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Gelfand Pairs: from self-similar Groups to Markov chains

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Gelfand Pairs: from self-similar Groups to Markov chains

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Introduction

This work is a collection of the main interests that I developed during my doctoral studies, that began on November 2003, and it presents the more interesting results that I got, mostly in joint works with Daniele D'Angeli, in my research activity.

As the title shows, the main subject of this thesis is constituted by the notion of Gelfand pair. In particular, I study here the finite Gelfand pairs arising from the action of automorphisms groups on the rooted homogeneous tree, but also on more general structures, namely the poset block structures.

Given a finite group G and a subgroup $K \leq G$ of G, then (G, K) is a *Gelfand pair* if the permutation representation of G on the space of complex functions $L(X)$ defined on the homogeneous space

$$X = G/K = \{gK : g \in G\}$$

is multiplicity-free, i.e. it decomposes into irreducible subrepresentations which are pairwise non isomorphic.

The finite Gelfand pairs theory is very fashinating because it is related to group theory, representation theory, harmonic analysis, combinatorics and to probability and statistics. There exists also a big literature for infinite Gelfand pairs: for instance the fundamental works by Faraut ([**30**]) and Helgason ([**41**]) and, more recently, by Grigorchuk ([**35**]) in connection with the theory of *branch groups*.

Moreover Persi Diaconis (see [**23**] and [**24**]) used Gelfand pairs in order to determine the rate of convergence to the stationary distribution of finite Markov chains. More precisely, given a Markov chain which is invariant under the action of a group G, its transition operator can be expressed as a convolution operator whose kernel can be written as a "Fourier series" where the classical exponentials $\exp(inx)$ are replaced by the irreducible representations of the group G.

I must mention also the names of Letac ([**42**], [**43**]), Delsarte ([**22**]), Dunkl ([**27**], [**28**], [**29**]) and Figà-Talamanca ([**32**]) for their contributions to the theory of finite Gelfand pairs.

Finally, in [**16**] T.Ceccherini-Silberstein, F.Scarabotti and F.Tolli largely develop the finite Gelfand pairs theory and investigate its connections with representation theory, but also with probability and statistics: this book really was a fundamental source in my studies.

In the first chapter of this thesis I study many examples of groups acting on the rooted homogeneous tree. Given a positive integer $q \geq 2$, I will denote by T_q the rooted homogeneous tree of degree q, i.e. the rooted tree in which each vertex has q children.

If $X = \{0, 1, \ldots, q-1\}$ is an alphabet of q elements and X^* is the set of all finite words in X, then each vertex in the $n-$th level L_n of T_q can be identified with a word of length n in the alphabet X. Moreover, the set of infinite words in X can be identified with the elements of the boundary ∂T_q of T_q.

For every $n \geq 1$, the set L_n is an ultrametric space, in particular a metric space, on which the full automorphisms group $Aut(T_q)$ acts isometrically.

A fundamental class of groups acting on T_q is the class of *self-similar* groups. A group G acting on T_q is self-similar if, for any $g \in G$ and $x \in X$, there exist $h \in G$ and $y \in X$ such that

$$g(xw) = yh(w),$$

for all $w \in X^*$.

Self-similarity was related in most cases with geometrical objects and only recently the notion of self-similar group appeared. The success of the development of the theory of self-similar group is due to the fact that many interesting examples of groups can be studied using their self-similar action on a rooted tree (see, for instance, [**35**] and [**37**]). Some examples of self-similar groups belong to the class of branch groups as, for example, the Grigorchuk group (see [**44**]).

An important class of examples are the *iterated monodromy groups* of postcritically finite rational functions, whose theory was largely developed by V. Nekrashevich ([**44**]). A fundamental example is given by the Basilica group, which is the iterated monodromy group associated with the complex polynomial $z^2 - 1$ and which has very interesting properties: it has exponential growth and it is the first example of an amenable group which cannot be constructed from groups of subexponential growth by using extensions and direct limits. Its amenability was proved by L.Bartholdi and B.Virág ([**10**]) using self-similarity of the random walk on it.

The groups that I study are the Adding Machine on the binary tree, the Basilica group, the group $IMG(z^2 + i)$ and the Baumslag-Solitar group $BSq =< s, t : t^{-1}st = s^q >$.

Let G be any of these groups. Fix $n \geq 1$ and consider the action of G on the level L_n of the tree, by setting

$$G_n = G/Stab_G(n),$$

where $Stab_G(n)$ is the subgroup of G constituted by the automorphisms acting trivially on L_n. Fix a vertex $x_0 \in L_n$ and let K_n be the subgroup of G_n stabilizing x_0, so that the quotient G_n/K_n can be identified with L_n.

For each example that I consider, I show that (G_n, K_n) is a Gelfand pair. The strategy used to prove that uses, in some cases, the fact that the action of G_n on L_n is $2-$points homogeneous or, equivalently, that the subgroup K_n acts transitively on each sphere of radius r centered at x_0, for $r = 0, 1, \ldots, n$.

Also the investigation of the structure of the rigid vertex stabilizers can be a useful criterion to get Gelfand pairs, as the example of the Basilica group shows.

In the second part of the first chapter I extend this study to the case of the generalized wreath products of permutations groups, introduced in [**5**], acting on the so-called *poset block structures*. These structures contain, as a particular case, the rooted tree. For these groups, one still gets Gelfand pair. More precisely, by using Gelfand's condition, one can prove that they give rise to symmetric Gelfand pairs.

In the second chapter of the thesis I leave the group theory and I change my point of view. More precisely, I study some reversible Markov chains which are defined on the cartesian product

$$X = X_1 \times \cdots \times X_n$$

of n finite sets, whose elements can be regarded as the leaves of a rooted tree of depth n with branch indices $(m_1, \ldots, m_n)$, where $|X_i| = m_i$.

In particular, I introduce the *crested product* of Markov chains (see [**18**]), which contains the crossed and nested product as particular cases and whose definition is inspired by the combinatoric theory of Association schemes ([**3**], [**4**]), to whom a section of this chapter is devoted.

The spectral analysis of the associated Markov operator is performed. The interesting fact is that the eigenspaces that one gets for the crossed and the nested product coincide, under some hypothesis, with the irreducible submodules of the action of the direct product and of the permutational wreath product of symmetric groups, respectively.

A particular example of nested product gives rise to the "Insect Markov chain" on the rooted homogeneous tree. This is a Markov chain defined on the $n-$th level of the tree, introduced by A. Figà-Talamanca in [**32**].

I study the Insect Markov chain also in relation with the *cut-off* phenomenon. This term was introduced in [**1**] by D. Aldous and P. Diaconis. The cut-off phenomenon occurs when the difference between the value of the probability measure $m^{(k)}$ given by the $k-$steps transition probability and the stationary distribution π is close to 0 only after a fixed number k_0 of steps, and it is large (close to 1) before k_0 steps.

In particular, I prove that the cut-off does not occur in the Insect Markov chain, using the spectral theory of the associated Markov operator and the Fourier analysis to get an expression for the $k-$steps transition probability $m^{(k)}(x) = p^{(k)}(x_0, x)$. This is possible since the

Markov chain considered is invariant with respect to the action of the full automorphisms group of the tree and then one can apply the Fourier analysis to the corresponding Gelfand pair.

Finally, the Insect Markov chain is generalized to the *block orthogonal structures*, which contain, as a particular case, the poset block structures. If one restricts the attention to the poset block structures, the spectral analysis shows that the eigenspaces associated with the corresponding Markov operator coincide with the irreducible submodules of the regular representation of the generalized wreath product of symmetric groups on the space of complex functions defined on the poset block structure.

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CHAPTER 1

Finite Gelfand Pairs

In this chapter the finite Gelfand Pairs theory is developed: we present the definition and the main properties. We consider then several examples of Gelfand pairs obtained considering the action of self-similar groups on homogeneous rooted trees. It is interesting to observe that some of these groups can also be regarded as iterated monodromy groups of complex polynomials. Finally, we study the Gelfand pairs obtained from the action of the generalized permutation wreath product on poset block structures.

1. Finite Gelfand Pairs

In this section the definition of finite Gelfand pairs and associated spherical functions is given. We present some basic results in Gelfand pairs theory. Our main source is [**16**].

1.1. Definition and main properties. Let G be a finite group and let $K \leq G$ a subgroup of G. Denote

$$X = G/K = \{gK : g \in G\}$$

the associated homogeneous space of right cosets of K in G. Set $L(G) = \{f : G \to \mathbb{C}\}$. Then the space $L(X)$ of all complex valued functions defined on X can be regarded as the subspace of $K-$invariant (on the right) functions of $L(G)$. The isomorphism is given by the map $f \mapsto \widetilde{f}$, where $f \in L(X)$ and $\widetilde{f}$ is the right$-K-$invariant function of $L(G)$ defined as $\widetilde{f}(g) = f(gx_0)$, where $x_0 \in X$ is the point stabilized by K.

A function $f \in L(G)$ is said *bi$-K-$invariant* if $f(kgk') = f(g)$, for all $g \in G$ and $k, k' \in K$. The space of bi$-K-$invariant functions can be identified with the space $L(K\backslash G/K) = \{f : K\backslash G/K \to \mathbb{C}\}$ of all complex valued functions defined on the set of double cosets KgK, for $g \in G$. It can also be regarded as the subspace $L(X)^K = \{f \in L(X) : f(kx) = f(x),\ \forall x \in X, k \in K\}$ of $K-$invariant functions on X.

The space $L(G)$ is an algebra with respect to the convolution product defined as

$$(f_1 * f_2)(g) = \sum_{h \in G} f_1(gh) f_2(h^{-1}).$$

It is easy to verify that $L(G)$ is commutative if and only if the group G is abelian. Moreover, both its subspaces $L(X)$ and $L(K\backslash G/K)$ are subalgebras of $L(G)$.

$L(G)$ can be endowed with a Hilbert space structure by setting, for $f_1, f_2 \in L(G)$,

$$\langle f_1, f_2 \rangle = \sum_{g \in G} f_1(g)\overline{f_2(g)};$$

analogously, the space $L(X)$ can be endowed with a Hilbert space structure by setting, for $f_1, f_2 \in L(X)$,

$$\langle f_1, f_2 \rangle = \sum_{x \in X} f_1(x)\overline{f_2(x)}.$$

Note that if $f_1, f_2 \in L(X)$ and $\widetilde{f_i}$ are the associated right$-K-$invariant functions in $L(G)$, then

$$\langle \widetilde{f_1}, \widetilde{f_2} \rangle_{L(G)} = |K| \langle f_1, f_2 \rangle_{L(X)}.$$

The left regular representation of G on $L(G)$ is given by the homomorphism $\lambda : G \to U(L(G))$ into the unitary group of $L(G)$ defined as

$$(\lambda(g)f)(h) = f(g^{-1}h), \quad \text{for } h, g \in G, \ f \in L(G).$$

The left regular representation of G on $L(X)$ is given by the homomorphism $\lambda : G \to U(L(X))$ into the unitary group of $L(X)$ defined as

$$(\lambda(g)f)(x) = f(g^{-1}x), \quad \text{for } g \in G, \ x \in X \text{ and } f \in L(X).$$

To indicate the left regular representation $\lambda(g)f$ of an element $g \in G$ on a function $f \in L(X)$ we will often use the notation f^g or $g(f)$.

Definition 1.1. *Let G be a finite group and $K \leq G$ a subgroup of G. The pair (G, K) is a **Gelfand pair** if the algebra $L(K \backslash G/K)$ is commutative.*

More generally, if G is a group acting transitively on a finite set X, then this action defines a Gelfand pair if (G, K) is a Gelfand pair, where K is the subgroup stabilizing a point $x_0 \in X$. Moreover, if $gx_0 = x$, then $K' = gKg^{-1}$ stabilizes x and (G, K) is a Gelfand pair if and only if (G, K') is a Gelfand pair.

A particular example of a Gelfand pair is given by the *symmetric Gelfand pairs*: this is the case if, for every $g \in G$, one has $g^{-1} \in KgK$. In fact, under this hypothesis, it is possible to show that the algebra $L(K \backslash G/K)$ is commutative. If $f \in L(K \backslash G/K)$, then we have $f(g) =$

$f(g^{-1})$. So, for $f_1, f_2 \in L(K\backslash G/K)$, we have:

$$\begin{aligned}(f_1 * f_2)(g) &= \sum_{h\in G} f_1(gh)f_2(h^{-1}) \\ &= \sum_{h\in G} f_1(gh)f_2(h) \\ &= \sum_{t\in G} f_1(t)f_2(g^{-1}t) \\ &= \sum_{t\in G} f_2(g^{-1}t)f_1(t^{-1}) \\ &= (f_2 * f_1)(g^{-1}) = (f_2 * f_1)(g),\end{aligned}$$

where we set $gh = t$.

If G acts on a finite set X, then the diagonal action of G on $X \times X$ is defined by

$$g(x, x') = (gx, gx'), \quad \text{for all } g \in G, x, x' \in X.$$

Theorem 1.2. *Consider the action of G on $X = G/K$. Let x_0 be the point stabilized by K and let $X = \Omega_0 \coprod \Omega_1 \dots \coprod \Omega_n$ be the decomposition of X into $K-$orbits, with $\Omega_0 = \{x_0\}$. For each $i = 0, 1, \dots, n$, choose $x_i \in \Omega_i$. Then the sets $G(x_i, x_0)$ are the orbits of the diagonal action of G on $X \times X$.*

Proof. Note that, for all $(x, y) \in X \times X$, there exist $g \in G$, $k \in K$ and $i \in \{0, 1, \dots, n\}$ such that

$$(x, y) = (x, gx_0) = (gg^{-1}x, gx_0) = (gkx_i, gkx_0) \in G(x_i, x_0),$$

where we used that G is transitive on X and we denoted $Kx_i = \Omega_i$ the $K-$orbit containing $g^{-1}x$. This shows that $X \times X = \bigcup_{i=0}^{n} G(x_i, x_0)$. Moreover, it is easy to verify that this is a disjoint union, what gives the assertion. $\square$

The following lemma is straightforward (see [**16**]). If $x, y \in X$ and G acts on X, we will use the notation $x \sim y$ to say that x and y are in the same $G-$orbit.

Lemma 1.3 (Gelfand's Condition). *Let G be a group acting transitively on a finite set X and set $K = \{k \in G : kx_0 = x_0\}$, with $x_0 \in X$. Then the following are equivalent:*

(1) for all $x, y \in X$, one has $(x, y) \sim (y, x)$ with respect to the diagonal action of G on $X \times X$;
(2) $g^{-1} \in KgK$ for all $g \in G$.

Now suppose that (X, d) is a finite metric space and that G isometrically acts on X. We say that this action is $2-$*points homogeneous* if, for all $(x, y), (x', y') \in X \times X$ such that $d(x, y) = d(x', y')$, there exists $g \in G$ such that $gx = x'$ e $gy = y'$. If K is the stabilizer of an element $x_0 \in X$, then Lemma 1.3 easily implies that, under these conditions, (G, K) is a symmetric Gelfand pair.

We can observe that the $K-$orbits under this action are the spheres centered at x_0 with radius j, for $j = 0, 1, \ldots$. Hence, a function $f \in L(X)$ is $K-$invariant if and only if it is constant on these spheres.

We want to give now a characterization of a Gelfand pair (G, K) in terms of the representation of the group G on $L(X)$, with $X = G/K$.

Definition 1.4. *A representation (ρ, V) of a group G is multiplicity-free is all its irreducible subrepresentations are pairwise non-equivalent.*

Given two representations (ρ_1, V_1) and (ρ_2, V_2) of G, we denote

$$\begin{aligned} Hom_G(V_1, V_2) \;\; = \;\; & \{T : V_1 \to V_2 \; : \; \rho_2(g)(Tv) = T(\rho_1(g)v) \;\text{ for all} \\ & g \in G, v \in V_1\} \end{aligned}$$

the space of operators intertwining the representations (ρ_1, V_1) and (ρ_2, V_2). We will say that T is $G-$equivariant. It is known that if $V_1 = V_2 = V$, then $Hom_G(V, V)$ is an algebra. The proof of the following proposition can be found in [**16**].

Proposition 1.5. *The following isomorphism holds:*

$$Hom_G(L(X), L(X)) \cong L(K\backslash G/K)$$

The following lemma can be proven by using character theory.

Lemma 1.6 (Wielandt's Lemma). *Let G be a finite group and $K \leq G$ a subgroup of G. Set $X = G/K$. Let $L(X) = \bigoplus_{i=0}^{N} m_i V_i$ a decomposition of $L(X)$ into irreducible $G-$subrepresentations, where m_i denotes the multiplicity of V_i. Then*

$$\sum_{i=0}^{N} m_i^2 = \text{ number of } G-\text{orbits on } X \times X = \text{ number of } K-\text{orbits on } X.$$

Theorem 1.7. *Let G be a finite group and $K \leq G$. Set $X = G/K$. Then the following are equivalent:*

1. *(G, K) is a Gelfand pair, i.e. $L(K\backslash G/K)$ is commutative;*
2. *$Hom_G(L(X), L(X))$ is commutative;*
3. *the decomposition of $L(X)$ into irreducible $G-$subrepresentations is multiplicity-free.*

Proof. The equivalence between (1) and (2) is given by Proposition 1.5.

(3)⇒(2) Suppose that the decomposition $L(X) = \bigoplus_{i=0}^{N} V_i$ into irreducible subrepresentations is multiplicity-free.

Let $T \in Hom_G(L(X), L(X))$ and denote T_i the restriction of T to V_i. If T_i is not trivial, then T_i is injective since V_i is irreducible and so

$\{Tv : v \in V_i\}$ is a subspace isomorphic to V_i. Hence it coincides with V_i and by Schur's Lemma (see, for instance, [**33**]) there exists $\lambda_i \in \mathbb{C}$ such that $Tv = \lambda_i v$, for all $v \in V_i$.

Since every $f \in L(X)$ decomposes uniquely in the form $f = \sum_{i=0}^N v_i$, with $v_i \in V_i$, we have that for every $T \in Hom_G(L(X), L(X))$ there exist $\lambda_0, \lambda_1, \ldots, \lambda_N \in \mathbb{C}$ such that

$$Tf = \sum_{i=0}^N \lambda_i v_i.$$

If $S \in Hom_G(L(X), L(X))$ is such that $Sf = \sum_{i=0}^N \mu_i v_i$, then

$$STf = \sum_{i=0}^N \mu_i \lambda_i v_i = TSf,$$

for all $f \in L(X)$ and so $Hom_G(L(X), L(X))$ is commutative.

(2)⇒(3) Suppose that $L(X)$ is not multiplicity-free, so that there exist two orthogonal irreducible isomorphic subrepresentations V and W in L(X). Let $\varphi : V \to W$ be such an isomorphism. Define U the orthogonal complement such that $L(X) = V \oplus W \oplus U$. We define two linear operators $S, T : L(X) \to L(X)$ by setting

$$T(v + w + u) = \varphi v \quad \text{and} \quad S(v + w + u) = \varphi^{-1} w,$$

for all $v \in V, w \in W$ and $u \in U$. It is easy to check that S and T are $G-$equivariant, but $ST \neq TS$ since, for instance, $(ST)|_W = 0$ and $(TS)|_W = I_W$. This implies that $Hom_G(L(X), L(X))$ is not commutative. $\square$

Consider the space $\mathbb{C}^{N+1}$ with the coordinatewise product

$$(\alpha_0, \alpha_1, \ldots, \alpha_N) \cdot (\beta_0, \beta_1, \ldots, \beta_N) = (\alpha_0\beta_0, \alpha_1\beta_1, \ldots, \alpha_N\beta_N),$$

for any $(\alpha_0, \alpha_1, \ldots, \alpha_N), (\beta_0, \beta_1, \ldots, \beta_N) \in \mathbb{C}^N$. This is an algebra of dimension $N + 1$. From the proof of Theorem 1.7 we get the following corollary.

Corollary 1.8. *Let (G, K) be a Gelfand pair and $L(X) = \bigoplus_{i=0}^N V_i$ the decomposition of $L(X)$ into irreducible inequivalent subrepresentations. Then*

(1) if $T \in Hom_G(L(X), L(X))$, then any V_i is an eigenspace of T;
(2) if $T \in Hom_G(L(X), L(X))$ and λ_i is the eigenvalue of the restriction of T to V_i, then the map

$$T \mapsto (\lambda_0, \lambda_1, \ldots, \lambda_N)$$

is an isomorphism between $Hom_G(L(X), L(X))$ and $\mathbb{C}^{N+1}$;
(3) $N + 1 = \dim(Hom_G(L(X), L(X))) = \dim(L(K\backslash G/K)) =$ number of $K-$orbits of X.

The following proposition gives a useful criterion for Gelfand pairs.

Proposition 1.9. *Let G be a finite group, K a subgroup of G and set $X = G/K$. If we have a decomposition $L(X) = \bigoplus_{t=0}^{h} Z_t$ into pairwise inequivalent $G-$subrepresentations with $h+1 =$ number of $K-$orbits of X. Then the Z_t's are irreducible and (G,K) is a Gelfand pair.*

Proof. We can refine, if necessary, the decomposition with the Z_t's into irreducibles as in the statement of Lemma 1.6. So we have

$$h+1 \leq \sum_{i=0}^{N} m_i \leq \sum_{i=0}^{N} m_i^2$$

and Lemma 1.6 forces $h = N$ and $m_i = 1$ for each $i = 0, 1, \ldots, N$. This gives the assertion. $\square$

1.2. Spherical functions. From now on suppose that (G,K) is a Gelfand pair.

Definition 1.10. *A bi$-K-$invariant function ϕ is called* ***spherical*** *if it has the following properties:*

(1) for all $f \in L(K\backslash G/K)$, there exists $\lambda_f \in \mathbb{C}$ such that $\phi * f = \lambda_f \phi$;
(2) $\phi(1_G) = 1$.

The constant function $\phi(g) \equiv 1$ is clearly spherical. The condition (1) tells us that ϕ is an eigenfunction for every convolution operator with a bi$-K-$invariant kernel, equivalently, by Proposition 1.5, for every $T \in Hom_G(L(X), L(X))$. The condition (2) tells us that the corresponding eigenvalue is the number $\lambda_f = (\phi * f)(1_G) \equiv T(\phi)(1_G)$.

Lemma 1.11. *Let ϕ a spherical function and let Φ be the linear functional on $L(G)$ defined by*

$$\Phi(f) = \sum_{g \in G} f(g)\phi(g^{-1}). \tag{1}$$

Then Φ is multiplicative on $L(K\backslash G/K)$, that is, for any $f_1, f_2 \in L(K\backslash G/K)$,

$$\Phi(f_1 * f_2) = \Phi(f_1)\Phi(f_2).$$

Viceversa every nontrivial multiplicative linear functional on $L(K\backslash G/K)$ is determined by a spherical function as in (1).

Corollary 1.12. *Let (G,K) a Gelfand pair. Then the number of distinct spherical functions equals the number of distinct irreducible subrepresentations in $L(X)$.*

Proof. Let $N+1$ the number of irreducible subrepresentations in $L(X)$. Then, by Corollary 1.8, $L(K\backslash G/K)$ and $\mathbb{C}^{N+1}$ are isomorphic as algebras. A linear multiplicative functional on $\mathbb{C}^{N+1}$ is always of the form

$$\Psi(\alpha_0, \alpha_1, \ldots, \alpha_N) = \alpha_j,$$

for some j. Therefore $L(K\backslash G/K)$ and $\mathbb{C}^{N+1}$ have exactly $N+1$ multiplicative linear functionals. By Lemma 1.11, the number of spherical functions is $N+1$. $\square$

From the definition, the following properties of the spherical functions easily follow.

PROPOSITION 1.13. *Let ϕ and ψ be two distinct spherical functions. Then*

(1) $\phi(g^{-1}) = \overline{\phi(g)}$ for all $g \in G$;
(2) $\phi * \psi = 0$;
(3) $\langle \lambda(g_1)\phi, \lambda(g_2)\psi \rangle = 0$ for all $g_1, g_2 \in G$;
(4) ϕ and ψ are orthogonal, i.e. $\langle \phi, \psi \rangle = 0$.

Denote $V_n =< \lambda(g)\phi_n : g \in G >$ the subspace of $L(X)$ spanned by the $G-$translates of ϕ_n, for $n = 0, 1, \ldots, N$.

THEOREM 1.14. *$L(X) = \bigoplus_{n=0}^{N} V_n$ is the decomposition of $L(X)$ into irreducible subrepresentations.*

Proof. By definition, each V_n is $G-$invariant and, by Proposition 1.13, V_n is orthogonal to V_m if $n \neq m$. The $V_i's$ are distinct and they exhaust $L(X)$. This gives the assertion. $\square$

The representation V_n is called **spherical representation** associated with the spherical function ϕ_n. In particular, V_0 is the trivial representation.

Let (ρ, V) a representation of G. If $K \leq G$, denote

$$V^K = \{v \in V : \rho(k)v = v, \text{ for all } k \in K\}$$

the space of $K-$invariant vectors in V.

THEOREM 1.15. *(G, K) is a Gelfand pair if and only if $\dim(V^K) \leq 1$ for each irreducible $G-$representation V. Moreover, V is spherical if and only if $\dim(V^K) = 1$.*

Proof. Let (G, K) be a Gelfand pair and let (ρ, V) a representation of G, with $\dim(V^K) \geq 1$. Fix a nontrivial vector $u \in V^K$. Let $T : V \to L(X)$ be the operator defined by $Tv(g) = \langle v, \rho(g)u \rangle_V$. Then $T \in Hom_G(V, L(X))$. Indeed Tv is a right $K-$invariant function and

$$\begin{aligned}(T\rho(h)v)(g) &= \langle \rho(h)v, \rho(g)u \rangle_V \\ &= \langle v, \rho(h^{-1}g)u \rangle_V \\ &= (\lambda(h)(Tv))(g),\end{aligned}$$

for all $v \in V$ and $h, g \in G$. By Schur's Lemma, $V \cong V_n$ for some spherical representation V_n.

But Corollary 1.8 tells us that $N+1$ equals the dimension of the space $L(X)^K$, which must be equal to $\oplus_{m=0}^{N} V_m^K$. Since, for every m, $\dim(V_m^K) \geq 1$ because every spherical function $\phi_m \in V_m^K$, we have $\dim(V_m^K) = 1$ for all m so that $\dim(V^K) = \dim(V_n^K) = 1$.

Conversely, suppose $\dim(V^K) \leq 1$ for all irreducible subrepresentations V. If $L(X) = \bigoplus_{h=0}^{H} m_h W_h$ is the decomposition into irreducible subrepresentations and $N+1$ denotes the number of $K-$orbits of X, then Lemma 1.6 gives

$$\sum_{h=0}^{H} m_h^2 = N+1 = \sum_{h=0}^{H} m_h \dim W_h^K \leq \sum_{h=0}^{H} m_h,$$

where the inequality follows from the hypothesis. This forces $m_h = 1$ for all h, so that $L(X)$ is multiplicity-free. $\square$

Since $L(X)^K = \bigoplus_{i=0}^{N} V_i^K$, we deduce that the spherical functions constitute a basis for the space of bi$-K-$invariant functions in $L(G)$.

The following theorem (see [**16**] for the proof) will be useful in what follows.

Theorem 1.16 (Garsia's Theorem). *A Gelfand pair is symmetric if and only if the associated spherical functions are real valued.*

2. Groups of automorphisms of homogeneous rooted trees

In this section we will study the Gelfand pairs associated with the action of groups of automorphisms of the homogeneous rooted tree. In particular, we focus our attention on the action of an automorphisms group G on the $n-$th level L_n of the tree. To do this, we consider the quotient group G_n of G modulo the stabilizer of L_n and we study the pair (G_n, K_n), where K_n denotes the subgroup of G_n stabilizing a leaf of L_n.

Let q be a positive integer, with $q \geq 2$. The case of the full automorphisms group $Aut(T_q)$ of the $q-$ary rooted tree T_q is studied, for instance, in [**16**]. The authors give there the decomposition of the space $L(L_n)$ into irreducible subrepresentations, together with the associated spherical functions.

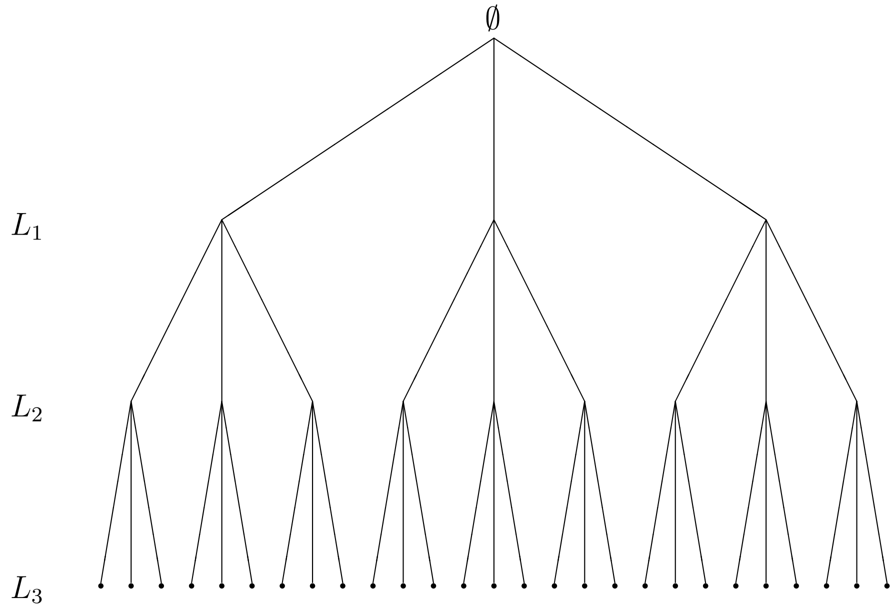


Fig.1. The ternary rooted tree of depth 3.

If $X = \{0, 1, \ldots, q-1\}$ is an alphabet of q elements and X^* is the set of all finite words in X, then each vertex in the $n-$th level L_n of T_q can be identified with a word of length n in the alphabet X. Moreover, we can identify the set of infinite words in X with the elements of the boundary ∂T_q of T_q.

The set L_n can be endowed with an ultrametric distance d, defined in the following way: if $x = x_1 \ldots x_n$ and $y = y_1 \ldots y_n$, then

$$d(x, y) = n - \max\{i : x_k = y_k, \ \forall k \leq i\}.$$

We observe that $d = d'/2$, where d' denotes the usual geodesic distance.

In this way (L_n, d) becomes an ultrametric space, in particular a metric space, on which the automorphisms group $Aut(T_q)$ isometrically acts. Note that the diameter of (L_n, d) is exactly n.

Fix $n \in \mathbb{N}$ and restrict our attention to the action of $Aut(T_q)$ on the level L_n. To indicate the action of an automorphism $g \in Aut(T_q)$ on a vertex x, we will use the notation $g(x)$ or x^g. Moreover, denote S_q the symmetric group on q elements.

Set

$$Aut(T_q)_n = Aut(T_q)/Stab_{Aut(T_q)}(n),$$

where $Stab_{Aut(T_q)}(n)$ denotes the subgroup of $Aut(T_q)$ stabilizing L_n. It is known that the following isomorphism holds:

$$Aut(T_q)_n \cong \underbrace{S_q \wr S_q \wr \cdots \wr S_q}_{n \text{ times}}.$$

If one considers the action of $Aut(T_q)_n$ on L_n one gets, for every n, a $2-$points homogeneous action, giving rise to the symmetric Gelfand pair $(Aut(T_q)_n, K_n)$, with $K_n = Stab_{Aut(T_q)_n}(0^n)$, where 0^n is the leftmost leaf of L_n. In fact, the following theorem holds.

THEOREM 2.1. *The action of $Aut(T_q)_n$ on (L_n, d) is $2-$points homogeneous.*

Proof. We use induction on the depth n of the tree T_q.

$n = 1$. The assertion follows from the $2-$transitivity of the group S_q.

$n > 1$. Let (x, y) and (x', y') be pairs of vertices in L_n with $d(x, y) = d(x', y')$. If $d(x, y) < n$, then vertices x and y belong to the same subtree of T and so $x_1 = y_1$. Analogously for x' and y'. Applying, if necessary, the transposition $(x_1 x_1') \in S_q$, we can suppose $x_1 = y_1 = x_1' = y_1'$, so that x, x', y and y' belong to the same subtree of depth less or equal to $n - 1$, and then induction works.

Finally, consider the case $d(x, y) = d(x', y') = n$. Consider the automorphism $g \in Aut(T_q)$ such that $g(x_1) = x_1'$ and $g(y_1) = y_1'$ and which acts trivially on the other vertices of L_1. Now we have that x and x' belong to the same subtree T'. Analogously y and y' belong to the same subtree T'', with $T' \neq T''$. The restriction of $Aut(T_q)_n$ to T' and T'' respectively acts transitively on each level. So there is an automorphism g' of T' carrying x to x' and acting trivially on T'' and analogously there is an automorphism g'' of T'' carrying y to y' and trivial on T'. The assertion is proved. $\square$

COROLLARY 2.2. *For all $n \geq 1$, $(Aut(T_q)_n, K_n)$ is a symmetric Gelfand pair.*

The decomposition of the space $L(L_n)$ under the action of $Aut(T_q)_n$ is known.

Denote $W_0 \cong \mathbb{C}$ the trivial representation and for every $j = 1, \ldots, n$, define the following subspace

$$W_j = \{f \in L(L_n) : f = f(x_1, \ldots, x_j), \quad \sum_{x=0}^{q-1} f(x_1, x_2, \ldots, x_{j-1}, x) \equiv 0\},$$

of dimension $q^{j-1}(q-1)$. One can verify that the W_j's are $Aut(T_q)_n-$ invariant, pairwise orthogonal and that the following decomposition holds

$$L(L_n) = \bigoplus_{j=0}^{n} W_j. \tag{2}$$

Since the spheres centered at $x_0 := 0^n$ (and so the K_n-orbits) are exactly $n+1$, we have from Proposition 1.9 that the subspaces W_j's are irreducible.

There exists a complete description of the corresponding spherical functions. For every $j = 0, \ldots, n$ we get

$$\phi_j(x) = \begin{cases} 1, & d(x, x_0) < n - j + 1; \\ \frac{1}{1-q}, & d(x, x_0) = n - j + 1; \\ 0, & d(x, x_0) > n - j + 1. \end{cases} \tag{3}$$

If we consider a countable subgroup of $Aut(T_q)$ and the relative action on L_n, we can ask if it is possible to find the same results about Gelfand pairs obtained for the full automorphisms group. In some cases the answer is positive.

In the next sections, we will consider the action of special finitely generated subgroups of $Aut(T_q)$, which belong to the class of self-similar groups and, in some cases, of iterated monodromy groups.

Remark. In [**14**] the authors consider a more general construction, namely they study the case of the action of the automorphisms group of the tree on the variety of special substructures of the tree.

Given an $n-$tuple $\mathbf{m}= (m_1, \ldots, m_n)$ of integers ≥ 2, a finite rooted tree T is of type $\mathbf{m}$ if each vertex at distance k from the root has exactly m_{k+1} sons, for every $k = 0, 1, \ldots, n-1$ (we also say that $(m_1, \ldots, m_n)$ are the branch indices of T).

If $\mathbf{r}= (r_1, \ldots, r_n)$ is another $n-$tuple of integers such that $1 \leq r_i \leq m_i$ for every $i = 1, \ldots, n$, one can consider the variety of subtrees of T whose branch indices are $(r_1, \ldots, r_n)$. It is easy to check that the substructures of type $\mathbf{r}$ in a rooted tree of type $\mathbf{m}$ are exactly

$$\binom{m_1}{r_1} \cdot \prod_{i=2}^{n} \binom{m_i}{r_i}^{r_1 r_2 \cdots r_{i-1}}.$$

Note that, if $\mathbf{r}$ =$(1, \ldots, 1)$, then a subtree of type $\mathbf{r}$ can be identified with a leaf of the $n-$th level of the tree of type $\mathbf{m}$.

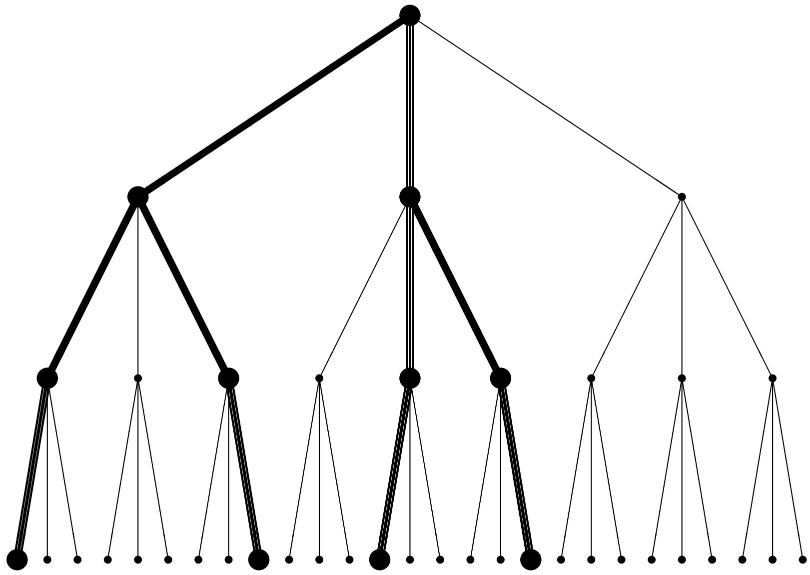


Fig.2 A tree of type (3,3,3) with a subtree of type (2,2,1).

The authors prove that the group $Aut(T) \cong S_{m_n} \wr \cdots \wr S_{m_1}$ transitively acts on this variety and then, using Gelfand's Condition, they show that the pair $(Aut(T), K(\mathbf{m}, \mathbf{r}))$ is a Gelfand pair, where $K(\mathbf{m}, \mathbf{r})$ denotes the stabilizer of a fixed substructure T'.

2.1. Self-similar Groups. Denote T_q the rooted $q-$ary tree. Every automorphism $g \in Aut(T_q)$ can be represented by its **labelling**. The labelling of $g \in Aut(T_q)$ is realized as follows: given a vertex $x = x_0x_1 \ldots x_{n-1} \in L_n$, we associate with x a permutation $g_x \in S_q$ giving the action of g on the q children of x. Formally, the action of g on the vertex labelled with the word $x = x_0x_1 \ldots x_{n-1}$ is

$$x^g = x_0^{g_\emptyset} x_1^{g_{x_0}} \ldots x_{n-1}^{g_{x_0 \ldots x_{n-2}}}.$$

Definition 2.3. *A group G acting on T_q is* ***self-similar*** *if, for any $g \in G$ and $x \in X$, there exist $h \in G$ and $y \in X$ such that*

$$g(xw) = yh(w), \tag{4}$$

for all $w \in X^*$.

The rule (4) tells us that a self-similar group G can be embedded into the permutational wreath product

$$G \wr S_q = G^X \rtimes S_q.$$

In particular if, for every $i = 0, 1, \ldots, q-1$, one has $g(x_iw) = y_ig_i(w)$ for all $w \in X^*$, then g can be written as

$$g = (g_0, g_1, \ldots, g_{q-1})\sigma, \tag{5}$$

where $\sigma \in S_q$ is the permutation such that $\sigma(x_i) = y_i$.

So the elements g_i are the restrictions of g to the subtree T_i rooted at the vertex $x_i \in L_1$, which is clearly isomorphic to the entire tree T_q. The iteration of this procedure leads to the notion of restriction g_v of g to each vertex v of T_q.

For an automorphisms group $G \leq Aut(T_q)$, the **vertex stabilizer** of $x \in T_q$ is the subgroup of G defined as

$$Stab_G(x) = \{g \in G : g(x) = x\};$$

the **level stabilizer** of L_n is given by

$$Stab_G(n) = \bigcap_{x \in L_n} Stab_G(x).$$

Observe that $Stab_G(n)$ is a normal subgroup of G of finite index for all $n \geq 1$. In particular, an automorphism $g \in Stab_G(1)$ can be identified with its restrictions $g_i, i = 0, 1, \ldots, q-1$ to the respective subtrees T_i. So we get the following embedding

$$\varphi : Stab_G(1) \longrightarrow \underbrace{Aut(T_q) \times Aut(T_q) \times \cdots \times Aut(T_q)}_{q \text{ times}} \tag{6}$$

that associates with g the $q-$ple $(g_0, g_1, \ldots, g_{q-1})$.

Definition 2.4. *G is **spherically transitive** if its action on L_n is transitive, for all $n \in \mathbb{N}$.*

Definition 2.5. *G is **fractal** if, for every vertex $x \in T_q$, one has $Stab_G(x)|_{T_x} \cong G$, where the isomorphism is given by identification of T_q with its subtree T_x rooted at x.*

Lemma 2.6. *G is fractal if and only if the embedding φ defined in (6) is a subdirect embedding into $G \times \cdots \times G$, i.e. if it is surjective on each factor.*

Proof. One implication is obvious. So we can suppose that φ is a subdirect embedding. We want to prove, by induction on $|x|$, that $Stab_G(x)|_{T_x} \cong G$ for all $x \in T_q$. The induction basis $|x| = 1$ is equivalent to the hypothesis. Now, by induction, $G \to G^{q^{n-1}}$ is a subdirect embedding and each factor G maps to G^q by φ. Since the composition of two subdirect embeddings is still subdirect, we get the assertion. $\square$

Observe that, if G is fractal, then it is spherically transitive if and only if its action on the first level of the tree is transitive.

In the next sections we will use the notion of rigid stabilizer to get Gelfand pairs. If G acts on T_q and $x \in T_q$, the **rigid vertex stabilizer** $Rist_G(x)$ is the subgroup of $Stab_G(x)$ consisting of those automorphisms of T_q that fix all vertices not having x as a prefix. Equivalently, the automorphisms in $Rist_G(x)$ have a trivial labelling at each vertex outside T_x. The **rigid level stabilizer** of L_n is defined as

$$Rist_G(n) = \prod_{x \in L_n} Rist_G(x).$$

The rigid level stabilizer $Rist_G(n)$ is normal in $Aut(T_q)$. In contrast to the level stabilizers, the rigid level stabilizers may have infinite index and may even be trivial. We observe that if the action of G on T_q is spherically transitive, then the subgroups $Stab_G(x)$, $x \in L_n$ are all conjugate, as well as the subgroups $Rist_G(x)$.

The following definitions hold for spherically transitive groups (see, for more details, [**8**]).

Definition 2.7. *G is **regular weakly branch** on K if there exists a normal subgroup $K \neq \{1\}$ in G, with $K \leq Stab_G(1)$, such that $\varphi(K) > K \times K \times \cdots \times K$. In particular, G is **regular branch** on K if it is regular weakly branch on K and K has finite index in G.*

We observe that this property for the subgroup K is stronger than fractalness, since the map φ is surjective on the whole product $K \times K \times \cdots \times K$.

Definition 2.8. *G is **weakly branch** if $Rist_G(x) \neq \{1\}$, for every $x \in T_q$ (this automatically implies $|Rist_G(x)| = \infty$ for every x). In particular, G is **branch** if $[G : Rist_G(n)] < \infty$ for every $n \geq 1$.*

2.1.1. *Example.* Consider the group G acting on the binary tree, generated by the automorphism having the following self-similar form:

$$a = (1, a)\varepsilon,$$

where ε is the nontrivial permutation of S_2.

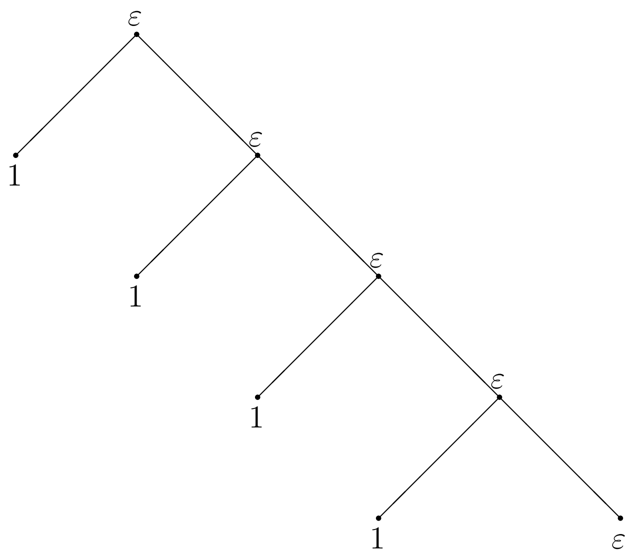


Fig.3. Labelling of a.

The group $G =< a >$ is isomorphic to $\mathbb{Z}$. It is called **Adding Machine** (Odometer) and it is defined also in the more general case (see [**37**] or [**39**]) of a $k-$ary tree as the group generated by the automorphism $a = (1, \dots, 1, a)\sigma$, where $\sigma = (0, 1, 2, \dots, q - 1)$ is the standard cycle that cyclically permutes the symbols in X. The automorphism a is called the odometer because of the way in which it acts on X^*. In particular, if we regard the word $w = x_1 \dots x_n \in X^n$ as the number $\sum_{i=1}^{n} x_i k^i$, then:

- $a(w) = w + 1$, for $w \neq (k - 1) \dots (k - 1)$;
- $a((k - 1) \dots (k - 1)) = 0 \dots 0$.

Consider now the binary case. It is easy to check that the following identities hold:

$$a^{2k} = (a^k, a^k), \qquad a^{2k+1} = (a^k, a^{k+1})\varepsilon. \tag{7}$$

In particular, the first level stabilizer is given by $Stab_G(1) =< a^2 >$, with $a^2 = (a, a)$. So G is a fractal group and its action on the binary tree is spherically transitive.

From (7) it follows that

$$Stab_G(n) =< a^{2^n} > .$$

Moreover, since G is abelian, one has $Stab_G(n) = Stab_G(x)$ for all $x \in L_n$. Formulas (7) tells us that the element a^{2^n} has the labelling $g_x = \varepsilon$ at each vertex $x \in L_n$ and the labelling $g_y = 1$ at each vertex $y \in L_i$, for $i < n$. Therefore $a^{2^n} \notin Rist_G(n)$ and all its powers do not belong to $Rist_G(n)$ too. So $Rist_G(n) = \{1\}$ for every $n \geq 1$. So this is an example where the subgroups $Stab_G(n)$ and $Rist_G(n)$ do not coincide, showing that $Rist_G(n)$ can also be trivial.

2.1.2. *Automaton Groups.*

DEFINITION 2.9. *An* ***automaton*** *is a quadruple* $\mathcal{A} = (S, X, \lambda, \pi)$, *where:*

(1) S *is a set, called set of states;*
(2) X *is an alphabet;*
(3) $\pi : S \times X \to S$ *is the transition map;*
(4) $\lambda : S \times X \to X$ *is the output map.*

The automaton $\mathcal{A}$ is said *finite* if S is finite and it is said *invertible* if, for all $s \in S$, the transformation $\lambda(s, \cdot) : X \to X$ is a permutation of X.

An automaton $\mathcal{A}$ can be represented by its **Moore diagram**: this is a directed labeled graph whose vertices are identified with the states of $\mathcal{A}$. For every state $s \in S$ and every letter $x \in X$, the diagram has an arrow from s to $\pi(s, x)$ labeled by $x|\lambda(s, x)$.

A natural action on the words over X is induced, so that the maps π and λ can be extended to $S \times X^*$ as:

$$\pi(s, xw) = \pi(\pi(s, x), w) \tag{8}$$

$$\lambda(s, xw) = \lambda(s, x)\lambda(\pi(s, x), w), \tag{9}$$

by setting $\pi(s, \emptyset) = s$ and $\lambda(s, \emptyset) = \emptyset$, for all $s \in S, x \in X$ and $w \in X^*$.

Moreover, the Equation (9) defines uniquely a map $\lambda : S \times X^\omega \to X^\omega$, where X^ω denotes the set of infinite words over X.

If we fix an initial state s in an automaton $\mathcal{A}$, then a transformation $\lambda(s, \cdot)$ on the set $X^* \cup X^\omega$ is defined: it is denoted by $\mathcal{A}_s$. The image of a word $x_1x_2\ldots$ can be easily found by using the Moore diagram. One has to consider the directed path starting at the state s with consecutive labels $x_1|y_1$, $x_2|y_2$ and so on, so that the image of the word $x_1x_2\ldots$ under the transformation $\mathcal{A}_s$ will be equal to $y_1y_2\ldots$.

Now if $X = \{0, 1, \ldots, q-1\}$ is an alphabet of q letters and G is a self-similar group on X^ω, then its action defines an automaton over the alphabet X whose states are the elements of G and such that the output and the transition maps λ and π are defined in such a way that

$$g(xw) = \lambda(g, x)w^{\pi(g,x)},$$

for all $w \in X^\omega$.

The automaton $\mathcal{A}$ that one gets has the property that the transformation $\mathcal{A}_g$ coincides with the action of g and it is called the *complete automaton* of the self-similar group G.

Since the complete automaton is infinite for infinite groups, it is more convenient to define the group generated by an automaton in the following way.

Given an invertible automaton $\mathcal{A} = (S, X, \lambda, \pi)$, the group generated by the transformations $\mathcal{A}_s$, for $s \in S$, is called the **automaton group** generated by $\mathcal{A}$ and is denoted by $G(\mathcal{A})$.

The following proposition holds.

PROPOSITION 2.10. *The action of a group on the set X^ω is self-similar if and only if it is generated by an automaton.*

The automaton groups $G(\mathcal{A})$, where $\mathcal{A}$ is a finite automaton, are are the most interesting. In Section 2.2.1 of this chapter a fundamental example will be presented.

2.2. Iterated Monodromy Groups. A particular class of self-similar groups is given by the so called itarated monodromy groups.

The Iterated Monodromy Groups theory has been mostly developed by V. Nekrashevych in [**44**]. See also [**38**] and [**9**].

In order to introduce the iterated monodromy groups, we need the following definition.

DEFINITION 2.11. *Let M be an arcwise connected and locally arcwise connected topological space. A $d-$**fold partial self-covering map** on the space M is a $d-$fold covering map $f : M_1 \longrightarrow M$, where M_1 is an open arcwise connected subset of M.*

It is known that a map $f : M_1 \longrightarrow M_2$ is a $d-$fold covering map if it is surjective and every point $x \in M_2$ has a neighborhood U such that the preimage $f^{-1}(U)$ is the disjoint union of d subsets $U_i \subseteq M_1$ such that $f : U_i \longrightarrow U$ is a homeomorphism.

So suppose we have a $d-$fold partial covering map $f : M_1 \longrightarrow M$ and let $\pi_1(M, t)$ be the fundamental group of M with base point t. It is clear that the set of iterated preimages of t naturally constitutes a $d-$ary rooted tree T, whose root is t and such that each point x has exactly d preimages $x_1, \ldots, x_d$ which are declared to be adjacent to x in T. In this way, the $n-$th level of the tree consists of d^n points belonging to $f^{-n}(t)$. Although the intersection of $f^{-n}(t)$ and $f^{-m}(t)$ can be non empty for $n \neq m$, the tree T has to be regarded as the disjoint union of the sets f^{-n}, for all $n \geq 0$.

There exists a natural action of the fundamental group $\pi_1(M, t)$ on T. Given a loop γ based at t, for each point $s \in f^{-n}(t)$ there exists a unique preimage $\gamma_{[s]}$ of γ starting at s and ending in some point $s' \in f^{-n}(t)$. The action of γ on T is defined as

$$\gamma(s) = s', \tag{10}$$

so it induces a permutation of $f^{-n}(t)$. The group of all permutations of $f^{-n}(t)$ induced by the action of $\pi_1(M,t)$ is called the $n-$**th monodromy group** of f. Moreover, γ acts on T as a tree automorphism. In fact, if $\gamma(s)=s'$, then $\gamma(f(s))=f(s')$, since $f(\gamma_{[s]})=\gamma_{[f(s)]}$.

DEFINITION 2.12. *The **Iterated Monodromy Group** of f is defined as the group*

$$IMG(f)=\pi_1(M,t)/N,$$

where N denotes the kernel of the action defined in (10).

One can show that, up to isomorphism, the group $IMG(f)$ does not depend from the base point t.

In order to better characterize the action of $\pi_1(M,t)$ on T, one can introduce an alphabet $X=\{0,1,\ldots,d-1\}$ and consider the set X^* of all finite words over X, which also has a $d-$ary rooted tree structure such that the word w is declared to be adjacent to wx, if $w\in X^*$ and $x\in X$.

In fact, it is possible to define an isomorphism $\Lambda: X^*\longrightarrow T$ in such a way that the action of $\pi_1(M,t)$ on X^* is **self-similar**.

The isomorphism Λ can be defined inductively. We set $\Lambda(\emptyset)=t$. For every word $w\in X^n$, we construct a path l_w in M from t to a point $s_w\in f^{-n}(t)$ and define $\Lambda(w)=s_w$. In this way, for each $x\in X$ the point t is connected by the path l_x to a point $s_x\in M$ belonging to $f^{-1}(t)$, with $s_x\neq s_{x'}$ for $x\neq x'$. We define $\Lambda(x)$ to be the endpoint of the path l_x.

Suppose we have already defined $\Lambda(w)$ for all $w\in X^m$, with $m\leq n$ and that it is an isomorphism between the first n levels of T and X^*. Let xw be a word of X^{n+1}, with $w\in X^n$ and $x\in X$. Define

$$l_{xw}=l_w f_{[w]}^{-n}(l_x),$$

where $f_{[w]}^{-n}(l_x)$ is the unique preimage of the path l_x under f^{-n} starting at w. Define $\Lambda(xw)$ to be the end of the path l_{xw}.

PROPOSITION 2.13. *The map $\Lambda: X^*\longrightarrow T$ defined above is an isomorphism.*

Proof. It suffices to prove that $f(\Lambda(xvx'))=\Lambda(xv)$, for all $x,x'\in X$ and $v\in X^*$. In fact,

$$f(l_{xvx'})=f(l_{vx'})f(f_{[vx']}^{-n}(l_x))=f(l_{vx'})f_{[v]}^{-(n-1)}(l_x).$$

By definition, $f_{[v]}^{-(n-1)}(l_x)$ is a path going from $\Lambda(v)$ to $\Lambda(xv)$, so f maps the end $\Lambda(xvx')$ of the path $l_{xvx'}$ to $\Lambda(xv)$. $\square$

DEFINITION 2.14. *The action of $IMG(f)$ on X^* induced by the isomorphism Λ is called the **standard action** of $IMG(f)$.*

THEOREM 2.15. *The standard action of $IMG(f)$ is self-similar. In particular, the restriction γ_x of $\gamma \in IMG(f)$ at $x \in X$ is given by*

$$\gamma_x = l_x \gamma_{[x]} (l_{\gamma(x)})^{-1}. \tag{11}$$

Proof. Let $v \in X^n$ and suppose $\gamma(xv) = x'u$, with $x' \in X$ and $u \in X^n$. Then the vertices v and u are connected by the path

$$\alpha = f_{[v]}^{-n}(l_x) \cdot \gamma_{[xv]} \cdot (f_{[u]}^{-n}(l_{x'}))^{-1},$$

which goes through the vertices $v \to xv \to x'u \to u$. So the loop $l = l_x \gamma_{[x]} l_{x'}^{-1}$ based at t is the element of $IMG(f)$ moving v to u and it is independent of v and u. This gives $\gamma_x = l_x \gamma_{[x]} (l_{\gamma(x)})^{-1}$. $\square$

A fundamental example is given by Iterated Monodromy Groups associated with rational functions $f \in \mathbb{C}(z)$. We need the following definition.

DEFINITION 2.16. *Let $\widehat{M}$ be a topological space. A map $f : \widehat{M} \longrightarrow \widehat{M}$ is a* ***branched covering*** *if there exists a set $R \subset \widehat{M}$ of branching points such that f is a local homeomorphism in each point $x \in \widehat{M} \setminus R$. The set $P = \bigcup_{n=0}^{\infty} f^n(R)$ is called the* ***postcritical set****. If the set $M = \widehat{M} \setminus \overline{P}$ is arcwise connected and locally arcwise connected, then $f : M_1 \longrightarrow M$ is a partial self-covering of the set M, with $M_1 = f^{-1}(M)$.*

In particular, let $f(z) = \frac{p(z)}{q(z)} \in \mathbb{C}(z)$ a non-constant **rational function**, with p, q co-prime. Then we have $deg(f) = \max\{deg(p), deg(q)\}$. The function f defines a branched $deg(f)-$fold self-covering of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. A point $z \in \widehat{\mathbb{C}}$ is critical if f is not a local homeomorphism on any neighborhood of z, i.e. if $f'(z) = 0$.

Let C_f be the set of the critical points of f. We denote by P_f the set of the post-critical points, i.e. $P_f = \bigcup_{n=0}^{\infty} f^n(C_f)$. If $\overline{P_f}$ is such that $M = \widehat{\mathbb{C}} \setminus \overline{P_f}$ is arcwise connected, then f defines a $deg(f)-$fold partial self-covering $f : M_1 \longrightarrow M$, with $M_1 = \widehat{\mathbb{C}} \setminus f^{-1}(\overline{P_f})$.

In particular, if P_f if finite, f is called **post-critically finite**. If this is the case, M and M_1 are punctured spheres and the fundamental group $\pi_1(M)$ is the free group of rank $|P_f| - 1$.

It is known (see [**44**]) that iterated monodromy groups of post-critically finite polynomials are amenable. Many problems about iterated monodromy groups are still open ([**7**]).

(1) When is the iterated monodromy group of a rational function torsion free?
(2) Can any be non amenable? Or contain a free subgroup of rank $k \geq 2$?

(3) Which rational functions have iterated monodromy groups of exponential growth?

2.2.1. *The Basilica Group.* The Basilica group, that we will denote B, was introduced by R. I. Grigorchuk and A. Żuk in [**40**] as the group of automorphisms of the binary tree generated by the three-state automaton having the following Moore diagram:

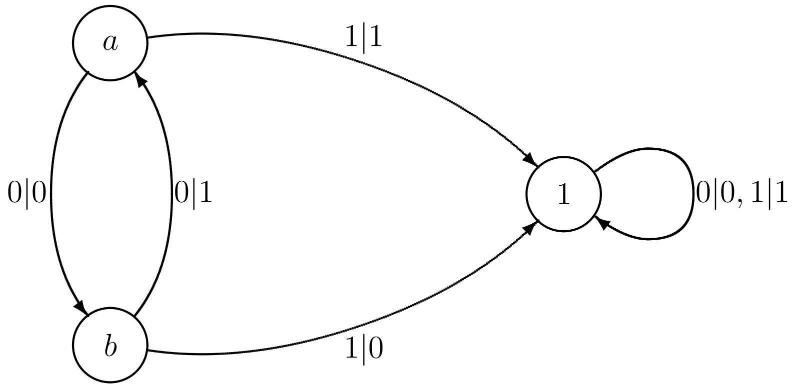


Fig.4. The automaton defining the Basilica group.

This is the first example of an amenable group (a highly non–trivial and deep result of Bartholdi and Virág [**10**]) not belonging to the class SG of subexponentially amenable groups, which is the smallest class containing all groups of subexponential growth and closed after taking subgroups, quotients, extensions and direct unions.

Studying the automaton above, we deduce that the Basilica group B is generated by the automorphisms a and b having the following self-similar form:

$$a = (b, 1) \tag{12}$$

and

$$b = (a, 1)\varepsilon, \tag{13}$$

where ε denotes the nontrivial permutation of S_2. In the following figure the labelling of generators a and b are presented. Observe that the nontrivial labellings are only in the leftmost branch of the tree.

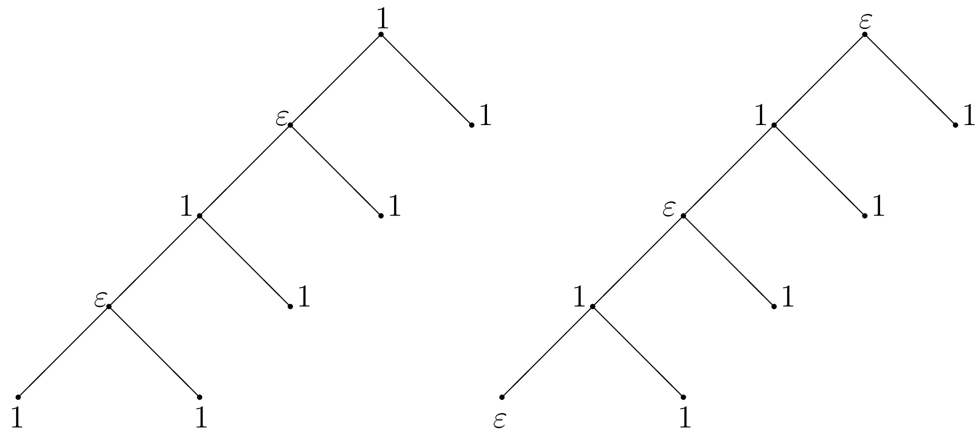


Fig.5. Labelling of the generators a and b.

One can easily verify that the first level stabilizer $Stab_B(1)$ is given by $Stab_B(1) = < a, a^b, b^2 >$. Since

$$a = (b, 1), \quad a^b = (1, b^a) \ \text{ and } b^2 = (a, a),$$

we can deduce from Lemma 2.6 that B is fractal.

It is obvious that the action of the Basilica group on the first level of T_2 is transitive. Since this group is fractal, it easily follows that the action is also spherically transitive, i.e. transitive on each level of the tree. Moreover, the Basilica group is weakly regular branch over its commutator subgroup B'. In fact, one can easily verify that

$$[a, b^2] = ([b, a], 1).$$

Using the fractalness of B, we get

$$B' \geq \langle [a, b^2] \rangle^B \geq \langle [b, a] \rangle^B \times \{1\} = B' \times \{1\}.$$

Moreover $(B' \times \{1\})^b = \{1\} \times B'$. So B' contains $B' \times B'$ and, since $B' \neq \{1\}$, the group B is regular weakly branch over B'.

It is a remarkable fact due to Nekrashevych [**44**] that this group can be described as the iterated monodromy group $IMG(z^2 - 1)$ of the complex polynomial $z^2 - 1$.

In fact, if we consider the complex polynomial $f(z) = z^2 - 1$, then it defines a 2−fold self-covering of the Riemann sphere $\widehat{\mathbb{C}}$. Using notation of section 2.2, the set of critical points of f is given by $C_f = \{0, \infty\}$, so that the set of post-critical points $P_f = \bigcup_{n=0}^{\infty} f^n(C_f)$ is $P_f = \{-1, 0, \infty\}$. So we have $M = \widehat{\mathbb{C}} \setminus \overline{P_f} = \mathbb{C} \setminus \{-1, 0\}$ and $M_1 = \widehat{\mathbb{C}} \setminus f^{-1}(\overline{P_f}) = \mathbb{C} \setminus \{-1, 0, 1\}$. In particular, f defines a 2−fold partial self-covering $f : M_1 \to M$ and the fundamental group $\pi_1(M)$ is the free group of rank 2. If a is a loop around 0 in M based, for instance, at $-\frac{1}{2}$, and b is a loop around -1 in M based at the same point $-\frac{1}{2}$, then it is easy verify that one gets the relations (12) and

(13), where the alphabet $X = \{0,1\}$ has to be regarded as the set of preimages of $-\frac{1}{2}$, with $-\frac{\sqrt{2}}{2}$ identified with 0 and $\frac{\sqrt{2}}{2}$ identified with 1.

2.3. Gelfand Pairs associated with groups of automorphisms of a homogeneous rooted tree. A first example of Gelfand pairs is given by the Adding Machine. In this context, denote this group by A. We already saw that this group is isomorphic to the group $\mathbb{Z}$ of integer numbers and so it is abelian. This implies that, considering its action on the $n-$th level L_n of the binary tree, setting $A_n = A/Stab_A(n)$ and $K_n = Stab_{A_n}(x_0)$, with $x_0 = 0^n$, then (A_n, K_n) is a Gelfand pair for every $n \in \mathbb{N}$. In particular, we have

$$A_n \cong \mathbb{Z}/2^n\mathbb{Z},$$

with generator a and such that $a^{2^n} = 1$ in A_n. Since the group is abelian, we have $Stab_A(x_0) = Stab_A(n)$ and so $K_n = \{1\}$.

The space $L(L_n)$ clearly has dimension 2^n. Since its decomposition into irreducible subrepresentations has to be multiplicity-free (see Theorem 1.7), then all irreducible (of dimension 1) representations of A_n occur in $L(L_n)$.

Denote V_h the representation of A_n corresponding to the character χ_h, defined as

$$\chi_h(a) = \omega_h = e^{\frac{2\pi i h}{2^n}}, \qquad \text{for } h = 0, \ldots, 2^n - 1.$$

So we get

$$L(L_n) = \bigoplus_{h=0}^{2^n-1} V_h.$$

For every $h = 0, 1, \ldots, 2^n - 1$, the corresponding spherical function ϕ_h coincides with the character χ_{2^n-h}. In fact, we have

$$\begin{aligned} \phi_h^a(a^l) &= \chi_{2^n-h}^a(a^l) = \chi_{2^n-h}(a^{l-1}) \\ &= e^{\frac{2\pi i(2^n-h)(l-1)}{2^n}} = e^{\frac{2\pi i(-hl+h)}{2^n}} \\ &= \chi_h(a) \cdot \chi_{2^n-h}(a^l) = \omega_h \cdot \phi_h(a^l) \end{aligned}$$

and so $\phi_h \in V_h$. Since the spherical functions coincide, in this case, with the characters of the cyclic group $\mathbb{Z}/2^n\mathbb{Z}$, Theorem 1.16 implies that the Gelfand pair $(A_n, 1)$ is not symmetric for $n \geq 3$.

This shows that the hypothesis of fractalness is not sufficient to get a $2-$points homogeneous action on L_n. A counterexample, in the case $n = 3$, is given by the pairs $(000, 101)$ and $(101, 011)$. We have $d(000, 101) = d(101, 011) = 3$. The vertex 000 is mapped into 101 by the automorphism a^5, but a^5 maps 101 into 010 and so the action is not $2-$points homogeneous.

In [**17**] I proved with Daniele D'Angeli that the action of the Basilica group B on the $n-$th level L_n of the rooted binary tree T_2 gives rise to symmetric Gelfand pairs.

As usual, for every $n \geq 1$, we can regard each vertex of the $n-$th level of T_2 as a word of length n in the alphabet $X = \{0,1\}$. Denote x_0 the vertex $\underbrace{00\ldots0}_{n \text{ times}}$ of L_n and set

$$B_n = B/Stab_B(n).$$

Let K_n the parabolic subgroup of B_n stabilizing x_0. The following general lemma holds.

Lemma 2.17. *Let G act spherically transitively on T_q. Denote G_n the quotient group $G/Stab_G(n)$ and K_n the stabilizer in G_n of a fixed leaf $x_0 \in L_n$. Then the action on L_n is $2-$points homogeneous if and only if K_n acts transitively on each sphere of L_n.*

Proof. Suppose that K_n acts transitively on each sphere of L_n and consider the elements x, y, x' and y' such that $d(x,y) = d(x',y')$. Since the action of G_n is transitive, there exists an automorphism $g \in G_n$ such that $g(x) = x'$. Now $d(x', g(y)) = d(x', y')$ and so $g(y)$ and y' are in the same sphere of center x' and radius $d(x', y')$. But K_n is conjugate with $Stab_{G_n}(x')$ and so there exists an automorphism $g' \in Stab_{G_n}(x')$ carrying $g(y)$ to y'. The composition of g and g' is the required automorphism.

Suppose now that the action of G_n on L_n is $2-$points homogeneous and consider two elements x and y in the sphere of center x_0 and radius i. Then $d(x_0, x) = d(x_0, y) = i$. So there exists an automorphism $g \in Stab_{G_n}(x_0)$ such that $g(x) = y$. This completes the proof. $\square$

We have the following theorem.

Theorem 2.18. *The action of the Basilica group B on L_n is $2-$points homogeneous, for all $n \in \mathbb{N}$.*

Proof. From Lemma 2.17 it suffices to show that the action of the parabolic subgroup $K_n = Stab_{B_n}(0^n)$ is transitive on each sphere.

Denote by u_j the vertex $0^{j-1}1$ for every $j = 1, \ldots, n$. Observe that the automorphisms

$$(b^2)^a = a^{-1}b^2a = (b^{-1},1)(a,a)(b,1) = (a^b, a) = ((1,b^a), a)$$

and

$$b^a b^{-1} a = (b^{-1},1)(a,1)\varepsilon(b,1)(1,a^{-1})\varepsilon(b,1) = (1,b)$$

belong to K_n for each n. Moreover, using the fractalness of B, it is possible to find elements $g_j \in K_n$ such that the restriction $g_j|T_{0^{j-1}}$ is $(b^2)^a = ((1,b^a),a)$ or $b^a b^{-1} a = (1,b)$. So, the action of such automorphisms on the subtree T_{u_j} corresponds to the action of the whole group $B =< a, b >$ on T. We can regard this action as the action of K_n on the spheres centered at x_0 and so we get that K_n acts transitively on these spheres. This implies that the action of B is 2−points homogeneous on L_n. $\square$

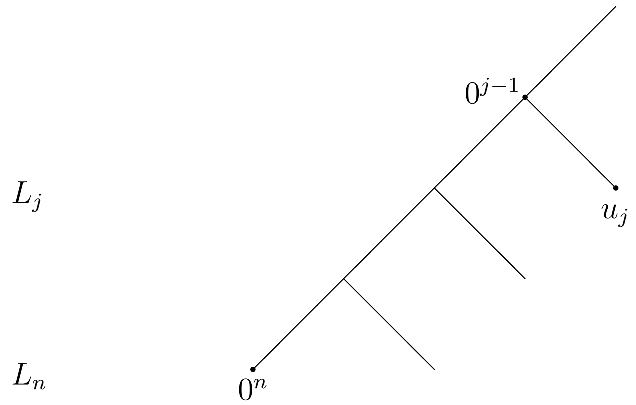


Fig.6. The 2−points homogeneous action of B.

COROLLARY 2.19. *For every $n \geq 1$, (B_n, K_n) is a symmetric Gelfand pair.*

We know that the number of K_n−orbits in $L(n)$ is exactly the number of the irreducible subrepresentations occurring in the decomposition of $L(L_n)$ under the action of B_n. Since the submodules W_j's described in the previous section are $n+1$ as the K_n−orbits, it follows that the Basilica group admits the same decomposition into irreducible subrepresentations and the same spherical functions that we gave for $Aut(T_2)_n$ in (3).

A similar argument can be used in the case of the **Grigorchuk group**, that we denote G. This group was introduced for the first time in [**34**] and it is the group of automorphisms of the rooted binary tree whose generators have the following self-similar form:

$$a = (1,1)\varepsilon, \quad b = (a,c), \quad c = (a,d), \quad d = (1,b),$$

where ε denotes, as usual, the nontrivial permutation in S_2. It is the first example of group with intermediate growth (in particular, it is amenable). It is a group belonging to $SG \setminus EG$, where EG denotes the smallest class containing all abelian and finite groups and closed after taking subgroups, quotients, extensions and direct unions.

It is a fractal group acting spherically transitively on T_2 and it is regular branch on its subgroup $K = < (ab)^2 >^G$. For more details see, for instance, [**36**].

The action of the Grigorchuk group on the binary rooted tree is 2−points homogeneous (see [**12**]) on the level L_n, for all $n \geq 1$. As a consequence, the decomposition of $L(L_n)$ under the action of this group into irreducible subrepresentations is still $L(L_n) = \bigoplus_{j=0}^{n} W_j$, where the W_j's are the subspaces introduced for $Aut(T_2)$.

Now consider the proof of Theorem 2.1 in the case $q = 2$. One can observe that the fundamental fact is that the automorphisms g' and g'' act transitively on the subtrees T' and T'', respectively, and trivially elsewhere. Moreover, the only hypothesis of fractalness does not guarantee that the action is 2−points homogeneous, as we have seen in the case of the Adding Machine, for which one gets symmetric Gelfand pairs only for $n = 1, 2$. On the other hand, if a fractal group G acts 2−transitively on L_1 and if it has the property that the rigid stabilizers of the vertices of the first level $Rist_G(i), i = 0, 1, \ldots, q-1$ are spherically transitive for each i, the proof of Theorem 2.1 works again by taking the automorphisms g' and g'' in the rigid vertex stabilizers. But this is not a necessary condition, as the example of the Grigorchuk group shows.

In fact, one can verify (see [**6**]) that, in this case, $Rist_G(0) =< d^a, d^{ac} >$, with $d^a = (b, 1)$ and $d^{ac} = (b^a, 1)$. So $Rist_G(0)$ fixes the vertices 00 and 01, and then it does not act transitively on the subtree T_0. This shows, for instance, that a fractal regular branch group does not need to have this property, which appears to be very strong.

On the other hand, a direct computation shows that the Basilica group has this property, what gives another proof that the action on each level L_n is 2−points homogeneous.

Theorem 2.20. *Let B be the Basilica group. Then the rigid vertex stabilizers $Rist_B(i)$, $i = 0, 1$, act spherically transitively on the corresponding subtrees T_i.*

Proof. Since B is spherically transitive and so $Rist_B(0) \simeq Rist_B(1)$, it suffices to prove the assertion only for $Rist_B(0)$. Consider the automorphisms $a = (b, 1)$ and $a^{b^2} = (b^a, 1)$ in $Rist_B(0)$. We want to show that the subgroup $< a, a^{b^2} >$ is spherically transitive on T_0, equivalently we will prove that the group $< b, b^a >$ is spherically transitive on T.

The latter is clearly transitive on the first level. To complete it suffices to prove its fractalness. We have

$$b^{-1}b^a = (1, a^{-1})\varepsilon(b^{-1}, 1)(a, 1)\varepsilon(b, 1) = (1, a^{-1}b^{-1})\varepsilon(a, b)\varepsilon = (b, (b^{-1})^a)$$

and

$$(b^{-1}b^a)^{b^2} = (a^{-1}, a^{-1})(b, (b^{-1})^a)(a, a) = (b^a, (b^{-1})^{a^2}),$$

and so the projection on the first factor gives both the generators b and b^a. The elements

$$(b^{-1}b^a)^{-1} = (b^{-1}, b^a), \quad ((b^{-1}b^a)^{-1})^{b^{-2}} = ((b^{-1})^{a^{-1}}, b)$$

fulfill the requirements for the projection on the second factor and this completes the proof. $\square$

In [**17**] we also studied the case of the group $I = IMG(z^2 + i)$, i.e. the iterated monodromy group defined by the complex polynomial $f(z) = z^2 + i$. This group has been introduced in [**7**] and later studied by K. U. Bux and R. Pérez ([**13**]), who proved that it has intermediate growth and so it is amenable.

The generators of I have the following self-similar form:

$$a = (1,1)\varepsilon, \quad b = (a, c), \quad c = (b, 1),$$

where ε denotes, as usual, the nontrivial permutation in S_2. The corresponding labellings are:

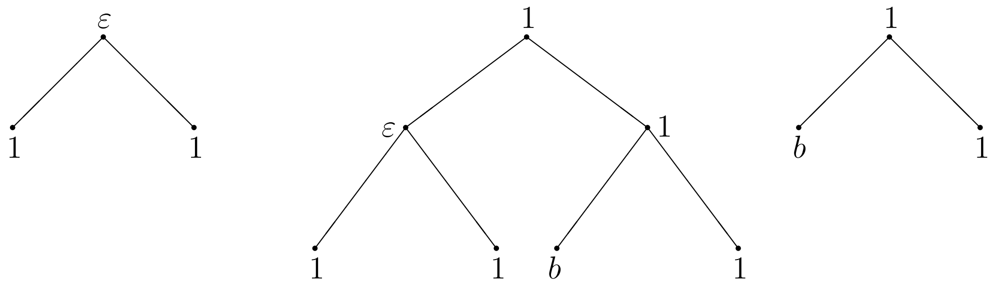


Fig.7. Labelling of the generators a, b and c of I.

By a direct computation one gets the following relations:

$$a^2 = b^2 = c^2 = (ac)^4 = (ab)^8 = (bc)^8 = 1.$$

Moreover, the first level stabilizer of I is $Stab_I(1) =< b, c, b^a, c^a >$. In particular, since

$$b^a = (c, a), \qquad c^a = (1, b),$$

we deduce that I is a fractal group. It is clear that I transitively acts on the first level of the rooted binary tree. Since I is fractal, it follows that this action is also spherically transitive.

Moreover, it is known (see [**38**]) that I is a regular branch group over its subgroup N defined by

$$N =< [a, b], [b, c] >^I .$$

For the group I it is possible to prove the same result proven for the Basilica group in Theorem 2.18. So set $I_n = I/Stab_I(n)$. In order to get an easy computation, this time we choose the vertex x_0 as $x_0 = 1^n \in L_n$ and we set $K_n = Stab_{I_n}(1^n)$. In the following theorem we will prove that the action of the parabolic subgroup K_n is transitive on each sphere.

THEOREM 2.21. *The action of the group I on L_n is $2-$points homogeneous for all $n \geq 1$.*

Proof. Denote by u_j the vertex $1^{j-1}0$ for every $j = 1, \dots, n$. Using the fractalness of I, it is possible to find an element $g_j \in K_n$ such that the restriction $g_j|T_{1^{j-1}}$ is b and an element $h_j \in K_n$ such that the restriction $h_j|T_{1^{j-1}}$ is c. Consider now the automorphism $b^a b b^a = (c,a)(a,c)(c,a) = (a^c, c^a)$. By fractalness it is possible to find an element $k_j \in K_n$ such that the restriction $k_j|T_{1^{j-1}}$ is $b^a b b^a$. The action of the subgroup generated by the automorphisms g_j, h_j, k_j on the subtree T_{u_j} corresponds to the action of the subgroup $H = < a, b, a^c >$ on T. It is easy to prove that this action is spherically transitive. In fact it is clear that H acts transitively on the first level, so it suffices to show that H is fractal. To show this consider, for instance, the elements

$$b = (a,c), \quad a^c a = (b,b), \quad b^a b b^a = (a^c, c^a)$$

and

$$b^a = (c,a), \quad a^c a = (b,b), \quad b b^a b = (c^a, a^c).$$

Now, the action of H on T_{u_j} can be regarded as the action of K_n on the spheres of center x_0, and so we get that K_n acts transitively on these spheres. This implies that the action of I on L_n is $2-$points homogeneous, as required. $\square$

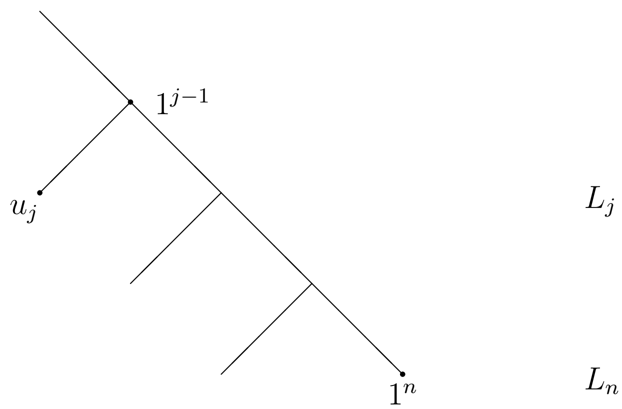


Fig.8. The $2-$points homogeneous action of I.

Corollary 2.22. *For every $n \geq 1$, (I_n, K_n) is a symmetric Gelfand pair.*

As in the case of the Basilica group, it follows that the group I_n admits the same decomposition into irreducible subrepresentations and the same spherical functions given in (3).

Remark. The interesting fact is that, in the case of $IMG(z^2+i)$, the rigid stabilizers of the vertices of the first level of the tree do not act spherically transitively on the corresponding subtrees T_0 and T_1. In fact, the rigid stabilizer of the first level is $Rist_I(1) = < c >^G$, so every automorphism in $Rist_I(1)$ is the product of elements of the form c^g, where $g = w(a,b,c)$ is a word in a, b and c, and of their inverses. Set $\varphi(c^g) = (g_0, g_1)$. We want to show, by induction on the length of the word $w(a,b,c)$, that we suppose reduced, that in both g_0 and g_1 the number of occurrences of a is even. This will imply that the action of $Rist_I(1)$ on the first level of the subtrees T_0 and T_1 cannot be transitive and will prove the assertion.

If $|w(a,b,c)| = 0$, then $c^g = c = (b,1)$. If $|w(a,b,c)| = 1$, then we can have $c^a = (1,b)$, $c^b = (b^a, 1)$ or $c^c = c = (b,1)$. Let us suppose the result to be true for $|w'(a,b,c)| = n-1$. Then we have $c^{w(a,b,c)} = c^{w'(a,b,c)x}$, with $x \in \{a,b,c\}$ and $c^{w'(a,b,c)} = (g_0', g_1')$ such that in both g_0' and g_1' the number of occurrences of a is even. If $x = a$, we get $c^{w(a,b,c)} = (g_1', g_0')$, if $x = b$, we get $c^{w(a,b,c)} = ((g_0')^a, (g_1')^b)$ and if $x = c$ then we get $c^{w(a,b,c)} = ((g_0')^b, g_1')$. In all cases, we get a pair (g_0, g_1) satisfying the condition that in both g_0 and g_1 the number of occurrences of a is even, as we wanted.

So the group I does not have the "rigid property" that Basilica group has, what shows that this property is not necessary to get symmetric Gelfand pairs.

In [**19**], I studied with D. D'Angeli a particular group of automorphisms of the rooted dyadic tree and the associated Gelfand pairs. In this context, we regard the binary rooted tree T_2 in the following way: the root of T_2 is identified with the group of integers $\mathbb{Z}$; each vertex, say at level L_n, can be regarded as a coset of $2^n\mathbb{Z}$ in $\mathbb{Z}$. Finally, the boundary ∂T_2 corresponds to the ring of dyadic integers $\mathbb{Z}_2$ (for more details see [**31**]).

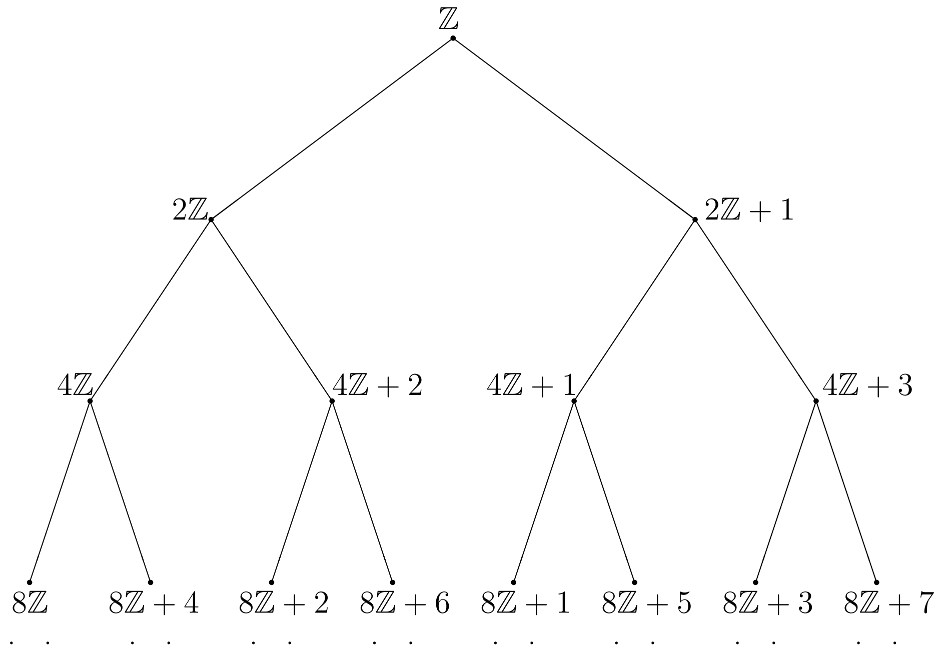


Fig.9. The dyadic tree.

We study the group G of automorphisms of T_2 generated by the sum of 1 and by the multiplication by an odd integer q for each vertex in T_2. Denote by a and b such automorphisms, respectively. The action of G on T_2 is self-similar: we directly prove that these automorphisms admit the following self-similar form:

$$a = (1, a)\varepsilon, \quad b = (b, ba^h),$$

with $q = 2h+1$ (observe that a is exactly the automorphism generating the Adding Machine). By using the self-similarity, we deduce that G is isomorphic to the Baumslag-Solitar group $BS_q =< s, t : t^{-1}st = s^q >$, introduced in [**11**]. Observe that, for $q = -1$, this group becomes the infinite dihedral group $D_\infty =< s, t : t^{-1}st = s^{-1} >$.

Denoting, as usual, G_n the finite homomorphic image of G acting faithfully on L_n and $K_n \leq G_n$ the parabolic subgroup stabilizing a fixed vertex in L_n, we prove there that (G_n, K_n) is a Gelfand pair for every $n \geq 1$. In particular, we show, by direct computations involving characters, that the decomposition of the corresponding permutation representation into irreducible $G-$representations is multiplicity-free and we give the relative spherical functions. Actually, the result can also be obtained from the general theory of representations of semidirect products developed in [**14**].

3. Groups of automorphisms of poset block structures

In this section we will study the Gelfand pairs associated with the action of groups on different structures, namely the **poset block structures**. These structures contain, as a particular case, the rooted binary trees that we considered in the previous sections. Moreover, they constitute a subclass of a more general class, given by the **orthogonal block structures** ([**3**] and [**4**]). We give here the definition of orthogonal block structure.

Let Ω be a finite set. Given a partition F of Ω, let R_F be the relation matrix of F, i.e.

$$R_F(\alpha,\beta) = \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are in the same part of } F \\ 0 & \text{otherwise.} \end{cases}$$

If $R_F(\alpha,\beta) = 1$, we usually write $\alpha \sim_F \beta$.

Definition 3.1. *A partition F of Ω is **uniform** if all its parts have the same size. This number is denoted k_F.*

The trivial partitions of Ω are the **universal** partition U, which has a single part and whose relation matrix is J_Ω, and the **equality partition** E, all of whose parts are singletons and whose relation matrix is I_Ω. We denote J_Ω the matrix of size $|\Omega|$ all of whose entries are 1 and I_Ω the identity matrix of size $|\Omega|$.

The partitions of Ω constitute a poset with respect to the relation $\preccurlyeq$, where $F \preccurlyeq G$ if every part of F is contained in a part of G. Given any two partitions F and G, their *infimum* is denoted $F \wedge G$ and is the partition whose parts are intersections of $F-$parts with $G-$parts; their *supremum* is denoted $F \vee G$ and is the partition whose parts are minimal subject to being unions of $F-$parts and $G-$parts.

Definition 3.2. *A set $\mathcal{F}$ of uniform partitions of Ω is an **orthogonal block structure** if:*

1. *$\mathcal{F}$ contains U and E;*
2. *for all F and $G \in \mathcal{F}$, $\mathcal{F}$ contains $F \wedge G$ and $F \vee G$;*
3. *for all F and $G \in \mathcal{F}$, the matrices R_F and R_G commute with each other.*

The groups that naturally act on the poset block structures are the generalized wreath products of permutation groups, introduced in [**5**]. We will show that they contain, as a particular case, the classical direct product and wreath product of permutation groups. In the next sections, we will give the definition and we will study the associated Gelfand pairs.

3.1. The generalized wreath product of permutation groups.

Let $(I, \leq)$ be a finite poset, with $|I| = n$. First of all, we need some definitions.

DEFINITION 3.3. *A subset $J \subseteq I$ is said*

- **ancestral** *if, whenever $i > j$ and $j \in J$, then $i \in J$;*
- **hereditary** *if, whenever $i < j$ and $j \in J$, then $i \in J$;*
- **a chain** *if, whenever $i, j \in J$, then either $i \leq j$ or $j \leq i$;*
- **an antichain** *if, whenever $i, j \in J$ and $i \neq j$, then neither $i \leq j$ nor $j \leq i$.*

In particular, for every $i \in I$, the following subsets of I are ancestral:

$$A(i) = \{j \in I : j > i\} \quad \text{and} \quad A[i] = \{j \in I : j \geq i\},$$

and the following subsets of I are hereditary:

$$H(i) = \{j \in I : j < i\} \quad \text{and} \quad H[i] = \{j \in I : j \leq i\}.$$

Given a subset $J \subseteq I$, we set

- $A(J) = \bigcup_{i \in J} A(i)$;
- $A[J] = \bigcup_{i \in J} A[i]$;
- $H(J) = \bigcup_{i \in J} H(i)$;
- $H[J] = \bigcup_{i \in J} H[i]$.

In what follows we will use the notation in [**5**].

For each $i \in I$, let $\Delta_i = \{\delta_0^i, \ldots, \delta_{m-1}^i\}$ be a finite set, with $m \geq 2$. For $J \subseteq I$, put $\Delta_J = \prod_{i \in J} \Delta_i$. In particular, we put $\Delta = \Delta_I$.

If $K \subseteq J \subseteq I$, let π_K^J denote the natural projection from Δ_J onto Δ_K. In particular, we put $\pi_J = \pi_J^I$. Moreover, we will use Δ^i for $\Delta_{A(i)}$ and π^i for $\pi_{A(i)}$.

For each $i \in I$, let G_i be a permutation group on Δ_i and let F_i be the set of all functions from Δ^i into G_i. For $J \subseteq I$, we put $F_J = \prod_{i \in J} F_i$ and set $F = F_I$. An element of F will be denoted $f = (f_i)$, with $f_i \in F_i$.

DEFINITION 3.4. *For each $f \in F$, the action of f on Δ is defined as follows: if $\delta = (\delta_i) \in \Delta$, then*

$$\delta f = \varepsilon, \quad \text{where } \varepsilon = (\varepsilon_i) \in \Delta \text{ and } \varepsilon_i = \delta_i(\delta \pi^i f_i). \tag{14}$$

It is easy to verify that this is a faithful action of F on Δ, i.e. if $f, h \in F$ and if $\delta f = \delta h$ for all $\delta \in \Delta$, then $f = h$.

In [**5**] it is proven that (F, Δ) is a permutation group with respect to the action defined in (14). This group is called the **generalized wreath product** of the permutation groups $(G_i, \Delta_i)_{i \in I}$ and it is denoted $\prod_{(I, \leq)}(G_i, \Delta_i)$.

The following theorem is given in [**5**]. We denote $Sym(\Delta_i)$ the symmetric group acting on the set Δ_i. We also use the notation $Sym(m)$ if $|\Delta_i| = m$.

THEOREM 3.5. *The generalized wreath product of the permutation groups $(G_i, \Delta_i)_{i \in I}$ is transitive on Δ if and only if (G_i, Δ_i) is transitive for each $i \in I$.*

In order to give the definition of poset block structure, we need to introduce some equivalence relations on Δ, that we will call **ancestral relations**.

Let $\mathcal{A}$ be the set of ancestral subsets of I. If $J \in \mathcal{A}$, then the equivalence relation $\sim_J$ on Δ associated with J is defined as

$$\delta \sim_J \epsilon \quad \Leftrightarrow \quad \delta\pi_J = \epsilon\pi_J,$$

for each $\delta, \epsilon \in \Delta$.

DEFINITION 3.6. *A* **poset block structure** *is a pair* $(\Delta, \sim_{\mathcal{A}})$, *where*

(1) $\Delta = \prod_{(I,\leq)} \Delta_i$, *with* $(I, \leq)$ *a finite poset and* $|\Delta_i| \geq 2$, *for each* $i \in I$;
(2) $\sim_{\mathcal{A}}$ *denotes the set of equivalence relations on* Δ *defined by the ancestral subsets of* I.

Note that the set $\sim_{\mathcal{A}}$ defines an orthogonal block structure on Δ.

DEFINITION 3.7. *An* **automorphism** *of a poset block structure* $(\Delta, \sim_{\mathcal{A}})$ *is a permutation* σ *of* Δ *such that, for every equivalence* $\sim_J$ *in* $\sim_{\mathcal{A}}$,

$$\delta \sim_J \varepsilon \qquad \Leftrightarrow \qquad (\delta\sigma) \sim_J (\varepsilon\sigma),$$

for all $\delta, \varepsilon \in \Delta$.

The following theorem is proven in [**5**].

THEOREM 3.8. *Let* $(\Delta, \sim_{\mathcal{A}})$ *be the poset block structure associated with the poset* $(I, \leq)$. *Let* F *be the generalized wreath product* $\prod_{(I,\leq)} Sym(\Delta_i)$. *Then* F *is the group of automorphisms of* (Δ, S).

Remark. We want to present an example of orthogonal block structure which cannot be obtained as the set $\sim_{\mathcal{A}}$ of ancestral relations associated with a poset $(I, \leq)$.

Consider the quaternion group $Q = \{1, -1, i, -i, j, -j, k, -k\}$. It has four (all normal) proper subgroups, three of them isomorphic to the Klein group and one isomorphic to the cyclic group $\mathbb{Z}/2\mathbb{Z}$. They are:

$$I = \{1, i, -1, -i\}, \quad J = \{1, j, -1, -j\}, \quad K = \{1, k, -1, -k\}$$

$$\text{and} \quad Z = \{1, -1\}.$$

Each subgroup $R \leq Q$ defines a uniform partition $\sim_R$ of Q into its cosets. Since in this group all subgroups commute pairwise, these partitions form an orthogonal block structure of height 3 that can be represented as

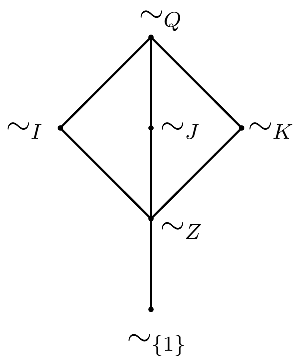


Fig.10. The orthogonal block structure associated with Q.

On the other hand, it is easy to check that this structure cannot be obtained as the ancestral poset of any poset $(I, \leq)$ with $|I| = 3$.

3.1.1. *The permutation direct product.* If $(I, \leq)$ is a finite poset, with $\leq$ the identity relation, then the generalized wreath product becomes the **permutation direct product**.

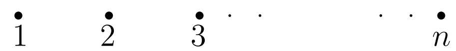


Fig.11. The poset I in the case of the permutation direct product.

In this case, we have $A(i) = \emptyset$ for each $i \in I$ and so an element f of F is given by $f = (f_i)_{i \in I}$, where f_i is a function from a singleton $\{*\}$ into G_i and so its action on δ_i does not depend from any other component of δ.

To fix our ideas, consider the case $n = 3$, with $\Delta_1 = \Delta_2 = \Delta_3 = \{0, 1\}$. The elements of Δ can be represented as the leaves of a rooted binary tree of depth three and so as words of length three in the alphabet $\{0, 1\}$.

The partitions of Δ given by the equivalences $\sim_J$, with $J \subseteq I$ ancestral, are:

- $\Delta = \{000, 001, 010, 011, 100, 101, 110, 111\}$ by the equivalence $\sim_\emptyset$;
- $\Delta = \{000, 001, 010, 011\} \coprod \{100, 101, 110, 111\}$ by the equivalence $\sim_{\{1\}}$;
- $\Delta = \{000, 001, 100, 101\} \coprod \{010, 011, 110, 111\}$ by the equivalence $\sim_{\{2\}}$;
- $\Delta = \{000, 010, 100, 110\} \coprod \{001, 011, 101, 111\}$ by the equivalence $\sim_{\{3\}}$;

- $\Delta = \{000, 001\} \coprod \{010, 011\} \coprod \{100, 101\} \coprod \{110, 111\}$ by the equivalence $\sim_{\{1,2\}}$;
- $\Delta = \{000, 010\} \coprod \{001, 011\} \coprod \{100, 110\} \coprod \{101, 111\}$ by the equivalence $\sim_{\{1,3\}}$;
- $\Delta = \{000, 100\} \coprod \{001, 101\} \coprod \{010, 110\} \coprod \{011, 111\}$ by the equivalence $\sim_{\{2,3\}}$;
- $\Delta = \{000\} \coprod \{001\} \coprod \{010\} \coprod \{011\} \coprod \{100\} \coprod \{101\} \coprod \{110\} \coprod \{111\}$ by the equivalence $\sim_I$;

The labelling in the picture describe the components of an automorphism $f \in F$ acting on Δ, where $f_1(*) = g_1 \in G_1$, $f_2(*) = g_2 \in G_2$ and $f_3(*) = g_3 \in G_3$.

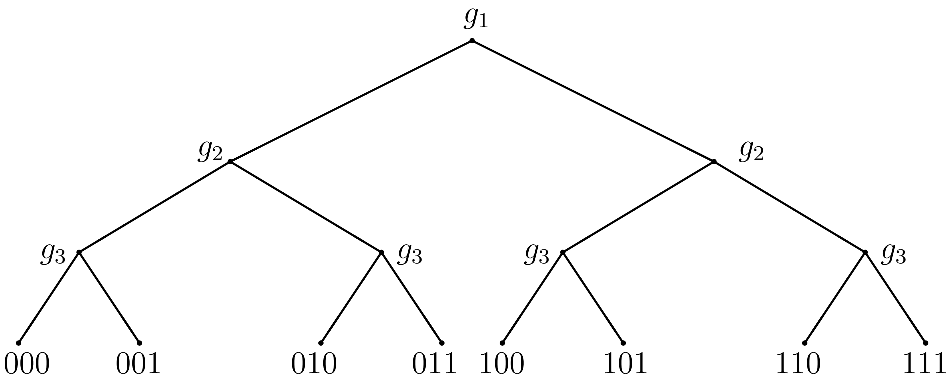


Fig.12. The labelling of an automorphism $f \in F$ acting on Δ.

3.1.2. *The permutation wreath product.* If $(I, \leq)$ is a finite chain as in the following picture,

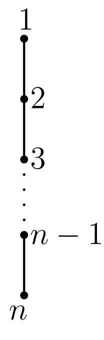


Fig.13. The poset I in the case of the permutation wreath product.

then the generalized wreath product becomes the **permutation wreath product**

$$(G_n, \Delta_n) \wr (G_{n-1}, \Delta_{n-1}) \wr \cdots \wr (G_1, \Delta_1).$$

In this case, we have $A(i) = \{1, 2, \ldots, i-1\}$ for each $i \in I$ and so an element f of F is given by $f = (f_i)_{i \in I}$, with

$$f_i : \Delta_1 \times \cdots \times \Delta_{i-1} \longrightarrow G_i$$

and so its action on δ_i depends on all the previous components of δ.

The partitions of Δ given by the equivalences $\sim_J$, with $J \subseteq I$ ancestral, are:

- $\Delta = \{000, 001, 010, 011, 100, 101, 110, 111\}$ by the equivalence $\sim_\emptyset$;
- $\Delta = \{000, 001, 010, 011\} \coprod \{100, 101, 110, 111\}$ by the equivalence $\sim_{\{1\}}$;
- $\Delta = \{000, 001\} \coprod \{010, 011\} \coprod \{100, 101\} \coprod \{110, 111\}$ by the equivalence $\sim_{\{1,2\}}$;
- $\Delta = \{000\} \coprod \{001\} \coprod \{010\} \coprod \{011\} \coprod \{100\} \coprod \{101\} \coprod \{110\} \coprod \{111\}$ by the equivalence $\sim_I$;

The labelling in the following picture describe the components of an automorphism $f \in F$ acting on Δ, where $f_1(*) = g_1 \in G_1$ and $f_2(0), f_2(1)$ are elements of G_2 and $f_3(00), f_3(01), f_3(10), f_3(11)$ are elements of G_3.

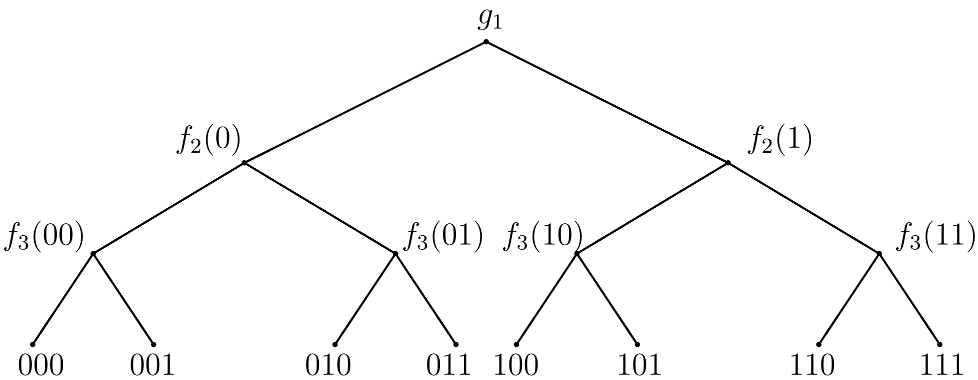


Fig.14. The labelling of an automorphism $f \in F$ acting on Δ.

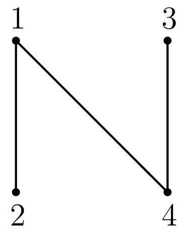
The representation of Δ by a rooted tree of depth n is not the best one. In [**21**] we give a better construction to represent a poset block structure $(\Delta, \sim_{\mathcal{A}})$, using the notion of ancestral poset. To understand it, let us introduce in $\mathcal{A}$ a partial order relation $\leq$ defined as

$$J_1 \leq J_2 \quad \Leftrightarrow J_1 \supseteq J_2,$$

for all $J_1, J_2 \in \mathcal{A}$. In particular, we write $J_1 \lessdot J_2$ if $J_1 \supseteq J_2$ and $J_1 \supseteq J_3 \supseteq J_2$ implies $J_1 = J_3$ or $J_2 = J_3$.

Its Hasse diagram is a poset $(\mathcal{A}, \leq)$. Observe that the empty set is always ancestral in I. A singleton $\{i\}$ constituted by a maximal element in I is still an ancestral set. Inductively, if J is an ancestral set, then $J \sqcup \{i\}$ is an ancestral set if i is a maximal element in $I \setminus J$. So the length of a maximal chain in $(\mathcal{A}, \leq)$ is n. We will call $(\mathcal{A}, \leq)$ the **ancestral poset**.

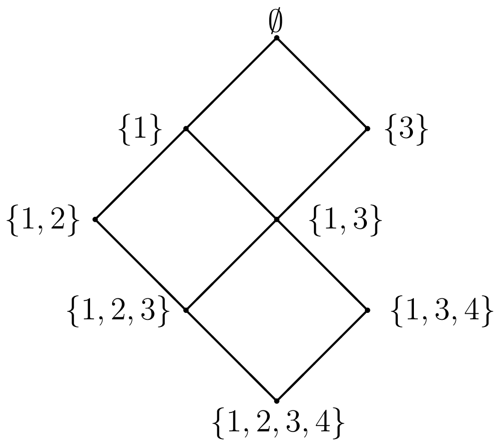
As an example, consider the poset $(I, \leq)$ given by



So we have

$$\mathcal{A} = \{\emptyset, \{1\}, \{3\}, \{1,2\}, \{1,3\}, \{1,2,3\}, \{1,3,4\}, \{1,2,3,4\}\}$$

and the poset $(\mathcal{A}, \leq)$ is given by



Let $C = \{I = J_0, J_1, \ldots, J_n = \emptyset\}$ be a maximal chain in $\mathcal{A}$, so that $|J_k| = |J_{k-1}| - 1$ for all $k = 1, \ldots, n$. In particular, let

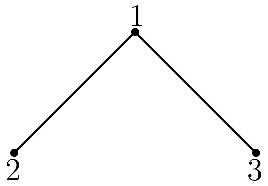
$$J_{k-1} = J_k \coprod \{i_k\},$$

for all $k = 1, \ldots, n$.

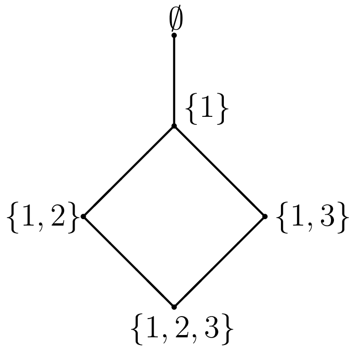
Let us design a rooted tree of depth n associated with C as follows: the $n-$th level is constituted by $|\Delta|$ vertices (each of these vertices constitutes a class of the equivalence $\sim_I$); the $(n-1)-$st level is constituted by $\frac{|\Delta|}{k_{\sim_{J_1}}}$ vertices. Each of these vertices is a father of $k_{\sim_{J_1}}$ sons that are in the same class of the equivalence $\sim_{J_1}$. Inductively, at the $i-$th level there are $\frac{|\Delta|}{k_{\sim_{J_{n-i}}}}$ vertices which are fathers of $k_{\sim_{J_{n-i}}}$ vertices of the $(i+1)-$st level belonging to the same class of the equivalence $\sim_{J_{n-i}}$.

We can perform the same construction for every maximal chain C in $(\mathcal{A}, \leq)$. The next step is to assemble the different structures identifying the vertices associated with the same relations. The resulting structure is a poset P, that represents the poset block structure $(\Delta, \sim_{\mathcal{A}})$.

3.1.3. *Example.* Consider the case of the following poset $(I, \leq)$:



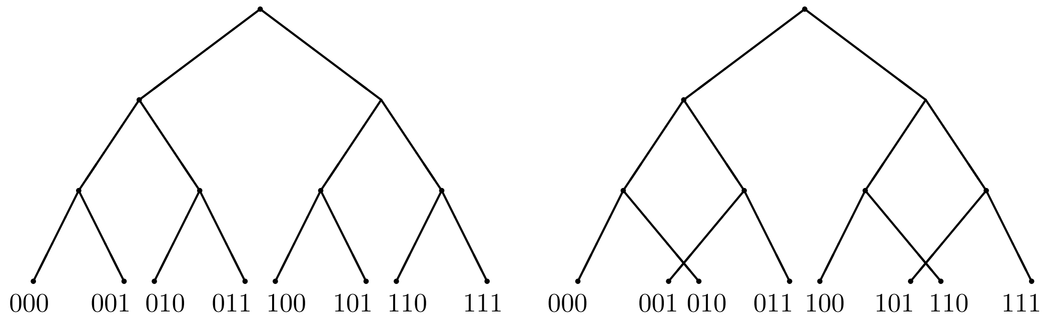
One can easily check that, in this case, the ancestral poset $(\mathcal{A}, \leq)$ is the following:



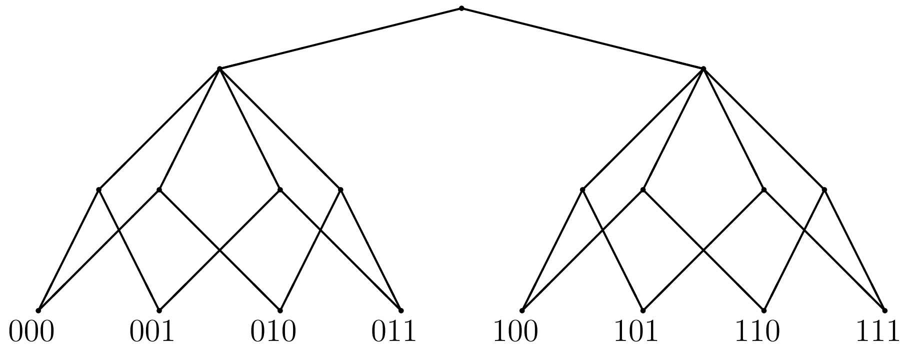
Suppose $m = 2$ and $\Delta_1 = \Delta_2 = \Delta_3 = \{0, 1\}$, so that we can think of Δ as the set of words of length 3 in the alphabet $\{0, 1\}$. The partitions of Δ given by the equivalences $\sim_J$, with $J \subseteq I$ ancestral, are:

- $\Delta = \{000, 001, 010, 011, 100, 101, 110, 111\}$ by the equivalence $\sim_\emptyset$;
- $\Delta = \{000, 001, 010, 011\} \coprod \{100, 101, 110, 111\}$ by the equivalence $\sim_{\{1\}}$;
- $\Delta = \{000, 001\} \coprod \{010, 011\} \coprod \{100, 101\} \coprod \{110, 111\}$ by the equivalence $\sim_{\{1,2\}}$;
- $\Delta = \{000, 010\} \coprod \{001, 011\} \coprod \{100, 110\} \coprod \{101, 111\}$ by the equivalence $\sim_{\{1,3\}}$;
- $\Delta = \{000\} \coprod \{001\} \coprod \{010\} \coprod \{011\} \coprod \{100\} \coprod \{101\} \coprod \{110\} \coprod \{111\}$ by the equivalence $\sim_I$.

Consider the chains $C_1 = \{I, \{1, 2\}, \{1\}, \emptyset\}$ and $C_2 = \{I, \{1, 3\}, \{1\}, \emptyset\}$ in A. The associated trees T_1 and T_2 are, respectively,



Assembling these trees, we get the following poset block structure.



3.2. Gelfand pairs associated with groups of automorphisms of a poset block structure. In what follows we will suppose $G_i = Sym(m)$, where $|\Delta_i| = m$ for all $i \in I$. Fix an element $\delta_0 = (\delta_0^1, \ldots, \delta_0^n)$ in Δ. Then the stabilizer $Stab_F(\delta_0)$ is the subgroup of F acting trivially on δ_0. We can think of an automorphism $f \in F$ as the $n-$tuple $(f_1, \ldots, f_n)$, with $f_i : \Delta^i \longrightarrow Sym(m)$. Set $\Delta_0^i = \prod_{j \in A(i)} \delta_0^j$. We have the following lemma.

LEMMA 3.9. *The stabilizer of* $\delta_0 = (\delta_0^1, \ldots, \delta_0^n) \in \Delta$ *in* F *is the subgroup*

$$\begin{aligned} K := Stab_F(\delta_0) \quad &= \quad \{f = (f_1, \ldots, f_n) \in F \ : \ f_i|_{\Delta_0^i} \in Stab_{Sym(m)}(\delta_0^i) \\ &\qquad \textit{whenever } \Delta^i = \Delta_0^i \textit{ or } A(i) = \emptyset\}. \end{aligned}$$

Proof. One can easily verify that K is a subgroup of F. If $i \in I$ is such that $A(i) = \emptyset$ then, by definition of generalized wreath product, it must be $f_i(*) \in Stab_{Sym(m)}(\delta_0^i)$. For the remaining indices $i \in I$ we have

$$\begin{aligned}\delta_0^i f = \delta_0^i &\iff \delta_0^i(\delta_0^{A(i)})f_i = \delta_0^i \\ &\iff (\delta_0^{A(i)})f_i \in \; Stab_{Sym(m)}(\delta_0^i) \\ &\iff f_i|_{\Delta_0^i} \in Stab_{Sym(m)}(\delta_0^i).\end{aligned}$$

This proves the lemma. $\square$

In the following lemma the $K-$orbits on Δ are described. We recall that the action of $Sym(m-1) \cong Stab_{Sym(m)}(\delta_0^i)$ on Δ_i has two orbits, i.e. $\Delta_i = \{\delta_0^i\} \coprod (\Delta_i \setminus \{\delta_0^i\})$. Set $\Delta_i^0 = \{\delta_0^i\}$ and $\Delta_i^1 = \Delta_i \setminus \{\delta_0^i\}$.

LEMMA 3.10. *The $K-$orbits on Δ have the following form:*

$$\left(\prod_{i\in I\setminus H[S]} \Delta_i^0\right) \times \left(\prod_{i\in S} \Delta_i^1\right) \times \left(\prod_{i\in H(S)} \Delta_i\right),$$

where S is any antichain in I.

Proof. First of all suppose that $\delta, \epsilon \in \left(\prod_{i\in I\setminus H[S]} \Delta_i^0\right) \times \left(\prod_{i\in S} \Delta_i^1\right) \times \left(\prod_{i\in H(S)} \Delta_i\right)$, for some antichain S. Then $\delta_{I\setminus H[S]} = \epsilon_{I\setminus H[S]} = \delta_0^{I\setminus H[S]}$. If $s \in S$ we have $A(s) \subseteq I\setminus H[S]$ and this implies $(A(s))f_s \in Stab_{Sym(m)}(\delta_0^s)$. So $\epsilon_s = \delta_s(\delta_0^{A(s)} f_s)$. If $i \in H(S)$ then $A(i) \neq \emptyset$ and $\Delta^i \neq \Delta_0^i$. This implies $(A(i))f_i \in Sym(m)$ and so $\epsilon_i = \delta_i(\delta_0^{A(i)} f_i)$. This shows that K acts transitively on each orbit.

On the other hand, let $S \neq S'$ be two distinct antichains and $\delta \in \left(\prod_{i\in I\setminus H[S]} \Delta_i^0\right) \times \left(\prod_{i\in S} \Delta_i^1\right) \times \left(\prod_{i\in H(S)} \Delta_i\right)$ and $\epsilon \in \left(\prod_{i\in I\setminus H[S']} \Delta_i^0\right) \times \left(\prod_{i\in S'} \Delta_i^1\right) \times \left(\prod_{i\in H(S')} \Delta_i\right)$. Suppose $s \in S \setminus (S \cap S')$ and so $I \setminus H[S] \neq I \setminus H[S']$. If $s \in I \setminus H[S']$ then $\delta_s \neq \delta_0^s = \epsilon_s$. But $(A(S))f_s \in Stab_{Sym(m)}(\delta_0^s)$ and so $\delta_s(A(S)f_s) \neq \epsilon_s$. If $s \in H(S')$ there exists $s' \in S' \setminus (S \cap S')$ such that $s < s'$. This implies that $s' \in I \setminus H[S]$ and we can proceed as above.

The proof follows from the fact that the orbits are effectively a partition of Δ. $\square$

Using Gelfand's condition (Lemma 1.3), the next proposition will prove that the generalized wreath product $F = \prod_{(I,\leq)}(Sym(\Delta_i), \Delta_i)$ acting on Δ and the stabilizer K of the element $\delta_0 = (\delta_0^1, \ldots, \delta_0^n)$ constitute a Gelfand pair.

PROPOSITION 3.11. *Given $(\delta, \epsilon) \in \Delta \times \Delta$, there exists an element $g \in F$ such that $g(\delta, \epsilon) = (\epsilon, \delta)$.*

Proof. Set $\delta = (\delta_i)_{i\in I}$ and $\epsilon = (\epsilon_i)_{i\in I}$. Let $i \in I$ such that $A(i) = \emptyset$. Then, by the $m-$transitivity of the symmetric group $Sym(m) \cong Sym(\Delta_i)$, there exists $g_i \in Sym(\Delta_i)$ such that $\delta_i g_i = \epsilon_i$ and $\epsilon_i g_i = \delta_i$. For every index i such that $A(i) \neq \emptyset$ define $f_i : \Delta^i \longrightarrow Sym(\Delta_i)$ as $\delta\pi^i f_i = \epsilon\pi^i f_i = \sigma_i$ where $\sigma_i \in Sym(\Delta_i)$ is a permutation such that $\delta_i\sigma_i = \epsilon_i$ and $\epsilon_i\sigma_i = \delta_i$. The element $g \in F$ that we get is the requested automorphism. $\square$

COROLLARY 3.12. *(F, K) is a symmetric Gelfand pair.*

Now set $L(\Delta) = \{f : \Delta \longrightarrow \mathbb{C}\}$. In [**5**] the authors give the decomposition of $L(\Delta)$ into $F-$irreducible subrepresentations. In particular, one has

$$L(\Delta) = \bigoplus_{S\subseteq I \ antichain} W_S \tag{15}$$

with

$$W_S = \left(\bigotimes_{i\in A(S)} L(\Delta_i)\right) \otimes \left(\bigotimes_{i\in S} V_i^1\right) \otimes \left(\bigotimes_{i\in I\setminus A[S]} V_i^0\right), \tag{16}$$

where, for each $i = 1, \dots, n$, $L(\Delta_i)$ is the space of the complex functions on Δ_i, whose decomposition into G_i-irreducible subrepresentations is

$$L(\Delta_i) = V_i^0 \bigoplus V_i^1,$$

where $V_i^0 \cong \mathbb{C}$ is the subspace of constant functions on Δ_i and $V_i^1 = \{f : \Delta_i \to \mathbb{C} : \sum_{x\in\Delta_i} f(x) = 0\}$. Moreover, one has $W_S = W_{S'}$ if and only if $S = S'$. In particular, this gives an alternative proof of the fact that (F, K) is a Gelfand pair, since the decomposition (15) is multiplicity-free.

Actually, the authors do not mention about Gelfand pairs theory, so in [**21**] we preferred to give a different proof, which appears in Proposition 3.11. On the other hand, the multiplicity-free decomposition of $L(\Delta)$ is not a sufficient condition to get a symmetric Gelfand pair.

In the next proposition we present the spherical functions associated with the symmetric Gelfand pair (F, K).

PROPOSITION 3.13. *For every antichain $S \subseteq I$, the spherical function ϕ_S belonging to the subspace W_S is*

$$\phi_S = \bigotimes_{i\in A(S)} \varphi_i \bigotimes_{i\in S} \psi_i \bigotimes_{i\in I\setminus A[S]} \rho_i, \tag{17}$$

where φ_i is the function defined on Δ_i as

$$\varphi_i(x) = \begin{cases} 1 & x = \delta_0^i \\ 0 & otherwise \end{cases}$$

and ψ_i is the function defined on Δ_i as

$$\psi_i(x) = \begin{cases} 1 & x = \delta_0^i \\ -\frac{1}{m-1} & otherwise \end{cases}$$

and ρ_i is the function on Δ_i such that $\rho_i(x) = 1$ for every $x \in \Delta_i$.

Proof. It is clear that $\phi_S \in W_S$ and that $\phi_S(\delta_0) = 1$, so it remains to show that each ϕ_S is $K-$invariant.

Set $B_1 = \{i \in A(S) : A(i) = \emptyset\}$. If there exists $i \in B_1$ such that $\delta_i \neq \delta_0^i$ then $\phi_S(\delta) = \phi_S^k(\delta) = 0$ for every $k \in K$, since $\delta_i \varphi_i = (\delta_i k^{-1})\varphi_i = 0$ because $k_i \in Stab_{G_i}(\delta_0^i)$. Hence ϕ and ϕ^k coincide on the elements $\delta \in \Delta$ satisfying this property. So we can suppose that $\delta_i = \delta_0^i$ for each $i \in B_1$.

Let B_2 be the set of maximal elements in $A(S) \setminus B_1$. If there exists $j \in B_2$ such that $\delta_j \neq \delta_0^j$ then one has $\phi_S(\delta) = \phi_S^k(\delta) = 0$ for every $k \in K$, since $\delta_j \varphi_j = (\delta_j k^{-1})\varphi_j = 0$ because $k_i \in Stab_{G_i}(\delta_0^i)$. Hence ϕ and ϕ^k coincide on $\delta \in \Delta$ satisfying this property. So we can suppose that $\delta_j = \delta_0^j$ for each $j \in B_2$. Iterating this argument, we can restrict our attention to the elements such that $\delta_{A(S)} = \delta_0^{A(S)}$. We have to prove that $\phi_S(\delta) = \phi_S^k(\delta)$, what means $(\delta_i)\psi_i = (\delta_i)\psi_i^k$ for every $i \in S$. This easily follows from the definition of K and of the function ψ_i. $\square$

3.3. The substructures of a poset block structures. As in the case of the rooted tree of type $\mathbf{m}$ and its subtrees of type $\mathbf{r}$, also in the case of the poset block structures it is possible to define some substructures and to consider the action of the generalized wreath product on the variety constituted by these substructures.

Consider the poset block structure associated with the poset $(I, \leq)$, with $|I| = n$.

For each $i \in I$, let $\Delta_i = \{\delta_0^i, \dots, \delta_{m_i-1}^i\}$ be a finite set, with $m_i \geq 2$ for all $i = 1, \dots, n$.

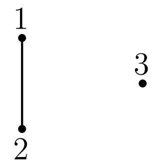
In order to understand how a substructure is done, we consider again the representation of Δ by a rooted tree of depth n and whose branch indices are $\mathbf{m} = (m_1, \dots, m_n)$.

We want to define a substructure with branch indices $\mathbf{r} = (r_1, \dots, r_n)$. If $i \in \{1, \dots, n\}$ is an index such that $A(i) = \emptyset$, then the choice of r_i elements in Δ_i does not depend from any other index.

If $i \in \{1, \dots, n\}$ is an index such that $\emptyset \neq A(i) = \{i_1, \dots, i_k\}$, then the choice of r_i elements in Δ_i depends on the choices performed for the indices $i_1, \dots, i_k$. We suppose here that $i_l < i$ in $\mathbb{N}$, for every $l = 1, \dots, k$. In other words, the $i-$th choice is the same for those substructures that coincide on the indices belonging to $A(i)$.

So the main difference that we have with respect to the case of the subtrees of a rooted tree is that, this time, the subtrees are not free and they have to be chosen following the conditions given by the ancestral sets $A(i)$.

Example. Consider the poset $(I, \leq)$ in the following figure:



We have $A(1) = A(3) = \emptyset$ and $A(2) = \{1\}$. Put now $\mathbf{m} = (3, 3, 3)$ and $\mathbf{r} = (2, 2, 2)$. A substructure can be represented as a subtree of type $\mathbf{r}$ of the rooted tree of depth 3 of type $\mathbf{m}$, with the condition that the choice of 2 elements in Δ_1 (first level) is free, the choice of 2 elements in Δ_2 (second level) depends on the first level and the choice of 2 elements of Δ_3 does not depend from any previous choice, so it must by the same starting from each vertex of the second level. For example, we can get the following substructure:

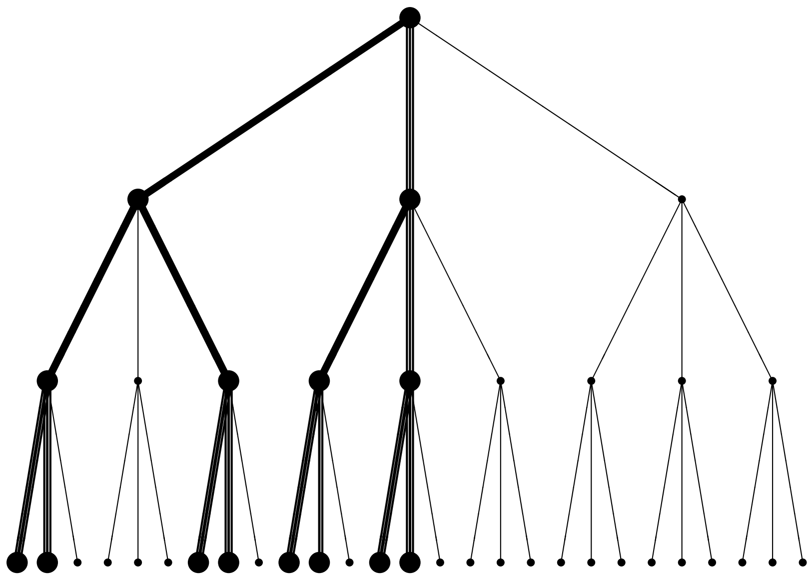


Fig.15. A substructure of type $(2, 2, 2)$.

If the poset $(I, \leq)$ is the chain in Figure 13, then we have $A(i) = \{1, \dots, i-1\}$. This implies that, for each $i \geq 2$, the choice of m_i elements in Δ_i is a function of all the previous coordinates and so it can be different starting from every vertex of the $(i-1)$−st level (that is the case of the usual subtrees of a rooted trees).

It is easy to check that the number of the substructures defined above is exactly

$$\prod_{i\in I: A(i)=\emptyset} \binom{m_i}{r_i} \cdot \prod_{i\in I: A(i)\neq\emptyset} \binom{m_i}{r_i}^{\prod_{j\in A(i)} r_j}.$$

In fact, for those indices $i \in I$ such that $A(i) = \emptyset$, we have $\binom{m_i}{r_i}$ possible choices; for those indices $i \in I$ such that $A(i) \neq \emptyset$, we have $\binom{m_i}{r_i}$ possible choices for each of the $\prod_{j\in A(i)} r_j$ vertices corresponding to (eventually) different choices for the coordinates in $A(i)$.

It is not difficult to verify that the generalized wreath product F of the symmetric groups of the sets Δ_i transitively acts on the variety of the substructures of a poset block structure.

We can also prove, using Gelfand's Condition, that (F, K) is a symmetric Gelfand pair, where K denotes the stabilizer of a fixed substructure. In fact, the following theorem holds.

THEOREM 3.14. *Let $(I, \leq)$ be a finite poset and let Δ be the associated poset block structure. Let F be the generalized wreath product of the symmetric groups $Sym(\Delta_i)$, with $|\Delta_i| = m_i \geq 2$ for all $i \in I$. Let $\boldsymbol{r}$ be an $n-$tuple of integers such that $1 \leq r_i \leq m_i$. If A and B are two substructures of type $\boldsymbol{r}$ in Δ, then there exists an automorphism $f \in F$ of Δ such that $f(A) = B$ and $f(B) = A$.*

Proof. We can suppose, without loss of generality, that $A(1) = \emptyset$. We want to get an automorphism $f = (f_i)_{i\in I} \in F$ such that $f(A) = B$ and $f(B) = A$. We will proceed by induction on the depth of the substructure.

Set $\pi_1(A) = \{i_1^A, \ldots, i_{r_1}^A\}$ and $\pi_1(B) = \{i_1^B, \ldots, i_{r_1}^B\}$.
By the m_1-transitivity of $Sym(\Delta_1)$, we can choose a permutation $f_1 \in Sym(\Delta_1)$ fixing $\pi_1(A) \cap \pi_1(B)$ such that $f_1(\pi_1(A) \setminus (\pi_1(A) \cap \pi_1(B))) = \pi_1(B) \setminus (\pi_1(A) \cap \pi_1(B))$ and $f_1(\pi_1(B) \setminus (\pi_1(A) \cap \pi_1(B))) = \pi_1(A) \setminus (\pi_1(A) \cap \pi_1(B))$.

Now let $2 \leq j \leq n$ and $A(j) = \{j_1, \ldots, j_k\}$, with $j_1 < \ldots < j_k < j$ in $\mathbb{N}$. Suppose that we have found an automorphism $f' \in F$ such that $f'(\pi_{\{1,\ldots,j-1\}}(A)) = \pi_{\{1,\ldots,j-1\}}(B)$ and $f'(\pi_{\{1,\ldots,j-1\}}(B)) = \pi_{\{1,\ldots,j-1\}}(A)$. We want to show that this result can be extended to the $j-$th level. For both A and B, the vertices at the $(j-1)-$st level are exactly $r_1 r_2 \cdots r_{j-1}$. Moreover f' maps vertices of the $(j-1)-$st level having the same choices for the coordinates in $A(j)$ into vertices that still have the same choices for the coordinates in $A(j)$, since f' is an automorphism of the poset block structure. Now for each possible ancestral situation $a_j \in \Delta^j$ for the vertices of the $(j-1)-$st level of A, we put $f_j(a_j) = g_j^A \in Sym(\Delta_j)$, where g_j^A maps the r_j elements starting from those vertices into the r_j elements in B starting from the image of those vertices by f'.

Analogously for each possible ancestral situation $b_j \in \Delta^j$ for the vertices of the $(j-1)-$st level of B.

If $a_j = b_j$, then f_j has to be defined has $f_j(a_j) = g_j^{AB} \in Sym(\Delta_j)$, where g_j^{AB} maps the r_j elements in A into the r_j elements of B and viceversa.

If we put $f'' = (1, \dots, 1, f_j, 1, \dots, 1)$, then the composition of f' and f'' gives the automorphism f required. $\square$

Now let K be the stabilizer of a fixed substructure. We get the following corollary.

COROLLARY 3.15. *(F, K) is a symmetric Gelfand pair.*

CHAPTER 2

Markov Chains

In this second chapter, we will change our point of view: we will leave the Group Theory to get a probabilistic approach. In particular, we will introduce some special Markov chains defined on finite sets: the associated spectral analysis will give interesting results, since the eigenspaces obtained will coincide with the irreducible submodules that one gets considering the action of a particular group on the space of the functions defined on the same set.

1. Reversible Markov Chains: general properties

In this section we recall some fundamental facts about finite Markov chains, that we will frequently use later. Our main source is [**16**].

Consider a finite set X, with $|X| = m$. Let $P = (p(x,y))_{x,y\in X}$ be a stochastic matrix of size m whose rows and columns are indexed by the elements of X, so that

$$\sum_{x\in X} p(x_0, x) = 1,$$

for every $x_0 \in X$. Consider the Markov chain on X with transition matrix P. We will use the notation P to indicate the Markov chain too.

Definition 1.1. *A* ***probability measure*** *(or distribution) on X is a function $\nu : X \to [0,1]$ such that $\sum_{x\in X} \nu(x) = 1$. It is called strict if $\nu(x) > 0$ for every $x \in X$.*

Definition 1.2. *The Markov chain P is* ***reversible*** *if there exists a strict probability measure π on X such that*

$$\pi(x)p(x,y) = \pi(y)p(y,x),$$

for all $x, y \in X$.

We will say that P and π are in detailed balance. For a complete treatment about these topics see [**2**].

Define a scalar product on $L(X) = \{f : X \longrightarrow \mathbb{C}\}$ in the following way:

$$\langle f_1, f_2\rangle_\pi = \sum_{x\in X} f_1(x)\overline{f_2(x)}\pi(x), \tag{18}$$

for all $f_1, f_2 \in L(X)$. Moreover, let P be the linear operator on $L(X)$ defined as:

$$(Pf)(x) = \sum_{y \in X} p(x,y) f(y). \tag{19}$$

PROPOSITION 1.3. *P and π are in detailed balance if and only if P is self-adjoint with respect to the scalar product $\langle \cdot, \cdot \rangle_\pi$.*

Proof. Suppose that P and π are in detailed balance and let $f_1, f_2 \in L(X)$. One has:

$$\begin{aligned}
\langle Pf_1, f_2 \rangle_\pi &= \sum_{x \in X} \left(\sum_{y \in X} p(x,y) f_1(y) \right) \overline{f_2(x)} \pi(x) \\
&= \sum_{x \in X} \sum_{y \in X} \pi(x) p(x,y) f_1(y) \overline{f_2(x)} \\
&= \sum_{x \in X} \sum_{y \in X} \pi(y) p(y,x) f_1(y) \overline{f_2(x)} \\
&= \langle f_1, Pf_2 \rangle_\pi.
\end{aligned}$$

Conversely, if we suppose that P is self-adjoint with respect to the scalar product $\langle \cdot, \cdot \rangle_\pi$, we get:

$$\pi(x) p(x,y) = \langle P\delta_y, \delta_x \rangle_\pi = \langle \delta_y, P\delta_x \rangle_\pi = \pi(y) p(y,x),$$

where, for every $x \in X$, the Dirac function δ_x is defined as:

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise.} \end{cases}$$

□

The following lemma gives a fundamental characterization of the spectrum of stochastic matrices.

LEMMA 1.4. *Let P be a stochastic matrix. Then 1 is always an eigenvalue of P. Moreover, if λ is another eigenvalue, then $|\lambda| \leq 1$.*

Proof. Let 1_X the function such that $1_X(x) = 1$, for all $x \in X$. Then $P1_X = 1_X$ and so 1 is an eigenvalue. Now let λ be another eigenvalue of P. Choose $x \in X$ such that $|f(x)| \geq |f(y)|$ for all $y \in X$ (it is possible since X is a finite set). Then

$$\begin{aligned}
|\lambda f(x)| &= |Pf(x)| = |\sum_{y \in X} p(x,y) f(y)| \leq \sum_{y \in X} p(x,y) |f(y)| \\
&\leq |f(x)| \sum_{y \in X} p(x,y) = |f(x)|,
\end{aligned}$$

which implies the assertion. □

Moreover it is known that, under the hypothesis that P is in detailed balance with π, it can be diagonalized over the reals.

Let λ_z be the eigenvalues of the matrix P, for every $z \in X$, with $\lambda_{z_0} = 1$. Then there exists an invertible unitary real matrix $U = (u(x,y))_{x,y\in X}$ such that $PU = U\Delta$, where $\Delta = (\lambda_x \delta_x(y))_{x,y\in X}$ is the diagonal matrix whose entries are the eigenvalues of P. This equation gives, for all $x, z \in X$,

$$\sum_{y\in X} p(x,y)u(y,z) = u(x,z)\lambda_z. \tag{20}$$

Moreover, we have $U^T DU = I$, where $D = (\pi(x)\delta_x(y))_{x,y\in X}$ is the diagonal matrix of coefficients of π. This second equation gives, for all $y, z \in X$,

$$\sum_{x\in X} u(x,y)u(x,z)\pi(x) = \delta_y(z). \tag{21}$$

Hence, the first equation tells us that each column of U is an eigenvector of P, the second one tells us that these columns are orthogonal with respect to the product $\langle\cdot,\cdot\rangle_\pi$.

If the spectral analysis is given, one can deduce the $k-$**step transition probability**, following the next proposition.

Proposition 1.5. *The $k-$th step transition probability is given by*

$$p^{(k)}(x,y) = \pi(y)\sum_{z\in X} u(x,z)\lambda_z^k u(y,z), \tag{22}$$

for all $x, y \in X$.

Proof. The proof is a consequence of (20) and (21). In fact, the matrix $U^T D$ is the inverse of U, so that $UU^T D = I$. This means

$$\sum_{y\in X} u(x,y)u(z,y) = \frac{1}{\pi(z)}\Delta_z(x).$$

From the equation $PU = U\Delta$ we get $P = U\Delta U^T D$, which gives

$$p(x,y) = \pi(y)\sum_{z\in X} u(x,z)\lambda_z u(y,z).$$

Iterating this argument we get

$$P^k = U\Delta^k U^T D,$$

which is the assertion. $\square$

Definition 1.6. *Let P be a stochastic matrix. P is* ***ergodic*** *if there exists $n_0 \in \mathbb{N}$ such that*

$$p^{(n_0)}(x,y) > 0, \quad \textit{for all } x, y \in X.$$

In order to study the ergodicity property, it is useful to recall that there exists a correspondence between reversible Markov chains and weighted graphs.

Definition 1.7. *A weight on a graph* $\mathcal{G} = (X, E)$ *is a function* $w : X \times X \longrightarrow [0, +\infty)$ *such that*

(1) $w(x, y) = w(y, x)$*;*
(2) $w(x, y) > 0$ *if and only if* $x \sim y$*.*

If $\mathcal{G}$ is a weighted graph, a stochastic matrix $P = (P(x, y))_{x,y\in X}$ on X can be associated with w by setting

$$p(x, y) = \frac{w(x, y)}{W(x)},$$

with $W(x) = \sum_{z\in X} w(x, z)$. The corresponding Markov chain is called the **random walk** on $\mathcal{G}$. It is easy to prove that the matrix P is in detailed balance with the distribution π defined, for every $x \in X$, as

$$\pi(x) = \frac{W(x)}{W},$$

with $W = \sum_{z\in X} W(z)$. Moreover, π is strictly positive if X does not contain isolated vertices.

The inverse construction can be performed. Namely, if we have a transition matrix P on X which is in detailed balance with the probability π, then we can define a weight w as $w(x, y) = \pi(x)p(x, y)$. This definition guarantees the symmetry of w and one gets a weighted graph by setting $E = \{\{x, y\} : w(x, y) > 0\}$.

There exist some interesting relations between the weighted graph associated with a transition matrix P and its spectrum. In fact, it is easy to prove that the multiplicity of the eigenvalue 1 of P equals the number of connected components of $\mathcal{G}$, as the following proposition shows.

Proposition 1.8. *Let* $\mathcal{G} = (X, E, w)$ *be a finite weighted graph. Then the multiplicity of the eigenvalue 1 of the transition matrix* P *equals the number of connected components of* $\mathcal{G}$*.*

Proof. By definition of the Markov operator P, it is obvious that if a function $f \in L(X)$ is constant in each connected component, then $Pf = f$.

Conversely, suppose $Pf = f$, with f real valued and non identically zero. Let $X_0 \subset X$ a connected component of $\mathcal{G}$ and let $x_0 \in X_0$ be such that $|f(x_0)| \geq |f(y)|$, for all $y \in X_0$. Up to replace f by $-f$, we can suppose $f(x_0) \geq 0$. We have $f(x_0) = \sum_{y\in X_0} p(x_0, y)f(y)$. Since $\sum_{y\in X_0} p(x_0, y) = 1$, we get

$$\sum_{y\in X_0} p(x_0, y)(f(x_0) - f(y)) = 0.$$

Since $p(x_0, y) \geq 0$ and $f(x_0) \geq f(y)$ for all $y \in X_0$, we deduce $f(y) = f(x_0)$ for all $y \sim x_0$. Consider now any vertex $z \in X_0$: by definition of X_0, there exists a path $p = (x_0, x_1, \ldots, x_n = z)$ connecting x_0 to z.

We have proven above that $f(x_1) = f(x_0) \geq f(y)$ for all $y \in X_0$. Iterating the same argument one gets

$$f(x_0) = f(x_1) = \cdots = f(x_n) = f(z)$$

and so f is constant on the connected components of $\mathcal{G}$, what completes the proof. $\square$

Definition 1.9. *A graph $\mathcal{G} = (X, E)$ is bipartite if there exists a nontrivial partition $X = X_1 \coprod X_2$ of its vertices such that $E \subseteq \{\{x_1, x_2\} : x_1 \in X_1,\ x_2 \in X_2\}$, i.e. every edge joins a vertex in X_1 with a vertex in X_2.*

The following propositions hold (see [**16**] for the proof).

Proposition 1.10. *Let $\mathcal{G} = (X, E, w)$ be a finite connected weighted graph and denote by P the corresponding transition matrix. Then the following are equivalent:*

1. *$\mathcal{G}$ is bipartite;*
2. *the spectrum $\sigma(P)$ is symmetric, i.e. $\lambda \in \sigma(P)$ if and only if $-\lambda \in \sigma(P)$;*
3. *$-1 \in \sigma(P)$.*

Proposition 1.11. *Let $\mathcal{G} = (X, E)$ be a finite graph. Then the following conditions are equivalent:*

1. *$\mathcal{G}$ is connected and not bipartite;*
2. *for every weight function on X, the associated transition matrix P is ergodic.*

So we can conclude that a reversible transition matrix P is ergodic if and only if the eigenvalue 1 has multiplicity one and -1 is not an eigenvalue.

2. Crested product of Markov Chains

In this section (see also [**18**]) we introduce a particular product of Markov chains defined on different sets. This idea is inspired to the definition of crested product for association schemes (see Section 4 of this chapter) given in [**4**]. In [**18**], we refer to it as the first crested product.

We need the following definition.

Definition 2.1. *A stochastic matrix P on a set X is **irreducible** if, for every $x_1, x_2 \in X$, there exists $n = n(x_1, x_2)$ such that $p^{(n)}(x_1, x_2) > 0$.*

In particular, it is clear that the irreducibility is equivalent to require that the graph associated with the probability P is connected, so that the eigenvalue 1 has multiplicity one.

Now for every $i = 1, \ldots, n$ let X_i be a finite set, with $|X_i| = m_i$, so that we can identify X_i with the set $\{0, 1, \ldots, m_i - 1\}$. Let P_i be an irreducible Markov chain on X_i and let p_i be the transition probability associated with P_i. Moreover, assume that p_i is in detailed balance with the strict probability measure σ_i on X_i, so that

$$\sigma_i(x)p_i(x,y) = \sigma_i(y)p_i(y,x),$$

for all $x, y \in X_i$.

Consider the cartesian product $X_1 \times \cdots \times X_n$. Let $\{1, \ldots, n\} = C \coprod N$ be a partition of the set $\{1, \ldots, n\}$ and let $p_1^0, p_2^0, \ldots, p_n^0$ be real numbers such that $p_i^0 > 0$ for every $i = 1, \ldots, n$ and $\sum_{i=1}^n p_i^0 = 1$.

Definition 2.2. *The* ***crested product*** *of Markov chains P_i's with respect to the partition $\{1, \ldots, n\} = C \coprod N$ is the Markov chain on the product $X_1 \times \cdots \times X_n$ whose transition matrix is*

$$\begin{aligned} P \; = \; & \sum_{i \in C} p_i^0 \left(I_1 \otimes \cdots \otimes I_{i-1} \otimes P_i \otimes I_{i+1} \otimes \cdots \otimes I_n \right) \\ & + \sum_{i \in N} p_i^0 \left(I_1 \otimes \cdots \otimes I_{i-1} \otimes P_i \otimes J_{i+1} \otimes \cdots \otimes J_n \right), \end{aligned} \tag{23}$$

where I_i denotes the identity matrix of size m_i and J_i denotes the uniform matrix on X_i, i.e. the matrix of size m_i all of whose entries are $\frac{1}{m_i}$, so that

$$J_i = \frac{1}{m_i} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix}.$$

In other words, we choose an index $i \in \{1, \ldots, n\}$ with probability p_i^0. If $i \in C$, then P acts on the $i-$th coordinate by the matrix P_i and fixes the remaining coordinates; if $i \in N$, then P fixes the coordinates corresponding to the indices $\{1, \ldots, i-1\}$, acts on the $i-$th coordinate by the matrix P_i and changes uniformly the remaining ones.

From (23) it follows that, for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in X_1 \times \cdots \times X_n$, the transition probability p associated with P is given by

$$p((x_1, \ldots, x_n), (y_1, \ldots, y_n)) =$$

$$\sum_{i \in C} p_i^0 (\delta_1(x_1, y_1) \cdots \delta_{i-1}(x_{i-1}, y_{i-1}) p_i(x_i, y_i) \delta_{i+1}(x_{i+1}, y_{i+1}) \cdots \delta_n(x_n, y_n))$$

$$+\sum_{i\in N} p_i^0 \left(\frac{\delta_1(x_1,y_1)\cdots\delta_{i-1}(x_{i-1},y_{i-1})p_i(x_i,y_i)}{\prod_{j=i+1}^n m_j} \right),$$

where δ_i is defined by

$$\delta_i(x_i,y_i) = \begin{cases} 1 & \text{if } x_i = y_i, \\ 0 & \text{otherwise.} \end{cases}$$

We want to investigate the spectral analysis of the operator P. We recall that the following isomorphism holds:

$$L(X_1 \times \cdots \times X_n) \cong \bigotimes_{i=1}^n L(X_i),$$

where $(f_1 \otimes \cdots \otimes f_n)(x_1,\ldots,x_n) := f_1(x_1)f_2(x_2)\cdots f_n(x_n)$, with $f_i \in L(X_i)$ and $x_i \in X_i$, for every $i = 1,\ldots,n$.

Assume that, for every $i = 1,\ldots,n$, the following spectral decomposition holds:

$$L(X_i) = \bigoplus_{j_i=0}^{r_i} V_{j_i}^i,$$

where $V_{j_i}^i$ is an eigenspace for P_i with associated eigenvalue λ_{j_i} and whose dimension is m_{j_i}. Observe that the hypothesis of reversibility implies that λ_{j_i} is real and that the hypothesis of irreducibility implies that the multiplicity of 1 as eigenvalue is one.

Now set $N = \{i_1,\ldots,i_l\}$ and $C = \{c_1,\ldots,c_h\}$, with $h + l = n$ and such that $i_1 < \ldots < i_l$ and $c_1 < \ldots < c_h$.

Theorem 2.3. *The probability P defined above is reversible if and only if P_k is symmetric for every $k > i_1$. If this is the case, P is in detailed balance with the strict probability measure π on $X_1 \times \cdots \times X_n$ given by*

$$\pi(x_1,\ldots,x_n) = \frac{\sigma_1(x_1)\sigma_2(x_2)\cdots\sigma_{i_1}(x_{i_1})}{m_{i_1+1}\cdots m_n}.$$

Proof. Consider the elements $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$ belonging to $X_1 \times \cdots \times X_n$. First, we want to prove that the condition $\sigma_k = \frac{1}{m_k}$, for every $k > i_1$, is sufficient. Let $k \in \{1,\ldots,n\}$ such that $x_i = y_i$ for every $i = 1,\ldots,k-1$ and $x_k \neq y_k$. Suppose $k < i_1$. Then we have

$$p(x,y) = p_k^0\left(p_k(x_k,y_k)\delta_{k+1}(x_{k+1},y_{k+1})\cdots\delta_n(x_n,y_n)\right).$$

If $x_i = y_i$ for every $i = k+1,\ldots,n$, we get

$$\begin{aligned} \pi(x)p(x,y) &= \sigma_1(x_1)\cdots\sigma_k(x_k)\cdots\sigma_{i_1}(x_{i_1})p_k^0\frac{p_k(x_k,y_k)}{m_{i_1+1}\cdots m_n} \\ &= \sigma_1(y_1)\cdots\sigma_k(y_k)\cdots\sigma_{i_1}(y_{i_1})p_k^0\frac{p_k(y_k,x_k)}{m_{i_1+1}\cdots m_n} \\ &= \pi(y)p(y,x), \end{aligned}$$

since $\sigma_k(x_k)p_k(x_k, y_k) = \sigma_k(y_k)p_k(y_k, x_k)$. If the condition $x_i = y_i$ is not satisfied for every $i = k+1, \ldots, n$, then the equality $\pi(x)p(x, y) = \pi(y)p(y, x) = 0$ easily follows.

If $k = i_1$, then we get

$$p(x, y) = p^0_{i_1}\left(p_{i_1}(x_{i_1}, y_{i_1})\frac{1}{m_{i_1+1}\cdots m_n}\right)$$

and so

$$\begin{aligned}\pi(x)p(x, y) &= \sigma_1(x_1)\cdots\sigma_{i_1}(x_{i_1})p^0_{i_1}\frac{p_{i_1}(x_{i_1}, y_{i_1})}{m^2_{i_1+1}\cdots m^2_n}\\ &= \sigma_1(y_1)\cdot\cdots\cdot\sigma_{i_1}(y_{i_1})p^0_{i_1}\frac{p_{i_1}(y_{i_1}, x_{i_1})}{m^2_{i_1+1}\cdots m^2_n}\\ &= \pi(y)p(y, x),\end{aligned}$$

since $\sigma_{i_1}(x_{i_1})p_{i_1}(x_{i_1}, y_{i_1}) = \sigma_{i_1}(y_{i_1})p_{i_1}(y_{i_1}, x_{i_1})$.

In the case $k > i_1$, we have

$$p(x, y) = \sum_{i\in N, i\leq k} p^0_i\frac{p_i(x_i, y_i)}{m_{i+1}\cdots m_n}$$

and so

$$\begin{aligned}\pi(x)p(x, y) &= \frac{\sigma_1(x_1)\cdots\sigma_{i_1}(x_{i_1})}{m_{i_1+1}\cdots m_n}\sum_{i\in N, i\leq k} p^0_i\frac{p_i(x_i, y_i)}{m_{i+1}\cdots m_n}\\ &= \frac{\sigma_1(y_1)\cdots\sigma_{i_1}(y_{i_1})}{m_{i_1+1}\cdots m_n}\sum_{i\in N, i\leq k} p^0_i\frac{p_i(y_i, x_i)}{m_{i+1}\cdots m_n}\\ &= \pi(y)p(y, x).\end{aligned}$$

In fact, the terms corresponding to an index $i < k$ satisfy $p_i(x_i, y_i) = p_i(y_i, x_i)$ since $x_i = y_i$, the term corresponding to the index k satisfies $p_k(x_k, y_k) = p_k(y_k, x_k)$ since the equality

$$p_k(x_k, y_k) = p_k(y_k, x_k)$$

holds by hypothesis.

Now we want to prove that the condition $\sigma_k = \frac{1}{m_k}$, for every $k > i_1$, is also necessary. Suppose that the equality $\pi(x)p(x, y) = \pi(y)p(y, x)$ holds. By hypothesis of irreducibility we can consider two elements $x^0, y^0 \in X_1 \times \cdots \times X_n$ such that $x^0_{i_1} \neq y^0_{i_1}$ and with the property that $p_{i_1}(x^0_{i_1}, y^0_{i_1}) \neq 0$. Now we have

$$\pi(x^0)p(x^0, y^0) = \pi(y^0)p(y^0, x^0) \Leftrightarrow \pi(x^0)p_{i_1}(x^0_{i_1}, y^0_{i_1}) = \pi(y^0)p_{i_1}(y^0_{i_1}, x^0_{i_1}).$$

This gives

$$\frac{\pi(x^0)}{\pi(y^0)} = \frac{p_{i_1}(y^0_{i_1}, x^0_{i_1})}{p_{i_1}(x^0_{i_1}, y^0_{i_1})} = \frac{\sigma_{i_1}(x^0_{i_1})}{\sigma_{i_1}(y^0_{i_1})}.$$

Consider now the element $x = (x_1^0, \ldots, x_{i_1}^0, y_{i_1+1}^0, \ldots, y_n^0)$. The equality $\pi(x)p(x, y^0) = \pi(y^0)p(y^0, x)$ implies

$$\frac{\pi(x)}{\pi(y^0)} = \frac{p_{i_1}(y_{i_1}^0, x_{i_1}^0)}{p_{i_1}(x_{i_1}^0, y_{i_1}^0)} = \frac{\sigma_{i_1}(x_{i_1}^0)}{\sigma_{i_1}(y_{i_1}^0)}.$$

So we get $\pi(x^0) = \pi(x)$, i.e. the probability π does not depend from the coordinates $i_1+1, \ldots, n$. Set now $x' = (x_1^0, \ldots, x_{i_1}^0, \ldots, x_{k-1}^0, x_k, \ldots, x_n)$. The equality $\pi(x^0)p(x^0, x') = \pi(x')p(x', x^0)$ gives

$$\pi(x^0)\left(\sum_{j\in N, j\leq k} p_j^0(p_j(x_j^0, x_j'))\right) = \pi(x')\left(\sum_{j\in N, j\leq k} p_j^0(p_j(x_j', x_j^0))\right).$$

Since the probability π does not depend from the coordinates $i_1 + 1, \ldots, n$, we get $p_k(x_k^0, x_k') = p_k(x_k', x_k^0)$. This implies $\sigma_k(x_k') = \sigma_k(x_k^0)$ and so the hypothesis of irreducibility guarantees that σ_k is uniform on X_k. This completes the proof. □

From now on, suppose that the matrix P_k is symmetric for every $k > i_1$. The following theorem describes the eigenspaces of P.

Theorem 2.4. *The eigenspaces of the operator P are given by*

- $W^1 \otimes \cdots \otimes W^{k-1} \otimes V_{j_k}^k \otimes V_0^{k+1} \otimes V_0^{k+2} \otimes \cdots \otimes V_0^n$, *with $j_k \neq 0$, for $k \in \{i_1 + 1, \ldots, n\}$ and where*

$$W^i = \begin{cases} L(X_i) & \text{if } i \in N, \\ V_{j_i}^i, \quad j_i = 0, \ldots, r_i & \text{if } i \in C, \end{cases}$$

with eigenvalue

$$\sum_{i\in C: i<k} p_i^0\lambda_{j_i} + p_k^0\lambda_{j_k} + \sum_{i>k} p_i^0.$$

- $V_{j_1}^1 \otimes \cdots \otimes V_{j_{i_1-1}}^{i_1-1} \otimes V_{j_{i_1}}^{i_1} \otimes V_0^{i_1+1} \otimes \cdots \otimes V_0^n$, *with $j_t = 0, \ldots, r_t$, for every $t = 1, \ldots, i_1$, with eigenvalue*

$$\sum_{i=1}^{i_1} p_i^0\lambda_{j_i} + \sum_{i=i_1+1}^{n} p_i^0.$$

Proof. Fix an index $k \in \{i_1 + 1, i_1 + 2, \ldots, n\}$ and consider the function φ in the space

$$W^1 \otimes \cdots \otimes W^{k-1} \otimes V_{j_k}^k \otimes V_0^{k+1} \otimes V_0^{k+2} \otimes \cdots \otimes V_0^n,$$

with $j_k \neq 0$ and

$$W^i = \begin{cases} L(X_i) & \text{if } i \in N, \\ V_{j_i}^i, \quad j_i = 0, \ldots, r_i & \text{if } i \in C, \end{cases}$$

so that $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_{k-1} \otimes \varphi_k \otimes \varphi_{k+1} \otimes \cdots \otimes \varphi_n$ with $\varphi_i \in W^i$ for $i = 1, \ldots, k-1$, $\varphi_k \in V^k_{j_k}$ and $\varphi_l \in V^l_0$ for $l = k+1, \ldots, n$. Set $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, then

$$
\begin{aligned}
(P\varphi)(x) &= \sum_y p(x,y)\varphi(y) \\
&= \sum_y \Bigg(\sum_{i \in C} p_i^0 \delta_1(x_1,y_1)\cdots\delta_{i-1}(x_{i-1},y_{i-1}) p_i(x_i,y_i) \delta_{i+1}(x_{i+1},y_{i+1}) \cdots \delta_n(x_n,y_n) \\
&+ \sum_{i \in N} p_i^0 \delta_1(x_1,y_1)\cdots\delta_{i-1}(x_{i-1},y_{i-1}) p_i(x_i,y_i) \frac{1}{m_{i+1}} \cdots \frac{1}{m_n} \Bigg) \\
&\times \varphi_1(y_1)\cdots\varphi_{k-1}(y_{k-1})\varphi_k(y_k)\varphi_{k+1}(y_{k+1})\cdots\varphi_n(y_n) \\
&= \sum_{i\in C,\ i\le k} \Bigg(\sum_{y_i} p_i^0 p_i(x_i,y_i)\varphi_i(y_i) \Bigg) \varphi_1(x_1)\cdots\varphi_{i-1}(x_{i-1})\varphi_{i+1}(x_{i+1})\cdots\varphi_n(x_n) \\
&+ \sum_{i\in C,\ i> k} \Bigg(\sum_{y_i} p_i^0 p_i(x_i,y_i)\varphi_i(y_i) \Bigg) \varphi_1(x_1)\cdots\varphi_{i-1}(x_{i-1})\varphi_{i+1}(x_{i+1})\cdots\varphi_n(x_n) \\
&+ \sum_{i\in N,\ i> k} \Bigg(\sum_{y_i,\ldots,y_n} p_i^0 p_i(x_i,y_i)\frac{1}{m_{i+1}}\cdots\frac{1}{m_n}\varphi_i(y_i)\cdots\varphi_n(y_n) \Bigg) \varphi_1(x_1)\cdots\varphi_{i-1}(x_{i-1}) \\
&+ \chi_N(k) \sum_{y_k,\ldots,y_n} p_k^0 p_k(x_k,y_k)\frac{1}{m_{k+1}}\cdots\frac{1}{m_n}\varphi_1(x_1)\cdots\varphi_{k-1}(x_{k-1})\varphi_k(y_k)\cdots\varphi_n(y_n) \\
&= \sum_{i\in C,\ i\le k} p_i^0\lambda_{j_i}\varphi(x) + \sum_{i\in C,\ i>k} p_i^0\cdot 1\cdot\varphi(x) \\
&+ \sum_{i\in N,\ i> k} \Bigg(\sum_{y_i} p_i^0 p_i(x_i,y_i)\varphi_i(y_i) \Bigg) \varphi_1(x_1)\cdots\varphi_{i-1}(x_{i-1})\varphi_{i+1}(x_{i+1})\cdots\varphi_n(x_n) \\
&+ \chi_N(k)\sum_{y_k} p_k^0 p_k(x_k,y_k)\varphi_1(x_1)\cdots\varphi_{k-1}(x_{k-1})\varphi_k(y_k)\varphi_{k+1}(x_{k+1})\cdots\varphi_n(x_n) \\
&= \sum_{i\in C,\ i\le k} p_i^0\lambda_{j_i}\varphi(x) + \sum_{i\in C,\ i>k} p_i^0\varphi(x) + \sum_{i\in N,\ i>k} p_i^0\varphi(x) + \chi_N(k)p_k^0\lambda_{j_k}\varphi(x) \\
&= \Bigg(\sum_{i\in C,\ i<k} p_i^0\lambda_{j_i} + p_k^0\lambda_{j_k} + \sum_{i>k} p_i^0 \Bigg)\varphi(x),
\end{aligned}
$$

where χ_N is the characteristic function of N. Note that in this case the summands corresponding to the indices $i < k$, $i \in N$, are equal to 0 since we have supposed $j_k \neq 0$.

Consider now the function φ in the space

$$V^1_{j_1} \otimes \cdots V^{i_1-1}_{j_{i_1-1}} \otimes V^{i_1}_{j_{i_1}} \otimes V^{i_1+1}_0 \otimes \cdots \otimes V^n_0,$$

with $j_t = 0, \ldots, r_t$, for every $t = 1, \ldots, i_1$. In this case we have

$$\begin{aligned}
(P\varphi)(x) &= \sum_y p(x,y)\varphi(y) \\
&= \sum_{i\in C,\ i<i_1} \left(\sum_{y_i} p_i^0 p_i(x_i,y_i)\varphi_i(y_i)\right)\varphi_1(x_1)\cdots\varphi_{i-1}(x_{i-1})\varphi_{i+1}(x_{i+1})\cdots\varphi_n(x_n) \\
&+ \sum_{i\in C,\ i>i_1} \left(\sum_{y_i} p_i^0 p_i(x_i,y_i)\varphi_i(y_i)\right)\varphi_1(x_1)\cdots\varphi_{i-1}(x_{i-1})\varphi_{i+1}(x_{i+1})\cdots\varphi_n(x_n) \\
&+ \sum_{i\in N,\ i>i_1} \left(\sum_{y_i,\ldots,y_n} p_i^0 p_i(x_i,y_i)\frac{1}{m_{i+1}}\cdots\frac{1}{m_n}\varphi_i(y_i)\cdots\varphi_n(x_n)\right)\varphi_1(x_1)\cdots\varphi_{i-1}(x_{i-1}) \\
&+ \sum_{y_{i_1},\ldots,y_n} \left(p_{i_1}^0 p_{i_1}(x_{i_1},y_{i_1})\frac{1}{m_{i_1+1}}\cdots\frac{1}{m_n}\varphi_{i_1}(y_{i_1})\cdots\varphi_n(x_n)\right)\varphi_1(x_1)\cdots\varphi_{i_1-1}(x_{i_1-1}) \\
&= \sum_{i\in C,\ i<i_1} p_i^0\lambda_{j_i}\varphi(x) + \sum_{i\in C,\ i>i_1} p_i^0\varphi(x) + \sum_{i\in N,\ i>i_1} p_i^0\varphi(x) + p_{i_1}^0\lambda_{j_{i_1}}\varphi(x) \\
&= \left(\sum_{i=1}^{i_1} p_i^0\lambda_{j_i} + \sum_{i=i_1+1}^{n} p_i^0\right)\varphi(x).
\end{aligned}$$

Observe that, by computing the sum of the dimensions of these eigenspaces, one gets

$$\sum_{k=i_1+1}^{n} m_1\cdots m_{k-1}(m_k-1) + m_1m_2\cdots m_{i_1} = m_1m_2\cdots m_n,$$

which is just the dimension of the space $X_1\times\cdots\times X_n$. $\square$

Remark. The expression of the eigenvalues of P given in the previous theorem tells us that if P_i is ergodic for every $i = 1, \ldots, n$, then also P is ergodic, since the eigenvalue 1 is obtained with multiplicity one and the eigenvalue -1 can never be obtained.

Now we can show the matrices U, D and Δ associated with P. For every i, let U_i, D_i and Δ_i be the matrices of eigenvectors, of the coefficients of σ_i and of eigenvalues for the probability P_i, respectively. The following proposition holds.

Proposition 2.5. *Let P be the crested product defined in (23). Then we have:*

- $U = \sum_{k=i_1+1}^{n} M_1 \otimes \cdots \otimes M_{k-1} \otimes (U_k - A_k) \otimes A_{k+1} \otimes \cdots \otimes A_n$ $+U_1 \otimes U_2 \otimes \cdots \otimes U_{i_1} \otimes A_{i_1+1} \otimes \cdots \otimes A_n$, *with*

$$M_i = \begin{cases} I_i^{\sigma_i - norm} & \text{if } i \in N \\ U_i & \text{if } i \in C \end{cases}$$

where

$$I_i^{\sigma_i - norm} = \begin{pmatrix} \frac{1}{\sqrt{\sigma_i(0)}} & & & \\ & \frac{1}{\sqrt{\sigma_i(1)}} & & \\ & & \ddots & \\ & & & \frac{1}{\sqrt{\sigma_i(m_i-1)}} \end{pmatrix}.$$

We denote A_i the matrix of size m_i whose entries on the first column are all 1 and the remaining ones are 0.

- $D = \bigotimes_{i=1}^{n} D_i$.
- $\Delta = \sum_{i \in C} p_i^0 (I_1 \otimes \cdots \otimes I_{i-1} \otimes \Delta_i \otimes I_{i+1} \otimes \cdots \otimes I_n)$ $+ \sum_{i \in N} p_i^0 \left(I_1 \otimes \cdots \otimes I_{i-1} \otimes \Delta_i \otimes J_{i+1}^{diag} \otimes \cdots \otimes J_n^{diag}\right)$, *where* J_i^{diag} *is the diagonal matrix of size m_i given by*

$$J_i^{diag} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}.$$

Proof. The expression of the matrix U, whose columns are an orthonormal basis of eigenvectors for P, is a consequence of Theorem 2.4. In order to get the diagonal matrix D, whose entries are the coefficients of π, it suffices to consider the tensor product of the corresponding matrices associated with the distribution σ_i, for every $i = 1, \ldots, n$, as it follows from Theorem 2.3. Finally, to get the matrix Δ of eigenvalues of P it suffices to replace, in the expression of the matrix P in (23), the matrix P_i by Δ_i and the matrix J_i by the corresponding diagonal matrix J_i^{diag}. $\square$

Remark. Observe that another matrix U' of eigenvectors for P is given by $U' = \bigotimes_{i=1}^{n} U_i$. The matrix U that we have given above seems to be more useful whenever one wants to compute the $k-$th step transition probability $p^{(k)}(0, x)$ using Formula (22), since it contains a greater number of 0 in the first row with respect to U' and so a small number of terms in the sum is nonzero. Here we denote 0 the element of $X_1 \times \cdots \times X_n$ given by the $n-$tuple $(0, \ldots, n)$.

Consider $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in X, where we set $X = X_1 \times \cdots \times X_n$. From (22) and Proposition 2.5, we get

$$\begin{aligned}
p^{(k)}(x,y) &= \pi(y)\Bigg[\sum_{z\in X}\left(\sum_{r=i_1+1}^{n} m_1(x_1,z_1)\cdots m_{r-1}(x_{r-1},z_{r-1})(u_r-a_r)(x_r,z_r)\right.\\
&\times\ a_{r+1}(x_{r+1},z_{r+1})\cdots a_n(x_n,z_n)\\
&+\ \left. u_1(x_1,z_1)\cdots u_{i_1}(x_{i_1},z_{i_1})a_{i_1+1}(x_{i_1+1},z_{i_1+1})\cdots a_n(x_n,z_n)\right)\lambda_z^k\\
&\times\ \left(\sum_{r=i_1+1}^{n} m_1(y_1,z_1)\cdots m_{r-1}(y_{r-1},z_{r-1})(u_r-a_r)(y_r,z_r)\right.\\
&\times\ a_{r+1}(y_{r+1},z_{r+1})\cdots a_n(y_n,z_n)\\
&+\ \left. u_1(y_1,z_1)\cdots u_{i_1}(y_{i_1},z_{i_1})a_{i_1+1}(y_{i_1+1},z_{i_1+1})\cdots a_n(y_n,z_n)\right)\Bigg],
\end{aligned}$$

where m_i, u_i, a_i are the probabilities associated, respectively, with the matrices M_i, U_i, A_i occurring in Proposition 2.5.

2.1. The crossed product. The crossed product of the Markov chains P_i's can be obtained as a particular case of the crested product, by choosing a special partition of $\{1,\ldots,n\}$, namely $C=\{1,\ldots,n\}$ and $N=\emptyset$. It is also called direct product. The analogous case for product of groups has been studied in [**25**].

In this case, we have

$$P=\sum_{i=1}^{n} p_i^0\,(I_1\otimes\cdots\otimes I_{i-1}\otimes P_i\otimes I_{i+1}\otimes\cdots\otimes I_n)\,; \tag{24}$$

the corresponding transition probability is

$$p((x_1,\ldots,x_n),(y_1,\ldots,y_n))=\sum_{i=1}^{n} p_i^0\delta_1(x_1,y_1)\cdots p_i(x_i,y_i)\cdots\delta_n(x_n,y_n),$$

for all $(x_1,\ldots,x_n),(y_1,\ldots,y_n)\in X_1\times\cdots\times X_n$. This is equivalent to choose the $i-$th coordinate with probability p_i^0 and to change it according with the probability transition P_i. In particular, we get

$$p((x_1,\ldots,x_n),(y_1,\ldots,y_n))=$$

$$\begin{cases} p_i^0 p_i(x_i,y_i) & \text{if } x_j=y_j \text{ for all } j\neq i\\ 0 & \text{otherwise.}\end{cases}$$

In the particular case $X_1=\cdots=X_n=X$, with $p_0^1=\cdots=p_n^0=\frac{1}{n}$, the probability p defines an analogous of the Ehrenfest model (see, for instance, [**15**]), where n is the number of balls and $|X|=m$ is the number of urns. In order to obtain a new configuration, we choose a ball with probability $1/n$ (let it be the $i-$th ball in the urn x_i) and with probability $p_i(x_i,y_i)$ we put it in the urn y_i.

As a consequence of Theorem 2.3, we get that if P_i is in detailed balance with π_i, then P is in detailed balance with the strict probability measure π on $X_1 \times \cdots \times X_n$ defined as

$$\pi(x_1, \ldots, x_n) = \pi_1(x_1)\pi_2(x_2)\cdots\pi_n(x_n).$$

The following proposition studies the spectral theory of the operator P and it is a straightforward consequence of Theorem 2.4.

PROPOSITION 2.6. *Let $\varphi_0^i = 1_{X_i}, \varphi_1^i, \ldots, \varphi_{m_i-1}^i$ be the eigenfunctions of P_i associated with the (not necessarily distinct) eigenvalues $\lambda_0^i = 1, \lambda_1^i, \cdots, \lambda_{m_i-1}^i$, respectively. Then the eigenvalues of P are the numbers*

$$\lambda_I = \sum_{k=1}^{n} p_k^0 \lambda_{i_k}^k,$$

with $I = (i_1, \ldots, i_n) \in \{0, \ldots, m_1 - 1\} \times \cdots \times \{0, \ldots, m_n - 1\}$. The corresponding eigenfunctions are defined as

$$\varphi_I((x_1, \ldots, x_n)) = \varphi_{i_1}^1(x_1)\cdots\varphi_{i_n}^n(x_n).$$

Moreover, the eigenspaces described in Theorem 2.4 become, in the case of the crossed product,

$$V_{j_1}^1 \otimes \cdots \otimes V_{j_n}^n, \tag{25}$$

where $j_i \in \{0, \ldots, r_i\}$ for every $i = 1, \ldots, n$.

As a consequence of Proposition 2.5, in order to get the matrices U, D and Δ associated with P, it suffices to consider the tensor product of the corresponding matrices associated with the probability P_i, for every $i = 1, \ldots, n$. If, for every i, we denote U_i, D_i and Δ_i the matrices of eigenvectors, of the coefficients of π_i and of eigenvalues for the probability P_i, respectively, then we get the following corollary.

COROLLARY 2.7. *Let P be the probability defined in (24). Then we have*

$$\begin{cases} PU = U\Delta \\ U^T D U = I, \end{cases}$$

where $U = \bigotimes_{i=1}^n U_i$, $\Delta = \bigotimes_{i=1}^n \Delta_i$ and $D = \bigotimes_{i=1}^n D_i$.

In particular, we can express the $k-$th step transition probability matrix as

$$P^k = \left(\bigotimes_{i=1}^n U_i\right)\left(\bigotimes_{i=1}^n \Delta_i\right)^k \left(\bigotimes_{i=1}^n U_i\right)^T \left(\bigotimes_{i=1}^n D_i\right).$$

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then we get

$$p^{(k)}(x, y) = \pi(y)\sum_I \varphi_I(x)\lambda_I^k \varphi_I(y) =$$

$$\pi_1(y_1)\cdots\pi_n(y_n)\sum_I \varphi^1_{i_1}(x_1)\cdots\varphi^n_{i_n}(x_n)\left(p^0_1\lambda^1_{i_1}+\cdots+p^0_n\lambda^n_{i_n}\right)^k\varphi^1_{i_1}(y_1)\cdots\varphi^n_{i_n}(y_n),$$

with $I=(i_1,\ldots,i_n)$.

If the matrix P_i is ergodic for every $i=1,\ldots,n$, then also the matrix P is ergodic, since the eigenvalue 1 can be obtained only by choosing $I=(0,\ldots,0)$ and the eigenvalue -1 can never be obtained.

Consider now the action of the symmetric group S_m on the set $X=\{0,1,2,\ldots,m-1\}$. Suppose $m\geq 2$. Choose the element $0\in X$ and consider the subgroup K of S_m stabilizing 0. It is well known that the subgroup K is isomorphic to the symmetric group S_{m-1}, so that we have

$$X\cong S_m/S_{m-1}.$$

Set $L(X)=\{f:X\to\mathbb{C}\}$ and, as usual, consider the action of S_m on $L(X)$ given by

$$(\pi f)(i)=f(\pi^{-1}i),$$

for all $\pi\in S_m$, $f\in L(X)$ and $i\in X$. It is well known (see, for instance, [**33**]) that the representation of S_m on $L(X)$ decomposes into two S_m-irreducible subrepresentations as

$$L(X)=V_0\oplus V_1, \tag{26}$$

where $V_0\cong\mathbb{C}$ is the space of constant functions on X and $V_1=\{f:X\longrightarrow\mathbb{C}:\sum_{i=0}^{m-1}f(i)=0\}$. In particular, we have $dim(V_0)=1$ and $dim(V_1)=m-1$.

Since the decomposition in (26) is multiplicity-free, it is clear that (S_m,S_{m-1}) is a Gelfand pair. The associated spherical functions are:

- $\phi_0\in V_0$, defined as $\phi_0(x)\equiv 1$ for all $i\in X$;
- $\phi_1\in V_1$, defined as

$$\phi_1(i)=\begin{cases}1 & \text{if } i=0\\ \frac{1}{1-m} & \text{otherwise.}\end{cases}$$

In particular, Theorem 1.16 implies that (S_m,S_{m-1}) is a symmetric Gelfand pair.

Observe that this is a particular case of the action of $Aut(T_q)$ on the $q-$ary rooted tree, studied in Chapter 1 of this work. The decomposition given in (26) is a particular case of (2), with $q=m$ and $n=1$.

Consider the cartesian product $X_1\times\cdots\times X_n$. To avoid confusion, suppose $X_1=\cdots=X_n=X$, with $|X|=m$. The symmetric group S_m acts on each factor X and on the space $L(X)$ as described above.

A natural action of the direct product $\underbrace{S_m \times \cdots \times S_m}_{n \text{ times}}$ on the product $X^n = \underbrace{X \times \cdots \times X}_{n \text{ times}}$ is defined by

$$(\pi_1, \ldots, \pi_n)(i_1, \ldots, i_n) = (\pi_1(i_1), \ldots, \pi_n(i_n)),$$

with $\pi_i \in S_m$ and $i_k \in X_k$. If $L(X) = V_0 \bigoplus V_1$ is the decomposition of the space $L(X)$ into S_m-irreducible subrepresentations, then the decomposition of $L(X \times \cdots \times X)$ into $(S_m)^n-$irreducible subrepresentations is given by

$$L(X^n) \cong L(X)^{\otimes^n} \cong \bigoplus_{l_i \in \{0,1\}} \left(\bigotimes_{i=1}^{n} V_{l_i}^i \right).$$

The interesting fact is that the same decomposition can be obtained by the spectral analysis of the operator P defined in (24), by choosing $P_i = J_i$, for every $i = 1, \ldots, n$. In fact, the eigenspaces given in (25) become, in this case,

$$V_{j_1}^1 \otimes \cdots \otimes V_{j_n}^n, \tag{27}$$

where $j_i \in \{0, 1\}$ for every $i = 1, \ldots, n$, since the operator J_i acting on $L(X_i)$ has two eigenspaces, given by the space of constant function V_0^i and the space $V_1^i = \{f : X_i \to \mathbb{C} : \sum_{k=0}^{m-1} f(k) = 0\}$, whose eigenvalues are 1 and 0, respectively.

This particularity will be remarked also in the next section devoted to the nested product: this time, the group which we will refer to will be the wreath product of symmetric groups.

2.2. The nested product. The nested product of the Markov chains P_i's is obtained as a particular case of the crested product, with respect to the partition $\{1, \ldots, n\} = C \coprod N$, with $C = \emptyset$ and $N = \{1, \ldots, n\}$. The term nested comes from the association schemes theory (see, for example, [**3**] and [**4**]).

The formula (23) becomes, in this case,

$$P = \sum_{i=1}^{n} p_i^0 \left(I_1 \otimes \cdots \otimes P_i \otimes J_{i+1} \otimes J_{i+2} \otimes \cdots \otimes J_n \right). \tag{28}$$

Theorem 2.3 tells us that P is reversible if and only if P_k is symmetric, for every $k > 1$, i.e. $\sigma_i \equiv \frac{1}{m_i}$ for every $i = 2, \ldots, n$. In this case, P is in detailed balance with the strict probability measure π on $X_1 \times \cdots \times X_n$ given by

$$\pi(x_1 \ldots, x_n) = \frac{\sigma_1(x_1)}{\prod_{i=2}^{n} m_i}.$$

Let us assume σ_i to be uniform, for every $i = 2, \ldots, n$. The transition probability associated with P is

$$\begin{aligned} p((x_1, \ldots, x_n), (y_1, \ldots, y_n)) &= \frac{p_1^0 p_1(x_1, y_1)}{m_2 m_3 \cdots m_n} \\ &+ \sum_{j=2}^{n-1} \frac{\delta((x_1, \ldots, x_{j-1}), (y_1, \ldots, y_{j-1})) p_j^0 p_j(x_j, y_j)}{m_{j+1} \cdots m_n} \\ &+ \delta((x_1, \ldots, x_{n-1}), (y_1, \ldots, y_{n-1})) p_n^0 p_n(x_n, y_n). \end{aligned}$$

Also in this case we can study the spectral theory of the operator P defined in (28).

Let

$$L(X_i) = \bigoplus_{j_i=0}^{r_i} V_{j_i}^i$$

be the spectral decomposition of $L(X_i)$, for all $i = 1, \ldots, n$ and let $\lambda_0^i = 1, \lambda_1^i, \ldots, \lambda_{r_i}^i$ the distinct eigenvalues of P_i associated with these eigenspaces. From Theorem 2.4 we get the following proposition.

PROPOSITION 2.8. *The eigenspaces of* $L(X_1 \times \cdots \times X_n)$ *are*

- $L(X_1) \otimes \cdots \otimes L(X_{n-1}) \otimes V_{j_n}^n$, *of eigenvalue* $p_n^0 \lambda_{j_n}^n$, *for* $j_n = 1, \ldots, r_n$, *with multiplicity* $m_1 \cdots m_{n-1} \cdot \dim(V_{j_n}^n)$;
- $L(X_1) \otimes \cdots \otimes L(X_k) \otimes V_{j_{k+1}}^{k+1} \otimes V_0^{k+2} \otimes \cdots \otimes V_0^n$, *of eigenvalue* $p_{k+1}^0 \lambda_{j_{k+1}}^{k+1} + p_{k+2}^0 + \cdots + p_n^0$, *with* $j_{k+1} = 1, \ldots, r_{k+1}$ *and for* $k = 1, \ldots, n-2$, *with multiplicity* $m_1 \cdots m_k \cdot \dim(V_{j_{k+1}}^{k+1})$;
- $V_{j_1}^1 \otimes V_0^2 \otimes \cdots \otimes V_0^n$, *of eigenvalue* $p_1^0 \lambda_{j_1}^1 + p_2^0 + \cdots + p_n^0$, *for* $j_1 = 0, 1, \ldots, r_1$, *with multiplicity* $\dim(V_{j_1}^1)$.

Moreover, as in the general case, one can verify that, under the hypothesis that the operators P_i are ergodic, for $i = 1, \ldots, n$, then also the operator P is ergodic.

Finally, Proposition 2.5 in the case of the nested product yields the following corollary.

COROLLARY 2.9. *Let* P *be the nested product of the probabilities* P_i, *with* $i = 1, \ldots, n$. *Then we have:*

- $U = U_1 \otimes A_2 \otimes \cdots \otimes A_n$
$+ \sum_{k=2}^n I_1^{\sigma_1 - norm} \otimes \cdots \otimes I_{k-1}^{\sigma_{k-1} - norm} \otimes (U_k - A_k) \otimes A_{k+1} \otimes \cdots \otimes A_n$.
- $D = \bigotimes_{i=1}^n D_i$.
- $\Delta = \sum_{i=1}^n p_i^0 \left(I_1 \otimes \cdots \otimes I_{i-1} \otimes \Delta_i \otimes J_{i+1}^{diag} \otimes \cdots \otimes J_n^{diag} \right)$.

Also in this case the interesting fact is that, for a particular choice of the matrices P_i's, the spectral decomposition given in Proposition 2.8 is the same that one gets by considering the action of a particular group on the space $L(X_1 \times \cdots \times X_n)$: this group is the iterated wreath product of the symmetric groups acting on the sets X_i.

For simplicity, suppose $X := X_1 = \cdots = X_n$, with $|X| = m$. So the elements of the cartesian product $X \times \cdots \times X$ can be regarded as the leaves of the rooted $m-$ary tree T_m of depth n. The group $\underbrace{S_m \wr \cdots \wr S_m}_{n \text{ times}}$ is the group of the automorphism of this tree and its action on $X \times \cdots \times X$ is described in Chapter 1, Section 2.

If we set $P_i = J_i$, for all $i = 1, \ldots, n$, then the irreducible subrepresentations in (2) are just the eigenspaces listed in Proposition 2.8: in particular, the space W_0 is the space of constant functions and, for every $j = 1, \ldots, n$, the space W_j in (2) coincides with the space

$$L(X_1) \otimes \cdots \otimes L(X_{j-1}) \otimes V_1^j \otimes V_0^{j+1} \otimes \cdots \otimes V_0^n,$$

where, as usual, $L(X_i) = V_0^i \oplus V_1^i$ is the decomposition of $L(X_i)$ into irreducible S_m-subrepresentations, with $V_0 \cong \mathbb{C}$ and $V_1 = \{f : X_i \longrightarrow \mathbb{C} : \sum_{k=0}^{m-1} f(k) = 0\}$.

2.2.1. *$k-$steps transition probability.* The formula that describes the transition probability after k steps in the case of nested product can be simplified using the base of eigenvectors given in Corollary 2.9 and supposing that the starting point is $0 = (0, \ldots, 0)$.

From the general formula, with the usual notations, we get

$$\begin{aligned}
p^{(k)}(0, y) &= \pi(y) \left[\sum_{z \in X} \left(\sum_{r=2}^{n} \delta_{\sigma_1}(0, z_1) \cdots \delta_{\sigma_{r-1}}(0, z_{r-1})(u_r - a_r)(0, z_r) \right.\right. \\
&\times \left. a_{r+1}(0, z_{r+1}) \cdots a_n(0, z_n) + u_1(0, z_1) a_2(0, z_2) \cdots a_n(0, z_n) \right) \lambda_z^k \\
&\times \left(\sum_{r=2}^{n} \delta_{\sigma_1}(y_1, z_1) \cdots \delta_{\sigma_{r-1}}(y_{r-1}, z_{r-1})(u_r - a_r)(y_r, z_r) \right. \\
&\times a_{r+1}(y_{r+1}, z_{r+1}) \cdots a_n(y_n, z_n) \\
&+ \left.\left. u_1(y_1, z_1) a_2(y_2, z_2) \cdots a_n(y_n, z_n) \right) \right] \\
&= \pi(y) \left[1 + \sum_{j=2}^{n} \sum_{\substack{z_j \neq 0 \\ z_i = 0,\ i \neq j}} \left(\sum_{r=j}^{n} \frac{1}{\sqrt{\sigma_1(0)} \cdots \sqrt{\sigma_{r-1}(0)}} (u_r - a_r)(0, z_r) \right) \right.
\end{aligned}$$

$$\times \quad \left(p_r^0\lambda_{z_r}^r + \sum_{m>r} p_m^0\right)^k \left(\sum_{r=j}^{n} \delta_{\sigma_1}(y_1, 0)\delta_{\sigma_2}(y_2, z_2)\cdots\delta_{\sigma_{r-1}}(y_{r-1}, z_{r-1})\right.$$
$$\times \quad (u_r - a_r)(y_r, z_r)a_{r+1}(y_{r+1}, z_{r+1})\cdots a_n(y_n, z_n))$$
$$\left. + \quad \sum_{\substack{z_1\neq 0 \\ z_i=0,\ i>1}} u_1(0, z_1)\left(p_1^0\lambda_{z_1}^1 + \sum_{m=2}^{n} p_m^0\right)^k u_1(y_1, z_1)\right].$$

Observe that in this case the sum consists of no more than

$$|X_1| + \sum_{i=2}^{n}(|X_i| - 1) = \sum_{i=1}^{n}|X_i| - n + 1$$

nonzero terms.

Example. We want to express the $k-$th step transition probability in the case $n = 2$. So consider the product $X \times Y$, with $X = \{0, 1, \ldots, m\}$ and $Y = \{0, 1, \ldots, n\}$. Let

$$L(X) = \bigoplus_{j=0}^{r} V_j \quad \text{and} \quad L(Y) = \bigoplus_{i=0}^{s} W_i$$

be the spectral decomposition of the spaces $L(X)$ and $L(Y)$, respectively. Let $\lambda_0 = 1, \lambda_1, \ldots, \lambda_r$ and $\mu_0 = 1, \mu_1, \ldots, \mu_s$ be the distinct eigenvalues of P_X and P_Y, respectively. Then the eigenspaces of $L(X \times Y)$ are $L(X) \otimes W_i$, for $i = 1, \ldots, s$, with dimension $(m + 1)\dim(W_i)$ and associated eigenvalue $p_Y^0\mu_i$, and $V_j \otimes W_0$, for $j = 0, \ldots, r$, with dimension $\dim(V_j)$ and associated eigenvalue $p_X^0\lambda_j + p_Y^0$.

The expression of U becomes

$$U = I_X^{\sigma_X-norm} \otimes (U_Y - A_Y) + U_X \otimes A_Y.$$

In particular, let $\{v^0, v_1^1, \ldots, v_{\dim(V_1)}^1, \ldots, v_1^r, \ldots, v_{\dim(V_r)}^r\}$ and $\{w^0, w_1^1, \ldots, w_{\dim(W_1)}^1, \ldots, w_1^s, \ldots, w_{\dim(W_s)}^s\}$ be the eigenvectors of P_X and P_Y, respectively, i.e. they represent the columns of the matrices U_X and U_Y.

Then, the columns of the matrix U corresponding to the elements $(i, 0) \in \{0, \ldots, m\} \times \{0, \ldots, n\}$ are the eigenvectors $v^i \otimes (1, \ldots, 1)$ with eigenvalue $p_X^0\lambda_i + p_Y^0$. On the other hand, the columns corresponding to the elements $(i, j) \in \{0, \ldots, m\} \times \{0, \ldots, n\}$, with $j = 1, \ldots, n$, are the eigenvectors $(0, \ldots, 0, \underbrace{\frac{1}{\sqrt{\sigma_X(i)}}}_{i-th\ \ place}, 0, \ldots, 0) \otimes w^j$ whose eigenvalue is $p_Y^0\mu_j$. As a consequence, only $m + 1 + n$ of these eigenvectors can be nonzero in the first coordinate, so the probability $p^{(k)}((0,0),(x,y))$ can be expressed as a sum of $m+1+n$ nonzero terms: moreover, these terms become $m+1$ if $x \neq 0$. We have

$$\begin{aligned} p^{(k)}((0,0),(x,y)) &= \pi((x,y))\left(\sum_{i=0}^{m} v^i(0)v^i(x)(p_X^0\lambda_i + p_Y^0)^k \right. \\ &+ \left. \frac{1}{\sqrt{\sigma_X(0)\sigma_X(x)}}\sum_{j=1}^{n} w^j(0)\delta_0(x)w^j(y)(p_Y^0\mu_j)^k\right) \\ &= \frac{\sigma_X(x)}{n+1}\left[\sum_{i=0}^{r}\left(\sum_{a=1}^{\dim(V_i)} v_a^i(0)v_a^i(x)\right)(p_X^0\lambda_i + p_Y^0)^k \right. \\ &+ \left. \sum_{j=1}^{s}\left(\frac{1}{\sqrt{\sigma_X(0)\sigma_X(x)}}\sum_{b=1}^{\dim(W_j)} w_b^j(0)\delta_0(x)w_b^j(y)\right)(p_Y^0\mu_j)^k\right]. \end{aligned}$$

2.3. The Insect. We describe here a particular Markov chain defined on the $n-th$ level of the rooted tree, introduced by A. Figà-Talamanca in [**32**] and called the “Insect” in [**18**], [**20**] and [**21**].

We already said above that if $|X_i| = m_i$, with $X_i = \{0,1,\ldots,m_i-1\}$, then the elements of the cartesian product $X_1 \times \cdots \times X_n$ can be regarded as the set of the leaves of the rooted tree T of depth n, such that the root has degree m_1, each vertex of the first level has m_2 children and in general each vertex of the $i-$th level of the tree has m_{i+1} children, for every $i = 1,\ldots,n-1$. As usual, we denote the $i-$th level of the tree by L_i. We recall that the group $Aut(T)$ of all automorphisms of T is given by the iterated wreath product

$$S_{m_n} \wr S_{m_{n-1}} \wr \cdots \wr S_{m_1}.$$

Moreover, $Aut(T)$ is also the group of isometries of T, with respect to the usual ultrametric distance.

The **Insect** is a Markov chain P on the level L_n of the tree, defined from the simple random walk on T starting in a vertex $x \in L_n$. In fact, it is possible to define a probability distribution μ_x on L_n such that, for every $y \in L_n$, $\mu_x(y)$ is the probability that y is the first point in L_n visited by the random walk. If we put $p(x,y) = \mu_x(y)$, then we get a stochastic matrix $P = (p(x,y))_{x,y\in L_n}$. Moreover, since the random walk is $Aut(T)-$invariant, we can suppose that the random walk starts at the leftmost leaf, that we will call $x_0 = (0,\ldots,0)$.

We use the notation of [**16**] in a more general context: the authors consider there the particular case $m_1 = \cdots m_n = q$.

Set $\xi_n = \emptyset$ and $\xi_i = \underbrace{00\ldots0}_{n-i \text{ times}}$. For $j \geq 0$, let α_j be the probability of ever reaching ξ_{j+1} given that ξ_j is reached at least once. This definition implies

$$\alpha_0 = 1 \qquad \text{and} \qquad \alpha_1 = \frac{1}{m_n + 1}.$$

In fact, with probability 1, the vertex ξ_1 is reached at the first step and, starting from ξ_1, with probability $\frac{1}{m_n+1}$ it reaches ξ_2, while with probability $\frac{m_n}{m_n+1}$ it returns to some vertex of L_n. Finally, one has $\alpha_n = 0$.

For $1 < j < n$, the following recursive relation holds:

$$\alpha_j = \frac{1}{m_{n+1-j}+1} + \alpha_{j-1}\alpha_j \frac{m_{n+1-j}}{m_{n+1-j}+1}. \tag{29}$$

In fact, starting at ξ_j, with probability $\frac{1}{m_{n+1-j}+1}$ the insect reaches in one step ξ_{j+1}, otherwise with probability $\frac{m_{n+1-j}}{m_{n+1-j}+1}$ it reaches ξ_{j-1} or one of its brothers; then, with probability α_{j-1} it reaches again ξ_j and one starts the recursive argument.

The solution of (29), for $1 \leq j \leq n-1$, is given by

$$\begin{aligned}
\alpha_j &= \frac{1 + m_n + m_n m_{n-1} + m_n m_{n-1} m_{n-2} + \cdots + m_n m_{n-1} m_{n-2} \cdots m_{n-j+2}}{1 + m_n + m_n m_{n-1} + m_n m_{n-1} m_{n-2} + \cdots + m_n m_{n-1} m_{n-2} \cdots m_{n-j+1}} \\
&= 1 - \frac{m_n m_{n-1} m_{n-2} \cdots m_{n-j+1}}{1 + m_n + m_n m_{n-1} + m_n m_{n-1} m_{n-2} + \cdots + m_n m_{n-1} m_{n-2} \cdots m_{n-j+1}}.
\end{aligned}$$

We already remarked that the random walk, and so the Insect Markov chain, is invariant with respect to the action of $Aut(T)$, which is the group of isometries of the tree. This implies that the probability $p(x_0, x)$, for $x \in L_n$, only depends on the ultrametric distance $d(x_0, x)$.

We get

$$\begin{aligned}
p(x_0, x_0) &= \frac{1}{m_n}(1-\alpha_1) + \frac{1}{m_n m_{n-1}}\alpha_1(1-\alpha_2) + \cdots \\
&+ \frac{1}{m_n m_{n-1} \cdots m_2}\alpha_1\alpha_2 \cdots \alpha_{n-2}(1-\alpha_{n-1}) + \frac{1}{m_n \cdots m_1}\alpha_1 \cdots \alpha_{n-1}.
\end{aligned}$$

In particular, the $j-$th summand is the probability of returning back to x_0 if the corresponding random walk in T reaches ξ_j but not ξ_{j+1}.

It is easy to compute $p(x_0, x)$, where x is a point at distance j from x_0. For $j = 1$, we clearly have $p(x_0, x_0) = p(x_0, x)$. We observe that, for $j > 1$, to reach x one is forced to first reach ξ_j, so that we have

$$\begin{aligned}
p(x_0, x) &= \frac{1}{m_n \cdots m_{n-j+1}}\alpha_1\alpha_2 \cdots \alpha_{j-1}(1-\alpha_j) + \cdots \\
&+ \frac{1}{m_n \cdots m_2}\alpha_1\alpha_2 \cdots \alpha_{n-2}(1-\alpha_{n-1}) + \frac{1}{m_n \cdots m_1}\alpha_1\alpha_2 \cdots \alpha_{n-1}.
\end{aligned}$$

Actually, the Insect Markov chain is a particular case of the nested product defined in (28), as the following proposition shows.

Proposition 2.10. *The transition probability*

$$\begin{aligned} p((x_1,\ldots,x_n),(y_1,\ldots,y_n)) &= \frac{p_1^0 p_1(x_1,y_1)}{m_2 m_3\cdots m_n} \\ &+ \sum_{j=2}^{n-1} \frac{\delta((x_1,\ldots,x_{j-1}),(y_1,\ldots,y_{j-1}))p_j^0 p_j(x_j,y_j)}{m_{j+1}\cdots m_n} \\ &+ \delta((x_1,\ldots,x_{n-1}),(y_1,\ldots,y_{n-1}))p_n^0 p_n(x_n,y_n), \end{aligned}$$

associated with the Markov chain in (28), gives rise to the Insect Markov chain defined on L_n, regarded as the sets of elements of the product $X_1\times\cdots\times X_n$, choosing $p_i^0 = \alpha_1\alpha_2\cdots\alpha_{n-i}(1-\alpha_{n-i+1})$ for $i = 1,\ldots,n-1$ and $p_n^0 = 1-\alpha_1$ and the transitions probabilities p_j's to be uniform for all $j = 1,\ldots,n$.

Proof. Set, for every $i = 1,\ldots,n-1$,

$$p_i^0 = \alpha_1\alpha_2\cdots\alpha_{n-i}(1-\alpha_{n-i+1})$$

and $p_n^0 = 1-\alpha_1$. Moreover, assume that the probability p_i on X_i is uniform, i.e.

$$P_i = J_i.$$

If $d(x_0,x) = n$, then we get

$$p(x_0,x) = \frac{\alpha_1\alpha_2\cdots\alpha_{n-1}}{m_1m_2\cdots m_n}.$$

If $d(x_0,x) = j > 1$, i.e. $x_i^0 = x_i$ for all $i = 1,\ldots,n-j$, then

$$p(x_0,x) = \frac{\alpha_1\alpha_2\cdots\alpha_{n-1}}{m_1m_2\cdots m_n} + \sum_{i=1}^{n-j}\frac{\alpha_1\cdots\alpha_{n-i-1}(1-\alpha_{n-i})}{m_n\cdots m_{i+2}m_{i+1}}.$$

Finally, if $x = x_0$, we get

$$p(x_0,x_0) = \frac{\alpha_1\alpha_2\cdots\alpha_{n-1}}{m_1m_2\cdots m_n} + \sum_{i=1}^{n-2}\frac{\alpha_1\cdots\alpha_{n-i-1}(1-\alpha_{n-i})}{m_n\cdots m_{i+2}m_{i+1}} + \frac{(1-\alpha_1)}{m_n}.$$

This completes the proof. □

Following the remark given after Corollary 2.9, we deduce that the eigenspaces of the operator P associated with the Insect Markov chain are exactly the W_j's of (2).

This fact can also be obtained as a consequence of Corollary 1.8, since the Markov operator P on $L(L_n)$ associated with the Insect Markov chain is $Aut(T)-$invariant, i.e. $P \in Hom_{Aut(T)}(L(L_n), L(L_n))$. This follows from the fact that the probability $p(x_0,x)$ only depends on the ultrametric distance $d(x_0,x)$. In formulae, we have

$$g(Pf) = P(g(f)),$$

for every $f \in L(L_n)$ and $g \in Aut(T)$. In fact, for $x \in L_n$, we have

$$(g(Pf))(x) = (Pf)(g^{-1}x) = \sum_{y \in L_n} p(g^{-1}x, y)f(y)$$

and

$$\begin{aligned}(P(g(f)))(x) &= \sum_{y \in L_n} p(x,y)(g(f))(y) = \sum_{y \in L_n} p(x,y)f(g^{-1}y) \\ &= \sum_{t \in L_n} p(x, gt)f(t),\end{aligned}$$

where we set $g^{-1}y = t$. Using that $p(g^{-1}x, y) = p(x, gy)$ since g is an isometry, we get the assertion.

In order to compute the corresponding eigenvalues we can use the formulas given in Proposition 2.8 for the eigenvalues of the nested product by setting:

- $\lambda_0^i = 1$ and $\lambda_1^i = 0$, for all $i = 1, \ldots, n$;
- $p_i^0 = \alpha_1\alpha_2 \cdots \alpha_{n-i}(1 - \alpha_{n-i+1})$, for $i = 1, \ldots, n-1$;
- $p_n^0 = 1 - \alpha_1$.

We get:

- $\lambda_0 = \sum_{i=1}^n p_i^0 = 1$;
- $\lambda_j = \sum_{i=j+1}^n p_i^0$, for every $j = 1, \ldots, n-1$;
- $\lambda_n = 0$.

It is easy to prove by induction on j that, for every $j = 1, \ldots, n-1$, the eigenvalue λ_j is equal to $1 - \alpha_1\alpha_2 \cdots \alpha_{n-j}$.

If $j = 1$, we have $\lambda_1 = \sum_{i=2}^n p_i^0 = 1 - p_1^0 = 1 - \alpha_1 \cdots \alpha_{n-1}$. Now suppose the assertion to be true for j and show that it holds also for $j+1$. We get

$$\begin{aligned}\lambda_{j+1} &= \sum_{i=j+2}^n p_i^0 = \sum_{i=j+1}^n p_i^0 - p_{j+1}^0 \\ &= \lambda_j - p_{j+1}^0 = 1 - \alpha_1\alpha_2 \cdots \alpha_{n-j} - \alpha_1 \cdots \alpha_{n-j-1}(1 - \alpha_{n-j}) \\ &= 1 - \alpha_1 \cdots \alpha_j \alpha_{n-j-1}.\end{aligned}$$

2.3.1. *Example.* Let us give an example in the case $n = 3$ with $m_1 = m_2 = m_3 = 2$ and

$$P_1 = P_2 = P_3 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

The tree associated with the product $X_1 \times X_2 \times X_3$ is the following:

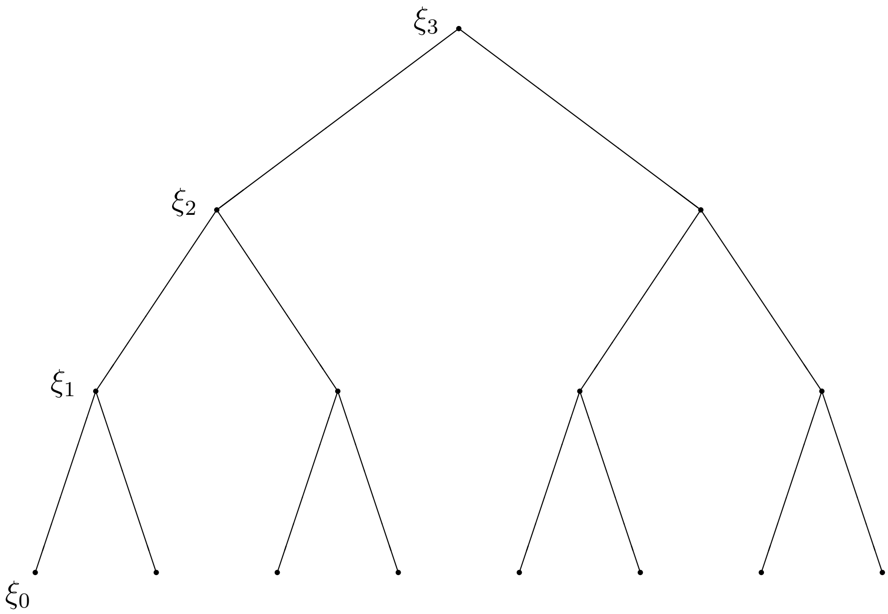


Fig.16. The rooted binary tree of depth 3.

Set

$$L(X_1) = V_0^1 \oplus V_1^1, \quad L(X_2) = V_0^2 \oplus V_1^2, \quad \text{and } L(X_3) = V_0^3 \oplus V_1^3.$$

The eigenspaces of P are:

- $W_0 = V_0^1 \otimes V_0^2 \otimes V_0^3$, of dimension 1;
- $W_1 = V_1^1 \otimes V_0^2 \otimes V_0^3$, of dimension 1;
- $W_2 = L(X_1) \otimes V_1^2 \otimes V_0^3$, of dimension 2;
- $W_3 = L(X_1) \otimes L(X_2) \otimes V_1^3$, of dimension 4.

We have $\alpha_0 = 1$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{3}{7}$, $\alpha_3 = 0$ and so $p_1^0 = \frac{1}{7}$, $p_2^0 = \frac{4}{21}$ and $p_3^0 = \frac{2}{3}$.

The eigenvalues of P are the following:

- $\lambda_0 = 1$, with multiplicity 1;
- $\lambda_1 = \frac{6}{7}$ with multiplicity 1;
- $\lambda_2 = \frac{2}{3}$ with multiplicity 2;
- $\lambda_3 = 0$, with multiplicity 4.

The matrix P is given by

$$P = p_1^0(J_1 \otimes J_2 \otimes J_3) + p_2^0(I_1 \otimes J_2 \otimes J_3) + p_3^0(I_1 \otimes I_2 \otimes J_3) =$$

$$= \frac{1}{168}\begin{pmatrix} 67 & 67 & 11 & 11 & 3 & 3 & 3 & 3 \\ 67 & 67 & 11 & 11 & 3 & 3 & 3 & 3 \\ 11 & 11 & 67 & 67 & 3 & 3 & 3 & 3 \\ 11 & 11 & 67 & 67 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 67 & 67 & 11 & 11 \\ 3 & 3 & 3 & 3 & 67 & 67 & 11 & 11 \\ 3 & 3 & 3 & 3 & 11 & 11 & 67 & 67 \\ 3 & 3 & 3 & 3 & 11 & 11 & 67 & 67 \end{pmatrix}.$$

3. The cut-off phenomenon

The rate of convergence of an ergodic Markov chain to the stationary distribution has been studied by Diaconis in relation with the following question: "How much does it take to converge to the stationary distribution π?". This is motivated by the fact that in many Markov chains the difference between the value of the probability measure $m^{(k)}$ given by the $k-$steps transition probability and π is close to 0 only after a fixed number k_0 of steps, and it is large (close to 1) before k_0 steps. So the distance exponentially fast breaks down in a small range. This phenomenon has been called "cut-off phenomenon". Actually, this term was introduced in [**1**]. Many applications are presented in the survey [**24**].

In [**20**] the cut-off phenomenon for the Insect Markov chain is investigated, using the spectral analysis of the associated Markov operator. In particular, we study the homogeneous case $m_1 = \cdots = m_n = q$ and we show that the cut-off phenomenon does not occur.

First of all, we need some definitions.

Definition 3.1. *Let $P = (p(x,y))_{x,y\in X}$ be a stochastic matrix. Then a* ***stationary distribution*** *for P is a probability measure π on X such that*

$$\pi(y) = \sum_{x\in X} \pi(x)p(x,y), \tag{30}$$

for all $y \in X$.

The following theorem gives a relation between stationary distributions and ergodicity. For a proof see, for example, Chapter 1 in [**16**].

Theorem 3.2. *Let P be a stochastic matrix on X. Then P is ergodic if and only if there exists a strict probability distribution on X such that*

$$\lim_{k\to\infty} p^{(k)}(x,y) = \pi(y) \quad \text{for all } x, y \in X.$$

This implies that the limits above exist, they are independent of x and they form a strict probability distribution. Moreover, π is the unique stationary distribution for P.

Note that if P is ergodic and in detailed balance with π, then its stationary distribution coincides with π. To show that, it suffices to sum over $x \in X$ the identity

$$\pi(x)p(x,y) = \pi(y)p(y,x),$$

what gives $\sum_{x\in X} \pi(x)p(x,y) = \pi(y)$, which is just (30).

Theorem 3.2 can be easily proven under the hypothesis of reversibility of the Markov chain P. In particular, we get the following theorem.

THEOREM 3.3. *Let P be a Markov chain in detailed balance with the distribution π. Suppose that P is ergodic. Then*

$$\lim_{k\to\infty} p^{(k)}(x,y) = \pi(y), \tag{31}$$

for all $x, y \in X$.

Proof. From Proposition 1.5, we have

$$p^{(k)}(x,y) = \pi(y) \sum_{z\in X} u(x,z)\lambda_z^k u(y,z).$$

Since P is ergodic, the eigenvalue 1 has multiplicity one, so there exists $z_0 \in X$ such that $\lambda_z < \lambda_{z_0} = 1$, for all $z \neq z_0$. Moreover, one has $u(x, z_0) = 1$ for all $x \in X$. The hypothesis of ergodicity implies $\lambda_z > -1$, for all $z \in X$, so that (31) can be rewritten as

$$p^{(k)}(x,y) = \pi(y) + \pi(y) \sum_{z\neq z_0} u(x,z)\lambda_z^k u(y,z).$$

Since $-1 < \lambda_z < 1$ for all $z \neq z_0$, we get $\lim_{k\to\infty} p^{(k)}(x,y) = \pi(y)$. □

The next definition will be useful later, because it introduces the notion of difference of two probability measures on X.

DEFINITION 3.4. *Let μ and ν two probability measures on X. Then their **total variation distance** is defined as*

$$\|\mu - \nu\|_{TV} = \max_{A\subseteq X} \left| \sum_{x\in A} \mu(x) - \nu(x) \right| \equiv \max_{A\subseteq X} |\mu(A) - \nu(A)|.$$

It is easy to prove that $\|\mu-\nu\|_{TV} = \frac{1}{2}\|\mu-\nu\|_{L^1(X)}$, where $\|\cdot\|_{L^1(X)}$ is the standard $L^1(X)$ distance given by

$$\|\mu - \nu\|_{L^1(X)} = \sum_{x\in X} |\mu(x) - \nu(x)|.$$

3.1. The cut-off phenomenon. Let $m_x^{(k)}(y) = p^{(k)}(x,y)$ the distribution probability after k steps. The total variation distance defined in Definition 3.4 allows to estimate how $m^{(k)}$ converges to the stationary distribution π.

There are interesting cases in which the total variation distance remains close to 1 for a long time and then tends to 0 in a very fast way (see, for some examples, [**24**] and [**26**]). This suggests the following definition (see [**16**]).

Suppose that X_n is a sequence of finite sets. Let m_n and p_n be a probability measure on X_n and an ergodic transition probability on X_n, respectively. Moreover, denote by π_n the stationary measure of p_n and by $m_n^{(k)}$ the distribution of (X_n, m_n, p_n) after k steps.

Now let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of positive real numbers such that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = 0.$$

DEFINITION 3.5. *The sequence of Markov chains (X_n, m_n, p_n) has a $(a_n, b_n)-$**cut-off** if there exist two functions $f_1, f_2 : [0, +\infty) \longrightarrow \mathbb{R}$ with*

- $\lim_{c\to+\infty} f_1(c) = 0$
- $\lim_{c\to+\infty} f_2(c) = 1$

such that, for any fixed $c > 0$, one has

$$\|m_n^{(a_n+cb_n)} - \pi_n\|_{TV} \leq f_1(c) \quad \textit{and} \quad \|m_n^{(a_n-cb_n)} - \pi_n\|_{TV} \geq f_2(c)$$

for a sufficiently large n.

The following proposition gives a necessary condition for the cut-off phenomenon.

PROPOSITION 3.6. *If (X_n, m_n, p_n) has an $(a_n, b_n)-$cut-off, then for any $0 < \epsilon_1 < \epsilon_2 < 1$ there exist $k_2(n) \leq k_1(n)$ such that*

(1) $k_2(n) \leq a_n \leq k_1(n)$;
(2) for n large, $k \geq k_1(n) \Rightarrow \|m_n^{(k)} - \pi_n\|_{TV} \leq \epsilon_1$;
(3) for n large, $k \leq k_2(n) \Rightarrow \|m_n^{(k)} - \pi_n\|_{TV} \geq \epsilon_2$;
(4) $\lim_{n\to\infty} \frac{k_1(n)-k_2(n)}{a_n} = 0$.

Proof. By hypothesis there exist c_1 and c_2 such that $f_2(c) \geq \epsilon_2$ for $c \geq c_2$ and $f_1(c) \leq \epsilon_1$ for $c \geq c_1$. So it suffices to take $k_1(n) = a_n + c_1 b_n$ and $k_2(n) = a_n - c_2 b_n$ to get the assertion. $\square$

3.2. The case of Insect Markov chain. Consider now the Insect Markov chain in the homogeneous case $m_1 = \cdots = m_n = q$. The indices α_i are the following:

$$\alpha_0 = 1, \qquad \alpha_1 = \frac{1}{q+1} \qquad \text{and} \quad \alpha_n = 0.$$

The recursive formula (29) becomes, in this case,

$$\alpha_j = \frac{1}{q+1} + \alpha_{j-1}\alpha_j \frac{1}{q+1},$$

for every $j = 1, \ldots, n-1$. The solution of this equation is given by

$$\alpha_j = \frac{q^j - 1}{q^{j+1} - 1}.$$

Fix the vertex $x_0 = 0^n$. Using the α_j's, for every $x \in L_n$, we can express the probability that x is the first vertex in L_n reached from x_0 in the Insect Markov chain. In particular, we have:

$$\begin{aligned} p(x_0, x_0) &= q^{-1}(1-\alpha_1) + q^{-2}\alpha_1(1-\alpha_2) + \cdots + \\ &+ q^{-n+1}\alpha_1\alpha_2\cdots\alpha_{n-2}(1-\alpha_{n-1}) + q^{-n}\alpha_1\alpha_2\cdots\alpha_{n-1}. \end{aligned}$$

It is clear that, if $d(x_0, x) = 1$, then $p(x_0, x) = p(x_0, x_0)$.

More generally, if $d(x_0, x) = j > 1$, one has:

$$\begin{aligned} p(x_0, x) &= q^{-j}\alpha_1\alpha_2\cdots\alpha_{j-1}(1-\alpha_j) + \cdots + \\ &+ q^{-n+1}\alpha_1\alpha_2\cdots\alpha_{n-2}(1-\alpha_{n-1}) + q^{-n}\alpha_1\alpha_2\cdots\alpha_{n-1}. \end{aligned}$$

The associated eigenvalues are:

$$\lambda_0 = 1, \qquad \lambda_n = 0;$$

more in general, for $1 \leq j < n$, we have

$$\lambda_j = 1 - \frac{q-1}{q^{n-j+1}-1}. \tag{32}$$

We already know that the Insect Markov chain is ergodic (it is the nested product of ergodic Markov chains).

Moreover, it is clear that P is in detailed balance with the uniform distribution π on L_n given by $\pi(x) = \frac{1}{q^n}$ for all $x \in L_n$.

An expression for $m^{(k)}(x) = p^{(k)}(x_0, x)$ can be easily obtained using the Fourier analysis. In [**16**], Chapter 4, it is proven that, if P is a $G-$invariant stochastic matrix on X, i.e. a stochastic matrix satisfying the condition

$$p(gx, gy) = p(x, y), \quad \forall x, y \in X, \; g \in G$$

and if (G, K) is a Gelfand pair, where K is the stabilizer of x_0, then

$$p^{(k)}(x_0, x) = \frac{1}{|X|}\sum_{i=0}^{n} d_i \lambda_i^k \phi_i(x), \tag{33}$$

where λ_i is the eigenvalue associated with the spherical function ϕ_i and d_i is the dimension of the corresponding spherical representation.

In our case, G is the full automorphisms group of the rooted $q-$ary tree of depth n and the ϕ_i's are the spherical functions given in (3).

Suppose now $n \geq 2$. In the following theorem, the cut-off phenomenon is detected thanks to a careful spectral analysis.

THEOREM 3.7. *The probability measure associated with the Insect Markov chain converges to the stationary distribution without a cut-off behavior.*

Proof. From (33) we get

- If $x = x_0$, then

$$m^{(k)}(x_0) = \frac{1}{q^n}\left\{1 + \sum_{j=1}^{n} q^{j-1}(q-1)\left[1 - \frac{q-1}{q^{n-j+1}-1}\right]^k\right\}.$$

- If $d(x_0,x)=h$, with $1\le h\le n-1$, then

$$\begin{aligned} m^{(k)}(x) &= \frac{1}{q^n}\left\{1+\sum_{j=1}^{n-h+1} q^{j-1}(q-1)\left[1-\frac{q-1}{q^{n-j+1}-1}\right]^k \phi_j(x)\right\} \\ &= \frac{1}{q^n}\left\{1+\sum_{j=1}^{n-h} q^{j-1}(q-1)\left[1-\frac{q-1}{q^{n-j+1}-1}\right]^k - q^{n-h}\left[1-\frac{q-1}{q^h-1}\right]^k\right\} \end{aligned}$$

- If $d(x_0,x)=n$, then

$$m^{(k)}(x)=\frac{1}{q^n}\left\{1-\left[1-\frac{q-1}{q^n-1}\right]^k\right\}.$$

Let π be the uniform distribution on L_n. Then we have

$$\begin{aligned} \|m^{(k)}-\pi\|_{L^1(L_n)} &= \frac{1}{q^n}\left\{\sum_{j=1}^{n} q^{j-1}(q-1)\lambda_j^k\right. \\ &+ \sum_{h=1}^{n-1}(q^h-q^{h-1})\left|\sum_{j=1}^{n-h} q^{j-1}(q-1)\lambda_j^k - q^{n-h}\lambda_{n-h+1}^k\right| \\ &+ \left. q^{n-1}(q-1)\lambda_1^k\right\}. \end{aligned}$$

Now observe that

$$\frac{1}{q^n}\sum_{h=1}^{n-1}(q^h-q^{h-1})\sum_{j=1}^{n-h} q^{j-1}(q-1)\lambda_j^k+\frac{1}{q^n}\sum_{j=1}^{n} q^{j-1}(q-1)\lambda_j^k =$$

$$\frac{1}{q^n}\sum_{j=1}^{n-1}\left[1+(q-1)+(q^2-q)+\cdots+(q^{n-j}-q^{n-j-1})\right]\cdot q^{j-1}(q-1)\lambda_j^k =$$

$$\frac{1}{q^n}\sum_{j=1}^{n-1} q^{n-1}(q-1)\lambda_j^k=\frac{q-1}{q}\sum_{j=1}^{n-1}\lambda_j^k$$

and

$$\frac{1}{q^n}\sum_{h=1}^{n-1}(q^h-q^{h-1})q^{n-h}\lambda_{n-h+1}^k+\frac{1}{q^n}(q^n-q^{n-1})\lambda_1^k=\frac{q-1}{q}\sum_{j=1}^{n-1}\lambda_j^k.$$

Using the trivial fact that $\sum_j|a_j-b_j|\le\sum_j(|a_j|+|b_j|)$, we conclude

$$\|m^{(k)}-\pi\|_{L^1(L_n)}\le\frac{2(q-1)}{q}\sum_{j=1}^{n-1}\lambda_j^k.$$

On the other hand

$$\begin{aligned} \|m^{(k)}-\pi\|_{L^1(L_n)} &\ge \sum_{x:d(x_0,x)=n}|m^{(k)}(x)-\pi(x)| \\ &= \frac{1}{q^n}(q^n-q^{n-1})\lambda_1^k=\frac{q-1}{q}\lambda_1^k. \end{aligned}$$

So we get the following estimate:

$$\frac{q-1}{q}\lambda_1^k \le \|m^{(k)} - \pi\|_{L^1(L_n)} \le \frac{2(q-1)}{q}\sum_{j=1}^{n-1}\lambda_j^k,$$

or, equivalently,

$$\frac{q-1}{2q}\lambda_1^k \le \|m^{(k)} - \pi\|_{TV} \le \frac{(q-1)}{q}\sum_{j=1}^{n-1}\lambda_j^k.$$

In what follows the following inequalities will be used:

(1) $(1-x)^k \le \exp(-kx)$ if $x \le 1$.
(2) $\frac{q^n-1}{q^{n-j+1}-1} \ge q^{j-1}$, for $j \ge 1$.
(3) $q^{j-1} \ge j$, for $q \ge 2$ and $j \ge 1$.

Choose $k_2(n) = \frac{q^n-1}{q-1}$, then

$$\begin{aligned}
\frac{q-1}{q}\sum_{j=1}^{n-1}\lambda_j^k &\le \frac{q-1}{q}\sum_{j=1}^{n-1}\exp\left(-\frac{q-1}{q^{n-j+1}-1}k\right) \le (\text{ if } k \ge k_2(n))\\
&\le \frac{q-1}{q}\sum_{j=1}^{n-1}\exp\left(-\frac{q-1}{q^{n-j+1}-1}k_2(n)\right)\\
&\le \frac{q-1}{q}\sum_{j=1}^{n-1}\exp(-q^{j-1}) \le \frac{(q-1)}{q}\sum_{j=1}^{n-1}(e^{-j})\\
&\le \frac{(q-1)}{q}\sum_{j=1}^{\infty}(e^{-1})^j = \frac{q-1}{q}\cdot\frac{1}{e-1} := \epsilon_2.
\end{aligned}$$

On the other hand, if $k_1(n) = 2\frac{q^n-1}{q-1}$, we get

$$\begin{aligned}
\frac{q-1}{2q}\lambda_1^k &= \frac{q-1}{2q}\left[1-\frac{q-1}{q^n-1}\right]^k \ge (\text{ if } k \le k_1(n))\\
&\ge \frac{q-1}{2q}\left[1-\frac{q-1}{q^n-1}\right]^{2\frac{q^n-1}{q-1}} := \epsilon_1.
\end{aligned}$$

Now $k_1(n) > k_2(n), \epsilon_1 < \epsilon_2$ and

- for $k \ge k_2(n)$ we have $\|m^{(k)} - \pi\|_{TV} \le \epsilon_2$,
- for $k \le k_1(n)$ we have $\|m^{(k)} - \pi\|_{TV} \ge \epsilon_1$.

This implies that cut-off phenomenon does not occur in this case by Proposition 3.6. In fact, the sequences $k_1(n)$ and $k_2(n)$ cannot satisfy condition (4) of Proposition 3.6. This gives the assertion. $\square$

3.2.1. *Remark.* Using the same strategy of Theorem 3.7 one can easily check that cut-off phenomenon does not occur also if we fix n and let $q \to +\infty$.

3.2.2. *Remark.* If $n = 1$ we get the simple random walk on the complete graph K_q on q vertices, in which each vertex has a loop. It is straightforward that it is performed choosing equiprobably one of the q vertices and so the probability measure $m^{(1)}$ equals the uniform distribution π on the set of the vertices.

4. Association Schemes

The definition of crested product given in Section 2 of this chapter for Markov chain is inspired to the definition of crested product of Association schemes introduced in [**4**]. Also the particular cases of crossed and nested products are inspired to the theory of association schemes (largely developed in [**3**]).

In this section we will present the definition of association scheme together with the main properties. Moreover, some particular examples on the rooted homogeneous tree will be described.

4.1. Definition and main properties. Association schemes are defined about relations between pairs of elements of a set Ω, supposed finite. Many equivalent definitions of association scheme can be given (see [**3**]): we want to give the definitions using partitions and matrices.

4.1.1. *A first definition.*

DEFINITION 4.1. *An **association scheme** with s associate classes on a finite set Ω is a partition of $\Omega\times\Omega$ into nonempty sets $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_s$, called the **associate classes**, such that*

(1) $\mathcal{C}_0 = Diag(\Omega) = \{(\omega, \omega) : \omega \in \Omega\}$.

(2) $\mathcal{C}_i$ *is symmetric for every* $i = 1, \ldots, s$, *i.e.* $\mathcal{C}_i = \mathcal{C}_i'$, *where* $\mathcal{C}_i'$ *denotes the dual of* $\mathcal{C}_i$ *defined as* $\mathcal{C}_i' = \{(\beta, \alpha) : (\alpha, \beta) \in \mathcal{C}_i\}$.

(3) For all $i, j, k \in \{0, 1, \ldots, s\}$ *there exists an integer* p_{ij}^k *such that, for all* $(\alpha, \beta) \in \mathcal{C}_k$,

$$|\{\gamma \in \Omega : (\alpha, \gamma) \in \mathcal{C}_i \text{ and } (\gamma, \beta) \in \mathcal{C}_j\}| = p_{ij}^k.$$

We will say that the rank of this association scheme is $s+1$. Observe that the conditions (2) and (3) imply $p_{ij}^k = p_{ji}^k$. The elements α and β are called $i-$th associates if $(\alpha, \beta) \in \mathcal{C}_i$. In particular, the set of $i-$th associates of α is denoted by

$$\mathcal{C}_i(\alpha) = \{\beta \in \Omega : (\alpha, \beta) \in \mathcal{C}_i\}.$$

Condition (2) implies $p^0_{ij} = 0$ if $i \neq j$. Similarly, $p^k_{0j} = 0$ if $j \neq k$ and $p^k_{i0} = 0$ if $i \neq k$, while $p^j_{0j} = p^i_{i0} = 1$. Moreover, the condition (3) implies that each element of Ω has $p^0_{ii} = a_i$ $i-$th associates.

Example. Let Ω be a finite set, with $|\Omega| = n$. Let $\mathcal{C}_0$ be the diagonal subset and set

$$\mathcal{C}_1 = \{(\alpha, \beta) \in \Omega \times \Omega : \alpha \neq \beta\} = (\Omega \times \Omega) \setminus \mathcal{C}_0.$$

This is the **trivial** association scheme, the only scheme on Ω having only one associate class. It has $a_1 = n - 1$ and it is denoted by $\underline{\underline{n}}$.

Example. Let Ω an $m \times n$ rectangular array, with $m, n \geq 2$. Set

- $\mathcal{C}_1 = \{(\alpha, \beta) : \alpha, \beta$ are in the same row but $\alpha \neq \beta\}$;
- $\mathcal{C}_2 = \{(\alpha, \beta) : \alpha, \beta$ are in the same column but $\alpha \neq \beta\}$;
- $\mathcal{C}_3 = \{(\alpha, \beta) : \alpha, \beta$ are in different rows and columns$\}$.

It is clear that $\mathcal{C}_3 = (\Omega \times \Omega) \setminus \mathcal{C}_0 \setminus \mathcal{C}_1 \setminus \mathcal{C}_2$. This is an association scheme with three associate classes and $a_1 = n - 1$, $a_2 = m - 1$, $a_3 = (m - 1)(n - 1)$. It is called the **rectangular** association scheme $R(m, n)$ and is also denoted by $\underline{\underline{m}} \times \underline{\underline{n}}$.

Example. Consider the partition $\Omega = \Delta_1 \sqcup \ldots \sqcup \Delta_m$ of the set Ω into m subsets of size n. These subsets are traditionally called *groups*. We declare α and β to be:

- first associates if they are in the same groups but $\alpha \neq \beta$;
- second associates if they are in different groups.

It is easy to verify that, if $\omega \in \Omega$, then it has $n - 1$ first associates and $(m - 1)n$ second associates. So this is an association scheme with $s = 2$ and $a_1 = n - 1$, $a_2 = (m - 1)n$. It is called the **group-divisible** association scheme, denoted by $GD(m, n)$ or also $\underline{\underline{m}}/\underline{\underline{n}}$.

4.1.2. *A second definition.* Given an association scheme with associate classes $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_s$, we can associate with each class $\mathcal{C}_i$ its **adjacency matrix** A_i, i.e. the matrix of size $|\Omega|$ defined as

$$(A_i)_{\alpha\beta} = \begin{cases} 1 & \text{if } (\alpha, \beta) \in \mathcal{C}_i \\ 0 & \text{otherwise.} \end{cases}$$

The following lemma holds.

LEMMA 4.2. *Given an association scheme with associate classes $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_s$, let A_i be the corresponding adjacency matrices. Then*

$$A_i A_j = \sum_{k=0}^{s} p_{ij}^k A_k. \tag{34}$$

Proof. Suppose $(\alpha, \beta) \in \mathcal{C}_k$. Then the $(\alpha, \beta)-$entry of the right-hand side of (34) is equal to p_{ij}^k, while the $(\alpha, \beta)-$entry of the left-hand side is equal to

$$\begin{aligned}(A_i A_j)(\alpha, \beta) &= \sum_{\gamma \in \Omega} A_i(\alpha, \gamma) A_j(\gamma, \beta) \\ &= |\{\gamma : (\alpha, \gamma) \in \mathcal{C}_i \text{ and } (\gamma, \beta) \in \mathcal{C}_j\}| \\ &= p_{ij}^k,\end{aligned}$$

because the product $A_i(\alpha, \gamma) A_j(\gamma, \beta)$ is zero unless $(\alpha, \gamma) \in \mathcal{C}_i$ and $(\gamma, \beta) \in \mathcal{C}_j$, in which case it is 1. $\square$

This lemma leads us to a new definition of association scheme, in terms of adjacency matrices.

DEFINITION 4.3. *An association scheme with s associate classes on a finite set Ω is a set of nonzero matrices $A_0, A_1, \ldots, A_s$, with rows and columns indexed by Ω, whose entries are equal to 0 or 1 and such that:*

(1) $A_0 = I_\Omega$, where I_Ω denotes the identity matrix of size $|\Omega|$;
(2) A_i is symmetric for every $i = 1, \ldots, s$;
(3) for all $i, j \in \{1, \ldots, s\}$, the product $A_i A_j$ is a linear combination of $A_0, A_1, \ldots, A_s$;
(4) $\sum_{i=0}^{s} A_i = J_\Omega$, where J_Ω denotes the all-1 matrix of size $|\Omega|$.

Observe that the condition (4) of this definition gives an analogue of the fact that the subsets $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_s$ constitute a partition of $\Omega \times \Omega$.

PROPOSITION 4.4. *If $A_0, A_1, \ldots, A_s$ are the adjacency matrices of an association scheme, then $A_i A_j = A_j A_i$ for all $i, j \in \{0, 1, \ldots, s\}$.*

Proof. We have

$$
\begin{aligned}
A_jA_i &= A_j^TA_i^T, \qquad \text{because the adjacency matrices are symmetric,}\\
&= (A_iA_j)^T\\
&= \left(\sum_k p_{ij}^k A_k\right)^T, \qquad \text{by Equation (34),}\\
&= \sum_k p_{ij}^k A_k^T\\
&= \sum_k p_{ij}^k A_k, \qquad \text{because the adjacency matrices are symmetric,}\\
&= A_iA_j. \qquad \square
\end{aligned}
$$

Example. Let $\prod$ be a **Latin square** of size n, i.e. an $n \times n$ array filled with n letters in such a way that each letter occurs once in each row and once in each column.

a	d	b	c
c	a	d	b
b	c	a	d
d	b	c	a

Fig.17. A Latin square of size 4.

Let Ω be the set of n^2 cells of the array. Consider $\alpha, \beta \in \Omega$, with $\alpha \neq \beta$. We declare α and β to be first associates if they are in the same row or in the same column or have the same letter. Otherwise, they are second associates. It is easy to check that so we get an association scheme on Ω, with two associate classes.

4.1.3. *The Bose-Mesner algebra.* Consider an association scheme with adjacency matrices $A_0, A_1 \ldots, A_s$. Let $\mathcal{A}$ be the space of all real linear combinations of these matrices. This is a real vector space of dimension $s+1$. In fact, the matrices $A_0, A_1, \ldots, A_s$ are linearly independent because, given α and β in Ω, there exists only one index i such that $A_i(\alpha, \beta) \neq 0$. It follows from Lemma 4.2 that $\mathcal{A}$ is closed under multiplication and so it is an algebra. Proposition 4.4 tells us that $\mathcal{A}$ is a commutative algebra, called the **Bose-Mesner algebra**.

Since every adjacency matrix is symmetric, a matrix $M \in \mathcal{A}$ is symmetric and so it is diagonalizable on $\mathbb{R}$, i.e. it has distinct real eigenvalues $\lambda_1, \dots, \lambda_r$ such that:

- $L(\Omega) = \bigoplus_{i=1}^{r} V_i$, where V_i is the eigenspace associated with the eigenvalue λ_i;
- the eigenspaces V_i and V_j are orthogonal, for $i \neq j$.

Here we denote $L(\omega)$ the space of the real functions defined on the set Ω.

The orthogonality of eigenspaces is with respect to the inner product on $L(\Omega)$ defined as

$$\langle f, g \rangle = \sum_{\omega \in \Omega} f(\omega) g(\omega), \qquad \text{for all } f, g \in L(\Omega).$$

Definition 4.5. *The orthogonal projector P on a subspace W is the map $P : L(\Omega) \longrightarrow L(\Omega)$ defined by*

$$Pv \in W \qquad \textit{and} \qquad v - Pv \in W^{\perp}.$$

Now put

$$P_1 = \frac{(M - \lambda_2 I) \cdots (M - \lambda_r I)}{(\lambda_1 - \lambda_2) \cdots (\lambda_1 - \lambda_r)}.$$

It is easy to check that, if $v \in V_1$, then $P_1 v = v$, while if $Mv = \lambda_i v$ for $i > 1$, then $P_1 v = 0$. So P_1 is the orthogonal projector onto V_1. Analogously for V_i, with $i > 1$.

Now let M_1 and M_2 be two matrices in $\mathcal{A}$ and let $P_1, \dots, P_r$ and $Q_1, \dots, Q_m$ be the respective eigenprojectors. They commute with each other, since they are polynomials in M_1 and M_2, respectively. The following properties of $P_i Q_j$'s hold:

- they are orthogonal, in fact $P_i Q_j P_{i'} Q_{j'} = P_i P_{i'} Q_j Q_{j'}$, which is zero unless $i = i'$ and $j = j'$;
- they are idempotents, in fact $P_i Q_j P_i Q_j = P_i P_i Q_j Q_j = P_i Q_j$;
- $\sum_i \sum_j P_i Q_j = (\sum_i P_i)(\sum_j Q_j) = I^2 = I$;
- the subspaces which they project onto are contained in eigenspaces of both M_1 and M_2.

If we apply this argument to $A_0, A_1, \dots, A_s$, we deduce that there exist mutually orthogonal subspaces $W_0, W_1, \dots, W_r$, with orthogonal projectors $S_0, S_1, \dots, S_r$, such that

- $L(\Omega) = W_0 \oplus W_1 \oplus \cdots \oplus W_r$;
- each W_i is contained in an eigenspace of every A_j;
- each S_i is a polynomial in $A_1, \dots, A_s$ and so in $\mathcal{A}$.

Thus there are unique constant $D(e, i)$ such that

$$S_e = \sum_i D(e, i) A_i.$$

On the other hand, if $C(i,e)$ is the eigenvalue of A_i on W_e, then

$$A_i = \sum_{e=0}^{r} C(i,e) S_e.$$

Moreover, the projectors $S_0, \ldots, S_r$ are linearly independent because $S_e S_f = \delta_{ef} S_e$ and so they constitute another basis for $\mathcal{A}$. Therefore we have $r = s$ and $D = C^{-1}$.

The subspaces W_e are called **strata**, while the matrices S_e are called **stratum projectors**. The matrix C is the **character table** of the association scheme.

4.1.4. *Crossed and nested product of association schemes.*

Definition 4.6. *Let $\mathcal{Q}_1$ be an association scheme on Ω_1 with classes $\mathcal{C}_i$, for $i \in \mathcal{K}_1$ and let $\mathcal{Q}_2$ be an association scheme on Ω_2 with classes $\mathcal{D}_j$, for $j \in \mathcal{K}_2$. Then $\mathcal{Q}_1$ is isomorphic to $\mathcal{Q}_2$ if there exist bijections*

$$\phi : \Omega_1 \longrightarrow \Omega_2 \quad \text{and} \quad \pi : \mathcal{K}_1 \longrightarrow \mathcal{K}_2$$

such that

$$(\alpha, \beta) \in \mathcal{C}_i \;\Leftrightarrow\; (\phi(\alpha), \phi(\beta)) \in \mathcal{D}_{\pi(i)}.$$

In this case, we say that the pair (ϕ, π) is an **isomorphism** between association schemes and write $\mathcal{Q}_1 \cong \mathcal{Q}_2$.

We can now introduce two special products of association schemes, called the crossed product and the nested product, respectively.

So let $\mathcal{Q}_1$ be an association scheme on the finite set Ω_1 with adjacency matrices $A_0, A_1, \ldots, A_m$, and let $\mathcal{Q}_2$ be an association scheme on the finite set Ω_2 with adjacency matrices $B_0, B_1, \ldots, B_r$.

Definition 4.7. *The **crossed product** of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ is the association scheme $\mathcal{Q}_1 \times \mathcal{Q}_2$ on $\Omega_1 \times \Omega_2$ whose adjacency matrices are*

$$A_i \otimes B_j,$$

for $i = 0, \ldots, m$ and $j = 0, \ldots, r$.

The crossed product of two association schemes is also called *direct product*. For example, one can easily verify that the rectangular association scheme $R(m,n)$ can be obtained as the crossed product of the schemes $\underline{\underline{m}}$ and $\underline{\underline{n}}$.

Definition 4.8. *The **nested product** of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ is the association scheme $\mathcal{Q}_1 / \mathcal{Q}_2$ on $\Omega_1 \times \Omega_2$ whose adjacency matrices are*

- *$A_i \otimes J_{\Omega_2}$, with $i \neq 0$;*
- *$I_{\Omega_1} \otimes B_j$, for every $j = 0, 1, \ldots, r$.*

The nested product of two association schemes is also called *wreath product*. For example, one can easily verify that the group-divisible association scheme $GD(m,n)$ can be obtained as the nested product of the schemes $\underline{\underline{m}}$ and $\underline{\underline{n}}$.

PROPOSITION 4.9. *The following properties of crossed and nested product hold:*

(1) *crossing is commutative, in the sense that* $\mathcal{Q}_1 \times \mathcal{Q}_2 \cong \mathcal{Q}_2 \times \mathcal{Q}_1$*;*
(2) *crossing is associative, in the sense that* $\mathcal{Q}_1 \times (\mathcal{Q}_2 \times \mathcal{Q}_3) \cong (\mathcal{Q}_1 \times \mathcal{Q}_2) \times \mathcal{Q}_3$*;*
(3) *nesting is associative, in the sense that* $\mathcal{Q}_1/(\mathcal{Q}_2/\mathcal{Q}_3) \cong (\mathcal{Q}_1/\mathcal{Q}_2)/\mathcal{Q}_3$.

Remarks. It is interesting to observe that the adjacency matrices of the nested product of association schemes remind the transition matrices occurring in the nested product of Markov chains (see Formula (28)). A similar consideration can be done for crossed product.

As in the case of reversible Markov chains, the crested product of association schemes, described in the following section, is a more general construction containing, as particular cases, the crossed and the nested product.

4.2. Crested product of association schemes. We introduce here the crested product of two association schemes $\mathcal{Q}_1$ and $\mathcal{Q}_2$, giving a new association scheme on the space $\Omega_1 \times \Omega_2$ that contains both crossed and nested products as special cases. Our main source is [**4**].

4.2.1. *Preliminaries.* Consider the definition of orthogonal block structures given in Definition 3.2 of Chapter 1. With a partition F belonging to an orthogonal block structure $\mathcal{F}$ on Ω, one can associate the adjacency matrix A_F defined as

$$A_F(\alpha,\beta) = \begin{cases} 1 & \text{if } F = \bigwedge\{G \in \mathcal{F} : R_G(\alpha,\beta) = 1\} \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to verify that the set $\{A_F : F \in \mathcal{F}, A_F \neq 0\}$ is an association scheme on Ω (see [**3**]).

Given two partitions F and G of two sets Ω_1 and Ω_2, respectively, denote $F \times G$ the partition of $\Omega_1 \times \Omega_2$ whose relation matrix is $R_F \otimes R_G$.

Now let $\mathcal{F}$ and $\mathcal{G}$ be two orthogonal block structures on Ω_1 and Ω_2, respectively. Then their *crossed product* is given by

$$\mathcal{F} \times \mathcal{G} = \{F \times G : F \in \mathcal{F}, G \in \mathcal{G}\}$$

and their *nested product* is given by

$$\mathcal{F}/\mathcal{G} = \{F \times U_2 : F \in \mathcal{F}\} \cup \{E_1 \times G : G \in \mathcal{G}\},$$

where E_i and U_i are the trivial partitions of Ω_i. One can show that the operation of deriving the association scheme from the orthogonal block structure commutes with both crossing and nesting.

DEFINITION 4.10. *For $i = 1, 2$, let $\mathcal{F}_i$ be an orthogonal block structure on a set Ω_i and choose $F_i \in \mathcal{F}_i$. The* ***crested product*** *of $\mathcal{F}_1$ and $\mathcal{F}_2$ with respect to F_1 and F_2 is the set $\mathcal{G}$ of partitions of $\Omega_1 \times \Omega_2$ given by*

$$\mathcal{G} = \{G_1 \times G_2 : G_1 \in \mathcal{F}_1, G_2 \in \mathcal{F}_2, G_1 \preccurlyeq F_1 \text{ or } G_2 \succcurlyeq F_2\}. \tag{35}$$

In [**4**] it is proven that the crested product of orthogonal block structures defined above is an orthogonal block structure on $\Omega_1 \times \Omega_2$.

Remarks.

- If $F_1 = U_1$ or $F_2 = E_2$, then $\mathcal{G}$ is the crossed product $\mathcal{F}_1 \times \mathcal{F}_2$.
- If $F_1 = E_1$ and $F_2 = U_2$, then $\mathcal{G}$ is the nested product $\mathcal{F}_1/\mathcal{F}_2$.

DEFINITION 4.11. *Let $\mathcal{Q}$ be an association scheme on Ω with adjacency matrices A_i, for $i \in \mathcal{K}$. Then a partition F of Ω is* ***inherent*** *in $\mathcal{Q}$ if its relation matrix R_F is in the Bose-Mesner algebra of $\mathcal{Q}$, i.e. if there exists a subset $\mathcal{L}$ of $\mathcal{K}$ such that $R_F = \sum_{i \in \mathcal{L}} A_i$.*

It is easy to check that the trivial partitions E and U are inherent in every association scheme.

Example. Consider the 12 edges of the cube and define an association scheme on the set Ω of these edges in the following way:

- two edges α and β are $1-$st associates if they meet at a vertex;
- two edges α and β are $2-$nd associates if they are diagonally opposite;
- two edges α and β are $3-$rd associates if they are parallel but not opposite;
- two edges α and β are $4-$th associates if they are skew.

The partitions inherent in this scheme have relation matrices $A_0 = I_\Omega$, $A_0 + A_2$, $A_0 + A_2 + A_3$ and $A_0 + A_1 + A_2 + A_3 + A_4 = J_\Omega$.

THEOREM 4.12. *If $\mathcal{Q}$ is an association scheme on Ω, then the set $\mathcal{F}$ of partitions of Ω which are inherent in $\mathcal{Q}$ is an orthogonal block structure on Ω.*

See [**4**] for the proof.

Now let $\mathcal{P}$ be a partition of $\Omega \times \Omega$ and let $V(\mathcal{P})$ be the real span of the adjacency matrices of its classes. It is clear that

$$\mathcal{Q} \preccurlyeq \mathcal{P} \quad \Longleftrightarrow \quad V(\mathcal{P}) \leq \mathcal{A},$$

where $\mathcal{A}$ is the Bose-Mesner algebra of $\mathcal{Q}$.

Definition 4.13. *Let $\mathcal{Q}$ be an association scheme on Ω. A partition $\mathcal{P}$ of $\Omega \times \Omega$ is* ***ideal*** *for $\mathcal{Q}$ if $V(\mathcal{P})$ is an ideal of $\mathcal{A}$, i.e. $V(\mathcal{P}) \leq \mathcal{A}$ and $AD \in V(\mathcal{P})$ whenever $A \in \mathcal{A}$ and $D \in V(\mathcal{P})$.*

Theorem 4.14. *Let $\mathcal{Q}$ be an association scheme with adjacency matrices A_i, for $i \in \mathcal{K}$. If $\mathcal{Q}$ has an inherent partition F with relation matrix R_F, then there exists an ideal partition $\vartheta(F)$ of $\mathcal{Q}$ whose adjacency matrices are scalar multiples of A_iR, for $i \in \mathcal{K}$.*

Proof. (Sketch) Let $\mathcal{L}$ be the subset of $\mathcal{K}$ such that $R_F = \sum_{i\in\mathcal{L}} A_i$. So there exist positive integers m_{ij} such that

$$R_F A_i = A_i R_F = \sum_{j\in\mathcal{K}} m_{ij} A_j.$$

It follows from the definition that

$$m_{ij} = (A_i R_F)(\alpha, \beta) = |\mathcal{C}_i(\alpha) \cap F(\beta)|,$$

where $F(\beta)$ denotes the $F-$class containing β. Put $i \sim j$ if $m_{ij} \neq 0$. One can check that $\sim$ is an equivalence relation. Define $[i] = \{j \in \mathcal{K} : j \sim i\}$ and $B_{[i]} = \sum_{j\sim i} A_j$. Then the distinct $B_{[i]}$ are the adjacency matrices of a partition $\mathcal{P}$ of $\Omega \times \Omega$ such that $\mathcal{Q} \preccurlyeq \mathcal{P}$. Moreover, it is easy to verify that $A_j B_{[i]} \in V(\mathcal{P})$. $\square$

Indeed, the inverse construction can be done, as the following theorem shows (see [**4**]).

Theorem 4.15. *Let $\mathcal{P}$ be an ideal partition for $\mathcal{Q}$. Let A_i be the adjacency matrices of $\mathcal{Q}$, for $i \in \mathcal{K}$, and let D_m be the adjacency matrices of $\mathcal{P}$, for $m \in \mathcal{M}$. Denote by σ the surjection from $\mathcal{K}$ to $\mathcal{M}$ such that class i of $\mathcal{Q}$ is contained in class $\sigma(i)$ of $\mathcal{P}$. Put $R = D_{\sigma(0)}$. Then R is the relation matrix of an inherent partition in $\mathcal{Q}$. Moreover, for all $i \in \mathcal{K}$, the matrix A_iR is an integer multiple of $D_{\sigma(i)}$.*

4.2.2. *Crested product of association schemes.* Let F be a partition in an orthogonal block structure $\mathcal{F}$, so that $R_F = \sum_{G\in\mathcal{L}} A_G$, where $\mathcal{L} = \{G \in \mathcal{F} : G \preccurlyeq F\}$. This implies that F is inherent in the association scheme derived from $\mathcal{F}$. Then $\{A_G : G \in \mathcal{L}\}$ and $\{R_G : G \in \mathcal{L}\}$ span the same subspace $\mathcal{A}|_F$ of $\mathcal{A}$, which is closed under matrix multiplication.

Let $\mathcal{P}$ be the ideal partition $\vartheta(F)$. For $G \in \mathcal{F}$, R_G is in the ideal of $\mathcal{A}$ generated by R_F if and only if $F \preccurlyeq G$, so $V(\mathcal{P})$ is the span of $\{R_G : G \in \mathcal{F},\ G \succcurlyeq F\}$. We denote $V(\vartheta(F))$ by $A|^F$.

Consider now the crested product $\mathcal{G}$ of the orthogonal block structures $\mathcal{F}_1$ and $\mathcal{F}_2$ with respect to the partitions F_1 and F_2. The span of the relation matrices of the partitions in $\mathcal{G}$ is

$$(\mathcal{A}_1|_{F_1} \otimes \mathcal{A}_2) + (\mathcal{A}_1 \otimes \mathcal{A}_2|^{F_2}),$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are the Bose-Mesner algebra of the association schemes derived by $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. The adjacency matrices of the association scheme derived by $\mathcal{G}$ are:

- $A_G \otimes A_H$, for $G \in \mathcal{L}$ and $H \in \mathcal{F}_2$;
- $A_G \otimes D$, for $G \in \mathcal{F}_1 \setminus \mathcal{L}$ and D an adjacency matrix of $\mathcal{P}$,

where $\mathcal{L} = \{G \in \mathcal{F}_1 : G \preccurlyeq F_1\}$ and $\mathcal{P} = \vartheta(F_2)$. This leads to the following definition.

Definition 4.16. *For $r = 1, 2$, let $\mathcal{Q}_r$ be an association scheme on a set Ω_r and let F_r be an inherent partition in $\mathcal{Q}_r$. Put $\mathcal{P} = \vartheta(F_2)$ and $\Omega = \Omega_1 \times \Omega_2$. Let the adjacency matrices of $\mathcal{Q}_1, \mathcal{Q}_2$ and $\mathcal{P}$ be A_i, for $i \in \mathcal{K}_1$, B_j, for $j \in \mathcal{K}_2$ and D_m, for $m \in \mathcal{M}$, respectively. Let $\mathcal{L}$ be the subset of $\mathcal{K}_1$ such that $R_{F_1} = \sum_{i\in\mathcal{L}} A_i$. The **crested product** of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ with respect to F_1 and F_2 is the association scheme $\mathcal{Q}$ on Ω whose adjacency matrices are*

- *$A_i \otimes B_j$, for $i \in \mathcal{L}$ and $j \in \mathcal{K}_2$;*
- *$A_i \otimes D_m$, for $i \in \mathcal{K}_1 \setminus \mathcal{L}$ and $m \in \mathcal{M}$.*

Observe that the crested product reduces to the crossed product if $F_1 = U_1$ or $F_2 = E_2$ (in which case $\mathcal{P} = \mathcal{Q}_2$) and it reduces to the nested product if $F_1 = E_1$ and $F_2 = U_2$ (in which case $\mathcal{P} = U_{\Omega_2\times\Omega_2}$).

Finally, the character table of the crested product $\mathcal{Q}$ can be described using the character table of the schemes $\mathcal{Q}_1$ and $\mathcal{Q}_2$. See [**4**] for more details.

4.2.3. *Some examples.* Let Q be an association scheme on a finite set Ω and let $A_0 = I_\Omega, A_1, \ldots, A_m$ be the adjacency matrices associated with Q. Consider also an association scheme Q' on a second finite set Ω', whose adjacency matrices are $A'_0 = I_{\Omega'}, A'_1, \ldots, A'_{m'}$.

We know that the nested product Q/Q' of the schemes Q and Q' is the association scheme on the set $\Omega \times \Omega'$ whose adjacency matrices are

- $A_i \otimes J_{\Omega'}$, for $i \neq 0$;
- $I_\Omega \otimes A'_j$, for $j = 0, 1, \ldots, m'$.

Consider now the inherent partition F of $\Omega \times \Omega'$ whose relation matrix is

$$R_F = \sum_{j=0}^{m'} (I_\Omega \otimes A'_j) = I_\Omega \otimes J_{\Omega'},$$

i.e. the partition $\Omega \times \Omega' = \bigsqcup_{\alpha\in\Omega}\{(\alpha, \alpha') : \alpha' \in \Omega'\}$. We can ask which is the ideal partition associated with F.

Theorem 4.14 tells us that the adjacency matrices of the ideal partition $\mathcal{P}$ of $X \times X$ associated with F are $D_i = \sum_{i \sim j} A_j$ (we will use also the notation $A_i \sim A_j$ to indicate $i \sim j$).

In our case we have $I_\Omega \otimes A'_j \sim I_\Omega \otimes A'_k$ for every $j, k = 0, 1, \dots, m'$. Moreover, it is easy to verify that, for $i, j \neq 0$, one has $A_i \otimes J_{\Omega'} \not\sim A_j \otimes J_{\Omega'}$ for $i \neq j$. So the adjacency matrices of the ideal partition $\mathcal{P}$ associated with F are

$$A_i \otimes J_{\Omega'}, \qquad \text{for } i = 0, 1, \dots, m.$$

Consider now an association scheme S on a finite set Θ with adjacency matrices $B_0 = I_\Theta, B_1, \dots, B_n$ and an association scheme S' on a finite set Θ' whose adjacency matrices are $B'_0 = I_{\Theta'}, B'_1, \dots, B'_n$. Take again the nested product S/S' on $\Theta \times \Theta'$, whose adjacency matrices are

- $B_i \otimes J_{\Theta'}$, for $i \neq 0$;
- $I_\Theta \otimes B'_j$, for $j = 0, 1, \dots, n'$.

We can consider the inherent partition G of $\Theta \times \Theta'$ whose relation matrix is

$$R_G = \sum_{j=0}^{n'} I_\Theta \otimes B'_j = I_\Theta \otimes J_{\Theta'},$$

which corresponds to the partition $\Theta \times \Theta' = \coprod_{\theta \in \Theta} \{(\theta, \theta') : \theta' \in \Theta'\}$.

We can now consider the crested product of the schemes S/S' and Q/Q' with respect to the inherent partitions G and F defined above. So we get a new association scheme on the set

$$\Theta \times \Theta' \times \Omega \times \Omega'$$

whose adjacency matrices are

- $(I_\Theta \otimes B'_j) \otimes (A_i \otimes J_{\Omega'})$, with $j = 0, 1, \dots, n'$ and $i \neq 0$;
- $(I_\Theta \otimes B'_j) \otimes (I_\Omega \otimes A'_k)$, with $j = 0, 1, \dots, n'$ and $k = 0, 1, \dots, m'$;
- $(B_i \otimes J_{\Theta'}) \otimes (A_j \otimes J_{\Omega'})$, with $i \neq 0$ and $j = 0, 1, \dots, m$.

Moreover, by choosing the inherent partition G for $\Theta \times \Theta'$ and the universal partition $U_{\Omega \times \Omega'}$ for $\Omega \times \Omega'$, i.e. the partition whose relation matrix is $R_{U_{\Omega \times \Omega'}} = J_\Omega \otimes J_{\Omega'}$, we can get a different crested product of the schemes S/S' and Q/Q'. Observe that the only adjacency matrix of the ideal partition $\mathcal{P}$ associated with $U_{\Omega \times \Omega'}$ is $J_\Omega \otimes J_{\Omega'}$. So the adjacency matrices of the crested product of the schemes S/S' and Q/Q' are

- $(I_\Theta \otimes B'_j) \otimes (A_i \otimes J_{\Omega'})$, with $j = 0, 1, \dots, n'$ and $i \neq 0$;
- $(I_\Theta \otimes B'_j) \otimes (I_\Omega \otimes A'_k)$, with $j = 0, 1, \dots, n'$ and $k = 0, 1, \dots, m'$;
- $(B_i \otimes J_{\Theta'}) \otimes (J_\Omega \otimes J_{\Omega'})$, with $i \neq 0$.

Finally, by choosing the identity partition $E_{\Theta \times \Theta'}$ for $\Theta \times \Theta'$ and the inherent partition F for $\Omega \times \Omega'$, we can get again a different crested product of the schemes S/S' and Q/Q', whose adjacency matrices are

- $(I_\Theta \otimes I_{\Theta'}) \otimes (A_i \otimes J_{\Omega'})$, with $i \neq 0$;

- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes A'_k)$, with $k = 0, 1, \ldots, m'$;
- $(I_\Theta \otimes B'_k) \otimes (A_i \otimes J_{\Omega'})$, with $i = 0, 1, \ldots, m$ and $k \neq 0$;
- $(B_j \otimes J_{\Theta'}) \otimes (A_i \otimes J_{\Omega'})$, with $j \neq 0$ and $i = 0, 1, \ldots, m$.

This completes the description of the nontrivial crested products that we can get from the schemes S/S' and Q/Q'. By choosing the identity partition $E_{\Theta\times\Theta'}$ as inherent partition of $\Theta \times \Theta'$ and the universal partition $U_{\Omega\times\Omega'}$ as inherent partition of $\Omega\times\Omega'$, we get the nested product

$$S/S'/Q/Q'.$$

This notation is correct because of the associativity of iterating the nested product of association schemes. The adjacency matrices of the scheme $S/S'/Q/Q'$ are, in this case,

- $(I_\Theta \otimes I_{\Theta'}) \otimes (A_i \otimes J_{\Omega'})$, with $i \neq 0$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes A'_k)$, with $k = 0, 1, \ldots, m'$;
- $(I_\Theta \otimes B'_k) \otimes (J_\Omega \otimes J_{\Omega'})$, with $k \neq 0$;
- $(B_j \otimes J_{\Theta'}) \otimes (J_\Omega \otimes J_{\Omega'})$, with $j \neq 0$.

The remaining choices for the inherent partitions of $\Theta \times \Theta'$ and $\Omega \times \Omega'$ give rise to the crossed product

$$(S/S') \times (Q/Q'),$$

i.e. the association scheme on $\Theta\times\Theta'\times\Omega\times\Omega'$ whose adjacency matrices are

- $(I_\Theta\otimes B'_j)\otimes(I_\Omega\otimes A'_k)$, with $j = 0, 1, \ldots, n'$ and $k = 0, 1, \ldots, m'$;
- $(I_\Theta \otimes B'_j) \otimes (A_i \otimes J_{\Omega'})$, with $j = 0, 1, \ldots, n'$ and $i \neq 0$;
- $(B_i \otimes J_{\Theta'}) \otimes (I_\Omega \otimes A'_k)$, with $i \neq 0$ and $k = 0, 1, \ldots, m'$;
- $(B_i \otimes J_{\Theta'}) \otimes (A_k \otimes J_{\Omega'})$, with $i, k \neq 0$.

As an easy example, we can consider the case when $\Theta = \Theta' = \Omega = \Omega' = \{1, 2\}$ and $S = S' = Q = Q' = \underline{\underline{2}}$. We recall that $\underline{\underline{2}}$ denotes the trivial association scheme on two elements, whose adjacency matrices are

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let us call these matrices B_0 and B_1 in the case of S, B'_0 and B'_1 in the case of S', A_0 and A_1 in the case of Q, A'_0 and A'_1 in the case of Q', respectively.

So the adjacency matrices of the nested product Q/Q' are

- $A_1 \otimes J_{\Omega'}$;
- $I_\Omega \otimes I_{\Omega'}$;
- $I_\Omega \otimes A'_1$.

Consider now the inherent partition F of $\Omega \times \Omega'$ whose relation matrix is

$$R_F = I_\Omega \otimes I_{\Omega'} + I_\Omega \otimes A'_1 = I_\Omega \otimes J_{\Omega'},$$

corresponding to the partition $\Omega\times\Omega' = \{(1,1),(1,2)\}\coprod\{(2,1),(2,2)\}$.

The adjacency matrices of the ideal partition $\mathcal{P}$ associated with F are

- $I_\Omega \otimes J_{\Omega'}$;
- $A_1 \otimes J_{\Omega'}$.

Analogously, the adjacency matrices associated with the nested product S/S' defined on the product $\Theta \times \Theta'$ are

- $B_1 \otimes J_{\Theta'}$;
- $I_\Theta \otimes I_{\Theta'}$;
- $I_\Theta \otimes B'_1$.

Consider the inherent partition G of $\Theta \times \Theta'$ whose relation matrix is, as above,

$$R_G = I_\Theta \otimes I_{\Theta'} + I_\Theta \otimes B'_1 = I_\Theta \otimes J_{\Theta'}.$$

We can now study the crested product of the schemes $\underline{\underline{2}}/\underline{\underline{2}}$ and $\underline{\underline{2}}/\underline{\underline{2}}$ with respect to the inherent partitions G and F defined above. So we get the association scheme on the set

$$\Theta \times \Theta' \times \Omega \times \Omega'$$

whose adjacency matrices are

- $(I_\Theta \otimes I_{\Theta'}) \otimes (A_1 \otimes J_{\Omega'})$;
- $(I_\Theta \otimes B'_1) \otimes (A_1 \otimes J_{\Omega'})$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes I_{\Omega'})$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes A'_1)$;
- $(I_\Theta \otimes B'_1) \otimes (I_\Omega \otimes I_{\Omega'})$;
- $(I_\Theta \otimes B'_1) \otimes (I_\Omega \otimes A'_1)$;
- $(B_1 \otimes J_{\Theta'}) \otimes (I_\Omega \otimes J_{\Omega'})$;
- $(B_1 \otimes J_{\Theta'}) \otimes (A_1 \otimes J_{\Omega'})$.

By choosing the inherent partition G for $\Theta \times \Theta'$ and the universal partition $U_{\Omega\times\Omega'}$ for $\Omega \times \Omega'$, i.e. the partition whose relation matrix is $R_{U_{\Omega\times\Omega'}} = J_\Omega \otimes J_{\Omega'}$, we get a different crested product of the schemes $\underline{\underline{2}}/\underline{\underline{2}}$ and $\underline{\underline{2}}/\underline{\underline{2}}$. The only adjacency matrix of the ideal partition $\mathcal{P}$ associated with $U_{\Omega\times\Omega'}$ is $J_\Omega \otimes J_{\Omega'}$. So the adjacency matrices of the crested product of the schemes $\underline{\underline{2}}/\underline{\underline{2}}$ and $\underline{\underline{2}}/\underline{\underline{2}}$ are

- $(I_\Theta \otimes I_{\Theta'}) \otimes (A_1 \otimes J_{\Omega'})$;
- $(I_\Theta \otimes B'_1) \otimes (A_1 \otimes J_{\Omega'})$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes I_{\Omega'})$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes A'_1)$;
- $(I_\Theta \otimes B'_1) \otimes (I_\Omega \otimes I_{\Omega'})$;
- $(I_\Theta \otimes B'_1) \otimes (I_\Omega \otimes A'_1)$;
- $(B_1 \otimes J_{\Theta'}) \otimes (J_\Omega \otimes J_{\Omega'})$.

Finally, by choosing the identity partition $E_{\Theta\times\Theta'}$ for $\Theta \times \Theta'$ and the inherent partition F for $\Omega \times \Omega'$, we get again a different crested product of the schemes $\underline{\underline{2}}/\underline{\underline{2}}$ and $\underline{\underline{2}}/\underline{\underline{2}}$, whose adjacency matrices are

- $(I_\Theta \otimes I_{\Theta'}) \otimes (A_1 \otimes J_{\Omega'})$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes I_{\Omega'})$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes A'_1)$;
- $(I_\Theta \otimes B'_1) \otimes (I_\Omega \otimes J_{\Omega'})$;
- $(I_\Theta \otimes B'_1) \otimes (A_1 \otimes J_{\Omega'})$;
- $(B_1 \otimes J_{\Theta'}) \otimes (I_\Omega \otimes J_{\Omega'})$;
- $(B_1 \otimes J_{\Theta'}) \otimes (A_1 \otimes J_{\Omega'})$.

This completes the description of the nontrivial crested products that we can get from the schemes $\underline{\underline{2}}/\underline{\underline{2}}$ and $\underline{\underline{2}}/\underline{\underline{2}}$. By choosing the identity partition $E_{\Theta\times\Theta'}$ as inherent partition of $\Theta \times \Theta'$ and the universal partition $U_{\Omega\times\Omega'}$ as inherent partition of $\Omega\times\Omega'$, we get the nested product

$$\underline{\underline{2}}/\underline{\underline{2}}/\underline{\underline{2}}/\underline{\underline{2}}.$$

The adjacency matrices of this scheme are

- $(I_\Theta \otimes I_{\Theta'}) \otimes (A_1 \otimes J_{\Omega'})$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes I_{\Omega'})$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes A'_1)$;
- $(I_\Theta \otimes B'_1) \otimes (J_\Omega \otimes J_{\Omega'})$;
- $(B_1 \otimes J_{\Theta'}) \otimes (J_\Omega \otimes J_{\Omega'})$.

The remaining choices of inherent partitions of $\Theta \times \Theta'$ and $\Omega \times \Omega'$ give rise to the crossed product

$$(\underline{\underline{2}}/\underline{\underline{2}}) \times (\underline{\underline{2}}/\underline{\underline{2}}),$$

whose adjacency matrices are

- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes I_{\Omega'})$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (I_\Omega \otimes A'_1)$;
- $(I_\Theta \otimes I_{\Theta'}) \otimes (A_1 \otimes J_{\Omega'})$;
- $(I_\Theta \otimes B'_1) \otimes (I_\Omega \otimes I_{\Omega'})$;
- $(I_\Theta \otimes B'_1) \otimes (I_\Omega \otimes A'_1)$;
- $(I_\Theta \otimes B'_1) \otimes (A_1 \otimes J_{\Omega'})$;
- $(B_1 \otimes J_{\Theta'}) \otimes (I_\Omega \otimes I_{\Omega'})$;
- $(B_1 \otimes J_{\Theta'}) \otimes (I_\Omega \otimes A'_1)$;
- $(B_1 \otimes J_{\Theta'}) \otimes (A_1 \otimes J_{\Omega'})$.

Remark. These products have also another interpretation from the orthogonal block structures point of view.

In fact, a ultrametric space has in a natural way an orthogonal block structure: if we fix a level L_i of the rooted tree of depth n, for $i = 1, \ldots, n$, then this level induces a partition in spheres on the $n-$th level: in particular, for any vertex $x \in L_i$, one sphere will be constituted by the vertices of L_n which have x as ancestor. Considering the partition in spheres induced by each level, one gets an orthogonal block structure.

Take now two rooted trees of depth 2 with branch indices (m, n) and (p, q), respectively. Consider the corresponding orthogonal block structures: each block consists of three partitions with sizes $1, n, mn$ and $1, q, pq$, respectively. We denote these partitions by F_0, F_1, F_2 for the first tree and by G_0, G_1, G_2 for the second tree. So the relation matrices in the case of the first tree are

- $R_0 = I_m \otimes I_n$;
- $R_1 = I_m \otimes J_n$;
- $R_2 = J_m \otimes J_n$

and in the case of the second tree are

- $S_0 = I_p \otimes I_q$;
- $S_1 = I_p \otimes J_q$;
- $S_2 = J_p \otimes J_q$.

The corresponding association schemes that we can get considering the matrices A_F defined above are Q, with adjacency matrices

- $A_0 = I_m \otimes I_n$;
- $A_1 = I_m \otimes (J_n - I_n)$;
- $A_2 = (J_m - I_m) \otimes J_n$

and Q', with adjacency matrices

- $A'_0 = I_p \otimes I_q$;
- $A'_1 = I_p \otimes (J_q - I_q)$;
- $A'_2 = (J_p - I_p) \otimes J_q$.

So we can observe that the association scheme Q is just the scheme $\underline{\underline{m}}/\underline{\underline{n}}$ and the association scheme Q' is just the scheme $\underline{\underline{p}}/\underline{\underline{q}}$. We can do the crested product of these schemes with respect to the possible inherent partitions, whose relation matrices are R_0 or S_0 in the case of the equality partition, then R_1 or S_1 and finally R_2 or S_2 in the case of the universal partition.

We can also do the crested product of orthogonal block structures and then we can associate to the block obtained a new association scheme by using the matrices A_F. Actually, we can show that the operation of deriving the association scheme from the orthogonal block structure commutes with cresting. Let us verify it in all cases.

The relation matrices of the block obtained by the crest product with respect to the partition F_1 and G_1 are

- $R_0 \otimes S_0$, with associated adjacency matrix $A_{0,0} = I_m \otimes I_n \otimes I_p \otimes I_q$;
- $R_0 \otimes S_1$, with $A_{0,1} = I_m \otimes I_n \otimes I_p \otimes (J_q - I_q)$;
- $R_0 \otimes S_2$, with $A_{0,2} = I_m \otimes I_n \otimes (J_p - I_p) \otimes J_q$;
- $R_1 \otimes S_0$, with $A_{1,0} = I_m \otimes (J_n - I_n) \otimes I_p \otimes I_q$;
- $R_1 \otimes S_1$, with $A_{1,1} = I_m \otimes (J_n - I_n) \otimes I_p \otimes (J_q - I_q)$;

- $R_1 \otimes S_2$, with $A_{1,2} = I_m \otimes (J_n - I_n) \otimes (J_p - I_p) \otimes J_q$;
- $R_2 \otimes S_1$, with $A_{2,1} = (J_m - I_m) \otimes J_n \otimes I_p \otimes J_q$;
- $R_2 \otimes S_2$, with $A_{2,2} = (J_m - I_m) \otimes J_n \otimes (J_p - I_p) \otimes J_q$

and these matrices $A_{i,j}$'s are just the adjacency matrices of the association scheme obtained by the crested product of the association schemes Q and Q' by choosing the partitions F_1 and G_1 as inherent partitions, respectively.

The relation matrices of the block obtained with the crest product with respect to the partition F_1 and G_2 are

- $R_0 \otimes S_0$, with associated adjacency matrix $A_{0,0} = I_m \otimes I_n \otimes I_p \otimes I_q$;
- $R_0 \otimes S_1$, with $A_{0,1} = I_m \otimes I_n \otimes I_p \otimes (J_q - I_q)$;
- $R_0 \otimes S_2$, with $A_{0,2} = I_m \otimes I_n \otimes (J_p - I_p) \otimes J_q$;
- $R_1 \otimes S_0$, with $A_{1,0} = I_m \otimes (J_n - I_n) \otimes I_p \otimes I_q$;
- $R_1 \otimes S_1$, with $A_{1,1} = I_m \otimes (J_n - I_n) \otimes I_p \otimes (J_q - I_q)$;
- $R_1 \otimes S_2$, with $A_{1,2} = I_m \otimes (J_n - I_n) \otimes (J_p - I_p) \otimes J_q$;
- $R_2 \otimes S_2$, with $A_{2,2} = (J_m - I_m) \otimes J_n \otimes J_p \otimes J_q$

and these matrices $A_{i,j}$'s are just the adjacency matrices of the association scheme obtained by the crested product of the association schemes Q and Q' by choosing the partitions F_1 and G_2 as inherent partitions, respectively.

The relation matrices of the block obtained with the crest product with respect to the partition F_0 and G_1 are

- $R_0 \otimes S_0$, with associated adjacency matrix $A_{0,0} = I_m \otimes I_n \otimes I_p \otimes I_q$;
- $R_0 \otimes S_1$, with $A_{0,1} = I_m \otimes I_n \otimes I_p \otimes (J_q - I_q)$;
- $R_0 \otimes S_2$, with $A_{0,2} = I_m \otimes I_n \otimes (J_p - I_p) \otimes J_q$;
- $R_1 \otimes S_1$, with $A_{1,1} = I_m \otimes (J_n - I_n) \otimes I_p \otimes J_q$;
- $R_2 \otimes S_1$, with $A_{2,1} = (J_m - I_m) \otimes J_n \otimes I_p \otimes J_q$
- $R_1 \otimes S_2$, with $A_{1,2} = I_m \otimes (J_n - I_n) \otimes (J_p - I_p) \otimes J_q$;
- $R_2 \otimes S_2$, with $A_{2,2} = (J_m - I_m) \otimes J_n \otimes (J_p - I_p) \otimes J_q$

and these matrices $A_{i,j}$'s are just the adjacency matrices of the association scheme obtained by the crested product of the association schemes Q and Q' by choosing the partitions F_0 and G_1 as inherent partitions, respectively.

The same result can be obtained by considering the crossed product and the nested product.

In fact, the relation matrices of the block obtained with the crest product with respect to the partition F_0 and G_2 are

- $R_0 \otimes S_0$, with associated adjacency matrix $A_{0,0} = I_m \otimes I_n \otimes I_p \otimes I_q$;
- $R_0 \otimes S_1$, with $A_{0,1} = I_m \otimes I_n \otimes I_p \otimes (J_q - I_q)$;
- $R_0 \otimes S_2$, with $A_{0,2} = I_m \otimes I_n \otimes (J_p - I_p) \otimes J_q$;
- $R_1 \otimes S_2$, with $A_{1,2} = I_m \otimes (J_n - I_n) \otimes J_p \otimes J_q$;

- $R_2 \otimes S_2$, with $A_{2,2} = (J_m - I_m) \otimes J_n \otimes J_p \otimes J_q$

and these matrices $A_{i,j}$'s are just the adjacency matrices of the association scheme obtained by the crested product of the association schemes Q and Q' by choosing the partitions F_0 and G_2 as inherent partitions, respectively. The remaining choices for the partitions give rise to the crossed product. The relation matrices of the block obtained with the crossed product are

- $R_0 \otimes S_0$, with associated adjacency matrix $A_{0,0} = I_m \otimes I_n \otimes I_p \otimes I_q$;
- $R_0 \otimes S_1$, with $A_{0,1} = I_m \otimes I_n \otimes I_p \otimes (J_q - I_q)$;
- $R_0 \otimes S_2$, with $A_{0,2} = I_m \otimes I_n \otimes (J_p - I_p) \otimes J_q$;
- $R_1 \otimes S_0$, with $A_{1,0} = I_m \otimes (J_n - I_n) \otimes I_p \otimes I_q$;
- $R_1 \otimes S_1$, with $A_{1,1} = I_m \otimes (J_n - I_n) \otimes I_p \otimes (J_q - I_q)$;
- $R_1 \otimes S_2$, with $A_{1,2} = I_m \otimes (J_n - I_n) \otimes (J_p - I_p) \otimes J_q$;
- $R_2 \otimes S_0$, with $A_{2,0} = (J_m - I_m) \otimes J_n \otimes I_p \otimes I_q$;
- $R_2 \otimes S_1$, with $A_{2,1} = (J_m - I_m) \otimes J_n \otimes I_p \otimes (J_q - I_q)$;
- $R_2 \otimes S_2$, with $A_{2,2} = (J_m - I_m) \otimes J_n \otimes (J_p - I_p) \otimes J_q$.

The interesting fact is that the nested product of the two original blocks gives an orthogonal block structure on a set with $mnpq$ elements, which is exactly the block of spherical partitions of the fourth level of the rooted tree of depth 4 and branch indices (m, n, p, q). The remaining crested product give other orthogonal block structures corresponding to different partitions which are not induced by the spheres of the trees.

5. A Markov chain on orthogonal block structures

In this section I will define a Markov chain on orthogonal block structures, introduced in [**21**], which reduces to the Insect Markov chain presented in Chapter 2, Section 2.3, if the orthogonal block is the poset block structure associated with a chain $(I, \leq)$.

In what follows, we will use the notation of Chapter 1.

Let $\mathcal{F}$ be an orthogonal block structure on a finite set Ω. We want to associate with $\mathcal{F}$ a Markov chain on Ω.

The ancestral poset defined in Chapter 1 is a particular case of the poset associated with the partitions of $\mathcal{F}$, as well as the poset block is a particular case of a poset $(P, \leq)$ that one can associate with $\mathcal{F}$.

We use the notation $F \lhd G$ if $F \preccurlyeq G$ and $F \preccurlyeq H \preccurlyeq G$ implies $H = F$ or $H = G$.

Let $C = \{E = F_0, F_1, \ldots, F_n = U\}$ be a maximal chain of partitions of $\mathcal{F}$ such that $F_i \lhd F_{i+1}$ for all $i = 0, \ldots, n-1$. We can define a rooted tree of depth n as follows: the $n-$th level is constituted by $|\Omega|$ vertices; the $(n-1)-$st by $\frac{|\Omega|}{k_{F_1}}$ vertices. Each of these vertices is a father of k_{F_1} sons that are in the same part of F_1. Inductively, at the $i-$th level of the tree there are $\frac{|\Omega|}{k_{F_{n-i}}}$ vertices, each of them is the father of the $k_{F_{n-i}}$ vertices of the $(i+1)-$st level belonging to the same part of F_{n-i}.

We can perform the same construction for every maximal chain C in $\mathcal{F}$. The next step is to glue the different trees identifying the vertices associated with the same partition. The resulting structure is the poset $(P, \leq)$.

For example, the poset block structure described in Chapter 1, Example 3.1.3, can be regarded as the orthogonal block structure on the set $\Omega = \{000, 001, 010, 011, 100, 101, 110, 111\}$ given by the set of partitions $\mathcal{F} = \{E, F_1, F_2, F_3, U\}$ where, as usually, E denotes the equality partition and U the universal partition of Ω, while the nontrivial partitions are defined as:

- $F_1 = \{000, 001, 010, 011\} \coprod \{100, 101, 110, 111\}$;
- $F_2 = \{000, 001\} \coprod \{010, 011\} \coprod \{100, 101\} \coprod \{110, 111\}$;
- $F_3 = \{000, 010\} \coprod \{001, 011\} \coprod \{100, 110\} \coprod \{101, 111\}$.

So the orthogonal block structure $\mathcal{F}$ can be represented by the following poset:

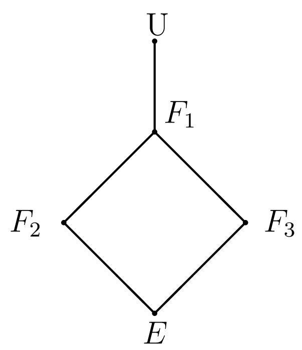


Fig.18. The orthogonal block structure $\mathcal{F} = \{E, F_1, F_2, F_3, U\}$.

The maximal chains in $\mathcal{F}$ have length 3 and they are:

- $C_1 = \{E, F_2, F_1, U\}$;
- $C_2 = \{E, F_3, F_1, U\}$.

The poset $(P, \leq)$ associated with $\mathcal{F}$ is

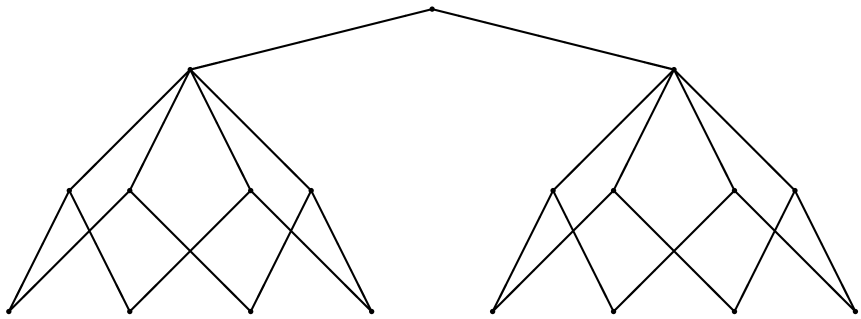


Fig.19. The poset $(P, \leq)$ associated with $\mathcal{F} = \{E, F_1, F_2, F_3, U\}$.

Observe that, if $F \lhd G$, then the number of $F-$classes contained in a $G-$class is k_F/k_G.

5.1. Definition of the Markov chain. The Markov chain that we want to describe is performed on the last level of the poset $(P, \leq)$ that we have just defined. We can think of an insect which, at the beginning of our process, lies on a fixed element ω_0 of Ω (this corresponds to the identity relation E, i.e. each element is in relation only with itself). The insect randomly moves reaching an adjacent vertex in $(P, \leq)$ (this corresponds, in the orthogonal block structure $\mathcal{F}$, to move from E to another relation F such that $E \lhd F$, i.e. ω_0 is identified with all the elements in the same $F-$class) and so on. At each step in $(P, \leq)$ (that does not correspond necessarily to a step in the Markov chain on Ω) the insect could randomly move from the $i-$th level of $(P, \leq)$ either to the $(i-1)-$st level or to the $(i+1)-$st level. Going up means to pass in $\mathcal{F}$ from a partition F to a partition L such that $F \lhd L$ (these are $|\{L \in \mathcal{F} : F \lhd L\}|$ possibilities in $(P, \leq)$), going down means to pass in $\mathcal{F}$ to a partition J such that $J \lhd F$ (these are $\sum_{J \in \mathcal{F}: J \lhd F} \frac{k_F}{k_J}$ possibilities in $(P, \leq)$). The next step of the random walk is whenever the insect reaches once again the last level in $(P, \leq)$. In order to formalize this idea let us introduce the following definitions.

Let $\alpha_{F,G}$ be the probability of moving from the partition F to the partition G. So the following relation is satisfied:

$$\begin{aligned} (36) \quad \alpha_{F,G} &= \frac{1}{\sum_{J \in \mathcal{F}: J \lhd F}(k_F/k_J) + |\{L \in \mathcal{F} : F \lhd L\}|} \\ &+ \sum_{J \in \mathcal{F}: J \lhd F} \frac{(k_F/k_J)\alpha_{J,F}\alpha_{F,G}}{\sum_{J \in \mathcal{F}: J \lhd F}(k_F/k_J) + |\{L \in \mathcal{F} : F \lhd L\}|}. \end{aligned}$$

In fact, the insect can directly pass from F to G with probability $\alpha_{F,G}$ or go down to any J such that $J \lhd F$ and then come back to F with probability $\alpha_{J,F}$ and one starts the recursive argument. From direct computations one gets

$$\alpha_{E,F} = \frac{1}{|\{L \in \mathcal{F} : E \lhd L\}|}. \tag{37}$$

Moreover, if $\alpha_{E,F} = 1$ we have, for all G such that $F \lhd G$

$$\alpha_{F,G} = \frac{1}{\sum_{J \in \mathcal{F}: J \lhd F}(k_F/k_J) + |\{L \in \mathcal{F} : F \lhd L\}|}; \tag{38}$$

if $\alpha_{E,F} \neq 1$, the coefficient $\alpha_{F,G}$ is defined as in (36).

Definition 5.1. *For every $\omega \in \Omega$, define*

$$p(\omega_0, \omega) = \sum_{\substack{E \neq F \in \mathcal{F} \\ \omega_0 \sim_F \omega}} \sum_{\substack{C \subseteq \mathcal{F} \text{ chain} \\ C = \{E, F_1, \ldots, F', F\}}} \frac{\alpha_{E,F_1} \cdots \alpha_{F',F}\left(1 - \sum_{F \lhd L} \alpha_{F,L}\right)}{k_F}.$$

The fact that p is effectively a transition probability on Ω will follow from Theorem 5.4. First define the following numbers:

$$p_F = \sum_{\substack{C \subseteq \mathcal{F} \text{ chain} \\ C = \{E, F_1, \ldots, F', F\}}} \alpha_{E,F_1} \cdots \alpha_{F',F} \left(1 - \sum_{F \lhd L} \alpha_{F,L}\right). \tag{39}$$

Observe that p_F expresses the probability of reaching the partition F but no partition L such that $F \prec L$ in $\mathcal{F}$.

Lemma 5.2. *The coefficients p_F's defined in (39) satisfy the following identity:*

$$\sum_{E \neq F \in \mathcal{F}} p_F = 1.$$

Proof. Using the definitions we have

$$\begin{aligned} \sum_{E \neq F \in \mathcal{F}} p_F &= \sum_{E \neq F \in \mathcal{F}} \sum_{\substack{C \subseteq \mathcal{F} \text{ chain} \\ C = \{E, F_1, \ldots, F', F\}}} \alpha_{E,F_1} \cdots \alpha_{F',F} \left(1 - \sum_{F \lhd L} \alpha_{F,L}\right) \\ &= \sum_{E \lhd F} \alpha_{E,F} = 1. \end{aligned}$$

In fact, for every $F \in \mathcal{F}$ such that $E \not\lhd F$, given a chain $C = \{E, F_1, \ldots, F', F\}$ we get the terms $\alpha_{E,F_1} \cdots \alpha_{F',F}\left(1 - \sum_{F \lhd L} \alpha_{F,L}\right)$. Since $C = \{E, F_1, \ldots, F', F, L\}$ is still a term of the sum one can check that only the summands $\sum_{E \lhd F} \alpha_{E,F}$ are not cancelled. The thesis follows from (37). $\square$

For every $F \in \mathcal{F}, F \neq E$ define M_F as the Markov operator whose transition matrix is

$$M_F = \frac{1}{k_F} R_F. \tag{40}$$

Definition 5.3. *Let M_F be the Markov operator defined in (40) and let p_F be the coefficient in (39). Set*

$$M = \sum_{E \neq F \in \mathcal{F}} p_F M_F. \tag{41}$$

By abuse of notation, we denote by M the stochastic matrix associated with the Markov operator M.

Theorem 5.4. *M coincides with the transition matrix of p.*

Proof. By direct computation we get:

$$\begin{aligned} M(\omega_0, \omega) &= \sum_{E \neq F \in \mathcal{F}} p_F M_F(\omega_0, \omega) = \sum_{\substack{E \neq F \in \mathcal{F} \\ \omega_0 \sim_F \omega}} p_F \cdot \frac{1}{k_F} \\ &= \sum_{\substack{E \neq F \in \mathcal{F} \\ \omega_0 \sim_F \omega}} \sum_{\substack{C \subseteq \mathcal{F} \ chain \\ C = \{E, F_1, \ldots, F', F\}}} \frac{\alpha_{E,F_1} \cdots \alpha_{F',F} \left(1 - \sum_{F \lhd L} \alpha_{F,L}\right)}{k_F} \\ &= p(\omega_0, \omega). \end{aligned}$$

□

5.2. Spectral analysis. We present here the spectral analysis of the operator M acting on the space $L(\Omega)$ of the complex functions defined on the set Ω endowed with the scalar product

$$\langle f_1, f_2 \rangle = \sum_{\omega \in \Omega} f_1(\omega) \overline{f_2(\omega)}.$$

First of all introduce (see, for example, [**3**]), for every $F \in \mathcal{F}$, the following subspaces of $L(\Omega)$:

$$V_F = \{f \in L(\Omega) : \ f(\alpha) = f(\beta) \text{ if } \alpha \sim_F \beta\}.$$

It is easy to show that the operator M_F defined in (40) is the projector onto V_F. In fact if $f \in L(\Omega)$, then $M_F f(\omega_0)$ is the average of the values that f takes on the elements ω such that $\omega \sim_F \omega_0$ and so $M_F f = f$ if $f \in V_F$ and $M_F f = 0$ if $f \in V_F^\perp$.

Set

$$W_G = V_G \cap (\sum_{G \prec F} V_F)^\perp.$$

In [**3**] it is proven that $L(\Omega) = \bigoplus_{G \in \mathcal{F}} W_G$. We can deduce the following proposition.

Proposition 5.5. *The W_G's are eigenspaces for the operator M with associated eigenvalue*

$$\lambda_G = \sum_{\substack{E \neq F \in \mathcal{F} \\ F \preccurlyeq G}} p_F. \tag{42}$$

Proof. By definition, $W_G \subseteq V_G$. This implies that, if $f \in W_G$,

$$M_F f = \begin{cases} f & \text{if } F \preccurlyeq G \\ 0 & \text{otherwise} \end{cases}$$

So, for $w \in W_G$, we get

$$\begin{aligned} M \cdot w &= \sum_{E \neq F \in \mathcal{F}} p_F M_F \cdot w \\ &= \Big(\sum_{\substack{E \neq F \in \mathcal{F} \\ F \preccurlyeq G}} p_F \Big) \cdot w. \end{aligned}$$

Hence the eigenvalue λ_G associated with the eigenspace W_G is

$$\lambda_G = \sum_{\substack{E \neq F \in \mathcal{F} \\ F \preccurlyeq G}} p_F.$$

and the assertion follows. □

5.2.1. *Example.* We can study now the transition probability p in the case of the orthogonal block structure $\mathcal{F}$ described in Fig.18. One can easily verify that we have:

- $\alpha_{E,F_2} = \alpha_{E,F_3} = \alpha_{F_2,F_1} = \alpha_{F_3,F_1} = \frac{1}{2}$;
- $\alpha_{F_1,U} = \frac{1}{3}$.

Let us compute the transition probability p on the last level of $(P, \leq)$:

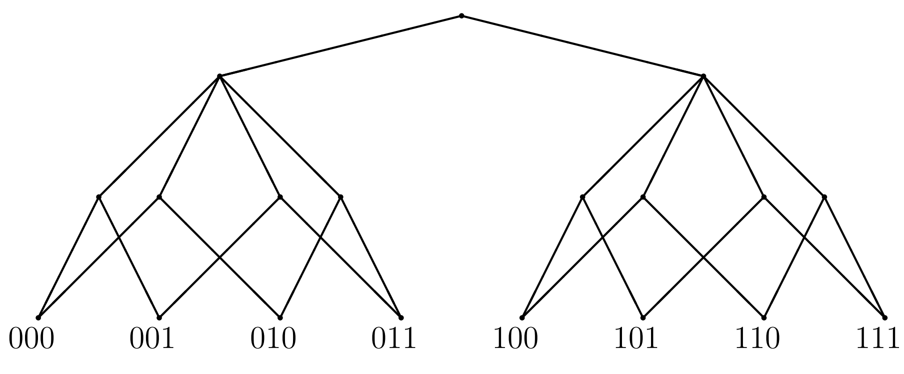


Fig.20. The poset $(P, \leq)$ associated with $\mathcal{F} = \{E, F_1, F_2, F_3, U\}$.

We have:

$$\begin{aligned}
p(000,000) &= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}+2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{4}+2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{8}=\frac{17}{48};\\
p(000,001) &= p(000,010)\\
&= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}+2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{4}+2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{8}=\frac{11}{48};\\
p(000,011) &= 2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{4}+2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{8}=\frac{5}{48};\\
p(000,100) &= p(000,101)=p(000,110)=p(000,111)\\
&= 2\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{8}=\frac{1}{48}.
\end{aligned}$$

The corresponding transition matrix is given by

$$P=\frac{1}{48}\begin{pmatrix}
17 & 11 & 11 & 5 & 1 & 1 & 1 & 1\\
11 & 17 & 5 & 11 & 1 & 1 & 1 & 1\\
11 & 5 & 17 & 11 & 1 & 1 & 1 & 1\\
5 & 11 & 11 & 17 & 1 & 1 & 1 & 1\\
1 & 1 & 1 & 1 & 17 & 11 & 11 & 5\\
1 & 1 & 1 & 1 & 11 & 17 & 5 & 11\\
1 & 1 & 1 & 1 & 11 & 5 & 17 & 11\\
1 & 1 & 1 & 1 & 5 & 11 & 11 & 17
\end{pmatrix}$$

The coefficients P_F, with $E \neq F$, are the following (see (39)):

- $p_U = 2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}=\frac{1}{6}$;
- $p_{F_1} = 2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{3}=\frac{1}{3}$;
- $p_{F_2} = \frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$;
- $p_{F_3} = \frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$.

The Markov operator M is given by (see (41) and (40)):

$$M=\frac{1}{4}M_{F_2}+\frac{1}{4}M_{F_3}+\frac{1}{3}M_{F_1}+\frac{1}{6}M_U$$

and its eigenvalues, according with formula (42), are the following:

- $\lambda_U = 1$;
- $\lambda_{F_1} = \frac{5}{6}$;
- $\lambda_{F_2} = \frac{1}{4}$;
- $\lambda_{F_3} = \frac{1}{4}$;
- $\lambda_E = 0$.

5.2.2. *Remark.* One can easily check that the Markov chain introduced in Definition 5.1 reduces to the Insect Markov chain presented in Section 2 of this chapter, whenever the orthogonal block is the poset block structure associated with a finite poset $(I, \leq)$ which is a chain.

In fact, in this case the ancestral poset is still a chain and the poset $(P, \leq)$ is a rooted tree whose depth is the cardinality of the set I.

5.2.3. *Remark.* In the case of poset block structures, the eigenspaces of the operator M coincide with the irreducible subrepresentations of the generalized wreath product of the groups $Sym(\Delta_i)$.

Actually, the subrepresentations given in (16) are indexed by the antichains of I. Instead in Proposition 5.5 they are indexed by the relations of the orthogonal block structure $\mathcal{F}$. The correspondence is the following.

Given a relation $G \in \mathcal{F}$, it can be regarded as an ancestral relation $\sim_J$, for some ancestral subset $J \subseteq I$. Set

$$S = \{i \in J : H(i) \cap J = \emptyset\}.$$

It is clear that S is an antichain of I. From the definition it follows that

$$A(S) = J \setminus S \text{ and } I \setminus A[S] = I \setminus J.$$

The corresponding eigenspace W_S becomes:

$$W_S = \left(\bigotimes_{i \in J \setminus S} L(\Delta_i) \right) \otimes \left(\bigotimes_{i \in S} V_i^1 \right) \otimes \left(\bigotimes_{i \in I \setminus J} V_i^0 \right).$$

It is easy to check that the functions in W_S are constant on the equivalence classes of the relation $\sim_J$. Moreover, these functions are orthogonal to the functions which are constant on the equivalence classes of the relation $\sim_{J'}$, with $\sim_{J'} \rhd \sim_J$ (where J' is obtained from J deleting an element of S). Since the orthogonality with the functions constant on $\sim_{J'}$ implies the orthogonality with all functions constant on $\sim_L$, where $\sim_L \succ \sim_J$, then we have $W_S \subseteq W_G$. On the other hand, it is easy to verify that

$$\dim(W_S) = \dim(W_G) = m^{|J \setminus S|} \cdot (m-1)^{|S|},$$

and so we have $W_S = W_G$.

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