \cdot h|\sigma$ is a function with positive values on $|\Delta|$. Moreover $w \cdot h|\sigma$ is the restriction of h^c to $w^{-1} \cdot \sigma$. Let v be a vector in the interior of σ, so it is a fortiori in the interior of $|\Delta|$. Because of the convexity of h^c, we have $(w \cdot h|\sigma)(v) \geq h(v) = (h|\sigma)(v)$, so $(w \cdot h|\sigma)(v) = (h|\sigma)(v)$. We have $w \cdot h|\sigma = h|\sigma$ because v is a vector inside the Weyl chamber $|\Delta|$. Thus $h|\sigma$ is dominant. If h^c is strictly dominant on Δ^c, then $h|\sigma$ is different from $w \cdot h|\sigma$ for each $w \in W^1$, so $h|\sigma$ is regular. We have proved the following proposition.

Proposition 7.2 *Let h be an almost spherical Δ-linear function, then*

1. *h^c is convex on Δ^c if and only if h is convex on Δ and $h|\sigma$ is dominant for each $\sigma \in \Delta(l)$.*

2. *h^c is strictly convex on Δ^c if and only if h is strictly convex on Δ and $h|\sigma$ is a regular weight for each $\sigma \in \Delta(l)$.*

8 Reduction to the complete toric variety

In the following we will always suppose that h is an almost spherical (Δ, Λ_X) linear function, unless we will explicitly say otherwise. We start to study the multiplication of sections of two line bundles on Y. First of all, we want to show that this problem is equivalent to the similar problem on the complete toric variety Z^c associated to Y. Let L_h and L_k be any two line bundles on Y generated by global sections. Let

$$M_{h,k} : H^0(Y, L_h) \otimes H^0(Y, L_k) \longrightarrow H^0(Y, L_{h+k})$$

be the product of sections on Y and let

$$m^c_{h,k} : H^0(Z^c, L_h|Z^c) \otimes H^0(Z^c, L_k|Z^c) \longrightarrow H^0(Z^c, L_{h+k}|Z^c)$$

be the product of sections of the restrictions to Z^c of these line bundles.

Theorem 8.1 *Suppose that h and k are two convex spherical Δ-linear function. Then $M_{h,k}$ is surjective if and only if $m^c_{h,k}$ is surjective.*

Proof. The necessity of the condition is implied by the surjectivity of the restriction maps from Y to Z^c. Indeed if $i : Z^c \to Y$ is the canonic inclusion then $m^c_{h,k} \circ (i^* \otimes i^*) = i^* \circ M_{h,k}$.

$$\begin{array}{ccc} H^0(Y, L_h) \otimes H^0(Y, L_k) & \overset{M_{h,k}}{\longrightarrow} & H^0(Y, L_{h+k}) \\ \downarrow i^* \otimes i^* & & \downarrow i^* \\ H^0(Z^c, L_h|_{Z^c}) \otimes H^0(Z^c, L_k|_{Z^c}) & \overset{m^c_{h,k}}{\longrightarrow} & H^0(Z^c, L_{h+k}|_{Z^c}) \end{array}$$

Now suppose that $m^c_{h,k}$ is surjective. It is sufficient to show that the image of $M_{h,k}$ contains a basis of semi-invariant sections. If h and k are linear then they are the pullbacks of two line bundles generated by global sections on the wonderful variety X, so $M_{h,k}$ is surjective by the theorem 3.2. In general, given $\nu \in \Pi(Y, h+k)$ there are $\lambda \in \Pi(Z^c, h)$ and $\mu \in \Pi(Z^c, k)$ such that $\nu = \lambda + \mu$. Moreover there are elements w_1 and w_2 in the Weyl group W^1 such that $w_1 \cdot \lambda$ and $w_2 \cdot \mu$ are dominant weights. Observe that $\nu \geq w_1 \cdot \lambda + w_2 \cdot \mu$ on $|\Delta|$. Moreover $w_1 \cdot \lambda \geq h$ and $w_2 \cdot \mu \geq k$ because h^c and k^c are convex and invariant for the action of W^1. Thus $s^{h-w_1\cdot\lambda} H^0(Y, L_{w_1\cdot\lambda}) \subset H^0(Y, L_h)$ and $s^{k-w_2\cdot\mu} H^0(Y, L_{w_2\cdot\mu}) \subset H^0(Y, L_k)$ (remember that if $h - w_1 \cdot \lambda = \sum_{\tau\in\Delta(1)} a_\tau d^\tau$ then $s^{h-w_1\cdot\lambda} \in H^0(Y, L_{h-w_1\cdot\lambda})$ is the section $\prod s_\tau^{a_\tau}$, where the $s_\tau \in H(Y, L_{d^\tau})$ are the sections of the lemma 6.1). Let $\varphi \in H^0(Y, L_{w_1\cdot\lambda+w_2\cdot\mu})$ be a lowest weight vector of weight $-\nu$. We know that φ is contained in $Im\, M_{w_1\cdot\lambda, w_2\cdot\mu}$. Thus $s^{h+k-w_1\cdot\lambda-w_2\cdot\mu}\varphi$ is contained in $s^{h+k-w_1\cdot\lambda-w_2\cdot\mu} Im\, M_{w_1\cdot\lambda,w_2\cdot\mu} \subset Im\, M_{h,k}$ and it is not zero. □

We can prove the following proposition without assuming the surjectivity of $m^c_{h,k}$. Given two convex spherical Δ-linear function, say h and k, let $\Pi(Y, h, k)$ be the set of the weights of the lowest weight vectors contained in $Im M_{h,k}$.

Proposition 8.1 *$\Pi(Y, h, k)$ is saturated with respect to the dominant order of the roots in $\widetilde{\phi}$.*

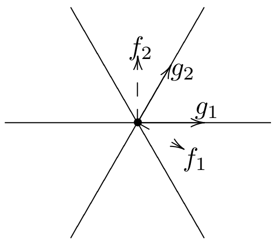
Proof. $\Pi(Y, h+k)$ is saturated because $-\alpha_i^s$ has positive values on $|\Delta|$ for each $i = 1, ..., l$. Indeed, given a spherical weight μ dominated by a spherical weight λ in $\Pi(Y, h+k)$ then $\mu = \lambda - \sum a_i \alpha_i^s$ where a_i is a positive integer for each i. So $\mu \geq \lambda \geq h$ and μ belongs to $\Pi(Y, h+k)$.

Given $\nu \in \Pi(Y, h, k)$ there are two weights $\lambda \in \Pi(Y, h)$ and $\mu \in \Pi(Y, k)$ such that $\nu = \lambda + \mu$. (By hypothesis there are sections $s_1^i \in H^0(Y, L_h)$ and sections $s_2^i \in H^0(Y, L_k)$ such that $s_\nu := M_{h,k}(\sum s_1^i \otimes s_2^i)$ is a semi-invariant section of weight ν. We define λ as the weight of s_1^j and μ as the weight of s_2^j for a suitable j). Moreover there are element w_1, w_2 in the Weyl group W^1 such that $w_1 \cdot \lambda$ and $w_2 \cdot \mu$ are dominant weights. Observe that $\nu \geq w_1 \cdot \lambda + w_2 \cdot \mu$ on $|\Delta|$, so $\nu \in \prod(Y, w_1 \cdot \lambda + w_2 \cdot \mu)$. Let ν' be a spherical weight dominated by ν, then $\nu' \in \Pi(Y, w_1 \cdot \lambda + w_2 \cdot \mu)$ because this set is saturated. Let φ be a lowest weight vector of weight ν'. Because of the surjectivity of $M_{w_1\cdot\lambda, w_2\cdot\mu}$ we have $\varphi \in s^{h+k-w_1\cdot\lambda-w_2\cdot\mu} ImM_{w_1\cdot\lambda,w_2\cdot\mu} \subset ImM_{h,k}$. $\square$

9 Reduction to the open toric variety

In this section we want to show that, given two ample line bundles, the product of sections on Z^c is surjective if and only if the product of sections on Z is surjective. Moreover we will study the relation between the sections of $L|Z$ and the sections of $L|Z^c$ for any ample line bundle L on Y. Remember that we have fixed a $\sigma \in \Delta(l)$ and we have set $v_h = h|\sigma$ for each $h \in SF(\Delta, \Lambda_X)$. Moreover $\prod(Z, h) = Q_h \cap (M + v_h)$ and $\Pi(Z^c, h) = P_h \cap (M + v_h)$.

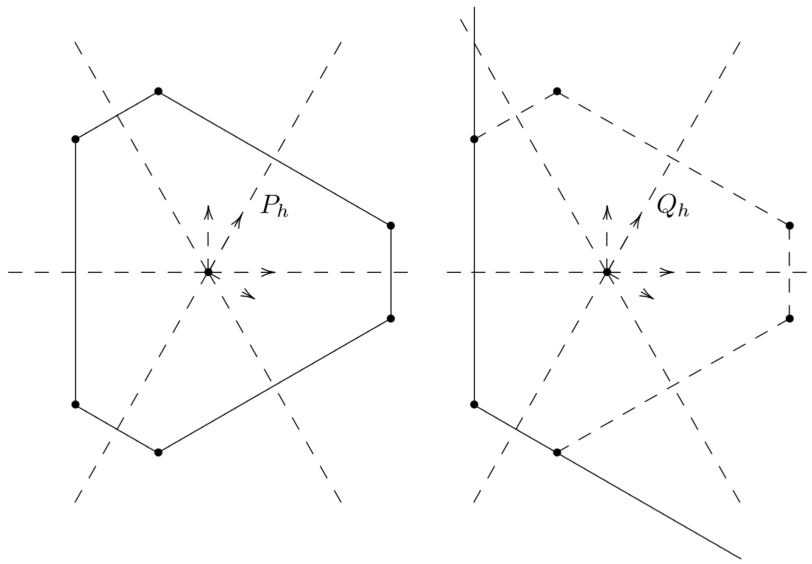
Now we want to prove some relations between P_h and Q_h, but before we have to define some notations. Recall that $\{e_1, ..., e_l\}$ is the basis of $N_{\mathbf{R}}$ dual to the basis $\{f_1, ..., f_l\}$ of $M_{\mathbf{R}}$. We have to define a second basis $\{g_1, ..., g_l\}$ of $M_{\mathbf{R}}$ because the fundamental Weyl chamber C^+ is more easily defined using the basis the fundamental weights than the basis of the simple roots. g_i is a positive multiple of $-\widetilde{\omega}_i$. Remember that there are positive constants a_i such $a_i\widetilde{\omega}_i$ is the i-th fundamental weight of $\widetilde{\phi}$. If $\widetilde{\phi}$ is reduced we define g_i as $-a_i\widetilde{\omega}_i$, while if the type of $\widetilde{\phi}$ is BC_l then $-g_i$ is the i-th fundamental weight of the root system of type B_l contained in $\widetilde{\phi}$. In general $-g_i$ is the i-th fundamental weight of the unique reduced root system contained in $\widetilde{\phi}$ which share a basis with $\widetilde{\phi}$. $g_1, ..., g_l$ generate a lattice which contains M. Let $\{\check{g}_1, ..., \check{g}_l\}$ be the dual basis of $\{g_1, ..., g_l\}$. We will seldom use this last basis. Given a point p in $M_{\mathbf{R}}$ we will use the following notations: $p = \sum x_i f_i = \sum \dot{x}_i g_i$, using the "normal" coordinates for the basis $\{f_1, ..., f_l\}$ and the "dotted" coordinates for the basis $\{g_1, ..., g_l\}$. (In the following figures we consider the case in which the restricted root system is of type A_2 and Z is $\mathbf{A}^2$).



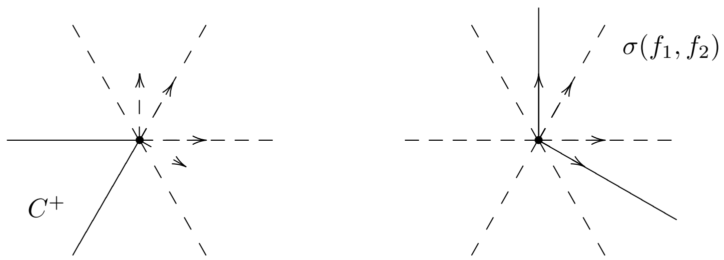
Observe that $C^+ = \{\sum \dot{x}_i g_i : \ \dot{x}_i \leq 0 \ \forall\, i\}$, namely $C^+ = \sigma(-g_1, ..., -g_l)$.

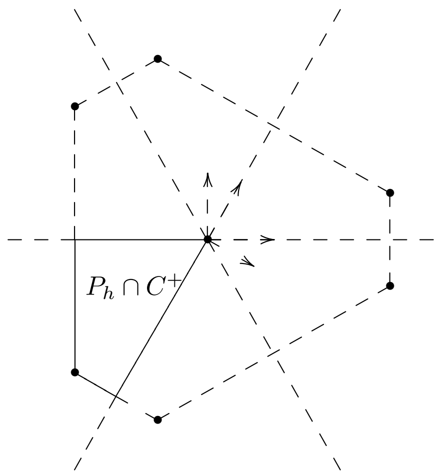
The equations of Q_h are of the form $\sum b_i x_i \geq b$ where the b_i are positive constants. So, given any $m \in Q_h$, we have $m + \bigoplus \mathbf{R}^+(f_i) \subset Q_h$, i.e. Q_h is stable by translation with respect to vectors in $\sigma(f_1, ..., f_l)$.

Let H_j be the hyperplane of $M_{\mathbf{R}}$ generated by $g_1, ..., \widehat{g}_j, ..., g_l$, so the intersection of H_j and C^+ is a Weyl wall. Let s_j be the orthogonal reflection with respect to H_j. Observe that, if P_h contains a point p, then it contains all the translates of p by W^1. Moreover, given any point $p \in P_h$, P_h contains the orthogonal projection $\frac{1}{2}(p + s_j p)$ of p to H_j for each j. Because h^c is strictly convex on Δ^c, there is no vertex of P_h contained in H_j. Indeed, given a vertex $h|\sigma$ of P_h, then $s_j \cdot h|\sigma$ is different from $h|\sigma$.



Proposition 9.1 *Let h be a spherical Δ-linear function such that h^c is strictly convex on Δ^c. Then $Q_h \cap C^+ = P_h \cap C^+$ and $Q_h = P_h \cap C^+ + \sigma(f_1, ..., f_l)$.*





Remember that the function associated to a polyhedron K is the piecewise linear function h_K such that $h_k(n) := inf\{m(n); m \in K\}$ for each $n \in N_{\mathbf{R}}$, so h_K has values in $\mathbf{R} \cup -\infty$. Moreover it has always finite values if and only if K is compact. There is a decomposition in convex cones of the convex set $\{n \in N_{\mathbf{R}} : h_K(n) \in \mathbf{R}\}$ such that there is an one-to-one correspondence between the cones of such decomposition and the faces of K. In particular there is an one-to-one correspondence between the 1-dimensional cones of such decomposition and the $(l-1)$-dimensional faces of K. This faces are in one-to-one correspondence with the semi-spaces that define K. Given a such cone τ the associated semi-space is $\{m \in M_{\mathbf{R}} : m(\varrho(\tau)) \geq h_K(\varrho(\tau))\}$.

First of all we will show that $P_h \cap C^+ = Q_h \cap C^+$. The function h_{Q_h} associated to Q_h is equal to h on $|\Delta|$ and it has value $-\infty$ on the complementary set. The semi-spaces defining $Q_h \cap C^+$ are $\{m \in M_{\mathbf{R}} : m(\varrho(\tau)) \geq h(\varrho(\tau))\}$ for each $\tau \in \Delta(1)$ and $\{m \in M_{\mathbf{R}} : m(\breve{g}_i) \leq 0\}$ for each i. It is evident that $P_h \cap C^+ \subseteq Q_h \cap C^+$, so it is sufficient to show that $Q_h \cap C^+ \subseteq P_h$. As a matter of fact it is sufficient to show that $Q_h \cap C^+ \cap M_{\mathbf{Q}} \subseteq P_h$ because P_h is closed. The semi-spaces defining P_h are $\{m \in M_{\mathbf{R}} : m(\varrho(\tau)) \geq h^c(\varrho(\tau))\}$ for each $\tau \in \Delta^c(1)$. Given any $m \in Q_h \cap C^+ \cup M_{\mathbf{Q}}$ and any $\tau \in \Delta^c(1)$ we have to show that $m(\varrho(\tau)) \geq h^c(\varrho(\tau))$. Because of the symmetry of Δ^c, there are $w \in W^1$ and $\tau' \in \Delta(1)$ such that $\varrho(\tau) = w \cdot \varrho(\tau')$. Observe that $w^{-1} \cdot m - m$ is a linear combination $\sum c_i f_i$ of the f_i with positive coefficients, so $m(\varrho(\tau)) = m(w \cdot \varrho(\tau')) = (w^{-1} \cdot m)(\varrho(\tau')) = m(\varrho(\tau')) + \sum c_i f_i(\varrho(\tau')) \geq m(\varrho(\tau')) \geq h(\varrho(\tau')) = h^c(\varrho(\tau))$.

Now we want to show that $Q_h = P_h \cap C^+ + \sigma(f_1, ..., f_l)$. The decomposition in cones of $N_{\mathbf{R}}$ associated to $h_{P_h \cap C^+}$ has 1-dimensional cones $\{\sigma(\breve{g}_1), ..., \sigma(\breve{g}_l)\} \cup \Delta(1)$. $h_{P_h \cap C^+}$ has finite values on all $N_{\mathbf{R}}$, it is equal to h on $|\Delta|$ and vanishes on the vectors $\breve{g}_1, ..., \breve{g}_l$. The function associated to $\sigma(f_1, ..., f_l)$ vanishes on $|\Delta|$ and has value $-\infty$ on the complementary set. Thus their sum is the function associated to Q_h. So the claim follows by the theorem 4.2, namely by the fact that $h_Q + h_{Q'} = h_{Q+Q'}$ for each polyhedrons Q and Q'.

□

We can prove a stronger statement on the "rational" points of Q_h and P_h.

Proposition 9.2 *Let h be a spherical Δ-linear function such that h^c is strictly convex on Δ^c. Then $Q_h \cap (v_h + M) = P_h \cap C^+ \cap (v_h + M) + \sum_{i=1}^{l} \boldsymbol{Z}^+ f_i$.*

Remark Observe that $H^0(Z, L|Z)$ is a $O_{Z^c}(Z^c)$-module through the restriction map $O_{Z^c}(Z^c) \to O_Z(Z)$ and $H^0(Z^c, L|Z^c)$ is a $O_{Z^c}(Z^c)$-submodule of $H^0(Z, L|Z)$. This proposition imply that $H^0(Z, L|Z)$ is generated by $H^0(Z^c, L|Z^c)$ as an $O_Z(Z)$-module.

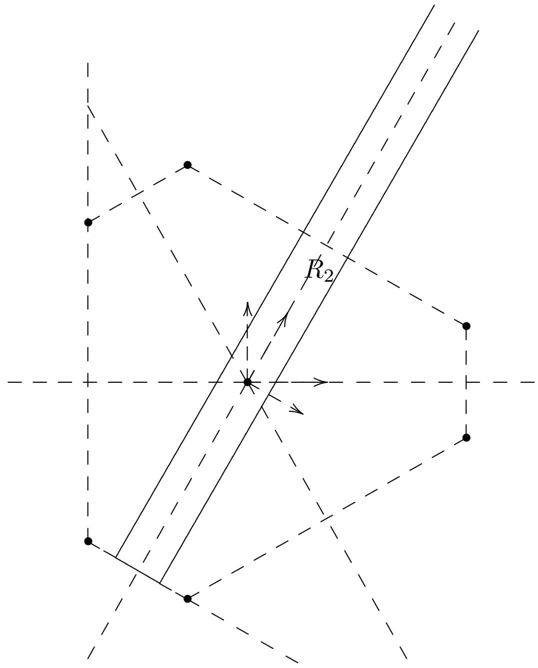
Proof. We need some lemmas. Recall that s_j is the orthogonal reflection with respect to H_j. Observe that f_j is orthogonal to H_j and let $\widetilde{f_i} = \frac{1}{2}(f_i + s_i f_j)$ for each $i \neq j$. Observe that $\widetilde{f_i} \in H_j$ for each $i \neq j$. Moreover $\{\widetilde{f_i}\}_{i \neq j}$ is a basis of H_j. $-f_i$ and $-f_j$ are distinct simple restricted roots, so they form an obtuse angle. Hence $\widetilde{f_i} = f_i + d_i f_j$ for a suitable positive integer d_i. We have the following easy consequence of the proposition 9.1.

Lemma 9.1 $Q_h \cap H_j = P_h \cap H_j \cap C^+ + \bigoplus_{i \neq j} \boldsymbol{R}^+ \tilde{f}_i$.

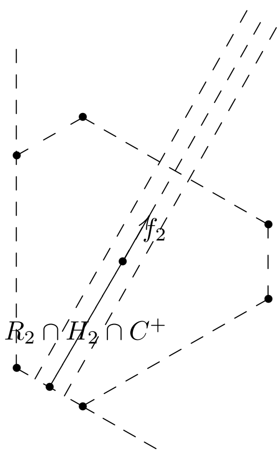
Proof. Let $p = p' + \sum r_i f_i \in Q_h \cap H_j$ with $p' \in P_h \cap C^+$ and r_i positive constants. Then $p = \frac{1}{2}(p' + s_j p') + \sum r_i \frac{1}{2}(f_j + s_j f_i)$. Hence is sufficient to observe that $\frac{1}{2}(p' + s_j p')$ belongs to $P_h \cap H_j \cap C^+$ (it is the projection of p' to H_j). □

Let $R_j = \{p + a f_i \mid p \in Q_h \cap H_i \text{ and } -1/2 \leq a \leq 1/2\}$. First of all we want to describe the conditions for a point $m \in M_{\mathbf{R}}$ to belong to R_j. Fixed any j, we define another basis $u_1, ..., u_l$ of $M_{\mathbf{R}}$ such that $u_j = f_j$ and $u_i = g_i$ if $i \neq j$. The conditions for a point $p = \sum y_i u_i$ to belong to $Q_h \cap H_j$ are $y_j = 0$ plus conditions of the form $\sum_{i \neq j} n_i y_i \geq n$. Thus the conditions for a point $p = \sum y_i u_i$ to belong to R_j are the inequalities of the form $\sum_{i \neq j} n_i y_i \geq n$ that define $Q_h \cap H_j$ plus the inequalities $-1/2 \leq y_j \leq 1/2$. A fundamental part of the proof is the following lemma on the R_j. This lemma is the unique part of the proof in which we will use the strictly convexity of h^c.

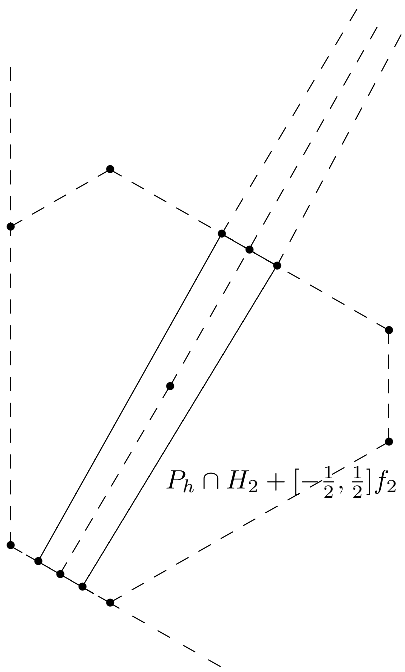
Lemma 9.2 *R_j is contained in Q for each j.*



Proof. Observe that it is sufficient to show that $P_h \cap H_j \cap C^+ + [-1/2, 1/2]f_j \subset Q_h$ because of the previous lemma.



Because of the convexity of Q_h it is sufficient to show that Q_h contains the points $p' \pm (1/2)f_j$ for each vertex p' of $P_h \cap H_j$.

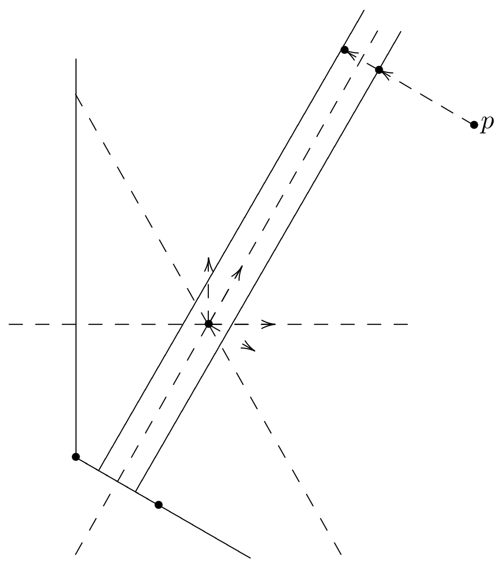


Observe that the vertices of $P_h \cap H_j$ are orthogonal projections to H_j of suitable vertices of P_h. Indeed let p' a vertex of $P_h \cap H_j$ and let p be the endpoint different by p' of the segment intersection of P_h with the semi-line outgoing from p' and parallel to f_j. If p is not a vertex of P_h then p is an interior point of a segment I contained in P_h. Thus p' is an interior point of the projection of I to H_j and this segment is contained in P_h by the symmetry of P_h, a contradiction.

If $q' + af_j$ with $q' \in Q_h \cap H_j$ belongs to $M + v_h$, then $s_j(q' + af_j) = q' - af_j$, so $2a \in \mathbf{Z}$. Observe that if q is a vertex of $P_h \cap H_j$, then there is a constant a such that $q + af_j$ is a vertex of P_h, so it is sufficient to show that the intersection of P_h with the line parallel to f_j and passing through any vertex of $P_h \cap H_j$ is not a point. If there is a vertex p of $P_h \cap H_j$ without such property, then p is vertex of P_h belonging to H_j, a contradiction. Indeed there is no segment contained in P_h that contains p as an internal point. (If I is a such segment then the projection to H_j of I would be contained in $P_h \cap H_j$ and would contain p, so it has to be p because p is a vertex of $P_h \cap H_j$. Hence I is parallel to f_j). $\square$

Now, we can conclude the proof of the proposition 9.2 (look to the following figure). Let p be a point contained in $Q_h \cap (M + v_h)$ and suppose that $p = \sum x_i f_i = \sum \dot{x}_i g_i$. If $\dot{x}_i \leq 0$ for each i, then $p \in P_h \cap C^+$. Otherwise there is an index j such that $\dot{x}_j > 0$. We know that $p = p' + \sum a_i f_i$ where $p' \in P \cap C^+$ and the a_i are positive constants, but the a_i may not be integers. If $\dot{x}_j \geq 2$ then $a_j \geq 1$. Indeed the j-th coordinate of f_i with respect to $\{g_1, ..., g_l\}$ is 2 if $i = j$ and it is negative otherwise. Thus the point $p - [a_j]f_j = p' + (a_j - [a_j])f_j + \sum_{i \neq j} a_i f_i$

belongs to $Q_h \cap (M + v_h)$ and it has j-th coordinate with respect to $\{g_1, ..., g_l\}$ strictly less than 2 ($[a_j]$ is the integral part of a_j). Moreover, this coordinate can be at most 1 because $p - [a_j]f_j$ is a weight. We can suppose that it is exactly 1, so $p - [a_j]f_j - (1/2)f_j$ belongs to $Q_h \cap H_j$ and it is the projection of $p - [a_j]f_j$ to H_j. Thus $p - [a_j]f_j$ belong to R_j, so also $p - [a_j]f_j - f_j$ belongs to R_j and its j-th coordinate with respect to $\{g_1, ..., g_l\}$ is negative. Moreover $p - (p - [a_j]f_j - f_j) = ([a_j] + 1)f_j$ is a linear combination of the f_i with positive integral coefficients. If there is an index k such that $p - [a_j]f_j - f_j$ has negative k-th coordinate with respect to $\{g_1, ..., g_l\}$, then we reiterate the process. The process has to end in a finite number of steps because Q_h is contained in the semi-space $\{\sum x_i \geq h(e_1 + ... + e_l)\}$. $\square$



Now we want to show a combinatorial condition equivalent to the suriectivity of the product of sections.

Lemma 9.3 *Let h and k be two spherical Δ-linear functions such that h^c and k^c are strictly convex on Δ^c. Then $m_{h,k}$ is surjective if and only*

$$Q_h \cap (v_h + M) + Q_k \cap (v_k + M) = Q_{h+k} \cap (v_{h+k} + M).$$

Moreover $m^c_{h,k}$ is surjective if and only

$$P_h \cap (v_h + M) + P_k \cap (v_k + M) = P_{h+k} \cap (v_{h+k} + M).$$

Proof. We prove only the first part of lemma because the proof of the second one is very similar. Suppose that $m_{h,k}$ is surjective and let $\nu \in Q_{h+k} \cap (v_{h+k} + M)$. Hence there is a seminvariant section $s \in H^0(Z, L_{h+k}|Z)$ of weight ν and there are seminvariant sections $t_i \in H^0(Z, L_h|Z)$ and $u_i \in H^0(Z, L_k|Z)$ such that $m_{h,k}(\sum t_i \otimes u_i) = s$. Let λ_i be the weight of t_i and let μ_i be the weight of u_i, we can suppose that $\lambda_1 + \mu_1 = \nu$ up to exchange the indices. Moreover $\lambda_1 \in Q_h \cap (v_h + M)$ and $\mu_1 \in Q_k \cap (v_k + M)$. Viceversa suppose that

$$Q_h \cap (v_h + M) + Q_k \cap (v_k + M) = Q_{h+k} \cap (v_{h+k} + M).$$

It sufficient to prove that the image of $m_{h,k}$ contains a basis of seminvariant sections. Given any seminvariant section s of weight ν, there are $\lambda \in Q_h \cap (v_h + M)$ and $\mu \in Q_k \cap (v_k + M)$ such that $\lambda + \mu = \nu$. Let $t \in H^0(Z, L_h|Z)$ be a seminvariant section of weight λ and let $u \in H^0(Z, L_h|Z)$ be a seminvariant section of weight μ, we known that $m_{h,k}(t \otimes u)$ is a not zero multiple of s. $\square$

Now we can prove the most important theorem on the product of sections.

Theorem 9.1 *Let h and k be two spherical Δ-linear functions such that h^c and k^c are strictly convex on Δ^c. Then $m_{h,k}$ is surjective if and only if $m^c_{h,k}$ is surjective.*

We have shown that the theorem is equivalent to the following more combinatorial statement:

$$Q_h \cap (v_h + M) + Q_k \cap (v_k + M) = Q_{h+k} \cap (v_{h+k} + M)$$

if and only if

$$P_h \cap (v_h + M) + P_k \cap (v_k + M) = P_{h+k} \cap (v_{h+k} + M).$$

Proof. The sufficiency of the condition is easy. Given a point $p \in Q_{h+k} \cap (M + v_{h+k})$ we know that $p = p' + \sum c_i f_i$ where $p' \in P_{h+k} \cap C^+ \cap (M + v_{h+k})$ and the c_i are positive integers. Moreover there are $p_h \in P_h \cap (M + v_h)$ and $p_k \in P_k \cap (M + v_k)$ such that $p' = p_h + p_k$. Thus $p = (p_h + \sum c_i f_i) + p_k$ and $p_h + \sum c_i f_i$ belongs to $Q_h \cap (M + v_h)$.

Suppose now that $Q_h \cap (v_h + M) + Q_k \cap (v_k + M) = Q_{h+k} \cap (v_{h+k} + M)$. Let $m = \sum z_i f_i = \sum \dot{z}_i g_i$ be a point in $P_{h+k} \cap (M + v_{h+k})$. We can suppose that m belongs to C^+ by the symmetry of the polytopes P_h and P_k. By hypothesis there are two points $p'_0 \in Q_h \cap (M + v_h)$ and $q'_0 \in Q_k \cap (M + v_k)$ such that $p'_0 + q'_0 = m$. First, we will show that we can choose p'_0 and q'_0 such that p'_0 belongs to P_h. Indeed we know that $p'_0 = p_0 + w$ where $p'_0 \in P_h \cap C^+ \cap (M + v_h)$ and $w \in \bigoplus \mathbf{Z}^+ f_i$, so $m = p_0 + q_0$ where $q_0 := q'_0 + w$ belongs to $Q_k \cap (M + v_k)$.

Proceeding as in the proposition 9.2, we can define a sequence of pairs of points $\{(p_i, q_i)\}_{i=0,\dots,r}$ with the following properties: 1) $p_i \in Q_h \cap (M + v_h)$ for each i; 2) $q_i \in Q_k \cap (M + v_h)$ for each i; 3) $m = p_i + q_i$ for each i; 4) (p_0, q_0) is as before; 5) $(p_{i+1}, q_{i+1}) = (p_i + f_{j_i}, q_i - f_{j_i})$ for a suitable j_i and 6) $q_r \in P_k$. Indeed we can define the $\{q_i\}$ as in the proposition 9.2 and then we set $p_i = m - q_i$. Now it is sufficient to show by induction that we can choose the indices j_i so that p_i belongs to P_h for each i. We known that $p_0 \in P_h$. Now suppose that p_n belongs to P_h by inductive hypothesis. Suppose that $p_n = \sum x_i f_i = \sum \dot{x}_i g_i$ and $q_n = \sum y_i f_i = \sum \dot{y}_i g_i$. If $q_n \in P_k$ we define $r = n$ and there is nothing to prove. Otherwise there is an index j_n such that $\dot{y}_{j_n} > 0$ and it is sufficient to prove that $p_n + f_{j_n}$ belongs to P_h. Observe that

$-\dot{x}_{i_j} = -(\dot{x}_{i_j} + \dot{y}_{i_j}) + \dot{y}_{i_j} > 0$, so $-\dot{x}_{i_j} \geq 1$ because it is an integer. Moreover $s_{j_n}p_n = p_n - (2 < p_n, f_{j_n} > / < f_{j_n}, f_{j_n} >)f_{j_n} = p_n - \dot{x}_{j_n} f_{j_n}$ belongs to P_h. P_h is convex and it contains the points p_n and $s_{j_n}p_n$, so it contains $p_n + f_{j_n}$. Thus we can choose $p_{n+1} = p_n + f_{j_n}$. $\square$

Remark. 1) The previous theorem is valid with the weaker hypotheses that h, k are convex and that $h|\sigma$, $k|\sigma$ are regular spherical weights for each $\sigma \in \Delta(l)$. Indeed these hypotheses implies that no vertex of P_h (respectively of P_k) is contained in a Weyl wall.

2) Suppose that $h = k$ is convex and that $h|\sigma$ is a regular spherical weights for each $\sigma \in \Delta(l)$. By the corollary 4.2 we have reduced ourselves to study the product of sections of an ample line bundle on a possibly singular toric variety Z' over $\mathbf{A}^l$. This suggests to consider only ample line bundles.

3) It is sufficient to consider the case in which $h|\sigma$ and $k|\sigma$ belongs to the lattice of roots for each $\sigma \in \Delta(l)$. Indeed if f is any spherical convex Δ-linear function, then the weights $f|\sigma$ are all contained in the orbit $v_f + M$ of the lattice Ω with respect to the action of M. For each $f \in SF(\Delta, \Lambda_X)$ let a_f be a positive integer such that $(a_f + 1)v_f$ belongs to the lattice M generated by the restricted roots. Recall that $h|\sigma$ and $k|\sigma$ are regular for each $\sigma \in \Delta(l)$. Then $h' = h + a_h v_h$ and $k' = k + a_k v_k$ are convex spherical Δ-linear function such that $h'|\sigma$ and $k'|\sigma$ are regular weights contained in M for each $\sigma \in \Delta(l)$. Moreover h' (respectively k') is strictly convex on Δ if and only if h (respectively k) is strictly convex on Δ. Because of the previous theorem, the surjectivity of $M_{h,k}$ is equivalent to the surjectivity of $M_{h',k'}$. Indeed the restriction of L_{v_f} to Z is trivial for each f.

4) Because of the previous proposition we can reduce ourselves to consider only completions of $\prod PSL(2)$.

10 Stable subvarieties

In some case we can reduce the study of the product of sections of two ample line bundles L_h and L_k on Z to the study of the product of sections of the restrictions of these line bundles to stable closed subvarieties. Indeed we will prove the following fact. Let s be a global section of L_{h+k} which does not vanish on a divisor Z_τ. If $s|Z_\tau$ is in the image of the product of sections of $L_h|Z_\tau$ and $L_k|Z_\tau$, then s is in the image of the product of sections of L_h and L_k. Before we will prove a proposition of independent importance, namely the surjectivity of the restriction of sections from a smooth complete toric variety Z^c to any closed S-stable subvariety. *In this sections we will allow Z^c to be any smooth complete toric variety, unless we say otherwise.*

Proposition 10.1 *Let Z^c any smooth complete S-toric variety and let L be any ample line bundle on Z^c. Given two cones $\gamma \subset \gamma'$ in Δ^c, the restriction*

$$H^0(Z^c_\gamma, L|Z^c_\gamma) \longrightarrow H^0(Z^c_{\gamma'}, L|Z^c_{\gamma'})$$

is surjective.

Proof. The fundamental part of the proof is the following lemma.

Lemma 10.1 *Let γ be a cone in Δ^c and let L_k be any linearizated ample line bundle on Z^c, then $H^i(Z^c_\gamma, L_{k-\sum_{\tau\in I} d^\tau}|Z^c_\gamma) = 0$ for each $i > 0$ and each subset I of $\{\tau : \ \tau \in \Delta^c(1), \ \tau \not\subset \gamma$ and $\tau + \gamma \in \Delta^c\}$.*

Proof. d^τ is the Δ^c-linear function such that $d^\tau(\tau') = -\delta_{\tau,\tau'}$ for each $\tau' \in \Delta^c(1)$. Observe that the lemma does not depend by the linearization of the line bundle L_k. Let $J_\gamma = \{\tau : \ \tau \in \Delta^c(1), \ \tau \not\subset \gamma$ and $\tau + \gamma \in \Delta^c\}$. Notice that there is an one-to-one correspondence between J_γ and the S-stable divisors of Z^c_γ. If x belongs to the open orbit of Z^c_γ, namely $Z^c_\gamma = \overline{S \cdot x}$, then Z^c_γ is a toric variety with respect to the torus $S' = S/\mathrm{Stab}(x)$. Observe that $\mathrm{Stab}(x)$ does not depend on the choice of x because S is abelian. Let N_γ be the sublattice of N generated by γ, then $N(\gamma) = N/N_\gamma$ is the group of one parameter subgroups of S' and $M \cap \gamma^\perp$ is the character group of S'. Moreover the fan of Z^c_γ in $N(\gamma)$ is composed of the cones $(\sigma + N_\gamma \otimes \mathbf{R})/N_\gamma \otimes \mathbf{R}$ where σ varies in the set of cones in Δ^c which contain γ. We can choose a S-linearization of L_k such that the induced S-linearization of $L_k|Z^c_\gamma$ is compatible with a unique S'-linearization of $L_k|Z^c_\gamma$ through the quotient map $S \mapsto S'$. This is equivalent to choose k that vanishes on γ. Indeed the character $k|\sigma$, with which S acts on any S-fixed point x_σ, is induced by a character of S' if and only if $k|\sigma$ vanishes on γ, namely $k|\sigma \in M \cap \gamma^\perp = \chi^*(S')$. In this case k induces a piecewise linear function on $N(\gamma) \otimes \mathbf{R}$, which is associated to the previous S'-linearization of $L_k|Z^c_\gamma$, so we call it k by abuse of notation. Let r be the dimension of Z^c_γ or, equivalently, the dimension of $N(\gamma) \otimes \mathbf{R}$. Observe that $\dim\,\gamma = l - r$.

We want to show the lemma by decreasing induction on the dimension of γ and on the cardinality $|I|$ of I. If $\dim\,\gamma = l$ then Z^c_γ is a point and the proposition is trivial. In the following we will suppose that $\dim\,\gamma < l$. Let $K = (L_{-\Sigma_{\tau\in J_\gamma} d^\tau})|Z^c_\gamma$ be the canonical bundle of Z^c_γ and let $L' = (L_{k-\Sigma_{\tau\in J_\gamma} d^\tau})|Z^c_\gamma$. By the Serre duality we have $H^i(Z^c_\gamma, L') = 0$ for each $i > 0$ if $H^i(Z^c_\gamma, (L')^{-1} \otimes K) = 0$ for each $i < \dim\,Z^c_\gamma$. $((L')^{-1} \otimes K)^{-1} = L_k|_{Z^c_\gamma}$ is very ample, so the Kodaira vanishing theorem implies that $H^i(Z^c_\gamma, (L')^{-1} \otimes K) = 0$ for each $i < \dim\,Z^c_\gamma$.

Thus we have showed the basis of induction. Finally we can suppose that there is $\varsigma \in J_\gamma - I$, so we have the following short exact sequence

$$0 \longrightarrow L_{k-\Sigma d^\tau - d^\varsigma}|Z^c_\gamma \xrightarrow{s_\varsigma} L_{k-\Sigma d^\tau}|Z^c_\gamma \longrightarrow L_{k-\Sigma d^\tau}|Z^c_{\gamma+\upsilon} \longrightarrow 0$$

where the second map is the restriction and the first one is the product by an invariant section s_ς of L_{d^ς} (observe that $0 \geq d^\varsigma$ over $N_{\mathbf{R}}$, so there is a seminvariant section of weight 0, namely an invariant section). The proposition is true on the first term by induction on $|I|$ and is true on the third one by induction on the dimension of γ, so it is true also on the second term. □

Now we can prove the proposition. By induction we can suppose that $\dim\ Z^c_\gamma = \dim\ Z^c_{\gamma'} + 1$, so there is $\tau \in \Delta^c(1)$ such that $\gamma' = \gamma + \tau$. We choose a linearization associated to a function k. There is the following short exact sequence

$$0 \longrightarrow L_{k-d^{\tau}}|Z^c_{\gamma} \xrightarrow{s_{\tau}} L_k|Z^c_{\gamma} \longrightarrow L_k|Z^c_{\gamma'} \longrightarrow 0$$

The proposition is implied by $H^1(Z^c_{\gamma}, L_{k-d^{\tau}}) = 0$. □

Now we want to prove a corollary about complete symmetric varieties, but we do not need it in the following sections.

Corollary 10.1 *Let L_h be an ample line bundle on Y such that h is spherical. Given two cones $\gamma \subset \gamma'$ in Δ, the restriction*

$$H^0(Y_{\gamma}, L_h|Y_{\gamma}) \longrightarrow H^0(Y_{\gamma'}, L_h|Y_{\gamma'})$$

is surjective.

Proof. It is sufficient to show that all the lowest weight vectors belong to the image. Let $v' \in H^0(Y_{\gamma'}, L_h|Y_{\gamma'})$ be any lowest weight vector and let $-\mu$ be the weight of v'. Let Z^c be the complete toric variety associate to Y. We know that v' does not vanish on $Z^c_{\gamma'}$ because $Y_{\gamma'}$ has a dense $U^- \times T$ orbit. Thus there is a seminvariant section $s \in H^0(Z^c_{\gamma}, L_h|Z^c_{\gamma})$ whose restriction to $Z^c_{\gamma'}$ is $v'|Z^c_{\gamma'}$. Because μ is dominant, there is a lowest weight vector $v \in H^0(Y_{\gamma}, L_h|Y_{\gamma})$ whose restriction to Z^c_{γ} is equal to s. Thus the restriction of v to $Z^c_{\gamma'}$ coincides with the restriction of v' to $Z^c_{\gamma'}$, so the restriction of v to $Y_{\gamma'}$ is v' because we are studying multiplicity-free representations. □

Now we want to prove a proposition that in some case allows ourselves to reduce the study of the product of sections of two lines bundles on Z to the study of the product of sections of the restrictions of the previous lines bundles to a suitable divisor. Before we prove a similar proposition on any complete smooth toric variety Z^c.

Proposition 10.2 *Let Z^c be a smooth complete toric variety and let L_h and L_k be two ample linearizated line bundles on Z^c. Let τ be a cone in $\Delta^c(1)$ and let s be a global section of L_{h+k} which does not vanish on Z^c_{τ}. If $s|Z^c_{\tau}$ belongs to the image of the product m^c_{τ} of sections of the restrictions of L_h and L_k to Z^c_{τ}, then s belongs to the image of the product m^c of sections of L_h and L_k.*

Proof. We can suppose that s is a semi-invariant section because there is a basis of semi-invariant sections. Indeed we can write $s = \sum s_{\nu_i}$ where s_{ν_i} is a semi-invariant section of weight ν_i for each i. Suppose that $s|Z^c_{\tau} = m^c_{\tau}(\sum t_{\lambda_j} \otimes r_{\mu_j})$ where t_{λ_j} is a semi-invariant section of weight λ_j for each j and r_{μ_j} is a semi-invariant section of weight μ_j for each j. Then $s_{\nu_i}|Z^c = m^c_{\tau}(\sum_{j:\lambda_j+\mu_j=\nu_i} t_{\lambda_j} \otimes r_{\mu_j})$, so $s_{\nu_i}|Z^c$ belongs to the image of m^c_{τ} for each i. Moreover, if s_{ν_i} belongs to the image of m^c for each i, then s belongs to the image of m.

Let s be a semi-invariant section of weight μ that does not vanish on Z^c_{τ}, so $\mu(\varrho(\tau)) = (h+k)(\varrho(\tau))$. Suppose that $s|Z^c_{\tau}$ belongs to the image of m^c_{τ}. Because of the previous proposition, there are sections $s'_i \in \Gamma(Z^c, L_h)$ and $s''_i \in \Gamma(Z^c, L_k)$ such that $m^c_\tau(\sum s'_i|Z^c_\tau \otimes s''_i|Z^c_\tau)$ is the restriction of s to Z^c_τ. Thus $m^c_\tau(\sum s'_i \otimes s''_i) = s$ because the space of sections is a multiplicity-free representation. $\square$

Now we can prove the proposition on Z.

Proposition 10.3 *Let Z be a smooth toric variety over $\boldsymbol{A}^l$. Let L_h and L_k be any two ample line bundles on Z. Let $\tau \in \Delta(1)$ and let s be a section of L_{h+k} that does not vanish on Z_τ. If $s|Z_\tau$ belongs to the image of the product $m^\tau_{h,k}$ of sections of $L_h|Z_\tau$ and $L_k|Z_\tau$, then s belongs to the image of the product $m_{h,k}$ of sections of L_h and L_k.*

Proof. We can proceed as in the previous proposition if we show that the restriction map is surjective.

Proposition 10.4 *Let Z be a smooth toric variety over $\boldsymbol{A}^l$ and let L be an ample line bundles on Z. Given any two cones γ, γ' in Δ with $\gamma' \subset \gamma$, then the restriction map*

$$H^0(Z_{\gamma'}, L|Z_{\gamma'}) \to H^0(Z_\gamma, L|Z_\gamma)$$

is surjective.

Proof. It is clearly sufficient to consider the case in which $\gamma' = \{0\}$, i.e. $Z = Z_{\gamma'}$. We want to use the proposition 10.1, so we will define a completion Z^c of Z and an ample line bundle on Z^c whose restriction to Z is L. We can think $\mathbf{A}^l$ as an open subvariety of $\prod_{i=1}^l \mathbf{P}^1$. We define a scalar product on $N_{\mathbf{R}}$ such that $\{e_1, ..., e_l\}$ is an orthonormal basis. Let W^1 be the group generated by the reflections with respect to the coordinate hyperplanes; it is isomorphic to $\prod_{i=1}^l \mathbf{Z}/2\mathbf{Z}$. The fan Δ_0^c of $\prod_{i=1}^l \mathbf{P}^1$ is invariant by W^1 and we can suppose that the fan Δ_0 of $\mathbf{A}^l$ is the intersection $\Delta_0^c \cap \sigma(e_1, ..., e_l) := \{\sigma \in \Delta_0^c : \sigma \subset \sigma(e_1, ..., e_l)\}$. Let Z^c be the toric variety over $\prod_{i=1}^l \mathbf{P}^1$ whose fan Δ^c is $W^1 \cdot \Delta = \{w \cdot \gamma : \ w \in W^1, \ \gamma \in \Delta\}$ (here Δ is the fan of Z). Notice that W^1 acts on M by duality.

Let h be the Δ-linear function associated to a linearization of L and let h^c be the Δ^c linear function defined as follows: $h^c(w \cdot v) = h(v)$ for each $w \in W^1$ and $v \in \sigma(e_1, ..., e_l)$. Because of the lemma 7.4 we can choose h such that the line bundle L_{h^c} is ample on Z^c.

We need a lemma that relates the sections over Z_γ with the sections over Z^c_γ. Before we will introduce some notations and we will do some observations on the fans corresponding respectively to Z^c_γ and Z_γ. Let $\Delta^c(\tau)$ (respectively $\Delta(\gamma)$) be the set of cones in Δ^c (respectively in Δ) which contain γ and let $\Delta^c(n)(\gamma)$ (respectively $\Delta(n)(\gamma)$) be the set of n-dimensional cones in Δ^c (respectively in Δ) which contain γ. Write $\gamma = \sum_{i=1}^m \mathbf{Z}^+ \varrho(\tau_i)$, where the τ_i are opportune cones in $\Delta(1)$.

Z_γ and Z^c_γ are toric varieties with respect to the torus S' associated to $\bigoplus_{i=1}^l \mathbf{Z}e_i / \sum_{i=1}^m \mathbf{Z}\varrho(\tau_i)$. The fan of Z_γ is $\{\sigma + \mathbf{R}\gamma/\mathbf{R}\gamma : \sigma \in \Delta(\tau)\}$ and the fan of Z^c_τ is $\{\sigma + \mathbf{R}\gamma/\mathbf{R}\gamma : \sigma \in \Delta^c(\tau)\}$. Up to reordering the indices we can suppose that γ is contained in $\sigma(e_{r+1},...,e_l)$, but it is not contained in any face of $\sigma(e_{r+1},...,e_l)$. Thus, for each i, we can write $\varrho(\tau_i) = \sum_{j=r+1}^{l} a_i^j e_j$ where the a_i^j are positive integers. For each i let $\widetilde{e}_i$ be the class of e_i modulo $\mathbf{R}\gamma$. Up to exchange the indices, we can suppose that $\{\widetilde{e}_1,...,\widetilde{e}_{l-m}\}$ is a basis of $\bigoplus_{i=1}^{l} \mathbf{R}e_i/\mathbf{R}\gamma$ and that the support of the fan corresponding to Z_τ is $\sigma(\widetilde{e}_1,...,\widetilde{e}_{l-m}, -\widetilde{e}_{r+1},...,-\widetilde{e}_{l-m})$.

For each j let $s_j \in W^1$ be the orthogonal reflection corresponding to e_j. Let $\widetilde{W}^1$ be the subgroup of W^1 generated by $s_1,...,s_r$. $\widetilde{W}^1$ fixes $\varrho(\tau_i)$ for each i, so it acts on $\bigoplus_{i=1}^{l} \mathbf{Z}e_i/\sum_{i=1}^{m} \mathbf{Z}\varrho(\tau_i)$. We have $\Delta(\gamma) = \Delta^c(\gamma) \cap \sigma(\widetilde{e}_1,...,\widetilde{e}_{l-m}, -\widetilde{e}_{r+1}, ..., -\widetilde{e}_{l-m}) := \{\sigma \in \Delta^c(\gamma) : \sigma + \mathbf{R}\gamma/\mathbf{R}\gamma \subset \sigma(\widetilde{e}_1,...,\widetilde{e}_{l-m}, -\widetilde{e}_{r+1},...,-\widetilde{e}_{l-m})\}$ and we will prove that $\Delta^c(\gamma) = \widetilde{W}^1 \cdot \Delta(\gamma)$. Indeed for each σ in $\Delta^c(l)(\gamma)$ there is $w \in W^1$ such that $w \cdot \sigma$ belong to $\Delta(l)(\gamma)$. Moreover $s_j\sigma$ contains $s_j\varrho(\tau_i) = \varrho(\tau_i)$ for each $j \leq r$ and for each i; instead, given $w \in W^1 - \mathrm{span}_{W^1}(s_1,...,s_r)$, there is i such that $w\varrho(\tau_i)$ is not contained in $\sigma(e_1,...,e_r)$, so $w \cdot \sigma$ does not contain $\varrho(\tau_i)$ because Z^c dominates $\prod_{i=}^{l} \mathbf{P}^1$.

One can show that $H^0(Z_\gamma, L_h|Z_\gamma)$ has a basis of semi-invariant sections for the action of S. The weights of such sections are opposite to the elements of $\Pi(Z_\gamma, h) := \{m \in \bigcap_{\sigma \in \Delta(l)(\gamma)} h|\sigma + M \cap \gamma^\perp \cap \check{\sigma}\}$. Also $H^0(Z_\gamma^c, L_{h^c}|Z_\gamma^c)$ has a basis of semi-invariant sections for the action of S and the weights of such sections are opposite to the elements of $\Pi(Z_\gamma^c, h) := \{m \in \bigcap_{\sigma \in \Delta^c(l)(\gamma)} h|\sigma + M \cap \gamma^\perp \cap \check{\sigma}\}$.

Up to change h^c by an element of $\bigoplus_{i=r+1}^{l} \mathbf{Z}f_i$, we can suppose that $h^c(\varrho(\tau_i)) = 0$ for each i, so that there is an action of S' on $L_{h^c}|Z_\tau^c$ compatible with the action of S through the quotient map. In this case h^c induces a piecewise linear function $\widetilde{h}^c$ on $(\bigoplus \mathbf{R}e_i)/\mathbf{R}\gamma$ and h induces a piecewise linear function $\widetilde{h}$ on $\sigma(\widetilde{e}_1,...,\widetilde{e}_{l-m}, -\widetilde{e}_{r+1},...,-\widetilde{e}_{l-m})$. h^c is $\widetilde{W}$ invariant, so $\tilde{h}^c$ is $\widetilde{W}$ invariant. Observe that now h^c may be not W^1 invariant. Moreover the restriction of $\tilde{h}^c$ to $\sigma(\widetilde{e}_1,...,\widetilde{e}_{l-m}, -\widetilde{e}_{e+r},...,-\widetilde{e}_{l-m})$ is equal to $\widetilde{h}$. Let $\widetilde{Q}_h$ be the polyhedron in $(M \cap \gamma^\perp) \otimes \mathbf{R}$ corresponding to $\widetilde{h}$ and let $\widetilde{P}_h$ be the polytope in $(M \cap \gamma^\perp) \otimes \mathbf{R}$ corresponding to $\widetilde{h}^c$. Notice that $\Pi(Z_\gamma, h) = \widetilde{Q}_h \cap M \cap \gamma^\perp$ and $\Pi(Z_\gamma^c, h) = \widetilde{P}_h \cap M \cap \gamma^\perp$.

We now want to prove some relations like the ones stated in the propositions 9.1 and 9.2. The proof will be much simpler because $\{e_1,...,e_l\}$ is an orthonormal basis. For each j in $\{r+1,...,l-m\}$ there is $\tilde{f}_j$ in $f_j + \sigma(-f_{l-m+1},...,-f_l)$ such that $\{f_1,...,f_r, \tilde{f}_{r+1},...,\tilde{f}_{l-m}\}$ is a basis of $(M \cap \gamma^\perp) \otimes \mathbf{R}$. Let $\widetilde{C}^+ = \sigma(-f_1,...,-f_r, -\tilde{f}_{r+1},..,-\tilde{f}_{l-m}, \tilde{f}_{r+1},...,\tilde{f}_{l-m})$.

Lemma 10.2 *1.* $\widetilde{Q}_h = \widetilde{P}_h \cap \widetilde{C}^+ + \sigma(f_1,...,f_r)$;

2. $\Pi(Z_\gamma, h) = \Pi(Z_\gamma^c, h) \cap \widetilde{C}^+ + \bigoplus_{i=1}^{r} \boldsymbol{Z}^+ f_i$.

Proof. Notice that, if r is equal to 0, then Z_τ is equal to Z_τ^c and there is nothing to prove. Thus we can suppose that $r \geq 1$.

For each $j \leq r$ let H_j be the hyperplane of $(M \cap \gamma^\perp) \otimes \mathbf{R}$ generated by $f_1,...,\widehat{f_j},...,f_r, \tilde{f}_{r+1},...,\tilde{f}_{l-m}$ and notice that the border of $\widetilde{C}^+$ is the union of the H_j. s_j acts on $(M \cap \gamma^\perp) \otimes \mathbf{R}$ as the orthogonal reflection with respect to H_j. Observe that, if $\widetilde{P}_h$ contains a point p, then it contains all the translates of p by $\widetilde{W}^1$ because $\tilde{h}^c$ is $\widetilde{W}^1$ invariant. Moreover, given any point $p \in \widetilde{P}_h$, $\widetilde{P}_h$ contains the orthogonal projection $\frac{1}{2}(p + s_j p)$ of p to H_j for each j. Since h^c is strictly convex on Δ^c, there is no vertex of $\widetilde{P}_h$ contained in H_j. Indeed, given a vertex $\widetilde{h}^c|\sigma$ of $\widetilde{P}_h$, then $s_j \cdot \widetilde{h}^c|\sigma$ is different from $\widetilde{h}^c|\sigma$.

The equations of $\widetilde{Q}_h$ are of the form $\sum_{i=1}^{l-m} d_i x_i \geq d$ where the d_i with $i \leq r$ are positive constants. So, given any $m \in \widetilde{Q}_h$, we have $m + \sigma(f_1, ..., f_r) \subset \widetilde{Q}_h$, i.e. $\widetilde{Q}_h$ is stable by translation with respect to vectors in $\sigma(f_1, ..., f_r)$.

We now show that $\widetilde{P}_h \cap \widetilde{C}^+ = \widetilde{Q}_h \cap \widetilde{C}^+$. It is evident that $\widetilde{P}_h \cap \widetilde{C}^+ \subseteq \widetilde{Q}_h \cap \widetilde{C}^+$, so it is sufficient to show that $\widetilde{Q}_h \cap \widetilde{C}^+ \subseteq \widetilde{P}_h$. As a matter of fact it is sufficient to show that $\widetilde{Q}_h \cap \widetilde{C}^+ \cap (M \cap \gamma^\perp) \otimes \mathbf{Q} \subseteq \widetilde{P}_h$ because $\widetilde{P}_h$ is closed. The semi-spaces defining $\widetilde{P}_h$ are $\{p \in (M \cap \gamma^\perp) \otimes \mathbf{R} : p(\varrho(\tau) + \mathbf{R}\gamma) \geq \widetilde{h}^c(\varrho(\tau) + \mathbf{R}\gamma)\}$ for each $\tau \in \Delta^c(\gamma)(1)$. Thus, given any $p \in \widetilde{Q}_h \cap \widetilde{C}^+ \cap (M \cap \gamma^\perp)_{\mathbf{Q}}$ and any $\tau \in \Delta^c(\gamma)(1)$, we have to show that $p(\varrho(\tau) + \mathbf{R}\gamma) \geq \widetilde{h}^c(\varrho(\tau) + \mathbf{R}\gamma)$. Because of the symmetry of $\Delta^c(\gamma)$, there are $w \in \widetilde{W}^1$ and $\tau' \in \Delta(1)(\gamma)$ such that $\varrho(\tau) = w \cdot \varrho(\tau')$. Observe that $w \cdot p - p$ belongs to $\sigma(f_1, ..., f_r)$. Write $w \cdot p - p = \sum c_i f_i$, so we have $p(\varrho(\tau) + \mathbf{R}\gamma) = p(w \cdot \varrho(\tau') + \mathbf{R}\gamma) = (w \cdot p)(\varrho(\tau') + \mathbf{R}\gamma) = p(\varrho(\tau') + \mathbf{R}\gamma) + \sum c_i f_i(\varrho(\tau') + \mathbf{R}\gamma) \geq p(\varrho(\tau') + \mathbf{R}\gamma) \geq \tilde{h}(\varrho(\tau') + \mathbf{R}\gamma) = \widetilde{h}^c(\varrho(\tau) + \mathbf{R}\gamma)$.

We can now show the first point of the lemma. The decomposition in cones of $(N(\gamma))_{\mathbf{R}}$ associated to $h_{\widetilde{P}_h \cap \widetilde{C}^+}$ has 1-dimensional cones $\{\sigma(-\tilde{e}_1), ..., \sigma(-\tilde{e}_r)\} \cup \Delta(1)(\gamma)$. $h_{\widetilde{P}_h \cap \widetilde{C}^+}$ has finite values on all $(N(\gamma))_{\mathbf{R}}$, it is equal to $\widetilde{h}$ on $\sigma(\tilde{e}_1, ..., \tilde{e}_{l-m}, -\tilde{e}_{r+1}, ..., -\tilde{e}_{l-m})$ and vanishes on the vectors $-\widetilde{e}_1, ..., -\widetilde{e}_r$. The function associated to $\sigma(f_1, ..., f_r)$ vanishes on $\sigma(\tilde{e}_1, ..., \tilde{e}_{l-1}, -\tilde{e}_{r+1}, ..., -\tilde{e}_{l-1})$ and has value $-\infty$ on the complementary set. Thus their sum is the function associated to $\widetilde{Q}_h$, so the claim follows by the theorem 4.2.

Now, we can conclude the proof of the lemma. Let p be a point contained in $\Pi(Z_\gamma, h)$. We know that $p = p' + \sum_{i=1}^r a_i f_i$ where $p' \in \widetilde{P}_h \cap \widetilde{C}^+$ and the a_i are positive constants, but the a_i may be not integers. Now it is sufficient to observe that $p' + \sum_{i=1}^r (a_i - [a_i]) f_i$ belongs to $\Pi(Z_\gamma^c, h) \cap \widetilde{C}^+$ (here $[a_i]$ is the integral part of a_i). Indeed, for each $j \leq r$, the j-th coordinate of p' is negative, $(a_j - [a_j]) < 1$ and $p' + \sum_{i=1}^r (a_i - [a_i]) f_i = p - \sum_{i=1}^r [a_i] f_i$ is a rational point. $\square$

Now we can conclude the proof of the propositions 10.4 and thus also the proof of the proposition 10.3. Clearly it is sufficient to show that the image of the restriction contains all the semi-invariant sections. Let s be any semi-invariant section of L on Z_γ and let p be its weight; we can write $p = p' + \sum_{i=1}^r a_i f_i$ where p' is the weight of a section s' of L on Z_γ^c and the a_i are positive integers. Observe that $p(\varrho(\tau_i) + \mathbf{R}\gamma) = \tilde{h}(\varrho(\tau_i) + \mathbf{R}\gamma) = h(\varrho(\tau_i))$ for each i, so $p'(\varrho(\tau_i) + \mathbf{R}\gamma) = \tilde{h}(\varrho(\tau_i) + \mathbf{R}\gamma) = h(\varrho(\tau_i))$ for each i. There is a section s'' on Z^c whose restriction on Z_τ^c is s' by the proposition 10.1. Thus $p' \in Q_{h^c} \cap M$, so $p \in Q_h \cap M$ and there is a semi-invariant section φ of L_h on Z with weight p. This section does not vanish on Z_τ because $p(\varrho(\tau)) = h(\varrho(\tau))$, so $\varphi|Z_\tau$ is a not zero multiple of

s. $\square$

11 Line bundles on an exceptional complete symmetric variety

Let Y be an exceptional complete symmetric variety, let Z be the associated open toric variety and let Δ be the fan of Z. Let h be a spherical strictly convex (Δ, Λ_X)-linear function such that $h|\sigma$ is regular for each $\sigma \in \Delta(l)$. We know that the multiplication $M_{h,h}$ of sections on Y is surjective if and only if the multiplication $m_{h,h}$ of sections on Z is surjective. In this section we want to generalize this fact to the h which are not spherical.

Remember that $Pic(X)$ is generated by the spherical weights and by the fundamental weights $\omega_{\alpha_1}, ..., \omega_{\alpha_s}$ corresponding to the exceptional roots $\alpha_1, ..., \alpha_s$.

Proposition 11.1 *Let $L_{h'}$ be an ample line bundle on Y such that $M_{h',h'}$ is surjective and let $a_1, ..., a_l$ be positive integers. If we define $h = h' + \sum a_i \omega_{\alpha_i}$ then the product $M_{h,h}$ of sections of L_h over Y is surjective.*

Proof. Observe that L_h is an ample bundle on Y. We will prove the proposition by induction on $\sum a_i$. $M_{h,h}$ is trivially surjective if $\sum a_i = 0$. We need a lemma on the maps $M_{h,\omega_{\alpha_i}}$.

Lemma 11.1 *Let L_h be an ample line bundle on Y and let $\omega \in \{\omega_{\alpha_1}, ..., \omega_{\alpha_s}\}$. Then $M_{h,\omega}$ is surjective.*

Proof. In the following V^*_λ is the unique subrepresentation of $H^0(Y, L_\lambda)$ which contains a lowest weight vector v_λ of weight $-\lambda$. We have $H^0(Y, L_h) = \bigoplus_{\lambda \in \Pi(Y,h)} s^{h-\lambda} V^*_\lambda$, $H^0(Y, L_{h+\omega}) = \bigoplus_{\mu \in \Pi(Y,h+\omega)} s^{h+\omega-\mu} V^*_\mu = \bigoplus_{\lambda \in \Pi(Y,h)} s^{h-\lambda} V^*_{\omega+\lambda}$ and $H^0(Y, L_\omega) = V^*_\omega$. The last equality is implied by the fact that ω is a minuscule weight i.e. it is non zero and there is no dominant weight λ such $\omega - \lambda \in \Lambda^+$ (see lemma 4.3 in [CS], proposition 1.12 in [S], pages 532 and following ones in [He]). The lemma is implied by the fact that, for each $\lambda \in \Pi(Y, h)$, $M_{h,\omega}(s^{h-\lambda} v_\lambda \otimes v_\omega)$ is a lowest weight vector of weight $-\lambda - \omega$. $\square$

We now go back to the proposition. Let j be an index such that $a_j > 0$ and define $\widetilde{h} = h - \omega_j$. We have the following commutative diagram

$$\begin{tikzcd}
H^0(Y, L_{\widetilde{h}}) \otimes H^0(Y, L_{\widetilde{h}}) \otimes H^0(Y, L_{\omega_j}) \otimes H^0(Y, L_{\omega_j}) \arrow[r] \arrow[d, "m_1"] & H^0(Y, L_h) \otimes H^0(Y, L_h) \arrow[dd, "M_{h,h}"] \\
H^0(Y, L_{2\widetilde{h}}) \otimes H^0(Y, L_{\omega_j}) \otimes H^0(Y, L_{\omega_j}) \arrow[d, "m_2"] & \\
H^0(Y, L_{2\widetilde{h}+\omega_j}) \otimes H^0(Y, L_{\omega_j}) \arrow[r, "M_{2\widetilde{h}+w_j, w_j}"] & H^0(Y, L_{2h}).
\end{tikzcd}$$

m_1 is surjective by induction, m_2 and $M_{2\widetilde{h}+w_j,w_j}$ are surjective because of the previous lemma, so $M_{h,h}$ is surjective. $\square$

Theorem 11.1 *Let L_h be an ample line bundle on Y. If $m_{h,h}$ is surjective then $M_{h,h}$ is surjective.*

Proof. We know that, up to exchange α_i with $\overline{\theta}(\alpha_i)$ for some i in $\{1,...,l\}$, there are positive integers $a_1,...,a_l$ such that $h' = h - \sum a_i w_i$ is a spherical piecewise linear function and $L_{h'}$ is ample. The restriction of L_h to Z is isomorphic to the restriction of $L_{h'}$ to Z, so $m_{h',h'}$ is surjective. Thus $M_{h',h'}$ is surjective because of the theorem 9.1. Hence $M_{h,h}$ is surjective by the previous proposition. $\square$

12 Open projectively normal toric varieties

Now we want to describe some families of open toric varieties such that, if L_h is an ample line bundle on a such variety, then the product $m_{h,h}$ of sections is surjective. One family is formed by all the varieties of dimension 2 proper over $\mathbf{A}^2$. Moreover we will find an infinite number of varieties that have such property for every given dimension. In some cases we will prove that, given any two ample line bundles L_h and L_k on a fixed variety, then the product $m_{h,k}$ is surjective. *In the following we will identify M with $\boldsymbol{Z}^l$.*

12.1 Blow-ups of $\mathbf{A}^l$

Now we study the class of varieties that are blow-ups of $\mathbf{A}^l$ along a stable closed subvariety. This is the unique case in which we will prove that given any two line bundles L_h and L_k generated by global sections then the product of sections is surjective.

Proposition 12.1 *Let Z be the blow-up of $\boldsymbol{A}^l$ along the stable closed subvariety associated to $\sigma(e_1,...,e_r)$. Let L_h and L_k be two line bundles generated by global sections on Z, then the product of sections $m_{h,k}$ is surjective.*

The inequalities for Q_h are $z_i \geq a_i$ for each $i = 1,..,l$ and $z_1+...+z_r \geq b$. The inequalities for Q_k are $z_i \geq c_i$ for each $i = 1,..,l$ and $z_1+...+z_r \geq d$. Here a_i, b, c_i and d are suitable integers. Let m be any point in $Q_{h+k} \cap M$, then there are $\tilde{m}_1 \in Q_h$ and $\tilde{m}_2 \in Q_k$ such that $\tilde{m}_1 + \tilde{m}_2 = m$, but they may have not integral coordinates. We want to move $\tilde{m}_1$ and $\tilde{m}_2$ a little, so that we will obtain two points with integral coordinates that belong respectively to Q_h and to Q_k. More precisely we will move $\tilde{m}_1$ by a vector v whose coordinates have values between -1 and 1, so we will have to move $\tilde{m}_2$ by the vector $-v$ whose coordinates have again values between -1 and 1. If $\tilde{m}_1 = (x_1,...,x_l)$ then $x_i \geq a_i$. Let $[x_i]$ be the integral part of x_i and let $\epsilon_i = -[([x_i] - x_i)]$ (ϵ_i is 0 if x_i is an integer and it is 1 otherwise). $[x_i] + \epsilon_i \geq [x_i] \geq a_i$ because the a_i are integers. Likewise, if $\tilde{m}_2 = (y_1,...,y_l)$ then $[y_i] + \epsilon_i \geq [y_i] \geq d_i$ (observe that $\epsilon_i = -[([y_i] - y_i)]$).

If $([y_1], ..., [y_l])$ belongs to Q_k, then we define $m_1 = ([x_1] + \epsilon_1, ..., [x_l] + \epsilon_l)$ and $m_2 = ([y_1], ..., [y_l])$. Clearly these points satisfy our requests. In the same way, if $([x_1], ..., [x_l])$ belongs to Q_h, then we define $m_1 = ([x_1], ..., [x_l])$ and $m_2 = ([y_1] + \epsilon_1, ..., [y_l] + \epsilon_l)$. Thus we can suppose that $\sum_{i=1}^{r}[x_i] \leq b$ and $\sum_{i=1}^{r}[y_i] \leq d$. We define $m_1 = ([x_1] + \epsilon_1, ..., [x_s] + \epsilon_s, [x_{s+1}], ..., [x_l])$ for an index s lesser than r and such that $b = h(e_1 + ... + e_r) = \sum_{i=1}^{r}[x_i] + \sum_{i=1}^{s}\epsilon_i = m_1(e_1 + ... + e_r)$. There is a such s because $\sum_{i=1}^{r}[x_i] - b$ is a negative integer, $\sum_{i=1}^{r}([x_i] + \epsilon_i) \geq b + d - \sum_{i=1}^{r}([y_i]) \geq b$ and $\epsilon_i \in \{0, 1\}$ for each i. Moreover we define $m_2 = m - m_1 = ([y_1], ..., [y_s], [y_{s+1}] + \varepsilon_{s+1}, ..., [y_l] + \varepsilon_l)$. To verify that $m_2 \in Q_k$ it is sufficient to show that $\sum_{i=1}^{s}[y_i] + \sum_{i=s+1}^{r}([y_i] + \epsilon_i) \geq d = k(e_1 + ... + e_r)$. This is implied by the inequality $m_2(e_1 + ... + e_r) = (m - m_1)(e_1 + ... + e_r) \geq (h + k)(e_1 + ... + e_r) - h(e_1 + ... + e_r)$. $\square$

Now we study a similar family of varieties, but we require that the two line bundles L_h and L_k are equal.

Corollary 12.1 *Let Z be the open toric variety obtained from $\boldsymbol{A}^l$ through the sequence of blow-ups along the subvarieties associated respectively to $\sigma(e_1, e_2)$, $\sigma(e_2, e_3)$,..., $\sigma(e_{r-1}, e_r)$. Let L_h be any line bundle generated by global sections on Z, then the product of sections $m_{h,h}$ is surjective.*

Proof. The inequalities for Q_h are: $z_i \geq a_i$ for each $i = 1, .., l$ and $z_{i-1} + z_i \geq b_i$ for each $i = 2, ..., r$, where the a_i and the b_i are suitable integers. Let $m = (x_1, ..., x_l) \in Q_{2h} \cap M = 2Q_h \cap M$. Observe that $m' = (x_1/2, ..., x_l/2) \in Q_h$ and $m' + m' = m$. We define $\epsilon_i = -[([x_i/2] - x_i/2)]$, $m_1 = ([x_1/2] + \epsilon_1, [x_2/2], [x_3/2] + \epsilon_3, ..., [x_s/2] + \epsilon_s, [x_{s+1}/2], ..., [x_l/2])$ and $m_2 = m - m_1$ for a suitable s. If r is odd then we choose $s = r$, otherwise we define $s = r - 1$. These points belong to $Q_h \cap M$ because $[x_{i-1}/2] + [x_i/2] + (\epsilon_{i-1} + \epsilon_i)/2 \geq b_i$ for each i. $\square$

12.2 Open toric varieties of dimension 2 and a singular family in dimension 3

Now we consider the family of smooth toric varieties proper over $\mathbf{A}^2$.

Theorem 12.1 *Let Z be any smooth toric variety proper over $\boldsymbol{A}^2$. Let L_{h_1} and L_{h_2} be two linearized line bundles generated by global sections and suppose that h_1 and h_2 are strictly convex on the same fan, then the product of sections m_{h_1,h_2} is surjective.*

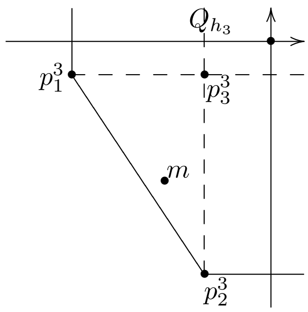
The hypotheses mean that there is a variety Z' and two ample line bundle L'_h and L'_k over Z' such that L_h is the pullback of L'_h and L_k is the pullback of L'_k. We want to remark that Z' may be singular.

Proof. Let $h_3 = h_1 + h_2$ and let Δ be the fan of Z. It is obviously sufficient to prove that the image of m_{h_1,h_2} contains a basis of semi-invariant sections, so it is sufficient to prove that

$$Q_{h_3} \cap M = Q_{h_1} \cap M + Q_{h_2} \cap M.$$

We want to decompose each Q_{h_i} in more simple polyhedrons. More precisely we will decompose each Q_{h_i} in two types of polyhedrons with vertices in M: 1) cones of form $p+\sigma(f_1, f_2)$ for a suitable point p and 2) triangles. These triangles will have a very particular form, indeed we require that the fan associated to any such triangle $\widetilde{T}$ has 1-dimensional cones generated respectively by $-v_1, -v_2$ and $v_1 + v_2$. Moreover we require that $\{v_1, v_2\}$ is a basis of M and that v_1 and v_2 are contained in $\sigma(e_1, e_2)$. This mean that $\widetilde{T}$ is a rectangular isosceles triangle with respect to the scalar product for which $\{v_1, v_2\}$ is an orthonormal basis. Observe that in general the other triangles are not rectangular isosceles triangles with respect to this scalar product.

Let $m = (x_1, x_2)$ be any point in $Q_{h_3} \cap M$. If there is a vertex p^3 of Q_{h_3} whose coordinates are both lesser than the corresponding coordinates of m then m is contained in the polyhedron $p^3 + \sigma(f_1, f_2)$, so we are reduce ourselves to study two polyhedrons associated to the pullbacks of two line bundles on $\mathbf{A}^2$. Indeed there is a maximal cone $\sigma \in \Delta$ such that $p^3 = h_3|\sigma$, so $p^3 = h_1|\sigma + h_2|\sigma$ where $h_1|\sigma$ is a vertex of Q_{h_1} and $h_2|\sigma$ is a vertex of Q_{h_2}. Thus $p^3 + \sigma(f_1, f_2) = (h_1|\sigma + \sigma(f_1, f_2)) + (h_2|\sigma + \sigma(f_1, f_2))$ where $h_j|\sigma + \sigma(f_1, f_2)$ is the polyhedron associated to the linearized line bundle $L_{h_j|\sigma}$. Observe that $L_{h_j|\sigma}$ is the pullback of a linearized line bundle on $\mathbf{A}^2$ because $h_j|\sigma$ is linear.



Otherwise for each j there are vertices p_1^j, p_2^j of Q_{h_j} with the following properties. Write $p_i^j = (z_1^{i,j}, z_2^{i,j})$ for each i and j and define $p_3^j = (y_1^{2,j}, y_2^{1,j})$ for each j. m belongs to the triangle T^3 with vertices p_1^3, p_2^3 and p_3^3. Moreover $T^3 = T^1 + T^2$ where T^j is the triangle with vertices p_1^j, p_2^j and p_3^j for each j. Indeed we can define p_1^3, p_2^3 as the two vertices $(z_1^{1,3}, z_2^{1,3})$, $(z_1^{2,3}, z_2^{2,3})$ of a side of Q_{h_3} such that $z_1^{1,3} \leq x_1 \leq z_1^{2,3}$, so $x_2 \leq z_2^{1,3}$ because otherwise m belongs to $(z_1^{1,3}, z_2^{1,3}) + \sigma(f_1, f_2)$. If $p_1^3 = h_3|\sigma_1$ and $p_2^3 = h_3|\sigma_2$ then we set $p_1^j = h_j|\sigma_1$ and $p_2^j = h_j|\sigma_2$ for each j. Observe that p_3^j belongs to Q_{h_j} for each j, so T^j is contained in Q_{h_j} for each j. Moreover the fans associated to these triangles are equal to the fan with 1-dimensional cones $\sigma(-e_1), \sigma(-e_1)$ and $\sigma(a_1e_1 + a_2e_2)$ for suitable integers a_1 and a_2. This means that, given i and j in $\{1, 2, 3\}$, for each side of T^i there is a side of T^j parallel to the previous one. We remark that this last fact is true because we have supposed that h_1 and h_2 are strictly convex on the same fan. (In the pictures we consider the case in which $h_1 = h_2$).

We need the following easy consequence of the proposition 12.1. Define a scalar product (,) such that $\{f_1, f_2\}$ is a orthonormal basis. In this proof, when we will say that a side L of a polytope P is orthogonal to a vector v, we will always suppose that $(p, v) \geq 0$ for each $p \in P$ (and $(x, v) = 0$ for each $x \in L$). Notice that a plane H is the locus of zeroes of $x_1 e_1 + x_2 e_2 \in N$ if and only if it is orthogonal to $x_1 f_1 + x_2 f_2$.

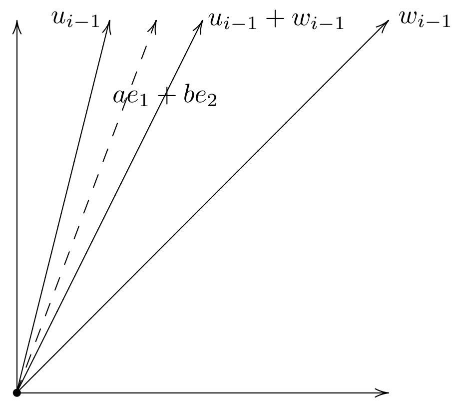
Lemma 12.1 *Let T^1 and T^2 be two triangles with sides orthogonal respectively to $-v_1$, $-v_2$ and $v_1 + v_2$. If $\{v_1, v_2\}$ is a bases of M then*

$$(T^1 + \sigma(v_1, v_2)) \cap M + (T^2 + \sigma(v_1, v_2)) \cap M = (T^1 + T^2 + \sigma(v_1, v_2)) \cap M$$

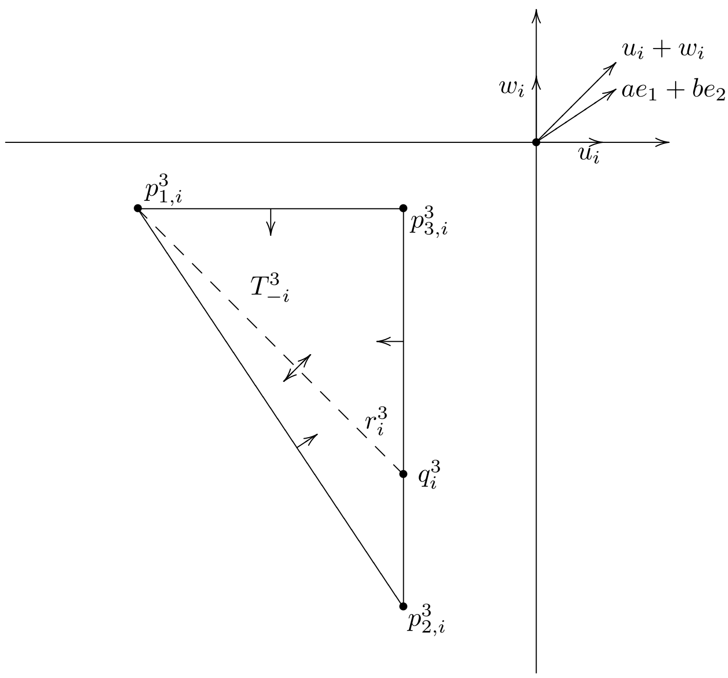
Proof. It is sufficient to observe that, for each j, $T^j + \sigma(v_1, v_2)$ is the polyhedron associated to a linearized ample line bundle on the blow-up of $\mathbf{A}^2$ in the stable point. (Here we think $\mathbf{A}^2$ as the toric variety associated to $\sigma(v_1, v_2)$). $\Box$

In general, we want decompose the triangles T^j in triangles that satisfy the hypothesis of the lemma. Moreover we require that, if T and $\sigma(v_1, v_2)$ are as in the lemma, then $\sigma(v_1, v_2)$ is contained in $\sigma(f_1, f_2)$. Thus if $T \subset T^j$ then $T + \sigma(v_1, v_2)$ is contained in $T^j + \sigma(f_1, f_2)$. Notice that $T^j + \sigma(f_1, f_2)$ is contained in Q_{h_j}. Thus it is sufficient to define such decompositions.

We will define a sequence of open toric varieties $Z_r \to Z_{r-1} \to \ldots \to Z_0$ such that: 1) they are toric varieties with respect to the torus $\mathrm{Spec}\mathbf{C}[N]$, 2) Z_i is the blow-up of Z_{i-1} in a stable point, 3) $Z_0 = \mathbf{A}^2$ and 4) Z_r dominates the toric variety whose fan is $\{\sigma(f_1, a_1 f_1 + a_2 f_2), \sigma(f_2, a_1 f_1 + a_2 f_2), \sigma(f_1), \sigma(f_2), \sigma(a_1 f_1 + a_2 f_2)\}$. We need these varieties only to define some triangles, but we are not interested to study line bundles on such varieties. Let Δ_i be the fan of Z_i and suppose that we have already defined Z_{i-1}. We can assume that there is an unique maximal cone $\sigma_{i-1} \in \Delta_{i-1}$ which contains $a_1 f_1 + a_2 f_2$ (otherwise we define $r = i - 1$). Let Z_i be the blow-up of Z_{i-1} in the stable point associated to σ_{i-1}. For each i let u_{i-1} and w_{i-1} be the two primitive vectors that generated σ_{i-1}. If $a_1 f_1 + a_2 f_2 = a_{i-1} u_{i-1} + b_{i-1} w_{i-1}$, then we claim that $a_i \leq a_{i-1}$, $b_i \leq b_{i-1}$ and $0 < a_i + b_i < a_{i-1} + b_{i-1}$. Hence the process has to stop in a finite number of steps and $a_r + b_r = 1$. Now we prove the claim. We can suppose,up to exchange u_i and w_i, that $a_1 f_1 + a_2 f_2 \in \sigma(u_{i-1}, u_{i-1} + w_{i-1})$. We define $u_i = u_{i-1}$ and $w_i = u_{i-1} + w_{i-1}$, so $a_i = a_{i-1} - b_{i-1} < a_{i-1}$ and $b_i = b_{i-1}$. Observe that $a_i \geq 0$ because $a_1 f_1 + a_2 f_2 \in \sigma(u_{i-1}, u_{i-1} + w_{i-1})$.



We want to decompose the triangles T^j in r triangles T_i^j with integral vertices. T_i^j will have sides orthogonal respectively to $-u_{i-1}$, $-w_{i-1}$ and $u_{i-1}+w_{i-1}$. For each j we define $T_0^j = T^j$. We define recursively T_{-i-1}^j as the set $T_{-i}^j - T_{i+1}^j$. We will prove inductively that T_{-i}^j is a triangle with sides orthogonal respectively to $-u_i$, $-w_i$ and $a_1f_1+a_2f_2$. For each j and for each $i<r-2$ we decompose T_{-i}^j in the two triangles T_{i+1}^j and T_{-i-1}^j. Moreover, for each j we decompose T_{-r+2}^j in the two triangles T_{r-1}^j and T_r^j. We want that $T_i^3 = T_i^1 + T_i^2$ for each i. Moreover T_i^1, T_i^2 and T_i^3 will be associated to the same fan for each i. Let $p_{1,i}^j$, $p_{2,i}^j$ and $p_{3,i}^j$ be the vertices of T_{-i}^j. We suppose that $p_{3,i}^j$ does not belong to the side of T_{-i}^j orthogonal to $a_1f_1+a_2f_2$. (In the figure we consider the case in which $h_1=h_2$, so $T_3 = 2T_1 = 2T_2$).



We decompose T^j_{-i} in two triangles by intersecting T^j_{-i} with a line r^j_i orthogonal to v_i+w_i and passing for a vertex $p^j_{k,i}$ for a suitable $k \in \{1,2\}$ independent by j. Let $\widetilde{T}^j_{i+1}$ be the triangle that contains $p^j_{3,i}$ and let $\widetilde{T}^j_{-i-1}$ be the other triangle. Observe that r^1_i, r^2_i and r^3_i are parallel, so the fans associated to $\widetilde{T}^1_{i+1}$, $\widetilde{T}^2_{i+1}$ and $\widetilde{T}^3_{i+1}$ (respectively to $\widetilde{T}^1_{-i-1}$, $\widetilde{T}^2_{-i-1}$ and $\widetilde{T}^3_{-i-1}$) are equal. Observe that the convex function associated to $\widetilde{T}^j_{i+1}$ is uniquely determined by the knowledge of the fan associated to $\widetilde{T}^j_{i+1}$ and by the knowledge of any two vertices of $\widetilde{T}^j_{i+1}$ (indeed the three 1-dimensional cones of the fan associated to $\widetilde{T}^j_{i+1}$ are contained in the union of any two different two-dimensional cones of the fan associated to $\widetilde{T}^j_{i+1}$). Thus $\widetilde{T}^3_{i+1} = \widetilde{T}^1_{i+1} + \widetilde{T}^2_{i+1}$ because each $\widetilde{T}^j_{i+1}$ share two vertices with T^j_{-i}. In the same way we can prove that $\widetilde{T}^3_{-i-1} = \widetilde{T}^1_{-i-1} + \widetilde{T}^2_{-i-1}$. We have to prove that $\widetilde{T}^j_{i+1} = T^j_{i+1}$ for each i and j.

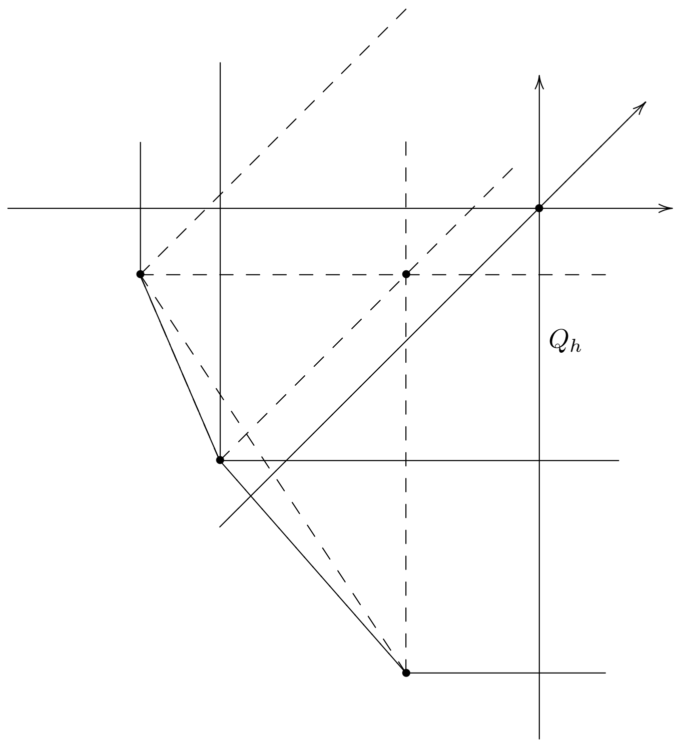
We can suppose that $a_1 f_1 + a_2 f_2 \in \sigma(u_i + w_i, u_i)$ up to exchange u_i and w_i. Let $p^j_{1,i} = (x^j_1, x^j_2)$ be the vertex T^j_{-i} not contained in the side orthogonal to u_i, let $p^j_{2,i}$ be the vertex of T^j_{-i} not contained in the side orthogonal to $-w_i$ and let $p^j_3 = (z^j_1, z^j_2)$ be the vertex of T^j_{-i} not contained in the side orthogonal to $a_1 f_1 + a_2 f_2$. Let r^j_i be the line orthogonal to $u_i + w_i$ and passing along $p^j_{1,i}$. Observe that r^j_i intersects the side orthogonal to $-u_i$ in the point $q^j_i = (z^j_1, z^j_2 + x^j_1 - z^j_1)$ and this point has integral coordinates. We have decomposed T^j_{-i} in two triangles: 1) the triangle T^j_{i+1} with sides orthogonal respectively to $-u_i$, $-w_i$,

$u_i + w_i$ and with vertices p_1^j, p_3^j, q^j; 2) the triangle T^j_{-i-1} with sides orthogonal respectively to $-u_i$, $-u_i - w_i$, $a_1 f_1 + a_2 f_2$ and with vertices p_1^j, q^j, p_2^j. For the case of T^j_{r-2} it is sufficient to observe that $a_1 f_1 + a_2 f_2 = u_{r-1} + (u_{r-1} + w_{r-1})$, up to exchange u_{r-1} and w_{r-1}. $\square$

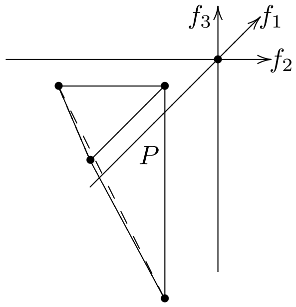
Now we consider a class of line bundles on varieties of dimension 3. This line bundles are the pullbacks of ample lines bundles on varieties which are usually singular.

Proposition 12.2 *Let h be a piecewise linear function which is strictly convex on the fan Δ with maximal cones $\sigma(e_1, e_2, ae_1 + ae_2 + e_3)$, $\sigma(e_1, e_3, ae_1 + ae_2 + e_3)$ and $\sigma(e_2, e_3, ae_1 + ae_2 + e_3)$. Here a is a strictly positive integer. Then $Q_{2h} \cap M = Q_h \cap M + Q_h \cap M$.*

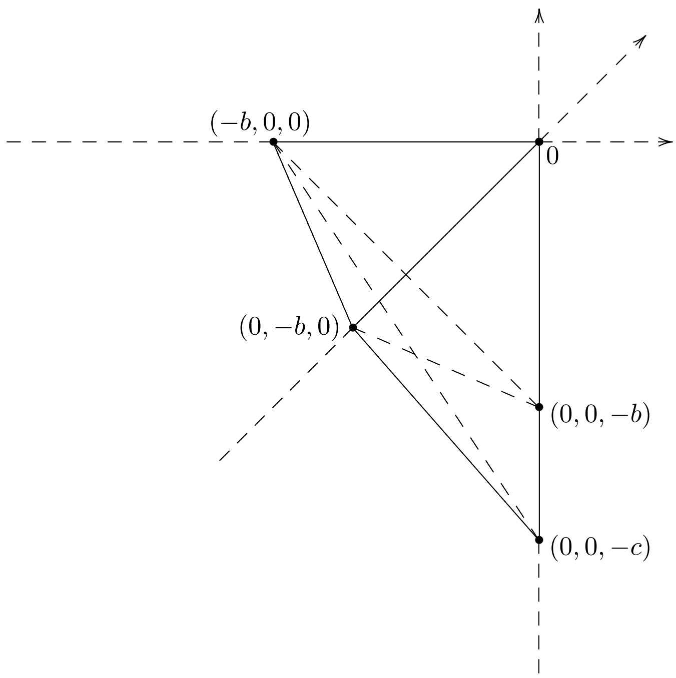
Remember that h defines a line bundle generated by global sections on every toric variety proper over the toric variety associated to Δ. Moreover the toric variety associated to Δ is proper over $\mathbf{A}^3$ and it is smooth if and only if $a = 1$. (Look to the figure for an example of Q_h).



Proof. We want to proceed as in the previous theorem. We can again decompose Q_{2h} in a simplex P and some cones $p + \sigma(f_1, f_2, f_3)$. So we can reduce ourselves to prove that $(2P + \sigma(f_1, f_2, f_3)) \cap M = (P + \sigma(f_1, f_2, f_3)) \cap M + (P + \sigma(f_1, f_2, f_3)) \cap M$. We again define a scalar product such that $\{f_1, f_2, f_3\}$ is an orthonormal basis.



P has three faces parallel to the coordinate planes and the fourth is orthogonal to $af_1+af_2+f_3$. We can suppose, up to a translation, that the origin 0 is the vertex of P which does not belong to the face orthogonal to $ae_1+ae_2+e_3$. Let $(-b,0,0)$, $(0,-b,0)$ and $(0,0,-c)$ be the other vertices of P. We have $c=ba$, so $c \geq b$. We want to decompose P in simplices with rational vertices. Let R be a such simplex. We suppose that there is a basis $\{v_1,v_2,v_3\}$ of M such that R is intersection of the semispaces $\{x \mid (x,v_i) \leq b_i\}$ with $i=1,2,3$ and $\{x \mid (x,v_1+v_2+v_3) \geq b\}$ where b_1,b_2,b_3,b are opportune integers. Moreover we require that $\sigma(v_1,v_2,v_3)$ is contained in $\sigma(f_1,f_2,f_3)$. It is again sufficient to define such decomposition because of the proposition 12.1. For simplicity we consider only the first step of the decomposition of P.



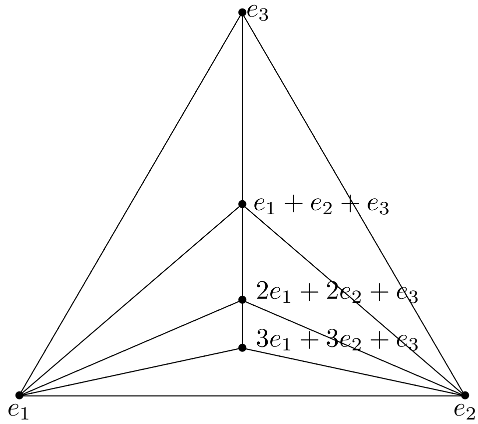
We decompose P intersecting it with the plane orthogonal to $f_1 + f_2 + f_3$ and passing through the vertices $(-b, 0, 0)$ and $(0, -b, 0)$. This plane intersects the side of P parallel to $\mathbf{R}f_3$ in $(0, 0, -b)$. We obtain two simplices with integral vertices. The first one has faces orthogonal respectively to $-f_1$, $-f_2$, $-f_3$ and $f_1+f_2+f_3$. This simplex has vertices $(0, 0, 0)$, $(-b, 0, 0)$, $(0, -b, 0)$ and $(0, 0, -b)$. The second simplex has faces orthogonal respectively to $-f_1$, $-f_2$, $-f_1-f_2-f_3$ and $af_1 + af_2 + f_3$. This simplex T has vertices $(-b, 0, 0)$, $(0, -b, 0)$, $(0, 0, -b)$ and $(0, 0, -c)$. Observe that $\sigma(f_1, f_2, f_1+f_2+f_3)$ is contained in $\sigma(f_1, f_2, f_3)$, so $T+\sigma(f_1, f_2, f_1+f_2+f_3)$ is contained in $P+\sigma(f_1, f_2, f_3)$. Moreover T is a simplex of the same type of P and $af_1+af_2+f_3 = (a-1)f_1+(a-1)f_2+(f_1+f_2+f_3)$, i.e. the coordinate with respect to the new basis are decreased. We will reiterate the process until we obtain a basis with respect to which $af_1 + af_2 + f_3$ has all coordinate equal to 1, so that we can use the proposition 12.1. $\square$

12.3 Two families of open toric varieties of dimension at least 3

Now we want to show that there is an infinite number of open toric varieties of any fixed dimension (greater than 2) such that the product of sections of any two ample line bundles is surjective. The principal instrument in what follows is the proposition 10.3. We will consider a very special class of varieties. Let L_h and L_k be any two ample line bundles on a variety of this family. Let s be a semi-invariant section of the product L_{h+k} such that its weight p has the following property: there is not a weight p' in $\prod(Z, h+k)$ such that $p \in p' + \sigma(f_1, ..., f_l)$. Then s does not vanish on a suitable divisor. This means that $H^0(Z, L_{h+k})$ is generated as a $O_Z(Z)$ module by the seminvariant sections that do not vanish on a suitable divisor.

Proposition 12.3 *Let Z be the open toric variety obtained from A^l through the sequence of blow-ups along the stable subvarieties associated respectively to $\sigma(e_1, ..., e_l)$, $\sigma(e_1, ..., e_{l-1}, (\sum_{i=1}^{l-1} e_i)+e_l)$, $\sigma(e_1, ..., e_{l-1}, 2(\sum_{i=1}^{l-1} e_i)+e_l)$,...,$\sigma(e_1, ..., e_{l-1}, i(\sum_{i=1}^{l-1} e_i)+e_l)$,...,$\sigma(e_1, ..., e_{l-1}, (n-1)(\sum_{i=1}^{l-1} e_i)+e_l)$. Let L_h and L_k be any two ample line bundles on Z, then the product of sections $m_{h,k}$ is surjective.*

Let Δ be the fan of Z. The l-dimensional cones in Δ are the following: $\sigma(e_1, ..., \widehat{e_j}, ..., e_l, \sum_{i=1}^{l} e_i)$ with $j = 1, ..., l-1$; $\sigma(e_1, ..., \widehat{e_j}, ..., e_{l-1}, (i-1)(\sum_{i=1}^{l-1} e_i) +e_l, i(\sum_{i=1}^{l-1} e_i)+e_l)$ with $j = 1, ..., l-1$ and $i = 2, ...n$; $\sigma(e_1, ..., e_{l-1}, n(\sum_{i=1}^{l-1} e_i)+ e_l)$. (In the figure we have drawn the 3-dimensional variety with $n = 3$).



Proof. Observe that we have already considered the case $n = 1$ in proposition 12.1, so we can suppose $n \geq 2$. Up to changing the linearizations of the line bundles we can suppose that $h(e_j) = k(e_j) = 0$ for each j. Observe that, if $(Q_h \cap M) + (Q_k \cap M)$ contains a weight p, then it contains any weight $p + \sum a_i f_i$ where the a_i are positive integers. So we can consider only the "minimal" weights. Let $p \in Q_{h+k} \cap M$ be any "minimal" weight, we claim that there is a cone $\tau \in \Delta(1)$ such that $p(\varrho(\tau)) = (h+k)(\varrho(\tau))$. This means that any semi-invariant section of weight p does not vanish on the divisor of Z associated to τ. This claim will allows ourselves to use the proposition 10.3. Thus will be sufficient to prove the surjectivity of the product of sections of the restrictions of L_h and L_k to any divisor of Z.

Claim 12.1 *Let p be any weight in $Q_{h+k} \cap M$ and suppose that there is not a weight p' in $Q_{h+k} \cap M$ such that $p \in p' + \sigma(f_1, ..., f_l)$. Then there is a cone $\tau \in \Delta(1)$ such that $p(\varrho(\tau)) = (h+k)(\varrho(\tau))$.*

Proof. The hypotheses imply that $p - f_l$ does not belong to Q_{h+k}. Hence there is an i such that $(p - f_l)(i(\sum_{i=1}^{l-1} e_i) + e_l) = p(i(\sum_{i=1}^{l-1} e_i) + e_l) - 1 < (h+k)(i(\sum_{i=1}^{l-1} e_i) + e_l)$ because $(p - f_l)(e_j) = p(e_j) \geq (h+k)(e_j)$ for each $j = 1, ..., l-1$. So $p(i(\sum_{i=1}^{l-1} e_i) + e_l) = (h+k)(i(\sum_{i=1}^{l-1} e_i) + e_l)$ and we have proved the claim. $\square$

Now it is sufficient to prove the surjectivity of the product of sections of the restrictions of L_h and L_k to the divisor Z_i associated to $\sigma(i(\sum_{i=j}^{l-1} e_j) + e_l)$ for each $i = 0, ..., n$. Z_i is a toric variety with respect to the torus whose group of 1-parameter subgroups is $(\bigoplus_{j=1}^{l} \mathbf{Z} e_j)/\mathbf{Z}(i \sum_{j=1}^{l-1} e_j + e_l)$. Observe that $(\bigoplus_{j=1}^{l} \mathbf{Z} e_j)/\mathbf{Z}(i \sum_{j=1}^{l-1} e_j + e_l)$ is freely generated by $\widetilde{e}_1, ..., \widetilde{e}_{l-1}$, where, for each j, $\widetilde{e}_j$ is the class of e_j modulo $i \sum_{j=1}^{l-1} e_j + e_l$. We have three possibilities: i) if $i = 0$ then the fan associated to Z_0 has 1-dimensional cones $\sigma(\widetilde{e}_1), ..., \sigma(\widetilde{e}_{l-1}), \sigma(\sum \widetilde{e}_i)$ and Z_0 is the blow-up of A^{l-1} in the stable point; ii) if $i = n$ then the fan associated to Z_n has 1-dimensional cones $\sigma(\widetilde{e}_1), ..., \sigma(\widetilde{e}_{l-1}), \sigma(-\sum \widetilde{e}_i)$ and Z_n is the projective space of dimension $l-1$; iii) if $0 < i < n$ then the fan associated to Z_0 has 1-dimensional cones $\sigma(\widetilde{e}_1), ..., \sigma(\widetilde{e}_{l-1}), \sigma(\sum \widetilde{e}_i)$, $\sigma(-\sum \widetilde{e}_i)$ and Z_i is the blow-up of the $(l-1)$-dimensional projective space in a stable point. We want to do some remark. Observe that we have already considered the variety Z_0 in proposition 12.1. The varieties Z_i with $0 < i < n$ are all isomorphic. Because Z_1 dominates Z_n it is sufficient to study the product of sections of any two line bundles $L_{h'}$ and $L_{k'}$ generated by global sections on Z_1.

Lemma 12.2 *Let $L_{h'}$ and $L_{k'}$ be any two line bundles on Z_1 generated by global sections. Then the multiplication of sections is surjective.*

Proof. We can suppose that $h'(\widetilde{e}_i) = k'(\widetilde{e}_i) = 0$ for each i, thus $L_{h'}$ is the pullback of a line bundle on Z_n if and only if $h'(\sum_{i=1}^{l-1} \widetilde{e}_i) = 0$. In the following we identify $\mathbf{Z}^{l-1}$ with the character group of the torus contained in Z_1. We proceed as in the proof of the proposition 12.1. Let m be any point in $Q_{h'+k'}$ with integral coordinates. There are $\tilde{m}_1 \in Q_{h'}$ and $\tilde{m}_2 \in Q_{k'}$ such that $\tilde{m}_1 + \tilde{m}_2 = m$ but they may not have integral coordinates. Let $a = h'(\sum_{i=1}^{l-1} \widetilde{e}_i)$, $b = -h'(-\sum_{i=1}^{l-1} \widetilde{e}_i)$, $c = k'(\sum_{i=1}^{l-1} \widetilde{e}_i)$ and $d = -k'(-\sum_{i=1}^{l-1} \widetilde{e}_i)$. The inequalities for $Q_{h'}$ are $u_i \geq 0$ for each $i = 1..., l-1$ and $a \leq \sum u_i \leq b$, while the inequalities for $Q_{k'}$ are $u_i \geq 0$ for each $i = 1..., l-1$ and $c \leq \sum u_i \leq d$. Suppose that $\tilde{m}_1 = (x_1, ..., x_{l-1})$, $\tilde{m}_2 = (y_1, ..., y_{l-1})$ and $m = (z_1, ..., z_{l-1})$, so $x_i \geq 0$, $a \leq \sum x_i \leq b$, $y_i \geq 0$ and $c \leq \sum y_i \leq d$. Let $[x_i]$ be the integral part of x_i and let $\epsilon_i = -[([x_i] - x_i)]$ (ϵ_i is 0 if x_i is an integer and it is 1 otherwise). Because 0 is an integer we have $[x_i] + \epsilon_i \geq [x_i] \geq 0$ for each $i = 1, ..., l$; likewise we have $[y_i] + \epsilon_i \geq [y_i] \geq 0$ for each $i = 1, ..., l$.

We define $m_1 = ([x_1] + \epsilon_1, ..., [x_{\bar{r}}] + \epsilon_{\bar{r}}, [x_{\bar{r}+1}], ..., [x_{l-1}])$ and $m_2 = ([y_1], ..., [y_{\bar{r}}], [y_{\bar{r}+1}] + \epsilon_{\bar{r}+1}, ..., [y_{l-1}] + \epsilon_{l-1})$ for a suitable $\bar{r}$. Now we want to simplify the notation. In particular we will be evident that the problem does not depend on the dimension. Let $t = \sum_{i=1}^{l-1} \epsilon_i$, $r = \sum_{i=1}^{\bar{r}} \epsilon_i$, $[x] = \sum_{i=1}^{l-1} [x_i]$, $x = \sum_{i=1}^{l-1} x_i$, $[y] = \sum_{i=1}^{l-1} [y_i]$ and $y = \sum_{i=1}^{l-1} y_i$. We known the following inequalities: i) $[x] \leq x \leq [x] + t$, $[y] \leq y \leq [y] + t$ and $0 \leq r \leq t$; ii) $a \leq x \leq b$ and $c \leq y \leq d$; iii) $a + c \leq [x] + [y] + t = x + y \leq b + d$. Observe that $\widetilde{m}_1(\sum_{i=1}^{l-1} \widetilde{e}_i) = x$, $m_1(\sum_{i=1}^{l-1} \widetilde{e}_i) = [x] + r$, $\widetilde{m}_2(\sum_{i=1}^{l-1} \widetilde{e}_i) = y$ and $m_2(\sum_{i=1}^{l-1} \widetilde{e}_i) = [y] + t - r$. It is sufficient to show that there is r such that $0 \leq r \leq t$, $a \leq [x] + r \leq b$ and $c \leq [y] + t - r \leq d$. Observe that r takes all the value between 0 and t when $\bar{r}$ varies between 0 and $l-1$.

1) If $t + [x] \leq b$ we define r as $\min\{[y] + t - c, t\}$. If $[y] \geq c$ then $r = t$, so $b \geq [x] + t = [x] + r \geq x \geq a$ and $c \leq [y] \leq y \leq d$. If $c \geq [y]$ then $b \geq [x] + t \geq [x] + r = [x] + [y] + t - c \geq a + c - c = a$ and $c = [y] + t - ([y] + t - c) = [y] + t - r \leq d$.

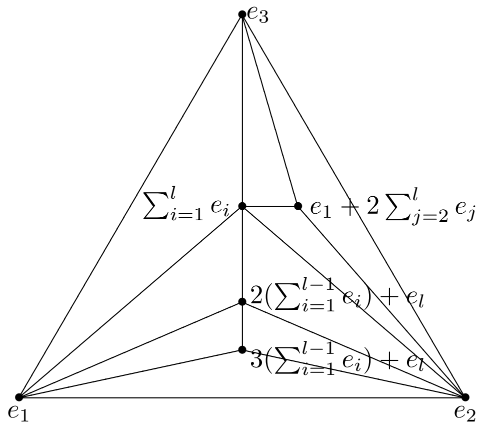
2) Suppose now that $[y] + [x] + t \leq b + c$. If $c - [y]$ is positive then we define $r = t + [y] - c$, so $t - r = c - [y]$ ($t + [y] \geq y \geq c$ so $r \geq 0$). In this case $c = [y] + t - r \leq d$ and $a \leq [x] + [y] + t - c = [x] + r \leq b$. If $c - [y]$ is negative then we define $r = t$, so $c \leq [y] = [y] + t - r \leq d$ and $a \leq x \leq [x] + t = [x] + r \leq c + b - [y] \leq b$.

3) Finally suppose that $t + [x] > b$ and $[y] + [x] + t > b + c$. We define $r = b - [x]$, so $a \leq [x] + r = b$ and $d \geq [y] + [x] + t - b = [y] + t - r \geq c$. $\square$

We have proved that the "minimal" weights of $\prod(Z, h+k)$ come from semi-invariant sections that do not vanish on a suitable divisor. Moreover we have not explicitly used the strictly convexity of L_h and L_k, indeed we have used it only to prove proposition 10.3. This fact are no longer true if we consider varieties whose fan is a little less symmetric. Notice that the fans of the varieties considered in this proposition are invariant for any automorphism of N which permutes the vectors of the basis, fixing e_l. In the following we define a class of varieties without such symmetry and obtained by blow-ups from varieties of the previous family.

Theorem 12.2 *Let Z be the open toric variety obtained from A^l through the sequence of blow-ups along the stable subvarieties associated respectively to $\sigma(e_1, ..., e_l)$, $\sigma(e_1, ..., e_{l-1}, (\sum_{i=1}^{l-1} e_i) + e_l)$,..., $\sigma(e_1, ..., e_{l-1}, i(\sum_{i=1}^{l-1} e_i) + e_l)$,...., $\sigma(e_1, ..., e_{l-1},\ (n-1)(\sum_{i=1}^{l-1} e_i) + e_l)$ and $\sigma(\sum_{j=1}^{l} e_j, e_2, ..., e_l)$. Let L_h be any ample line bundles on Z, then the product of sections $m_{h,h}$ is surjective.*

The fan of Z has maximal cones: $\sigma(e_1, ..., \widehat{e_j}, ..., e_l, \sum_{i=1}^{l} e_i)$ for each $j = 2, ..., l-1$; $\sigma(e_1, ..., \widehat{e_j}, ..., e_{l-1}, (i-1)(\sum_{i=1}^{l-1} e_i) + e_l, i(\sum_{i=1}^{l-1} e_i) + e_l)$ for each $j = 1, ..., l-1$ and $i = 2, ...n$; $\sigma(e_1, ..., e_{l-1}, n(\sum_{i=1}^{l-1} e_i) + e_l)$; $\sigma(e_2, ..., \widehat{e_j}, ..., e_l, \sum_{i=1}^{l} e_i, e_1 + 2\sum_{j=2}^{l} e_j)$ for each $j = 2, ..., l$; $\sigma(e_2, ..., e_l,\ e_1 + 2\sum_{j=2}^{l} e_j)$ (In the figure we have drawn the case in which $l = 3$ and $n = 3$).



Proof. Observe that Z is the blow-up of a variety Z' of the previous proposition. Z' is obtained from A^l through the sequence of blow-ups along the stable subvarieties associated respectively to $\sigma(e_1, ..., e_l)$, $\sigma(e_1, ..., e_{l-1}, (\sum_{i=1}^{l-1} e_i) + e_l)$,..., $\sigma(e_1, ..., e_{l-1}, i(\sum_{i=1}^{l-1} e_i) + e_l)$,...., $\sigma(e_1, ..., e_{l-1}, (n-1)(\sum_{i=1}^{l-1} e_i) + e_l)$. Z is the blow-up of Z' along the subvariety associated to $\sigma(\sum_{i=1}^{l} e_i, e_2, ..., e_l)$.

We introduce some notation to simplify the counts: $w := e_1 + 2\sum_{j=2}^{l} e_j$ and $v_i := i(\sum_{i=1}^{l-1} e_i) + e_l$ for each i. In the proof we allow L_h to be the pullback of an ample linearized line bundle on Z'. In this case $h(w) = h(v_1) + \sum_{i=2}^{l} h(e_i)$, while if L_h is ample on Z then $h(w) > h(v_1) + \sum_{i=2}^{l} h(e_i)$. We want to prove the proposition by induction on $h(w)$ and on the dimension of Z. Observe that if $h(w) = h(v_1) + \sum_{i=2}^{l} h(e_i)$ then $m_{h,h}$ is surjective because of the previous proposition. If the dimension of Z is 2, then $m_{h,h}$ is surjective because of the theorem 12.1, so the basis of the induction is proved. Suppose now that $h(w) > h(v_1) + \sum_{i=2}^{l} h(e_i)$. We want to proceed in a similar way to the previous proposition. We can suppose that $h(e_j) = 0$ for each j. Let $a_i = h(v_i)$ and $b = h(w)$. It is sufficient to show that any $m \in Q_{2h} \cap M$ belongs to $(Q_h \cap M) + (Q_h \cap M)$. As before, if p belongs to $(Q_h \cap M) + (Q_h \cap M)$ then $p + \bigoplus \mathbf{Z}^+ f_i$ is contained in $(Q_h \cap M) + (Q_h \cap M)$. Thus we can suppose that $m - f_l$ does not belong to $(Q_{2h} \cap M)$, so either there is an i such that $m(v_i) = a_i$ or $m(w) - 2h(w) \in \{0, 1\}$.

In the first case we have reduced ourselves to study a divisor because of proposition 10.3. If $m(w) = 2b$ we again have to study a divisor. We have the following possibilities for the divisor Z_τ associated to a cone τ: i) if $\varrho(\tau) = v_i$ with $1 < i < n$, then Z_τ is isomorphic to a divisor of Z', more precisely it is the blow-up of the projective space in a stable point; ii) if $\varrho(\tau)$ is equal to w or to v_n, then Z_τ is the projective space; iii) if $\varrho(\tau)$ is equal to e_1 or to e_l, then Z_τ is a variety considered in the previous proposition; iv) if $\varrho(\tau) = e_i$ with $i \neq 1, l$, then Z_τ is variety as in the hypotheses of this proposition, but with dimension $l-1$; v) if $\varrho(\tau) = v_1$, then Z_τ is the blow-up of the projective space in two S-stable points. If the fan of the projective space has maximal cones $\sigma(u_1, ..., u_{l-1})$ and $\sigma(-\sum u_i, u_1, ..., \widehat{u}_i, ..., u_{l-1})$ for each $i = 1, ..., l-1$, then $\{u_1, ..., u_{l-1}\}$ is a basis of the lattice and $Z_{\sigma(v_1)}$ is the blow-up centered in the points associated respectively to $\sigma(u_1, ..., u_{l-1})$ and $\sigma(-\sum u_i, u_2, ..., u_{l-1})$. The 1-dimensional cones of the fan of $Z_{\sigma(v_1)}$ are generated respectively by $u_1, ..., u_{l-1}, \sum u_i, -\sum u_i$ and $-u_1$. The unique case which we have not already examined is the last one. Let M' be the character group of the torus contained in $Z_{\sigma(v_1)}$ and let P be the polytope associated to any ample linearized line bundle on $Z_{\sigma(v_1)}$, we have to show that $2P \cap M' = P \cap M' + P \cap M'$. P has inequalities: $0 \leq z_1 \leq a$; $0 \leq z_j$ for each j; $b \leq \sum z_j \leq c$ (a, b and c are suitable integers). Let $m = (x_1, ..., x_{l-1})$ be an integral point in $2P$. We can proceed as done in the previous proposition for the divisor $Z'_{\sigma(v_1)}$ of the varieties Z'. Indeed $m = m/2 + m/2$, $m/2 = (x_1/2, ..., x_{l-1}/2)$ is in P and $0 \leq [x_1/2] \leq x_1/2 \leq [x_1/2] + \epsilon_1 \leq a$. Let P' be the polytope with equations $0 \leq z_j$ for each j and $b \leq \sum z_j \leq c$, then $m \in 2P'$ and $m/2 \in P'$. Notice that P' is the polytope corresponding to an ample line bundle on the divisor $Z'_{\sigma(v_1)}$ of Z', thus we can use the lemma 12.2. Moreover any point $(x_1/2 + \epsilon_1, ..., x_r/2 + \epsilon_r, x_{r+1}/2, ..., x_{l-1}/2)$ belongs to P if and only if it belongs to P' ($[x_1/2] + \epsilon_1$ is the least integer greater of $x_1/2$).

Thus we can suppose that $m(w) = 2b + 1$. We now want to write some necessary conditions to the strictly convexity of h on the fan Δ associated to Z. The condition $(h|\sigma(v_1, w, e_2, ..., e_{l-1}))(v_i) > h(v_i)$ implies

$$a_i + (i-1)b < (2i-1)a_1$$

for each $i > 1$. The conditions $h|\sigma(w, e_2, ..., e_l)(v_1) > h(v_1)$, $h|\sigma(v_1, e_1, e_3, ..., e_l)(e_2) > h(e_2)$, $h|\sigma(v_1, e_1, e_3, ..., e_l)(w) > h(w)$ and $h|\sigma(v_1, e_1, e_3, ..., e_l)(v_i) > h(v_i)$ imply:

$$b > a_1 > 0, \quad 2a_1 > b$$

and

$$ia_1 > a_i$$

(Indeed $v_i = (2i-1)v_1 - (i-1)w + (i-1)(e_2 + ... + e_{l-1})$, $v_1 = w - e_2 - ... - e_l$, $e_2 = v_1 - e_1 - e_3 - ... - e_l$, $w = 2v_1 - e_1$ and $v_i = iv_1 - (i-1)e_l$). These inequalities imply $b > 1$ and $ib > a_i$.

Let Δ' be the fan of Z' and let h' be the piecewise linear function on Δ such that $h'(e_i) = 0$, $h'(v_i) = h(v_i)$ and $h'(w) = h(w) - 1$. We need the following lemma on h'.

Lemma 12.3 *h' is convex on Δ and is strictly convex either on Δ or on Δ'.*

Proof. Observe that $h \geq h'$.

i) Let σ be a l-dimensional cone which does not contain w and let τ be an 1-dimensional cone not contained in σ, then $(h'|\sigma)(\varrho(\tau)) = (h|\sigma)(\varrho(\tau)) > h(\varrho(\tau)) \geq h'(\varrho(\tau))$.

We now consider the maximal cones that contain w.

ii) Consider $\sigma(e_2, ..., e_l, w)$. We have $h'|\sigma(e_2, ..., e_l, w) = (b-1)f_1$, so $(h'|\sigma(e_2, ..., e_l, w))\ (v_i) = i(b-1) \geq ia_1 > a_i$ for each $i > 1$, $(h'|\sigma(e_2, ..., e_l, w))(v_1) = b - 1 \geq a_1$ and $(h'|\sigma(e_2, ..., e_l, w))(e_1) = b - 1 > 0$.

iii) Consider the cone $\sigma(e_2, ..., e_{l-1}, v_1, w)$. We have $h'|\sigma(e_2, ..., e_{l-1}, v_1, w) = h|\sigma(e_2, ..., e_{l-1}, v_1, w) - \varphi$ where φ is the linear function such that $\varphi(w) = 1$ and $\varphi(e_j) = \varphi(v_1) = 0$ for each $j = 2, ..., l-1$, namely $\varphi = f_l - f_1$. Thus $(h'|\sigma(e_2, ..., e_{l-1}, v_1, w))(v_i) = (h|\sigma(e_2, ..., e_{l-1}, v_1, w))(v_i) - \varphi(v_i) = (h|\sigma(e_2, ..., e_{l-1}, v_1, w))\ (v_i) + i - 1 > h(v_i) + i - 1 > h'(v_i)$ for each $i > 1$. Moreover $(h'|\sigma(e_2, ..., e_{l-1}, v_1, w))(e_1) = (h|\sigma(e_2, ..., e_{l-1}, v_1, v_i))(e_1) - \varphi(e_1) = (h|\sigma(e_2, ..., e_{l-1}, v_1, w))(e_1) + 1 > 0$ and $(h'|\sigma(e_2, ..., e_{l-1}, v_1, w))\ (e_l) = (h|\sigma(e_2, ..., e_{l-1}, v_1, w))\ (e_l) - \varphi(e_l) = (h|\sigma(e_2, ..., e_{l-1}, v_1, w))\ (e_l) - 1 \geq 0$.

iv) Finally we have to consider the cones $\sigma_j = \sigma(e_2, ..., \widehat{e_j}, ..., e_{l-1}, v_1, e_l, w)$ for each $j = 2, ..., l-1$. We have $h'|\sigma_j = h|\sigma_j - \psi_j$ where ψ_j is the linear function such that $\psi_j(w) = 1$ and $\psi(e_i) = \psi(v_1) = 0$ for each $i \neq 1, j$, namely $\psi_j = f_j - f_1$. For each $i > 1$ we have $(h'|\sigma_j)(v_i) = (h|\sigma_j)(v_i) - \psi_j(v_i) = (h|\sigma_j)(v_i) > h(v_i) = h'(v_i)$. Moreover $(h'|\sigma_j)(e_1) = (h|\sigma_j)(e_1) + 1 > 0$ and $(h|\sigma_j)(e_j) = (h'|\sigma_j)(e_j) - 1 \geq 0$. $\square$

By induction we can suppose that $m_{h',h'}$ is surjective, so we can suppose that there are two points $m_1 \in Q_h \cap M$ and $m_2 \in Q_{h'} \cap M$ such that $m_1 + m_2 = m$ (at least one point must belongs to Q_h because otherwise m does not belong to Q_{2h}). If m_2 belong to Q_h then $m \in Q_h \cap M + Q_h \cap M$. Otherwise we have $m_2(w) = b - 1$ and $m_1(w) = b + 2$. Write $m_1 = (x_1, ..., x_l)$ and $m_2 = (y_1, ..., y_l)$ (we have identified M with $\mathbf{Z}^l$).

We can suppose that $m_1 - f_l \notin Q_h$ because $m_2 + f_l \in Q_h$. Thus there is i such that $m_1(v_i) = a_i$. Moreover we can suppose that $(m_1 + f_1 - f_j, m_2 - f_1 + f_j)$

does not belong to $Q_h \times Q_{h'}$ for any $j = 2, ..., l-1$, so $x_j = 0$ or $y_1 = 0$. If $y_1 = 0$ then $2a_1 - 1 \geq b - 1 = m_2(w) = 2\sum y_j = 2m_2(v_1) \geq 2a_1$, so we have obtained a contradiction. Hence $y_1 \neq 0$ and $x_j = 0$ for each $j = 2, ..., l-1$. Suppose that there is $i > 1$ such that $m_1(v_i) = ix_1 + x_l = a_i$, then we have

$$(2i-1)a_1 \leq (2i-1)(x_1 + x_l) = m_1(v_i) + (i-1)m_1(w) =$$
$$= a_i + (i-1)(b+2) \leq (2i-1)a_1 + 2i - 2,$$

so $0 \leq (2i-1)(x_1 + x_l - a_1) \leq 2i-2$ (remember that $a_i + (i-1)b < (2i-1)a_1$). We have $x_1 + x_l = a_1$ because $x_1 + x_l - a_1$ is an integer. Observe that we have showed that $m_1(v_1) = x_1 + x_l = a_1$ or $m_1(e_l) = x_l = 0$. In the last case we have $x_2 = ... = x_l = 0$ and $x_1 = b + 2$. We can suppose that $(m_1 - f_1, m_2 + f_1)$ does not belong to $Q_h \times Q_{h'}$, so there is $s > 0$ such that $m_1(v_s) - a_s < s$. Observe that $m_1(v_s) = sx_1 = sb + 2s$, so $a_s \leq sb = m_1(v_s) - 2s < a_s - s < a_s$, so we have obtained a contradiction. (The inequality $a_s \leq sb$ is one of the inequalities that we have obtained by the strictly convexity of h).

Finally we can suppose that $x_j = 0$ for each $j = 2, ..., l-1$, $x_1 + x_l = a_1$ and $x_1 + 2x_l = b + 2$, so $x_1 = 2a_1 - b - 2$ and $x_l = b + 2 - a_1$. Moreover we can suppose that $(m_1 + f_1 - f_l, m_2 - f_1 + f_l)$ does not belong to $Q_h \times Q_{h'}$, so $x_l = 0$, $y_1 = 0$ or there is $i > 1$ such that $\varepsilon := m_2(v_i) - a_i < i$. Observe that we have already considered the first two cases.

We have $a_i \leq m_1(v_i) = ix_1 + x_l = (2i-1)a_1 - (i-1)b - 2(i-1)$, so $(2i-1)a_1 \geq a_i + (i-1)b + 2(i-1)$. Finally

$$(2i-1)a_1 \leq (2i-1)(\sum y_j) \leq (2i-1)y_1 + (3i-2)\sum_{j \neq 1,l} y_j + (2i-1)y_l =$$
$$= m_2(v_i) + (i-1)m_2(w) = (i-1)(b-1) + a_i + \varepsilon =$$
$$= (i-1)b + a_i + 2(i-1) - 3(i-1) + \varepsilon \leq (2i-1)a_1 - 3(i-1) + \varepsilon,$$

so $3(i-1) \leq \varepsilon \leq i-1$, a contradiction. $\square$

Part III
Fano varieties

Now we want to study the Fano complete symmetric variety. The Fano toric variety are already studied. See [VK] for a finiteness theorem of smooth toric variety of arbitrarily fixed dimension. See [Ba2] and [WW] for the classification of Fano toric variety of dimension at most 3. See [Ba3] and [Sa] for a classification of Fano toric 4-folds. See also [Ba1], [Bo], [Re], [Ca1] and [Ca2].

13 Wonderful Fano symmetric varieties

We will say that a variety whose anticanonical bundle is generated by global sections is an almost Fano variety. We now want to show that the wonderful symmetric varieties are almost Fano varieties. Moreover we will classify the Fano wonderful symmetric varieties. Before we will explain the combinatorial conditions implying that a complete symmetric variety is an (almost) Fano variety. Let Y be a complete symmetric variety and let L_{k_Y} be the anticanonical bundle of Y, i.e. L_{-k_Y} is the maximal exterior power of the tangent bundle of Y. Let Z and Z^c be respectively the open toric variety and the complete toric variety associated to Y. If there is no confusion we will use the notation k instead of k_Y. We want to write k as a sum of two functions "associated" respectively to the anticanonical bundle of Z and to the anticanonical bundles of the closed orbits.

Lemma 13.1 *Let $k_1 = \sum_{\alpha\in\phi_1^+}\alpha$ and let $k_2 = k - k_1$, then k_2 is the unique Δ-linear function such that $k_2(\rho(\tau)) = -1$ for each cone $\tau \in \Delta(1)$.*

Proof. Let k' be the unique Δ-linear function such that $k'(\rho(\tau)) = 1$ for each cones $\tau \in \Delta(1)$. For example, if Y is wonderful then k' is the restriction of $-\sum_{i=1}^{l}\alpha_i^s$ to $|\Delta|$. We want to show that $k_2 = -k'$. We observe that the restriction of $L_{k'}$ to Z^c is the canonical bundle of Z^c (see page 70 in [O]) and the restriction of L_{-k_1} to any closed orbit O is the canonical bundle of O. Moreover k' and k_1 are characterized by such proprieties. For k_1 it follows because of the theorem 6.1. For k' it is true because k' is the unique Δ linear function such that the associated Δ^c-linear function $(k')^c$ is, up to a linear function, the Δ^c function with value 1 on $\rho(\tau)$ for each $\tau \in \Delta^c(1)$ (actually $(k')^c$ is exactly equal to such function). Moreover, given a T-fixed point x_σ in Z, the restriction of $L_{k'}$ to $U^- \times x_\sigma$ is the normal bundle of $U^- \times x_\sigma$ in $U^- \times Z$. Remember that the T-fixed points of Z^c are the translates of the T-fixed points of Z by the action of $N_{H^0}(T)$, so $L_{k'}$ is characterized by this property. It follows that $k' = -k_2$ because the restriction of L_{-k_2} to any closed G-orbit O is the normal bundle of O. □

Remark. 1) Observe that we can write $k_1 = 2\delta - 2\delta_0$ where $\delta = \sum_{\alpha\in\Gamma}\omega_\alpha$ is the sum of all the positive roots of ϕ and δ_0 is the sum of the roots in ϕ_0^+ i.e. it is the sum of all the positive roots of the root system ϕ_0 in $\bigoplus_{\alpha\in\Gamma_0}\mathbf{R}\alpha$ (recall that ϕ_0 is the set of the roots fixed by θ).

2) We want to point out that k_1 depends only on the open orbit G/H of Y, while k_2 depends only on the (open) toric variety Z. Moreover k is always almost spherical.

3) The restriction of L_k to Z is the anticanonical bundle of Z, indeed the restriction of L_{k_1} to Z is trivial. Instead the restriction of L_k to Z^c is not the anticanonical bundle of Z^c, except in the case in which the involution is trivial, i.e. Y is a point. Indeed, this is the unique case in which $k_1 = 0$.

Usually one asks that a Fano variety should be complete, but in order to use the following proposition (whose proof is trivial), we shall also consider the not complete case extending the definition in the obvious way.

Proposition 13.1 *If Y is a Fano variety, then Z is a Fano variety.*

We now can consider the wonderful symmetric varieties. In this case the associated open toric variety is $\mathbf{A}^l$, so k is linear and strictly convex on $\sigma(e_1,...,e_l)$. Therefore we have only to verify if the weight $2\delta - 2\delta_0 + \sum_{i=1}^l \alpha_i^s$ is (strongly) dominant. We can suppose that the variety is simple, because the product of two complete symmetric varieties is an (almost) Fano variety if and only if each factor is an (almost) Fano variety.

Remember that the lattice Ω generated by the spherical weights is the lattice of the integral weights of the root system $\widetilde{\phi}$. Moreover, the fundamental weights $\widetilde{\omega}_i'$ of $\widetilde{\phi}$ are such that $\widetilde{\omega}_i' = a_i(w_{\alpha_i} + w_{\bar{\theta}(\alpha_i)})$ where $a_i \in \{0,1\}$ for any i. More precisely $\widetilde{\omega}_i' = (w_{\alpha_i} + w_{\bar{\theta}(\alpha_i)})$ if $\theta(\alpha_i) \neq -\alpha_i$ and $\widetilde{\omega}_i' = 2(w_{\alpha_i} + w_{\bar{\theta}(\alpha_i)})$ if $\theta(\alpha_i) = -\alpha_i$. Observe that k is a special weight, so $< k, \beta >= 0$ for any simple root β fixed by θ. For each $i \in \{1,...l\}$ we want to show that $< k, \alpha_i^s > \geq 0$.

Notice that $< k, \alpha_i^s >= 2 < k, \alpha_i >= 2 < k, \alpha_{\overline{\theta}(i)} >$. We can write k_1 as the sum of the two spherical weights $2\widetilde{\delta} = 2\sum_{\alpha_j \in \Gamma_1} \omega_{\alpha_j}$ and $-2\widetilde{\delta_0} = 2\sum_{\alpha_j \in \Gamma_0} \omega_{\alpha_j} - 2\delta_0$. Observe that $< -2\widetilde{\delta_0}, \alpha_i^s >= 2 < 2\sum_{\alpha_j \in \Gamma_0} \omega_{\alpha_j} - 2\delta_0, \alpha_i >= 4 < -\delta_0, \alpha_i > \geq 0$ because δ_0 is a positive sum of simple roots fixed by θ and $< \beta, \alpha_i > \leq 0$ for any simple root β fixed by θ. Moreover we can prove that $2 < \sum_{j=1}^l \alpha_j^s, \alpha_i^s > / < \alpha_i^s, \alpha_i^s > \geq -1$ by looking to the Cartan matrix of $\widetilde{\phi}$. Therefore, if $\theta(\alpha_i) \neq -\alpha_i$ we have $2 < k, \alpha_i^s > / < \alpha_i^s, \alpha_i^s > \geq 2+0-1 = 1$. Suppose now that $\theta(\alpha_i) = -\alpha_i$, so $\alpha_i^s = 2\alpha_i$ and $\widetilde{\omega}_i' = 2w_i$. In this case $2 < 2\widetilde{\delta} - 2\widetilde{\delta_0}, \alpha_i^s > / < \alpha_i^s, \alpha_i^s >= 1 + 2 < -2\widetilde{\delta_0}, \alpha_i^s > / < \alpha_i^s, \alpha_i^s > \geq 1$, so $(2 < k, \alpha_i^s > / < \alpha_i^s, \alpha_i^s >) \geq 0$. Therefore the anticanonical bundle of any wonderful symmetric variety is without base points. Moreover if the anticanonical bundle of a wonderful symmetric variety X is not ample, then there is an (unique) j such that $\theta(\alpha_j) = -\alpha_j$, $< \widetilde{\delta_0}, \alpha_j^s >= 0$ and $< \sum_{i=1}^l \alpha_i^s, \alpha_j^s >= -1$. This implies that $< \beta, \alpha_j^s >= 0$ for any simple root β fixed by θ and that the restricted root system is reduced and different from A_n and B_n. We have three possibilities: 1) there are i_1 e i_2 such that $\alpha_{i_1}^s, \alpha_j^s, \alpha_{i_2}^s$ generate a root system of type C_3; 2) there are i_1, i_2 e i_3 such that $\alpha_{i_1}^s, \alpha_j^s, \alpha_{i_2}^s, \alpha_{i_3}^s$ generate a root system of type D_4 and 3) there is i_1 such that $\alpha_{i_1}^s, \alpha_j^s$ generate a root system of type G_2. Moreover G is simple, i.e. the wonderful symmetric variety is not the completion of a group. Studying the Satake diagram we obtain the following theorem.

Theorem 13.1 *Let X be a wonderful symmetric variety. Then:*

- *The anticanonical bundle of X is generated by global sections.*
- *X is a Fano variety if and only if its simple factors are Fano varieties.*
- *A simple wonderful symmetric variety is not a Fano variety if and only if the involution induced on $M_{\boldsymbol{R}}$ is $-id$ and the (restricted) root system is different from A_n and B_n.*

- *Explicitly, the simple wonderful symmetric varieties whose anticanonical bundle is not ample are associated to:*
  1. *the involution of type CI;*
  2. *the involution of type DI, such the rank of the restricted root system $\widetilde{\phi}$ is equal to the rank of the root system ϕ;*
  3. *the involution of type EI;*
  4. *the involution of type EV;*
  5. *the involution of type $EVIII$;*
  6. *the involution of type FI;*
  7. *the involution of type G.*

14 A finiteness theorem for Fano complete symmetric varieties

Now we want to show that there is only a finite number of Fano complete symmetric varieties for each homogeneous symmetric variety.

Theorem 14.1 *For each G/H there is only a finite number of Fano complete symmetric varieties whose open orbit is isomorphic to G/H.*

Proof. Let Y be a Fano complete symmetric variety and let k be the Δ linear function associated to the anticanonical bundle of Y, then k is strictly convex on Δ and Z is a Fano variety. Thus the polar polyhedron Q_k° of Q_k is the convex hull of the points $\{\varrho(\tau)\ :\ \tau \in \Delta(1)\} \cup \{0\}$ and Δ consists of the cones generated by the faces of Q_k° which does not contain 0.

First of all, we want to show that there is an upper bound C to the number of l-cones in Δ and this bound depends only on G/H. Because k_2 is strictly convex on Δ, there is a injective map $\Delta(l) \to \Lambda_X$ that takes σ to $k_2|\sigma$. Thus it is sufficient to show that there are only a finite number of possibilities for these weights. Because k_2 is strictly convex on Δ, we have the inequality $(k_2|\sigma)(e_i) \geq -1$ for each i. Moreover $k|\sigma$ is dominant, so $(k|\sigma)(-e_i) \geq 0$ for each i. Thus $(k_2|\sigma)(e_i) \leq (2\delta - 2\delta_0)(-e_i)$ for each i. Therefore the $k_2|\sigma$ belong to the intersection of a fixed polytope and a lattice, so there is only a finite number of them.

Now we want to prove that the volume of Q_k^0 is bounded. (We can define a measure such that , given a basis $\{v_1, ..., v_l\}$ of M, the parallelepiped $\{\sum x_i v_i : 0 \leq x_i \leq 1\ \forall i\}$ has volume one). Q_k^0 has at most C faces of codimension 1 not containing 0, so the volume of Q_k^0 is at most $C/l!$. Indeed a simplex with vertices $\{v_1, ..., v_l, 0\}$ has volume $1/l!$ if $\{v_1, ..., v_l\}$ is a basis of M.

Now we can prove that there is only a finite number of possible 1-dimensional cones. Let P be the convex hull of $0, e_1, ..., e_l$. Let τ be any such 1-cone, then the convex hull of P and $\rho(\tau)$ is contained in Q_k^0, so its volume is smaller than C/l'. The set of the vectors v, such that the volume of the convex hull of

P and v is smaller than C/l', is a bounded set. Since the $\rho(\tau)$ belong to N, the number of the possible τ is finite. Hence there is only a finite number of complete symmetric varieties. $\square$

Remark 1) We have not proved that there is a finite number of open Fano toric varieties.

2) We have proved that the rank of the Picard group of a Fano complete symmetric variety is lesser than $l+s+l\prod_1^l((2\delta-2\delta_0)(-e_i)+2)$ where $2s$ is the number of exceptional roots. Thus we have proved the following proposition.

Proposition 14.1 *Let G/H be a (homogeneous) symmetric variety, then there is a constant C such that the rank of $Pic(Y)$ is lesser of C for each Fano complete symmetric variety Y whose open orbit is isomorphic to G/H.*

15 (Almost) Fano open toric varieties of dimension 2

Now we want to classify the toric varieties proper over $\mathbf{A}^2$ with anticanonical bundle ample, respectively without base points. We start with a lemma which is false in higher dimension.

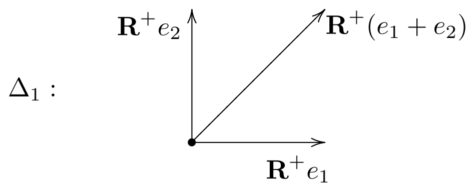
Lemma 15.1 *Let Z and Z' be any two (smooth) toric varieties of dimension 2. Suppose that the maximal cones of the fan Z are 2-dimensional and that Z' is proper over Z. If Z is not a Fano variety, then Z' is not a Fano variety.*

Proof. Let Δ and Δ' be the fans respectively of Z and Z'. Let k and k' be the functions associated to the anticanonical bundles respectively of Z and Z'. k is not strictly convex on Δ, so there are cones $\tau\in\Delta(1)$ and $\sigma\in\Delta(2)$ such that $(k|\sigma)(\rho(\tau))\leq k(\rho(\tau))=-1$ and τ is not contained in σ. We know that Z' is obtained from Z through a sequence of blow-ups (see theorem 1.28 in [O]), thus we can suppose, by the inductive hypothesis, that Z' is the blow-up of Z along the fixed point associated to a cone $\sigma'\in\Delta(2)$. Observe that τ belongs to the fan Δ' of Z'. If $\sigma'\neq\sigma$ then $\sigma\in\Delta'$ and $(k'|\sigma)(\rho(\tau))=(k|\sigma)(\rho(\tau))\leq -1=k'(\rho(\tau))$. Suppose now that $\sigma'=\sigma=\sigma(v_1,v_2)$ and let $\check{\sigma}=(\varphi_1,\varphi_2)$ be the dual cone, so $\varphi_i(v_j)=\delta_{i,j}$ and $k|\sigma=-(\varphi_1+\varphi_2)$. We know that $\Delta'(2)=(\Delta(2)\backslash\{\sigma\})\cup\{\sigma(v_1,v_1+v_2),\sigma(v_2,v_1+v_2)\}$. If k' is strictly convex on Δ' then $(k'|\sigma(v_1,v_1+v_2))(\rho(\tau))=(-\varphi_1)(\rho(\tau))\geq 0$ and $(k'|\sigma(v_2,v_1+v_2))(\rho(\tau))=(-\varphi_2)(\rho(\tau))\geq 0$, so $(k|\sigma)(\rho(\tau))=(-\varphi_1-\varphi_2)(\rho(\tau))\geq 0$, a contradiction. $\square$.

Now we can classify the Fano toric varieties proper over $\mathbf{A}^2$.

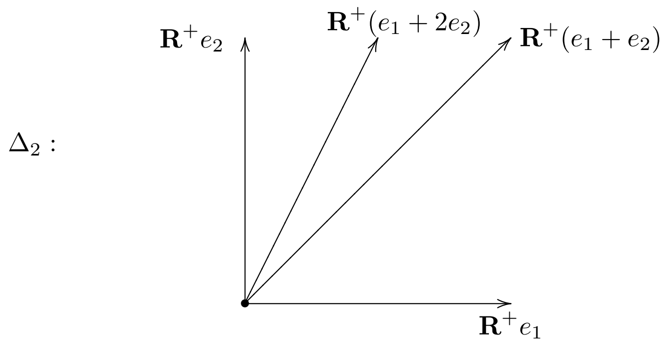
Proposition 15.1 *Let Z be a Fano toric variety proper over $\boldsymbol{A}^2$, then Z is $\boldsymbol{A}^2$ or it is the blow-up of $\boldsymbol{A}^2$ in the unique fixed point.*

Proof. $\mathbf{A}^2$ is clearly a Fano variety. Let Z_1 be the blow-up of $\mathbf{A}^2$ in the unique fixed point, let Δ_1 be its fan and let k_1 be the function associated to its anticanonical bundle.



We have $\Delta_1(2) = \{\sigma(e_1, e_1+e_2), \sigma(e_2, e_1+e_2)\}$. Thus $k_1|\sigma(e_1, e_1+e_2) = -f_1$ and $k_1|\sigma(e_2, e_1+e_2) = -f_2$, so $(k_1|\sigma(e_1, e_1+e_2))(e_2) = -f_1(e_2) = 0$ and $(k_1|\sigma(e_2, e_1+e_2))(e_1) = -f_2(e_1) = 0$. Therefore Z_1 is a Fano variety.

Now we show that these are the only Fano varieties using the previous lemma. Let Z_2 be a blow-up of Z_1. We can suppose, up to isomorphisms, that Z_2 is the blow-up of Z_1 in the point associated to $\sigma(e_2, e_1+e_2)$.

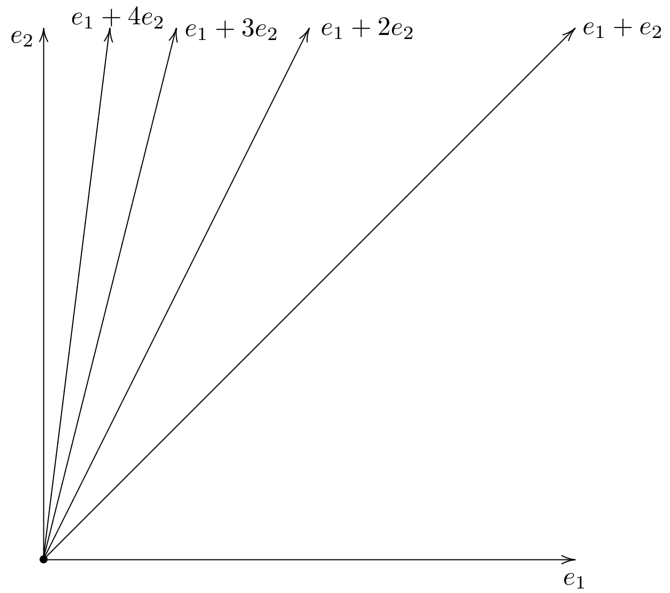


Let Δ_2 be the fan of Z_2, so $\Delta_2(2) = \{\sigma(e_1, e_1+e_2), \sigma(e_1+e_2, e_1+2e_2), \sigma(e_1+2e_2, e_2)\}$. Let k_2 be the function associated to the anticanonical bundle of Z_2, we have $k_2|\sigma(e_1, e_1+e_2) = k_2|\sigma(e_1+e_2, e_1+2e_2) = -f_1$, so k_2 is not strictly convex on Δ_2 and Z_2 is not a Fano variety. The proposition is implied by the previous lemma. □

Now we want to classify the almost Fano toric varieties proper over $\mathbf{A}^2$.

Proposition 15.2 *The almost-Fano toric varieties proper over $\boldsymbol{A}^2$ are, up to isomorphisms, $\boldsymbol{A}^2$ and the varieties Z_n, whose fan Δ_n is such that* $\Delta_n(2) = \{\sigma(e_1, e_1+e_2), \sigma(e_1+e_2, e_1+2e_2), ..., \sigma(e_1+(n-1)e_2, e_1+ne_2), \sigma(e_1+ne_2, e_2)\}$.

Δ_4 :



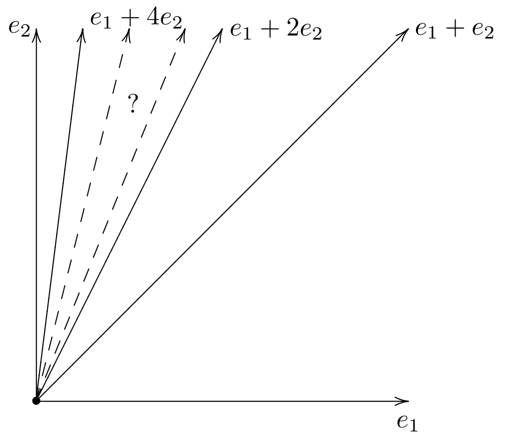
Proof. First of all we will show that the varieties Z_n are almost-Fano. Let k_n be the function associated to the anticanonical bundle of Z_n. k_n is linear on $\sigma(e_1, e_1 + ne_2)$ and $k_n|\sigma(e_1, e_1 + ne_2) = -f_1$; indeed $(-f_1)(e_1 + me_2) = -1$ for any m (observe that $\sigma(e_1, e_1 + ne_2)$ does not belong to Δ_n if $n > 1$). Moreover $(-f_1)(e_2) = 0 > -1$. We have $k_n|\sigma(e_1 + ne_2, e_2) = (n-1)f_1 - f_2$ and $(k_n|\sigma(e_1 + ne_2, e_2))(e_1 + re_2) = ((n-1)f_1 - f_2)(e_1 + re_2) = n - 1 - r > -1$ if $r < n$. Therefore k is convex and Z_n is almost-Fano.

Now we want to show the viceversa. Let Z be an almost-Fano toric variety proper over $\mathbf{A}^2$ and let Δ be its fan. (Recall that Z is obtained from $\mathbf{A}^2$ through a sequence of blow-ups). We can suppose that Z is different from $\mathbf{A}^2$ and Z_1, so $\tau = \mathbf{R}^+(e_1 + e_2)$ belongs to the fan Δ. We want to show that, up to isomorphisms, Δ contains the cones $\sigma(e_1, e_1 + e_2)$ and $\sigma(e_1 + e_2, e_1 + 2e_2)$. First of all we will determine the restrictions of k to the cones containing τ and afterwards we will determine the cones themselves. Let $\sigma \in \Delta(2)$ be a maximal cone containing τ and write $k|\sigma = a_1 f_1 + a_2 f_2$, so $(k|\sigma)(e_1 + e_2) = a_1 + a_2 = -1$, $(k|\sigma)(e_1) = a_1 \geq -1$ and $(k|\sigma)(e_2) = a_2 \geq -1$. This implies that the unique possibilities for $k|\sigma$ are $-f_1$ and $-f_2$. If $k|\sigma = -f_1$ and $\sigma = \sigma(e_1 + e_2, b_1 e_1 + b_2 e_2)$, then $-1 = (k|\sigma)(b_1 e_1 + b_2 e_2) = -b_1$ and $\sigma = \sigma(e_1 + e_2, e_1 + b_2 e_2)$. In the same way, if $\lambda = -f_2$ then we have $\sigma = \sigma(e_1 + e_2, b_1 e_1 + e_2)$. Because of the non-singularity of σ the only possibilities for σ are $\sigma(e_1 + e_2, e_1)$, $\sigma(e_1 + e_2, e_2)$, $\sigma(e_1 + e_2, e_1 + 2e_2)$ and $\sigma(e_1 + e_2, 2e_1 + e_2)$. ($b_1 - b_2$ is the determinant of the matrix of the change of basis from the basis $\{e_1 + e_2, b_1 e_1 + b_2 e_2\}$ to the basis $\{e_1, e_2\}$). We have to show that Δ cannot contain both $\sigma(e_1 + e_2, e_1 + 2e_2)$ and $\sigma(e_1 + e_2, 2e_1 + e_2)$. This would imply that $(k|\sigma(e_1 + e_2, e_1 + 2e_2))(2e_1 + e_2) = (-f_1)(2e_1 + e_2) = -2 > -1$, a contradiction. Observe that if Δ contains $\sigma(e_1 + e_2, 2e_1 + e_2)$, then

Z is isomorphic to a variety whose fan contains $\sigma(e_1+e_2, e_1+2e_2)$ through the isomorphism induced by the automorphism of N that exchanges e_1 and e_2. So we can suppose that Δ contains $\sigma(e_1+e_2, e_1)$ and $\sigma(e_1+e_2, e_1+2e_2)$. Notice that Δ contains either $\sigma(e_1+e_2, e_1+2e_2)$ or $\sigma(e_1+e_2, 2e_1+e_2)$ because Z is not Z_1.

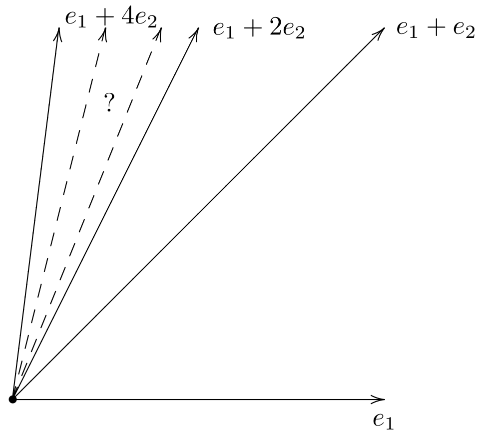
Because of the non-singularity of Z, Δ contains a cone $\sigma = \sigma(e_1+ne_2, e_2)$ for a suitable integer n; we want to show that Z is Z_n. (In the following figures we have drawn the rest of the proof in the case $n = 4$).

Δ :



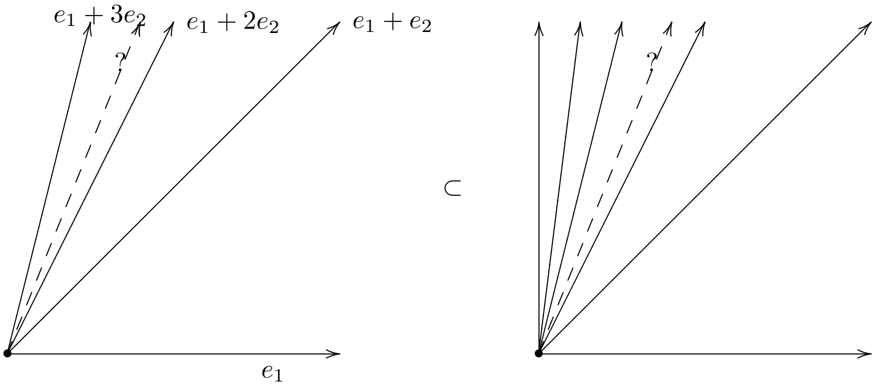
Let Z' be the open subvariety of Z whose fan Δ' is $\Delta \backslash \{\sigma(e_1+ne_2, e_2), \sigma(e_2)\}$. Z is an almost-Fano variety, so also Z' is an almost Fano variety.

Δ' :



We claim that, for all $m > 1$, there is an unique variety $\widetilde{Z}'_m$ with the two following properties: 1) the fan $\widetilde{\Delta}'_m$ of $\widetilde{Z}'_m$ has support $\sigma(e_1, e_1+me_2)$; 2) $\widetilde{Z}'_m$ is an open subvariety of an almost Fano variety $\widetilde{Z}_m$ whose fan $\widetilde{\Delta}_m$ has support $\sigma(e_1, e_2)$. Notice that the hypotheses imply that $\widetilde{Z}'_m$ is an almost-Fano variety. The open subvariety Z'_m of Z_m whose fan is $\Delta_m \backslash \{\sigma(e_1+me_2, e_2), \sigma(e_2)\}$, satisfies these properties. Observe that also Z' satisfies such properties with $m = n$, so it is sufficient to prove the claim. Indeed if the claim is true then $Z' = Z'_n$ and $\Delta' = \Delta_n \backslash \{\sigma(e_1 + ne_2, e_2), \sigma(e_2)\}$. Therefore $\Delta = \Delta_n$, so Z is Z_n.

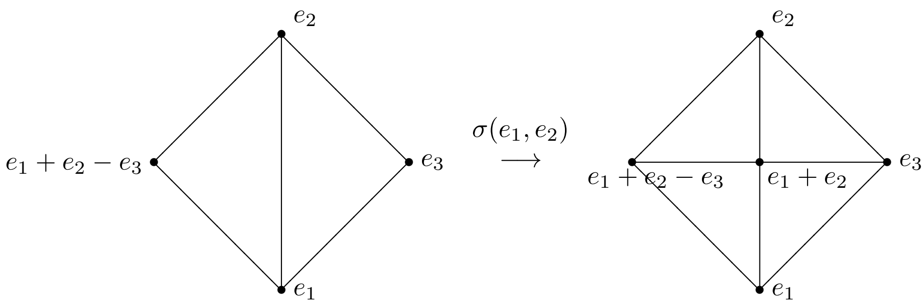
We show the claim for induction on m. We have already verified the basis of induction. Let $\widetilde{Z}'_m$ be a variety that satisfies the hypotheses of the claim and let σ' be the unique cone in $\widetilde{\Delta}'_m(2)$ which contains $e_1 + me_2$. Because of the inductive hypothesis it is sufficient to show that $\sigma' = \sigma(e_1+me_2, e_1+(m-1)e_2)$. In this case the fan $\widetilde{\Delta}'_m \backslash \{\sigma(e_1 + me_2, e_1 + (m-1)e_2), \sigma(e_1 + me_2)\}$ has support $\sigma(e_1, e_1 + (m-1)e_2)$ and the corresponding variety is an open subvariety of the almost-Fano toric variety $\widetilde{Z}_m$, so it is $\widetilde{Z}'_{m-1}$ by the inductive hypothesis. Therefore $\widetilde{\Delta}'_m \backslash \{\sigma(e_1+me_2, e_1+(m-1)e_2), \sigma(e_1+me_2)\} = \Delta_m \backslash \{\sigma(e_1+me_2, e_2), \sigma(e_1+me_2, e_1+(m-1)e_2), \sigma(e_2), \sigma(e_1+me_2)\}$, so $\widetilde{\Delta}'_m = \Delta_m \backslash \{\sigma(e_1+me_2e_2), \sigma(e_2))\}$.



Let k be the function associated to the anticanonical bundle of a fixed $\widetilde{Z}_m$. Notice that the restriction of k to the support of the fan $\widetilde{\Delta}'_m$ of $\widetilde{Z}'_m$ is the function associated to the anticanonical bundle of $\widetilde{Z}'_m$. Let $(k|\sigma') = a_1 f_1 + a_2 f_2$, we have $-1 = (k|\sigma')(e_1 + me_2) = a_1 + ma_2$, $(k|\sigma')(e_1) = a_1 \geq -1$ and $(k|\sigma')(e_2) = a_2 \geq -1$, so the unique possibilities for $(k|\sigma')$ are $-f_1$ and $(m-1)f_1 - f_2$. We have to determine the constants c and d such that $\sigma' = \sigma(e_1 + me_2, ce_1 + de_2)$. Suppose that $(k|\sigma') = -f_1$, then $c = -(k|\sigma')(ce_1 + de_2) = 1$. Because of the smoothness of σ' we have $d - m = \pm 1$ ($d - m$ is the determinant of the matrix of change of basis from $\{e_1 + me_2, e_1 + de_2\}$ to $\{e_1, e_2\}$). Thus we have two possibilities: either $\sigma' = \sigma(e_1 + me_2, e_1 + (m-1)e_2)$ or $\sigma' = \sigma(e_1 + me_2, e_1 + (m+1)e_2)$. We exclude the last one because $e_1 + (m+1)e_2$ does not belong to $\sigma(e_1 + me_2, e_1) = |\widetilde{\Delta}'_m|$. If $(k|\sigma') = (m-1)f_1 - f_2$, then $d = (m-1)c + 1$ because $((m-1)f_1 - f_2)(ce_1 + de_2) = -1$. Because of the non-singularity of σ' we have $c - 1 = \pm 1$, so there are two possibilities: either $\sigma' = \sigma(e_1 + me_2, e_2)$ or $\sigma' = \sigma(e_1 + me_2, 2e_1 + (2m-1)e_2)$. Again we exclude the first one because e_2 does not belong to $\sigma(e_1 + me_2, e_1)$. We exclude also the second one because $-1 = k(ce_1 + de_2) \leq (k|\sigma(e_1, e_1 + e_2))(ce_1 + de_2)$ and $(k|\sigma(e_1, e_1 + e_2))(2e_1 + (2m-1)e_2) = (-f_1)(2e_1 + (2m-1)e_2) = -2 < -1$. Thus we have proved that $ce_1 + de_2 = e_1 + (m-1)e_2$. $\square$

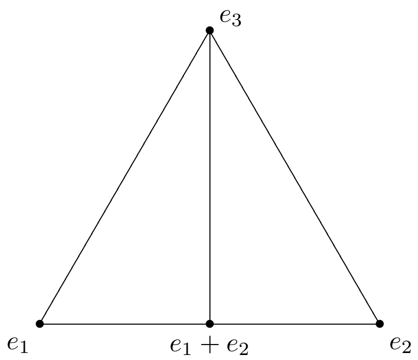
16 Fano toric varieties of dimension at least 3

We want to prove a generalization of the lemma 15.1. For example, we could try to prove that given a Fano toric variety Z, which is the blow-up of a toric variety Z', then also Z' is a Fano variety. Unluckily this is already false in dimension three. Let $\overline{Z}$ be the toric variety of dimension three whose fan $\overline{\Delta}$ has maximal cones $\sigma(e_1, e_2, e_3)$ and $\sigma(e_1, e_2, e_1 + e_2 - e_3)$. The function associated to the anticanonical bundle of $\overline{Z}$ is the restriction of $-f_1 - f_2 - f_3$ to $|\overline{\Delta}|$, so the anticanonical bundle is the trivial bundle and it is not ample. But the blow-up $\overline{Z}'$ of $\overline{Z}$ along the closed subvariety $\overline{Z}_{\sigma(e_1,e_2)}$ associated to $\sigma(e_1, e_2)$ is a Fano variety. Indeed the fan $\overline{\Delta}'$ of $\overline{Z}'$ has maximal cones $\sigma(e_1, e_3, e_1 + e_2)$, $\sigma(e_2, e_3, e_1 + e_2)$, $\sigma(e_1, e_1 + e_2, e_1 + e_2 - e_3)$ and $\sigma(e_2, e_1 + e_2 - e_3, e_1 + e_2)\}$.



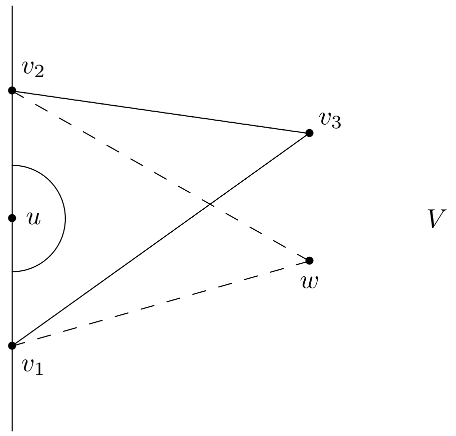
Let k be the function associated to the anticanonical bundle of $\overline{Z}'$. We have $(k|\sigma(e_1, e_3, e_1 + e_2))(e_2) = (-f_1 - f_3)(e_2) = 0 > -1$, $(k|\sigma(e_1, e_3, e_1 + e_2))(e_1 + e_2 - e_3) = (-f_1 - f_3)(e_1 + e_2 - e_3) = 0 > -1$, $(k|\sigma(e_2, e_3, e_1 + e_2))(e_1) = (-f_2 - f_3)(e_1) = 0 > -1$, $(k|\sigma(e_2, e_3, e_1 + e_2))(e_1 + e_2 - e_3) = (-f_2 - f_3)(e_1 + e_2 - e_3) = 0 > -1$, $(k|\sigma(e_1, e_1 + e_2, e_1 + e_2 - e_3))(e_2) = (-f_1)(e_2) = 0 > -1$, $(k|\sigma(e_1, e_1 + e_2, e_1 + e_2 - e_3))(e_3) = (-f_1)(e_3) = 0 > -1$, $(k|\sigma(e_2, e_1 + e_2, e_1 + e_2 - e_3))(e_1) = (-f_2)(e_1) = 0 > -1$ and $(k|\sigma(e_2, e_1 + e_2, e_1 + e_2 - e_3))(e_3) = (-f_2)(e_3) = 0 > -1$. Therefore k is strictly convex on $\overline{\Delta}'$ and $\overline{Z}'$ is a Fano variety. It is easy to make higher dimensional example like $\overline{Z}$, for example we can take $\overline{Z} \times \mathbf{A}^{l-2}$ and its blow-up along $\overline{Z}_{\sigma(e_1,e_2)} \times \mathbf{A}^{l-2}$. In these examples we always have considered blow-ups along subvarieties of positive dimension. This observation suggests to consider only blow-ups in S-fixed points. Indeed, we will classify the Fano toric variety obtained from $\mathbf{A}^l$ through a sequence of blow-ups in S-fixed points. Notice that in the lemma 15.1 the variety Z' is always obtained from Z through a sequence of blow-ups in fixed points.

In this section we will prove a generalization of the lemma 15.1 on a particular class of varieties of arbitrarily fixed dimension l. We consider the class of the smooth toric varieties whose fan contains two cones σ and σ' with the following properties: 1) the intersection $\sigma \cap \sigma'$ is a cone of dimension $l - 1$, so we can suppose $\sigma = \sigma(v_1, ..., v_{l-1}, v_l)$ and $\sigma' = \sigma(v_1, ..., v_{l-1}, w)$; 2) $v_l + w$ belongs to the intersection $\sigma \cap \sigma'$ and it is not zero. Now we want to show that this class contains "many" varieties. First of all it is not empty, for example it contains the blow-up of $\mathbf{A}^l$ along a stable subvariety of codimension 2. (In the following figure we draw a 3 dimensional example).



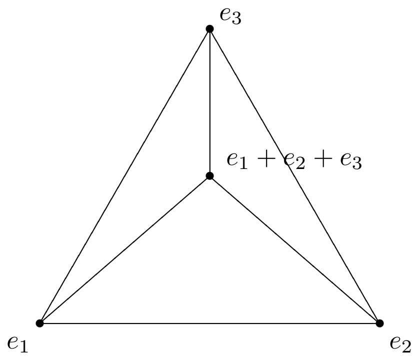
Observe that the hypotheses imply that w is not a multiple of v_l. We can show that the first hypothesis is a very weak request, for example it is satisfied by all varieties proper over $\mathbf{A}^l$ and different from $\mathbf{A}^l$. Indeed, let Z be any toric variety whose fan Δ contains a $(l-1)$-dimensional cone τ which is not contained in the border of $|\Delta|$, then there are exactly two cones σ and σ' in Δ which contain τ.

Moreover, given two cones $\sigma = \sigma(v_1, ..., v_{l-1}, v_l)$ and $\sigma' = \sigma(v_1, ..., v_{l-1}, w)$ with $w = \sum a_i v_i$, we have $a_l = \pm 1$ because of the smoothness of Z (a_l is the determinant of the matrix of basis change from the basis $\{v_1, ..., v_{l-1}, w\}$ to the basis $\{v_1, ..., v_{l-1}, v_l\}$). If $a_l = 1$, then σ and σ' are contained in the same semi-space V with border $\mathbf{R}(\sigma \cap \sigma')$. Thus, given any vector u is in the relative interior of $\sigma \cap \sigma'$, u is in the interior of σ (respectively of σ') respect to the relative topology of V. Hence $\sigma \cap \sigma'$ contains an open set of V, a contradiction. Therefore $a_l = -1$.



Observe that $a_l = -1$ implies that $v_l + w$ is contained in the vector space generated by $\sigma \cap \sigma'$ and the second hypothesis is equivalent to the request that $a_i \geq 0$ for any i different from l. This hypothesis is more restrictive, indeed

$\sigma \cup \sigma'$ is convex if and only if $v_l + w \in \sigma \cup \sigma'$. Notice that all the smooth toric varieties proper over $\mathbf{A}^2$ and different from $\mathbf{A}^2$ belong to our class. In higher dimension it is easy to construct varieties whose cones do not satisfy the second hypothesis, for example it is not satisfied by the blow-up of $\mathbf{A}^l$ in the fixed point.

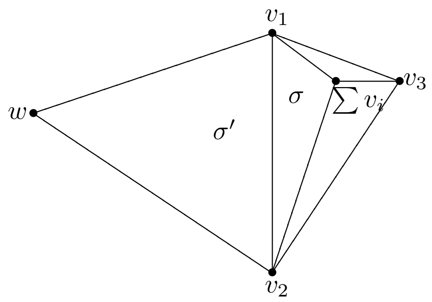


The hypotheses imply that $\sigma \cup \sigma' = \sigma(v_1, v_2, v_3, w)$ and $\sigma \cap \sigma' = \sigma(v_1, v_2)$. Moreover this class of varieties is stable by blow-ups centered in fixed points. Indeed, let Z be a variety which satisfies the hypotheses with respect to σ and σ' and let Z' be a blow-up of Z in a S-fixed point. Then the fan of Z' contain two cones $\varsigma \subseteq \sigma$ and $\varsigma' \subseteq \sigma'$ of dimension l such that $\varsigma \cap \varsigma' = \sigma \cap \sigma'$. These cones are univocally defined by these conditions and they satisfy our request, so Z' belongs to our class. Now we can prove a generalization of the lemma 15.1.

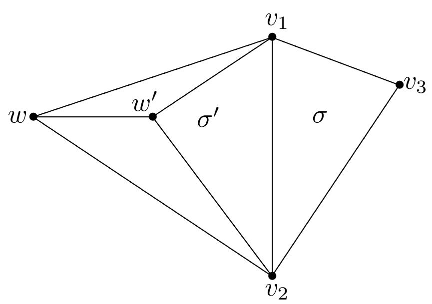
Lemma 16.1 *Let Z be a toric variety with fan Δ and let k be the function associated to the anticanonical bundle of Z. Let Z' be any toric variety obtained from Z through a sequence of blow-ups centred in S-fixed points. Suppose that there are two cones $\sigma = \sigma(v_1, ..., v_{l-1}, v_l)$ and $\sigma' = \sigma(v_1, ..., v_{l-1}, w)$ in $\Delta(l)$ such that $v_l + w$ belongs to the intersection $\sigma \cap \sigma' = \sigma(v_1, ..., v_{l-1})$ and $v_l \neq -w$. If $(k|\sigma(v_1, ..., v_{l-1}, v_l))(w) \leq -1$, then Z' is not a Fano variety.*

Observe that $(k|\sigma(v_1, ..., v_{l-1}, v_l))(w) \leq -1$ implies that Z is not a Fano variety. Moreover, we know that $w = \sum_{i=1}^{l-1} a_i v_i - v_l$ where the a_i are positive integers.

Proof. We can suppose $l \geq 3$ because of the lemma 15.1. We will show the lemma by induction, so we can suppose that Z' is the blow-up of Z centred in the fixed point associated to a cone $\widetilde{\sigma}$ in $\Delta(l)$. Let k' be the function associated to the anticanonical bundle of Z'. If $\widetilde{\sigma}$ is different from $\sigma(v_1, ..., v_{l-1}, v_l)$ and $\sigma(v_1, ..., v_{l-1}, w)$, then Z' satisfies the hypotheses of the lemma for $\sigma(v_1, ..., v_{l-1}, v_l)$ and $\sigma(v_1, ..., v_{l-1}, w)$. Suppose that $\widetilde{\sigma} = \sigma(v_1, ..., v_{l-1}, v_l)$,



then $\sigma(v_1, ..., v_{l-1}, \sum_{i=1}^{l} v_i)$ and $\sigma(v_1, ..., v_{l-1}, w)$ belong to the fan of Z'. For each j in $\{1, ..., l\}$, let φ_j be the linear function such that $\varphi_j(v_i) = \delta_{i,j}$. We have $w + (\sum_{i=1}^{l} v_i) = \sum_{i=1}^{l-1}(a_i + 1)v_i$, $k|\sigma(v_1, ..., v_{l-1}, v_l) = -\sum \varphi_i$ and $(k'|\sigma(v_1, ..., v_{l-1}, \sum_{i=1}^{l} v_i)(w) = (-\sum_{i=1}^{l-1} \varphi_i + (l-2)\varphi_l)(w) = (k|\sigma(v_1, ..., v_{l-1}, v_l))(w) + (l-1)\varphi_l(w) = (k|\widetilde{\sigma})(w) - (l-1) < -1$, so Z' satisfies the hypotheses of the lemma for $\sigma(v_1, ..., v_{l-1}, \sum_{i=1}^{l} v_i)$ and $\sigma(v_1, ..., v_{l-1}, w)$. Finally we suppose $\widetilde{\sigma} = \sigma'$.



The fan of Z' contains the cones $\sigma(v_1, ..., v_{l-1}, v_l)$ and $\sigma(v_1, ..., v_{l-1}, \sum_{i=1}^{l-1} v_i + w)$. We have $v_l + (\sum_{i=1}^{l-1} v_i + w) = \sum_{i=1}^{l-1}(a_i + 1)v_i$ and $(k|\sigma(v_1, ..., v_{l-1}, v_l))$ $(\sum_{i=1}^{l-1} v_i + w) = (k|\sigma(v_1, ..., v_{l-1}, v_l))(w) - (l-1) < -1$, so Z' satisfy the hypotheses of the lemma for $\sigma(v_1, ..., v_{l-1}, v_l)$ and $\sigma(v_1, ..., v_{l-1}, \sum_{i=1}^{l-1} v_i + w)$. □

Now we can classify the Fano toric varieties obtained from $\mathbf{A}^l$ through a sequence of blow-ups centred in S-fixed points.

Proposition 16.1 *Let Z be a Fano toric variety obtained from $\boldsymbol{A}^l$ through a sequence of blow-ups centred in S-fixed points, then either Z is $\boldsymbol{A}^l$ or Z is the blow-up of $\boldsymbol{A}^l$ in the origin. Moreover these varieties are Fano.*

Proof. Let Z_1 be the blow-up of $\mathbf{A}^l$ in the S-stable point, let Δ_1 be the fan of Z_1 and let k_1 be the function associated to the anticanonical bundle of Z_1. The maximal cones in Δ_1 are $\{\sigma(e_1, ..., \widehat{e_j}, ..., e_l, \sum e_i)\}$ where j varies in $\{1, ..., l\}$, while the 1-dimensional cones are $\sigma(e_1), ..., \sigma(e_l)$ and $\sigma(\sum e_i)$. We have $(k|\sigma(e_1, ..., \widehat{e_j}, ..., e_l, \sum e_i))(e_j) = (-\sum_{i \neq j} f_j + (l-2)f_j)(e_j) = l - 2 > -1$. $\mathbf{R}^+ e_j$ is the unique 1-dimensional cone not contained in $\sigma(e_1, ..., \widehat{e_j}, ..., e_l, \sum e_i)$, so k_1 is strictly convex on Δ_1 and Z_1 is a Fano variety.

The blow-ups of Z_1 in a S fixed point are all isomorphic to the blow-up Z_2 of Z_1 in the point associated to $\sigma(e_1, ..., e_{l-1}, \sum e_i)$. Let Δ_2 be the fan of Z_2 and let k_2 be the function associated to the anticanonical bundle of Z_2. The maximal cones in Δ_2 are: $\sigma(e_1, ..., \widehat{e_j}, ..., e_l, \sum_{i=1}^{l} e_i)$, $\sigma(e_1, ..., \widehat{e_j}, ..., e_{l-1}, \sum_{i=1}^{l} e_i, 2\sum_{i=1}^{l-1} e_i + e_l)$ with $j = 1, ..., l-1$ and $\sigma(e_1, ..., e_{l-1}, 2\sum_{i=1}^{l-1} e_i + e_l)$. It is sufficient to show that Z_2 satisfies the hypotheses of the lemma 16.1 respect to $\sigma(e_2, ..., e_{l-1}, \sum_{i=1}^{l} e_i, e_l)$ and $\sigma(e_2, ..., e_{l-1}, \sum_{i=1}^{l} e_i, 2\sum_{i=1}^{l-1} e_i + e_l)$. We have $(2\sum_{i=1}^{l-1} e_i + e_l) + e_l = 2(\sum_{i=1}^{l} e_i)$ and $(k_2|\sigma(e_2, ..., e_{l-1}, \sum_{i=1}^{l} e_i, e_l))(2\sum_{i=1}^{l-1} e_i + e_l) = ((l-2)f_1 - \sum_{i=2}^{l-1} f_i - f_l)(2e_1 + 2\sum_{i=2}^{l-1} e_i + e_l) = 2(l-2) - 2(l-2) - 1 = -1$.
□

17 Introduction to the open Fano toric varieties of dimension 3

For varieties of dimension 3 we can consider also blow-ups along varieties of positive dimension. The proof will be similar to the previous one, but much more difficult. We study again the class of varieties of the previous section and we will prove a lemma similar to the lemma 16.1, but with stronger hypotheses on the beginning variety Z. Thus we consider the class of varieties Z whose fans Δ contain two 3-dimensional cones $\sigma(v_1, v_2, v_3)$ and $\sigma(v_1, v_2, w)$ with $w = a_1 v_1 + a_2 v_2 - v_3$. We suppose that a_1 and a_2 are positive integers such that $a_1 + a_2 > 0$. Up to reordering the indices we can suppose that $a_1 \geq a_2$. Recall that $\sigma(v_1, v_2, v_3) \cup \sigma(v_1, v_2, w)$ is the convex cone $\sigma(v_1, v_2, v_3, w)$ (but it does not belong to Δ).

Let $\widetilde{Z}$ be the open subvariety of Z whose fan $\widetilde{\Delta}$ has maximal cones $\sigma(v_1, v_2, v_3)$ and $\sigma(v_1, v_2, w)$. We want to find the conditions for $\widetilde{Z}$ to be an (almost) Fano variety. The function associated to the anticanonical bundle of $\widetilde{Z}$ is the restriction to the support $|\widetilde{\Delta}|$ of the function k associated to the anticanonical bundle of Z. For each j, let φ_j be the linear function such that $\varphi_j(v_i) = \delta_{i,j}$, so $(k|\sigma(v_1, v_2, v_3)) = -\varphi_1 - \varphi_2 - \varphi_3$. If $\widetilde{Z}$ is a Fano variety then $-a_1 - a_2 + 1 = (k|\sigma(v_1, v_2, v_3))(w) \geq 0$, so $a_1 + a_2 = 1$. Therefore $v_3 + w = v_1$ and $\widetilde{Z}$ is the blow up of $\mathbf{A}^3$ along a stable subvariety of dimension 1. This is a Fano variety, indeed we have $(k|\sigma(v_1, v_2, v_3)(w) = (-\varphi_1 - \varphi_2 - \varphi_3)(v_1 - v_3) = 0 > -1$ and $(k|\sigma(v_1, v_2, w)(v_3) = (-\varphi_1 - \varphi_2)(v_3) = 0 > -1$. If $\widetilde{Z}$ is an almost Fano variety then either $\widetilde{Z}$ is a Fano variety or k is linear on $|\widetilde{\Delta}|$. If k is linear then $-a_1 - a_2 + 1 = (k|\sigma(v_1, v_2, v_3)(w) = -1$, so $a_1 + a_2 = 2$. We have two possibilities: either $w = v_1 + v_2 - v_3$ or $w = 2v_1 - v_3$. In the first case we obtain a variety isomorphic to the variety $\overline{Z}$ of the previous paragraph. This is the case in which we will have more problems, so we will study it in a second time.

Lemma 17.1 *Let Z be a toric variety which contains an open toric subvariety $\widetilde{Z}$ whose fan has maximal cones $\sigma(v_1, v_2, v_3)$ and $\sigma(v_1, v_2, w)$. Write $w + v_3 = a_1 v_1 + a_2 v_2$ where a_1 and a_2 are positive integers with $a_1 \geq a_2$. We suppose that:*

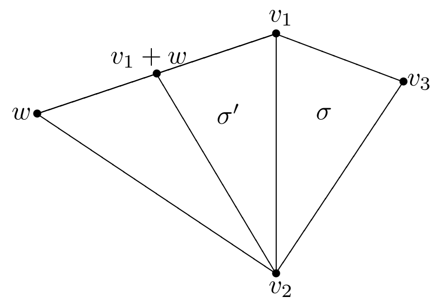
1. $a_1 + a_2 \geq 2$,

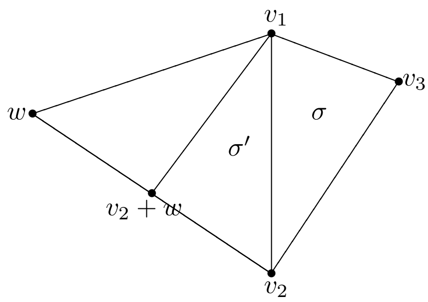
2. $v_3 + w \neq v_1 + v_2$.

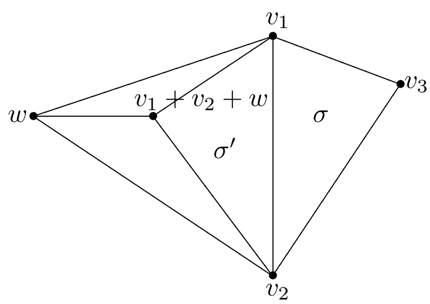
Let Z' be a toric variety obtained from Z through a sequence of blow-ups, then Z' is not a Fano variety.

Proof. First of all we want to do some considerations on the hypotheses. i) The inequality $a_1 + a_2 \geq 2$ is equivalent to the condition $(k|\sigma(v_1, v_2, v_3))(w) \leq -1$, so $\widetilde{Z}$ and Z are not Fano varieties. ii) Because of the previous observations the two hypotheses $a_1 + a_2 \geq 2$ and $v_3 + w \neq v_1 + v_2$ are equivalent to the inequality $a_1 \geq 2$. An useful observation is that the inequality $a_1 + a_2 > 2$ implies $a_1 \geq 2$. We will say that a variety satisfies the hypotheses of the lemma weakly if either it satisfies the hypotheses of the lemma or it contains a variety $\widetilde{Z}$ isomorphic to $\overline{Z}$, while we will sometimes say that a variety satisfies the hypotheses properly if it satisfies the hypotheses. We could try to prove this lemma as the lemma 16.1 by induction. Indeed we will prove that any blow-up of Z satisfies the hypotheses of the lemma weakly, but unluckily we cannot prove that $v_3 + w \neq v_1 + v_2$. However, first we consider the case in which Z' is a blow up of Z along a closed subvariety and afterwards we try to resolve the problem. We will demonstrate that if Z' is a blow up of Z along the subvariety associated to a cone τ, then Z' satisfies always the hypotheses weakly and it satisfies the hypotheses of the lemma properly if $\tau \neq \sigma(v_1, v_2)$. In general we have a sequence $Z = Z_0 \leftarrow Z_1 \leftarrow ... \leftarrow Z_i \leftarrow ... \leftarrow Z_r = Z'$ where Z_{i+1} is the blow-up of Z_i along the stable subvariety associated to a suitable cone τ_i. If Z_j satisfies the hypotheses, then Z_{j+1} will satisfies the hypotheses weakly, in particular Z_{j+1} is not a Fano variety. If Z_{j+1} satisfies the hypotheses properly we can proceed by induction. Otherwise Z_{j+1} contains a variety isomorphic to $\overline{Z}$. Let $\underline{\Delta}$ be the fan of such variety. In this case we have two possibilities: either this fan $\underline{\Delta}$ is contained in the fan of Z_i for all $i > j$ or there are $j < h < r$ such that $\underline{\Delta}$ is not contained in the fan of Z_{h+1}, but it is contained in the fan of Z_i for all $j < i \leq h$. In the first case Z_i satisfies the hypotheses weakly for all $i > j$, in particular Z' is not a Fano variety. In the second case we will prove that Z_{h+1} satisfies the hypotheses of the lemma.

Now we suppose that Z' is the blow-up of Z along the closed subvariety associated to a cone τ. Let Δ' be the fan of Z'. If τ is not contained in $\sigma(v_1, v_2, v_3, w)$ then there is nothing to prove because $\sigma(v_1, v_2, v_3)$ and $\sigma(v_1, v_2, w)$ belong to Δ'. We now suppose that τ is contained in $\sigma(v_1, v_2, v_3, w)$ but it is not contained in $\sigma(v_1, v_2)$. Observe that the hypotheses are symmetric in the two cones $\sigma(v_1, v_2, v_3)$ and $\sigma(v_1, v_2, w)$, so we can suppose that $\tau \subset \sigma(v_1, v_2, w)$. We have three possibilities: $\tau = \sigma(v_1, w)$, $\tau = \sigma(v_2, w)$ and $\tau = \sigma(v_1, v_2, w)$.

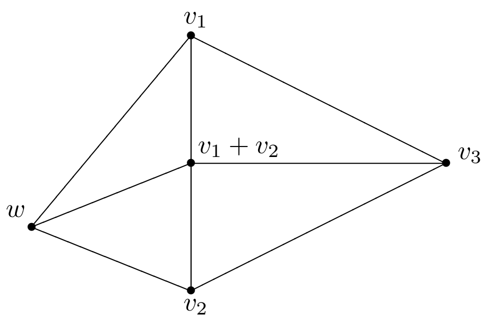






We always have $\Delta'(1) = \Delta(1) \cup \{\mathbf{R}^+(w + b_1v_1 + b_2v_2)\}$ with $b_1, b_2 \in \{0, 1\}$. Δ' contains the cones $\sigma(v_1, v_2, v_3)$ and $\sigma(v_1, v_2, w + b_1v_1 + b_2v_2)$. We have $(w + b_1v_1 + b_2v_2) + v_3 = (a_1 + b_1)v_1 + (a_2 + b_2)v_1$ with $(a_1 + b_1) + (a_2 + b_2) > 2$, so Z' satisfies the hypotheses of the lemma. We want to remark that this part of the prove does not require the last hypothesis.

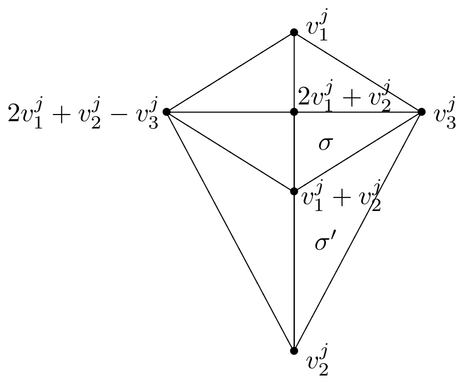
Finally let $\tau = \sigma(v_1, v_2)$.



The fan of Z' contains the cones $\sigma(v_1, v_1 + v_2, v_3)$ and $\sigma(v_1, v_1 + v_2, w)$. We have $v_3 + w = (a_1 - a_2)v_1 + a_2(v_1 + v_2)$ with $(a_1 - a_2) + a_2 = a_1 \geq 2$. So Z' satisfies the hypotheses of the lemma weakly. We want to emphasize that this is the first case in which we have used the last hypothesis.

Now we can consider the general case. We have a sequence $Z = Z_0 \leftarrow Z_1 \leftarrow \ldots \leftarrow Z_i \leftarrow \ldots \leftarrow Z_r = Z'$ where Z_{i+1} is the blow-up of Z_i along the subvariety associated to a suitable cone τ_i. Let Δ_i be the fan of Z_i for each i. Suppose that Z_j satisfies the hypotheses of the lemma with respect to $\sigma(v_1^j, v_2^j, v_3^j)$

and $\sigma(v_1^j, v_2^j, w^j)$. If $\tau_j \neq \sigma(v_1^j, v_2^j)$ then Z_{j+1} satisfies the hypotheses of the lemma, in particular it is not a Fano variety. Hence we can suppose that $\tau_j = \sigma(v_1^j, v_2^j)$. Let $\sigma(v_1^{j+1}, v_2^{j+1}, v_3^{j+1})$ and $\sigma(v_1^{j+1}, v_2^{j+1}, w^{j+1})$ in Δ_{j+1} be two cones with respect to which Z_{j+1} satisfies the hypotheses weakly. If Z_{j+1} satisfies the hypotheses with respect to $\sigma(v_1^{j+1}, v_2^{j+1}, v_3^{j+1})$ and $\sigma(v_1^{j+1}, v_2^{j+1}, w^{j+1})$, then we can proceed by induction. Otherwise $\overline{Z}$ is isomorphic to the variety whose fan has maximal cones $\sigma(v_1^{j+1}, v_2^{j+1}, v_3^{j+1})$ and $\sigma(v_1^{j+1}, v_2^{j+1}, w^{j+1})$. If $\sigma(v_1^{j+1}, v_2^{j+1}, v_3^{j+1})$ and $\sigma(v_1^{j+1}, v_2^{j+1}, w^{j+1})$ belong to the fan of Z', then Z' is not a Fano variety. Otherwise there is a h such that: 0) $j < h < r$; 1) $\sigma(v_1^{j+1}, v_2^{j+1}, v_3^{j+1})$ and $\sigma(v_1^{j+1}, v_2^{j+1}, w^{j+1})$ belong to Δ_i for all $j+1 \leq i \leq h$; 2) either $\sigma(v_1^{j+1}, v_2^{j+1}, v_3^{j+1})$ or $\sigma(v_1^{j+1}, v_2^{j+1}, w^{j+1})$ does not belong to Δ_{h+1}. We can suppose that $\sigma(v_1^{j+1}, v_2^{j+1}, v_3^{j+1})$ does not belong to Δ_{h+1}, otherwise we exchange $\sigma(v_1^{j+1}, v_2^{j+1}, v_3^{j+1})$ with $\sigma(v_1^{j+1}, v_2^{j+1}, w^{j+1})$. In the first part of the proof we have proved that, if $\tau_h \neq \sigma(v_1^{j+1}, v_2^{j+1})$, then Z_{h+1} satisfies the hypotheses of the lemma, so we can suppose that $\tau_h = \sigma(v_1^{j+1}, v_2^{j+1})$. Moreover we can suppose $\sigma(v_1^{j+1}, v_2^{j+1}, v_3^{j+1}) = \sigma(v_1^j, v_1^j + v_2^j, v_3^j)$ and $\sigma(v_1^{j+1}, v_2^{j+1}, w^{j+1}) = \sigma(v_1^j, v_1^j + v_2^j, w^j)$. Let $\widetilde{Z}_j$ be the toric variety whose fan has maximal cones $\sigma(v_1^j, v_2^j, v_3^j)$ and $\sigma(v_1^j, v_2^j, w^j)$; it is an open subvariety of Z_j. The inverse image $\widetilde{Z}_{j+1}$ of $\widetilde{Z}_j$ in Z_{j+1} is the blow-up of $\widetilde{Z}_j$ along the subvariety associated to $\sigma(v_1^j, v_2^j)$. (Observe that the closed subvariety of Z_j associated to $\sigma(v_1^j, v_2^j)$ is contained in the open subvariety $\widetilde{Z}_j$). In the same way the inverse image $\widetilde{Z}_{h+1}$ in Z_{h+1} of $\widetilde{Z}_{j+1}$ is the blow-up of $\widetilde{Z}_{j+1}$ along the subvariety associated to $\sigma(v_1^j, v_1^j + v_2^j)$. We want to show that $\widetilde{Z}_{h+1}$ satisfies the hypotheses of the lemma with respect to two suitable cones, so Z_{h+1} satisfies the hypotheses of the lemma with respect to the same cones.



Observe that $w^j = 2v_1^j + v_2^j - v_3^j$, because $\overline{Z}$ is isomorphic to the variety whose fan has maximal cones $\sigma(v_1^j, v_1^j + v_2^j, v_3^j)$ and $\sigma(v_1^j, v_1^j + v_2^j, w^j)$. The fan of $\widetilde{Z}_j$ has maximal cones $\sigma(v_1^j, v_2^j, v_3^j)$ and $\sigma(v_1^j, v_2^j, 2v_1^j + v_2^j - v_3^j)$. The fan of $\widetilde{Z}_{j+1}$ has maximal cones $\sigma(v_1^j, v_1^j + v_2^j, v_3^j)$, $\sigma(v_2^j, v_1^j + v_2^j, v_3^j)$, $\sigma(v_1^j, v_1^j + v_2^j, 2v_1^j + v_2^j - v_3^j)$ and $\sigma(v_2^j, v_1^j + v_2^j, 2v_1^j + v_2^j - v_3^j)$. Finally the fan of $\widetilde{Z}_{h+1}$ has maximal cones $\sigma(v_2^j, v_1^j + v_2^j, v_3^j)$, $\sigma(v_2^j, v_1^j + v_2^j, 2v_1^j + v_2^j - v_3^j)$, $\sigma(v_1^j + v_2^j, 2v_1^j + v_2^j, v_3^j)$, $\sigma(v_1^j +$

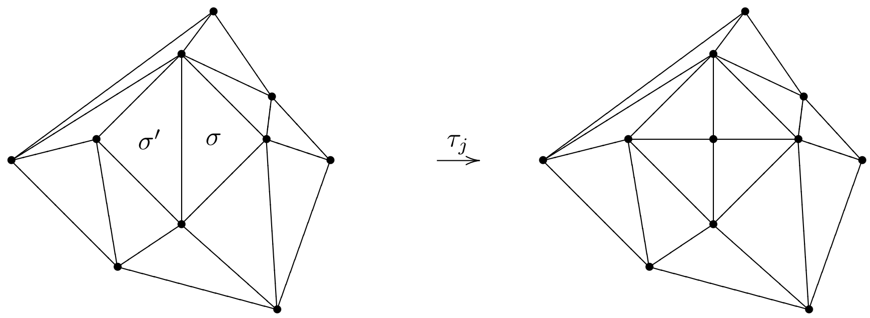
$v_2^j, 2v_1^j + v_2^j, 2v_1 + v_2 - v_3)$, $\sigma(v_1^j, 2v_1^j + v_2^j, v_3^j)$ and $\sigma(v_1^j, 2v_1^j + v_2^j, 2v_1^j + v_2^j - v_3^j)$. Observe that $\widetilde{Z}_{r+1}$ satisfies the hypotheses of the lemma with respect to $\sigma(v_1^j + v_2^j, v_3^j, 2v_1^j + v_2^j)$ and $\sigma(v_2^j, v_3^j, v_1^j + v_2^j)$; indeed we have $(2v_1^j + v_2^j) + v_2^j = 2(v_1^j + v_2^j)$. □

Now we want to study the varieties which contain an open subvariety isomorphic to $\overline{Z}$. Observe that these varieties are never Fano varieties. Let Z be a such variety and let Z' be the blow-up of Z along the subvariety isomorphic to the subvariety of $\overline{Z}$ associated to $\sigma(e_1, e_2)$. We will show that, if Z' satisfies the hypotheses of the lemma 17.1, then there are not Fano varieties obtainable from Z through a sequence of blow-ups.

Lemma 17.2 *Let Z be a 3 dimensional toric variety whose fan contains two cones $\sigma(v_1, v_2, v_3)$ and $\sigma(v_1, v_2, v_1 + v_2 - v_3)$. Let Z' be the blow-up of Z along the stable subvariety associated to $\sigma(v_1, v_2)$ and let Z'' be a toric variety obtained from Z through a sequence of blow-ups. If Z'' is a Fano variety, then Z'' is obtainable from Z' through a sequence of blow-ups.*

Proof. We cannot proceed as in the previous lemma, because we do not know the other cones of Z. We have a sequence $Z = Z_0 \leftarrow Z_1 \leftarrow ... \leftarrow Z_i \leftarrow ... \leftarrow Z_h = Z''$ where $\pi_{i+1} : Z_{i+1} \to Z_i$ is the blow-up along the subvariety of Z_i associated to a suitable cone τ_{i+1}. Let Δ_i be the fan of Z_i for each i. First of all we want to show that there is a cone τ_j equal to $\sigma(v_1, v_2)$. If τ_i is not contained in $\sigma(v_1, v_2, v_3, v_1 + v_2 - v_3)$ for any i, then $\sigma(v_1, v_2, v_3)$ and $\sigma(v_1, v_2, v_1 + v_2 - v_3)$ belong to Δ_i for all i. Thus Z'' is not a Fano variety, a contradiction. Let j be the first index such that τ_j is contained in $\sigma(v_1, v_2, v_3, v_1 + v_2 - v_3)$, so $\tau_i \nsubseteq \sigma(v_1, v_2, v_3, v_1 + v_2 - v_3)$ for all $i < j$. If $\tau_j \neq \sigma(v_1, v_2)$ we know that Z_j satisfies the hypotheses of the lemma 17.1, so Z'' is not a Fano variety, a contradiction. Therefore there is j such that $\tau_j = \sigma(v_1, v_2)$ and $\tau_i \nsubseteq \sigma(v_1, v_2, v_3, v_1 + v_2 - v_3)$ for all $i < j$. We want to reorder the cones associated to the subvarieties along which we are blowing-up. Observe that this is not possible in general. We will show that Z'' is obtainable from Z' through the sequence of blow-ups along the subvarieties associated to the cones $\tau_1, ..., \hat{\tau}_j, ..., \tau_h$.

We want to consider the following sequence of blow-ups: $Z = Z'_0 \leftarrow Z'_1 \leftarrow ... \leftarrow Z'_i \leftarrow ... \leftarrow Z'_j$, where $\pi'_1 : Z'_1 \to Z'_0$ is the blow-up along the subvariety of Z'_0 associated to τ_j and $\pi'_{i+1} : Z'_{i+1} \to Z'_i$ is the blow up along the subvariety of Z'_i associated to τ_i for each $i \geq 1$. Let Δ'_i be the fan of Z'_i. We want to show that these blow-ups are well defined and that $Z'_j = Z_j$. For the first point we have to show that τ_i belongs to Δ'_i for each $i < j$. Because $\tau_j = \sigma(v_1, v_2)$ the elements of $\Delta'_1(3)$ not contained in $\sigma(v_1, v_2, v_3, v_1 + v_2 - v_3)$ are exactly the elements of $\Delta_0(3)$ not contained in $\sigma(v_1, v_2, v_3, v_1 + v_2 - v_3)$, i.e. the elements of $\Delta_0(3)$ different from $\sigma(v_1, v_2, v_3)$ and $\sigma(v_1, v_2, v_1 + v_2 - v_3)$.



The first claim follows because τ_i is contained in $|\Delta|\backslash\sigma(v_1, v_2, v_3, v_1+v_2-v_3)$ for each $i < j$. Recall that, given $\varsigma \in \Delta(l)$, U_ς is the open subvariety of Z associated to the cone ς. Z is the union of the open sets U_1 and U_2 defined as follows: U_1 is the union of $U_{\sigma(v_1,v_2,v_3)}$ and $U_{\sigma(v_1,v_2,v_1+v_2-v_3)}$, while U_2 is the union of the U_ς where the ς varies in all the other maximal cones. The blow-up π'_1 induces an isomorphism between U_2 and its inverse image, because the subvariety of Z associated to $\sigma(v_1, v_2)$ does not intersect any U_ς with $\varsigma \neq \sigma(v_1, v_2, v_3), \sigma(v_1, v_2, v_1 + v_2 - v_3)$. In the same way π_j induces an isomorphism between the inverse image of U_2 in Z_{j-1} and its inverse image in Z_j. So the inverse image of U_2 in Z_j is isomorphic to the the inverse image of U_2 in Z'_j. Moreover $\pi'_j \circ ... \circ \pi'_2$ induces an isomorphism between $(\pi'_1)^{-1}(U_1)$ and its inverse image. In the same way $\pi_{j-1} \circ ... \circ \pi_1$ induces an isomorphism between U_1 and its inverse image. So the inverse image of U_1 in Z_j is isomorphic to the the inverse image of U_1 in Z'_j. Observe that the restrictions of these isomorphisms to the torus are always the identity. Thus the second claim follows because there is at most a morphism between two toric varieties such that its restriction to the torus is the identity.

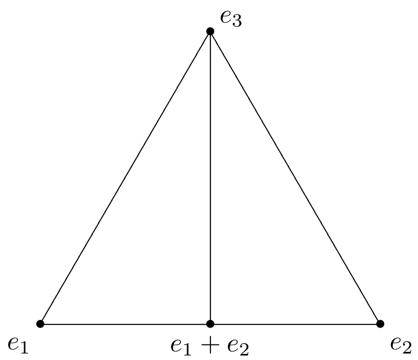
It is now sufficient to observe that $Z'_1 = Z'$. $\square$

18 Open Fano toric varieties of dimension 3

Now we have the instruments to classify, up to isomorphisms, the toric Fano varieties obtainable from $\mathbf{A}^3$ through a sequence of blow-ups. We want to find a finite number of varieties satisfying the lemma 17.1 and such that there are only a finite number of toric varieties obtainable from $\mathbf{A}^3$ through a sequence of blow-ups, but not obtainable from any of the previous varieties through a sequence of blow-ups. We will proceed as follows: $\mathbf{A}^3$ is a Fano variety, so we consider all the possible blow-ups of $\mathbf{A}^3$. Let Z be a blow-up of $\mathbf{A}^3$: 1) if Z satisfies the hypothesis of lemma 17.1 we know that there are not Fano variety obtainable from Z through a sequence of blow-ups; 2) if Z satisfies the hypotheses of the lemma 17.2 we will study the variety Z' of that lemma; 3) finally if Z is a Fano variety we reiterate the procedure. It is a priori possible that Z belongs to none of the previous cases, but this will not happen for the varieties which we will study. In the following, if two blow-up of a given variety are isomorphic we will examine only one of them. In our cases the isomorphism will be induced by an isomorphism of N that exchanges the vector of the basis $\{e_1, e_2, e_3\}$. We want to observe that, in this paragraph, we include a variety in the class of varieties obtainable from the variety itself through a sequence of blow-ups. We want to remember how verify that a toric variety Z is a Fano variety. Let Δ be its fan, let k be the function associated to its anticanonical bundle and suppose that all the maximal cones in Δ are 3-dimensional. Then Z is a Fano variety if and only if, given any cone $\sigma \in \Delta(3)$ and any cone $\tau \in \Delta(1)$ with $\tau \nsubseteq \sigma$, $(k|\sigma)(\rho(\tau)) > -1$ (here $\rho(\tau)$ is the primitive vector of τ).

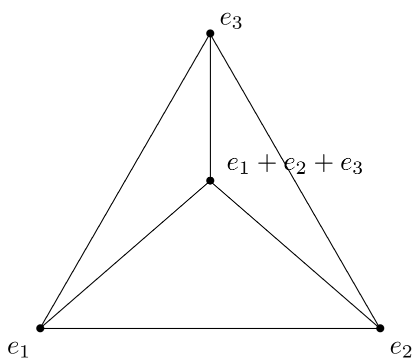
$\mathbf{A}^3$ is a Fano variety. Up to isomorphisms there are two blow-ups of $\mathbf{A}^3$: i) the variety "1" which is the blow-up of $\mathbf{A}^3$ along the subvariety associated to $\sigma(e_1, e_2)$

"1"



and ii) the variety "2" which is the blow-up of $\mathbf{A}^3$ along the subvariety associated to $\sigma(e_1, e_2, e_3)$.

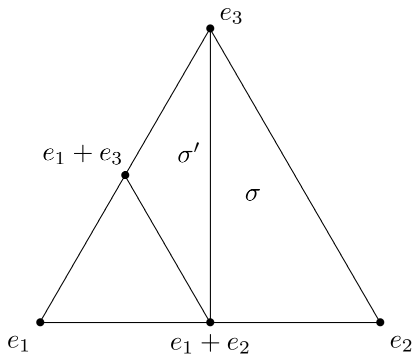
"2"



Now we study the variety "1". Its fan has maximal cones $\sigma(e_1, e_3, e_1 + e_2)$ and $\sigma(e_2, e_3, e_1 + e_2)$. The 1-dimensional cones are generated respectively by e_1, e_2, e_3 and $e_1 + e_2$. Let k_1 be the function associated to its anticanonical bundle. We have $(k_1|\sigma(e_1, e_3, e_1 + e_2))(e_2) = (-f_1 - f_3)(e_2) = 0 > -1$ and $(k_1|\sigma(e_2, e_3, e_1 + e_2))(e_1) = (-f_2 - f_3)(e_1) = 0 > -1$, so this variety is a Fano variety.

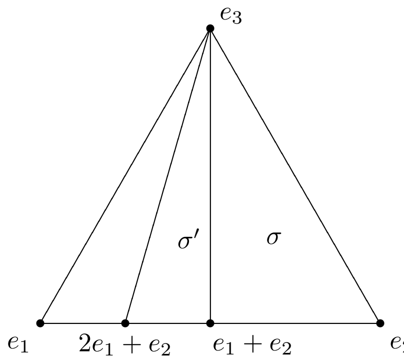
The blow-ups of the variety "1" are, up to isomorphisms: i) the variety "3" which is the blow-up of the variety "1" along the subvariety associated to $\sigma(e_1, e_3)$;

"3"



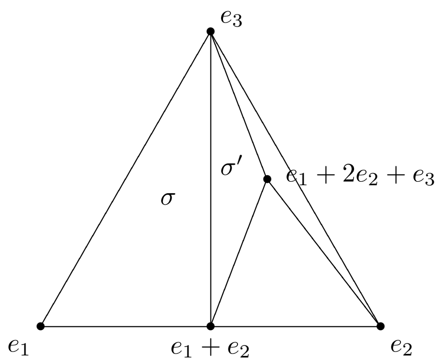
ii) the variety "4" which is the blow-up of the variety "1" along the subvariety associated to $\sigma(e_1, e_1 + e_2)$;

"4"



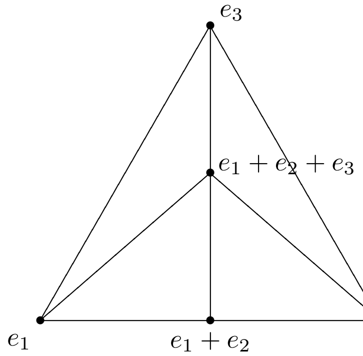
iii) the variety "5" which is the blow-up of the variety "1" along the subvariety associated to $\sigma(e_1 + e_2, e_2, e_3)$;

"5"



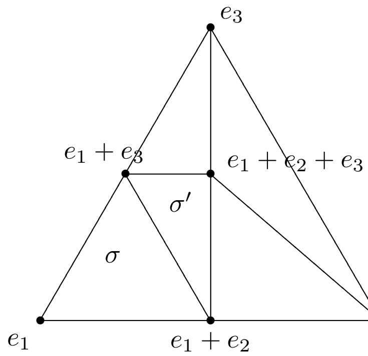
and iv) the variety "6" which is the blow-up of the variety "1" along the subvariety associated to $\sigma(e_1 + e_2, e_3)$.

"6"



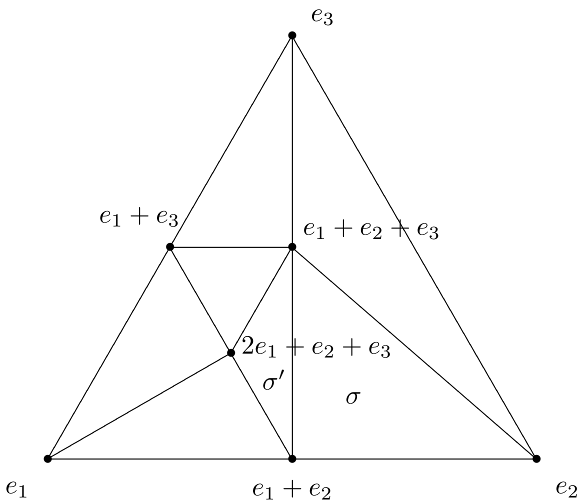
The fan of the variety "3" has maximal cones $\sigma(e_1, e_1+e_2, e_1+e_3)$, $\sigma(e_3, e_1+e_2, e_1+e_3)$ and $\sigma(e_2, e_3, e_1+e_2)$. We have $(e_1+e_3)+e_2 = e_3+(e_1+e_2)$, so this variety satisfies the hypotheses of the lemma 17.2 for $\sigma(e_3, e_1+e_2, e_1+e_3)$ and $\sigma(e_2, e_3, e_1+e_2)$. Hence we have to study the variety "7" obtained blowing-up "3" along the subvariety associated to $\sigma(e_3, e_1+e_2)$.

"7"



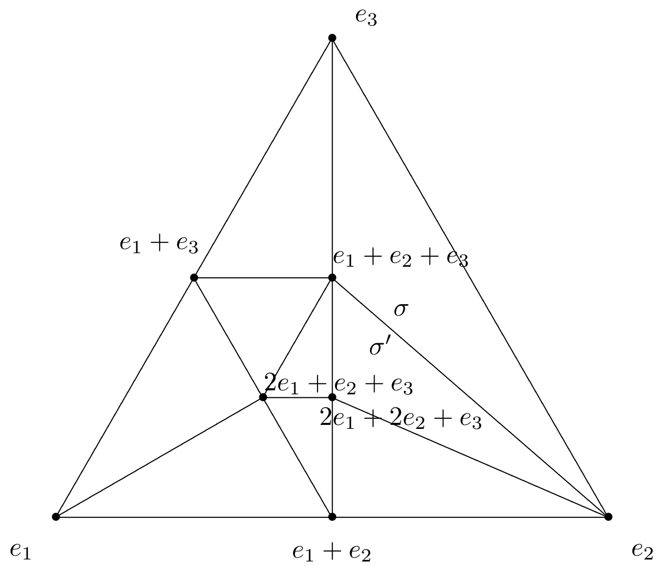
The fan of "7" has maximal cones $\sigma(e_1, e_1+e_2, e_1+e_3)$, $\sigma(e_1+e_2, e_1+e_3, e_1+e_2+e_3)$, $\sigma(e_3, e_1+e_3, e_1+e_2+e_3)$, $\sigma(e_2, e_3, e_1+e_2+e_3)$ and $\sigma(e_2, e_1+e_2, e_1+e_2+e_3)$. We have $(e_1+e_2+e_3)+e_1 = (e_1+e_2)+(e_1+e_3)$, so this variety satisfies the hypotheses of the lemma 17.2 for $\sigma(e_1, e_1+e_2, e_1+e_3)$ and $\sigma(e_1+e_2, e_1+e_3, e_1+e_2+e_3)$. Hence we have to study the variety "8" obtained blowing-up "7" along the subvariety associated to $\sigma(e_1+e_2, e_1+e_3)$.

"8"



The fan of this variety has maximal cones $\sigma(e_1, e_1+e_3, 2e_1+e_2+e_3)$, $\sigma(e_1 + e_3, e_1 + e_2 + e_3, 2e_1 + e_2 + e_3)$, $\sigma(e_1, e_1 + e_2, 2e_1 + e_2 + e_3)$, $\sigma(e_1 + e_2, 2e_1 + e_2 + e_3, e_1 + e_2 + e_3)$, $\sigma(e_2, e_1 + e_2, e_1 + e_2 + e_3)$, $\sigma(e_3, e_1 + e_3, e_1 + e_2 + e_3)$ and $\sigma(e_2, e_3, e_1+e_2+e_3)$. We have $(2e_1+e_2+e_3)+e_2 = (e_1+e_2)+(e_1+e_2+e_3)$, so this variety satisfies the hypotheses of the lemma 17.2 for $\sigma(e_2, e_1 + e_2, e_1 + e_2 + e_3)$ and $\sigma(e_1+e_2, e_1+e_2+e_3, 2e_1+e_2+e_3)$. Hence we have to study the variety "9" obtained blowing-up "8" along the subvariety associated to $\sigma(e_1+e_2, e_1+e_2+e_3)$.

"9"



The fan of this variety has maximal cones $\sigma(e_2, e_1 + e_2, 2e_1 + 2e_2 + e_3)$, $\sigma(e_2, 2e_1+2e_2+e_3, e_1+e_2+e_3)$, $\sigma(e_3, e_2, e_1+e_2+e_3)$, $\sigma(e_3, e_1+e_3, e_1+e_2+e_3)$, $\sigma(e_1 + e_3, e_1 + e_2 + e_3, 2e_1 + e_2 + e_3)$, $\sigma(e_1 + e_2 + e_3, 2e_1 + e_2 + e_3, 2e_1 + 2e_2 +$

e_3), $\sigma(e_1+e_2, 2e_1+e_2+e_3, 2e_1+2e_2+e_3)$, $\sigma(e_1, e_1+e_2, 2e_1+e_2+e_3)$ and $\sigma(e_1, e_1+e_3, 2e_1+e_2+e_3)$. We have $(2e_1+2e_2+e_3)+e_3 = 2(e_1+e_2+e_3)$, so "9" satisfies the hypotheses of the lemma 17.1 for $\sigma(e_3, e_2, e_1+e_2+e_3)$ and $\sigma(e_2, e_1+e_2+e_3, 2e_1+2e_2+e_3)$. Thus there are not Fano varieties obtained from "3" through a sequence of blow-ups.

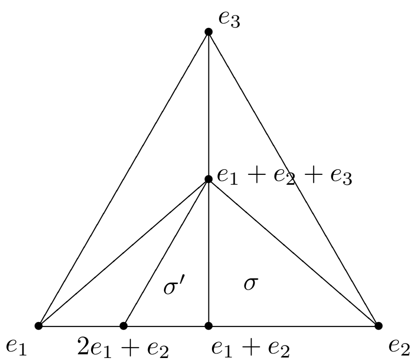
Now we examine the variety "4". Its fan has maximal cones $\sigma(e_1, e_3, 2e_1+e_2)$, $\sigma(e_3, 2e_1+e_2, e_1+e_2)$ and $\sigma(e_2, e_1+e_2, e_3)$. We have $(2e_1+e_2)+e_2 = 2(e_1+e_2)$, so "4" satisfies the hypotheses of the lemma 17.1 for $\sigma(e_3, 2e_1+e_2, e_1+e_2)$ and $\sigma(e_2, e_1+e_2, e_3)$.

Now we examine the variety "5". Its fan has maximal cones $\sigma(e_1, e_3, e_1+e_2)$, $\sigma(e_1+2e_2+e_3, e_1+e_2, e_3)$, $\sigma(e_1+e_2, e_2, e_1+2e_2+e_3)$ and $\sigma(e_1+2e_2+e_3, e_2, e_3)$. We have $(e_1+2e_2+e_3)+e_1 = 2(e_1+e_2)+e_3$, so "5" satisfies the hypotheses of the lemma 17.1 with respect to $\sigma(e_1, e_1+e_2, e_3)$ and $\sigma(e_3, e_1+e_2, e_1+2e_2+e_3)$.

Observe that we have demonstrate that if a toric Fano variety is obtained from the variety "1" through a sequence of blow-ups, then either it is the variety "1" or it is obtained from the variety "6" through a sequence of blow-ups. Now we want to show that the variety "6" is a Fano variety. Let k_6 be the function associated to the anticanonical bundle of "6" and let Δ_6 be the fan associated to "6". We have $\Delta_6(3) = \{\sigma(e_1+e_2, e_1, e_1+e_2+e_3), \sigma(e_1, e_3, e_1+e_2+e_3), \sigma(e_2, e_3, e_1+e_2+e_3), \sigma(e_1+e_2, e_2, e_1+e_2+e_3)\}$ and $\Delta_6(1) = \{\sigma(e_1), \sigma(e_2), \sigma(e_3), \sigma(e_1+e_2), \sigma(e_1+e_2+e_3)\}$. So $(k_6|\sigma(e_1+e_2, e_1, e_1+e_2+e_3))(e_2) = (-f_1)(e_2) = 0 > -1$, $(k_6|\sigma(e_1+e_2, e_1, e_1+e_2+e_3))(e_3) = (-f_1)(e_3) = 0 > -1$, $(k_6|\sigma(e_1, e_3, e_1+e_2+e_3))(e_2) = (-f_1+f_2-f_3)(e_2) = 1 > -1$, $(k_6|\sigma(e_1, e_3, e_1+e_2+e_3))(e_1+e_2) = (-f_1+f_2-f_3)(e_1+e_2) = 0 > -1$, $(k_6|\sigma(e_2, e_3, e_1+e_2+e_3))(e_1) = (f_1-f_2-f_3)(e_1) = 1 > -1$, $(k_6|\sigma(e_2, e_3, e_1+e_2+e_3))(e_1+e_2) = (f_1-f_2-f_3)(e_1+e_2) = 0 > -1$, $(k_6|\sigma(e_1+e_2, e_2, e_1+e_2+e_3))(e_1) = (-f_2)(e_1) = 0 > -1$ and $(k_6|\sigma(e_1+e_2, e_2, e_1+e_2+e_3))(e_3) = (-f_2)(e_3) = 0 > -1$, so "6" is a Fano variety.

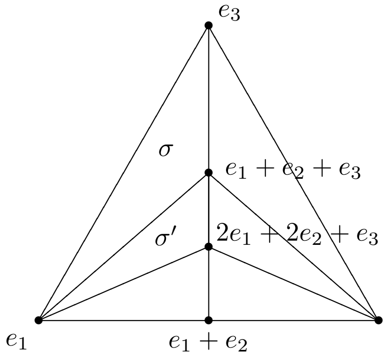
The blow-ups of the variety "6" are, up to isomorphisms: i) the variety "10" which is the blow-up of the variety "6" along the subvariety associated to $\sigma(e_1, e_1+e_2)$;

"10"



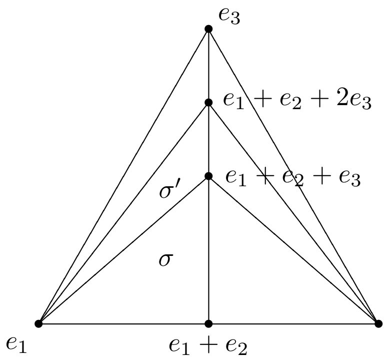
ii) the variety "11" which is the blow-up of the variety "6" along the subvariety associated to $\sigma(e_1+e_2, e_1+e_2+e_3)$;

"11"



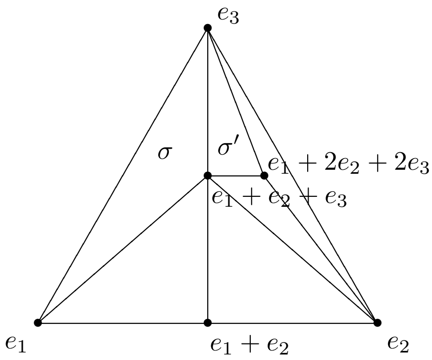
iii) the variety "12" which is the blow-up of the variety "6" along the subvariety associated to $\sigma(e_3, e_1 + e_2 + e_3)$;

"12"



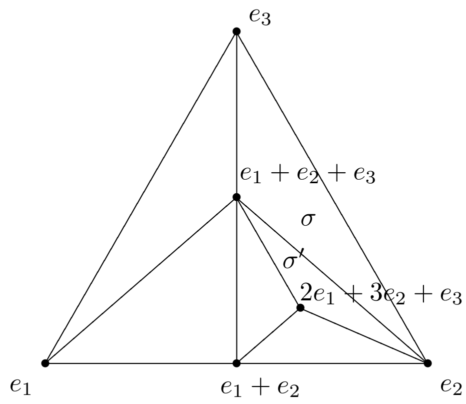
iv) the variety "13" which is the blow-up of the variety "6" along the subvariety associated to $\sigma(e_2, e_3, e_1 + e_2 + e_3)$;

"13"



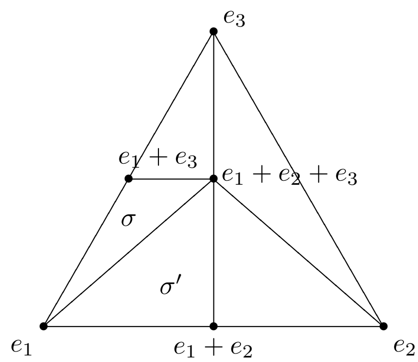
v) the variety "14" which is the blow-up of the variety "6" along the subvariety associated to $\sigma(e_2, e_1 + e_2, e_1 + e_2 + e_3)$;

"14"



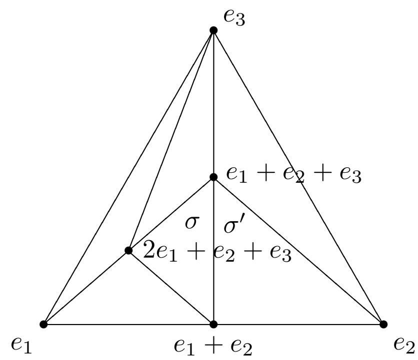
vi) the variety "15" which is the blow-up of the variety "6" along the subvariety associated to $\sigma(e_1, e_3)$;

"15"



and vii) the variety "16" which is the blow-up of the variety "6" along the subvariety associated to $\sigma(e_1, e_1 + e_2 + e_3)$.

"16"



We begin studying the variety "10". Its fan has maximal cones $\sigma(e_2, e_1 + e_2, e_1 + e_2 + e_3)$, $\sigma(e_2, e_3, e_1 + e_2 + e_3)$, $\sigma(e_1, e_3, e_1 + e_2 + e_3)$, $\sigma(e_1, e_1 + e_2 + e_3, 2e_1 + e_2)$ and $\sigma(2e_1 + e_2, e_1 + e_2, e_1 + e_2 + e_3)$. We have $(2e_1 + e_2) + e_2 =$

$2(e_1 + e_2)$, so "10" satisfies the hypotheses of the lemma 17.1 with respect to $\sigma(e_2, e_1 + e_2, e_1 + e_2 + e_3)$ and $\sigma(2e_1 + e_2, e_1 + e_2, e_1 + e_2 + e_3)$.

Now we examine the variety "11". Its fan has maximal cones $\sigma(e_1, e_1 + e_2, 2e_1 + 2e_2 + e_3)$, $\sigma(e_1, 2e_1 + 2e_2 + e_3, e_1 + e_2 + e_3)$, $\sigma(e_1, e_3, e_1 + e_2 + e_3)$, $\sigma(e_2, e_3, e_1 + e_2 + e_3)$, $\sigma(e_2, 2e_1 + 2e_2 + e_3, e_1 + e_2 + e_3)$ and $\sigma(e_2, e_1 + e_2, 2e_1 + 2e_2 + e_3)$. We have $(2e_1 + 2e_2 + e_3) + e_3 = 2(e_1 + e_2 + e_3)$, so "11" satisfies the hypotheses of the lemma 17.1 with respect to $\sigma(e_1, 2e_1 + 2e_2 + e_3, e_1 + e_2 + e_3)$ and $\sigma(e_1, e_3, e_1 + e_2 + e_3)$.

The fan of the variety "12" has maximal cones $\sigma(e_1, e_1 + e_2, e_1 + e_2 + e_3)$, $\sigma(e_1, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$, $\sigma(e_1, e_3, e_1 + e_2 + 2e_3)$, $\sigma(e_2, e_3, e_1 + e_2 + 2e_3)$, $\sigma(e_2, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$ and $\sigma(e_2, e_1 + e_2, e_1 + e_2 + e_3)$. We have $(e_1 + e_2 + 2e_3) + (e_1 + e_2) = 2(e_1 + e_2 + e_3)$, so "12" satisfies the hypotheses of the lemma 17.1 with respect to $\sigma(e_1, e_1 + e_2, e_1 + e_2 + e_3)$ and $\sigma(e_1, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$.

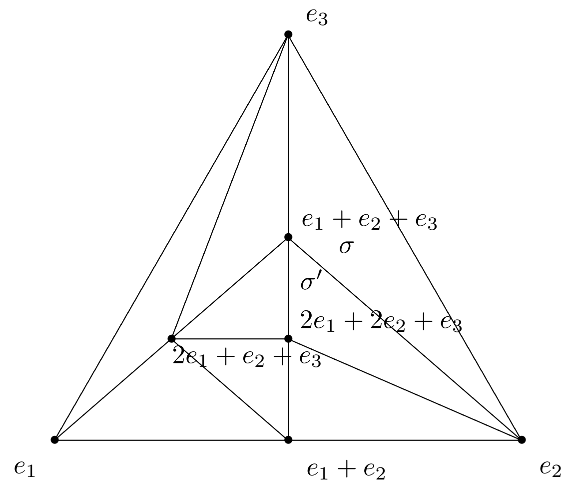
The fan of the variety "13" has maximal cones $\sigma(e_1, e_1 + e_2, e_1 + e_2 + e_3)$, $\sigma(e_2, e_1+e_2, e_1+e_2+e_3)$, $\sigma(e_2, e_1+e_2+e_3, e_1+2e_2+2e_3)$, $\sigma(e_2, e_3, e_1+2e_2+2e_3)$, $\sigma(e_3, e_1 + e_2 + e_3, e_1 + 2e_2 + 2e_3)$ and $\sigma(e_1, e_3, e_1 + e_2 + e_3)$. We have $(e_1 + 2e_2 + 2e_3) + e_1 = 2(e_1 + e_2 + e_3)$, so "13" satisfies the hypotheses of the lemma 17.1 with respect to $\sigma(e_1, e_3, e_1 + e_2 + e_3)$ and $\sigma(e_3, e_1 + e_2 + e_3, e_1 + 2e_2 + 2e_3)$.

The fan of the variety "14" has maximal cones $\sigma(e_1, e_1 + e_2, e_1 + e_2 + e_3)$, $\sigma(e_1, e_3, e_1 + e_2 + e_3)$, $\sigma(e_2, e_3, e_1 + e_2 + e_3)$, $\sigma(e_2, e_1 + e_2 + e_3, 2e_1 + 3e_2 + e_3)$, $\sigma(e_1 + e_2, e_1 + e_2 + e_3, 2e_1 + 3e_2 + e_3)$ and $\sigma(e_2, e_1 + e_2, 2e_1 + 3e_2 + e_3)$. We have $(2e_1 + 3e_2 + e_3) + e_3 = 2(e_1 + e_2 + e_3) + e_2$, so "14" satisfies the hypotheses of the lemma 17.1 with respect to $\sigma(e_1, e_2, e_1+e_2+e_3)$ and $\sigma(e_2, e_1+e_2+e_3, 2e_1 + 3e_2 + e_3)$.

The fan of the variety "15" has maximal cones $\sigma(e_2, e_1 + e_2, e_1 + e_2 + e_3)$, $\sigma(e_2, e_3, e_1 + e_2 + e_3)$, $\sigma(e_3, e_1 + e_3, e_1 + e_2 + e_3)$, $\sigma(e_1, e_1 + e_3, e_1 + e_2 + e_3)$ and $\sigma(e_1, e_1 + e_2, e_1 + e_2 + e_3)$. We have $(e_1 + e_2) + (e_1 + e_3) = e_1 + (e_1 + e_2 + e_3)$, so "15" satisfies the hypotheses of the lemma 17.2 with respect to $\sigma(e_1, e_1 + e_3, e_1 + e_2 + e_3)$ and $\sigma(e_1, e_1 + e_2, e_1 + e_2 + e_3)$. Hence we have to study the variety obtained blowing-up "15" along the subvariety associated to $\sigma(e_1, e_1+e_2+e_3)$, but this one is the variety that we have called "8". So there are not Fano varieties obtained from "15" through a sequence of blow-ups, because "8" satisfies the hypotheses of the lemma 17.1.

The fan of the variety "16" has maximal cones $\sigma(e_1 + e_2, e_1 + e_2 + e_3, 2e_1 + e_2 + e_3)$, $\sigma(e_1, e_1 + e_2, 2e_1 + e_2 + e_3)$, $\sigma(e_1, e_3, 2e_1 + e_2 + e_3)$, $\sigma(e_3, e_1 + e_2 + e_3, 2e_1 + e_2 + e_3)$, $\sigma(e_2, e_3, e_1 + e_2 + e_3)$ and $\sigma(e_2, e_1 + e_2, e_1 + e_2 + e_3)$. We have $(2e_1+e_2+e_3)+e_2 = (e_1+e_2)+(e_1+e_2+e_3)$, so "16" satisfies the hypotheses of the lemma 17.2 with respect to $\sigma(e_2, e_1+e_2, e_1+e_2+e_3)$ and $\sigma(e_1+e_2, e_1+e_2+e_3, 2e_1+e_2+e_3)$. Hence we have to study the variety "17" obtained blowing-up "16" along the subvariety associated to $\sigma(e_1 + e_2, e_1 + e_2 + e_3)$.

"17"



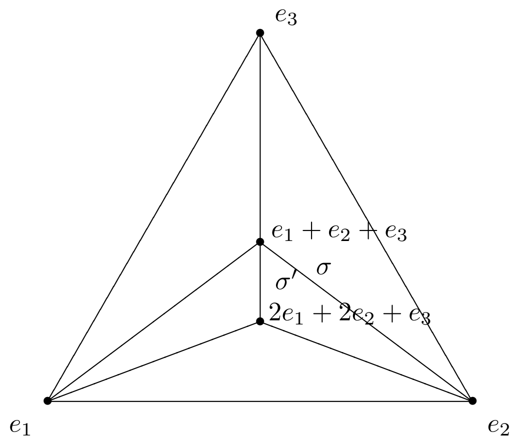
The fan of this variety has maximal cones $\sigma(e_2, e_1 + e_2, 2e_1 + 2e_2 + e_3)$, $\sigma(e_2, e_1+e_2+e_3, 2e_1+2e_2+e_3)$, $\sigma(e_2, e_3, e_1+e_2+e_3)$, $\sigma(e_3, e_1+e_2+e_3, 2e_1+e_2+e_3)$, $\sigma(e_1+e_2+e_3, 2e_1+e_2+e_3, 2e_1+2e_2+e_3)$, $\sigma(e_1+e_2, 2e_1+e_2+e_3, 2e_1+2e_2+e_3)$, $\sigma(e_1, e_1 + e_2, 2e_1 + e_2 + e_3)$ and $\sigma(e_1, e_3, 2e_1 + e_2 + e_3)$. We have $(2e_1 + 2e_2 + e_3) + e_3 = 2(e_1 + e_2 + e_3)$, so "17" satisfies the hypotheses of the lemma 17.1 with respect to $\sigma(e_2, e_3, e_1 + e_2 + e_3)$ and $\sigma(e_2, e_1 + e_2 + e_3, 2e_1 + 2e_2 + e_3)$. Observe that we have classified the toric Fano varieties obtainable form the variety "1" through a sequence of blow-ups, so we have only to study the varieties dominating the variety "2".

Let k_2 be the function associated to the anticanonical bundle of "2" and let Δ_2 be the fan associated to "2". We have $\Delta_2(3) = \{\sigma(e_2, e_3, e_1 + e_2 + e_3), \sigma(e_1, e_3, e_1+e_2+e_3), \sigma(e_1, e_2, e_1+e_2+e_3)\}$ and $\Delta_2(1) = \{\sigma(e_1), \sigma(e_2), \sigma(e_3), \sigma(e_1 + e_2 + e_3)\}$. We have already showed that this variety is a Fano variety. Indeed, we have $(k_2|\sigma(e_2, e_3, e_1 + e_2 + e_3))(e_1) = (f_1 - f_2 - f_3)(e_1) = 1 > -1$, $(k_2|\sigma(e_1, e_3, e_1 + e_2 + e_3))(e_2) = (-f_1 + f_2 - f_3)(e_2) = 1 > -1$ and $(k_2|\sigma(e_1, e_2, e_1 + e_2 + e_3))(e_3) = (-f_1 - f_2 + f_3)(e_3) = 1 > -1$. This inequalities prove again that "2" is a Fano variety.

The blow-up of the variety "2" are, up to isomorphisms: i) the variety "6" that we have already examined;

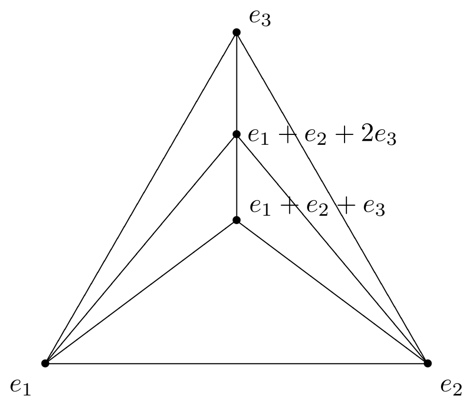
ii) the variety "18" which is the blow-up of the variety "2" along the subvariety associated to $\sigma(e_1, e_2, e_1 + e_2 + e_3)$;

"18"



and iii) the variety "19" which is the blow-up of the variety "6" along the subvariety associated to $\sigma(e_3, e_1+e_2+e_3)$.

"19"



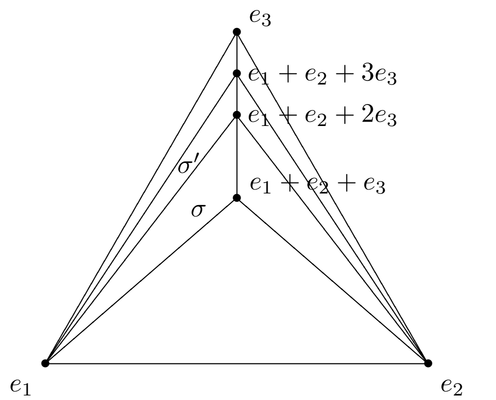
The fan of the variety "18" has maximal cones $\sigma(e_1, e_2, 2e_1+2e_2+e_3)$, $\sigma(e_1, e_1+e_2+e_3, 2e_1+2e_2+e_3)$, $\sigma(e_1, e_3, e_1+e_2+e_3)$, $\sigma(e_2, e_3, e_1+e_2+e_3)$ and $\sigma(e_2, e_1+e_2+e_3, 2e_1+2e_2+e_3)$. Observe that the blow-up of "18" along the subvariety associated to $\sigma(e_1, e_3)$ is isomorphic to the variety "13" through the isomorphism φ given by the action of $(1,2,3)$ $\in$Sym$_3$ on $\{e_1, e_2, e_3\}$. We have showed that such variety satisfies the hypotheses of the lemma 17.1. Indeed we have $(2e_1+2e_2+e_3)+e_3 = 2(e_1+e_2+e_3)$, so "18" satisfies the hypotheses of the lemma 17.1 with respect to $\sigma(e_2, e_3, e_1+e_2+e_3)$ and $\sigma(e_2, e_1+e_2+e_3, 2e_1+2e_2+e_3)$.

Now we want to show that the variety "19" is a Fano variety. Let k_{19} be the function associated to the anticanonical bundle of "19" and let Δ_{19} be the fan associated to "19". We have $\Delta_{19}(3) = \{\sigma(e_1, e_3, e_1+e_2+2e_3), \sigma(e_1, e_1+e_2+e_3, e_1+e_2+2e_3), \sigma(e_1, e_2, e_1+e_2+e_3), \sigma(e_2, e_1+e_2+e_3, e_1+e_2+2e_3)\sigma(e_2, e_3, e_1+e_2+2e_3)\}$ and $\Delta_{19}(1) = \{\sigma(e_1), \sigma(e_2), \sigma(e_3), \sigma(e_1+e_2+e_3), \sigma(e_1+e_2+2e_3)\}$. We have $(k_{19}|\sigma(e_1, e_3, e_1+e_2+2e_3))(e_1+e_2+e_3) = (-f_1+2f_2-f_3)(e_1+e_2+e_3) = 0 > -1$, $(k_{19}|\sigma(e_1, e_3, e_1+e_2+2e_3))(e_2) = (-f_1+2f_2-f_3)(e_2) = 2 > -1$,

$(k_{19}|\sigma(e_1, e_1+e_2+e_3, e_1+e_2+2e_3))(e_2) = (-f_1)(e_2) = 0 > -1$, $(k_{19}|\sigma(e_1, e_1+e_2+e_3, e_1+e_2+2e_3))(e_3) = (-f_1)(e_3) = 0 > -1$, $(k_{19}|\sigma(e_1, e_2, e_1+e_2+e_3))(e_1+e_2+2e_3) = (-f_1-f_2+f_3)(e_1+e_2+2e_3) = 0 > -1$, $(k_{19}|\sigma(e_1, e_2, e_1+e_2+e_3))(e_3) = (-f_1-f_2+f_3)(e_3) = 1 > -1$, $(k_{19}|\sigma(e_2, e_1+e_2+e_3, e_1+e_2+2e_3))(e_2) = (-f_2)(e_1) = 0 > -1$, $(k_{19}|\sigma(e_2, e_1+e_2+e_3, e_1+e_2+2e_3))(e_3) = (-f_2)(e_3) = 0 > -1$, $(k_{19}|\sigma(e_2, e_3, e_1+e_2+2e_3))(e_1+e_2+e_3) = (2f_1-f_2-f_3)(e_1+e_2+e_3) = 0 > -1$ and $(k_{19}|\sigma(e_2, e_3, e_1+e_2+2e_3))(e_1) = (2f_1-f_2-f_3)(e_1) = 2 > -1$, so this variety is a Fano variety.

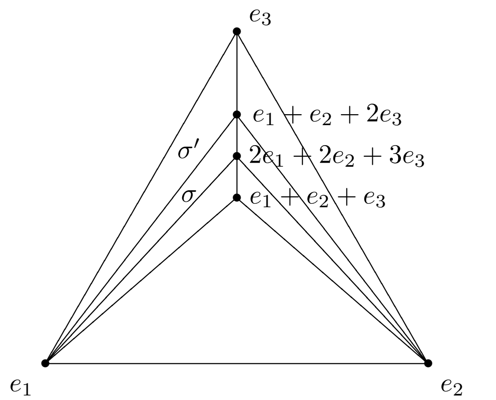
The blow-up of the variety "19" are, up to isomorphisms: i) the variety "12" which satisfies the hypotheses of the lemma 17.1; ii) the variety "20" which is the blow-up of the variety "19" along the subvariety associated to $\sigma(e_3, e_1+e_2+2e_3)$;

"20"

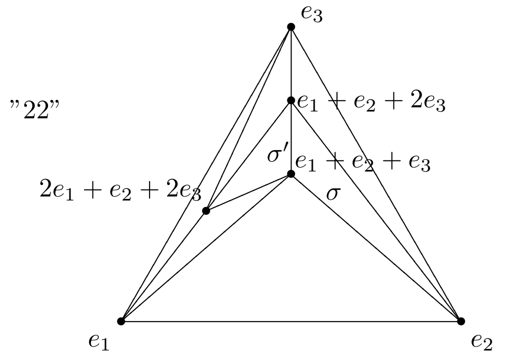


iii) the variety "21" which is the blow-up of the variety "19" along the subvariety associated to $\sigma(e_1+e_2+e_3, e_1+e_2+2e_3)$;

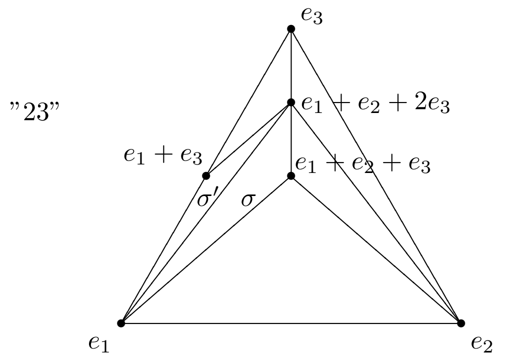
"21"



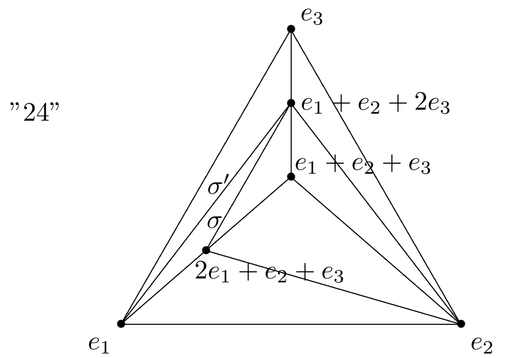
iv) the variety "22" which is the blow-up of the variety "19" along the subvariety associated to $\sigma(e_1, e_1+e_2+2e_3)$;



v) the variety "23" which is the blow-up of the variety "19" along the subvariety associated to $\sigma(e_1, e_3)$;

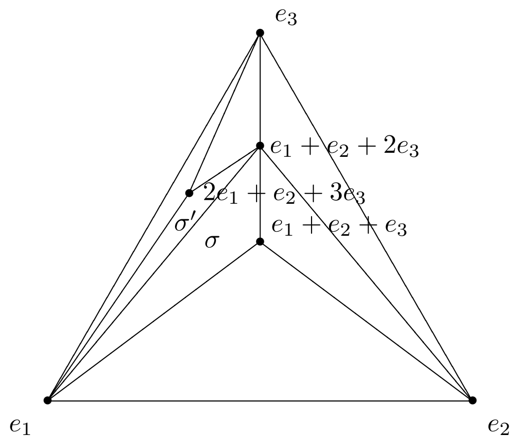


vi) the variety "24" which is the blow-up of the variety "19" along the subvariety associated to $\sigma(e_1, e_1+e_2+e_3)$;



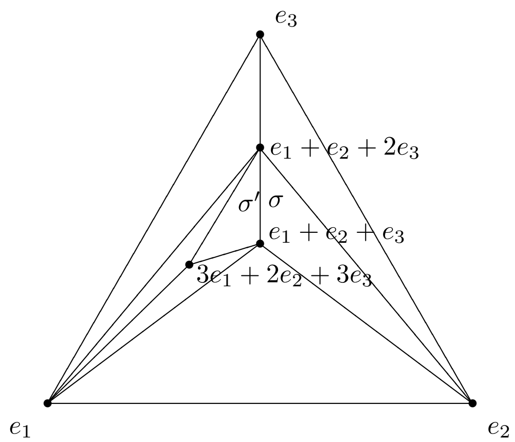
vii) the variety "25" which is the blow-up of the variety "19" along the subvariety associated to $\sigma(e_1, e_3, e_1+e_2+2e_3)$;

"25"



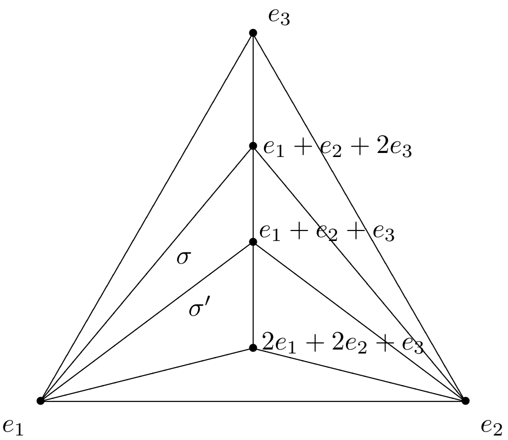
viii) the variety "26" which is the blow-up of the variety "19" along the subvariety associated to $\sigma(e_1, e_1+e_2+e_3, e_1+e_2+2e_3)$;

"26"



and ix) the variety "27" which is the blow-up of the variety "19" along the subvariety associated to $\sigma(e_1, e_2, e_1+e_2+e_3)$.

"27"



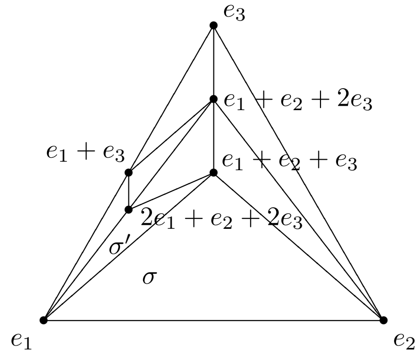
The fan of the variety "20" has maximal cones $\sigma(e_1, e_3, e_1 + e_2 + 3e_3)$, $\sigma(e_1, e_1 + e_2 + 3e_3, e_1 + e_2 + 2e_3)$, $\sigma(e_1, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$, $\sigma(e_1, e_2, e_1 + e_2 + e_3)$, $\sigma(e_2, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$, $\sigma(e_2, e_1 + e_2 + 2e_3, e_1 + e_2 + 3e_3)$ and $\sigma(e_2, e_3, e_1 + e_2 + 3e_3)$. We have $(e_1 + e_2 + e_3) + (e_1 + e_2 + 3e_3) = 2(e_1 + e_2 + 2e_3)$, so "20" satisfies the hypotheses of the lemma 17.1 with respect to $\sigma(e_1, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$ and $\sigma(e_1, e_1 + e_2 + 2e_3, e_1 + e_2 + 3e_3)$.

The fan of the variety "21" has maximal cones $\sigma(e_1, e_3, e_1 + e_2 + 2e_3)$, $\sigma(e_1, e_1+e_2+2e_3, 2e_1+2e_2+3e_3)$, $\sigma(e_1, e_1+e_2+e_3, 2e_1+2e_2+3e_3)$, $\sigma(e_1, e_2, e_1+e_2+e_3)$, $\sigma(e_2, e_1+e_2+e_3, 2e_1+2e_2+3e_3)$, $\sigma(e_2, e_1+e_2+2e_3, 2e_1+2e_2+3e_3)$ and $\sigma(e_2, e_3, e_1+e_2+2e_3)$. We have $(2e_1+2e_2+3e_3)+e_3 = 2(e_1+e_2+2e_3)$, so "21" satisfies the hypotheses of the lemma 17.1 with respect to $\sigma(e_1, e_3, e_1+e_2+2e_3)$ and $\sigma(e_1, e_1 + e_2 + 2e_3, 2e_1 + 2e_2 + 3e_3)$.

The fan of the variety "22" has maximal cones $\sigma(e_2, e_3, e_1 + e_2 + 2e_3)$, $\sigma(e_2, e_1+e_2+e_3, e_1+e_2+2e_3)$, $\sigma(e_1, e_2, e_1+e_2+e_3)$, $\sigma(e_1, e_1+e_2+e_3, 2e_1+e_2+2e_3)$, $\sigma(e_1+e_2+e_3, e_1+e_2+2e_3, 2e_1+e_2+2e_3)$, $\sigma(e_3, e_1+e_2+2e_3, 2e_1+e_2+2e_3)$ and $\sigma(e_1, e_3, 2e_1 + e_2 + 2e_3)$. We have $(2e_1 + e_2 + 2e_3) + e_2 = 2(e_1 + e_2 + e_3)$, so "22" satisfies the hypotheses of the lemma 17.1 with respect to $\sigma(e_2, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$ and $\sigma(e_1 + e_2 + e_3, e_1 + e_2 + 2e_3, 2e_1 + e_2 + 2e_3)$.

The fan of the variety "23" has maximal cones $\sigma(e_3, e_1 + e_3, e_1 + e_2 + 2e_3)$, $\sigma(e_1, e_1+e_3, e_1+e_2+2e_3)$, $\sigma(e_1, e_1+e_2+e_3, e_1+e_2+2e_3)$, $\sigma(e_1, e_2, e_1+e_2+e_3)$, $\sigma(e_2, e_1+e_2+e_3, e_1+e_2+2e_3)$ and $\sigma(e_2, e_3, e_1+e_2+2e_3)$. We have $(e_1+e_2+e_3)+(e_1+e_3) = (e_1+e_2+2e_3)+e_1$, so "23" satisfies the hypotheses of the lemma 17.2 with respect to $\sigma(e_1, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$ and $\sigma(e_1, e_1 + e_3, e_1 + e_2 + 2e_3)$. Hence we have to study the variety "28" obtained blowing-up "23" along the subvariety associated to $\sigma(e_1, e_1 + e_2 + 2e_3)$.

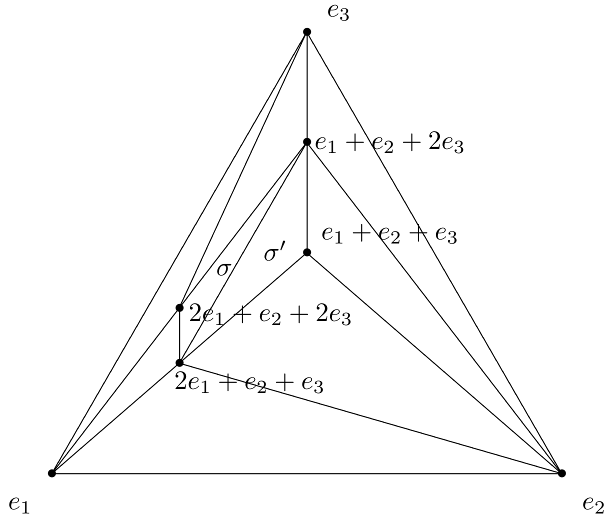
"28"



The fan of this variety has maximal cones $\sigma(e_1, e_2, e_1+e_2+e_3)$, $\sigma(e_2, e_1+e_2+e_3, e_1+e_2+2e_3)$, $\sigma(e_2, e_3, e_1+e_2+2e_3)$, $\sigma(e_3, e_1+e_3, e_1+e_2+2e_3)$, $\sigma(e_1+e_3, e_1+e_2+2e_3, 2e_1+e_2+2e_3)$, $\sigma(e_1, e_1+e_3, 2e_1+e_2+2e_3)$, $\sigma(e_1, e_1+e_2+e_3, 2e_1+e_2+2e_3)$ and $\sigma(e_1+e_2+e_3, e_1+e_2+2e_3, 2e_1+e_2+2e_3)$. We have $(2e_1+e_2+2e_3)+e_2 = 2(e_1+e_2+e_3)$, so "28" satisfies the hypotheses of the lemma 17.1 with respect to $\sigma(e_1, e_2, e_1+e_2+e_3)$ and $\sigma(e_1, e_1+e_2+e_3, 2e_1+e_2+2e_3)$.

The fan of the variety "24" has maximal cones $\sigma(e_2, e_3, e_1+e_2+2e_3)$, $\sigma(e_2, e_1+e_2+e_3, e_1+e_2+2e_3)$, $\sigma(e_2, e_1+e_2+e_3, 2e_1+e_2+e_3)$, $\sigma(e_1, e_2, 2e_1+e_2+e_3)$, $\sigma(e_1, 2e_1+e_2+e_3, e_1+e_2+2e_3)$, $\sigma(e_1+e_2+e_3, e_1+e_2+2e_3, 2e_1+e_2+e_3)$ and $\sigma(e_1, e_3, e_1+e_2+2e_3)$. We have $(2e_1+e_2+e_3)+e_3 = (e_1+e_2+2e_3)+e_1$, so "24" satisfies the hypotheses of the lemma 17.2 with respect to $\sigma(e_1, 2e_1+e_2+e_3, e_1+e_2+2e_3)$ and $\sigma(e_1, e_3, e_1+e_2+2e_3)$. Hence we have to study the variety "29" obtained blowing-up "24" along the subvariety associated to $\sigma(e_1, e_1+e_2+2e_3)$.

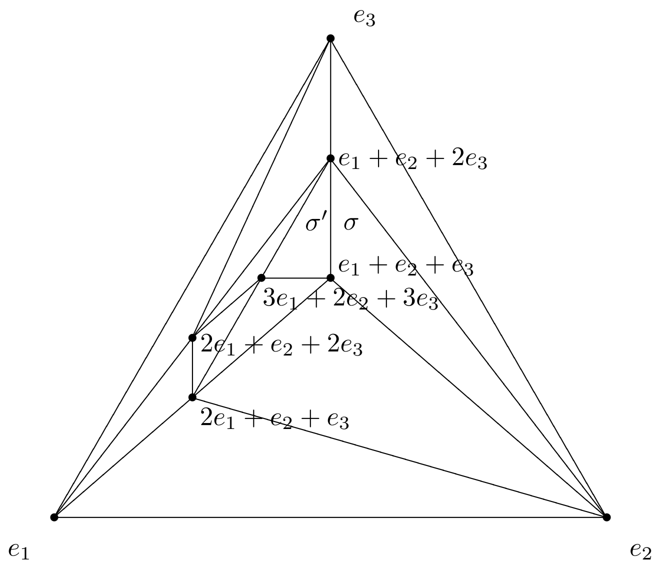
"29"



This variety has maximal cones $\sigma(e_2, e_3, e_1+e_2+2e_3)$, $\sigma(e_2, e_1+e_2+e_3, e_1+e_2+2e_3)$, $\sigma(e_2, e_1+e_2+e_3, 2e_1+e_2+e_3)$, $\sigma(e_1, e_2, 2e_1+e_2+e_3)$, $\sigma(e_1, 2e_1+e_2+$

$e_3, 2e_1+e_2+2e_3)$, $\sigma(2e_1+e_2+2e_3, 2e_1+e_2+e_3, e_1+e_2+2e_3)$, $\sigma(e_1+e_2+e_3, 2e_1+e_2+e_3, e_1+e_2+2e_3)$, $\sigma(e_3, e_1+e_2+2e_3, 2e_1+e_2+2e_3)$ and $\sigma(e_1, e_3, 2e_1+e_2+2e_3)$. We have $(2e_1+e_2+2e_3)+(e_1+e_2+e_3) = (2e_1+e_2+e_3)+(e_1+e_2+2e_3)$, so "29" satisfies the hypotheses of the lemma 17.2 with respect to $\sigma(e_1+e_2+e_3, 2e_1+e_2+e_3, e_1+e_2+2e_3)$ and $\sigma(2e_1+e_2+2e_3, 2e_1+e_2+e_3, e_1+e_2+2e_3)$. Hence we have to study the variety "30" obtained blowing-up "29" along the subvariety associated to $\sigma(2e_1+e_2+e_3, e_1+e_2+2e_3)$.

"30"



The fan of this variety has maximal cones $\sigma(e_2, e_3, e_1+e_2+2e_3)$, $\sigma(e_2, e_1+e_2+e_3, e_1+e_2+2e_3)$, $\sigma(e_2, e_1+e_2+e_3, 2e_1+e_2+e_3)$, $\sigma(e_1, e_2, 2e_1+e_2+e_3)$, $\sigma(e_1, 2e_1+e_2+e_3, 2e_1+e_2+2e_3)$, $\sigma(2e_1+e_2+2e_3, 2e_1+e_2+e_3, 3e_1+2e_2+3e_3)$, $\sigma(e_1+e_2+e_3, 2e_1+e_2+e_3, 3e_1+2e_2+3e_3)$, $\sigma(3e_1+2e_2+3e_3, e_1+e_2+2e_3, e_1+e_2+e_3)$, $\sigma(e_1+e_2+2e_3, 2e_1+e_2+2e_3, 3e_1+2e_2+3e_3)$, $\sigma(e_3, e_1+e_2+2e_3, 2e_1+e_2+2e_3)$ and $\sigma(e_1, e_3, 2e_1+e_2+2e_3)$. We have $(3e_1+2e_2+3e_3)+e_2 = 3(e_1+e_2+e_3)$, so "30" satisfies the hypotheses of the lemma 17.1 with respect to $\sigma(e_2, e_1+e_2+e_3, e_1+e_2+2e_3)$ and $\sigma(e_1+e_2+e_3, e_1+e_2+2e_3, 3e_1+2e_2+3e_3)$.

The fan of the variety "25" has maximal cones $\sigma(e_1, e_2, e_1+e_2+e_3)$, $\sigma(e_2, e_1+e_2+e_3, e_1+e_2+2e_3)$, $\sigma(e_2, e_3, e_1+e_2+2e_3)$, $\sigma(e_3, e_1+e_2+2e_3, 2e_1+e_2+3e_3)$, $\sigma(e_1, e_3, 2e_1+e_2+3e_3)$, $\sigma(e_1, e_1+e_2+2e_3, 2e_1+e_2+3e_3)$ and $\sigma(e_1, e_1+e_2+e_3, e_1+e_2+2e_3)$. We have $(2e_1+e_2+3e_3)+(e_1+e_2+e_3) = 2(e_1+e_2+2e_3)+e_1$, so "25" satisfies the hypotheses of the lemma 17.1 with respect to $\sigma(e_1, e_1+e_2+e_3, e_1+e_2+2e_3)$ and $\sigma(e_1, e_1+e_2+2e_3, 2e_1+e_2+3e_3)$.

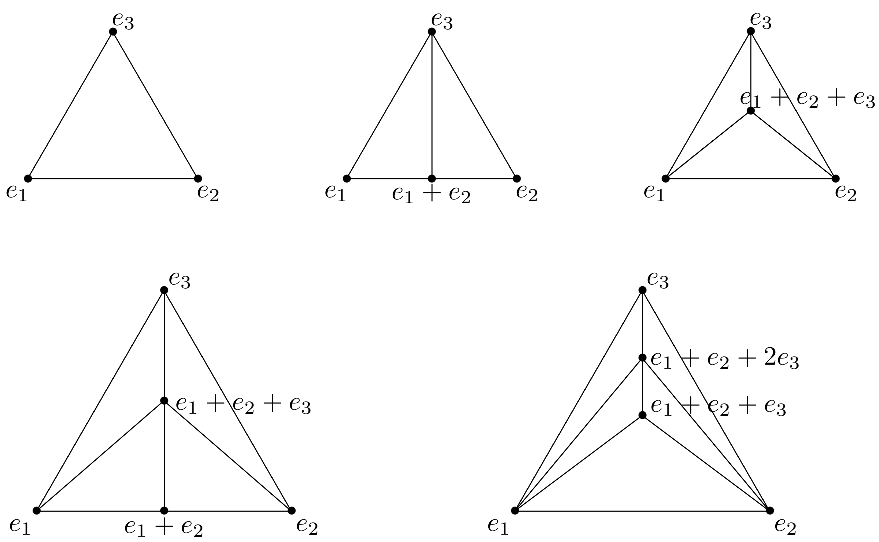
The fan of the variety "26" has maximal cones $\sigma(e_2, e_3, e_1+e_2+2e_3)$, $\sigma(e_2, e_1+e_2+e_3, e_1+e_2+2e_3)$, $\sigma(e_1, e_2, e_1+e_2+e_3)$, $\sigma(e_1, e_1+e_2+e_3, 3e_1+2e_2+3e_3)$, $\sigma(e_1, e_1+e_2+2e_3, 3e_1+2e_2+3e_3)$, $\sigma(3e_1+2e_2+3e_3, e_1+e_2+e_3, e_1+e_2+2e_3)$ and $\sigma(e_1, e_3, e_1+e_2+2e_3)$. We have $(3e_1+2e_2+3e_3)+e_2 = 3(e_1+e_2+e_3)$, so "26" satisfies the hypotheses of the lemma 17.1 with respect to $\sigma(e_2, e_1+e_2+e_3, e_1+e_2+2e_3)$ and $\sigma(e_1+e_2+e_3, e_1+e_2+2e_3, 3e_1+2e_2+3e_3)$.

The fan of the variety "27" has maximal cones $\sigma(e_1, e_3, e_1 + e_2 + 2e_3)$, $\sigma(e_1, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$, $\sigma(e_1, e_1 + e_2 + e_3, 2e_1 + 2e_2 + e_3)$, $\sigma(e_1, e_2, 2e_1 + 2e_2 + e_3)$, $\sigma(e_2, e_1 + e_2 + e_3, 2e_1 + 2e_2 + e_3)$, $\sigma(e_2, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$ and $\sigma(e_1, e_3, e_1 + e_2 + 2e_3)$. We have $(e_1 + e_2 + 2e_3) + (2e_1 + 2e_2 + e_3) = 3(e_1 + e_2 + e_3)$, so "27" satisfies the hypotheses of the lemma 17.1 with respect to $\sigma(e_1, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$ and $\sigma(e_1, e_1 + e_2 + e_3, 2e_1 + 2e_2 + e_3)$.

Therefore we have proved the following theorem.

Proposition 18.1 *The Fano toric varieties obtainable from $\boldsymbol{A}^3$ through a sequence of blow-up are, up to isomorphisms:*

1. $\boldsymbol{A}^3$
2. *a 2-blow-up of* $\boldsymbol{A}^3$
3. *the 3-blow-up of* $\boldsymbol{A}^3$
4. *the variety whose fan has maximal cones* $\sigma(e_1, e_1 + e_2, e_1 + e_2 + e_3)$, $\sigma(e_1, e_3, e_1 + e_2 + e_3)$, $\sigma(e_2, e_3, e_1 + e_2 + e_3)$ *and* $\sigma(e_2, e_1 + e_2, e_1 + e_2 + e_3)$. *This variety is obtainable from* $\boldsymbol{A}^3$ *through two consecutive blow-ups along subvarieties of codimension 2,*
5. *the variety whose fan has maximal cones* $\sigma(e_1, e_3, e_1 + e_2 + 2e_3)$, $\sigma(e_1, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$, $\sigma(e_1, e_2, e_1 + e_2 + e_3)$, $\sigma(e_2, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$ *and* $\sigma(e_2, e_3, e_1 + e_2 + 2e_3)$. *This variety is obtainable from* $\boldsymbol{A}^3$ *through a 3-blow up followed by a 2-blow up.*



We now want to explain why is too difficult to generalize the proof to 3-dimensional almost Fano varieties or to Fano varieties of arbitrary dimension.

It is likely that one can prove a lemma similar to the lemma 17.1 for the almost Fano varieties. We suppose that we have to replace the hypothesis $a_1 \geq 2$ with the hypothesis $a_1 \geq 3$. Unluckily, it is too difficult to generalize the explicit part of the proof that we have done in this paragraph, because there are too much varieties to study. Indeed we know that there are an infinite number of almost Fano variety of dimension 3, for example the varieties $\mathbf{A}^1 \times Z_n$ where the Z_n are the varieties of the proposition 15.2. Moreover, most of the varieties that we have explicitly studied in this section are quasi-Fano varieties, so it is difficult to find a family of varieties that hopefully contains all the almost-Fano varieties.

We have two problem to generalize the proof to Fano varieties of arbitrary dimension. First, it is difficult to generalize lemma 17.1. Observe that we have used that there is only one troublesome variety, namely $\overline{Z}$, and there is only one "bad" blow-up of $\overline{Z}$. But this is false in higher dimension. For example, in dimension 4 we have to consider two varieties: i) the variety whose fan has maximal cones $\sigma(v_1, v_2, v_3, v_4)$ and $\sigma(v_1, v_2, v_3, v_1 + v_2 + v_3 - v_4)$ and ii) the variety whose fan has maximal cones $\sigma(v_1, v_2, v_3, v_4)$ and $\sigma(v_1, v_2, v_3, v_1 + v_2 - v_4)$. Moreover we have to consider the blow-ups of any of such varieties along the subvarieties associated to any cone of dimension at least 2 contained in $\sigma(v_1, v_2, v_3)$. So we have to consider five case up to isomorphisms. Second, we cannot reiterate the explicit part of the proof for every dimension.

19 Complete symmetric varieties of rank at least 3

In this section we begin to classify the Fano complete symmetric varieties of rank at least 3 which are obtained from the wonderful variety by a sequence of blow ups along closed orbits. Let Y be a such variety, then either Y is the wonderful variety X or it is the blow-up of X along the closed orbit because of the proposition 16.1. Recall that we have already classified the wonderful Fano symmetric varieties in the theorem 13.1. We will use again the notation used in the proof of that theorem. We want to prove that the blow-up of the wonderful varieties along the closed orbit is not Fano if G/H has a simple factor of rank at least 3. Remember that the weights associated to the anti-canonical bundle are $\lambda_i = 2\delta - 2\delta_0 - (l-2)\alpha_i^s + \sum_{j \neq i} \alpha_j^s$ with $i = 1, ..., l$. It is sufficient to prove that there is always an index i such that $< \lambda_i, (\alpha_i^s)^\vee >$ is negative. Recall that $(\alpha_i^s)^\vee = (2/ < \alpha_i^s, \alpha_i^s >)\alpha_i^s \in M_{\mathbf{R}}$ is the coroot corresponding to α_i^s. Suppose that G/H is not simple and let G'/H' be a simple factor of rank l' at least 3. Let λ_i' be the weights of G' defined in a similar way to the λ_i, then $\lambda_i = \lambda_i' - (l - l')\alpha_i^s + \omega$ where ω is a spherical weight which vanishes on the restricted roots of G'. Thus $< \lambda_i, (\alpha_i^s)^\vee > \leq < \lambda_i', (\alpha_i^s)^\vee >$, so it is sufficient to consider the simple symmetric varieties of rank at least 3.

We consider two cases. First, suppose that there is an $i \in \{1, ..., l\}$ such that we have $< \beta, \alpha_i >= 0$ for each root β fixed by θ, so $< \beta, \theta(\alpha_i) >= 0$. In this case

$< 2\widetilde{\delta}, (\alpha_i^s)^\vee > \leq 2$, $< 2\widetilde{\delta}_0, (\alpha_i^s)^\vee >= 0$, $< \alpha_i^s, (\alpha_i^s)^\vee >= 2$ and $< \alpha_j^s, (\alpha_i^s)^\vee > \leq 0$ for each $j \neq i$. Hence $< \lambda_i, (\alpha_i^s)^\vee > \leq 2 + 0 - 2 - 0 = 0$. Observe that this case include the compactifications of a group. Moreover, the only simple involutions of rank at least 3 which are not include in this case are: 1) the involutions of type AII; 2) the involutions of type CII; 3) the involutions of type $DIII$ if the rank of G is odd.

Second, we suppose that there are simple roots $\alpha_1,\ \alpha_2,\ \alpha_3,\ \beta_1$ and β_2 with the following properties: 1) $\alpha_1,\ \alpha_2,\ \alpha_3\ \in \phi_1$ and $\beta_1,\ \beta_2 \in \phi_0$; 2) $< \beta_i, \beta_i >=< \alpha_j, \alpha_j >$ for each i and j; 3) $< \beta_1, \beta_2 >= 0$ and $< \beta_i, (\alpha_2)^\vee >=< \alpha_j^s, (\alpha_2^s)^\vee >= -1$ for each i and for each j different from 2; 4) $\alpha_2^s = 2\alpha_2 + \beta_1 + \beta_2$. For simplicity we suppose that $< \alpha_2, \alpha_2 >= 2$. Observe that for each $\beta \in \Gamma_0$ we have $< \beta, \alpha_2^s >=< \theta(\beta), \theta(\alpha_2^s) >= - < \beta, \alpha_2^s >$, so $< \beta, \alpha_2^s >= 0$. Moreover we have $< 2\delta, \alpha_2 >=< 2\delta, \beta_1 >=< 2\delta, \beta_2 >= 2$, so $< 2\delta - 2\delta_0, \alpha_2^s >=< 2\delta, \alpha_2^s >= 8$. Observe that $< \alpha_2^s, \alpha_2^s >=< \alpha_2^s, 2\alpha_2 >= 4$, so $(\alpha_2^s)^\vee = \frac{1}{2}\alpha_2^s$ and $< 2\delta - 2\delta_0, (\alpha_2^s)^\vee >= 4$. Hence $< \lambda_2, (\alpha_2^s)^\vee > \leq < 2\delta - 2\delta_0 - (l-2)\alpha_2^s + \alpha_1^s + \alpha_3^s, (\alpha_2^s)^\vee > \leq 4 - 2(l-2) - 1 - 1 \leq 0$. Thus we have proved the following lemma.

Lemma 19.1 *Suppose that $(G,\theta) = (G_1,\theta_1) \times (G_2,\theta_2)$, where (G_1,θ_1) is simple of rank at least 3. If Y is a Fano complete symmetric variety then it is wonderful.*

20 Complete Fano symmetric varieties I

Now we want conclude the classification of the (almost) Fano complete symmetric varieties (with the suitable hypotheses). We have already classified the associated Fano toric varieties, so we have only to calculate some weights. First of all we will do some remark on the not simple symmetric variety and we will introduce some notations. Suppose that $(G,\theta) = (G_1,\theta_1) \times (G_2,\theta_2)$ and let ϕ_i be the root system of G_i, so $\phi = \phi_1 \cup \phi_2$. Let Ω_i be the lattice generated by the spherical weights of G_i, so $\Omega = \Omega_1 \oplus \Omega_2$. Given a weight λ in Ω, we can write $\lambda = \lambda_1 + \lambda_2$ where λ_i belongs to Ω_i. Observe that λ is (strongly) dominant if and only if both λ_1 and λ_2 are (strongly) dominant. Thus we can reduce ourselves to study some weights of simple symmetric varieties, even if the complete symmetric variety may not be the product of a completion of G_1/H_1 and of a completion of G_2/H_2. Because of the previous section we can suppose that the rank of G_i/H_i is at most 3 for each i.

Now we list all the weights which we have to determine. If $l = 3$ we have to study the weights

$$\lambda_0 = 2\delta - 2\delta_0$$
$$\lambda_1 = 2\delta - 2\delta_0 + \alpha_1^s$$
$$\lambda_2 = 2\delta - 2\delta_0 + \alpha_2^s$$
$$\lambda_3 = 2\delta - 2\delta_0 + \alpha_3^s$$
$$\lambda_4 = 2\delta - 2\delta_0 + \alpha_1^s + \alpha_2^s$$

$$\lambda_5 = 2\delta - 2\delta_0 + \alpha_1^s + \alpha_3^s$$
$$\lambda_6 = 2\delta - 2\delta_0 + \alpha_2^s + \alpha_3^s$$
$$\lambda_7 = 2\delta - 2\delta_0 - \alpha_1^s + \alpha_2^s + \alpha_3^s$$
$$\lambda_8 = 2\delta - 2\delta_0 + \alpha_1^s - \alpha_2^s + \alpha_3^s$$
$$\lambda_9 = 2\delta - 2\delta_0 + \alpha_1^s + \alpha_2^s - \alpha_3^s$$
$$\lambda_{10} = 2\delta - 2\delta_0 - 2\alpha_1^s + \alpha_2^s + \alpha_3^s$$
$$\lambda_{11} = 2\delta - 2\delta_0 + \alpha_1^s - 2\alpha_2^s + \alpha_3^s$$

and

$$\lambda_{12} = 2\delta - 2\delta_0 + \alpha_1^s + \alpha_2^s - 2\alpha_3^s.$$

If $l = 2$ we have to determine the weights

$$\mu_n = 2\delta - 2\delta_0 - (n-1)\alpha_1^s + \alpha_2^s$$
$$\nu_n = 2\delta - 2\delta_0 + \alpha_1^s - (n-1)\alpha_2^s$$
$$\eta = 2\delta - 2\delta_0$$

for each $n \geq 0$. Observe that $\mu_0 = \nu_0$.

If $l = 1$ we have to study the weights $\psi_n = 2\delta - 2\delta_0 - (n-1)\alpha_1^s$ for each $n \geq 0$.

Suppose that $l = 3$ and $(G, \theta) = (G_1, \theta_1) \times (G_2, \theta_2)$ where the rank of (G_1, θ_1) is 2 and the rank of (G_2, θ_2) is 1. Then

$$\lambda_0 = \eta + \psi_1$$
$$\lambda_1 = \nu_1 + \psi_1$$
$$\lambda_2 = \mu_1 + \psi_1$$
$$\lambda_3 = \eta + \psi_0$$
$$\lambda_4 = \mu_0 + \psi_1$$
$$\lambda_5 = \nu_1 + \psi_0$$
$$\lambda_6 = \mu_1 + \psi_0$$
$$\lambda_7 = \mu_2 + \psi_0$$
$$\lambda_8 = \nu_2 + \psi_0$$
$$\lambda_9 = \mu_0 + \psi_2$$
$$\lambda_{10} = \mu_3 + \psi_0$$
$$\lambda_{11} = \nu_3 + \psi_0$$

and

$$\lambda_{12} = \mu_0 + \psi_3.$$

Suppose that $(G,\theta)=(G_1,\theta_1)\times(G_2,\theta_2)\times(G_3,\theta_3)$ where the rank of (G_i,θ_i) is 1 for each i. Let ψ_n^i be the weights of G_i defined as before. We have

$$\lambda_0=\psi_1^1+\psi_1^2+\psi_1^3$$
$$\lambda_1=\psi_0^1+\psi_1^2+\psi_1^3$$
$$\lambda_2=\psi_1^1+\psi_0^2+\psi_1^3$$
$$\lambda_3=\psi_1^1+\psi_1^2+\psi_0^3$$
$$\lambda_4=\psi_0^1+\psi_0^2+\psi_1^3$$
$$\lambda_5=\psi_0^1+\psi_1^2+\psi_0^3$$
$$\lambda_6=\psi_1^1+\psi_0^2+\psi_0^3$$
$$\lambda_7=\psi_2^1+\psi_0^2+\psi_0^3$$
$$\lambda_8=\psi_0^1+\psi_2^2+\psi_0^3$$
$$\lambda_9=\psi_0^1+\psi_0^2+\psi_2^3$$
$$\lambda_{10}=\psi_3^1+\psi_0^2+\psi_0^3$$
$$\lambda_{11}=\psi_0^1+\psi_3^2+\psi_0^3$$

and

$$\lambda_{12}=\psi_0^1+\psi_0^2+\psi_3^3.$$

Suppose $l=2$ and $(G,\theta)=(G_1,\theta_1)\times(G_2,\theta_2)$ where the rank of (G_i,θ_i) is 1 for each i. Let ψ_n^i be the weights of G_i defined as before. We have

$$\mu_n=\psi_n^1+\psi_0^2$$
$$\nu_n=\psi_0^1+\psi_n^2$$
$$\eta=\psi_1^1+\psi_1^2.$$

Now we will write the weights of each complete symmetric variety whose associated toric variety is Fano, respectively almost Fano.

Let Y be a complete symmetric variety of rank 2 such that the fan of associated toric variety has 1-dimensional cones generated respectively by $e_1, e_1+e_2, e_1+2e_2, ..., e_1+ie_2, ..., e_1+me_2, e_2$. The weights associated to the anticanonical bundle of Y are ν_1 and μ_m.

Let Y be a complete symmetric variety of rank 2 such that the fan of associated toric variety has 1-dimensional cones generated respectively by $e_1, me_1+e_2, (m-1)e_1+e_2, ..., ie_1+e_2, ..., e_1+e_2, e_2$. The weights associated to the anticanonical bundle of Y are μ_1 and ν_m.

Let Y be the complete symmetric variety of rank 3 which is the the blow-up of X along $X_{\sigma(e_1,e_2)}$. The weights associated to the anticanonical bundle of Y are λ_5 and λ_6.

Let Y be the complete symmetric variety of rank 3 which is the the blow-up of X along $X_{\sigma(e_1,e_3)}$. The weights associated to the anticanonical bundle of Y are λ_4 and λ_6.

Let Y be the complete symmetric variety of rank 3 which is the the blow-up of X along $X_{\sigma(e_2,e_3)}$. The weights associated to the anticanonical bundle of Y are λ_4 and λ_5.

Let Y be the complete symmetric variety of rank 3 which is the the blow-up of X along $X_{\sigma(e_1,e_2,e_3)}$. The weights associated to the anticanonical bundle of Y are λ_7, λ_8 and λ_9.

Let Y be the complete symmetric variety of rank 3 which is obtained by a blow-up of X along $X_{\sigma(e_1,e_2,e_3)}$ followed by a blow-up along $Y_{\sigma(e_1,e_2)}$. The weights associated to the anticanonical bundle of Y are λ_1, λ_2, λ_7 and λ_8.

Let Y be the complete symmetric variety of rank 3 which is obtained by a blow-up of X along $X_{\sigma(e_1,e_2,e_3)}$ followed by a blow-up along $Y_{\sigma(e_1,e_3)}$. The weights associated to the anticanonical bundle of Y are λ_1, λ_3, λ_7 and λ_9.

Let Y be the complete symmetric variety of rank 3 which is obtained by a blow-up of X along $X_{\sigma(e_1,e_2,e_3)}$ followed by a blow-up along $Y_{\sigma(e_2,e_3)}$. The weights associated to the anticanonical bundle of Y are λ_2, λ_3, λ_8 and λ_9.

Let Y be the complete symmetric variety of rank 3 which is obtained by a blow-up of X along $X_{\sigma(e_1,e_2,e_3)}$ followed by a blow-up along $Y_{\sigma(e_1,e_1+e_2+e_3)}$. The weights associated to the anticanonical bundle of Y are λ_2, λ_3, λ_7, λ_{11} and λ_{12}.

Let Y be the complete symmetric variety of rank 3 which is obtained by a blow-up of X along $X_{\sigma(e_1,e_2,e_3)}$ followed by a blow-up along $Y_{\sigma(e_2,e_1+e_2+e_3)}$. The weights associated to the anticanonical bundle of Y are λ_1, λ_3, λ_8, λ_{10} and λ_{12}.

Let Y be the complete symmetric variety of rank 3 which is obtained by a blow-up of X along $X_{\sigma(e_1,e_2,e_3)}$ followed by a blow-up along $Y_{\sigma(e_3,e_1+e_2+e_3)}$. The weights associated to the anticanonical bundle of Y are λ_1, λ_2, λ_9, λ_{10} and λ_{11}.

Finally let Y be a complete symmetric variety of rank l at least 3 which is the blow-up of X along $X_{\sigma(e_1,\ldots,e_l)}$. Let (G',θ') be a simple factor of (G,θ). By the lemma 19.1 we can suppose that the rank l' of (G',θ') is at most 2. If $l'=2$ we are interested to the weights μ_0, μ_{l-1}, ν_{l-1}. If $l'=1$ we are interested to the weights ψ_0 and ψ_{l-1}.

Observe that it is not necessary to determine λ_0, but its knowledge is useful to determine the other weights.

To calculate these weights is useful to use another notation for the roots in ϕ. We define $\{\beta_1,...,\beta_r\}$ as a reordering of the basis $\{\alpha_1,...,\alpha_r\}$ of ϕ with the same notation of [Hu]. Let ω'_i be the fundamental weight dual to $\beta_i^\vee$ and let $\overline{\omega}_i=\omega'_i-\theta(\omega'_i)$. Remember that $\delta=\sum_{i=1}^m\omega'_i$. To calculate δ_0, it is useful to write $\sum\omega'_i$ as a linear combination of roots $\{\beta_1,...,\beta_r\}$ for each root system of a classic Lie algebra. We have

$$2\delta = \sum_{i=1}^{m} i(m+1-i)\alpha_i \text{ if the type of } \phi \text{ is } A_m,$$

$$2\delta = \sum_{i=1}^{m} i(2m-i)\alpha_i \text{ if the type of } \phi \text{ is } B_m,$$

$$2\delta = \sum_{i=1}^{m-1} i(2m+1-i)\alpha_i + \frac{m(m+1)}{2}\alpha_m \text{ if the type of } \phi \text{ is } C_m,$$

and

$$2\delta = \sum_{i=1}^{m-2} i(2m-1-i)\alpha_i + \frac{m(m-1)}{2}(\alpha_{m-1}+\alpha_m)$$

if the type of ϕ is D_m.

Now we want to write the simple restricted roots and the previous weights respect to the $\overline{\omega}_i$. In the first column we indicate the type of the involution while, if the homogeneous symmetric variety is a group, we write the type of its root system. In the second one we will write the rank m of G. In the third column we write the α_i^s as a linear combination of the $\{\beta_1^s, ..., \beta_r^s\}$ and as a linear combination of the $\{\overline{\omega}_1, ..., \overline{\omega}_r\}$. In the forth one we write the weights that we have defined in the begin of this section. Afterwards we will write a table in which we indicated what weights are dominant, respectively regular. First we consider the case in which $l = 1$.

A_1	2	$\alpha_1^s = \beta_1^s = 2\overline{\omega}_1$	$\psi_n = (4-2n)\overline{\omega}_1$
AI	1	$\alpha_1^s = \beta_1^s = 4\overline{\omega}_1$	$\psi_n = (6-4n)\overline{\omega}_1$
AII	3	$\alpha_1^s = \beta_2^s = 2\overline{\omega}_2$	$\psi_n = (6-2n)\overline{\omega}_2$
AIV	m	$\alpha_1^s = \beta_1^s = \overline{\omega}_1$	$\psi_n = (m+1-n)\overline{\omega}_1$
BII	m	$\alpha_1^s = \beta_1^s = 2\overline{\omega}_1$	$\psi_n = (2m-2n+1)\overline{\omega}_1$
CII	m	$\alpha_1^s = \beta_2^s = \overline{\omega}_2$	$\psi_n = (2m-n)\overline{\omega}_2$
DII	m	$\alpha_1^s = \beta_1^s = 2\overline{\omega}_1$	$\psi_n = (2m-2n)\overline{\omega}_1$
FII	4	$\alpha_1^s = \beta_4^s = \overline{\omega}_4$	$\psi_n = (12-n)\overline{\omega}_1$

A_1	2	ψ_n if $n \leq 2$	ψ_n if $n \leq 1$
AI	1	ψ_n if $n \leq 1$	ψ_n if $n \leq 1$
AII	3	ψ_n if $n \leq 3$	ψ_n if $n \leq 2$
AIV	m	ψ_n if $n \leq m+1$	ψ_n if $n \leq m$
BII	m	ψ_n if $n \leq m$	ψ_n if $n \leq m$
CII	m	ψ_n if $n \leq 2m$	ψ_n if $n \leq 2m-1$
DII	m	ψ_n if $n \leq m$	ψ_n if $n \leq m-1$
FII	4	ψ_n if $n \leq 12$	ψ_n if $n \leq 11$

Observe that ψ_0 and ψ_1 are always regular. We now consider the case in which $l = 2$.

A_2	4	$\alpha_1^s = \beta_1^s = 2\overline{\omega}_1 - \overline{\omega}_2$ $\alpha_2^s = \beta_2^s = -\overline{\omega}_1 + 2\overline{\omega}_2$	$\eta = 2\overline{\omega}_1 + 2\overline{\omega}_2$ $\mu_n = (3-2n)\overline{\omega}_1 + (3+n)\overline{\omega}_2$ $\nu_n = (3+n)\overline{\omega}_1 + (3-2n)\overline{\omega}_2$
B_2	4	$\alpha_1^s = \beta_1^s = 2\overline{\omega}_1 - 2\overline{\omega}_2$ $\alpha_2^s = \beta_2^s = -\overline{\omega}_1 + 2\overline{\omega}_2$	$\eta = 2\overline{\omega}_1 + 2\overline{\omega}_2$ $\mu_n = (3-2n)\overline{\omega}_1 + (2+2n)\overline{\omega}_2$ $\nu_n = (3+n)\overline{\omega}_1 + (2-2n)\overline{\omega}_2$
G_2	4	$\alpha_1^s = \beta_1^s = 2\overline{\omega}_1 - \overline{\omega}_2$ $\alpha_2^s = \beta_2^s = -3\overline{\omega}_1 + 2\overline{\omega}_2$	$\eta = 2\overline{\omega}_1 + 2\overline{\omega}_2$ $\mu_n = (1-2n)\overline{\omega}_1 + (3+n)\overline{\omega}_2$ $\nu_n = (1+3n)\overline{\omega}_1 + (3-2n)\overline{\omega}_2$
AI	2	$\alpha_1^s = \beta_1^s = 4\overline{\omega}_1 - 2\overline{\omega}_2$ $\alpha_2^s = \beta_2^s = -2\overline{\omega}_1 + 4\overline{\omega}_2$	$\eta = 2\overline{\omega}_1 + 2\overline{\omega}_2$ $\mu_n = (4-4n)\overline{\omega}_1 + (4+2n)\overline{\omega}_2$ $\nu_n = (4+2n)\overline{\omega}_1 + (4-4n)\overline{\omega}_2$
AII	5	$\alpha_1^s = \beta_2^s = 2\overline{\omega}_2 - \overline{\omega}_4$ $\alpha_2^s = \beta_4^s = -\overline{\omega}_2 + 2\overline{\omega}_4$	$\eta = 4\overline{\omega}_2 + 4\overline{\omega}_4$ $\mu_n = (5-2n)\overline{\omega}_2 + (5+n)\overline{\omega}_4$ $\nu_n = (5+n)\overline{\omega}_2 + (5-2n)\overline{\omega}_4$
$AIII$	$m \geq 4$	$\alpha_1^s = \beta_1^s = 2\overline{\omega}_1 - \overline{\omega}_2$ $\alpha_2^s = \beta_2^s = -\overline{\omega}_1 + \overline{\omega}_2$	$\eta = 2\overline{\omega}_1 + (m-2)\overline{\omega}_2$ $\mu_n = (3-2n)\overline{\omega}_1 + (m-2+n)\overline{\omega}_2$ $\nu_n = (3+n)\overline{\omega}_1 + (m-2-n)\overline{\omega}_2$
$AIII$	3	$\alpha_1^s = \beta_1^s = 2\overline{\omega}_1 - 2\overline{\omega}_2$ $\alpha_2^s = \beta_2^s = -2\overline{\omega}_1 + 4\overline{\omega}_2$	$\eta = 2\overline{\omega}_1 + 2\overline{\omega}_2$ $\mu_n = (2-2n)\overline{\omega}_1 + (4+2n)\overline{\omega}_2$ $\nu_n = (2+2n)\overline{\omega}_1 + (4-4n)\overline{\omega}_2$
BI	$m \geq 3$	$\alpha_1^s = \beta_1^s = 4\overline{\omega}_1 - 2\overline{\omega}_2$ $\alpha_2^s = \beta_2^s = -2\overline{\omega}_1 + 2\overline{\omega}_2$	$\eta = 2\overline{\omega}_1 + (2m+1)\overline{\omega}_2$ $\mu_n = (4-4n)\overline{\omega}_1 + (2m+2n+1)\overline{\omega}_2$ $\nu_n = (4+2n)\overline{\omega}_1 + (2m-2n+1)\overline{\omega}_2$
BI	2	$\alpha_1^s = \beta_1^s = 4\overline{\omega}_1 - 4\overline{\omega}_2$ $\alpha_2^s = \beta_2^s = -2\overline{\omega}_1 + 4\overline{\omega}_2$	$\eta = 2\overline{\omega}_1 + 2\overline{\omega}_2$ $\mu_n = (4-4n)\overline{\omega}_1 + (2+4n)\overline{\omega}_2$ $\nu_n = (4+2n)\overline{\omega}_1 + (2-4n)\overline{\omega}_2$
CII	$m \geq 5$	$\alpha_1^s = \beta_2^s = 2\overline{\omega}_2 - \overline{\omega}_4$ $\alpha_2^s = \beta_4^s = -\overline{\omega}_2 + \overline{\omega}_4$	$\eta = 4\overline{\omega}_2 + (2m-5)\overline{\omega}_4$ $\mu_n = (5-2n)\overline{\omega}_2 + (2m+n-5)\overline{\omega}_4$ $\nu_n = (5+n)\overline{\omega}_2 + (2m-n-5)\overline{\omega}_4$
CII	4	$\alpha_1^s = \beta_2^s = 2\overline{\omega}_2 - \overline{\omega}_4$ $\alpha_2^s = \beta_4^s = -2\overline{\omega}_2 + 2\overline{\omega}_4$	$\eta = 4\overline{\omega}_2 + 3\overline{\omega}_4$ $\mu_n = (4-2n)\overline{\omega}_2 + (4+n)\overline{\omega}_4$ $\nu_n = (4+2n)\overline{\omega}_2 + (4-2n)\overline{\omega}_4$
DI	m	$\alpha_1^s = \beta_1^s = 4\overline{\omega}_1 - 2\overline{\omega}_2$ $\alpha_2^s = \beta_2^s = -2\overline{\omega}_1 + 2\overline{\omega}_2$	$\eta = 2\overline{\omega}_1 + (2m-4)\overline{\omega}_2$ $\mu_n = (4-4n)\overline{\omega}_1 + (2m+2n-4)\overline{\omega}_2$ $\nu_n = (4+2n)\overline{\omega}_1 + (2m-2n-4)\overline{\omega}_2$
$DIII$	4	$\alpha_1^s = \beta_2^s = 2\overline{\omega}_2 - 2\overline{\omega}_4$ $\alpha_2^s = \beta_4^s = -2\overline{\omega}_2 + 4\overline{\omega}_4$	$\eta = 4\overline{\omega}_2 + 2\overline{\omega}_4$ $\mu_n = (4-2n)\overline{\omega}_2 + (4+2n)\overline{\omega}_4$ $\nu_n = (4+2n)\overline{\omega}_2 + (4-4n)\overline{\omega}_4$
$DIII$	5	$\alpha_1^s = \beta_1^s = 2\overline{\omega}_1 - \overline{\omega}_4$ $\alpha_2^s = \beta_4^s = -\overline{\omega}_1 + \overline{\omega}_4$	$\eta = 4\overline{\omega}_1 + 3\overline{\omega}_4$ $\mu_n = (5-2n)\overline{\omega}_1 + (3+n)\overline{\omega}_4$ $\nu_n = (5+n)\overline{\omega}_1 + (3-n)\overline{\omega}_4$

$EIII$	6	$\alpha_1^s = \beta_2^s = 2\overline{\omega}_2 - \overline{\omega}_1$ $\alpha_2^s = \beta_1^s = -\overline{\omega}_2 + \overline{\omega}_1$	$\eta = 6\overline{\omega}_2 + 5\overline{\omega}_1$ $\mu_n = (7-2n)\overline{\omega}_2 + (5+n)\overline{\omega}_1$ $\nu_n = (7+n)\overline{\omega}_2 + (5-n)\overline{\omega}_1$
EIV	6	$\alpha_1^s = \beta_1^s = 2\overline{\omega}_1 - \overline{\omega}_6$ $\alpha_2^s = \beta_6^s = -\overline{\omega}_1 + 2\overline{\omega}_6$	$\eta = 8\overline{\omega}_1 + 8\overline{\omega}_6$ $\mu_n = (9-2n)\overline{\omega}_1 + (9+n)\overline{\omega}_6$ $\nu_n = (9+n)\overline{\omega}_1 + (9-2n)\overline{\omega}_6$
G	2	$\alpha_1^s = \beta_1^s = 4\overline{\omega}_1 - 2\overline{\omega}_2$ $\alpha_2^s = \beta_2^s = -6\overline{\omega}_1 + 4\overline{\omega}_2$	$\eta = 2\overline{\omega}_1 + 2\overline{\omega}_2$ $\mu_n = -4n\overline{\omega}_1 + (4+2n)\overline{\omega}_2$ $\nu_n = 6n\overline{\omega}_1 + (4-4n)\overline{\omega}_2$

A_2	4	μ_n if $n \le 1$ ν_n if $n \le 1$ η	μ_n if $n \le 1$ ν_n if $n \le 1$ η
B_2	4	μ_n if $n \le 1$ ν_n if $n \le 1$ η	μ_n if $n \le 1$ ν_0 η
G_2	4	μ_0 ν_n if $n \le 1$ η	μ_0 ν_n if $n \le 1$ η
AI	2	μ_n if $n \le 1$ ν_n if $n \le 1$ η	μ_0 ν_0 η
AII	5	μ_n if $n \le 2$ ν_n if $n \le 2$ η	μ_n if $n \le 2$ ν_n if $n \le 2$ η
$AIII$	$m \ge 4$	μ_n if $n \le 1$ ν_n if $n \le m-2$ η	μ_n if $n \le 1$ ν_n if $n \le m-3$ η
$AIII$	3	μ_n if $n \le 1$ ν_n if $n \le 1$ η	μ_0 ν_0 η
BI	$m \ge 3$	μ_n if $n \le 1$ ν_n if $n \le m$ η	μ_0 ν_n if $n \le m$ η
BI	2	μ_n if $n \le 1$ ν_0 η	μ_0 ν_0 η
CII	$m \ge 5$	μ_n if $n \le 2$ ν_n if $n \le 2m-5$ η	μ_n if $n \le 2$ ν_n if $n \le 2m-6$ η

CII	4	μ_n if $n \leq 2$ ν_n if $n \leq 2$ η	μ_n if $n \leq 1$ ν_n if $n \leq 1$ η
DI	m	μ_n if $n \leq 1$ ν_n if $n \leq m-2$ η	μ_0 ν_n if $n \leq m-3$ η
$DIII$	4	μ_n if $n \leq 2$ ν_n if $n \leq 1$ η	μ_n if $n \leq 1$ ν_0 η
$DIII$	5	μ_n if $n \leq 2$ ν_n if $n \leq 3$ η	μ_n if $n \leq 2$ ν_n if $n \leq 2$ η
$EIII$	6	μ_n if $n \leq 3$ ν_n if $n \leq 5$ η	μ_n if $n \leq 3$ ν_n if $n \leq 4$ η
EIV	6	μ_n if $n \leq 4$ ν_n if $n \leq 4$ η	μ_n if $n \leq 4$ ν_n if $n \leq 4$ η
G	2	μ_0 ν_n if $n \leq 1$ η	η

Finally we can consider the case of rank $l = 3$.

A_3	6	$\alpha_1^s = \beta_1^s = 2\overline{\omega}_1 - \overline{\omega}_2$ $\alpha_2^s = \beta_2^s = -\overline{\omega}_1 + 2\overline{\omega}_2 - \overline{\omega}_3$ $\alpha_3^s = \beta_3^s = -\overline{\omega}_2 + 2\overline{\omega}_3$	$\lambda_0 = 2\overline{\omega}_1 + 2\overline{\omega}_2 + 2\overline{\omega}_3$ $\lambda_1 = 4\overline{\omega}_1 + \overline{\omega}_2 + 2\overline{\omega}_3$ $\lambda_2 = \overline{\omega}_1 + 4\overline{\omega}_2 + \overline{\omega}_3$ $\lambda_3 = 2\overline{\omega}_1 + \overline{\omega}_2 + 4\overline{\omega}_3$ $\lambda_4 = 3\overline{\omega}_1 + 3\overline{\omega}_2 + \overline{\omega}_3$ $\lambda_5 = 4\overline{\omega}_1 + 4\overline{\omega}_3$ $\lambda_6 = \overline{\omega}_1 + 3\overline{\omega}_2 + 3\overline{\omega}_3$ $\lambda_7 = -\overline{\omega}_1 + 4\overline{\omega}_2 + 3\overline{\omega}_3$ $\lambda_8 = 5\overline{\omega}_1 - 2\overline{\omega}_2 + 5\overline{\omega}_3$ $\lambda_9 = 3\overline{\omega}_1 + 4\overline{\omega}_2 - \overline{\omega}_3$ $\lambda_{10} = -3\overline{\omega}_1 + 5\overline{\omega}_2 + 3\overline{\omega}_3$ $\lambda_{11} = 6\overline{\omega}_1 - 4\overline{\omega}_2 + 6\overline{\omega}_3$ $\lambda_{12} = 3\overline{\omega}_1 + 5\overline{\omega}_2 - 3\overline{\omega}_3$

B_3	6	$\alpha_1^s=\beta_1^s=2\overline{\omega}_1-\overline{\omega}_2$ $\alpha_2^s=\beta_2^s=-\overline{\omega}_1+2\overline{\omega}_2-2\overline{\omega}_3$ $\alpha_3^s=\beta_3^s=-\overline{\omega}_2+2\overline{\omega}_3$	$\lambda_0=2\overline{\omega}_1+2\overline{\omega}_2+2\overline{\omega}_3$ $\lambda_1=4\overline{\omega}_1+\overline{\omega}_2+2\overline{\omega}_3$ $\lambda_2=\overline{\omega}_1+4\overline{\omega}_2$ $\lambda_3=2\overline{\omega}_1+\overline{\omega}_2+4\overline{\omega}_3$ $\lambda_4=3\overline{\omega}_1+3\overline{\omega}_2$ $\lambda_5=4\overline{\omega}_1+4\overline{\omega}_3$ $\lambda_6=\overline{\omega}_1+3\overline{\omega}_2+2\overline{\omega}_3$ $\lambda_7=-\overline{\omega}_1+4\overline{\omega}_2+2\overline{\omega}_3$ $\lambda_8=5\overline{\omega}_1-2\overline{\omega}_2+6\overline{\omega}_3$ $\lambda_9=3\overline{\omega}_1+4\overline{\omega}_2-2\overline{\omega}_3$ $\lambda_{10}=-3\overline{\omega}_1+5\overline{\omega}_2+2\overline{\omega}_3$ $\lambda_{11}=6\overline{\omega}_1-4\overline{\omega}_2+8\overline{\omega}_3$ $\lambda_{12}=3\overline{\omega}_1+5\overline{\omega}_2-4\overline{\omega}_3$
C_3	6	$\alpha_1^s=\beta_1^s=2\overline{\omega}_1-\overline{\omega}_2$ $\alpha_2^s=\beta_2^s=-\overline{\omega}_1+2\overline{\omega}_2-\overline{\omega}_3$ $\alpha_3^s=\beta_3^s=-2\overline{\omega}_2+2\overline{\omega}_3$	$\lambda_0=2\overline{\omega}_1+2\overline{\omega}_2+2\overline{\omega}_3$ $\lambda_1=4\overline{\omega}_1+\overline{\omega}_2+2\overline{\omega}_3$ $\lambda_2=\overline{\omega}_1+4\overline{\omega}_2+\overline{\omega}_3$ $\lambda_3=2\overline{\omega}_1+4\overline{\omega}_3$ $\lambda_4=3\overline{\omega}_1+3\overline{\omega}_2+\overline{\omega}_3$ $\lambda_5=4\overline{\omega}_1-\overline{\omega}_2+4\overline{\omega}_3$ $\lambda_6=\overline{\omega}_1+2\overline{\omega}_2+3\overline{\omega}_3$ $\lambda_7=-\overline{\omega}_1+3\overline{\omega}_2+3\overline{\omega}_3$ $\lambda_8=5\overline{\omega}_1-3\overline{\omega}_2+5\overline{\omega}_3$ $\lambda_9=3\overline{\omega}_1+5\overline{\omega}_2-\overline{\omega}_3$ $\lambda_{10}=-3\overline{\omega}_1+4\overline{\omega}_2+3\overline{\omega}_3$ $\lambda_{11}=6\overline{\omega}_1-5\overline{\omega}_2+6\overline{\omega}_3$ $\lambda_{12}=3\overline{\omega}_1+7\overline{\omega}_2-3\overline{\omega}_3$
AI	3	$\alpha_1^s=\beta_1^s=4\overline{\omega}_1-2\overline{\omega}_2$ $\alpha_2^s=\beta_2^s=-2\overline{\omega}_1+4\overline{\omega}_2-2\overline{\omega}_3$ $\alpha_3^s=\beta_3^s=-2\overline{\omega}_2+4\overline{\omega}_3$	$\lambda_0=2\overline{\omega}_1+2\overline{\omega}_2+2\overline{\omega}_3$ $\lambda_1=6\overline{\omega}_1+2\overline{\omega}_3$ $\lambda_2=6\overline{\omega}_2$ $\lambda_3=2\overline{\omega}_1+6\overline{\omega}_3$ $\lambda_4=4\overline{\omega}_1+4\overline{\omega}_2$ $\lambda_5=6\overline{\omega}_1-2\overline{\omega}_2+6\overline{\omega}_3$ $\lambda_6=4\overline{\omega}_2+4\overline{\omega}_3$ $\lambda_7=-4\overline{\omega}_1+6\overline{\omega}_2+4\overline{\omega}_3$ $\lambda_8=8\overline{\omega}_1-6\overline{\omega}_2+8\overline{\omega}_3$ $\lambda_9=4\overline{\omega}_1+6\overline{\omega}_2-4\overline{\omega}_3$ $\lambda_{10}=-8\overline{\omega}_1+8\overline{\omega}_2+4\overline{\omega}_3$ $\lambda_{11}=10\overline{\omega}_1-10\overline{\omega}_2+10\overline{\omega}_3$ $\lambda_{12}=4\overline{\omega}_1+8\overline{\omega}_2-8\overline{\omega}_3$

AII	7	$\alpha_1^s = \beta_2^s = 2\overline{\omega}_2 - \overline{\omega}_4$ $\alpha_2^s = \beta_4^s = -\overline{\omega}_2 + 2\overline{\omega}_4 - \overline{\omega}_6$ $\alpha_3^s = \beta_6^s = -\overline{\omega}_4 + 2\overline{\omega}_6$	$\lambda_0 = 4\overline{\omega}_2 + 4\overline{\omega}_4 + 4\overline{\omega}_6$ $\lambda_1 = 6\overline{\omega}_2 + 3\overline{\omega}_4 + 4\overline{\omega}_6$ $\lambda_2 = 3\overline{\omega}_2 + 6\overline{\omega}_4 + 3\overline{\omega}_6$ $\lambda_3 = 4\overline{\omega}_2 + 3\overline{\omega}_4 + 6\overline{\omega}_6$ $\lambda_4 = 5\overline{\omega}_2 + 5\overline{\omega}_4 + 3\overline{\omega}_6$ $\lambda_5 = 6\overline{\omega}_2 + 2\overline{\omega}_4 + 6\overline{\omega}_6$ $\lambda_6 = 3\overline{\omega}_2 + 5\overline{\omega}_4 + 5\overline{\omega}_6$ $\lambda_7 = \overline{\omega}_2 + 6\overline{\omega}_4 + 5\overline{\omega}_6$ $\lambda_8 = 7\overline{\omega}_2 + 7\overline{\omega}_6$ $\lambda_9 = 5\overline{\omega}_2 + 6\overline{\omega}_4 + \overline{\omega}_6$ $\lambda_{10} = -\overline{\omega}_2 + 7\overline{\omega}_4 + 5\overline{\omega}_6$ $\lambda_{11} = 8\overline{\omega}_2 - 2\overline{\omega}_4 + 8\overline{\omega}_6$ $\lambda_{12} = 5\overline{\omega}_2 + 7\overline{\omega}_4 - \overline{\omega}_6$
$AIII$	$m \geq 6$	$\alpha_1^s = \beta_1^s = 2\overline{\omega}_1 - \overline{\omega}_2$ $\alpha_2^s = \beta_2^s = -\overline{\omega}_1 + 2\overline{\omega}_2 - \overline{\omega}_3$ $\alpha_3^s = \beta_3^s = -\overline{\omega}_2 + \overline{\omega}_3$	$\lambda_0 = 2\overline{\omega}_1 + 2\overline{\omega}_2 + (m-4)\overline{\omega}_3$ $\lambda_1 = 4\overline{\omega}_1 + \overline{\omega}_2 + (m-4)\overline{\omega}_3$ $\lambda_2 = \overline{\omega}_1 + 4\overline{\omega}_2 + (m-5)\overline{\omega}_3$ $\lambda_3 = 2\overline{\omega}_1 + \overline{\omega}_2 + (m-3)\overline{\omega}_3$ $\lambda_4 = 3\overline{\omega}_1 + 3\overline{\omega}_2 + (m-5)\overline{\omega}_3$ $\lambda_5 = 4\overline{\omega}_1 + (m-3)\overline{\omega}_3$ $\lambda_6 = \overline{\omega}_1 + 3\overline{\omega}_2 + (m-4)\overline{\omega}_3$ $\lambda_7 = -\overline{\omega}_1 + 4\overline{\omega}_2 + (m-4)\overline{\omega}_3$ $\lambda_8 = 5\overline{\omega}_1 - 2\overline{\omega}_2 + (m-2)\overline{\omega}_3$ $\lambda_9 = 3\overline{\omega}_1 + 4\overline{\omega}_2 + (m-6)\overline{\omega}_3$ $\lambda_{10} = -3\overline{\omega}_1 + 5\overline{\omega}_2 + (m-4)\overline{\omega}_3$ $\lambda_{11} = 6\overline{\omega}_1 - 4\overline{\omega}_2 + (m-1)\overline{\omega}_3$ $\lambda_{12} = 3\overline{\omega}_1 + 5\overline{\omega}_2 + (m-7)\overline{\omega}_3$
$AIII$	5	$\alpha_1^s = \beta_1^s = 2\overline{\omega}_1 - \overline{\omega}_2$ $\alpha_2^s = \beta_2^s = -\overline{\omega}_1 + 2\overline{\omega}_2 - 2\overline{\omega}_3$ $\alpha_3^s = \beta_3^s = -2\overline{\omega}_2 + 4\overline{\omega}_3$	$\lambda_0 = 2\overline{\omega}_1 + 2\overline{\omega}_2 + 2\overline{\omega}_3$ $\lambda_1 = 4\overline{\omega}_1 + \overline{\omega}_2 + 2\overline{\omega}_3$ $\lambda_2 = \overline{\omega}_1 + 4\overline{\omega}_2$ $\lambda_3 = 2\overline{\omega}_1 + 6\overline{\omega}_3$ $\lambda_4 = 3\overline{\omega}_1 + 3\overline{\omega}_2$ $\lambda_5 = 4\overline{\omega}_1 - \overline{\omega}_2 + 6\overline{\omega}_3$ $\lambda_6 = \overline{\omega}_1 + 2\overline{\omega}_2 + 4\overline{\omega}_3$ $\lambda_7 = -\overline{\omega}_1 + 3\overline{\omega}_2 + 4\overline{\omega}_3$ $\lambda_8 = 5\overline{\omega}_1 - 3\overline{\omega}_2 + 8\overline{\omega}_3$ $\lambda_9 = 3\overline{\omega}_1 + 5\overline{\omega}_2 - 4\overline{\omega}_3$ $\lambda_{10} = -3\overline{\omega}_1 + 4\overline{\omega}_2 + 4\overline{\omega}_3$ $\lambda_{11} = 6\overline{\omega}_1 - 5\overline{\omega}_2 + 10\overline{\omega}_3$ $\lambda_{12} = 3\overline{\omega}_1 + 7\overline{\omega}_2 - 8\overline{\omega}_3$

BI	$m \geq 4$	$\alpha_1^s = \beta_1^s = 4\overline{\omega}_1 - 2\overline{\omega}_2$ $\alpha_2^s = \beta_2^s = -2\overline{\omega}_1 + 4\overline{\omega}_2 - 2\overline{\omega}_3$ $\alpha_3^s = \beta_3^s = -2\overline{\omega}_2 + 2\overline{\omega}_3$	$\lambda_0 = 2\overline{\omega}_1 + 2\overline{\omega}_2 + (2m+1)\overline{\omega}_3$ $\lambda_1 = 6\overline{\omega}_1 + (2m+1)\overline{\omega}_3$ $\lambda_2 = 6\overline{\omega}_2 + (2m-1)\overline{\omega}_3$ $\lambda_3 = 2\overline{\omega}_1 + (2m+3)\overline{\omega}_3$ $\lambda_4 = 4\overline{\omega}_1 + 4\overline{\omega}_2 + (2m-1)\overline{\omega}_3$ $\lambda_5 = 6\overline{\omega}_1 - 2\overline{\omega}_2 + (2m+3)\overline{\omega}_3$ $\lambda_6 = 4\overline{\omega}_2 + (2m+1)\overline{\omega}_3$ $\lambda_7 = -4\overline{\omega}_1 + 6\overline{\omega}_2 + (2m+1)\overline{\omega}_3$ $\lambda_8 = 8\overline{\omega}_1 - 6\overline{\omega}_2 + (2m+5)\overline{\omega}_3$ $\lambda_9 = 4\overline{\omega}_1 + 6\overline{\omega}_2 + (2m-3)\overline{\omega}_3$ $\lambda_{10} = -8\overline{\omega}_1 + 8\overline{\omega}_2 + (2m+1)\overline{\omega}_3$ $\lambda_{11} = 10\overline{\omega}_1 - 10\overline{\omega}_2 + (2m+7)\overline{\omega}_3$ $\lambda_{12} = 4\overline{\omega}_1 + 8\overline{\omega}_2 + (2m-5)\overline{\omega}_3$
BI	3	$\alpha_1^s = \beta_1^s = 4\overline{\omega}_1 - 2\overline{\omega}_2$ $\alpha_2^s = \beta_2^s = -2\overline{\omega}_1 + 4\overline{\omega}_2 - 4\overline{\omega}_3$ $\alpha_3^s = \beta_3^s = -2\overline{\omega}_2 + 4\overline{\omega}_3$	$\lambda_0 = 2\overline{\omega}_1 + 2\overline{\omega}_2 + 2\overline{\omega}_3$ $\lambda_1 = 6\overline{\omega}_1 + 2\overline{\omega}_3$ $\lambda_2 = 6\overline{\omega}_2 - 2\overline{\omega}_3$ $\lambda_3 = 2\overline{\omega}_1 + 6\overline{\omega}_3$ $\lambda_4 = 4\overline{\omega}_1 + 4\overline{\omega}_2 - 2\overline{\omega}_3$ $\lambda_5 = 6\overline{\omega}_1 - 2\overline{\omega}_2 + 6\overline{\omega}_3$ $\lambda_6 = 4\overline{\omega}_2 + 2\overline{\omega}_3$ $\lambda_7 = -4\overline{\omega}_1 + 6\overline{\omega}_2 + 2\overline{\omega}_3$ $\lambda_8 = 8\overline{\omega}_1 - 6\overline{\omega}_2 + 10\overline{\omega}_3$ $\lambda_9 = 4\overline{\omega}_1 + 6\overline{\omega}_2 - 6\overline{\omega}_3$ $\lambda_{10} = -8\overline{\omega}_1 + 8\overline{\omega}_2 + 2\overline{\omega}_3$ $\lambda_{11} = 10\overline{\omega}_1 - 10\overline{\omega}_2 + 14\overline{\omega}_3$ $\lambda_{12} = 4\overline{\omega}_1 + 8\overline{\omega}_2 - 10\overline{\omega}_3$
CI	3	$\alpha_1^s = \beta_1^s = 4\overline{\omega}_1 - 2\overline{\omega}_2$ $\alpha_2^s = \beta_2^s = -2\overline{\omega}_1 + 4\overline{\omega}_2 - 2\overline{\omega}_3$ $\alpha_3^s = \beta_3^s = -4\overline{\omega}_2 + 4\overline{\omega}_3$	$\lambda_0 = 2\overline{\omega}_1 + 2\overline{\omega}_2 + 2\overline{\omega}_3$ $\lambda_1 = 6\overline{\omega}_1 + 2\overline{\omega}_3$ $\lambda_2 = 6\overline{\omega}_2$ $\lambda_3 = 2\overline{\omega}_1 - 2\overline{\omega}_2 + 6\overline{\omega}_3$ $\lambda_4 = 4\overline{\omega}_1 + 4\overline{\omega}_2$ $\lambda_5 = 6\overline{\omega}_1 - 4\overline{\omega}_2 + 6\overline{\omega}_3$ $\lambda_6 = 2\overline{\omega}_2 + 4\overline{\omega}_3$ $\lambda_7 = -4\overline{\omega}_1 + 4\overline{\omega}_2 + 4\overline{\omega}_3$ $\lambda_8 = 8\overline{\omega}_1 - 8\overline{\omega}_2 + 8\overline{\omega}_3$ $\lambda_9 = 4\overline{\omega}_1 + 8\overline{\omega}_2 - 4\overline{\omega}_3$ $\lambda_{10} = -8\overline{\omega}_1 + 6\overline{\omega}_2 + 4\overline{\omega}_3$ $\lambda_{11} = 10\overline{\omega}_1 - 12\overline{\omega}_2 + 10\overline{\omega}_3$ $\lambda_{12} = 4\overline{\omega}_1 + 12\overline{\omega}_2 - 8\overline{\omega}_3$

CII	$m \geq 7$	$\alpha_1^s = \beta_2^s = 2\overline{\omega}_2 - \overline{\omega}_4$ $\alpha_2^s = \beta_4^s = -\overline{\omega}_2 + 2\overline{\omega}_4 - \overline{\omega}_6$ $\alpha_3^s = \beta_6^s = -\overline{\omega}_4 + \overline{\omega}_6$	$\lambda_0 = 4\overline{\omega}_2 + 4\overline{\omega}_4 + (2m-9)\overline{\omega}_6$ $\lambda_1 = 6\overline{\omega}_2 + 3\overline{\omega}_4 + (2m-9)\overline{\omega}_6$ $\lambda_2 = 3\overline{\omega}_2 + 6\overline{\omega}_4 + (2m-10)\overline{\omega}_6$ $\lambda_3 = 4\overline{\omega}_2 + 3\overline{\omega}_4 + (2m-8)\overline{\omega}_6$ $\lambda_4 = 5\overline{\omega}_2 + 5\overline{\omega}_4 + (2m-10)\overline{\omega}_6$ $\lambda_5 = 6\overline{\omega}_2 + 2\overline{\omega}_4 + (2m-8)\overline{\omega}_6$ $\lambda_6 = 3\overline{\omega}_2 + 5\overline{\omega}_4 + (2m-9)\overline{\omega}_6$ $\lambda_7 = \overline{\omega}_2 + 6\overline{\omega}_4 + (2m-9)\overline{\omega}_6$ $\lambda_8 = 7\overline{\omega}_2 + (2m-7)\overline{\omega}_6$ $\lambda_9 = 5\overline{\omega}_2 + 6\overline{\omega}_4 + (2m-11)\overline{\omega}_6$ $\lambda_{10} = -\overline{\omega}_2 + 7\overline{\omega}_4 + (2m-9)\overline{\omega}_6$ $\lambda_{11} = 8\overline{\omega}_2 - 2\overline{\omega}_4 + (2m-6)\overline{\omega}_6$ $\lambda_{12} = 5\overline{\omega}_2 + 7\overline{\omega}_4 + (2m-12)\overline{\omega}_6$
CII	6	$\alpha_1^s = \beta_2^s = 2\overline{\omega}_2 - \overline{\omega}_4$ $\alpha_2^s = \beta_4^s = -\overline{\omega}_2 + 2\overline{\omega}_4 - \overline{\omega}_6$ $\alpha_3^s = \beta_6^s = -2\overline{\omega}_4 + 2\overline{\omega}_6$	$\lambda_0 = 4\overline{\omega}_2 + 4\overline{\omega}_4 + 3\overline{\omega}_6$ $\lambda_1 = 6\overline{\omega}_2 + 3\overline{\omega}_4 + 3\overline{\omega}_6$ $\lambda_2 = 3\overline{\omega}_2 + 6\overline{\omega}_4 + 2\overline{\omega}_6$ $\lambda_3 = 4\overline{\omega}_2 + 2\overline{\omega}_4 + 5\overline{\omega}_6$ $\lambda_4 = 5\overline{\omega}_2 + 5\overline{\omega}_4 + 2\overline{\omega}_6$ $\lambda_5 = 6\overline{\omega}_2 + \overline{\omega}_4 + 5\overline{\omega}_6$ $\lambda_6 = 3\overline{\omega}_2 + 4\overline{\omega}_4 + 4\overline{\omega}_6$ $\lambda_7 = \overline{\omega}_2 + 5\overline{\omega}_4 + 4\overline{\omega}_6$ $\lambda_8 = 7\overline{\omega}_2 - \overline{\omega}_4 + 6\overline{\omega}_6$ $\lambda_9 = 5\overline{\omega}_2 + 7\overline{\omega}_4$ $\lambda_{10} = -\overline{\omega}_2 + 6\overline{\omega}_4 + 4\overline{\omega}_6$ $\lambda_{11} = 8\overline{\omega}_2 - 3\overline{\omega}_4 + 7\overline{\omega}_6$ $\lambda_{12} = 5\overline{\omega}_2 + 9\overline{\omega}_4 - 2\overline{\omega}_6$
DI	m	$\alpha_1^s = \beta_1^s = 4\overline{\omega}_1 - 2\overline{\omega}_2$ $\alpha_2^s = \beta_2^s = -2\overline{\omega}_1 + 4\overline{\omega}_2 - 2\overline{\omega}_3$ $\alpha_3^s = \beta_3^s = -2\overline{\omega}_2 + 2\overline{\omega}_3$	$\lambda_0 = 2\overline{\omega}_1 + 2\overline{\omega}_2 + (2m-6)\overline{\omega}_3$ $\lambda_1 = 6\overline{\omega}_1 + (2m-6)\overline{\omega}_3$ $\lambda_2 = 6\overline{\omega}_2 + (2m-8)\overline{\omega}_3$ $\lambda_3 = 2\overline{\omega}_1 + (2m-4)\overline{\omega}_3$ $\lambda_4 = 4\overline{\omega}_1 + 4\overline{\omega}_2 + (2m-8)\overline{\omega}_3$ $\lambda_5 = 6\overline{\omega}_1 - 2\overline{\omega}_2 + (2m-4)\overline{\omega}_3$ $\lambda_6 = 4\overline{\omega}_2 + (2m-6)\overline{\omega}_3$ $\lambda_7 = -4\overline{\omega}_1 + 6\overline{\omega}_2 + (2m-6)\overline{\omega}_3$ $\lambda_8 = 8\overline{\omega}_1 - 6\overline{\omega}_2 + (2m-2)\overline{\omega}_3$ $\lambda_9 = 4\overline{\omega}_1 + 6\overline{\omega}_2 + (2m-10)\overline{\omega}_3$ $\lambda_{10} = -8\overline{\omega}_1 + 8\overline{\omega}_2 + (2m-6)\overline{\omega}_3$ $\lambda_{11} = 10\overline{\omega}_1 - 10\overline{\omega}_2 + 2m\overline{\omega}_3$ $\lambda_{12} = 4\overline{\omega}_1 + 8\overline{\omega}_2 + (2m-12)\overline{\omega}_3$

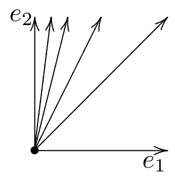
$DIII$	6	$\alpha_1^s=\beta_2^s=2\overline{\omega}_2-\overline{\omega}_4$ $\alpha_2^s=\beta_4^s=-\overline{\omega}_2+2\overline{\omega}_4-2\overline{\omega}_6$ $\alpha_3^s=\beta_6^s=-2\overline{\omega}_4+4\overline{\omega}_6$	$\lambda_0=4\overline{\omega}_2+4\overline{\omega}_4+2\overline{\omega}_6$ $\lambda_1=6\overline{\omega}_2+3\overline{\omega}_4+2\overline{\omega}_6$ $\lambda_2=3\overline{\omega}_2+6\overline{\omega}_4$ $\lambda_3=4\overline{\omega}_2+2\overline{\omega}_4+6\overline{\omega}_6$ $\lambda_4=5\overline{\omega}_2+5\overline{\omega}_4$ $\lambda_5=6\overline{\omega}_2+\overline{\omega}_4+6\overline{\omega}_6$ $\lambda_6=3\overline{\omega}_2+4\overline{\omega}_4+4\overline{\omega}_6$ $\lambda_7=\overline{\omega}_2+5\overline{\omega}_4+4\overline{\omega}_6$ $\lambda_8=7\overline{\omega}_2-\overline{\omega}_4+8\overline{\omega}_6$ $\lambda_9=5\overline{\omega}_2+7\overline{\omega}_4-4\overline{\omega}_6$ $\lambda_{10}=-\overline{\omega}_2+6\overline{\omega}_4+4\overline{\omega}_6$ $\lambda_{11}=8\overline{\omega}_2-3\overline{\omega}_4+10\overline{\omega}_6$ $\lambda_{12}=5\overline{\omega}_2+9\overline{\omega}_4-8\overline{\omega}_6$
$DIII$	7	$\alpha_1^s=\beta_2^s=2\overline{\omega}_2-\overline{\omega}_4$ $\alpha_2^s=\beta_4^s=-\overline{\omega}_2+2\overline{\omega}_4-\overline{\omega}_6$ $\alpha_3^s=\beta_6^s=-\overline{\omega}_4+\overline{\omega}_6$	$\lambda_0=4\overline{\omega}_2+4\overline{\omega}_4+3\overline{\omega}_6$ $\lambda_1=6\overline{\omega}_2+3\overline{\omega}_4+3\overline{\omega}_6$ $\lambda_2=3\overline{\omega}_2+6\overline{\omega}_4+2\overline{\omega}_6$ $\lambda_3=4\overline{\omega}_2+3\overline{\omega}_4+4\overline{\omega}_6$ $\lambda_4=5\overline{\omega}_2+5\overline{\omega}_4+2\overline{\omega}_6$ $\lambda_5=6\overline{\omega}_2+2\overline{\omega}_4+4\overline{\omega}_6$ $\lambda_6=3\overline{\omega}_2+5\overline{\omega}_4+3\overline{\omega}_6$ $\lambda_7=\overline{\omega}_2+6\overline{\omega}_4+3\overline{\omega}_6$ $\lambda_8=7\overline{\omega}_2+5\overline{\omega}_6$ $\lambda_9=5\overline{\omega}_2+6\overline{\omega}_4+\overline{\omega}_6$ $\lambda_{10}=-\overline{\omega}_2+7\overline{\omega}_4+3\overline{\omega}_6$ $\lambda_{11}=8\overline{\omega}_2-2\overline{\omega}_4+6\overline{\omega}_6$ $\lambda_{12}=5\overline{\omega}_2+7\overline{\omega}_4$
$EVII$	7	$\alpha_1^s=\beta_1^s=2\overline{\omega}_1-\overline{\omega}_6$ $\alpha_2^s=\beta_6^s=-\overline{\omega}_1+2\overline{\omega}_6-2\overline{\omega}_7$ $\alpha_3^s=\beta_7^s=-2\overline{\omega}_6+4\overline{\omega}_7$	$\lambda_0=8\overline{\omega}_1+8\overline{\omega}_6+2\overline{\omega}_7$ $\lambda_1=10\overline{\omega}_1+7\overline{\omega}_6+2\overline{\omega}_7$ $\lambda_2=7\overline{\omega}_1+10\overline{\omega}_6$ $\lambda_3=8\overline{\omega}_1+6\overline{\omega}_6+6\overline{\omega}_7$ $\lambda_4=9\overline{\omega}_1+9\overline{\omega}_6$ $\lambda_5=10\overline{\omega}_1+5\overline{\omega}_6+6\overline{\omega}_7$ $\lambda_6=7\overline{\omega}_1+8\overline{\omega}_6+4\overline{\omega}_7$ $\lambda_7=5\overline{\omega}_1+9\overline{\omega}_6+4\overline{\omega}_7$ $\lambda_8=11\overline{\omega}_1+3\overline{\omega}_6+8\overline{\omega}_7$ $\lambda_9=9\overline{\omega}_1+11\overline{\omega}_6-4\overline{\omega}_7$ $\lambda_{10}=3\overline{\omega}_1+10\overline{\omega}_6+4\overline{\omega}_7$ $\lambda_{11}=12\overline{\omega}_1+\overline{\omega}_6+10\overline{\omega}_7$ $\lambda_{12}=9\overline{\omega}_1+13\overline{\omega}_6-8\overline{\omega}_7$

A_3	6	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ $\lambda_4,\ \lambda_5,\ \lambda_6$	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ $\lambda_4,\ \lambda_6$
B_3	6	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ $\lambda_4,\ \lambda_5,\ \lambda_6$	$\lambda_0,\ \lambda_1,\ \lambda_3$ λ_6
C_3	6	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ $\lambda_4,\ \lambda_6$	$\lambda_0,\ \lambda_1,\ \lambda_2$ $\lambda_4,\ \lambda_6$
AI	3	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ $\lambda_4,\ \lambda_6$	λ_0
AII	7	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ $\lambda_4,\ \lambda_5,\ \lambda_6$ $\lambda_7,\ \lambda_8,\ \lambda_9$	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ $\lambda_4,\ \lambda_5,\ \lambda_6$ $\lambda_7,\ \lambda_9$
$AIII$	$m \geq 6$	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ $\lambda_4,\ \lambda_5,\ \lambda_6$ λ_9 λ_{12} if $m \geq 7$	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ $\lambda_4,\ \lambda_6$ λ_9 if $m \geq 7$ λ_{12} if $m \geq 8$
$AIII$	5	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ $\lambda_4,\ \lambda_6$	$\lambda_0,\ \lambda_1$ λ_6
BI	$m \geq 4$	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ $\lambda_4,\ \lambda_6,\ \lambda_9,\ \lambda_{12}$	λ_0 $\lambda_4,\ \lambda_9,\ \lambda_{12}$
BI	3	$\lambda_0,\ \lambda_1,\ \lambda_3,\ \lambda_6$	λ_0
CI	3	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_4$ λ_6	λ_0
CII	$m \geq 7$	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ $\lambda_4,\ \lambda_5,\ \lambda_6$ $\lambda_7,\ \lambda_8,\ \lambda_9$ λ_{12}	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ $\lambda_4,\ \lambda_5,\lambda_6$ $\lambda_7,\ \lambda_9$ λ_{12}
CII	6	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ $\lambda_4,\ \lambda_5,\ \lambda_6$ $\lambda_7,\ \lambda_9$	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ $\lambda_4,\ \lambda_5,\lambda_6$ λ_7
DI	m	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ $\lambda_4,\ \lambda_6$ λ_9 if $m \geq 5$ λ_{12} if $m \geq 6$	λ_0 λ_4 if $m \geq 5$ λ_9 if $m \geq 6$ λ_{12} if $m \geq 7$
$DIII$	6	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ $\lambda_4,\ \lambda_5,\ \lambda_6$ λ_7	$\lambda_0,\ \lambda_1,\ \lambda_3$ $\lambda_5,\ \lambda_6$ λ_7
$DIII$	7	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ $\lambda_4,\ \lambda_5,\ \lambda_6$ $\lambda_7,\ \lambda_8,\ \lambda_9$ λ_{12}	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ $\lambda_4,\ \lambda_5,\ \lambda_6$ $\lambda_7,\ \lambda_9$
$EVII$	7	$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3$ $\lambda_4,\ \lambda_5,\ \lambda_6$ $\lambda_7,\ \lambda_8$ $\lambda_{10},\ \lambda_{11}$	$\lambda_0,\ \lambda_1,\ \lambda_3$ $\lambda_5,\ \lambda_6$ $\lambda_7,\ \lambda_8$ $\lambda_{10},\ \lambda_{11}$

21 Complete Fano symmetric varieties II

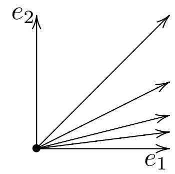
Now we can conclude the classification. *In this section we always suppose that* $\{\alpha_1, ..., \alpha_l\}$ *is ordered so that* $\{\alpha_1^s, ..., \alpha_l^s\}$ *is a basis of* ϕ^s *with the notations of [Hu].* We have to introduce some definitions. Let Y be a complete symmetric variety of rank 2. We will say that the type of Y is $V(n)$ if the fan of the associated toric variety has 1-dimensional cones generated respectively by $e_1, e_1 + e_2, e_1 + 2e_2, ..., e_1 + ie_2, ..., e_1 + ne_2, e_2$.

$V(n)$:



Instead we will say that the type of Y is $W(n)$ if the fan of the associated toric variety has 1-dimensional cones generated respectively by $e_1, ne_1 + e_2, (n-1)e_1 + 2e_2, ..., ie_1 + e_2, ..., e_1 + e_2, e_2$.

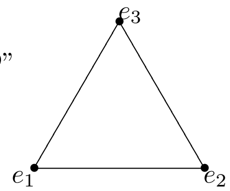
$W(n)$:



Observe that Y is of type V_1 if and only if it is of type $W(1)$. We will say that the type of Y is O if Y is wonderful. Finally, if Y has not type $V(n)$, $W(n)$ or O, we will say that the type of Y is P.

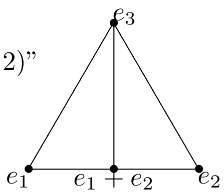
Let Y be a complete symmetric variety of rank 3. We will say that the type of Y is O if Y is the wonderful variety X.

"O"



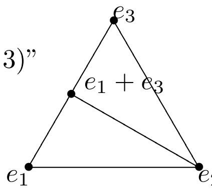
We will say that the type of Y is $Q(1,2)$ if Y is the blow-up of X along $X_{\sigma(e_1,e_2)}$.

"$Q(1,2)$"



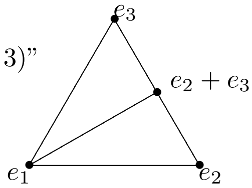
We will say that the type of Y is $Q(1,3)$ if Y is the blow-up of X along $X_{\sigma(e_1,e_3)}$.

"$Q(1,3)$"



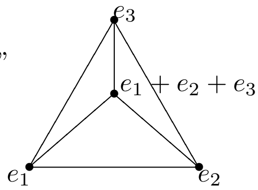
We will say that the type of Y is $Q(2,3)$ if Y is the blow-up of X along $X_{\sigma(e_2,e_3)}$.

"$Q(2,3)$"



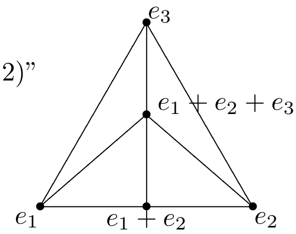
We will say that the type of Y is R if Y is the blow-up of X along $X_{\sigma(e_1,e_2,e_3)}$.

"R"



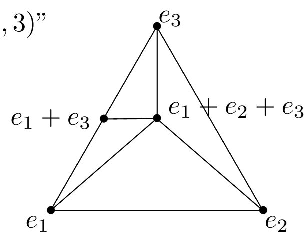
We will say that the type of Y is $S(1,2)$ if Y is the blow-up of the variety Y' of type R along $Y'_{\sigma(e_1,e_2)}$.

"$S(1,2)$"

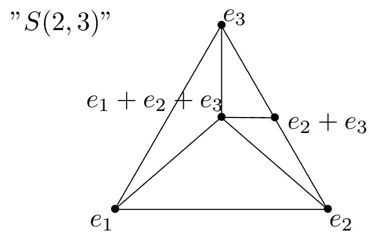


We will say that the type of Y is $S(1,3)$ if Y is the blow-up of the variety Y' of type R along $Y'_{\sigma(e_1,e_3)}$.

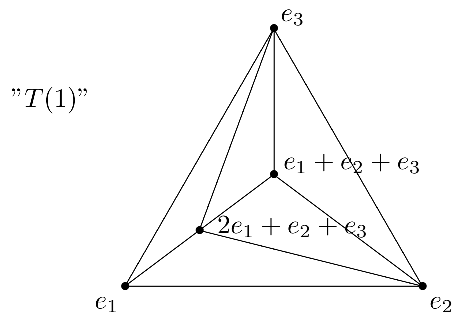
"$S(1,3)$"



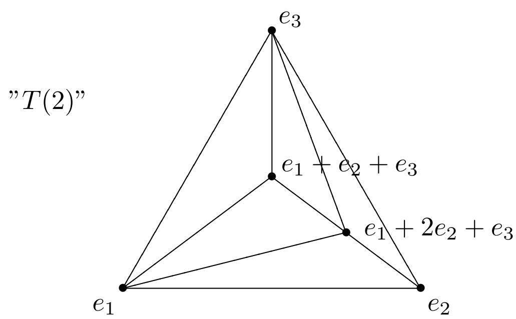
We will say that the type of Y is $S(2,3)$ if Y is the blow-up of the variety Y' of type R along $Y'_{\sigma(e_2,e_3)}$.



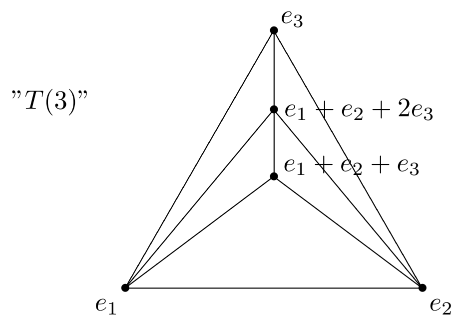
We will say that the type of Y is $T(1)$ if Y is the blow-up of the variety Y' of type R along $Y'_{\sigma(e_1,e_1+e_2+e_3)}$.



We will say that the type of Y is $T(2)$ is the blow-up of the variety Y' of type R along $Y'_{\sigma(e_2,e_1+e_2+e_3)}$.



We will say that the type of Y is $T(3)$ if Y is the blow-up of the variety Y' of type R along $Y'_{\sigma(e_3,e_1+e_2+e_3)}$.



Finally, if Y is none of previous ones, we will say that the type of Y is P. Now we can write the classification.

Theorem 21.1 *Let Y be a complete symmetric variety associated to an open toric variety Z. Suppose that Y is a completion of the symmetric variety G/H associated to an involution (G, θ) of rank 2.*

1. *If Y is a Fano variety then either it is the wonderful symmetric variety X or it is the blow-up of X along the closed orbit, so there at most two Fano completions of G/H.*

2. *If (G, θ) is not simple then Y is a Fano variety if and only if it is the wonderful symmetric variety X or it is the blow-up of X along the closed orbit.*

3. *If Y is a wonderful simple variety then it is Fano if and only if (G, θ) has not type G.*

4. *If Y is a simple Fano variety and it is not wonderful, we have exactly the following possibilities for (G, θ) (let m be the rank of G):*

   - *G/H is the adjoint group of type A_2.*
   - *(G, θ) has type AII and $m = 5$;*
   - *(G, θ) has type $AIII$ and $m \geq 4$;*
   - *(G, θ) has type CII;*
   - *(G, θ) has type $DIII$ and $m = 5$;*
   - *(G, θ) has type $EIII$;*
   - *(G, θ) has type EIV;*

5. *If Y is almost-Fano then it has type O, $V(n)$ or $W(n)$ for a suitable n.*

6. *There is a finite number of almost Fano complete symmetric varieties, but this number can be arbitrarily large.*

7. *Given an (almost) Fano toric variety Z proper over $\boldsymbol{A}^2$, there is an involution (G, θ) such that the associated complete symmetric variety $Y = X_Z$ is an (almost) Fano variety.*

8. *The classification of the simple almost-Fano varieties Y is as in the figure 1.*

9. *Suppose that $G/H = G_1/H_1 \times G_2/H_2$. Then the completion of $G_1/H_1 \times G_2/H_2$ of type $V(n)$ is an almost Fano variety if and only if the completion of $G_2/H_2 \times G_1/H_1$ of type $W(n)$ is an almost Fano variety. Moreover, if Y has type $V(n)$, then the property of being almost Fano depends only on G_1 (and not on G_2). Likewise if Y has type $W(n)$ then the property of being almost Fano depends only on G_2 (and not on G_1).*

(G,θ)	rank G	type of Y
A_2	4	O, $V(1) = W(1)$
B_2	4	O, $V(1) = W(1)$
G_2	4	O
AI	2	O, $V(1) = W(1)$
AII	5	O,$V(1) = W(1)$, $V(2)$, $W(2)$
$AIII$	$m \geq 4$	O,$V(1) = W(1)$, $W(n)$ if $n \leq m-2$
$AIII$	3	O, $V(1) = W(1)$
BI	$m \geq 3$	O, $V(1) = W(1)$ $W(n)$ if $n \leq m$
BI	2	O
CII	$m \geq 5$	O, $V(1) = W(1)$, $V(2)$ $W(n)$ if $n \leq 2m-5$
CII	4	O, $V(1) = W(1)$, $V(2)$, $W(2)$
DI	m	O, $V(1) = W(1)$ $W(n)$ if $n \leq m-2$
$DIII$	4	O, $V(1) = W(1)$, $V(2)$
$DIII$	5	O, $V(1) = W(1)$, $V(2)$, $W(2)$, $W(3)$
$EIII$	6	O, $V(1) = W(1)$, $V(2)$, $V(3)$, $W(n)$ if $n \leq 5$
EIV	6	O,$V(1) = W(1)$, $V(n)$ if $n \leq 4$ $W(n)$ if $n \leq 4$
G	2	O

Figure 1: Simple almost Fano varieties

10. *Suppose that $(G,\theta) = (G_1,\theta_1) \times (G_2,\theta_2)$. If Y is wonderful then it is almost-Fano, while the classification of the almost-Fano Y of type $V(n)$ is as in the figure 2.*

For the case of rank 3 we need a definition to simplify the notations.

Definition 21.1 *Let (G,θ) be an involution of rank 1. We will say that (G,θ) is of type 2 if ψ_2 is regular, while we will say that (G,θ) is of type 3 if ψ_3 is regular. If (G,θ) is not of type 2 then we will say that (G,θ) is of type 0.*

We have the following classification:

Lemma 21.1 *Let (G,θ) be an involution of rank 1.*

1. *If (G,θ) is of type 3 then it is of type 2.*
2. *The involutions of type 0 are the following:*
   - *(G,θ) such that G/H is the adjoint group of type A_1.*
   - *(G,θ) of type AI with $m = 1$.*

(G_1, θ_1)	rank G_1	type of Y
A_1	2	$V(1)$,$V(2)$
AI	1	$V(1)$
AII	3	$V(n)$ if $n \leq 3$
AIV	m	$V(n)$ if $n \leq m+1$
BII	m	$V(n)$ if $n \leq m$
CII	m	$V(n)$ if $n \leq 2m$
DII	m	$V(n)$ if $n \leq m$
FII	(4,1)	$V(n)$ if $n \leq 12$

Figure 2: Not-simple almost Fano varieties

3. *The involutions of type 2 are the following:*
   - (G, θ) *of type* AII *with* $m = 3$;
   - (G, θ) *of type* AIV;
   - (G, θ) *of type* BII;
   - (G, θ) *of type* CII;
   - (G, θ) *of type* DII;
   - (G, θ) *of type* FII.

4. *The involutions of type 3 are the following:*
   - (G, θ) *of type* AIV *with* $m \geq 3$;
   - (G, θ) *of type* BII *with* $m \geq 3$;
   - (G, θ) *of type* CII;
   - (G, θ) *of type* DII;
   - (G, θ) *of type* FII.

Theorem 21.2 *Let* Y *be a complete symmetric variety obtained from the wonderful variety* X *by a sequence of blow-ups along stable subvarieties. Suppose that* Y *is a completion of the symmetric variety* G/H *associated to an involution* (G, θ) *of rank 3.*

1. *If* Y *is a Fano variety then it has type* O, $Q(1,2)$, $Q(1,3)$, $Q(2,3)$, R, $S(1,2)$, $S(1,3)$, $S(2,3)$, $T(1)$, $T(2)$ *or* $T(3)$. *Thus there are at most eleven Fano completions of* G/H.

2. *There is an involution* (G, θ) *such that* Y *is a Fano variety if and only if it is of type* O, $Q(1,2)$, $Q(1,3)$, $Q(2,3)$, R, $S(1,2)$, $S(1,3)$, $S(2,3)$, $T(1)$, $T(2)$ *or* $T(3)$.

3. *If* Y *is simple and Fano then it is of type* O, $Q(1,2)$, $Q(1,3)$, $Q(2,3)$ *or* $S(1,3)$.

4. *The classification of the simple Fano varieties Y is as in the figure 3.*

5. *Suppose that $(G,\theta) = (G_1,\theta_1)\times(G_2,\theta_2)$, where (G_1,θ_1) is a simple involution of rank 2 and (G_2,θ_2) is a involution of rank 1, then the classification of the Fano varieties Y is as in the figure 4 (let r be the type of (G_2,θ_2)).*

6. *Suppose that $(G,\theta) = (G_1,\theta_1) \times (G_2,\theta_2) \times (G_3,\theta_3)$ where (G_i,θ_i) is a involution of rank 1 for each i. Let r be the number of (G_i,θ_i) of type 2 and let s be the number of (G_i,θ_i) of type 3, so $3 \geq r \geq s \geq 0$. Then the classification of the Fano varieties Y is as follows:*

   - *If $r \leq 1$, then Y is a Fano variety if and only if it is of type O, $Q(1,2)$, $Q(1,3)$ or $Q(2,3)$. In particular there are four Fano varieties.*
   - *If $r = 2$, then there are five Fano varieties. Let i and j be the indices such that G_i and G_j are of type 2. We can suppose that $i < j$. Y is a Fano variety if and only if it is of type O, $Q(1,2)$, $Q(1,3)$, $Q(2,3)$ or $S(i,j)$.*
   - *If (r,s) is equal to $(3,0)$ or to $(3,1)$, then there are eight Fano varieties. Y is a Fano variety if and only if it is of type O, $Q(1,2)$, $Q(1,3)$, $Q(2,3)$, R, $S(1,2)$, $S(1,3)$ or $S(2,3)$.*
   - *If $(r,s) = (3,2)$, then there are nine Fano varieties. Suppose that G_i is not of type 3, then Y is a Fano variety if and only if it is of type O, $Q(1,2)$, $Q(1,3)$, $Q(2,3)$, R, $S(1,2)$, $S(1,3)$, $S(2,3)$ or $T(i)$.*
   - *If $(r,s) = (3,3)$, then there are eleven Fano varieties. Y is a Fano variety if and only if it is of type O, $Q(1,2)$, $Q(1,3)$, $Q(2,3)$, R, $S(1,2)$, $S(1,3)$, $S(2,3)$, $T(1)$, $T(2)$ or $T(3)$.*

Theorem 21.3 *Let Y be a complete symmetric variety obtained from the wonderful variety X by a sequence of blow-ups along closed orbits. Suppose that Y is a completion of the symmetric variety G/H associated to the involution (G,θ) of rank l ($l > 1$).*

1. *If Y is a Fano variety then either it is the wonderful variety X or it is the blow-up of X along the closed orbit.*

2. *If there is a simple factor of (G,θ) of rank at least 3 and Y is a Fano variety, then Y is wonderful.*

3. *Suppose that $l \geq 6$. If there is a simple factor of (G,θ) of rank at least 2 and Y is a Fano variety, then Y is wonderful.*

4. *Suppose that Y is a Fano variety and it is not wonderful then we have the following possibilities for a simple factor (G',θ') of (G,θ) of rank 2 (let m be the rank of G'):*

   - *G/H is the adjoint group of type A_2 and $l = 2$;*

(G,θ)	rank G	type of Y
A_3	6	O, $Q(1,3)$
B_3	6	O
C_3	6	O, $Q(1,3)$
AI	3	O
AII	7	O, $Q(1,2)$, $Q(1,3)$, $Q(2,3)$, $S(1,3)$
$AIII$	$m \geq 6$	O, $Q(1,3)$
$AIII$	5	O
BI	$m \geq 3$	O
CI	3	$\nexists$
CII	$m \geq 7$	O, $Q(1,2)$, $Q(1,3)$, $Q(2,3)$, $S(1,3)$
CII	6	O, $Q(1,2)$, $Q(1,3)$, $Q(2,3)$
DI	m	O
$DIII$	6	O, $Q(1,2)$
$DIII$	7	O, $Q(1,2)$, $Q(1,3)$, $Q(2,3)$, $S(1,3)$
$EVII$	7	O, $Q(1,2)$

Figure 3: Simple Fano varieties of rank 3

- *(G',θ') is of type AII, $m=5$ and $l \leq 3$;*
- *(G',θ') is of type $AIII$, $m \geq 4$ and $l=2$;*
- *(G',θ') is of type CII, $m \geq 5$ and $l \leq 3$;*
- *(G',θ') is of type CII, $m=4$ and $l=2$;*
- *(G',θ') is of type $DIII$, $m=5$ and $l \leq 3$;*
- *(G',θ') is of type $EIII$, $m=6$ and $l \leq 4$;*
- *(G',θ') is of type EIV, $m=6$ and $l \leq 5$;*

5. *Suppose that Y is a Fano variety and it is not wonderful then we have the following possibilities for a simple factor (G',θ') of (G,θ) of rank 1 (let m be the rank of G'):*

   - *G/H is the adjoint group of type A_1 and $l=2$;*
   - *(G',θ') is of type AI and $l=2$;*
   - *(G',θ') is of type AII, $m=3$ and $l \leq 3$;*
   - *(G',θ') is of type AIV and $l \leq m+1$;*
   - *(G',θ') is of type BII and $l \leq m+1$;*
   - *(G',θ') is of type CII and $l \leq 2m$;*
   - *(G',θ') is of type DII and $l \leq m$;*
   - *(G',θ') is of type FII and $l \leq 12$;*

6. *Let (G_i,θ_i) be involutions as in the previous two points, then the blow-up of the wonderful completion of $\prod G_i/H_i$ along the closed orbit is a Fano variety.*

(G_1, θ_1)	rank G_1	type r of (G_2, θ_2)	type of Y
A_2	4	$\forall r$	O, $Q(1,2)$, $Q(1,3)$, $Q(2,3)$
B_2	4	$\forall r$	O, $Q(1,3)$
G_2	4	$\forall r$	O, $Q(2,3)$
AI	2	$\forall r$	O
AII	5	$\forall r$ if $r = 2$	O, $Q(1,2)$, $Q(1,3)$, $Q(2,3)$, $S(1,2)$ R, $S(1,3)$, $S(2,3)$
$AIII$	$m \geq 4$	$\forall r$ if $r = 2$ and $m \geq 5$	O, $Q(1,2)$, $Q(1,3)$, $Q(2,3)$ $S(2,3)$
$AIII$	3	$\forall r$	O
BI	$m \geq 2$	$\forall r$	O
CII	$m \geq 5$	$\forall r$ if $r = 2$ if $r = 3$	O, $Q(1,2)$, $Q(1,3)$, $Q(2,3)$, $S(1,2)$ R, $S(1,3)$, $S(2,3)$ $T(1)$
CII	4	$\forall r$	O, $Q(1,2)$, $Q(1,3)$, $Q(2,3)$
DI	m	$\forall r$	O, $Q(2,3)$
$DIII$	4	$\forall r$	O, $Q(1,3)$
$DIII$	5	$\forall r$ if $r = 2$	O, $Q(1,2)$, $Q(1,3)$, $Q(2,3)$, $S(1,2)$ R, $S(1,3)$, $S(2,3)$
$EIII$	6	$\forall r$ if $r = 2$ if $r = 3$	O, $Q(1,2)$, $Q(1,3)$, $Q(2,3)$, $S(1,2)$ R, $S(1,3)$, $S(2,3)$, $T(3)$ $T(1)$, $T(2)$
EIV	6	$\forall r$ if $r = 2$ if $r = 3$	O, $Q(1,2)$, $Q(1,3)$, $Q(2,3)$, $S(1,2)$ R, $S(1,3)$, $S(2,3)$, $T(3)$ $T(1)$, $T(2)$
G	2	$\forall r$	$\nexists$

Figure 4: Not simple Fano varieties of rank 3

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