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Dispersive equations in Quantum Mechanics

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Tesi di Dottorato in Matematica

Dirpersive equations in Quantum Mechanics

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Introduction

It is a remarkable fact that evolution equations with different algebraic structures have solutions with similar behaviors; throughout the standard classification of Partial Differential Equations in hyperbolic, parabolic and elliptic, the class off **dispersive equations** is an interesting family, which presents some typical and characterizing phenomena.

Among the others, dispersive equations include the following ones:

- the **Schrödinger equation**

$$\begin{cases} iu_t(t,x) + \Delta_x u(t,x) = 0 & \text{in } \mathbb{R}\times\mathbb{R}^n \\ u(0,x) = f(x); \end{cases} \tag{0.1}$$

- the **wave equation**

$$\begin{cases} u_{tt}(t,x) - \Delta_x u(t,x) = 0 & \text{in } \mathbb{R}\times\mathbb{R}^n \\ u(0,x) = f(x) \\ u_t(0,x) = g(x); \end{cases} \tag{0.2}$$

- the **Klein-Gordon equation**

$$\begin{cases} u_{tt}(t,x) - \Delta_x u(t,x) + u(t,x) = 0 & \text{in } \mathbb{R}\times\mathbb{R}^n \\ u(0,x) = f(x) \\ u_t(0,x) = g(x); \end{cases} \tag{0.3}$$

- the **Dirac equation**

$$\begin{cases} iu_t(t,x) + \mathcal{H}u(t,x) = 0 & \text{in } \mathbb{R}\times\mathbb{R}^3 \\ u(0,x) = f(x). \end{cases} \tag{0.4}$$

For the Schrödinger equation, the unknown $u : \mathbb{R}^{1+n} \to \mathbb{C}$ is a complex-valued function; for the wave and Klein-Gordon equations $u : \mathbb{R}^{1+n} \to \mathbb{R}$ is real-valued. Finally, for the Dirac equation, the unknown $u : \mathbb{R}^{1+3} \to \mathbb{C}^4$ is a spinor-valued function. The Dirac operator $\mathcal{H}$ is defined by

$$\mathcal{H} = -i\sum_{j=1}^{3} \alpha_j \partial_j + \beta, \tag{0.5}$$

where the coefficients $\alpha_j, \beta \in \mathcal{M}_{4\times 4}(\mathbb{C})$ are the standard Hermitian 4×4 *Dirac matrices*, which are explicitly defined by

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{0.6}$$

$$\alpha_4 := \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{0.7}$$

We also recall that α_j, β, for $j = 1, 2, 3$, satisfy the well known anti-commutation rules

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I_4, \qquad j, k = 1, \ldots, 4$$

(see e.g. [**108**] for more details on the derivation of the equation). We will usually denote by $\mathcal{D}$ the massless Dirac operator defined by

$$\mathcal{D} = -i \sum_{j=1}^{3} \alpha_j \partial_j.$$

In the above mentioned Cauchy problems, the unknown u has the physical meaning of a quantum *wavefunction*. The Schrödinger equation describes the free motion of a non-relativistic particle; the wave and Klein-Gordon equations describe the free motion of a relativistic particle with spin-0, while the Dirac equation describes the motion of spin-$\frac{1}{2}$ particles (electrons, neutrinos).

From a strictly algebraic point of view, the Schrödinger equation (0.1) is different from (0.2), (0.3) and (0.4): actually wave, Klein-Gordon and Dirac equations are hyperbolic, hence they have finite speed of propagation, while the Schrödinger equation has infinite speed of propagation (hence it is not hyperbolic). On the other hand, the solutions of all these equations present some peculiar and common behaviors, which are summarized in the terminology of **dispersion**.

Let us also observe that all the above examples can be written in the form

$$u_t + ih(D)u = 0, \qquad h(D) = \mathcal{F}^{-1}\left(h(\xi)\mathcal{F}\right),$$

where $\mathcal{F}$ is the Fourier transform with respect to x; as a consequence, the solutions can be defined by $u = e^{ith(D)} f$, once we impose the initial conditions. Hence it is not unnatural to think that these equations show some common properties, which have to be related to the structure of the propagators $e^{ith(D)}$.

In order to introduce dispersion from a physical point of view, let us consider the wave equation (0.2); in space dimension $n = 3$, when the initial datum f is null, it is well known that the solution of

$$\begin{cases} u_{tt} - \Delta u = 0 \\ u(0, x) = 0 \\ u_t(0, x) = g(x) \end{cases}$$

can be uniquely expressed via the *Kirchhoff formula*

$$u(t, x) = \fint_{\partial B(x,t)} t g(y)\, dS(y), \tag{0.8}$$

for $t > 0$, where $B(x, t)$ denotes the ball in $\mathbb{R}^3$ of center x and radius t, $\partial B(x, t)$ its boundary and dS the two-dimensional surface-measure on $\partial B(x, t)$. By the Stokes Theorem, we can pass to a volume integral in (0.8) and immediately obtain the following estimate

$$\sup_{x \in \mathbb{R}^3} |u(t, x)| \leq \frac{C}{t} \|\nabla g\|_{L^1}, \tag{0.9}$$

for some $C > 0$ and for all times $t > 0$. Estimate (0.9) is interesting for large times, in fact it says that the L^∞-norm of u decays to 0 when $t \to +\infty$. This is a very well known phenomenon, and it has a physical explanation. The energy

$$E(u,t) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|u_t|^2 + |\nabla u|^2\right) dx$$

is conserved under the wave flow, namely the identity

$$E(u,t) = E(u,0)$$

holds for all times $t \in \mathbb{R}$, on each solution $u \in H^1$ of the equation. On the other hand, due to the finite speed of propagation, if we start with a compactly supported initial datum, at each time t the solution is compactly supported (in space) in a bounded region whose diameter increases as t. As a consequence, the solution tends to spread over this increasing region, and the energy conservation forces the intensity necessarily to decay.

This fact seems to be strictly related to the finite speed of propagation, but it is in fact (from a mathematical point of view) a consequence of the functional properties of the propagator of the equation.

For example, also the Schrödinger equation (0.1), which has infinite speed of propagation, has this property. The solution of (0.1) is uniquely determined by the propagator $S(t) = e^{it\Delta}$, which is a unitary group of operators on L^2, hence it conserves the mass (spacial L^2-norm). It can be directly represented by its explicit convolution kernel, or by Fourier transforming with respect to the space variable, we obtain the representation

$$u(t,x) \simeq \int_{\mathbb{R}^n} e^{i(t|\xi|^2 + x\cdot\xi)} \widehat{f}(\xi)\, d\xi \tag{0.10}$$

for the solution of (0.1). The right hand side in (0.10) is an oscillatory integral (of the 1^{st} kind), and by standard *stationary phase methods* (see e.g. [**100**]) it is possible to prove the dispersive estimate

$$\sup_{x\in\mathbb{R}^n} |u(t,x)| \leq \frac{C}{t^{n/2}} \|f\|_{L^1} \tag{0.11}$$

(which is also immediate if we look at the kernel of $e^{it\Delta}$). The last estimate is very similar to (0.9); in dimension $n = 3$ the Schrödinger solution decays faster than the wave one, with no loss of regularity with respect to the data. It is a fact that, if the initial datum is compactly supported, the solution of the Schrödinger equation looses istantaneously this property, due to the infinite speed of propagation; on the other hand, we can observe that most of the mass stays localized in a finite region, with increasing diameter. This, together with the L^2-conservation, causes the same physical phenomenon which has been shown for the wave equation, and which is described by estimate (0.11).

In both cases of wave and Schrödinger equation, for more general initial data, we can decompose them into elementary wavepackets and observe that, during the evolution, the single packets travel independently with different speeds, but the same dispersive phenomena occur for the total L^∞-norm of the solution.

In the last 30 years, dispersion has rapidly become one of the crucial tools in evolution equations. This kind of physical phenomena can be summarized in a hierarchy of linear a priori estimates for the solutions of the equations.

In this PhD thesis, we are interested in the above equations; we will focus our attention on the following family of linear estimates:

- **decay estimates**
- **Strichartz estimates**
- **Kato-smoothing estimates**.

We give some examples, using the Schrödinger equation as a model.

0.1. Decay Estimates. Let us consider a solution u of (0.1); as we previously said, u satisfies the L^1-L^∞ decay estimate (0.11). By interpolation between (0.1) and the L^2-conservation, we immediately obtain the whole family of decay estimates for the Schrödinger equation

$$\|u(t)\|_{L^p} \leq Ct^{-\frac{n}{2}+\frac{n}{p}}\|f\|_{L^{p'}}, \qquad p \geq 2, \qquad \frac{1}{p}+\frac{1}{p'}=1. \tag{0.12}$$

Similar estimates hold for the wave, Klein-Gordon and Dirac equations, with a loss of derivatives on the initial data (see Chapter 1).

These estimates are well known and very interesting from a physical point of view. Their interest was historically motivated by the phenomenon itself; in the last years, once it was discovered that decay estimates are basic to prove Strichartz estimates, they also became a mathematical topic.

0.2. Strichartz Estimates. Strichartz estimates were introduced by R. Strichartz in [**101**], as a consequence of Fourier restriction theorems. In the fundamental papers [**45**], [**47**] by J. Ginibre and G. Velo, using the so called TT^* they proved Strichartz estimates as a consequence of decay estimates. In the paper [**66**] by M. Keel and T. Tao the program was completed with the proofs of the endpoint estimates.

The natural norms which are considered in this family of estimates are of mixed type, namely we deal with $L^p_tL^q_x$-spaces. If u is the unique solution of the Schrödinger equation (0.1), then the following estimates

$$\|u\|_{L^p_tL^q_x} = \|e^{it\Delta}f\|_{L^p_tL^q_x} \leq C\|f\|_{L^2} \tag{0.13}$$

hold for any couple (p,q) satisfying the Schrödinger admissibility condition

$$\begin{cases} \frac{2}{p}=\frac{n}{2}-\frac{n}{q} \\ p \geq 2 \qquad \text{if } n \geq 3 \\ p > 2 \qquad \text{if } n = 2. \end{cases}$$

For the wave, Klein-Gordon and Dirac equation, similar estimates hold, with different admissibility conditions and different initial spaces (see the Appendix of Chapter 2).

Strichartz estimates represent the crucial instrument to perform fixed point arguments in the study of nonlinear problems. One of the first examples of nonlinear application of Strichartz estimates was given in [**46**] for the nonlinear Schrödinger equation. Later, also for the nonlinear wave equation some critical problems were solved by this technique (see e.g. [**52**], [**94**]).

0.3. Kato-smoothing and local smoothing estimates. The last family of estimates we introduce are commonly referred to as smoothing effects. It is frequent for equations with infinite speed of propagation that the solution is more regular then the initial data. The gain of derivatives, which is in fact related to the algebraic structure of the equations, is a very interesting fact, and is often a crucial improvement for nonlinear techniques.

The smoothing effect was discovered by T. Kato for the Kortweg-de Vries equation; for the Schrödinger equation, Kato and Yajima in [**65**] proved the well known inequality

$$\|\langle x\rangle^{-\frac{1}{2}-}|D|^{\frac{1}{2}}e^{it\Delta}f\|_{L^2_tL^2_x} \leq C\|f\|_{L^2}; \tag{0.14}$$

a stronger local version of the previous inequality (see the standard references [**23**], [**109**], [**110**]) is the following:

$$\sup_{R\in(0,+\infty)}\frac{1}{R}\int_{-\infty}^{\infty}\int_{B_R}\left|\nabla\left(e^{it\Delta}f\right)\right|^2\,dx\,dt \leq C\|f\|_{\dot{H}^{\frac{1}{2}}}. \tag{0.15}$$

Here $|D|^{\frac{1}{2}} = \mathcal{F}^{-1}(|\xi|^{\frac{1}{2}}\mathcal{F})$, $\mathcal{F}$ is the standard Fourier transform, B_R is the ball of radius R centered in 0, and $\dot{H}^{\frac{1}{2}}$ is the usual homogeneous Sobolev space with the norm

$$\|f\|_{\dot{H}^{\frac{1}{2}}} = \||D|^{\frac{1}{2}}f\|_{L^2}.$$

Both estimates (0.14) and (0.15) show that the unique solution of the free Schrödinger equation with initial datum f gains half derivative in L^2 with respect to f, if we look to a weighted $L^2_tL^2_x$ norm.

For the wave, Klein-Gordon and Dirac equations, we cannot expect a gain of derivatives on the solutions, because of the finite speed of propagation; on the other hand, estimates which are analogous to the previous ones for Schrödinger hold also for these equations, with the same regularity for all times (see Chapter 2).

An example of application of Kato-smoothing-type estimates is given in Chapter 2, where they are a tool to prove Strichartz estimates.

0.4. Aim of the thesis and plan of the work. The aim of this PhD work was to study dispersive phenomena for some perturbations of the above mentioned equations. In particular, we are interested in:

- **electromagnetic and electrostatic potentials** (linear perturbations)
- **nonlinear perturbations**.

When we deal with physical models in which particles interact with some external source, it is a fundamental problem to compare the asymptotic behavior of free and perturbed solutions. This is a physical motivation to investigate on dispersive estimates for equations with external potentials of electromagnetic type, which are usually represented by lower order terms in the equations.

From a mathematical point of view, it is clear that dispersive-type estimates have to be considered as a tool, more than a goal: they turn out to be crucial to prove existence and uniqueness results. In particular, the well-posedness problem for some nonlinear equations can be easily solved by standard fixed point arguments in which Strichartz estimates are involved. In a sense, Strichartz estimates play the role which have the Sobolev embeddings in the nonlinear elliptic theory.

The first part of this thesis (Chapters 1,2,3) is devoted to linear equations with potentials. In Chapter 1 we prove some decay estimates for the wave and massless Dirac equations with magnetic potentials (see also [**31**]), using functional techniques based on the Spectral Theorem and resolvent estimates. In Chapter 2, with similar techniques, we study Strichartz and Kato-smoothing estimates for Schrödinger, wave, Klein-Gordon, massless and massive Dirac equations with magnetic potentials (see also [**32**]). Chapter 3 concludes the linear part of the work; here we present a different and indirect technique to prove linear dispersive estimates for 1D Schrödinger, wave and Klein-Gordon equations with an electric potential. It is based on the mapping properties of the **wave operators**, which are at the core of Scattering Theory (see [**30**]). Chapter 4 completes the work, with two nonlinear applications. The first is a nonlinear Schrödinger equation with time dependent coefficients, possibly vanishing; we are interested in existence and uniqueness results, and some Lorentz spaces version of Strichartz estimates are also proved (see [**38**]). The second example is a system of two coupled nonlinear Schrödinger equations; here we prove existence, uniqueness and blow-up results, and we explicitly estimate a blow-up threshold for the the initial data (see also [**40**]).

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CHAPTER 1

Decay estimates for the magnetic Dirac and wave equations

1. Introduction

The first part of this thesis is devoted to the study of dispersive a priori estimates for solutions of the perturbed equations. A very natural problem consists in extending the known estimates for the free equations to some typical perturbative equations coming out from physical models. In particular, we are interested to the case of electrostatic or electromagnetostatic potentials, for which some examples are given here and in the following chapters. In the present chapter we treat Dirac and wave equations with electromagnetic potentials, and we prove some decay estimates for both the equations; the proofs of the main Theorem were given in [**31**].

Dispersive properties of evolution equations play a crucial role in the study of nonlinear problems, and for this reason they have attracted a great deal of attention in recent years. In particular, for the Schrödinger and the wave equation a well established theory exists, see [**47**] and [**66**]. On the other hand, in the variable coefficient case the theory is very far from complete. The simplest situtation is a perturbation with a term of order zero; this is already very interesting from the physical point of view (electrostatic potential). Several results are available for the equations

$$i\partial_t u - \Delta u + V(x)u = 0, \qquad \Box u + V(x)u = 0.$$

We cite among the others [**17**], [**50**], [**49**], [**60**], [**84**] [**90**] and the recent survey [**92**] for Schrödinger; and [**11**], [**12**], [**24**], [**33**], [**44**] for the wave equation. We must also mention the wave operator approach of Kato and Yajima (see [**62**], [**5**], [**115**], [**116**], [**117**]) which permits to deal with the above equations in a unified way, although under nonoptimal assumptions on the potential in dimensions 1 and 3.

The next step in generality is a perturbation with a first order differential operator $a(x)\cdot\nabla$; from the physical point of view this corresponds to a magnetic potential. In this case only a handful of results are available: Strichartz estimates for the 3D wave equation [**25**], provided the coefficients are small and in the Schwartz class; and smoothing estimates for the 3D Schrödinger and wave operators [**106**]. The most general case of variable coefficients has been studied in [**53**], [**88**] and [**97**], where local Strichartz estimates have been proved, in various degrees of complexity; see also [**13**].

In the present chapter, our main focus will be on the three dimensional wave equation with an electromagnetic potential

$$u_{tt} - (\nabla + iA(x))^2 u + B(x)u = 0, \qquad u : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}, \tag{1.1}$$

and the closely related massless Dirac system with a potential:

$$iu_t - \mathcal{D}u + V(x)u = 0, \qquad u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4. \tag{1.2}$$

Here $A : \mathbb{R}^3 \to \mathbb{R}^3$, $B : \mathbb{R}^3 \to \mathbb{R}$, $V(x) = V^*(x)$ is a 4×4 complex matrix on $\mathbb{R}^3$, and the symbol $\mathcal{D}$ denotes the constant coefficient, elliptic, L^2 selfadjoint operator

$$\mathcal{D} = \frac{1}{i}\sum_{j=1}^{3} \alpha_k \partial_k,$$

where the *Dirac matrices* α_1, α_2, α_3 have the following structure:

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{1.3}$$

We neglect the physical constants (i.e., we set $c = \hbar = 1$), and we consider the zero mass case exclusively; the case of a positive mass, whose second order counterpart is the Klein-Gordon equation, has an additional term $\alpha_4 u$ with

$$\alpha_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.4}$$

The relation between massless Dirac and wave equation is readily explained: indeed, the Dirac matrices satisfy the commutation rules

$$\alpha_\ell \alpha_k + \alpha_k \alpha_\ell = 2\delta_{kl} I_4$$

which imply immediately

$$\mathcal{D}^2 = -\Delta I_4,$$

where I_4 is the 4×4 identity matrix. Thus we have the fundamental relation

$$(i\partial_t - \mathcal{D})(i\partial_t + \mathcal{D}) = (\Delta - \partial_{tt}^2) I_4,$$

which can be intepreted as follows: squaring the Dirac system produces a diagonal system of wave equations (or, conversely: taking the square root of a wave equation produces a Dirac system. According to the folklore, this was the route that led Dirac to his equation). When a potential is present in the Dirac system, the above reduction produces an electromagnetic wave equation in a natural way. A discussion of this can be found e.g. in [**67**] (Volume 4, Chapter 4); see also section 6 below.

Our goal here is to establish the decay rate of the spatial L^∞ norm of the solution, with minimal assumptions on the potentials. The expected decay rate is t^{-1}, both for the wave equation and the Dirac system. Indeed, known results for hyperbolic systems (for constant coefficients see e.g. [**70**], [**73**], and for C_0^∞ perturbations thereof see [**61**]) suggest a $t^{-\frac{n-1}{2}}$ decay rate in n space dimensions.

Before stating our first result we introduce some basic notations. Under the assumptions of Theorem 1.1 below, the perturbed laplacian

$$H := -(\nabla + iA(x))^2 + B(x), \tag{1.5}$$

where $A(x) = (A_1(x), A_2(x), A_3(x)) : \mathbb{R}^3 \to \mathbb{R}^3$ and $B(x) : \mathbb{R}^3 \to \mathbb{R}$, is a selfadjoint unbounded operator on $\mathbb{R}^3$; the explicit standard construction is recalled in Section 2. Spectral calculus allows us to define the operators $\psi(H)$ for any well behaved function $\psi(s)$.

In particular, consider a *(non-homogeneous) Paley-Littlewood partition of unity* on $\mathbb{R}^3$, defined as follows: fix a radial nonnegative function $\psi(r) \in C_0^\infty$ with $\psi(r) = 1$ for $r < 1$, $\psi(r) = 0$ for $r > 2$, define $\phi_j(r) = \psi(2^{-j+2}r) - \psi(2^{-j+1}r)$ for all $j \geq 1$, and $\phi_0 = \psi$. Then $1 = \sum_{j\geq 0} \phi_j$ is the required partition of unity on $\mathbb{R}^3$. The operators $\phi_j(\sqrt{H})$ will be used in the following to define suitable norms associated to the operator H. We shall also use the notations

$$\langle x \rangle = (1 + |x|^2)^{1/2}, \qquad \langle D \rangle^s f = (1 - \Delta)^{s/2} f \equiv \mathcal{F}^{-1}(\langle \xi \rangle^s \widehat{f})$$

Our first result concerns the Cauchy problem for the wave equation perturbed with a small rough electromagnetic potential

$$u_{tt}(t, x) - (\nabla + iA(x))^2 u + B(x)u = 0, \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \tag{1.6}$$

$$u(0, x) = 0, \quad u_t(0, x) = g(x). \tag{1.7}$$

We can prove:

THEOREM 1.1. *Assume the potentials* $A : \mathbb{R}^3 \to \mathbb{R}^3$, $B : \mathbb{R}^3 \to \mathbb{R}$ *satisfy*

$$|A_j| \leq \frac{C_0}{|x|\langle x\rangle(|\log |x|| + 1)^\beta}, \qquad \sum_{j=1}^{3} |\partial_j A_j| + |B| \leq \frac{C_0}{|x|^2(|\log |x|| + 1)^\beta}, \tag{1.8}$$

for some constant $C_0 > 0$ *sufficiently small and some* $\beta > 1$. *Then any solution of the Cauchy problem* (1.6), (1.7) *satisfies the decay estimate*

$$|u(t, x)| \leq \frac{C}{t} \sum_{j\geq 0} 2^{2j} \|\langle x\rangle w_\beta^{1/2} \varphi_j(\sqrt{H}) g\|_{L^2}, \tag{1.9}$$

where $w_\beta(x) := |x|(|\log |x|| + 1)^\beta$. *If in addition we assume that, for some* $\epsilon > 0$,

$$\langle D\rangle^{1+\epsilon} A_j \in L^\infty, \qquad \langle D\rangle^\epsilon B \in L^\infty \tag{1.10}$$

then u *satisfies for any* $\delta > 0$ *the estimate*

$$|u(t, x)| \leq \frac{C}{t} \|\langle x\rangle^{3/2+\delta} g\|_{H^{2+\epsilon}}. \tag{1.11}$$

REMARK 1.1. Under out assumptions, the potentials A, B are close to the scale invariant case $A \sim |x|^{-1}$, $B \sim |x|^{-2}$, both from the point of view of singularity and decay at infinity. This is the main source of difficulty in the proof of Theorem 1.1, and requires the use of nonstandard Lorentz space techniques in conjuction with the classical spectral methods.

REMARK 1.2. The norm appearing in (1.9) can be regarded as a distorted analogue of a standard Besov norm, generated by the operator H. Similar norms already appeared in [**25**] for magnetic potentials with coefficients in the Schwartz class; in that case, however, it was possible to prove the equivalence with standard Besov norms (see also [**33**], [**44**] for the analogous norms generated by $-\Delta + V(x)$, which are also equivalent to the nondistorted norms). Under the slightly stronger assumptions (1.10) on the coefficients, it is possible to prove an estimate like (1.11) expressed in terms of standard weighted Sobolev norms.

Moreover, we remark that in our estimate we lose 2 derivatives; we do not know if this is optimal. Recall that in the corresponding dispersive estimate for the free wave equation on $\mathbb{R}^3$, the derivative loss is exactly 1.

REMARK 1.3. As an essential step in the proof of Theorem 1.1, we need to establish the *limiting absorption principle* (LAP) for the operator H. This is obtained in Section 3 through several steps: starting from the "weak" LAP of [**6**] for the free resolvent, we first prove a strong version of the LAP for the free operator in the weighted spaces

$$L^2(w_\beta(x)dx), \qquad w_\beta(x) := |x|(|\log|x|| + 1)^\beta$$

and then we get the LAP for the perturbed operator. For the precise statements see Proposition 1.4. See also [**106**] for related results.

REMARK 1.4. When the initial data are of the form

$$u(0,x) = f, \qquad u_t(0,x) = 0,$$

Theorem 1.1 implies, by standard arguments, the estimate

$$|u(t,x)| \leq \frac{C}{t} \sum_{j\geq 0} 2^{3j} \|\langle x\rangle w_\beta^{1/2} \varphi_j(\sqrt{H}) f\|_{L^2} \tag{1.12}$$

with an additional loss of one derivatives as expected. If in addition we assume that for some $\epsilon > 0$

$$\langle D\rangle^{2+\epsilon} A_j \in L^\infty, \qquad \langle D\rangle^{1+\epsilon} B \in L^\infty \tag{1.13}$$

then also the simpler estimate

$$|u(t,x)| \leq \frac{C}{t} \|\langle x\rangle^{3/2+\delta} f\|_{H^{3+\epsilon}}. \tag{1.14}$$

holds for all $\delta > 0$.

Our second result concerns the perturbed Dirac system

$$iu_t - \mathcal{D}u + V(x)u = 0, \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^3, \tag{1.15}$$

$$u(0,x) = f(x). \tag{1.16}$$

By exploiting the above mentioned relation between the magnetic wave equation and the Dirac system, we can prove the following Theorem as a direct consequence of Theorem 1.1:

THEOREM 1.2. *Assume the 4×4 complex valued matrix* $V(x) = V^*(x)$ *satisfies*

$$|V(x)| \leq \frac{C_0}{|x|\langle x\rangle(|\log|x|| + 1)^\beta}, \qquad |DV(x)| \leq \frac{C_0}{|x|^2(|\log|x|| + 1)^\beta}, \tag{1.17}$$

for some $C_0 > 0$ *small enough and some* $\beta > 1$*. Then the solution of the Cauchy problem* (1.15), (1.16) *satisfies the decay estimate*

$$|u(t,x)| \leq \frac{C}{t} \sum_{j\geq 0} 2^{3j} \|\langle x\rangle w_\beta^{1/2} \varphi_j(\mathcal{D} + V) f\|_{L^2}, \tag{1.18}$$

where $w_\beta(x) = |x|(|\log|x|| + 1)^\beta$*. If in addition we assume that, for some* $\epsilon > 0$*,*

$$\langle D\rangle^{2+\epsilon} V \in L^\infty, \tag{1.19}$$

then u satisfies for any $\delta > 0$ the estimate

$$|u(t,x)| \leq \frac{C}{t} \|\langle x \rangle^{3/2+\delta} f\|_{H^{3+\epsilon}}. \tag{1.20}$$

We remark that for the unperturbed Dirac system, with vanishing mass, the loss of derivatives is exactly 2 (see Proposition 1.7).

Since Theorem 1.2 is proved essentially by "squaring" the perturbed Dirac operator, a condition on the derivative DV is essential in order to apply Theorem 1.1 to the resulting wave equation. On the other hand, we can study the Cauchy problem (1.15), (1.16) by a direct application of the spectral calculus for the selfadjoint operator $\mathcal{D} + V(x)$; this alternative approach allows us to consider much rougher potentials $V(x)$ (see (1.21)). The price to pay is an additional loss of one derivative, so that the total loss is 4 derivatives in our last result:

THEOREM 1.3. *Assume the 4×4 complex valued matrix $V(x) = V^*(x)$ satisfies*

$$|V(x)| \leq \frac{C_0}{|x|^{1/2}\langle x \rangle^{3/2}(|\log|x|| + 1)^{\beta/2}}, \tag{1.21}$$

for some $C_0 > 0$ small enough and some $\beta > 1$. Then the solution of the Cauchy problem (1.15), (1.16) *satisfies for any $\epsilon > 0$ the decay estimate*

$$|u(t,x)| \leq \frac{C}{t} \sum_{j \geq 0} 2^{4j} \|\langle x \rangle^{3/2+\epsilon} \varphi_j(\mathcal{D} + V) f\|_{L^2}. \tag{1.22}$$

REMARK 1.5. As a byproduct of our method of proof, we obtain the *limiting absorption principle* for the perturbed Dirac operator under assumption (1.21) (see Section 3.2). The LAP had been proved earlier for the free Dirac equation by Yamada [**119**], and for the Dirac equation with potential (and with mass) in [**82**] under quite stronger assumptions.

2. The self-adjointness of the perturbed operators

In this section we check the selfadjointness of the perturbed operators Δ_W and $\mathcal{D}_V$ under quite general assumptions on the potentials A, B, V, which in particular are implied by the assumptions of Theorems 1.1, 1.2 and 1.3. Most of the material here is standard; however we decided to include a sketch of the proof for the sake of completeness. Moreover, the use of Lorentz spaces techniques (see the Appendix for a short review) makes the proofs quite straightforward.

It will be useful sometimes to express the magnetic laplacian both in the covariant form

$$H = -(\nabla + iA(x))^2 + B(x) \tag{1.23}$$

and in the expanded form

$$H = -\Delta + W(x, D), \qquad W(x, D) = \sum_{j=1}^{3} a_j(x)\partial_j + b(x) \tag{1.24}$$

where

$$a_j(x) = -2iA_j(x), \qquad b(x) = -i\sum_{j=1}^{3} \partial_j A_j(x) + |A(x)|^2 + B(x), \qquad A_j, B \in \mathbb{R}. \tag{1.25}$$

Then we have the following:

PROPOSITION 1.1. *Consider the operator on* $C_0^\infty(\mathbb{R}^n)$

$$H = -(\nabla + iA(x))^2 + B(x), \tag{1.26}$$

where $A(x) : \mathbb{R}^n \to \mathbb{R}^n$ *and* $B(x) : \mathbb{R}^n \to \mathbb{R}$ *are measurable functions. Assume that the Lorentz (weak Lebesgue) norms of the coefficients*

$$\|A\|_{L^{n,\infty}} \leq C_0, \qquad \|B\|_{L^{n/2,\infty}} \leq C_0 \tag{1.27}$$

are bounded by some constant $C_0 > 0$ *small enough. Then* H *has a (unique) self-adjoint extension to* $H^2(\mathbb{R}^n)$.

PROOF. Our proof is based on the standard results on quadratic forms, see e.g. the standard reference [**86**]. First of all we notice that by (1.27) we have immediately

$$|A(x)|^2 \in L^{n/2,\infty}$$

with a small norm. Now, the quadratic form $q(\phi, \psi)$ given by

$$q(\varphi, \psi) = ((\nabla + iA)\varphi, (\nabla + iA)\psi)_{L^2} + (B\varphi, \psi)_{L^2}$$

is well defined on the form domain H^1 under assumptions (1.27). Indeed, consider the identity

$$q(\psi, \psi) = \|\nabla\psi\|_{L^2}^2 + ((|A|^2 + B)\psi, \psi)_{L^2} + 2\Im(A\nabla\psi, \psi)_{L^2}; \tag{1.28}$$

using the embedding $H^1 \subset L^{2n/(n-2),2}$, the Hölder inequality in Lorentz spaces (see the Appendix at the end of the paper for a quick synopsis of the relevant results), and recalling assumption (1.27), we have easily

$$\begin{aligned}|q(\psi, \psi)| \leq &\|\nabla\psi\|_{L^2}^2 + C\||A|^2 + B\|_{L^{n/2,\infty}}\|\psi\overline{\psi}\|_{L^{\frac{n}{n-2},1}} + C\|A\|_{L^{n,\infty}}\|\nabla\psi \cdot \overline{\psi}\|_{L^{\frac{n}{n-1},1}} \\ \leq &\|\nabla\psi\|_{L^2}^2 + CC_0\|\psi\|_{L^{\frac{2n}{n-2},2}}^2 + CC_0\|\nabla\psi\|_{L^{2,2}}\|\psi\|_{L^{\frac{2n}{n-2},2}} \leq C\|\nabla\psi\|_{L^2}^2.\end{aligned}$$

It is clear that the form is symmetric, since A and B are real valued. Now, recalling Theorem VIII.15 in [**86**], in order to prove that q is the form associated to a (uniquely defined) self-adjoint operator, it will be sufficient to show that it is *closed*, i.e., its domain $H^1(\mathbb{R}^n)$ is complete under the norm

$$|\!|\!|\psi|\!|\!|^2 = q(\psi, \psi) + C\|\psi\|_{L^2}^2 \tag{1.29}$$

for some $C > 0$, and that it is *semibounded*, i.e.,

$$q(\psi, \psi) \geq -C\|\psi\|_{L^2}^2 \tag{1.30}$$

for some $C > 0$. Both properties follow from the identity (1.28); indeed, by estimating as above we obtain easily

$$q(\psi, \psi) \geq \|\nabla\psi\|_{L^2}^2 - CC_0\|\nabla\psi\|_{L^2}^2.$$

In particular this implies that the norm (1.29) is *equivalent* to the $H^1(\mathbb{R}^n)$ norm, provided C_0 is small enough, so that the form is closed; and this implies also that (1.30) is satisfied with $C = 0$. ☐

For the perturbed Dirac operator we have a similar result:

PROPOSITION 1.2. *Let $V(x) = V^*(x)$ be a 4×4 complex valued matrix on $\mathbb{R}^3$. Assume that*

$$\|V\|_{L^{3,\infty}} \leq C_0, \tag{1.31}$$

for some $C_0 > 0$ sufficiently small. Then the perturbed Dirac operator $\mathcal{D}_V = \mathcal{D}+V$ is self-adjoint on $H^1(\mathbb{R}^3, \mathbb{C}^4)$.

PROOF. The proof is analogous to the proof of Theorem 1.1. We define the quadratic form $q : H^{1/2} \times H^{1/2} \to \mathbb{C}$ associated to the operator $\mathcal{D}_V$ as

$$q(\varphi, \psi) := (\mathcal{D}\varphi, \psi) + (V\varphi, \psi).$$

First we prove that the domain of q is $H^{1/2}$. With the same arguments of the previous theorem we estimate

$$\begin{aligned} |q(\varphi,\varphi)| &\leq \|\varphi\|^2_{H^{1/2}} + C\|V\|_{L^{3,\infty}} \|\varphi^2\|_{L^{n/(n-1),1}} \\ &\leq \|\varphi\|^2_{H^{1/2}} + C\|V\|_{L^{3,\infty}} \|\varphi\|^2_{L^{2n/(n-1),2}} \\ &\leq (1 + C\|V\|_{L^{3,\infty}})\, \|\varphi\|_{H^{1/2}} \end{aligned}$$

(where we used the embedding $H^{1/2} \subset L^{2n/(n-1),2}$). From this point on, the proof proceeds exactly as in Proposition 1.1 □

3. The limiting absorption principle

The essential tool in our proof will be the spectral theorem in the following version: given a selfadjoint (unbounded) operator A on L^2 and a continuous bounded function $f(\lambda)$ on $\mathbb{R}$, the operator $f(A)$ can be defined as the L^2 limit

$$f(A)\phi = -\frac{1}{\pi} \cdot \lim_{\epsilon \downarrow 0} \int f(\lambda) \Im R(\lambda + i\epsilon)\phi d\lambda \tag{1.32}$$

for any $\phi \in L^2$. Here $R(z) = (A-z)^{-1}$ denotes the resolvent operator of A (see e.g. [**107**]). Under suitable assumptions on H, the limit operators $R(\lambda \pm i0) = \lim_{\epsilon\downarrow 0} R(\lambda \pm i\epsilon)$ are well defined as bounded operators in weighted L^2 spaces; this is usually called the *limiting absorption principle* (see below for details). Thus we have also the simpler representation

$$f(A)\phi = -\frac{1}{\pi} \cdot \int f(\lambda) \Im R(\lambda + i0)\phi d\lambda. \tag{1.33}$$

Recalling the definition (1.25), consider now the operators

$$H = -\Delta + W(x, D) \equiv -\Delta + \sum_{j=1}^{3} a_j(x)\partial_j + b(x)$$

and

$$\mathcal{D}_V = \mathcal{D} + V(x).$$

In Section 2 we proved that, under assumptions (1.27) on a_j, b and $V(x)$, both H and $\mathcal{D}_V$ are selfadjoint operators on L^2. In particular, the spectral formula (1.32) holds for both. We shall use the following notations: the free resolvents will be written as

$$R_0(z) = (-z - \Delta)^{-1}, \qquad R_{\mathcal{D}}(z) = (-zI_4 + \mathcal{D})^{-1}$$

while we shall use the notation $R(z)$ for both perturbed resolvents:

$$R(z) = (-z - \Delta + W)^{-1}, \qquad R(z) = (-z + \mathcal{D} + V)^{-1}.$$

From the context the meaning of $R(z)$ will always be clear. Note that $R_0(z)$ is defined for all $z \notin \mathbb{R}^+$ while $R_{\mathcal{D}}(z)$ is defined for $z \notin \mathbb{R}$, and the same properties hold for the perturbed resolvents.

Our first task will be to show that the stronger representation (1.33), i.e., the limiting absorption principle, holds also for the perturbed operators. For $A = -\Delta$ this is a classical result (see e.g. Agmon [**2**]); here we shall use a very precise version of the principle, due to Barcelo, Ruiz and Vega [**6**]. On the other hand, for the Dirac operator only a few results are available, which concern the case with a nonzero mass term (see [**82**], [**119**]).

The classical results on R_0 (see [**2**]) state that the limits

$$\lim_{\epsilon \downarrow 0} R_0(\lambda \pm i\epsilon) = R_0(\lambda \pm i0) \tag{1.34}$$

exist in the norm of bounded operators from $L^2(\langle x\rangle^s dx)$ to $H^2(\langle x\rangle^{-s} dx)$ for any $s > 1$; the convergence is uniform for λ belonging to any compact subset of $]0, +\infty[$, and the following estimate holds

$$\|\langle x\rangle^{-s} R_0(\lambda \pm i0)\langle x\rangle^{-s} f\|_{L^2} \leq \frac{C(s)}{\sqrt{\lambda}} \|f\|_{L^2} \qquad \forall \lambda > 0,\ s > \frac{1}{2}. \tag{1.35}$$

In $n = 3$ dimensions, the operators $R_0(\lambda \pm i0)$ have the explicit representation

$$R_0(\lambda \pm i0) g(x) = \frac{1}{4\pi} \int \frac{e^{\pm i\sqrt{\lambda}|x-y|}}{|x-y|} g(y) dy, \quad \lambda \geq 0. \tag{1.36}$$

Recall also that for $\lambda < 0$ we have the similar formula

$$R_0(\lambda) g(x) = \frac{1}{4\pi} \int \frac{e^{-\sqrt{|\lambda|}\,|x-y|}}{|x-y|} g(y) dy, \quad \lambda \leq 0. \tag{1.37}$$

These results were extended in [**6**] to more general weights. Introduce the norm

$$\|\!|a(x)|\!\| = \sup_{\mu > 0} \int_\mu^{+\infty} \frac{h(r) r}{(r^2 - \mu^2)^{1/2}} dr \quad \text{where} \quad h(r) \equiv \sup_{|x| = r} |a(x)|. \tag{1.38}$$

For any measurable function on $\mathbb{R}^n$ such that $\operatorname{supp} f \subseteq \operatorname{supp} a$, we can consider the (semi-)norm

$$\|f\|_{L^2(a(x)dx)} \equiv \|a(x)^{1/2} f\|_{L^2} < \infty$$

and we can define a Hilbert space $L^2(a(x)dx)$ as the closure in this norm of the subspace of C_0^∞ functions with support contained in $\operatorname{supp} a$. Then we can summarize Theorems 1 and 2 in [**6**] as follows:

THEOREM 1.4 ([**6**]). *Let $a(x)$ be a nonnegative function on $\mathbb{R}^n$ with $\|\!|a|\!\| < \infty$, and denote by $R_0(\lambda \pm i0)$ the limit operators* (1.34). *Then the operators $R_0(z)$ for $z \notin \mathbb{R}^+$ and $R_0(\lambda \pm i0)$ can be extended to bounded operators from $L^2(a(x)^{-1}dx)$ to $L^2(a(x)dx)$, and the following estimates hold:*

$$\|R_0(\lambda \pm i0) f\|_{L^2(a(x)dx)} \leq \frac{C}{\sqrt{|\lambda|}} \|\!|a|\!\| \cdot \|f\|_{L^2(a(x)^{-1}dx)}, \qquad \lambda \neq 0 \tag{1.39}$$

(here of course $R_0(\lambda \pm i0) \equiv R_0(\lambda)$ for $\lambda < 0$)

$$\|\nabla R_0(\lambda \pm i0) f\|_{L^2(a(x)dx)} \leq C \|a\| \cdot \|f\|_{L^2(a(x)^{-1}dx)}. \tag{1.40}$$

Moreover, the limiting absorption principle holds in the weak form: for all $f, g \in L^2(a(x)^{-1}dx)$

$$\lim_{\epsilon \downarrow 0} (R_0(\lambda \pm i\epsilon) f, g) = (R_0(\lambda \pm i0) f, g). \tag{1.41}$$

Remark 1.6. It is not difficult to extend the estimates (1.39) and (1.40) to the whole complex plane. Indeed, fix two functions $f, g \in C_0^\infty$ with support contained in $\operatorname{supp} a$ and consider on the half plane

$$S = \{z \colon \Im z > 0\}$$

the holomorphic function

$$F(z) = z^{1/2} (R_0(z) f, g). \tag{1.42}$$

It is clear that $F(z)$ is continuous on $\overline{S}$ up to the boundary, moreover it satisfies the estimate

$$|F(x)| \leq C \|a\| \cdot \|f\|_{L^2(a(x)^{-1}dx)} \|g\|_{L^2(a(x)^{-1}dx)} \tag{1.43}$$

on the boundary $\Im z = 0$, and finally it has a polynomial growth for $|z| \to +\infty$, as it easily follows from the explicit expression of $R_0(z)$ as a convolution operator (see [**6**]). By the Phragmén-Lindelöf Theorem (see e.g. [**100**]) on the half plane we immediately obtain that estimate (1.43) holds on all of $\overline{S}$. A similar argument can be applied in the lower half plane $\Im z < 0$. In conclusion we obtain

$$\|R_0(z) f\|_{L^2(a(x)dx)} \leq \frac{C}{\sqrt{|z|}} \|a\| \cdot \|f\|_{L^2(a(x)^{-1}dx)} \tag{1.44}$$

for all $f \in L^2(a(x)^{-1}dx)$ (see also part (ii) in Theorem 1, [**6**]). Notice that this estimate holds on the whole complex plane, in the sense that we apply it to $R_0(\lambda \pm i0)$ when $z \in \mathbb{R}^+$.

If we apply the same argument to the function

$$G(z) = (\nabla R_0(z) f, g)$$

we obtain in an analogous way the estimate

$$\|\nabla R_0(z) f\|_{L^2(a(x)dx)} \leq C \|a\| \cdot \|f\|_{L^2(a(x)^{-1}dx)}, \qquad z \in \mathbb{C}. \tag{1.45}$$

We now specialize the theorem to a particular choice of weights. Precisely, consider the family of functions

$$w_\beta(x) = |x| (|\log |x|| + 1)^\beta, \qquad \beta > 1. \tag{1.46}$$

As it is proved in [**6**] (see Proposition 1), the norms

$$\|w_\beta^{-1}\| < +\infty$$

are finite for all $\beta > 1$, hence we can apply 1.4 with the choice

$$a(x) = (w_\beta(x))^{-1} = \frac{1}{|x|(|\log|x|| + 1)^\beta}.$$

In this case it is possible to improve the above result and to obtain a stronger version of the limiting absorption principle. To this end, we need the following Lemma, which is inspired by [**2**]:

LEMMA 1.1. *Let H be a Hilbert space, H' its dual, and H_0 a second Hilbert space compactly embedded in H'. Let T_j, T $(j = 1, 2, \dots)$ be bounded operators in $\mathcal{L}(H, H')$ such that*

(i) *T_j, T are symmetric for the pairing $\langle \cdot, \cdot \rangle_{H' \times H}$, i.e.,*

$$\langle Tf, g \rangle_{H' \times H} = \langle Tg, f \rangle_{H' \times H} \qquad \forall f, g \in H;$$

(ii) *$T_j, T \in \mathcal{L}(H, H_0)$ and, for some constant C independent of j,*

$$\|T_j\|_{\mathcal{L}(H, H_0)} \leq C.$$

Assume that

$$T_j f \rightharpoonup Tf \quad \text{weakly in } H' \text{ for all } f \in H. \tag{1.47}$$

Then $T_j \to T$ in the operator norm of $\mathcal{L}(H, H')$.

PROOF. Fix an $f \in H$; the sequence $T_j f$ converges weakly to Tf in H', and is bounded in H_0 by (ii), hence it admits a subsequence which converges in the norm of H', and the limit must be the same i.e. Tf. By applying the same argument to any subsequence of $T_j f$, we conclude that the entire sequence $T_j f$ converges to Tf in the norm of H.

Now, let f_j be any sequence which converges to f weakly in H. Then we have for all $g \in H$

$$\langle T_j f_j, g \rangle = \langle T_j g, f_j \rangle \to \langle Tf, g \rangle$$

since $T_j g \to Tg$ strongly in H' and $f_j \rightharpoonup f$ weakly in H. In other words, for any $f_j \rightharpoonup f$ weakly in H we have that $T_j f_j \rightharpoonup Tf$ weakly in H'. But, as in the first step, we can remark that the sequence $T_j f_j$ is bounded in H_0 and by compact embedding we obtain that the convergence is strong: $T_j f_j \to Tf$ in the norm of H'.

By the same argument we obtain that, for any $f_j \rightharpoonup f$ weakly in H, the sequence Tf_j converges to Tf in the norm of H'.

Finally, assume by contradiction that T_j does not converge to T in the operator norm of $\mathcal{L}(H, H')$. This means that we can find a sequence $f_j \in H$ with norm $\|f_j\|_H = 1$ such that

$$\|T_j f_j - Tf_j\|_{H'} > \epsilon > 0$$

for some ϵ independent of j. By extracting a subsequence we can assume that $f_j \rightharpoonup f$ weakly in H, and by the above steps we immediately obtain a contradiction. □

Then we can prove:

PROPOSITION 1.3. *Let $w_\beta(x)$, $x \in \mathbb{R}^n$ be one of the radial weights* (1.46) *for some fixed $\beta > 1$. Then, for all $\lambda \neq 0$, the limits*

$$\lim_{\epsilon \downarrow 0} R_0(\lambda \pm i\epsilon) = R_0(\lambda \pm i0) \tag{1.48}$$

exist in the norm of bounded operators from $L^2(w_\beta(x)dx)$ to $H^2(w_\beta(x)^{-1}dx)$ and satisfy the estimates

$$\|R_0(\lambda \pm i0)f\|_{L^2(w_\beta^{-1}dx)} \leq \frac{C(b)}{\sqrt{|\lambda|}} \|f\|_{L^2(w_\beta dx)}, \qquad \forall \lambda \neq 0, \tag{1.49}$$

$$\|\nabla R_0(\lambda \pm i0)f\|_{L^2(w_\beta^{-1}dx)} \leq C(b) \|f\|_{L^2(w_\beta dx)}. \tag{1.50}$$

PROOF. We apply Lemma 1.1 with the choices: $H = L^2(w_\beta(x)dx)$, and hence $H' = L^2(w_\beta(x)^{-1}dx)$ with the standard L^2 pairing; $H_0 = H^1(w_{\beta_0}(x)^{-1}dx)$ for some arbitrary β_0 with $\beta > \beta_0 > 1$; the norm of H_0 of course is

$$\|f\|^2_{H_0} = \|w_{\beta_0}^{-1/2} f\|^2_{L^2} + \|w_{\beta_0}^{-1/2} \nabla f\|^2_{L^2}.$$

Finally, as operators T_j we shall take (any subsequence of) the resolvent operators $R_0(\lambda \pm i\epsilon)$ as $\epsilon \downarrow 0$, while $T = R_0(\lambda \pm i0)$, for some fixed $\lambda \in \mathbb{R}$.

We now check the assumptions of the lemma. The compact embedding of H_0 into H' is clear. Also the symmetry of the operators in the sense of (i) is evident. The uniform bounds on T_j, T as bounded operators from H to H' are simply the estimates (1.44), (1.45) applied with the choice $a(x) = w_\beta(x)^{-1}$. But it is clear that the estimate (1.44) implies also the following estimate

$$\|R_0(z)f\|_{L^2(w_{\beta_0}^{-1}dx)} \leq \frac{C(\beta_0)}{\sqrt{|z|}} \|f\|_{L^2(w_\beta dx)}, \qquad \forall z \neq 0, \tag{1.51}$$

which is only apparently stronger, in view of the trivial embedding

$$L^2(w_\beta dx) \subseteq L^2(w_{\beta_0} dx).$$

In a similar way we have

$$\|\nabla R_0(z)f\|_{L^2(w_{\beta_0}^{-1}dx)} \leq C(\beta_0) \|f\|_{L^2(w_\beta dx)}. \tag{1.52}$$

These inequalities show that assumption (ii) of the Lemma is satisfied. Finally, assumption (1.47) is nothing but the weak limiting absorption principle of Barcelo, Ruiz, Vega (see (1.41)).

In conclusion, Lemma 1.1 implies that the limit (1.48) exists in the norm of bounded operators from $L^2(w_\beta dx)$ to $L^2(w_\beta^{-1}dx)$. Moreover, by the identity

$$\Delta R_0(z) = -I - zR_0(z)$$

we obtain that the limit exists also in the norm of bounded operators from $L^2(w_\beta dx)$ to $H^2(w_\beta^{-1}dx)$. The estimates (1.49) and (1.50) follow from the corresponding estimates for general z. □

3.1. The limiting absorption principle for the magnetic laplacian. In what follows, we shall focus on the case $n = 3$ exclusively. We follow the standard approach, based on the resolvent identity

$$R(z) = (-z - \Delta + W(x,D))^{-1} = R_0(z)(I + WR_0(z))^{-1}.$$

Thus the main step of the proof will consist in inverting the operator $I + WR_0$ in suitable weighted spaces. We shall assume that the coefficients $a_j(x)$ and $b(x)$ in $W(x,D)$, defined as in (1.25), satisfy the assumptions

$$|a_j(x)| \leq \frac{C_0}{|x|\langle x\rangle^s(|\log|x\|+1)^\beta}, \qquad |b(x)| \leq \frac{C_0}{|x|^2(|\log|x\|+1)^\beta} \tag{1.53}$$

for some $s \in [0,1]$, $\beta > 1$ and some constant C_0 small enough.

Our result is the following:

PROPOSITION 1.4. *Assume the coefficients of* $W(x,D) = \sum a_j(x)\partial_j + b(x)$ *satisfy* (1.53) *for some* C_0 *small enough, some* $s \in [0,1]$ *and some* $\beta > 1$.

Then the operator $I + WR_0$ *is invertible on the weighted space* $L^2(w_\beta(x)\langle x\rangle^{2s}dx)$, *and the inverse operators* $(I + WR_0(z))^{-1}$ *are uniformly bounded for all* $z \in \mathbb{C}$. *Moreover, the strong limiting absorption principle holds for* $R(z)$, *in the following sense:*

(i) *the boundary values*

$$\lim_{\epsilon\downarrow 0} R(\lambda \pm i\epsilon) = R(\lambda \pm i0) \tag{1.54}$$

exist in the norm of bounded operators from $L^2(w_\beta(x)dx)$ *to* $H^2(w_\beta^{-1}(x)dx)$;

(ii) *the following estimate*

$$\|R(z)f\|_{L^2(w_\beta(x)dx)} \le \frac{C(\beta)}{\sqrt{|z|}} \cdot \|f\|_{L^2(w_\beta(x)^{-1}dx)} \tag{1.55}$$

holds for all $z \in \mathbb{C}$, $z \ne 0$.

REMARK 1.7. In the case $s = 0$ we recover exactly the strong limiting absorption principle proved in Proposition 1.3 above for the free operator R_0. The additional weight $\langle x\rangle^s$ was considered in view of the estimates that will be needed in the following section.

PROOF. Consider the operator

$$W(x,D)R_0(z)f = \sum a_j(x)\partial_j R_0(z)f + b(x)R_0(z)f;$$

we estimate the two terms separately.

First of all we have

$$\|w_\beta^{1/2}\langle x\rangle^s a_j(x)\partial_j R_0 f\|_{L^2} \le \|w_\beta\langle x\rangle^s a_j\|_{L^\infty}\|w_\beta^{-1/2}\partial_j R_0 f\|_{L^2} \le C_0\|w_\beta^{1/2}f\|_{L^2}$$

by estimate (1.52), and this implies trivially

$$\|w_\beta^{1/2}\langle x\rangle^s a_j(x)\partial_j R_0 f\|_{L^2} \le C_0\|w_\beta^{1/2}\langle x\rangle^s f\|_{L^2}. \tag{1.56}$$

In order to estimate the electric term, we recall that, from the explicit expression of the free resolvent, we can write

$$|R_0(z)f| \le \frac{1}{4\pi}\left|\frac{1}{|x|} * |f|\right|.$$

Then we have

$$\|w_\beta^{1/2}b(x)R_0(z)f\|_{L^2} \le \|w_\beta^{1/2}b(x)\|_{L^2}\|R_0(z)f\|_{L^\infty} \le \|w_\beta^{1/2}b(x)\|_{L^2}\cdot C\left\|\frac{1}{|x|} * |f|\right\|_{L^\infty}. \tag{1.57}$$

Recalling Young and Hölder inequalities in Lorentz spaces (see Theorems 1.5, 1.6), we have

$$\left\|\frac{1}{|x|} * |f|\right\|_{L^\infty} \le C\|f\|_{L^{3/2,1}} = C\|w_\beta^{-1/2}w_\beta^{1/2}f\|_{L^{3/2,1}} \le C\|w_\beta^{-1/2}\|_{L^{6,2}}\|w_\beta^{1/2}f\|_{L^2}.$$

Since $w_\beta^{-1/2} \in L^{6,2}$ for any $\beta > 1$ (Proposition 1.8), (1.57) gives

$$\|w_\beta^{1/2}b(x)R_0(z)f\|_{L^2} \le C\|w_\beta^{1/2}b(x)\|_{L^2}\cdot\|w_\beta^{1/2}f\|_{L^2}.$$

Now, by assumption (1.53) on $b(x)$ we have easily

$$\|w_\beta^{1/2}b(x)\|_{L^2} \le CC_0$$

and we conclude that

$$\|w_\beta^{1/2} b(x) R_0(z) f\|_{L^2} \le CC_0 \cdot \|w_\beta^{1/2} f\|_{L^2}. \tag{1.58}$$

In a similar way we have

$$\|w_\beta^{1/2}\langle x\rangle b R_0(z) f\|_{L^2} \le \|w_\beta^{1/2}\langle x\rangle b\|_{L^6} \|R_0(z) f\|_{L^3} \le \|w_\beta^{1/2}\langle x\rangle b\|_{L^6} \cdot C \left\| \frac{1}{|x|} * |f| \right\|_{L^3} \tag{1.59}$$

and

$$\begin{aligned} \left\| \frac{1}{|x|} * |f| \right\|_{L^3} \le C\|f\|_{L^1} =& C\|w_\beta^{-1/2}\langle x\rangle^{-1} w_\beta^{1/2}\langle x\rangle f\|_{L^1} \\ \le& C\|w_\beta^{-1/2}\langle x\rangle^{-1}\|_{L^2} \|w_\beta^{1/2}\langle x\rangle f\|_{L^2}. \end{aligned}$$

As above, we notice that $w_\beta^{-1/2}\langle x\rangle^{-1} \in L^2$ for any $\beta > 1$, hence we have from (1.59)

$$\|w_\beta^{1/2}\langle x\rangle b R_0(z) f\|_{L^2} \le C\|w_\beta^{1/2}\langle x\rangle b\|_{L^6} \cdot \|w_\beta^{1/2}\langle x\rangle f\|_{L^2}.$$

Assumption (1.53) guarantees that

$$\|w_\beta^{1/2}\langle x\rangle b(x)\|_{L^6} \le CC_0$$

and, in conclusion,

$$\|\langle x\rangle w_\beta^{1/2} b(x) R_0(z) f\|_{L^2} \le CC_0 \cdot \|\langle x\rangle w_\beta^{1/2} f\|_{L^2} \tag{1.60}$$

If we interpolate between (1.58) and (1.60), we obtain the estimate

$$\|\langle x\rangle^s w_\beta^{1/2} b(x) R_0(z) f\|_{L^2} \le CC_0 \cdot \|\langle x\rangle^s w_\beta^{1/2} f\|_{L^2} \tag{1.61}$$

Summing up, from estimates (1.56) and (1.61) we get for all $z \in \mathbb{C}$

$$\|\langle x\rangle^s w_\beta^{1/2} W R_0(z) f\|_{L^2} \le CC_0 \cdot \|\langle x\rangle^s w_\beta^{1/2} f\|_{L^2}. \tag{1.62}$$

Then it is clear that we can invert the operator $I + WR_0$ by a Neumann series on the space $L^2(\langle x\rangle^{2s} w_\beta dx)$. Hence, the standard representation

$$R(z) = R_0(z)(I + WR_0(z))^{-1} \tag{1.63}$$

is valid. To conclude the proof of the Proposition, it is now sufficient to remark that, from property (1.48) of Proposition 1.3 and the uniform bounds on the norm of $(I + WR_0(z))^{-1}$ we have just obtained (for $s = 0$), the limits in (1.54) exist in a weak sense. Proceeding as in the proof of Proposition 1.3, using Lemma 1.1, we deduce (i). Finally, (ii) is a consequence of (1.63) and the corresponding estimate (1.51) for R_0. □

Remark 1.8. Note that the assumptions of the preceding proposition can be expressed in terms of the original coefficients A, B as follows:

$$|A(x)| \le \frac{C_0}{|x|\langle x\rangle^s (|\log|x|| + 1)^\beta}, \qquad |\nabla A(x)| + |B(x)| \le \frac{C_0}{|x|^2 (|\log|x|| + 1)^\beta} \tag{1.64}$$

for some $\beta > 1$ and a constant $C_0 > 0$ small enough.

3.2. The limiting absorption principle for the Dirac operator and its perturbation. In this section we will study the limiting absorption principle for the massless Dirac operator $\mathcal{D}$; this property was studied by Yamada in [**119**] for the operator with mass. Moreover, as in the case of the magnetic Laplacian, we will extend this result to the perturbed operator $\mathcal{D}_V = \mathcal{D} + V(x)$, under a suitable assumption on the potential V.

It is well known that the spectrum of the free operator $\mathcal{D}$ is the whole real line. Due to the relation $\mathcal{D}^2 = -\Delta I_4$, we immediately obtain the representation

$$R_{\mathcal{D}}(z) = R_0(z^2)(\mathcal{D} + zI_4), \tag{1.65}$$

for all $z \in \mathcal{C}$ with $\Re z = 0$. Using this formula and the Proposition 1.3, we easily prove the following:

Proposition 1.5. *Let $w_\beta(x)$, $x \in \mathbb{R}^3$ be defined as in* (1.46)*, for some fixed $\beta > 1$. Then, for all $\lambda \in \mathbb{R}$, the limits*

$$\lim_{\epsilon\downarrow 0} R_{\mathcal{D}}(\lambda \pm i\epsilon) = R_{\mathcal{D}}(\lambda \pm i0) := R_0(\lambda^2 \pm i0)(\mathcal{D} + \lambda I_4) \tag{1.66}$$

exist in the norm of bounded operators from $L^2(w_\beta(x)dx)$ to $H^1(w_\beta(x)^{-1}dx)$ and satisfy the estimate

$$\|R_{\mathcal{D}}(z)f\|_{L^2(w_\beta(x)^{-1}dx)} \leq \|f\|_{L^2(w_\beta(x)dx)}, \tag{1.67}$$

for all $z \in \mathbb{C}$. Moreover, we have the explicit representation

$$\begin{aligned} R_{\mathcal{D}}(\lambda \pm i0)f \;&=\; \frac{i|\lambda|}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|\lambda|\cdot|x-y|}}{|x-y|}\left(I_4 - \sum_{j=1}^{3}\alpha_j \frac{x_j - y_j}{|x-y|}\right) f(y)\,\mathrm{d}y \\ &\quad + \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|\lambda|\cdot|x-y|}}{|x-y|^2}\sum_{j=1}^{3}\alpha_j \frac{x_j - y_j}{|x-y|} f(y)\,\mathrm{d}y.. \end{aligned} \tag{1.68}$$

Proof. The strong convergence of $R_{\mathcal{D}}(\lambda \pm i\epsilon)$ to $R_{\mathcal{D}}(\lambda \pm i0)$ in the space of bounded operators from $L^2(w_\beta(x)dx)$ to $H^1(w_\beta(x)^{-1}dx)$ is obtained by interpolation using the property (1.48) and the representation (1.65); estimate (1.67) immediately follows from (1.65) and the estimates (1.49), (1.50), (1.51), (1.52). In conclusion, recalling the explicit representation (1.36) for $R_0(\lambda \pm i0)$, after an integration by parts we get the formula (1.68) and this concludes the proof. □

At this point, we will proceed in a similar way to the case of the perturbed Laplacian and we will prove that it is possible to extend the above result to small electric perturbations of the free Dirac operator. As for the magnetic coefficients of $W(x, D)$, we need to assume that the potential V satisfies

$$|V(x)| \leq \frac{C_0}{|x|\langle x\rangle^s(|\log|x|| + 1)^\beta}, \tag{1.69}$$

for some $s \in [0, 1]$, $\beta > 1$ and some constant C_0 small enough. We prove the following result:

Proposition 1.6. *Assume the potential V satisfies* (1.69) *for some C_0 sufficiently small, some $s \in [0, 1]$ and some $\beta > 1$.*

Then the operator $I + VR_{\mathcal{D}}$ *is invertible on the weighted space* $L^2(w_\beta(x)\langle x\rangle^{2s}dx)$*, and the inverse operators* $(I + VR_{\mathcal{D}}(z))^{-1}$ *are uniformly bounded for all* $z \in \mathbb{C}$*. Moreover, the strong limiting absorption principle holds for* $R(z)$*, in the following sense:*

(i) *the limits*

$$\lim_{\epsilon\downarrow 0} R(\lambda \pm i\epsilon) = R(\lambda \pm i0) \tag{1.70}$$

exist in the norm of bounded operators from $L^2(w_\beta(x)dx)$ *to* $H^1(w_\beta^{-1}(x)dx)$*;*

(ii) *the following estimate*

$$\|R(z)f\|_{L^2(w_\beta(x)^{-1}dx)} \leq C(\beta)\cdot \|f\|_{L^2(w_\beta(x)dx)} \tag{1.71}$$

holds for all $z \in \mathbb{C}$*,* $z \neq 0$*.*

Proof. The argument is the same of the proof of Proposition 1.4 for the magnetic part of W. First we observe that, by hypothesis (1.69), we have

$$\|w_\beta^{1/2}\langle x\rangle^s V(x)R_{\mathcal{D}}f\|_{L^2} \leq \|w_\beta\langle x\rangle^s V(x)\|_{L^\infty}\|w_\beta^{-1/2}R_{\mathcal{D}}f\|_{L^2} \leq C_0\cdot \|w_\beta^{-1/2}f\|_{L^2}.$$

Hence we obtain the estimate

$$\|w_\beta^{1/2}\langle x\rangle^s V(x)R_{\mathcal{D}}(z)f\|_{L^2} \leq \|w_\beta^{1/2}\langle x\rangle^s f\|_{L^2},$$

uniformly in $z \in \mathbb{C}$; thus we can invert the operator $I + VR_{\mathcal{D}}$ by a Neumann series on the space $L^2(w_\beta dx)$. Again, we can exploit the representation

$$R(z) = R_{\mathcal{D}}(z)(I + VR_{\mathcal{D}}(z))^{-1}. \tag{1.72}$$

By property (1.66) of Proposition 1.5 and the uniform bounds of $(I + VR_{\mathcal{D}})^{-1}$, it follows that the limits in (1.70) exist in a weak sense. Then we can procede as in the previous cases, using Lemma 1.1 and obtain (i). In conclusion, the estimate (ii) is an immediate consequence of (1.72) and the inequality (1.67). This concludes the proof. □

In the following we shall also need a weaker version of the last result: we shall require that V satisfies

$$|V(x)| \leq \frac{C_0}{|x|^{1/2}\langle x\rangle^s(|\log|x\|+1)^{\beta/2}}, \tag{1.73}$$

for some $s > \frac{1}{2}$, $\beta > 1$ and some constant C_0 small enough. Then we have

Corollary 1.1. *Assume the potential* V *satisfies* (1.53) *for some* C_0 *sufficiently small,* $s > \frac{1}{2}$ *and* $\beta > 1$*.*

Then the operators $I+VR_{\mathcal{D}}$ *are invertible on the space* $L^2(\langle x\rangle^{2s}dx)$*, and the inverse operators* $(I + VR_{\mathcal{D}}(z))^{-1}$ *are uniformly bounded for all* $z \in \mathbb{C}$*. Moreover, the strong limiting absorption principle holds for* $R(z)$*, in the following sense:*

(i) *the limits*

$$\lim_{\epsilon\downarrow 0} R(\lambda \pm i\epsilon) = R(\lambda \pm i0) \tag{1.74}$$

exist in the norm of bounded operators from $L^2(\langle x\rangle^{2s}dx)$ *to* $H^1(\langle x\rangle^{-2s}dx)$*;*

(ii) *the following estimate*

$$\|R(z)f\|_{L^2(\langle x\rangle^{-2s}dx)} \leq C \cdot \|f\|_{L^2(\langle x\rangle^{2s}dx)} \tag{1.75}$$

holds for all $z \in \mathbb{C}$, $z \neq 0$.

PROOF. The proof is analogous to the proof of Proposition 1.6. Indeed, from estimate (1.67) and assumption (1.73) we have immediately

$$\|\langle x\rangle^s V R_{\mathcal{D}}\|_{L^2} \leq \|\langle x\rangle^s w_\beta^{1/2} V\|_{L^\infty} \|w_\beta^{-1/2} R_{\mathcal{D}} f\|_{L^2} \leq C_0 \|w_\beta^{1/2} f\|_{L^2}$$

and by the trivial inequality

$$w_\beta^{1/2} \leq C_s \langle x\rangle^s,$$

valid for all $s > 1/2$, we conclude that

$$\|\langle x\rangle^s V R_{\mathcal{D}}\|_{L^2} \leq C_0 \|\langle x\rangle^s f\|_{L^2}.$$

Thus we can again invert $(I + VR_{\mathcal{D}})$ with a Neumann series, and proceeding exactly as before we obtain the proof of the Corollary. □

4. Resolvent Estimates

In this section we prepare the crucial resolvent estimates that will be used in the proof of the main results. In order to use the spectral formula, we need estimates on the perturbed resolvent operators and their derivatives with respect to λ as bounded operators from suitable weighted L^p spaces to L^∞. We shall use the Hölder and Young inequalities in Lorentz spaces extensively; for the convenience of the reader, we give a sketch of the main useful results in the Appendix.

We consider first the resolvent of the magnetic laplacian. We recall that, by Proposition 1.4, the operators $R(\lambda \pm i0) = R_0(\lambda \pm i0)(I + W(x,D)R_0(\lambda \pm i0))^{-1}$ are well defined as bounded operators from $L^2(w_\beta(x)dx)$ to $H^2(w_\beta(x)^{-1}dx)$; moreover, we have the explicit representation (1.36). Our first result is the following:

LEMMA 1.2. *Let* $R(\lambda \pm i0) = R_0(\lambda \pm i0)(I + W(x,D)R_0(\lambda \pm i0))^{-1}$ *be the resolvent of* $-\Delta + W$ *and assume the coefficients of* $W(x,D) = \sum a_j(x)\partial_j + b(x)$ *satisfy* (1.53). *Then, for all* $\lambda \geq 0$, *the following estimates hold:*

$$\|R(\lambda \pm i0)f\|_{L^\infty} \leq C \|w_\beta^{1/2} f\|_{L^2}, \tag{1.76}$$

$$\|\partial_\lambda R(\lambda \pm i0)f\|_{L^\infty} \leq C \left(1 + \frac{1}{\sqrt{\lambda}}\right) \|\langle x\rangle w_\beta^{1/2} f\|_{L^2}. \tag{1.77}$$

PROOF. The estimate (1.76) is the easiest one. In fact, by formula (1.63) and the explicit representation (1.36) for R_0, we obtain

$$\|R(\lambda \pm i0)f\|_{L^\infty} \leq C \cdot \|\frac{1}{|x|} * |(I + WR_0)^{-1} f|\|_{L^\infty};$$

using Young inequality in Lorentz spaces, we get

$$\begin{aligned}
\|R(\lambda \pm i0)f\|_{L^\infty} &\leq \|(I + WR_0)^{-1} f\|_{L^{3/2,1}} \\
&\leq \|w_\beta(x)^{-1/2} w_\beta(x)^{1/2} (I + WR_0)^{-1} f\|_{L^{3/2,1}} \\
&\leq \|w_\beta(x)^{-1/2}\|_{L^{6,2}} \|w_\beta(x)^{1/2} (I + WR_0)^{-1} f\|_{L^2}.
\end{aligned}$$

The uniform bound for the operators $(I+WR_0)^{-1}$ proved in Proposition 1.4 and the observation that $w_\beta^{-1/2} \in L^{6,2}$, for all $\beta > 1$ (see Proposition 1.8) are sufficient now to conclude the proof of estimate (1.76).

In order to proceed with the proof of (1.77) we observe that from (1.36) we immediately obtain the following explicit representations, for all $\lambda > 0$:

$$\partial_\lambda R_0(\lambda \pm i0)f = R_0^2(\lambda \pm i0)f = \pm\frac{i}{8\pi\sqrt{\lambda}}\int_0^\infty e^{\pm i\sqrt{\lambda}|x-y|}f(y)dy, \tag{1.78}$$

$$\partial_j R_0^2(\lambda \pm i0)f = \pm\frac{1}{8\pi}\int_0^\infty e^{\pm i\sqrt{\lambda}|x-y|}\sum\frac{x_j-y_j}{|x-y|}f(y)dy. \tag{1.79}$$

At this point, differentiating in (1.63) we get

$$\partial_\lambda R(\lambda \pm i0) = A + B \tag{1.80}$$

where

$$A = R_0^2(\lambda \pm i0)(I + WR_0(\lambda \pm i0))^{-1}$$

and

$$B = R_0(\lambda \pm i0)(I + WR_0(\lambda \pm i0))^{-1}WR_0^2(\lambda \pm i0)(I + WR_0(\lambda \pm i0))^{-1}.$$

We treat separately the two terms. By (1.78), we estimate

$$\begin{aligned}\|Af\|_{L^\infty} &\le \frac{C}{\sqrt{\lambda}}\|(I+WR_0)^{-1}f\|_{L^1}\\ &\le \frac{C}{\sqrt{\lambda}}\|\langle x\rangle^{-1}w_\beta(x)^{-1/2}\|_{L^2}\|\langle x\rangle w_\beta(x)^{1/2}(I+WR_0)^{-1}f\|_{L^2}.\end{aligned}$$

We observe (Proposition 1.8) that $\langle x\rangle^{-1}w_\beta(x)^{-1/2} \in L^2$ for all $\beta > 1$ and, by the uniform bound for the norms of $(I+WR_0)^{-1}$ in the space of bounded operators onto $L^2(\langle x\rangle w_\beta(x)dx)$ for (see Proposition 1.4), we conclude that, for some $C > 0$

$$\|Af\|_{L^\infty} \le \frac{C}{\sqrt{\lambda}}\|\langle x\rangle w_\beta(x)^{1/2}f\|_{L^2}. \tag{1.81}$$

For the estimate of the term B, we start with some computation on the operator WR_0^2. Using the representation (1.79), we obtain

$$\|w_\beta^{1/2}a_j\partial_j R_0^2 f\|_{L^2} \le \|w_\beta^{1/2}a_j\|_{L^2}\|\partial_j R_0^2 f\|_{L^\infty} \le C\cdot\|w_\beta^{1/2}a_j\|_{L^2}\|f\|_{L^1}.$$

By the above observation that

$$\|f\|_{L^1} \le \|\langle x\rangle w_\beta^{1/2}(x)f\|_{L^2},$$

it turns out that, if $w_\beta^{1/2}a_j \in L^2$, then

$$\|w_\beta(x)^{1/2}a_j(x)\partial_j R_0^2 f\|_{L^2} \le C\cdot\|\langle x\rangle w_\beta^{1/2}(x)f\|_{L^2}. \tag{1.82}$$

In a similar way, using (1.78), we have

$$\|w_\beta^{1/2}bR_0^2 f\|_{L^2} \le \|w_\beta^{1/2}b(x)\|_{L^2}\|R_0^2 f\|_{L^\infty} \le \frac{C}{\sqrt{\lambda}}\|w_\beta^{1/2}b\|_{L^2}\|f\|_{L^1}.$$

If we assume that $w_\beta^{1/2} b \in L^2$, we conclude that

$$\|w_\beta(x)^{1/2} b(x) R_0^2 f\|_{L^2} \leq \frac{C}{\sqrt{\lambda}} \cdot \|\langle x\rangle w_\beta^{1/2}(x) f\|_{L^2}. \tag{1.83}$$

Inequalities (1.82) and (1.83) can be unified now, to show that, under the assumptions

$$w_\beta^{1/2} a_j \in L^2, \qquad w_\beta^{1/2} b \in L^2, \tag{1.84}$$

the estimate

$$\|w_\beta(x)^{1/2} W(x,D) R_0^2(\lambda \pm i0) f\|_{L^2} \leq C\left(1 + \frac{1}{\sqrt{\lambda}}\right) \|\langle x\rangle w_\beta(x)^{1/2} f\|_{L^2} \tag{1.85}$$

holds, for some $C > 0$. Observe that assumptions (1.84) are weaker than (1.53), so that they are obviously satisfied by the hypothesis of the Lemma.

Now we are ready for the estimate of the term B. First, we use the representation (1.36) for R_0 to obtain

$$\begin{aligned} \|Bf\|_{L^\infty} &\leq \left\| \frac{1}{|x|} * |(I + WR_0)^{-1} W R_0^2 (I + WR_0)^{-1} f| \right\|_{L^\infty} \\ &\leq \|(I + WR_0)^{-1} W R_0^2 (I + WR_0)^{-1} f\|_{L^{3/2,1}} =: \|Tf\|_{L^{3/2,1}}. \end{aligned}$$

As before, we use the properties of the weights $w_\beta(x)$ to observe that

$$\|g\|_{L^{3/2,1}} \leq \|w_\beta(x)^{1/2} g\|_{L^2}.$$

Then, the last series of inequalities gives

$$\|Bf\|_{L^\infty} \leq \|w_\beta(x)^{1/2} Tf\|_{L^2}.$$

Now we use the uniform bounds for the inverse operators $(I + WR_0)^{-1}$ (see Proposition 1.4) to proceed with

$$\|Bf\|_{L^\infty} \leq \|w_\beta(x)^{1/2} W R_0^2 (I + WR_0)^{-1} f\|_{L^2};$$

finally, by inequality (1.85) and the above mentioned estimates on the norms of $(I + WR_0)^{-1}$ in the space of bounded operators onto $L^2(\langle x\rangle w_\beta(x)^{1/2} dx)$, we obtain the estimate

$$\|Bf\|_{L^\infty} \leq C\left(1 + \frac{1}{\sqrt{\lambda}}\right) \|\langle x\rangle w_\beta(x)^{1/2} f\|_{L^2}. \tag{1.86}$$

In conclusion, estimates (1.81), (1.86) and the representation (1.80) conclude the proof of (1.77) and the Lemma. □

REMARK 1.9. The limiting absorption principle allows us to rewrite the spectral formula in the following way: for any (smooth, compactly supported) function $\phi(\lambda)$ on $\mathbb{R}$, and any test function f,

$$\phi(-\Delta + W) f = \int_0^{+\infty} \phi(\lambda) \Im R(\lambda + i0) f d\lambda. \tag{1.87}$$

where the integral is restricted to the positive real axis since of course $\Im R(\lambda) = 0$ for negative λ.

The resolvent estimates just proved imply that we can integrate by parts in the above formula, i.e., if

$$\phi(\lambda) = \psi'(\lambda)$$

then

$$\phi(-\Delta+W)f=\int_0^{+\infty}\psi'(\lambda)\Im R(\lambda+i0)fd\lambda \tag{1.88}$$
$$=-\int_0^{+\infty}\psi(\lambda)\partial_\lambda\Im R(\lambda+i0)fd\lambda$$

The problems arising from the singularity at $\lambda=0$ are easily overcome. To prove this, consider a cutoff function $\chi(\lambda)$ supported in $[-L,L]$, and write

$$\phi(-\Delta+W)f=\lim_{L\to+\infty}\int_0^{+\infty}\phi(\lambda)(1-\chi(\lambda L))\Im R(\lambda+i0)fd\lambda$$

whence

$$\begin{aligned}\phi(-\Delta+W)f=&-\lim_{L\to+\infty}L\int_0^{1/L}\psi(\lambda)\chi'(\lambda L)\Im R(\lambda+i0)fd\lambda\\&-\lim_{L\to+\infty}\int_0^{+\infty}(1-\chi(\lambda L))\psi(\lambda)\partial_\lambda\Im R(\lambda+i0)fd\lambda.\\=&u_L+v_L.\end{aligned}$$

The last term v_L converges to (1.88) uniformly, thanks to estimate (1.77) (and Lebesgue's dominated convergence theorem), hence it is clear that $u_L=\phi(-\Delta+W)f-v_L$ also converges uniformly, and it will be sufficient to show that its limit is 0, e.g., in distribution sense. To estimate the integral

$$u_L=-L\int_0^{1/L}\psi(\lambda)\chi'(\lambda L)\Im R(\lambda+i0)fd\lambda$$

we can use the identity

$$\Im R(\lambda+i0)=(I+R_0(\lambda-i0)W)^{-1}\Im R_0(\lambda+i0)(I+WR_0(\lambda+i0))^{-1}. \tag{1.89}$$

Consider then the L^2 product

$$(\Im R(\lambda+i0)f,g)=(\Im R_0(\lambda+i0)(I+WR_0(\lambda+i0))^{-1}f,(I+WR_0(\lambda+i0))^{-1}g).$$

From the explicit formula

$$\Im R_0(\lambda+i0)h=C\int\frac{\sin(\sqrt{\lambda}|x-y|)}{|x-y|}h(y)dy$$

we have

$$|\Im R_0(\lambda+i0)h|\le C\sqrt{\lambda}\int|h(y)|dy$$

which implies

$$\|\Im R_0(\lambda+i0)h\|_{L^\infty}\le C\sqrt{\lambda}\|h\|_{L^1}\le C\sqrt{\lambda}\|\langle x\rangle w_\beta^{1/2}h\|_{L^2}$$

for any $\beta>1$. Recalling now the uniform bound for $(I+WR_0(\lambda+i0))^{-1}$ in Proposition 1.4 in the weighted L^2 norms with weight $\langle x\rangle w_\beta^{1/2}$, we obtain easily

$$|(\Im R(\lambda+i0)f,g)|\le C\sqrt{\lambda}\|\langle x\rangle w_\beta^{1/2}f\|_{L^2}\|\langle x\rangle w_\beta^{1/2}g\|_{L^2}.$$

From this estimate it is easy to prove that

$$(u_L,g)=-L\int_0^{1/L}\psi(\lambda)\chi'(\lambda L)(\Im R(\lambda+i0)f,g)d\lambda\to 0$$

as $L \to +\infty$, which concludes the argument.

We will prove now an analogue of Lemma 1.2 for the Dirac operator. In what follows, $R(z) = (-zI_4 + \mathcal{D} + V)^{-1}$ denotes the resolvent of the perturbed Dirac operator. Our approach here will be slightly different: we shall use the formula

$$R(z) = R_{\mathcal{D}}(z) + R_{\mathcal{D}}(z)V(x)R_{\mathcal{D}}(z)(I + V(x)R_{\mathcal{D}}(z))^{-1}, \tag{1.90}$$

valid for all $z \in \mathbb{C}$ (to be interpreted of course, for $z = \lambda \in \mathbb{R}$, as the extended resolvents $R(\lambda) := R(\lambda \pm i0)$ on the weighted L^2 spaces, as given by Proposition 1.6 and Corollary 1.1). When inserted in the spectral formula, the first term $R_{\mathcal{D}}$ at the right hand side reproduces the solution to the free Dirac equation, and the main part of our proof will be the estimate of second term

$$Q := R_{\mathcal{D}}VR_{\mathcal{D}}(I + VR_{\mathcal{D}})^{-1}. \tag{1.91}$$

To this end, we shall need an explicit representation for $R_{\mathcal{D}}(\lambda \pm i0)$, which is easily obtained from the formula

$$R_{\mathcal{D}}(\lambda \pm i0) = R_0(\lambda^2 \pm i0)(\mathcal{D} + \lambda I_4). \tag{1.92}$$

Recalling (1.36), after an integration by parts we obtain

$$\begin{aligned} R_{\mathcal{D}}(\lambda \pm i0)f &= \frac{i\lambda}{4\pi}\int_{\mathbb{R}^3} \frac{e^{\pm i\lambda|x-y|}}{|x-y|}\left(I_4 \mp \sum_{j=1}^{3} \alpha_j \frac{x_j - y_j}{|x-y|}\right) f(y)dy \\ &+ \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{\pm i\lambda|x-y|}}{|x-y|^2}\sum_{j=1}^{3}\alpha_j \frac{x_j - y_j}{|x-y|} f(y)dy. \end{aligned} \tag{1.93}$$

¿From here we derive immediately an analogous representation for

$$R^2_{\mathcal{D}}(\lambda) = \frac{\partial}{\partial\lambda}R_{\mathcal{D}}(\lambda);$$

indeed, differentiating (1.93) with respect to λ, we get

$$\begin{aligned} R^2_{\mathcal{D}}(\lambda \pm i0)f &= \frac{\lambda}{4\pi}\int_{\mathbb{R}^3} e^{\pm i\lambda|x-y|}\left(\mp I_4 + \sum_{j=1}^{3} \alpha_j \frac{x_j - y_j}{|x-y|}\right) f(y)dy \\ &\pm \frac{i}{4\pi}\int_{\mathbb{R}^3} \frac{e^{\pm i\lambda|x-y|}}{|x-y|}\sum_{j=1}^{3}\alpha_j \frac{x_j - y_j}{|x-y|} f(y)dy. \end{aligned} \tag{1.94}$$

We collect all the necessary estimates in the following lemma (we write for simplicity $R_{\mathcal{D}}(\lambda)$ instead of $R_{\mathcal{D}}(\lambda \pm i0)$ since the estimates are the same):

Lemma 1.3. *Suppose that*

$$|V(x)| \le \frac{C_0}{|x|^{1/2}\langle x\rangle^s(|\log|x|| + 1)^{\beta/2}}, \tag{1.95}$$

for some $s > \frac{3}{2}$, $\beta > 1$, $C_0 > 0$. *Then the following estimates hold for all* $\epsilon > 0$ *small enough and all* $\lambda \in \mathbb{R}$:

$$\|\langle x\rangle^{1/2+\epsilon}VR^2_{\mathcal{D}}(\lambda)f\|_{L^2} \le C_\epsilon \cdot \langle\lambda\rangle \cdot \|\langle x\rangle^{3/2+\epsilon}f\|_{L^2}, \tag{1.96}$$

$$\|R_{\mathcal{D}}(\lambda)VR_{\mathcal{D}}(\lambda)f\|_{L^\infty} \leq C_\epsilon \cdot \langle\lambda\rangle^2 \cdot \|\langle x\rangle^{1/2+\epsilon} f\|_{L^2} \tag{1.97}$$

$$\|R^2_{\mathcal{D}}(\lambda)VR_{\mathcal{D}}(\lambda)f\|_{L^\infty} + \|R_{\mathcal{D}}(\lambda)VR^2_{\mathcal{D}}(\lambda)f\|_{L^\infty} \leq C_\epsilon \cdot \langle\lambda\rangle^2 \cdot \|\langle x\rangle^{3/2+\epsilon} f\|_{L^2} \tag{1.98}$$

for some $C = C_\epsilon$ independent of λ.

Proof. In the following we shall use the shorthand notation, for $s \in \mathbb{R}$,

$$\|f\|_{L^2_\gamma} := \|\langle x\rangle^\gamma f\|_{L^2} \tag{1.99}$$

From the explicit representations (1.93) and (1.94) we have the simple pointwise estimates

$$|R_{\mathcal{D}}(\lambda)f| \leq C(|\lambda|\cdot|x|^{-1} + |x|^{-2}) * f, \qquad |R^2_{\mathcal{D}}(\lambda)f| \leq C(|\lambda| + |x|^{-1}) * f. \tag{1.100}$$

Since $|x|^{-1} \in L^{3,\infty}$, by the Young inequality in Lorentz spaces (see the Appendix) we get

$$\begin{aligned}\|VR^2_{\mathcal{D}}(\lambda)f\|_{L^2_\gamma} &\leq \|V\|_{L^2_\gamma}\cdot|\lambda|\cdot\|1 * f\|_{L^\infty} + \|V\|_{L^2_\gamma}\||x|^{-1} * f\|_{L^\infty} \\ &\leq \|V\|_{L^2_\gamma}\left(|\lambda|\cdot\|f\|_{L^1} + \|f\|_{L^{3/2,1}}\right).\end{aligned}$$

By the obvious inequalities valid for all $\epsilon > 0$

$$\|f\|_{L^1} \leq C(\epsilon)\|f\|_{L^2_{3/2+\epsilon}}, \qquad \|f\|_{L^{3/2,1}} \leq C(\epsilon)\|f\|_{L^2_{1/2+\epsilon}}, \tag{1.101}$$

we arrive at the first estimate

$$\|VR^2_{\mathcal{D}}(\lambda)f\|_{L^2_\gamma} \leq C(\epsilon)\|V\|_{L^2_\gamma}\langle\lambda\rangle\|f\|_{L^2_{3/2+\epsilon}}. \tag{1.102}$$

Since $\|V\|_{L^2_\gamma} < \infty$ by assumption (1.95) as soon as $\gamma = 1/2 + \epsilon < s - 1$, we see that (1.96) follows provided ϵ is suitably small.

In a similar way, in order to prove (1.97) we use again (1.100) and we write (recall that $|x|^{-2} \in L^{3/2,\infty}$)

$$\begin{aligned}\|R_{\mathcal{D}}(\lambda)VR_{\mathcal{D}}(\lambda)f\|_{L^\infty} &\leq C\left(|\lambda|\cdot\||x|^{-1} * VR_{\mathcal{D}}f\|_{L^\infty} + \||x|^{-2} * VR_{\mathcal{D}}f\|_{L^\infty}\right) \\ &\leq C\left(|\lambda|\cdot\|VR_{\mathcal{D}}(\lambda)f\|_{L^{3/2,1}} + \|VR_{\mathcal{D}}(\lambda)f\|_{L^{3,1}}\right).\end{aligned}$$

For the first term we can write, recalling again (1.100),

$$\begin{aligned}\|VR_{\mathcal{D}}(\lambda)f\|_{L^{3/2,1}} &\leq \|V\|_{L^{3/2,1}}|\lambda|\cdot\||x|^{-1} * f\|_{L^\infty} + \|V\|_{L^2}\||x|^{-2} * f\|_{L^{6,2}} \\ &\leq \|V\|_{L^{3/2,1}}|\lambda|\cdot\|f\|_{L^{3/2,1}} + \|V\|_{L^2}\|f\|_{L^2} \\ &\leq \left(\|V\|_{L^{3/2,1}}|\lambda| + \|V\|_{L^2}\right)\|f\|_{L^2_{3/2+\epsilon}}\end{aligned} \tag{1.103}$$

(see (1.101)), while for the second term we have

$$\begin{aligned}\|VR_{\mathcal{D}}(\lambda)f\|_{L^{3,1}} &\leq \|V\|_{L^{3,1}}|\lambda|\cdot\||x|^{-1} * f\|_{L^\infty} + \|V\|_{L^{6,2}}\||x|^{-2} * f\|_{L^{6,2}} \\ &\leq \|V\|_{L^{3,1}}|\lambda|\cdot\|f\|_{L^{3/2,1}} + \|V\|_{L^{6,2}}\|f\|_{L^2} \\ &\leq \left(\|V\|_{L^{3,1}}|\lambda| + \|V\|_{L^{6,2}}\right)\|f\|_{L^2_{3/2+\epsilon}}\end{aligned} \tag{1.104}$$

where we have used (1.101) and the trivial inequality $\|f\|_{L^2} \leq \|f\|_{L^2_\gamma}$, $\forall\gamma > 0$. Summing up, we get

$$\|R_{\mathcal{D}}(\lambda)VR_{\mathcal{D}}(\lambda)f\|_{L^\infty} \leq C\cdot C(V)\langle\lambda\rangle^2\|f\|_{L^2_{3/2+\epsilon}} \tag{1.105}$$

where the quantity

$$C(V) := \|V\|_{L^{3/2,1}} + \|V\|_{L^{3,1}} + \|V\|_{L^{6,2}} + \|V\|_{L^2} < \infty \tag{1.106}$$

is finite by assumption (1.95) (see also the Appendix 8).

The proof of (1.98) is similar: by (1.100) we get

$$\begin{aligned}\|R^2_{\mathcal{D}}(\lambda)VR_{\mathcal{D}}(\lambda)f\|_{L^\infty} &\le C\left(|\lambda|\cdot\|1*VR_{\mathcal{D}}f\|_{L^\infty}+\||x|^{-1}*VR_{\mathcal{D}}f\|_{L^\infty}\right)\\ &\le C\left(|\lambda|\cdot\|VR_{\mathcal{D}}(\lambda)f\|_{L^1}+\|VR_{\mathcal{D}}(\lambda)f\|_{L^{3/2,1}}\right).\end{aligned}$$

We have already estimated the second term in (1.103), and for the first one we have

$$\begin{aligned}\|VR_{\mathcal{D}}(\lambda)f\|_{L^1} &\le \|V\|_{L^{3/2}}|\lambda|\cdot\||x|^{-1}*f\|_{L^3}+\|V\|_{L^3}\||x|^{-2}*f\|_{L^{3/2}}\\ &\le \left(\|V\|_{L^{3/2}}|\lambda|+\|V\|_{L^3}\right)\|f\|_{L^1}\\ &\le \left(\|V\|_{L^{3/2}}|\lambda|+\|V\|_{L^3}\right)\|f\|_{L^2_{3/2+\epsilon}}\end{aligned} \tag{1.107}$$

and hence

$$\|R^2_{\mathcal{D}}(\lambda)VR_{\mathcal{D}}(\lambda)f\|_{L^\infty}\le C\cdot C'(V)\langle\lambda\rangle^2\|f\|_{L^2_{3/2+\epsilon}} \tag{1.108}$$

where the quantity

$$C'(V):=\|V\|_{L^{3/2}}+\|V\|_{L^{3/2,1}}+\|V\|_{L^3}+\|V\|_{L^2}<\infty \tag{1.109}$$

is finite again by assumption (1.95).

Finally, the last estimate can be obtained as follows:

$$\begin{aligned}\|R_{\mathcal{D}}(\lambda)VR^2_{\mathcal{D}}(\lambda)f\|_{L^\infty} &\le C\left(|\lambda|\cdot\||x|^{-1}*VR^2_{\mathcal{D}}f\|_{L^\infty}+\||x|^{-2}*VR^2_{\mathcal{D}}f\|_{L^\infty}\right)\\ &\le C\left(|\lambda|\cdot\|VR_{\mathcal{D}}(\lambda)f\|_{L^{3/2,1}}+\|VR_{\mathcal{D}}(\lambda)f\|_{L^{3,1}}\right).\end{aligned}$$

Proceeding as above, we estimate

$$\begin{aligned}\|VR^2_{\mathcal{D}}(\lambda)f\|_{L^{3/2,1}} &\le \|V\|_{L^{3/2,1}}|\lambda|\cdot\|1*f\|_{L^\infty}+\|V\|_{L^{3/2,1}}\||x|^{-1}*f\|_{L^\infty}\\ &\le \|V\|_{L^{3/2,1}}4\langle\lambda\rangle\left(\|f\|_{L^1}+\|f\|_{L^{3/2,1}}\right)\\ &\le \|V\|_{L^{3/2,1}}4\langle\lambda\rangle\|f\|_{L^2_{3/2+\epsilon}}\end{aligned} \tag{1.110}$$

and

$$\begin{aligned}\|VR^2_{\mathcal{D}}(\lambda)f\|_{L^{3,1}} &\le \|V\|_{L^{3,1}}|\lambda|\cdot\|1*f\|_{L^\infty}+\|V\|_{L^{3,1}}\||x|^{-1}*f\|_{L^\infty}\\ &\le \|V\|_{L^{3,1}}4\langle\lambda\rangle\left(\|f\|_{L^1}+\|f\|_{L^{3/2,1}}\right)\\ &\le \|V\|_{L^{3,1}}4\langle\lambda\rangle\|f\|_{L^2_{3/2+\epsilon}}\end{aligned} \tag{1.111}$$

whence

$$\|R_{\mathcal{D}}(\lambda)VR^2_{\mathcal{D}}(\lambda)f\|_{L^\infty}\le C\cdot C''(V)\langle\lambda\rangle^2\|f\|_{L^2_{3/2+\epsilon}} \tag{1.112}$$

where the quantity

$$C''(V):=\|V\|_{L^{3/2,1}}+\|V\|_{L^{3,1}}<\infty \tag{1.113}$$

is finite by assumption (1.95). □

Remark 1.10. The same remark concerning the simpler version of the spectral formula (1.87) and the integration by parts formula (1.88) applies also to the Dirac resolvent, with obvious modifications in the proof.

5. Proof of Theorem 1.1

Let $(\varphi_j)_{j=0,1,\dots}$ be a standard Paley-Littlewood partition of the unity, with the properties

$$\varphi_j(\lambda) = \varphi_0(2^{-j}\lambda), \qquad \varphi_0 + \sum_{j\geq 1} \varphi_j = 1, \tag{1.114}$$

for a suitable $\varphi_0 \in \mathcal{C}_0^\infty$. We consider the Cauchy problem

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) + W(x,D)u = 0 \\ u(0,x) = 0, \quad u_t(0,x) = \varphi_j(\sqrt{-\Delta + W})g(x), \end{cases} \tag{1.115}$$

The solution can be represented using the spectral formula as follows:

$$u(t,x) = \frac{1}{2\pi i} \int_0^{+\infty} \varphi_j(\sqrt{\lambda}) \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}} R(\lambda) g d\lambda, \tag{1.116}$$

and after an integration by parts (see Remark 1.9) this gives

$$u(t,x) = \frac{C}{t} \int_0^{+\infty} \cos(t\sqrt{\lambda}) \left[\partial_\lambda \varphi_j(\sqrt{\lambda}) R(\lambda) g + \varphi_j(\sqrt{\lambda}) \partial_\lambda R(\lambda) g \right] d\lambda. \tag{1.117}$$

Thus, recalling estimates (1.76) and (1.77), we have

$$|u(t,x)| \leq \frac{C}{t} \| \langle x \rangle w_\beta^{1/2} g \|_{L^2} \int_0^{+\infty} \left(|\partial_\lambda \varphi_j(\sqrt{\lambda})| + \left(1 + \frac{1}{\sqrt{\lambda}}\right) |\varphi_j(\sqrt{\lambda})| \right) d\lambda$$

and a change of variables $\lambda = 2^{2j}\mu$ in the integral gives immediately

$$|u(t,x)| \leq \frac{C}{t} 2^{2j} \| \langle x \rangle w_\beta^{1/2} g \|_{L^2} \tag{1.118}$$

with some constant C independent of j and g.

If we now define as usual

$$\widetilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}, \quad \varphi_{-1} = 0,$$

so that $\varphi_j \equiv \varphi_j \widetilde{\varphi}_j$, we see that the Cauchy problem (1.115) can be written equivalently

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) + W(x,D)u = 0 \\ u(0,x) = 0, \quad u_t(0,x) = \varphi_j(\sqrt{-\Delta_W}) \widetilde{\varphi}_j(\sqrt{-\Delta_W}) g(x), \end{cases} \tag{1.119}$$

hence our estimate (1.118) implies also the estimate

$$|u(t,x)| \leq \frac{C}{t} 2^{2j} \| \langle x \rangle w_\beta^{1/2} \widetilde{\varphi}_j(\sqrt{-\Delta_W}) g \|_{L^2}. \tag{1.120}$$

Finally, consider the original Cauchy problem (1.6), and decompose g as a sum

$$g = \sum_{j\geq 0} \varphi_j(\sqrt{-\Delta_W}) g(x).$$

By estimate (1.120) we obtain easily estimate (1.9).

$$|u(t,x)| \leq \frac{C}{t} \sum_{j\geq 0} 2^{2j} \| \langle x \rangle w_\beta^{1/2} \varphi_j(\sqrt{-\Delta_W}) g \|_{L^2}. \tag{1.121}$$

The computations in the case of initial data of the form

$$u(0,x) = f, \qquad u_t(0,x) = 0$$

are completely analogous, and we thus obtain estimate (1.12).

REMARK 1.11. In view of the application to the Dirac system, the following remark will be useful. If the initial datum g has the form

$$g = (-\Delta_W)^s h \tag{1.122}$$

for some $s > 0$, a direct application of estimate (1.121) would give only

$$|u(t,x)| \leq \frac{C}{t} \sum_{j \geq 0} 2^{2j} \| \langle x \rangle w_\beta^{1/2} \varphi_j(\sqrt{-\Delta_W})(-\Delta_W)^s h \|_{L^2}. \tag{1.123}$$

Actually, if we go back to the spectral formula (1.117), we see that the solution can be written

$$u(t,x) = \frac{C}{t} \int_0^{+\infty} \lambda^{s/2} \cos(t\sqrt{\lambda}) \left[\partial_\lambda \varphi_j(\sqrt{\lambda}) R(\lambda) h + \varphi_j(\sqrt{\lambda}) \partial_\lambda R(\lambda) h \right] d\lambda. \tag{1.124}$$

with an additional factor $\lambda^{s/2}$. Thus, proceeding as above, we arrive at the simpler estimate

$$|u(t,x)| \leq \frac{C}{t} \sum_{j \geq 0} 2^{(2+s)j} \| \langle x \rangle w_\beta^{1/2} \varphi_j(\sqrt{-\Delta_W}) h \|_{L^2}. \tag{1.125}$$

We now prove estimate (1.11) under the stronger assumption (1.10) on the potential $W(x,D)$. Consider first the case of initial data of the form

$$u(0,x) = 0, \qquad u_t(0,x) = g.$$

We can write g as follows:

$$g = (1 - \Delta + W)^{-1-\epsilon} (1 - \Delta + W)^{1+\epsilon} g$$

for some fixed $\epsilon > 0$. Then the solution u can be represented as

$$u(t,x) = \frac{1}{2\pi i} \int_0^{+\infty} \psi(\sqrt{\lambda}) \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}} R(\lambda) h d\lambda$$

where

$$h = (1 - \Delta + W)^{1+\epsilon} g, \qquad \psi(\sqrt{\lambda}) = (1 + \lambda)^{1+\epsilon}.$$

Proceeding as above, after an integration by parts we arrive at

$$|u(t,x)| \leq \frac{C}{t} \| \langle x \rangle w_\beta^{1/2} h \|_{L^2} \int_0^{+\infty} ((1+\lambda)^{-1-\epsilon} + (1+\lambda)^{-2-\epsilon}) d\lambda$$

and hence

$$|u(t,x)| \leq \frac{C}{t} \| \langle x \rangle w_\beta^{1/2} (1 - \Delta + W)^{1+\epsilon} g \|_{L^2} \leq \frac{C}{t} \| \langle x \rangle^{3/2+\epsilon} (1 - \Delta + W)^{1+\epsilon} g \|_{L^2}. \tag{1.126}$$

To conclude the proof of the Theorem, it remains to show that

$$\| \langle x \rangle^{3/2+\epsilon} (1 - \Delta + W)^{1+\epsilon} g \|_{L^2} \leq \| \langle x \rangle^{3/2+\epsilon} g \|_{H^{2+2\epsilon}}. \tag{1.127}$$

We start from the inequality

$$\| \langle x \rangle^s (1 - \Delta + W) f \|_{L^2} \leq \| \langle x \rangle^s f \|_{H^2}$$

which is obviously valid for any $s \geq 0$. By a standard complex interpolation argument, interpolating with the trivial inequality

$$\| \langle x \rangle^s f \|_{L^2} \leq \| \langle x \rangle^s f \|_{L^2}$$

we obtain that

$$\| \langle x \rangle^s (1 - \Delta + W)^\epsilon f \|_{L^2} \leq \| \langle x \rangle^s f \|_{H^{2\epsilon}}$$

for all $0 \le \epsilon \le 1$ and all $s \ge 0$. This implies

$$\|\langle x\rangle^s (1-\Delta+W)^{1+\epsilon} f\|_{L^2} \le \|\langle x\rangle^s (1-\Delta+W) f\|_{H^{2\epsilon}} \le \|\langle x\rangle^s f\|_{H^{2+2\epsilon}} + \|\langle x\rangle^s W f\|_{H^{2\epsilon}}. \tag{1.128}$$

The last term is of the form

$$\|\langle x\rangle^s W(x,D) f\|_{H^{2\epsilon}} \le \|\langle x\rangle^s a(x) D f\|_{H^{2\epsilon}} + \|\langle x\rangle^s b(x) f\|_{H^{2\epsilon}}; \tag{1.129}$$

in order to estimate it, we recall the Kato-Ponce inequality (see [**64**])

$$\|\langle D\rangle^q (vw)\|_{L^p} \le C\|\langle D\rangle^q v\|_{L^{p_1}} \|w\|_{L^{p_2}} + C\|v\|_{L^{p_3}} \|\langle D\rangle^q w\|_{L^{p_4}} \tag{1.130}$$

which is valid for all $q \ge 0$, $p^{-1} = p_1^1 + p_2^{-1} = p_3^{-1} + p_4^{-1}$. With the choices $v(x) = a(x)$, $w(x) = \langle x\rangle^s Df(x)$, $q = 2\epsilon$, $p_1 = p_3 = \infty$ and $p_2 = p_4 = 2$, we obtain

$$\|\langle D\rangle^{2\epsilon} \langle x\rangle^s a(x) D f\|_{L^2} \le C\|\langle D\rangle^{2\epsilon} a\|_{L^\infty} \|\langle x\rangle^s D f\|_{L^2} + C\|a\|_{L^\infty} \|\langle D\rangle^{2\epsilon} (\langle x\rangle^s D f)\|_{L^2}.$$

Now it is clear that

$$\|\langle D\rangle^{2\epsilon} (\langle x\rangle^s D f)\|_{L^2} \le C\|\langle x\rangle^s f\|_{H^{1+2\epsilon}}$$

(use again complex interpolation between the cases $\epsilon = 0$ and $\epsilon = 1$) and in conclusion we obtain

$$\|\langle D\rangle^{2\epsilon} \langle x\rangle^s a(x) D f\|_{L^2} \le C\|\langle D\rangle^{2\epsilon} a\|_{L^\infty} \|\langle x\rangle^s f\|_{H^{1+2\epsilon}}.$$

Here we have used the simple fact that

$$\|a\|_{L^\infty} \le C\|\langle D\rangle^{2\epsilon} a\|_{L^\infty}.$$

The corresponding estimate for the electric term is analogous (actually simpler):

$$\|\langle D\rangle^{2\epsilon} \langle x\rangle^s b(x) f\|_{L^2} \le C\|\langle D\rangle^{2\epsilon} b\|_{L^\infty} \|\langle x\rangle^s f\|_{H^{2\epsilon}}.$$

Recalling now (1.128) and (1.129) we conclude the proof of estimate (1.11).

On the other hand, when the data are of the form

$$u(0,x) = f, \qquad u_t(0,x) = 0$$

the computations are completely analogous and we obtain estimate (1.14) under the stronger assumptions (1.13) on the coefficients.

6. Proof of Theorem 1.2

Remark 1.12. We notice that Theorem 1.1 (and Remark 1.4) can be trivially extended to a *system* of wave equations of the form

$$u_{tt} - (\nabla + iA(x))^2 u + B(x) u = 0 \tag{1.131}$$

where $u(t,x)$ is a $\mathbb{C}^N$ valued function and $A_1(x)$, $A_2(x)$, $A_3(x)$, $B(x)$ are $\mathbb{C}^{N\times N}$ matrices whose coefficients satisfy the assumptions of the Theorem. The resulting dispersive estimates have exactly the same form as in the scalar case.

Consider now the Cauchy problem

$$\begin{cases} iu_t - \mathcal{D}u - V(x)u = 0 \\ u(0,x) = f(x). \end{cases} \tag{1.132}$$

If we apply to the pertubed Dirac system the operator $i\partial_t + \mathcal{D} + V$ we obtain that u is also a solution of a 4×4 system of perturbed wave equations of the form (1.131) with

$$A_j(x) = -\frac{1}{2}(\alpha_j V(x) + V(x)\alpha_j), \tag{1.133}$$

$$B(x) = \mathcal{D}V(x) + V(x)^2 + A_1^2 + A_2^2 + A_3^2 + i\sum \partial_j A_j \tag{1.134}$$

and initial data

$$u(0,x) = f, \qquad u_t(0,x) = i^{-1}(\mathcal{D} + V)f. \tag{1.135}$$

Note that the perturbed operator

$$-\Delta_W = -(\nabla + iA(x))^2 + B(x) \tag{1.136}$$

is exactly the square of the operator $\mathcal{D} + V$:

$$-\Delta_W = (\mathcal{D} + V)^2 \tag{1.137}$$

and hence the initial data for (1.131) can be written

$$u(0,x) = f, \qquad u_t(0,x) = i^{-1}(-\Delta_W)^{1/2} f. \tag{1.138}$$

We are in position to apply to the solution u the estimates already proved in Theorem 1.1; keeping Remark 1.11 into account, we arrive easily at the estimate

$$|u(t,x)| \leq \frac{C}{t}\sum_{j\geq 0} 2^{3j}\|\langle x\rangle w_\beta^{1/2}\varphi_j(\mathcal{D}+V)f\|_{L^2}, \tag{1.139}$$

provided the coefficients $a_j(x)$ and $b(x)$ satisfy the assumptions (1.8). Recalling the explicit form (1.133) of the coefficients in terms of $V(x)$, we see that V must satisfy the conditions

$$|V(x)| \leq \frac{C_0}{|x|\langle x\rangle(|\log|x|| + 1)^\beta}$$

from the magnetic term, and

$$|V(x)^2| + |DV(x)| \leq \frac{C_0}{|x|^2(|\log|x|| + 1)^\beta},$$

from the electric term, for some $\beta > 1$ and some small constant C_0. Summing up, we obtain that (1.139) holds under assumption (1.17).

The estimate in terms of the Sobolev norm can be obtained in exactly the same way as for the perturbed wave equation. Indeed, proceeding as in (1.126) we arrive at the estimate

$$|u(t,x)| \leq \frac{C}{t}\|\langle x\rangle^{3/2+\epsilon}(-\Delta_W)^{3/2+\epsilon}f\|_{L^2}. \tag{1.140}$$

The same arguments used at the end of Section 5 give here

$$|u(t,x)| \leq \frac{C}{t}\|\langle x\rangle^{3/2+\epsilon}f\|_{H^{3+2\epsilon}} \tag{1.141}$$

provided

$$\langle D\rangle^{1+2\epsilon}A_j \in L^\infty, \quad \langle D\rangle^{1+2\epsilon}B \in L^\infty,$$

which is implied by

$$\langle D\rangle^{2+2\epsilon}V \in L^\infty.$$

7. Proof of Theorem 1.3

By exploiting the connection between the massless Dirac and the wave equation, it is easy to obtain an optimal dispersive estimate in the unperturbed case. In order to state it, we recall the definition of the *homogeneous Besov space* $\dot{B}^s_{1,1}$. The Besov norm is given by

$$\|v\|_{\dot{B}^s_{1,1}} = \sum_{j\in\mathbb{Z}} 2^{js}\|\phi_j(\sqrt{-\Delta})v\|_{L^1},$$

where ϕ_j is a *homogeneous* Paley-Littlewood sequence, i.e., fixed a test function $\psi(r) \in C_0^\infty$ such that $\psi(r) = 1$ for $r < 1$, $\psi(r) = 0$ for $r > 2$, we have $\phi_j(r) = \psi(2^{-j+2}r) - \psi(2^{-j+1}r)$ for all $j \in \mathbb{Z}$.

Now, let $u(t,x)$ be a smooth solution of the free massless Dirac equation

$$iu_t(t,x) = \mathcal{D}u(t,x) \tag{1.142}$$

with initial data

$$u(0,x) = f(x). \tag{1.143}$$

Then we have:

Proposition 1.7. *The solution $u(t,x)$ of problem* (1.142),(1.143) *satisfies the dispersive estimate*

$$|u(t,x)| \le \frac{C}{t}\|f\|_{\dot{B}^2_{1,1}}. \tag{1.144}$$

Proof. Recall the identity

$$(i\partial_t + \mathcal{D})(i\partial_t - \mathcal{D}) = (\Delta - \partial^2_{tt})I_4;$$

if we apply the operator $i\partial_t + \mathcal{D}$ to the system (1.142) we see that u solves the Cauchy problem for the wave equation

$$u_{tt} - \Delta u = 0$$

with initial data

$$u(0,x) = f, \qquad u_t(0,x) = i^{-1}\mathcal{D}f.$$

Then, as a consequence of the well known decay estimates for solutions to the free wave equation (see e.g. [**95**]), we obtain

$$|u(t,x)| \le \frac{C}{t}\left(\|f\|_{\dot{B}^2_{1,1}} + \|Df\|_{\dot{B}^1_{1,1}}\right)$$

whence (1.144) follows immediately. □

The proof of Theorem 1.3 follows the same lines as the proof of Theorem 1.1. Consider the Cauchy problem with frequency truncated data

$$\begin{cases} iu_t(t,x) = \mathcal{D}_V u(t,x) \\ u(0,x) = \varphi_j(\mathcal{D}_V)f, \end{cases} \tag{1.145}$$

where $(\varphi_j(\lambda))_{j=0,1,\ldots}$ is the standard Paley-Littlewood partition of the unity defined in (1.114). By means of spectral formula, we can represent the solution of (1.145) as

$$u(t,x) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \varphi_j(\lambda)e^{i\lambda t}\mathcal{I}[R_V(\lambda)]f\,d\lambda. \tag{1.146}$$

Using the identity

$$R_V(\lambda) = R_{\mathcal{D}} - R_{\mathcal{D}} V R_{\mathcal{D}} (I + V R_{\mathcal{D}})^{-1}, \tag{1.147}$$

which is valid thanks to Corollary 1.1, we can split the integrals in (1.146) into two terms, the first one containing the contribution of the free resolvent $R_{\mathcal{D}}$ and the second one containing the contribution of the operator $R_{\mathcal{D}} V R_{\mathcal{D}} (I + V R_{\mathcal{D}})^{-1}$. The first term

$$A := \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \varphi_j(\lambda) e^{i\lambda t} \Im \left[R_{\mathcal{D}}(\lambda) \right] f \, \mathrm{d}\lambda$$

was estimated above (see (1.144)); it remains to estimate the term

$$B = -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \varphi_j(\lambda) e^{i\lambda t} \Im \left[Q(\lambda) \right] f \, \mathrm{d}\lambda, \tag{1.148}$$

where

$$Q(\lambda) := R_{\mathcal{D}}(\lambda) V R_{\mathcal{D}}(\lambda) (I + V R_{\mathcal{D}}(\lambda))^{-1}.$$

After an integration by parts, we obtain

$$B = -\frac{1}{2\pi t} \left[\int_{-\infty}^{+\infty} \varphi_j(\lambda) e^{i\lambda t} \frac{\partial}{\partial \lambda} \mathcal{I}(Q(\lambda)) f \, \mathrm{d}\lambda + \int_{-\infty}^{+\infty} \varphi_j'(\lambda) e^{i\lambda t} \mathcal{I}[Q(\lambda)] f \, \mathrm{d}\lambda \right]; \tag{1.149}$$

an explicit computation shows that

$$\begin{aligned} \frac{\partial Q}{\partial \lambda} &= R_{\mathcal{D}}^2 V R_{\mathcal{D}} (I_4 + V R_{\mathcal{D}})^{-1} + R_{\mathcal{D}} V R_{\mathcal{D}}^2 (I_4 + V R_{\mathcal{D}})^{-1} \\ &\quad + R_{\mathcal{D}} V R_{\mathcal{D}} (I_4 + V R_{\mathcal{D}})^{-1} V R_{\mathcal{D}}^2 (I_4 + V R_{\mathcal{D}})^{-1}. \end{aligned}$$

Now we can apply Lemma 1.3: under assumption (1.21), estimates (1.96), (1.97) and (1.98) are satisfied, and the Lemma gives

$$\| Q(\lambda) f \|_{L^\infty} \leq C \langle \lambda \rangle^2 \| \langle x \rangle^{3/2+\epsilon} f \|_{L^2}, \tag{1.150}$$

$$\left\| \frac{\partial}{\partial \lambda} Q(\lambda) f \right\|_{L^\infty} \leq C \langle \lambda \rangle^3 \| \langle x \rangle^{3/2+\epsilon} f \|_{L^2}, \tag{1.151}$$

for some $C > 0$. Using (1.150) and (1.151) in (1.149) we arrive at the estimate

$$|B| \leq \frac{C}{t} \| f \|_{L^2_{3/2+\epsilon}} \left[\int_{-\infty}^{+\infty} \left(\langle \lambda \rangle^3 |\varphi_j(\lambda)| + \langle \lambda \rangle^2 \left| \varphi_j'(\lambda) \right| \right) \mathrm{d}\lambda \right].$$

Recalling that $\phi_j(\lambda) = \phi_0(2^{-j}\lambda)$, after a change of variables $2^{-j}\lambda = \mu$ we easily obtain

$$|B| \leq \frac{C}{t} 2^{4j} \| \langle x \rangle^{3/2+\epsilon} f \|_{L^2}. \tag{1.152}$$

From this point on, we can proceed as in the proof of Theorem 1.1 and complete the proof of Theorem 1.3.

8. An appendix on Lorentz spaces

For the convenience of the reader, we recall here the definitions and the main properties of the Lorentz spaces $L^{p,q}$, in view of the applications needed in the proof of our results.

For any measurable function $f:\mathbb{R}^n\to\mathbb{C}$ and any $s\geq 0$ we define the *upper-level* E^f_s as the set

$$E^f_s := \{x : |f(x)| > s\}.$$

The *non-increasing rearrangement* of f is then the function

$$f^*(t) := \inf\{s>0 : |E^f_s|\leq t\}, \qquad t\in(0,+\infty).$$

It is also useful to consider the average of f^* defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(r)\,dr.$$

The standard definition of the Lorentz spaces is the following:

Definition 1.1. For any $1\leq p<\infty$ and $1\leq q\leq\infty$ we define the quasinorm $\|f\|_{L^{p,q}}$ as follows:

$$\|f\|_{L^{p,q}} = \begin{cases} \left[\int_0^\infty (t^{1/p}f^*(t))^q \frac{dt}{t}\right]^{1/q}, & 1\leq q<\infty \\ \sup_{t>0} t^{1/p}f^*(t), & q=\infty. \end{cases} \tag{1.153}$$

When $p\neq 1$, if we replace f^* with f^{**} in the above definitions we obtain an equivalent quasinorm which is actually a norm (see [**10**], [**19**]). The *Lorentz space* $L^{p,q}$ is defined by

$$L^{p,q} = \{f : \|f\|_{L^{p,q}}<\infty\}. \tag{1.154}$$

Moreover we define

$$L^{1,1} := L^1, \qquad L^{\infty,\infty} = L^\infty.$$

The spaces $L^{\infty,q}$ for $1\leq q<\infty$ are usually left undefined (although $L^{\infty,1}$ is defined in [**19**] as the closure of L^∞ compactly supported functions in the L^∞ norm).

With the above definitions, one obtains the elementary properties

$$L^{p,p} = L^p, \qquad 1\leq p\leq\infty;$$

$$L^{p,q_1}\subseteq L^{p,q_2}, \qquad 1<p<\infty, \quad 1\leq q_1\leq q_2\leq\infty$$

(with continuous embedding). When the second index is ∞ we obtain the weak Lebesgue spaces (Marcinkiewicz spaces):

$$L^{p,\infty} = L^p_w, \qquad 1\leq p\leq\infty.$$

Moreover, the Lorentz spaces can be obtained by an equivalent construction using real interpolation:

$$L^{p,q} = (L^{p_0}, L^{p_1})_{\theta,q}, \qquad p^{-1} = (1-\theta)p_0^{-1}+\theta p_1^{-1}$$

provided

$$p_0<p_1, \qquad p_0<q\leq\infty, \qquad 0<\theta<1.$$

An alternative characterization of the Lorentz norm can be given using the so-called *atomic decomposition*:

LEMMA 1.4. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be a measurable function and let $1 \leq p < \infty$, $1 \leq q \leq \infty$; then $f \in L^{p,q}$ if and only if there exist a sequnce of sets $(E_j)_{j\in\mathbb{Z}}$ and a sequence of numbers $a = (a_j)_{j\in\mathbb{Z}}$ such that $|E_j| = O(2^j)$, $a \in l^q$ and the following estimate*

$$|f(x)| \leq C \sum_{j\in\mathbb{Z}} a_j 2^{-j/p} \chi_{E_j}(x) \tag{1.155}$$

holds, for some $C > 0$.

It is possible to see that the best constant C in (1.155) is equivalent to the Lorentz norm of the function f.

The most useful properties of Lorentz spaces are the Hölder and Young inequalities, which extend the classical ones for Lebesgue spaces. These were originally proved by O'Neil in [**80**]. We collect them in the following theorems:

THEOREM 1.5 (Hölder inequality). *Let $f \in L^{p_1,q_1}$, $g \in L^{p_2,q_2}$. The following estimates hold:*

- *if $p_1, p_2, p \in]1, \infty[$, $q_1, q_2, q \in [1, \infty]$, then*

$$\|fg\|_{L^{p,q}} \leq C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}, \qquad 1 > p_1^{-1} + p_2^{-1} = p^{-1}, \quad q_1^{-1} + q_2^{-1} \geq q^{-1}; \tag{1.156}$$

- *if $p_1, p_2 \in [1, \infty[$, $q_1, q_2 \in [1, \infty]$, then*

$$\|fg\|_{L^1} \leq C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}, \qquad p_1^{-1} + p_2^{-1} = 1, \quad q_1^{-1} + q_2^{-1} \geq 1. \tag{1.157}$$

We remark that the above statement does not cover the trivial inequality

$$\|fg\|_{L^{p,q}} \leq \|f\|_{L^\infty}\|g\|_{L^{p,q}} \tag{1.158}$$

which is evidently true whenever $L^{p,q}$ is defined.

THEOREM 1.6 (Young inequality). *Let $f \in L^{p_1,q_1}$, $g \in L^{p_2,q_2}$. Then the following estimates hold:*

- *if $p_1, p_2, p \in]1, \infty[$, $q_1, q_2, q \in [1, \infty]$, then*

$$\|f * g\|_{L^{p,q}} \leq C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}, \qquad p_1^{-1} + p_2^{-1} = 1 + p^{-1}, \quad q_1^{-1} + q_2^{-1} \geq q^{-1}; \tag{1.159}$$

- *if $p_1, p_2 \in]1, \infty[$, $q_1, q_2 \in [1, \infty]$, then*

$$\|f * g\|_{L^\infty} \leq C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}, \qquad p_1^{-1} + p_2^{-1} = 1, \quad q_1^{-1} + q_2^{-1} \geq 1. \tag{1.160}$$

As before, we remark that the above statement does not cover the inequality

$$\|f * g\|_{L^{p,q}} \leq C\|f\|_{L^1}\|g\|_{L^{p,q}} \tag{1.161}$$

which is easily seen to be true in all cases when $L^{p.q}$ is defined (e.g., by real interpolation).

We conclude this section by studying the weight functions $w_\beta(x) = |x|(|\log|x|| + 1)^\beta$, with $\beta > 1$ which plays a crucial role in our results; in the following proposition we determine precisely to which Lorentz the powers w_β^{-s} belong.

PROPOSITION 1.8. *For any $s > 0$, $q \in [1, \infty]$ we have $w_\beta^{-s} \in L^{n/s,q}$, provided $\beta > 1/sq$.*

Proof. We will use the equivalent Lorentz norm (1.155). For any $j \in \mathbb{Z}$ consider the ball $B^j := B_{2^{j/n}} = \{x : |x| \le 2^{j/n}\}$ and the rings $E_j := B^{j+1} \setminus B^j$; it is clear that $|E_j| = C_n 2^j$, where C_n depends only on the dimension n. Then, for all $x \in \mathbb{R}^n$ we have the estimate

$$|w_\beta^{-s}(x)| = \sum_{j\in\mathbb{Z}} \frac{1}{|x|^s(|\log|x|| + 1)^{\beta s}} \chi_{E_j}(x) \le C \sum_{j\in\mathbb{Z}} (|j|\log 2 + 1)^{-\beta s} 2^{-js/n} \chi_{E_j}(x).$$

The proof is concluded by the remark that the sequence $a_j = (|j|\log 2 + 1)^{-\beta s}$ is in l^q if and only if $\beta > 1/sq$. $\square$

CHAPTER 2

Strichartz estimates and Kato-smoothing effect

1. Introduction

Following the previous chapter, we continue the investigation on equations with electromagnetic potentials; here we point our attention to the other dispersive-type estimates we presented in the Introduction, namely Strichartz and Kato-smoothing estimates. In this chapter we present some results in this direction, concerning Schrödinger, Wave, Klein-Gordon and Dirac equations. The reference for the results in the present chapter is [**32**].

Strichartz estimates have become a standard tool in the study of linear and nonlinear evolution equations. They are available for a large class of constant coefficient equations, by the methods of [**47**] and [**66**]. In a sense, they represent the modern energy estimates, and are especially effective for problems of low regularity and global existence for nonlinear equations.

Using the notations $L^pL^q = L^p(\mathbb{R}_t; L^q(\mathbb{R}^n_x))$, $\|f\| \lesssim \|g\|$ to mean $\|f\| \leq C\|g\|$, and H^s_q and $\dot{H}^s_q$ to denote the spaces with norms

$$\|f\|_{\dot{H}^s_q} = \|\langle D\rangle^s f\|_{L^q}, \qquad \|f\|_{\dot{H}^s_q} = \||D|^s f\|_{L^q}.$$

where $\langle D\rangle = (1-\Delta)^{1/2}$, $|D| = (-\Delta)^{1/2}$, the Strichartz estimates for the Schrödinger equation take the following form: for $n \geq 2$,

$$\|e^{it\Delta} f\|_{L^pL^q} \lesssim \|f\|_{L^2},$$

provided the couple (p, q) is *Schrödinger admissible*:

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \qquad 2 \leq p \leq \infty, \qquad \frac{2n}{n-2} \geq q \geq 2, \qquad q \neq \infty. \tag{2.1}$$

The couple $(p, q) = (2, 2n/n-2)$ is called the *endpoint* and is allowed when $n > 2$.

For the wave equation the estimates can be written as follows: for $n \geq 3$,

$$\|e^{it|D|} f\|_{L^p\dot{H}_q^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} \lesssim \|f\|_{L^2},$$

provided the couple (p, q) is *wave admissible*:

$$\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}, \qquad 2 \leq p \leq \infty, \qquad \frac{2(n-1)}{n-3} \geq q \geq 2, \qquad q \neq \infty. \tag{2.2}$$

The wave equation endpoint is $(p, q) = (2, 2(n-1)/(n-3))$ and is allowed in dimension $n > 3$.

Finally for the Klein-Gordon equation we have: for $n \geq 2$,

$$\|e^{it\langle D\rangle} f\|_{L^pH_q^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} \lesssim \|f\|_{L^2},$$

provided (p, q) is Schrödinger admissible (see Section 4 for a proof of the last estimate, for which a reference is not immediately available).

We shall also be interested in the decay properties of the Dirac equation, which has been already introduced in the previous chapters. We recall that this is a 4×4 constant coefficient system of the form

$$iu_t + \mathcal{D}u = 0$$

in the *massless* case, and

$$iu_t + \mathcal{D}u + \beta u = 0$$

in the *massive* case. Here $u : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{C}^4$, the operator $\mathcal{D}$ is defined as

$$\mathcal{D} = \frac{1}{i} \sum_{k=1}^{3} \alpha_k \partial_k$$

and α_j, β are the 4×4 *Dirac matrices.* Then the solution $u(t,x) = e^{it\mathcal{D}} f$ of the massless Dirac system with initial value $u(0,x) = f(x)$ satisfies the Strichartz estimate:

$$\|e^{it\mathcal{D}} f\|_{L^p \dot{H}_q^{\frac{1}{q} - \frac{1}{p} - \frac{1}{2}}} \lesssim \|f\|_{L^2}, \qquad n = 3,$$

for all wave admissible (p,q), while in the massive case we have

$$\|e^{it(\mathcal{D}+\beta)} f\|_{L^p H_q^{\frac{1}{q} - \frac{1}{p} - \frac{1}{2}}} \lesssim \|f\|_{L^2}, \qquad n = 3,$$

for all Schrödinger admissible (p,q) (see Section 4 for more details).

In view of the applications, it is an important problem to extend Strichartz estimates to more general equations with variable coefficients, possibly of low regularity in order to retain the advantages over classical energy methods. Indeed, in recent years a large number of works have investigated this kind of problem. In the case of potential perturbations like

$$iu_t - \Delta u + V(x)u = 0, \qquad \Box u + V(x)u = 0,$$

Strichartz estimates are now fairly well understood. We mention among the many works [**18**], [**49**], [**50**], [**90**], [**93**] and the survey [**92**] for the Schrödinger equation, and [**24**], [**44**], [**33**] for the wave equation. We also mention the wave operator approach of Yajima ([**115**], [**116**], [**117**], [**5**]), which was recently optimized in dimension 1 in [**30**].

Results are much less complete in the case of first order perturbations i.e. *magnetic potentials*

$$iu_t + \Delta u + a \cdot \nabla u + bu = 0, \qquad \Box u + a \cdot \nabla u + bu = 0.$$

Concerning Strichartz estimates for the Schrödinger equation with small potentials a, b we recall at least the papers [**98**], [**43**]; in 3D the recent work [**37**] handles for the first time the case of large magnetic potentials. For the wave equation with small magnetic potentials, partial Strichartz estimates were obtained in 3D in [**25**] in the case of smooth, rapidly decaying coefficients. The dispersive estimate in 3D was proved in [**31**] for the magnetic wave equation with small singular potentials and for the massless Dirac system with a small singular matrix potential. We must also mention the papers [**97**], [**88**], [**42**] containig some local estimates in the fully variable coefficient case. Only in the one dimensional case the optimal dispersive estimates for the case of fully variable singular coefficients have been proved in [**30**].

A method of proof which is very efficient in the case of electric potentials was introduced in [**90**] and further developed in [**18**], [**17**]. The main idea is to combine Strichartz estimates for the free equation with Kato smoothing estimates for the perturbed equation. The same method is used in [**37**] for the 3D Schrödinger equation with a large magnetic potential.

Our goal here is to apply a suitable modification of this method in a systematic way to several equations perturbed with magnetic potentials: Schrödinger, wave and Klein-Gordon equations, and the Dirac system with and without mass.

Thus consider a magnetic Schrödinger operator

$$H = -(\nabla + iA(x))^2 + B(x), \tag{2.3}$$

which is selfadjoint under the following assumptions: A_j and B are real valued, and, denoting by B_- (resp. B_+) the negative (resp. positive) part of B,

$$\|B\|_{L^{n/2,\infty}} < \infty, \qquad \|B_-\|_{L^{n/2,\infty}} < \delta, \qquad \|A\|_{L^{n,\infty}} < \delta \tag{2.4}$$

for some δ sufficiently small (see Lemma 2.1 below). Here $L^{p,\infty} = L^p_w$ denotes the Lorentz or weak Lebesgue space. However, in order to state our results, it is more convenient to represent the operator in the form

$$H \equiv -\Delta + W(x, D) \equiv -\Delta + a(x) \cdot \nabla + b(x) \tag{2.5}$$

and to make the abstract assumption that H is selfadjoint. In view of (2.4), the following explicit conditions on a, b are sufficient (but not necessary) for the selfadjointness of H:

$$a(x) \text{ is pure imaginary}, \quad \Im b = -i\nabla \cdot a \tag{2.6}$$

and

$$\|\nabla a\|_{L^{n/2,\infty}} + \|b\|_{L^{n/2,\infty}} < \infty, \qquad \|\Re b_-\|_{L^{n/2,\infty}} < \delta, \qquad \|a\|_{L^{n,\infty}} < \delta \tag{2.7}$$

for a small enough δ.

In what follows, we shall always assume that the coefficients a, b are measurable functions; the assumptions on their decay at infinity and singularity at the origin are specified in each single statement of our main results.

Our first result concerns smoothing estimates of Kato-Yajima type for the scalar Schrödinger, wave and Klein-Gordon equations. Besides being a necessary tool to prove the Strichartz estimates, they have also an independent interest (see e.g. [**8**], [**62**], [**65**]). Notice in particular that we allow a singularity at 0 in the coefficient, and that the electric potential can be large, while the magnetic term must satisfy a smallness condition. We shall use the following weight functions:

$$\tau_\epsilon(x) = \begin{cases} |x|^{\frac{1}{2}-\epsilon} + |x| & \text{if } n \geq 3, \\ |x|^{\frac{1}{2}-\epsilon} + |x|^{1+\epsilon} & \text{if } n = 2 \end{cases}$$

and

$$w_\sigma(x) = |x|(1 + |\log|x||)^\sigma, \qquad \sigma > 1.$$

Then we have:

Proposition 2.1 (Smoothing estimates for scalar equations). *Let $n \geq 2$. Assume the operator*

$$-\Delta + W(x, D) = -\Delta + a(x) \cdot \nabla + b_1(x) + b_2(x)$$

is selfadjoint with

$$|a(x)| \leq \frac{\delta}{\tau_\epsilon w_\sigma^{1/2}}, \qquad |b_1(x)| \leq \frac{\delta}{\tau_\epsilon^2}, \qquad 0 \leq b_2(x) \leq \frac{C}{\tau_\epsilon^2} \tag{2.8}$$

for some $\delta, \epsilon > 0$ sufficiently small and some $\sigma > 1$, $C > 0$. Moreover assume that 0 is not a resonance for $-\Delta + b_2$.

Then the following smoothing estimates hold: for the Schrödinger equation

$$\|\tau_\epsilon^{-1} e^{it(-\Delta+W)} f\|_{L^2L^2} + \|\tau_\epsilon^{-1}|D|^{1/2} e^{it(-\Delta+W)} f\|_{L^2L^2} \lesssim \|f\|_{L^2}$$

while for the wave and Klein-Gordon equations

$$\|\tau_\epsilon^{-1} e^{it\sqrt{-\Delta+W}} f\|_{L^2L^2} + \|\tau_\epsilon^{-1} e^{it\sqrt{1-\Delta+W}} f\|_{L^2L^2} \lesssim \|f\|_{L^2}.$$

The assumption that 0 is not a resonance for $-\Delta + b_2(x)$ here means: if $(-\Delta + b_2)f = 0$ and $\langle x\rangle^{-1} f \in L^2$ then $f \equiv 0$.

We can then prove Strichartz estimates for the perturbed scalar equations as a consequence of the above smoothing properties. Notice that we must require some additional regularity on the magnetic coefficient $a(x)$. Moreover, the use of the Christ-Kiselev lemma (see Section 3 for details) prevents us from reaching the endpoint.

THEOREM 2.1 (Strichartz for Schrödinger). *Let $n \geq 2$, $-\Delta + W$ be as in Proposition 2.1 and assume in addition that*

$$\langle x\rangle^{1+3\epsilon} \chi(x) a_j(x) \in C^{\frac{1}{2}+2\epsilon} \qquad \text{for some function} \quad \chi \gtrsim w_\sigma^{1/2}. \tag{2.9}$$

Then, for any non-endpoint Schrödinger admissible couple (p, q), the following Strichartz estimate holds:

$$\|e^{it(-\Delta+W)} f\|_{L^pL^q} \lesssim \|f\|_{L^2}. \tag{2.10}$$

THEOREM 2.2 (Strichartz for wave). *Let $n \geq 3$, $-\Delta+W$ be as in Proposition 2.1 and assume in addition that*

$$|a(x)| \leq \frac{C}{\tau_\epsilon^2}, \qquad |b_1 + b_2 - \nabla \cdot a| \leq \frac{C}{|x|\tau_\epsilon}. \tag{2.11}$$

Then, for any non-endpoint wave admissible couple (p, q) the following Strichartz estimate holds:

$$\|e^{it\sqrt{-\Delta+W}} f\|_{L^p \dot{H}_q^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} \lesssim \|f\|_{L^2}. \tag{2.12}$$

THEOREM 2.3 (Strichartz for Klein-Gordon). *Let $n \geq 2$, $-\Delta + W$ be as in Proposition 2.1 and assume in addition that*

$$|a(x)| \leq \frac{C}{\tau_\epsilon^2}, \qquad |b_1 + b_2 - \nabla \cdot a| \leq \frac{C}{\langle x\rangle\tau_\epsilon}. \tag{2.13}$$

Then, for any non-endpoint Schrödinger admissible couple (p, q), the following Strichartz estimate holds:

$$\|e^{it\sqrt{-\Delta+1+W}}\|_{L^p H_q^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} \leq C\|f\|_{L^2}. \tag{2.14}$$

Our final results concern the Dirac system:

THEOREM 2.4 (Massless Dirac). *Let $n = 3$, and let $V(x) = V(x)^*$ be a 4×4 complex valued matrix such that*

$$|V(x)| \leq \frac{\delta}{w_\sigma(x)} \tag{2.15}$$

for some δ sufficiently small and some $\sigma > 1$. Then the following smoothing estimate holds:

$$\|w_\sigma^{-1/2} e^{it(\mathcal{D}+V)} f\|_{L^2L^2} \lesssim \|f\|_{L^2} \tag{2.16}$$

and, for any non-endpoint wave admissible couple (p, q), the following Strichartz estimate holds:

$$\|e^{it(\mathcal{D}+V)} f\|_{L^p \dot{H}^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} \lesssim \|f\|_{L^2}. \tag{2.17}$$

THEOREM 2.5 (Massive Dirac). *Let $n = 3$, and let $V(x) = V(x)^*$ be a 4×4 complex valued matrix such that*

$$|V(x)| \leq \frac{\delta}{\tau_\epsilon(x)} \tag{2.18}$$

for some $\delta, \epsilon > 0$ sufficiently small. Then the following smoothing estimate holds:

$$\|\tau_\epsilon^{-1} e^{it(\mathcal{D}+\beta+V)} f\|_{L^2L^2} \lesssim \|f\|_{L^2} \tag{2.19}$$

and, for any non-endpoint Schrödinger admissible couple (p, q), the following Strichartz estimate holds:

$$\|e^{it(\mathcal{D}+\beta+V)} f\|_{L^p H^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} \lesssim \|f\|_{L^2}. \tag{2.20}$$

The paper is organized as follows: in Section 2 we prove resolvent estimates for the perturbed operator, which are equivalent to smoothing estimates for the corresponding flow via Kato theory, while Section 3 is devoted to the proof of the main theorems. A short Appendix collects the estimates for the free Klein-Gordon and Dirac equations; these can be obtained by a standard application of the Ginibre-Velo and Keel-Tao methods, and we decided to include a sketch of the proof for the sake of completeness.

2. Resolvent Estimates

In this section we shall prove the basic resolvent estimates for the perturbed operators, which are the crucial step in the proof. As an immediate consequence we shall obtain smoothing estimates for the corresponding evolution operators, by a standard application of the well-known result of Kato (see [**87**]):

THEOREM 2.6 (Kato smoothing Theorem, [**62**]). *Let X, Y be Hilbert spaces, let $H : X \to X$ be a self-adjoint operator whose resolvent we denote by $R(\lambda) = (H - \lambda)^{-1}$, and let $A : X \to Y$ be a closed, densely defined operator, which may be unbounded. Assume that*

$$\|AR(\lambda)A^* g\|_Y \leq M\|g\|_Y \qquad \forall g \in D(A^*),\ \lambda \notin \mathbb{R}. \tag{2.21}$$

Then the operator A is H-smooth, i.e., $e^{itH} f \in D(A)$ for all $f \in X$ and a.e. t, and

$$\int_{-\infty}^{\infty} \|Ae^{-itH} f\|_Y^2\, dt \leq \frac{2}{\pi} M^2 \|f\|_X^2 \qquad \forall f \in X. \tag{2.22}$$

2.1. The magnetic Schrödinger operator. The following lemma gives sufficient conditions for the magnetic Schrödinger operator $H = -(\nabla + iA(x))^2 + B(x)$ to be selfadjoint. We sketch a proof since the assumptions on the coefficients are not completely standard:

LEMMA 2.1. *Let $A_j(x)$, $A = (A_1, \dots, A_n)$ and $B(x)$ be real valued functions satisfying*

$$\|B_+\|_{L^{n/2,\infty}} < C, \qquad \|B_-\|_{L^{n/2,\infty}} < \delta, \qquad \|A\|_{L^{n,\infty}} < \delta \tag{2.23}$$

for some $C, \delta > 0$. Then, if δ is sufficiently small, the operator

$$H = -(\nabla + iA(x))^2 + B(x) \tag{2.24}$$

can be uniquely defined as a selfadjoint nonnegative operator in L^2, with form domain $H^1(\mathbb{R}^n)$. Moreover we have

$$\|H^{1/2} g\|_{L^2} \simeq \|g\|_{\dot H^1}. \tag{2.25}$$

PROOF. The quadratic form

$$q(\phi, \psi) = ((\nabla + iA(x))\phi, (\nabla + iA(x))\psi)_{L^2} + (B(x)\phi, \psi)_{L^2}$$

is well defined on $H^1 \times H^1$ under assumptions (2.23). Indeed, using the embedding $\dot H^1 \subset L^{2n/(n-2),2}$, Hölder's inequality in Lorentz spaces [**80**] and assumptions (2.23), we have

$$\begin{aligned} |q(\varphi, \varphi)| \le & \|\nabla\varphi\|_{L^2}^2 + 2\|A\|_{L^{n,\infty}} \|\nabla\varphi \cdot \overline{\varphi}\|_{L^{\frac{n}{n-1},1}} + \||A|^2 + |B|\|_{L^{\frac{n}{2},\infty}} \|\varphi^2\|_{L^{\frac{n}{n-2},1}} \\ \lesssim & \|\nabla\varphi\|_{L^2}^2. \end{aligned}$$

The form q is symmetric since A and B are real valued. By standard results (see e.g. [**87**], Theorem VIII.15), q is the form associated to a unique defined self-adjoint operator provided the form is *closed*, i.e. its domain $H^1(\mathbb{R}^n)$ is complete under the norm

$$|||\varphi|||^2 = q(\varphi, \varphi) + C\|\varphi\|_{L^2}^2, \tag{2.26}$$

for some $C > 0$, and it is *semibounded*, i.e.

$$q(\varphi, \varphi) \ge -C\|\varphi\|_{L^2}^2, \tag{2.27}$$

for some $C > 0$. To prove this we estimate the form from below as follows

$$\begin{aligned} q(\varphi, \varphi) &= \|\nabla\varphi\|_{L^2}^2 + 2\Im(A \cdot \nabla\varphi, \varphi)_{L^2} + ((|A|^2 + B_+)\varphi, \varphi)_{L^2} - (B_-\varphi, \varphi)_{L^2} \\ &\ge \|\nabla\varphi\|_{L^2}^2 + 2\Im(A \cdot \nabla\varphi, \varphi)_{L^2} - (B_-\varphi, \varphi)_{L^2}. \end{aligned}$$

Proceeding as for the upper bound we obtain

$$q(\varphi, \varphi) \ge \|\nabla\varphi\|_{L^2}^2 - C\delta\|\nabla\varphi\|_{L^2}^2 \gtrsim \|\nabla\varphi\|_{L^2}^2 \tag{2.28}$$

for δ small enough. This proves the semiboundedness of the form and (2.25), which implies that the norm (2.26) is equivalent to the norm of H^1 and hence the form is closed. $\square$

We now investigate in some detail the properties of the resolvent operators

$$R(z) = (-\Delta + W - z)^{-1} \tag{2.29}$$

$$R_0(z) = (-\Delta - z)^{-1}, \qquad R_{b_2}(z) = (-\Delta + b_2(x) - z)^{-1}.$$

The following weight functions will appear in our resolvent estimates ($\epsilon > 0, \sigma > 1$):

$$\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}, \qquad w_\sigma(x) = |x|(1 + |\log|x||)^\sigma, \tag{2.30}$$

and

$$\tau_\epsilon(x) = \begin{cases} |x|^{\frac{1}{2}-\epsilon} + |x| & \text{if } n \geq 3, \\ |x|^{\frac{1}{2}-\epsilon} + |x|^{1+\epsilon} & \text{if } n = 2. \end{cases} \tag{2.31}$$

Notice that

$$|x| \leq \tau_\epsilon(x), \qquad w_\sigma^{\frac{1}{2}}(x) \leq C\tau_\epsilon(x),$$

and

$$\tau_\epsilon(x) \leq C\langle x\rangle, \text{ for } n \geq 3, \qquad \tau_\epsilon(x) \leq C\langle x\rangle^{1+\epsilon}, \text{ for } n = 2$$

for some constant $C = C(\epsilon, \sigma)$.

In order to estimate the resolvent R we shall use the formal identity

$$R = R_0(I + b_2 R_0)^{-1}(I + (a \cdot \nabla + b_1)R_{b_2})^{-1}. \tag{2.32}$$

Our first goal will be to prove that the operators $(I + b_2 R_0)^{-1}$ and $(I + (a \cdot \nabla + b_1)R_{b_2})^{-1}$ are well defined and uniformly bounded in suitable weighted L^2 spaces. In the following lemma, the assumption that 0 is not a resonance of $-\Delta + b(x)$ means that the only distribution solution f of the equation $-\Delta f + bf = 0$ belonging to $L^2(\langle x\rangle^{-2}dx)$ is $f \equiv 0$.

Lemma 2.2. *Let $b(x)$ be real valued and such that, for some $\epsilon, \delta > 0$ small enough (recall (2.31)),*

$$\|\tau_\epsilon^2 b_+\|_{L^\infty} < \infty, \qquad \|\tau_\epsilon^2 b_-\|_{L^\infty} < \delta. \tag{2.33}$$

Assume that 0 is not a resonance for $-\Delta + b(x)$. Then $I + bR_0(z)$ is invertible with a uniformly bounded inverse on $L^2(\tau_\epsilon^2 dx)$:

$$\|\tau_\epsilon(I + bR_0(z))^{-1} f\|_{L^2} \leq C\|\tau_\epsilon f\|_{L^2}. \tag{2.34}$$

Proof. We recall the following estimates for the free resolvent R_0: fix any $\sigma > 1$, then for all $z \in \mathbb{C}$

$$\|w_\sigma^{-\frac{1}{2}} R_0(z) f\|_{L^2} \leq \frac{C}{\sqrt{|z|}} \|w_\sigma^{\frac{1}{2}} f\|_{L^2}, \tag{2.35}$$

$$\|w_\sigma^{-\frac{1}{2}} \nabla R_0(z) f\|_{L^2} \leq C\|w_\sigma^{\frac{1}{2}} f\|_{L^2}, \tag{2.36}$$

$$\| |x|^{-1} R_0 f\|_{L^2} \leq C\| |x| f\|_{L^2}, \qquad n \geq 3 \tag{2.37}$$

$$\| |x|^{-1+\epsilon} |D|^\epsilon R_0 f\|_{L^2} \leq C\| |x|^{1-\epsilon} |D|^{-\epsilon} f\|_{L^2}, \qquad n = 2 \quad (0 < \epsilon < 1/2) \tag{2.38}$$

(see [**6**], [**31**] for (2.35), (2.36), and [**65**] for (2.37)-(2.38)). As usual, for $\lambda \in \mathbb{R}^+$ the resolvent $R_0(z)$ must be replaced with the limit operators $R_0(\lambda \pm i0)$. By the elementary inequalities $|x| \leq \tau_\epsilon(x)$, $w_\sigma^{\frac{1}{2}}(x) \leq C\tau_\epsilon(x)$, we can condense the estimates (2.35) and (2.37) in the following (weaker) one for $n \geq 3$:

$$\|\tau_\epsilon^{-1} R_0(z) f\|_{L^2} \leq \frac{C}{\sqrt{\langle z\rangle}} \|\tau_\epsilon f\|_{L^2}, \qquad \text{for all } z \in \mathbb{C}. \tag{2.39}$$

In dimension $n = 2$ we deduce by duality from (2.38) the following

$$\| |D|^\epsilon |x|^{-1+\epsilon} R_0 f\|_{L^2} \leq C\| |D|^{-\epsilon} |x|^{1-\epsilon} f\|_{L^2},$$

which implies, via Sobolev embedding and Hölder inequality,

$$\|\langle x\rangle^{-\sigma}|x|^{-1+\epsilon}R_0 f\|_{L^2} \le C\|\langle x\rangle^{\sigma}|x|^{1-\epsilon}f\|_{L^2}, \qquad \sigma > \epsilon$$

and hence (2.39) follows also for $n = 2$ (recall (2.31))

Now, using assumption (2.33), we have

$$\|\tau_\epsilon b R_0(z) f\|_{L^2} \le \|\tau_\epsilon^2 b\|_{L^\infty}\|\tau_\epsilon^{-1}R_0(z)f\|_{L^2} \le \frac{C}{\sqrt{\langle z\rangle}}\|\tau_\epsilon^2 b\|_{L^\infty}\|\tau_\epsilon f\|_{L^2}, \tag{2.40}$$

with C as in (2.39); hence, if z is sufficiently large, namely so large that

$$\langle z\rangle > C^2\|\tau_\epsilon^2 b\|^2_{L^\infty},$$

we can invert the operator $I + bR_0$ by a Neumann series in the weighted space $L^2(\tau_\epsilon^2 dx)$, with a uniform bound on the norm of the inverse.

In the low frequency case

$$\langle z\rangle \le C^2\|\tau_\epsilon^2 b\|^2_{L^\infty}, \tag{2.41}$$

the family of operators $(I + bR_0(z))$ is uniformly bounded in $L^2(\tau_\epsilon^2 dx)$ by (2.40). We also notice that bR_0 is a compact operator on $L^2(\tau_\epsilon^2 dx)$; indeed, R_0 is a compact operator from $L^2(\tau_\epsilon^2 dx)$ to $L^2(\tau_\epsilon^{-2}dx)$ (see (2.35)–(2.36)), while multiplication by b is bounded from $L^2(\tau_\epsilon^{-2}dx)$ to $L^2(\tau_\epsilon^2 dx)$. Thus by standard analytic Fredholm theory we can invert $I + bR_0(z)$ uniformly in z, provided $I + bR_0(z)$ is injective on $L^2(\tau_\epsilon^2 dx)$ for each fixed z. This is obvious for z outside $\overline{\mathbb{R}^+}$, since by our assumptions the operator $-\Delta + b$ is nonnegative and selfadjoint, and is true by assumption for $z = 0$, hence we need only check the case $z = \lambda > 0$.

Thus let $\lambda \ge 0$ and $f \in L^2(\tau_\epsilon^2 dx)$ such that $f + b(x)R_0(\lambda + i0)f = 0$ (the $-i0$ case is identical). We notice that estimate (2.36) implies that $R_0(z)f \in H^1_{\text{loc}}$ and hence in particular $R_0(z)f$ is in $L^{2n/(n-2)}$ locally. Since $|b| \lesssim \tau_\epsilon^{-2}$ which is locally in L^n, we conclude that $f = -bR_0(\lambda)f$ is locally in L^2. Recalling that $f \in L^2(\tau_\epsilon^2 dx)$ this implies $f \in L^2(\langle x\rangle^2 dx)$. Thus we are in the framework of the standard Agmon theory and we deduce that λ is an eigenvalue of $-\Delta + b(x)$; but this is excluded under our assumptions on b, for instance by the results of [**58**] (Theorem 2.1).

In conclusion, we can invert $(I + bR_0(z))$ in $L^2(\tau_\epsilon^2 dx)$, for $z \ne 0$, with an uniform bound for the inverse $(I + bR_0)^{-1}$, and this completes the proof. □

The preceding lemma allows us to construct the resolvent operator

$$R_b(z) = R_0(z)(I + bR_0(z))^{-1}, \tag{2.42}$$

which, in view of (2.34) and (2.39), is a bounded operator from $L^2(\tau_\epsilon^2 dx)$ to $L^2(\tau_\epsilon^{-2}dx)$ for all $z \in \mathbb{C}$.

We have next:

LEMMA 2.3. *Consider the operator $-\Delta + a(x)\cdot\nabla + b_1(x) + b_2(x)$ under the following assumptions: the operator is selfadjoint, b_2 is real valued and nonnegative, and for some $\delta, \epsilon > 0$ small enough, $\sigma > 1$,*

$$\|\tau_\epsilon w_\sigma^{\frac12} a\|_{L^\infty} + \|\tau_\epsilon^2 b_1\|_{L^\infty} < \delta, \qquad \|\tau_\epsilon^2 b_2\|_{L^\infty} < \infty. \tag{2.43}$$

Moreover assume that 0 is not a resonance for $-\Delta + b_2(x)$. *Then* $I + (a\cdot\nabla + b_1)R_{b_2}$ *is invertible with a bounded inverse on* $L^2(\tau_\epsilon^2 dx)$:

$$\|\tau_\epsilon (I + (a\cdot\nabla + b_1)R_{b_2})^{-1} f\|_{L^2} \leq C\|\tau_\epsilon f\|_{L^2}. \tag{2.44}$$

PROOF. Using assumptions (2.43), Hölder inequality and estimate (2.36), we can write

$$\begin{aligned}\|\tau_\epsilon a\cdot\nabla R_{b_2} f\|_{L^2} &\leq \|\tau_\epsilon a\cdot\nabla R_0(I+b_2R_0)^{-1}f\|_{L^2}\\ &\leq \|\tau_\epsilon w_\sigma^{\frac12} a\|_{L^\infty}\|w_\sigma^{-\frac12}\nabla R_0(I+b_2R_0)^{-1}f\|_{L^2}\\ &\lesssim \delta\cdot\|w_\sigma^{\frac12}(I+b_2R_0)^{-1}f\|_{L^2}\\ &\lesssim \delta\cdot\|\tau_\epsilon(I+b_2R_0)^{-1}f\|_{L^2}\end{aligned}$$

and Lemma 2.2 gives finally

$$\|\tau_\epsilon a\cdot\nabla R_{b_2} f\|_{L^2} \lesssim \delta\cdot\|\tau_\epsilon f\|_{L^2}.$$

On the other hand, by (2.43) and estimate (2.39)

$$\begin{aligned}\|\tau_\epsilon b_1 R_{b_2} f\|_{L^2} &\leq \|\tau_\epsilon^2 b_1\|_{L^\infty}\|\tau_\epsilon^{-1}R_0(I+b_2R_0)^{-1}f\|_{L^2}\\ &\lesssim \delta\cdot\|\tau_\epsilon(I+b_2R_0)^{-1}f\|_{L^2}\end{aligned}$$

and again by Lemma 2.2 we have

$$\|\tau_\epsilon b_1 R_{b_2} f\|_{L^2} \lesssim \delta\cdot\|\tau_\epsilon f\|_{L^2}.$$

Thus, if δ is sufficiently small, we can invert $I + (a\cdot\nabla + b_1)R_{b_2}$ via a Neumann series, and we obtain (2.44). ☐

We collect and complete the above estimates in the following

PROPOSITION 2.2. *Consider the operator* $-\Delta + W(x, D) \equiv -\Delta + a(x)\cdot\nabla + b_1(x) + b_2(x)$ *under the assumptions: the operator is selfadjoint,* b_2 *is real valued and nonnegative, and for some* $\delta, \epsilon > 0$ *small enough,* $\sigma > 1$,

$$\|\tau_\epsilon w_\sigma^{\frac12} a\|_{L^\infty} + \|\tau_\epsilon^2 b_1\|_{L^\infty} < \delta, \qquad \|\tau_\epsilon^2 b_2\|_{L^\infty} < \infty. \tag{2.45}$$

Moreover assume that 0 is not a resonance for $-\Delta + b_2(x)$. *Then the resolvent operator* $R(z) = (-\Delta + W - z)^{-1}$ *satisfies the following estimates for all* $z \in \mathbb{C}$:

$$\|\tau_\epsilon^{-1} R(z) f\|_{L^2} \leq \frac{C}{\sqrt{\langle z\rangle}}\|\tau_\epsilon f\|_{L^2}, \tag{2.46}$$

$$\|\tau_\epsilon^{-1}\nabla R(z) f\|_{L^2} \leq C\|\tau_\epsilon f\|_{L^2}. \tag{2.47}$$

and

$$\|\langle x\rangle^{-1} R(z) f\|_{H^1} \leq C\|\langle x\rangle f\|_{L^2}, \qquad n \geq 3; \tag{2.48}$$

replace the weights $\langle x\rangle^{-1}, \langle x\rangle$ *by* $\langle x\rangle^{-1-\epsilon}, \langle x\rangle^{1+\epsilon}$ *respectively in dimension 2. As a consequence, the Schrödinger flow* $e^{it(-\Delta+W)}f$ *has the smoothing property*

$$\|\tau_\epsilon^{-1} e^{it(-\Delta+W)} f\|_{L^2L^2} + \|\tau_\epsilon^{-1}|D|^{1/2} e^{it(-\Delta+W)} f\|_{L^2L^2} \leq C\|f\|_{L^2}. \tag{2.49}$$

Remark 2.1. For the following applications it will be convenient to rewrite the (second) smoothing estimate above in the equivalent form

$$\|\tau_\epsilon^{-1}\nabla|D|^{-1/2}e^{it(-\Delta+W)}f\|_{L^2L^2}\leq C\|f\|_{L^2}. \tag{2.50}$$

This follows immediately from the fact that $\partial_j|D|^{-1/2}=iR_j|D|^{1/2}$, where $R_j=i^{-1}\partial_j|D|^{-1}$ is the j-th Riesz operator, and on the other hand τ_ϵ^{-1} is an A_2 weight, as proved in Lemma 2.4 below.

Proof. Estimates (2.46) and (2.47) are immediate consequences of (2.32), (2.36) and of Lemmas 2.2, 2.3. Moreover, (2.46) implies in particular

$$\|\tau_\epsilon^{-1}R(z)f\|_{L^2}\leq C\|\tau_\epsilon f\|_{L^2},$$

and the Kato smoothing theorem with the choices $A=\tau_\epsilon^{-1}$, $X=Y=L^2$ gives the first estimate in (2.49).

To prove (2.48), write

$$\begin{aligned}\|\langle x\rangle^{-1}Rf\|_{H^1}&\lesssim\|\langle x\rangle^{-1}Rf\|_{L^2}+\|\langle x\rangle^{-2}Rf\|_{L^2}+\|\langle x\rangle^{-1}\nabla Rf\|_{L^2}\\&\lesssim\|\langle x\rangle^{-1}Rf\|_{L^2}+\|\langle x\rangle^{-1}\nabla Rf\|_{L^2}\end{aligned}$$

The first term at the right hand side can be estimated by (2.46)

$$\|\langle x\rangle^{-1}Rf\|_{L^2}\leq C\|\tau_\epsilon^{-1}Rf\|_{L^2}\leq C\|\tau_\epsilon f\|_{L^2}\leq C\|\langle x\rangle f\|_{L^2}, \tag{2.51}$$

while the third term is bounded using (2.47):

$$\|\langle x\rangle^{-1}\nabla Rf\|_{L^2}\leq\|\tau_\epsilon^{-1}\nabla Rf\|_{L^2}\leq C\|\tau_\epsilon f\|_{L^2}\leq C\|\langle x\rangle f\|_{L^2} \tag{2.52}$$

and this proves (2.48).

Now write (2.48) in the equivalent forms

$$\|\langle D\rangle\langle x\rangle^{-1}R(z)\langle x\rangle^{-1}f\|_{L^2}\leq C\|f\|_{L^2}$$

and, by duality,

$$\|\langle x\rangle^{-1}R(z)\langle x\rangle^{-1}\langle D\rangle f\|_{L^2}\leq C\|f\|_{L^2}.$$

The last two estimates state that the operator $\langle x\rangle^{-1}R(z)\langle x\rangle^{-1}$ is bounded, uniformly in $z\in\mathbb{C}$, from L^2 to H^1 and from H^{-1} to L^2. By complex interpolation this implies that it is also bounded from $H^{-1/2}$ to $H^{1/2}$, i.e.,

$$\|\langle D\rangle^{1/2}\langle x\rangle^{-1}R(z)\langle x\rangle^{-1}\langle D\rangle^{1/2}f\|_{L^2}\leq C\|f\|_{L^2}$$

Then by Kato smoothing we obtain also the second estimate in (2.49).

The proof for the case $n=2$ is completely analogous. □

2.2. The wave and Klein-Gordon generators. We consider now the operator $\sqrt{-\Delta+W}$, where as usual

$$W=W(x,D)=a\cdot\nabla+b,\qquad b=b_1+b_2$$

which generates the flow $e^{it\sqrt{-\Delta+W}}$ of the perturbed wave equation. The free operator $|D|:=\sqrt{-\Delta}$ is self-adjoint and nonnegative on L^2, and can be handled as follows. If we denote its resolvent by $R_{|D|}(z)=(|D|-z)^{-1}$, we have

$$R_{|D|}(z)=(|D|+z)R_0(z^2). \tag{2.53}$$

This simple identity allows us to estimate $R_{|D|}$ using some standard techniques from harmonic analysis. We need a lemma:

LEMMA 2.4. *Let $n \geq 2$. For any $\sigma > 1$, the weight $w_\sigma = |x|(1 + |\log|x||)^\sigma$ is an A_2 weight, i.e., there exist a constant A such that, for any ball $B = B(x_0, R)$,*

$$A(x_0, R) \equiv \left[\frac{1}{|B|}\int_B w_\sigma dx\right] \cdot \left[\frac{1}{|B|}\int_B w_\sigma^{-1} dx\right] \leq A < \infty. \tag{2.54}$$

Obviously, we have also $w_\sigma^{-1} \in A_2$. The same property holds for the weights τ_ϵ, τ_ϵ^{-1} defined in (2.31).

PROOF. The bound for the function $A(x_0, R)$ is trivial if $R \leq |x_0|/2$, indeed it is sufficient to write

$$A(x_0, R) \leq C \max_B w_\sigma \cdot \max_B w_\sigma^{-1} \leq C'$$

since the ball B is at a distance greater than $|x_0|/2$ from the origin.

If, on the other hand, $R \geq |x_0|/2$, it is easy to check that $A(x_0, R)$ is bounded by a constant (depending only on the space dimension n) times $A(0, 3R)$. Thus we are reduced to the case of balls $B(0, R)$ centered in 0.

For *small* $R \leq 10$ the function $A(0, R)$ is bounded. Indeed, Hôpital's theorem gives

$$\lim_{\epsilon \downarrow 0} \int_0^\epsilon \frac{r^{n-2} dr}{(1 + |\log r|)^\sigma} \cdot \frac{(1 + |\log \epsilon|)^\sigma}{\epsilon^{n-1}} = \frac{1}{n-1}$$

which implies for small R

$$\int_0^R \frac{r^{n-2} dr}{(1 + |\log r|)^\sigma} \sim \frac{R^{n-1}}{(1 + |\log R|)^\sigma} \tag{2.55}$$

and similarly

$$\int_0^R r^n (1 + |\log r|)^\sigma dr \sim R^{n+1} (1 + |\log R|)^\sigma$$

whence we get $A(0, R) \leq C$.

For *large* $R > 10$ we rescale and obtain

$$A(0, R) = \int_0^1 \frac{\tau^{n-2} d\tau}{(1 + |\log R + \log \tau|)^\sigma} \cdot \int_0^1 \tau^n (1 + |\log R + \log \tau|)^\sigma d\tau$$

The second integral is clearly bounded by $C(\log R)^\sigma$. The first integral can be split into

$$\int_0^{1/\sqrt{R}} \frac{\tau^{n-2} d\tau}{(1 + |\log R + \log \tau|)^\sigma} \leq \int_0^{1/\sqrt{R}} \frac{\tau^{n-2} d\tau}{(1 + |\log \tau|)^\sigma} \sim \frac{R^{-\frac{n-1}{2}}}{(1 + \frac{1}{2}\log R)^\sigma} \leq R^{-\frac{1}{2}}$$

where we used again (2.55), and

$$\int_{1/\sqrt{R}}^1 \frac{\tau^{n-2} d\tau}{(1 + |\log R + \log \tau|)^\sigma} \leq \int_{1/\sqrt{R}}^1 \frac{\tau^{n-2} d\tau}{(1 + \frac{1}{2}\log R)^\sigma} \leq C(\log R)^{-\sigma}.$$

Putting everything together, we obtain the required bound also for large R, and this concludes the proof of the Lemma.

The proof for τ_ϵ is much simpler. We reduce as above to the case of spheres $B(0,R)$ centered in the origin. For $R \le 1$ we can use the equivalence $\tau_\epsilon \simeq |x|^{1/2-\epsilon}$ and the bound follows from the well-known fact that $|x|^{1/2-\epsilon}$ is an A_2 weight. For $R>1$ we use the estimate

$$A(0,R) \lesssim \frac{1}{|B|}\int_B (1+|x|)dx \cdot \frac{1}{|B|}\int_B \frac{dx}{|x|}$$

(replace $|x|$ with $|x|^{1+\epsilon}$ for $n=2$) whence the bound follows easily. □

Knowing that $w_\sigma^{-1} \in A_2$, we see that the Riesz operators

$$R_j = i^{-1}\frac{\partial_j}{|D|}$$

are bounded on the space $L^2(w_\sigma^{-1}dx)$ by standard results (see e.g. the Corollary to Theorem 2, §V.4.2 of [**99**]). Writing $|D| = i^{-1}\sum R_j\partial_j$, we have

$$\|w_\sigma^{-1/2}|D|g\|_{L^2} \le \sum_j \|w_\sigma^{-1/2}R_j\partial_j g\|_{L^2} \le C\|w_\sigma^{-1/2}\nabla g\|_{L^2}.$$

Thus estimate (2.53) implies

$$\|w_\sigma^{-\frac{1}{2}}R_{|D|}(z)f\|_{L^2} \le C\|w_\sigma^{-\frac{1}{2}}\nabla R_0(z^2)f\|_{L^2} + C|z|\cdot\|w_\sigma^{-\frac{1}{2}}R_0(z^2)f\|_{L^2}. \tag{2.56}$$

Then, inequalities (2.35) and (2.36) yield immediately the following estimate for the free resolvent: for any fixed $\sigma > 1$,

$$\|w_\sigma^{-\frac{1}{2}}R_{|D|}(z)f\|_{L^2} \le C\|w_\sigma^{\frac{1}{2}}f\|_{L^2}, \tag{2.57}$$

uniformly in $z \in \mathbb{C}$.

We are ready to prove a corresponding estimate for the resolvent of the perturbed operator

$$R(z) = (\sqrt{-\Delta + W} - z)^{-1}, \qquad W = a(x)\cdot\nabla + b(x),$$

following the same approach as in the preceding cases.

LEMMA 2.5. *Consider the operator $-\Delta + W(x,D) \equiv -\Delta + a(x)\cdot\nabla + b_1(x) + b_2(x)$ under the assumptions: the operator is selfadjoint, b_2 is real valued and nonnegative, and for some $\delta, \epsilon > 0$ small enough, $\sigma > 1$,*

$$\|\tau_\epsilon w_\sigma^{\frac{1}{2}}a\|_{L^\infty} + \|\tau_\epsilon^2 b_1\|_{L^\infty} < \delta, \qquad \|\tau_\epsilon^2 b_2\|_{L^\infty} < \infty. \tag{2.58}$$

Moreover assume that 0 is not a resonance for $-\Delta + b_2(x)$. Then the resolvent operator $R(z) = (\sqrt{-\Delta+W} - z)^{-1}$ satisfies

$$\|\tau_\epsilon^{-1}R(z)f\|_{L^2} \le C\|\tau_\epsilon f\|_{L^2}, \tag{2.59}$$

for all $z \in \mathbb{C}$. As a consequence, the perturbed wave flow $e^{it\sqrt{\Delta+W}}$ satisfies the smoothing estimate

$$\|\tau_\epsilon^{-1}e^{it\sqrt{-\Delta+W}}f\|_{L^2L^2} \le C\|f\|_{L^2}. \tag{2.60}$$

PROOF. We write for brevity

$$|D_W| = \sqrt{-\Delta + W(x,D)}.$$

By the (Phragmén-Lindelöf) maximum principle, it is sufficient to prove estimate (2.59) for real $z = \lambda$. We notice that by the same arguments used in the proof of Lemma 2.1, we have

$$\||D_W|g\|_{L^2} \simeq \|g\|_{\dot{H}^1};$$

thus for $\lambda \leq 0$ we can write

$$\|(|D_W| - \lambda)g\|_{L^2}^2 = \||D_W|g\|_{L^2}^2 + \lambda^2\|g\|_{L^2}^2 - 2\lambda(|D_W|g, g)_{L^2} \gtrsim \|g\|_{\dot{H}^1}$$

by the nonnegativity of $|D_W|$. This implies for all $\lambda \leq 0$

$$\|R(\lambda)g\|_{\dot{H}^1} \lesssim \|g\|_{L^2},$$

whence by duality we have also

$$\|R(\lambda)g\|_{L^2} \lesssim \|g\|_{\dot{H}^{-1}},$$

and interpolating we obtain

$$\|R(\lambda)g\|_{\dot{H}^{1/2}} \lesssim \|g\|_{\dot{H}^{-1/2}}, \qquad \lambda \leq 0.$$

Now, using the Hardy's inequalities

$$\||x|^{-1/2}f\|_{L^2} \lesssim \|f\|_{\dot{H}^{1/2}} \qquad \text{or equivalently} \qquad \|f\|_{\dot{H}^{-1/2}} \lesssim \||x|^{1/2}f\|_{L^2}$$

we obtain the estimate

$$\||x|^{-1/2}R(\lambda)g\|_{L^2} \lesssim \||x|^{1/2}g\|_{L^2}, \qquad \lambda \leq 0 \tag{2.61}$$

which implies (2.59) for $z = -\lambda \leq 0$ (and is actually stronger).

Consider now $R(\lambda)$, $\lambda \geq 0$; we use the identity

$$R(\lambda) = (|D_W| - \lambda)^{-1} = 2\lambda R_W(\lambda^2) + (|D_W| + \lambda)^{-1}$$

where $R_W(\lambda) = (-\Delta + W - \lambda)^{-1}$. The second term at the right hand side has already been estimated, while the first one can be estimated using (2.46), and this concludes the proof of (2.59). The last inequality (2.60) is an application of Kato's theorem as usual. $\square$

We conclude this section with a study of the operator $\sqrt{-\Delta + 1 + W}$ associated with the perturbed Klein-Gordon flow $e^{it\sqrt{-\Delta+1+W}}$. In the free case $W = 0$ the operator reduces to $\langle D\rangle = (1 - \Delta)^{1/2}$ and its resolvent

$$R_{\langle D\rangle}(z) = (\langle D\rangle - z)^{-1}$$

can be handled in a similar way as $R_{|D|}$.

We start from estimates (2.36) and (2.39) which imply

$$\langle z\rangle^{1/2}\|\tau_\epsilon^{-1}R_0(z)f\|_{L^2} + \|w_\sigma^{-1/2}\nabla R_0(z)f\|_{L^2} \lesssim \|\tau_\epsilon f\|_{L^2}.$$

As above, using the fact that w_σ^{-1} is an A_2 weight, we can replace ∇ with $|D|$ in the left hand side and hence (recalling that $w_\sigma^{\frac{1}{2}} \lesssim \tau_\epsilon$) we arrive at

$$\langle z\rangle^{1/2}\|\tau_\epsilon^{-1}R_0(z)f\|_{L^2} + \|\tau_\epsilon^{-1}\langle D\rangle R_0(z)f\|_{L^2} \lesssim \|\tau_\epsilon f\|_{L^2}. \tag{2.62}$$

Then using the identity

$$R_{\langle D\rangle}(z) = (\langle D\rangle + z) \cdot R_0(1 - z^2)$$

we obtain from (2.62) the estimate

$$\|\tau_\epsilon^{-1}R_{\langle D\rangle}(z)f\|_{L^2} \lesssim \|\tau_\epsilon f\|_{L^2}. \tag{2.63}$$

For the perturbed operator we have:

LEMMA 2.6. *Consider the operator $-\Delta+W(x,D)\equiv-\Delta+a(x)\cdot\nabla+b_1(x)+b_2(x)$ under the assumptions: the operator is selfadjoint, b_2 is real valued and nonnegative, and for some $\delta,\epsilon>0$ small enough, $\sigma>1$,*

$$\|\tau_\epsilon w_\sigma^{\frac{1}{2}}a\|_{L^\infty}+\|\tau_\epsilon^2 b_1\|_{L^\infty}<\delta,\qquad \|\tau_\epsilon^2 b_2\|_{L^\infty}<\infty. \tag{2.64}$$

Moreover assume that 0 is not a resonance for $-\Delta+b_2(x)$. Then the resolvent operator $R(z)=(\sqrt{1-\Delta+W}-z)^{-1}$ satisfies

$$\|\tau_\epsilon^{-1}R(z)f\|_{L^2}\leq C\|\tau_\epsilon f\|_{L^2}. \tag{2.65}$$

As a consequence, the perturbed Klein-Gordon flow $e^{it\sqrt{\Delta+1+W}}$ satisfies the smoothing estimate

$$\|\tau_\epsilon^{-1}e^{it\sqrt{-\Delta+1+W}}f\|_{L^2L^2}\leq C\|f\|_{L^2}. \tag{2.66}$$

PROOF. Writing

$$|D_W|=\sqrt{-\Delta+W(x,D)},\qquad \langle D_W\rangle=\sqrt{1-\Delta+W(x,D)}$$

we notice that

$$\|\langle D_W\rangle f\|_{L^2}\simeq\|f\|_{L^2}+\||D_W|f\|_{L^2}\simeq\|f\|_{H^1}$$

by the same arguments used in the proof of Lemma 2.1 and the identity

$$\|\langle D_W\rangle f\|_{L^2}^2=((1-\Delta+W)f,f).$$

Proceeding as in the proof of Lemma 2.5, we arrive at

$$\|R(\lambda)g\|_{H^{1/2}}\lesssim\|g\|_{H^{-1/2}},\qquad \lambda\leq 0$$

for the resolvent $R=(\langle D_W\rangle-z)^{-1}$, and by Hardy inequality as before we obtain half of(2.66).

For positive λ we write

$$R(\lambda)=(\langle D_W\rangle-\lambda)^{-1}=2\lambda R_W(\lambda^2-1)+(\langle D_W\rangle+\lambda)^{-1}$$

where $R_W(z)=(-\Delta+W-z)^{-1}$, and by (2.46) and the first part of the proof we obtain (2.65). Kato's theorem gives (2.66) as usual. ☐

2.3. The magnetic Dirac operators. We now consider the resolvent of a perturbed Dirac operator $\mathcal{D}+V(x)$. The proofs here will be short since we shall rely on a few results proved in [**31**]; in particular, we recall that if $V=V^*$ has a sufficiently small $L^{3,\infty}$ norm, hence under the assumptions of Theorem 2.4, the operator $\mathcal{D}+V$ is self-adjoint on $L^2(\mathbb{R}^3,\mathbb{C}^4)$, with form domain $H^1(\mathbb{R}^3,\mathbb{C}^4)$ and spectrum $\mathbb{R}$. The same holds for the operator with nonzero mass $\mathcal{D}+\beta+V$, but the spectrum is $\mathbb{R}\setminus]-1,1[$.

Let us consider the massless case first. We shall use the notations

$$R_{\mathcal{D}}(z)=(-\mathcal{D}-zI_4)^{-1},\qquad R(z)=(-\mathcal{D}+V-zI_4)^{-1} \tag{2.67}$$

where I_4 denotes the identity 4×4-matrix. The following result is contained in Proposition 3.6 of [**31**], apart from the smoothing estimate which is a standard consequence of Kato's theorem as above:

PROPOSITION 2.3. *Assume that the 4×4 matrix $V(x) = V^*(x)$ satisfies*

$$\|w_\sigma V\|_{L^\infty} < \delta, \tag{2.68}$$

for some δ sufficiently small and some $\sigma > 0$. Then $\mathcal{D} + V$ satisfies the limiting absorption principle, i.e., the limit operators $R(\lambda \pm i0)$ exist in the topology of bounded operators from $L^2(w_\sigma^{1/2}dx)$ to $H^1(w_\sigma^{1/2}dx)$. Moreover the resolvent operator $R = (-\mathcal{D} + V - zI_4)^{-1}$ satisfies the estimate

$$\|w_\sigma^{-1/2} R(z) f\|_{L^2} \leq C\|w_\sigma^{1/2} f\|_{L^2}, \qquad z \in \mathbb{C}. \tag{2.69}$$

As a consequence, the Dirac flow satisfies the smoothing estimate

$$\|w_\sigma^{-1/2} e^{it(\mathcal{D}+V)} f\|_{L^2 L^2} \leq C\|f\|_{L^2}. \tag{2.70}$$

We consider now the operators with mass $\mathcal{D} + \beta$ and $\mathcal{D} + \beta + V$. We shall use the notations

$$R_\beta(z) = (\mathcal{D} + \beta - zI_4)^{-1}, \qquad R(z) = (\mathcal{D} + \beta + V - zI_4)^{-1}.$$

From the identities

$$\mathcal{D}^2 = -\Delta I_4, \qquad (\mathcal{D} + \beta)^2 = (1 - \Delta) I_4,$$

we obtain the following representations in terms of $R_0(z) = (-\Delta - z)^{-1}$

$$R_{\mathcal{D}}(z) = R_0(z^2)(\mathcal{D} + zI_4), \qquad R_\beta(z) = R_0(z^2 - 1)(\mathcal{D} + \beta + zI_4);$$

and hence we can write

$$R_\beta(z) = R_0(z^2 - 1)\mathcal{D} + R_0(z^2 - 1)(\beta + zI_4). \tag{2.71}$$

Then a straightforward application of estimate (2.62) gives

$$\|\tau_\epsilon^{-1} R_\beta(z) f\|_{L^2} \leq C\|\tau_\epsilon f\|_{L^2}. \tag{2.72}$$

uniformly in $z \in \mathbb{C}$.

In the perturbed case we can prove

PROPOSITION 2.4. *Assume that the 4×4 matrix $V(x) = V^*(x)$ satisfies*

$$\|\tau_\epsilon^2 V\|_{L^\infty} < \delta, \tag{2.73}$$

for some δ sufficiently small and $\epsilon > 0$. Then the perturbed resolvent operator $R(z) = (\mathcal{D} + \beta + V - zI_4)^{-1}$ satisfies

$$\|\tau_\epsilon^{-1} R(z) f\|_{L^2} \leq C\|\tau_\epsilon f\|_{L^2}. \tag{2.74}$$

As a consequence, the flow $e^{it(\mathcal{D}+\beta+V)}$ satisfies the smoothing estimate

$$\|\tau_\epsilon^{-1} e^{it(\mathcal{D}+\beta+V)} f\|_{L^2 L^2} \leq C\|f\|_{L^2}. \tag{2.75}$$

PROOF. The operator $VR_\beta(z)$ is bounded on $L^2(\tau_\epsilon^2 dx)$ with norm bounded by $C\delta$ since

$$\|\tau_\epsilon V R_\beta(z)\|_{L^2} \leq \|\tau_\epsilon^2 V\|_{L^\infty} \|\tau_\epsilon^{-1} R_\beta(z)\|_{L^2} \leq C\delta \|\tau_\epsilon f\|_{L^2}$$

by (2.73) and (2.72). Thus for δ small a Neumann expansion shows that $(I + VR_\beta(z))^{-1}$ is well defined and uniformly bounded on $L^2(\tau_\epsilon^2 dx)$. Hence the usual representation

$$R(z) = R_\beta(z)(I + VR_\beta(z))^{-1}$$

together with (2.72) gives (2.74), and (2.75) follows. □

3. Proof of the Strichartz Estimates

The method we shall follow is inspired by [**90**], [**17**] and consists in mixing Strichartz and smoothing estimates for the free operator with smoothing estimates for the perturbed operator. The main tool will be the well-known Christ-Kiselev lemma [**22**], which can be stated as follows: given two Banach spaces X, Y and a bounded integral operator $Tf = \int_{\mathbb{R}} K(t,s)f(s)ds$ from $L^p(\mathbb{R}, X)$ to $L^{\tilde{p}}(\mathbb{R}, Y)$, then its truncated version $Sf = \int_0^t K(t,s)f(s)ds$ is also bounded on the same spaces, provided $p < \tilde{p}$ (the Hilbert transform being a trivial counterexample for $p = \tilde{p}$). Thus to prove an estimate of the form

$$\left\| \int_0^t e^{i(t-s)A} F(s)ds \right\|_{L_t^p L_x^q} \lesssim \|F\|_{L_t^{\tilde{p}} L_x^{\tilde{q}}}$$

it is sufficient to prove the untruncated estimate

$$\left\| \int_{\mathbb{R}} e^{i(t-s)A} F(s)ds \right\|_{L_t^p L_x^q} \lesssim \|F\|_{L_t^{\tilde{p}} L_x^{\tilde{q}}}$$

but only if $p < \tilde{p}$, which in particular excludes endpoint-endpoint estimates where $p = \tilde{p} = 2$.

3.1. Schrödinger equation: proof of Theorem 2.1. Notice that $u(t,x) = e^{it(-\Delta+W)}f$ satisfies the equation $iu_t - \Delta u = -Wu$, hence we can write

$$e^{it(\Delta - W)}f = e^{it\Delta}f - \int_0^t e^{i(t-s)\Delta} W(x,D)u\,ds = I - II - III$$

with

$$I = e^{it\Delta}f, \qquad II = \int_0^t e^{i(t-s)\Delta} b(x)u\,ds, \qquad III = \int_0^t e^{i(t-s)\Delta} a(x)\cdot\nabla u\,ds.$$

The first term I can be estimated directly with standard Strichartz estimates:

$$\|e^{it\Delta}f\|_{L_t^p L_x^q} \le C\|f\|_{L^2} \tag{2.76}$$

for any admissible couple (p,q). In order to estimate the second term we resort to the Christ-Kiselev lemma and we are reduced to estimate the untruncated integral

$$II_1 = e^{it\Delta}\int e^{-is\Delta} b(x)u\,ds.$$

To this end we apply first the Strichartz estimates for the free group, then the dual of the smoothing estimate from Proposition 2.2 in the special case $W = 0$, i.e.,

$$\left\| \int e^{-is\Delta} F(s)ds \right\|_{L^2} \lesssim \|\tau_\epsilon F\|_{L^2 L^2}$$

obtaining

$$\|II_1\|_{L^p L^q} \lesssim \left\| \int e^{-is\Delta} bu\,ds \right\|_{L^2} \lesssim \|\tau_\epsilon bu\|_{L^2L^2} \le \|\tau_\epsilon^2 b\|_{L^\infty}\|\tau_\epsilon u\|_{L^2L^2}.$$

Then by assumption (2.45) and again the smoothing estimate (2.49) we conclude

$$\|II\|_{L^pL^q} \lesssim \|f\|_{L^2} \tag{2.77}$$

for any non-endpoint admissible couple (p,q).

The last term III is more delicate. We reduce it as above to the untruncated form

$$III_1 = e^{it\Delta}\int e^{-is\Delta} a\cdot\nabla u\,ds$$

and we apply to it the free Strichartz estimate and then the following dual smoothing estimate:

$$\left\| \int e^{-is\Delta} F(s)ds \right\|_{L^2} \lesssim \||D|^{-1/2}\chi F\|_{L^2L^2}, \tag{2.78}$$

valid for any function $\chi(x) \gtrsim w_\sigma(x)^{1/2}$. Estimate (2.78) is proved as follows: from (2.36) we deduce, using the fact that w_σ is an A_2 weight, the equivalent property

$$\|w_\sigma^{-1/2}|D|^{1/2}R_0(z)f\|_{L^2} \leq C\|w_\sigma^{1/2}|D|^{-1/2}f\|_{L^2}$$

which implies, via Kato smoothing,

$$\|w_\sigma^{-1/2}|D|^{1/2}e^{it\Delta}f\|_{L^2L^2} \leq \|f\|_{L^2}.$$

Since $\chi \gtrsim w_\sigma^{1/2}$ this gives also

$$\|\chi^{-1}|D|^{1/2}e^{it\Delta}f\|_{L^2L^2} \leq \|f\|_{L^2}$$

and by duality we get (2.78). Thus we arrive at

$$\|III_1\|_{L^pL^q} \lesssim \||D|^{-1/2}\chi a(x)\cdot\nabla u\|_{L^2L^2} \tag{2.79}$$

Now assume we can prove the inequality

$$\||D|^{-1/2}\chi a(x)\cdot\nabla g\|_{L^2} \lesssim \|\tau_\epsilon^{-1}\nabla|D|^{-1/2}g\|_{L^2}; \tag{2.80}$$

then from (2.79) and the smoothing estimate (2.50) we finally obtain

$$\|III_1\|_{L^pL^q} \lesssim \|\tau_\epsilon^{-1}\nabla|D|^{-1/2}u\|_{L^2} \lesssim \|f\|_{L^2} \tag{2.81}$$

which, together with (2.76) and (2.77), concludes the proof of the Theorem.

It remains to check inequality (2.80). We rewrite it in the equivalent form

$$\||D|^{-1/2}\chi a(x)|D|^{1/2}\tau_\epsilon h\|_{L^2} \lesssim \|h\|_{L^2},$$

i.e., we need to prove that the operator

$$T = |D|^{-1/2}\chi a(x)|D|^{1/2}\tau_\epsilon \tag{2.82}$$

is bounded on L^2. We shall use the following lemma, where we shall make use of several properties of Lorentz spaces $L^{p,q}$ (see [**80**]).

LEMMA 2.7. *Let $\alpha(x),\beta(x)$ be measurable functions on $\mathbb{R}^n$ such that for some $0<\delta<1/2$, some $\rho\in[0,n/2-\delta[$, and a radial function $\gamma(|x|)$, with $\gamma(s)$ decreasing, we have*

(i) $|\alpha(x)-\alpha(y)| \lesssim |x-y|^{1/2+\delta}(\gamma(|x|)+\gamma(|y|))$ and $\gamma\in L^{\frac{2n}{1+2\rho+2\delta},\infty}$

(ii) $\alpha\beta\in L^\infty$, $|x|^{-\rho}\beta(x)\in L^\infty$ and $|x|^\rho\gamma(|x|)\in L^{\frac{2n}{1+2\delta},\infty}$

Then the operator $T=|D|^{-1/2}\alpha(x)|D|^{1/2}\beta(x)$ is bounded on L^2.

The same result holds in the range $\rho\in[0,n/2+\delta[$ if we replace (i) with

(i') $|\alpha(x)-\alpha(y)| \leq \langle x-y\rangle^{-2\delta}|x-y|^{1/2+\delta}(\gamma(|x|)+\gamma(|y|))$ and $\gamma\in L^{\frac{2n}{1+2\rho-2\delta},\infty}$.

PROOF. Since $\alpha\beta$ is bounded, we can equivalently prove that the modified operator

$$\widetilde{T} = T - \alpha\beta = |D|^{-1/2}\cdot[\alpha,|D|^{1/2}]\cdot\beta$$

is bounded on L^2. Moreover, by the Sobolev embedding in Lorentz spaces (proved e.g. by real interpolation)

$$\||D|^{-1/2}g\|_{L^2} \lesssim \|g\|_{L^{\frac{2n}{n+1},2}}$$

it is sufficient to prove that the following reduced operator S satisfies

$$S = [\alpha, |D|^{1/2}] \cdot \beta : L^{\frac{2n}{n+1},2} \to L^2.$$

Now we observe that the commutator $[\alpha, |D|^{1/2}]$ admits an explicit representation of the form

$$[\alpha, |D|^{1/2}]f = c(n) \int_{\mathbb{R}^n} \frac{\alpha(x) - \alpha(y)}{|x-y|^{n+1/2}} f(y) dy$$

for a constant $c(n)$ depending only on the space dimension. Indeed, by standard Fourier transform techniques we see that

$$[\alpha, |D|^z]f = c(z) \int_{\mathbb{R}^n} \frac{\alpha(x) - \alpha(y)}{|x-y|^{n+z}} f(y) dy$$

and this formula is valid for $\Re z < 0$ under quite general assumptions on α; moreover our assumptions show that the right hand side is a well defined and analytic function of z for $\Re z < 1/2 + \delta$ (as proved below), hence by analytic continuation the representation is valid also in this larger region and in particular for $z = 1/2$.

In order to estimate S we split it as $S = S_1 + S_2$ with

$$S_1 f = c \int_{|y| \geq 2|x|} \frac{\alpha(x) - \alpha(y)}{|x-y|^{n+1/2}} \beta(y) f(y) dy$$

$$S_2 f = c \int_{|y| \leq 2|x|} \frac{\alpha(x) - \alpha(y)}{|x-y|^{n+1/2}} \beta(y) f(y) dy$$

In the region $|y| \geq 2|x|$ we deduce by assumption (i) that

$$|\alpha(x) - \alpha(y)| \leq 2|x-y|^{1/2+\delta} \gamma(|x|)$$

since γ is decreasing; moreover we have $|x-y| \simeq |y|$, hence

$$\left| \frac{\alpha(x) - \alpha(y)}{|x-y|^{n+1/2}} \beta(y) f(y) \right| \lesssim \gamma(|x|) \frac{|\beta(y)|}{|y|^\rho} \frac{|f(y)|}{|x-y|^{n-\rho-\delta}} \lesssim \gamma(|x|) \frac{|f(y)|}{|x-y|^{n-\rho-\delta}}$$

using (ii). Thus, by Hölder inequality in Lorentz spaces, we get

$$\|S_1 f\|_{L^{\frac{2n}{n+1},2}} \lesssim \|\gamma\|_{L^{\frac{2n}{1+2\rho+2\delta},\infty}} \left\| \int \frac{|f(y)|}{|x-y|^{n-\rho-\delta}} dy \right\|_{L^{\frac{2n}{n-2\rho-2\delta},2}}$$

(provided $\rho < n/2 - \delta$) and by (i) and Young inequality we arrive at

$$\|S_1 f\|_{L^{\frac{2n}{n+1},2}} \lesssim \||y|^{-n+\rho+\delta}\|_{L^{\frac{n}{n-\rho-\delta},\infty}} \|f\|_{L^2}$$

which concludes. the estimate of the first piece S_1.

In the region $|y| \leq 2|x|$, on the other hand, we can write

$$\left| \frac{\alpha(x) - \alpha(y)}{|x-y|^{n+1/2}} \beta(y) f(y) \right| \lesssim \frac{|\beta(y)|}{|y|^\rho} \frac{|y|^\rho |\gamma(|y|/2) f(y)|}{|x-y|^{n-\delta}} \lesssim \frac{|y|^\rho |\gamma(|y|/2) f(y)|}{|x-y|^{n-\delta}}$$

so that by Young inequality

$$\|S_2 f\|_{L^{\frac{2n}{n+1},2}} \lesssim \left\| \int \frac{|y|^\rho |\gamma(|y|/2) f(y)|}{|x-y|^{n-\delta}} \right\|_{L^{\frac{2n}{n+1},2}} \lesssim \||y|^{\delta-n}\|_{L^{\frac{n}{n-\delta},\infty}} \|\gamma |y|^\rho f\|_{L^{\frac{2n}{n+1+2\delta},2}}$$

and by Hölder inequality we get

$$\|S_2 f\|_{L^{\frac{2n}{n+1},2}} \lesssim \||y|^\rho \gamma\|_{L^{\frac{2n}{1+2\delta},\infty}} \|f\|_{L^2}$$

and this concludes the proof under assumptions (i)-(ii).

The case of assumptions (i')-(ii) is almost identical. No change is necessary in the estimate of $S_2 f$, while for $S_1 f$ it is sufficient to write

$$\|S_1 f\|_{L^{\frac{2n}{n+1},2}} \lesssim \|\gamma\|_{L^{\frac{2n}{1+2\rho-2\delta},\infty}} \left\| \int \frac{|f(y)|}{|x-y|^{n-\rho+\delta}} dy \right\|_{L^{\frac{2n}{n-2\rho+2\delta},2}}$$

which is true if $\rho < n/2 + \delta$, and then proceed as above. □

Notice that if we restrict to the special choice $\beta = |x|^\rho$, $\gamma(x) = \langle x \rangle^{-\lambda}$, $\alpha(x) = \chi(x)a(x)$, the following conditions imply that (i), (ii), (i') are all satisfied:

$$0 < \delta < \frac{1}{2}, \qquad 0 \le \rho < \frac{n}{2} + \delta, \qquad \lambda \ge \frac{1}{2} + \rho + \delta \tag{2.83}$$

and

$$\langle x \rangle^\lambda \chi(x) a(x) \in C^{1/2+\delta} \tag{2.84}$$

(recall that $\|f\|_{C^\mu} = \|f\|_{L^\infty} + \sup_{x \neq y} |x-y|^{-\mu} |f(x) - f(y)|$). All conditions in (i), (ii), (i') are trivial to check apart from Hölder continuity; actually we shall now see that the following stronger inequality holds:

$$|\alpha(x) - \alpha(y)| \lesssim \min\{1, |x-y|\}^{1/2+\delta} (\langle x \rangle^{-\lambda} + \langle y \rangle^{-\lambda}). \tag{2.85}$$

Indeed, when $|x-y| \ge 1$ condition (2.85) follows from $\langle x \rangle^\lambda \chi(x) a(x) \in L^\infty$ which is contained in (2.84). When $|x-y| \le 1$, we write

$$|\alpha(x) - \alpha(y)| \le A + B,$$

where

$$A = \chi(x) a(x) \langle x \rangle^\lambda |\langle x \rangle^{-\lambda} - \langle y \rangle^{-\lambda}|,$$

and

$$B = \langle y \rangle^{-\lambda} |\langle x \rangle^\lambda \chi(x) a(x) - \langle y \rangle^\lambda \chi(y) a(y)|.$$

Then we have directly from (2.84)

$$B \lesssim \langle y \rangle^{-\lambda} |x-y|^{1/2+\delta} \le (\langle x \rangle^{-\lambda} + \langle y \rangle^{-\lambda}) |x-y|^{1/2+\delta}$$

while for A we use the elementary inequality

$$|\langle x \rangle^{-\lambda} - \langle y \rangle^{-\lambda}| \lesssim \sup_{\xi \in [x,y]} |\nabla \langle z \rangle^{-\lambda}|_{z=\xi} \cdot |x-y| \lesssim (\langle x \rangle^{-\lambda} + \langle y \rangle^{-\lambda}) |x-y|^{1/2+\delta}$$

together with the bound $\langle x \rangle^\lambda \chi(x) a(x) \in L^\infty$.

We can finally apply the lemma to the operator (2.82); since $\tau_\epsilon = |x|^{1/2-\epsilon} + |x|$ for $n \ge 3$ and $\tau_\epsilon = |x|^{1/2-\epsilon} + |x|^{1+\epsilon}$ for $n = 2$, by the above computation it is sufficient to check conditions (2.83), (2.84) for $\rho = 1/2 - \epsilon$ and $\rho = 1$ ($\rho = 1/2 - \epsilon$ and $\rho = 1 + \epsilon$ in dimension 2). We see that the choices $\delta = 2\epsilon$ and $\lambda = 1 + 3\epsilon$ work in all cases, thus it is sufficient to assume $\langle x \rangle^{1+3\epsilon} \chi(x) a(x) \in C^{1/2+2\epsilon}$ i.e. assumption (2.9). The proof is concluded.

3.2. Wave and Klein-Gordon equations: proof of Theorems 2.2, 2.3. Since $u(t,x) = e^{it\sqrt{-\Delta+W}}f$ solves the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = -Wu \\ u(0,x) = f(x) \\ u_t(0,x) = i\left(\sqrt{-\Delta+W}\right) f(x), \end{cases} \tag{2.86}$$

we have the alternative representation

$$e^{it\sqrt{-\Delta+W}}f = \cos(t|D|)f + i\frac{\sin(t|D|)}{|D|}\sqrt{-\Delta+W}f - \int_0^t \frac{\sin((t-s)|D|)}{|D|}Wuds. \tag{2.87}$$

The first two terms satisfy the standard Strichartz estimates for the free wave equation (see (2.2) in the Introduction, and recall also (2.25)). For the third term we apply as usual the Christ-Kiselev lemma and we are reduced to the untruncated integral

$$\int \frac{\sin((t-s)|D|)}{|D|}Wuds = I + II$$

where, writing $c(x) = -\nabla \cdot a + b_1 + b_2$,

$$I = \int \frac{\sin((t-s)|D|)}{|D|}\nabla \cdot (a(x)u)ds, \qquad II = \int \frac{\sin((t-s)|D|)}{|D|}c(x)uds.$$

Consider I; clearly, it is sufficient and actually stronger to estimate the integral

$$I_1 = |D|^{-1}e^{it|D|}\int e^{-is|D|}\nabla \cdot (a(x)u)ds = |D|^{-1}\nabla \cdot e^{it|D|}\int e^{-is|D|}a(x)uds.$$

To this end we recall the standard Strichartz estimate

$$\|e^{it|D|}f\|_{L^p\dot{H}_q^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} \lesssim \|f\|_{L^2} \tag{2.88}$$

valid for any wave admissible couple (p,q). Moreover, the smoothing estimate (2.60) holds also in the free case $W \equiv 0$

$$\|\tau_\epsilon^{-1}e^{it|D|}f\|_{L^2L^2} \lesssim \|f\|_{L^2} \tag{2.89}$$

and by duality is equivalent to

$$\left\| \int e^{-is|D|}F(s)ds \right\|_{L^2} \lesssim \|\tau_\epsilon F\|_{L^2L^2}. \tag{2.90}$$

Applying (2.88) and (2.90) to I_1 we obtain, since the Riesz operators are bounded in all L^p with $1 < p < \infty$,

$$\|I_1\|_{L^p\dot{H}_q^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} \lesssim \|\tau_\epsilon a(x)u\|_{L^2L^2} \leq \|\tau_\epsilon^2 a(x)\|_{L^2}\|\tau_\epsilon^{-1}u\|_{L^2L^2}.$$

Using again the smoothing estimate (2.60) and assumption (2.11), we conclude

$$\|I_1\|_{L^p\dot{H}_q^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} \lesssim \|f\|_{L^2}.$$

Consider now the second term II, or more generally

$$II_1 = e^{it|D|}\int |D|^{-1}e^{-is|D|}c(x)uds.$$

Proceeding as in [**18**], we shall use the following estimate from [**8**] (see also [**56**])

$$\||x|^{-1}|D|^{-1}e^{it|D|}f\|_{L^2L^2} \lesssim \|f\|_{\dot{H}^{-1/2}}$$

in the dual form:

$$\left\| \int |D|^{-1} e^{-is|D|} F(s) ds \right\|_{\dot{H}^{1/2}} \lesssim \||x|F\|_{L^2 L^2}. \tag{2.91}$$

Then, applying the Strichartz estimate for the wave equation (2.88) in the form

$$\|e^{it|D|} f\|_{L^p \dot{H}_q^{\frac{1}{q}-\frac{1}{p}}} \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}}$$

followed by (2.91), we obtain

$$\|II_1\|_{L^p \dot{H}_q^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} \lesssim \||x|c(x)u\|_{L^2 L^2} \lesssim \||x|\tau_\epsilon c(x)\|_{L^\infty} \|\tau_\epsilon^{-1} u\|_{L^2 L^2}.$$

Recalling assumption (2.11) and the smoothing estimate (2.60) we finally obtain

$$\|II_1\|_{L^p \dot{H}_q^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} \lesssim \|f\|_{L^2}$$

which concludes the proof of Theorem 2.2.

The proof of Theorem 2.3 is completely analogous, using the Strichartz estimate for the free equation

$$\|e^{it\langle D\rangle} f\|_{L^p H_q^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} \lesssim \|f\|_{L^2},$$

which is valid for all Schrödinger admissible couple (p, q), and the following estimate from [**8**]:

$$\|\langle x\rangle^{-1} e^{it\langle D\rangle} f\|_{L^2 L^2} \lesssim \|f\|_{L^2}$$

which implies by duality

$$\left\| \int e^{-is\langle D\rangle} F(s) ds \right\|_{L^2} \lesssim \|\langle x\rangle F\|_{L^2 L^2}$$

and hence also

$$\left\| \int \langle D\rangle^{-1} e^{-is\langle D\rangle} F(s) ds \right\|_{H^{1/2}} \lesssim \|\langle x\rangle F\|_{L^2 L^2}.$$

This estimate replaces (2.91) in the above computation.

3.3. Dirac equation: proof of Theorems 2.4, 2.5. As proved in the Appendix, the Strichartz estimate for the free massless Dirac equation is the following:

$$\|e^{it\mathcal{D}} f\|_{L^p \dot{H}_q^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} \lesssim \|f\|_{L^2} \tag{2.92}$$

for any wave admissible couple (p, q). On the other hand, as a special case of the smoothing estimate (2.70), we have

$$\|w_\sigma^{-\frac{1}{2}} e^{it\mathcal{D}} f\|_{L^2 L^2} \lesssim \|f\|_{L^2} \tag{2.93}$$

and by duality we obtain

$$\left\| \int e^{-is\mathcal{D}} F(s) ds \right\|_{L^2} \lesssim \|w_\sigma^{\frac{1}{2}} F\|_{L^2 L^2}. \tag{2.94}$$

Consider now the perturbed Dirac flow $u = e^{it(\mathcal{D}+V)} f$. An alternative representation of u is the following:

$$u(t, x) = e^{it\mathcal{D}} f - e^{it\mathcal{D}} \int_0^t e^{-is\mathcal{D}} V u(s) ds. \tag{2.95}$$

The term $e^{it\mathcal{D}}f$ satisfies the free Strichartz estimates (2.97); in order to estimate the Duhamel term as usual we apply the Christ-Kiselev lemma and switch to the untruncated integral. Then, using (2.20), (2.94) and Hölder inequality, we have

$$\begin{aligned} \left\| e^{it\mathcal{D}} \int e^{-is\mathcal{D}} V u ds \right\|_{L^p \dot{H}_q^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} &\lesssim \left\| \int e^{-is\mathcal{D}} V u ds \right\|_{L^2} \\ &\lesssim \|w_\sigma^{\frac{1}{2}} V u\|_{L^2 L^2} \le \|w_\sigma V\|_{L^\infty} \cdot \|w_\sigma^{-\frac{1}{2}} u\|_{L^2 L^2}. \end{aligned} \tag{2.96}$$

Recalling the smoothing estimate (2.70) we obtain

$$\left\| e^{it\mathcal{D}} \int e^{-is\mathcal{D}} V u ds \right\|_{L^p \dot{H}_q^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} \lesssim \|f\|_{L^2}$$

and this completes the proof of 2.4.

The proof of Theorem 2.5 is completely analogous.

4. Strichartz estimates for the free flows: an Appendix

Strichartz estimates for the free Schrödinger and wave equations are well known, see the Introduction for the precise statements. It is less easy to find in the literature optimal results for Klein-Gordon and Dirac equations. Hence we devote this appendix to a quick proof of the estimates in these cases.

The massless Dirac flow is trivial since it can be reduced to the wave equation:

PROPOSITION 2.5. *Let $n = 3$. The following Strichartz estimates hold:*

$$\|e^{it\mathcal{D}} f\|_{L^p \dot{H}_q^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} \lesssim \|f\|_{L^2} \tag{2.97}$$

for any wave admissible couple (p, q).

PROOF. By the identity

$$(i\partial_t + \mathcal{D})(i\partial_t - \mathcal{D}) = -\Box I_4,$$

we obtain that $u(t, x) = e^{it\mathcal{D}}f$ satisfies the Cauchy problem

$$\begin{cases} u_{tt} - \Delta I_4 u = 0 \\ u(0, x) = f(x) \\ u_t(0, x) = i\mathcal{D}f(x) \end{cases} \tag{2.98}$$

and hence each component of u satisfies the same Strichartz estimates as for the 3D wave equation. □

The Klein-Gordon and massive Dirac equations need some work. We begin by the free Klein-Gordon flow $u = e^{it\langle D\rangle}f$. We shall apply a precise stationary phase result due to Hörmander [**55**]:

LEMMA 2.8. *Assume that $\phi : \mathbb{R}^n \to \mathbb{R}$ has a Fourier transform $\widehat{\phi} \in \mathcal{C}^\infty$ with the decay property*

$$\left| D^\alpha \widehat{\phi}(\xi) \right| \le C_\alpha \langle \xi \rangle^{-\frac{n}{2}-1-|\alpha|} \qquad \forall \xi \in \mathbb{R}^n, \ \alpha \in \mathbb{N}^n. \tag{2.99}$$

Then the following estimate holds: for some $C > 0$,

$$\left|e^{it\langle D\rangle}\phi\right| \leq C(|t|+|x|)^{-\frac{n}{2}}. \tag{2.100}$$

Now, using an inhomogeneous dyadic decomposition $\{\psi_0, \varphi_j(D)\}_{j\geq 1}$ with the usual properties: $\psi_0(\xi)$ supported in $B(0,1)$, $\varphi_0(\xi) = \psi_0(\xi/2) - \psi_0(\xi)$,

$$\varphi_j(\xi) = \varphi_0(2^{-j}\xi), \qquad \psi_0 + \sum_{j\geq 1}\varphi_j = 1$$

we can localize the estimate as follows:

LEMMA 2.9. *The flow $e^{it\langle D\rangle}f$ satisfies the localized dispersive estimate*

$$|e^{it\langle D\rangle}\varphi_j(D)f| \leq C|t|^{-\frac{n}{2}}2^{j(\frac{n}{2}+1)}\|\widetilde{\varphi}_j(D)f\|_{L^1}, \tag{2.101}$$

for each $t\in\mathbb{R}$, $x\in\mathbb{R}^n$, $j\geq 0$ and some $C>0$; here $\widetilde{\varphi}_j$ denotes $\varphi_{j-1}+\varphi_j+\varphi_{j+1}$, with $\varphi_{-1}=0$.

PROOF. We can write

$$e^{it\langle D\rangle}\varphi_j(D)f = e^{it\langle D\rangle}\langle D\rangle^{-\frac{n}{2}-1}\langle D\rangle^{\frac{n}{2}+1}\varphi_j(D)f = e^{it\langle D\rangle}\mathcal{F}^{-1}\left(\langle\xi\rangle^{-\frac{n}{2}-1}\right)*(\varphi_j(D)f),$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Then, applying Lemma 2.8 with $\phi = \mathcal{F}^{-1}\left(\langle\xi\rangle^{-\frac{n}{2}-1}\right)$, we obtain

$$\left|e^{it\langle D\rangle}\varphi_j(D)f\right| \leq C|t|^{-\frac{n}{2}}\|\langle D\rangle^{\frac{n}{2}+1}\varphi_j(D)f\|_{L^1}. \tag{2.102}$$

Since

$$\langle D\rangle^{\frac{n}{2}+1}\varphi_j(D)f = \mathcal{F}^{-1}\left(\langle\xi\rangle^{\frac{n}{2}+1}\varphi_j(\xi)\right)*f,$$

Young inequality gives

$$\|\langle D\rangle^{\frac{n}{2}+1}\varphi_j(D)f\|_{L^1} \leq \|\mathcal{F}^{-1}\left(\langle\xi\rangle^{\frac{n}{2}+1}\varphi_j(\xi)\right)\|_{L^1}\|f\|_{L^1}. \tag{2.103}$$

Notice that we can replace in this computation f with $\widetilde{\varphi}_j(D)f$ since $\varphi_j(D)\widetilde{\varphi}_j(D) = \varphi_j(D)$. Thus to conclude the proof it is sufficient to get the following estimate:

$$\|\mathcal{F}^{-1}\left(\langle\xi\rangle^{\frac{n}{2}+1}\varphi_j(\xi)\right)\|_{L^1} \leq C2^{j(\frac{n}{2}+1)}. \tag{2.104}$$

Using the scaling operators $S_\lambda\phi(x) = \phi(\lambda x)$, we can write

$$\begin{aligned}\mathcal{F}^{-1}\left(\langle\xi\rangle^{\frac{n}{2}+1}\varphi_j(\xi)\right) &= \mathcal{F}^{-1}\left(\langle\xi\rangle^{\frac{n}{2}+1}S_{2^{-j}}\varphi_0(\xi)\right)\\ &= 2^{j(\frac{n}{2}+1)}2^{jn}S_{2^j}\mathcal{F}^{-1}\left((2^{-2j}+|\xi|^2)^{\frac{n}{2}+1}\varphi_0(\xi)\right)\end{aligned}$$

and hence

$$\|\mathcal{F}^{-1}\left(\langle\xi\rangle^{\frac{n}{2}+1}\varphi_j(\xi)\right)\|_{L^1} \leq 2^{j(\frac{n}{2}+1)}\|\mathcal{F}^{-1}\left((2^{-2j}+|\xi|^2)^{\frac{n}{2}+1}\varphi_0(\xi)\right)\|_{L^1}.$$

Moreover, multiplying and dividing by $\langle x\rangle^{2m}$ for some integer m, we obtain

$$\begin{aligned}\|\mathcal{F}^{-1}\left((2^{-2j}+|\xi|^2)^{\frac{n}{2}+1}\varphi_0(\xi)\right)\|_{L^1} &\leq C\|\langle x\rangle^{2m}\mathcal{F}^{-1}\left((2^{-2j}+|\xi|^2)^{\frac{n}{2}+1}\varphi_0(\xi)\right)\|_{L^2}\\ &= C\|(1-\Delta)^m\left((2^{-2j}+|\xi|^2)^{\frac{n}{2}+1}\varphi_0(\xi)\right)\|_{L^2},\end{aligned} \tag{2.105}$$

provided

$$m > \frac{n}{4}. \tag{2.106}$$

We shall choose m as the smallest integer satisfying (2.106). We are interested in the growth with respect to j of the quantity

$$I := (1-\Delta)^m \left((2^{-2j}+|\xi|^2)^{\frac{n}{2}+1}\varphi_0(\xi)\right).$$

When n is even, $\left(2^{-2j}+|\xi|^2\right)^{\frac{n}{2}+1}$ is a polynomial, and hence we obtain

$$\|I\|_{L^2} \leq C\|\varphi_0\|_{L^2}$$

with C independent of j. When n is odd, it is clear that almost all the terms in the expansion of I are uniformly bounded in j, apart from the (possibly) worst one

$$II = \Delta^m \left(2^{-2j}+|\xi|^2\right)^{\frac{n}{2}+1}.$$

We have the two possibilities

$$n = 4k+3 \qquad \text{or} \qquad n = 4k+1,$$

with $m = k+1$. If $n = 4k+3$, we have

$$|II| \simeq \left|D^{2k+2}\left((2^{-2j}+|\xi|^2)^{2k+\frac{5}{2}}\right)\right|$$

which expands in a sum of bounded terms. If $n = 4k+1$, we have

$$|II| \simeq \left|D^{2k+2}\left((2^{-2j}+|\xi|^2)^{2k+\frac{3}{2}}\right)\right| \lesssim \left(2^{-2j}+|\xi|^2\right)^{-1/2}|\xi|^{2k+2} + \text{ bounded terms,}$$

and also in this case we have a uniform bound in j. In conclusion, we have proved that

$$\|(1-\Delta)^m \left((2^{-2j}+|\xi|^2)^{\frac{n}{2}+1}\varphi_0(\xi)\right)\|_{L^2} \leq C,$$

for some $C > 0$, which implies (2.104), and the proof is complete. ☐

REMARK 2.2. By interpolation between estimate (2.101) and the localized L^2 conservation

$$\|e^{it\langle D\rangle}\varphi_j(D)f\|_{L^2} \leq \|\varphi_j(D)f\|_{L^2}, \tag{2.107}$$

we obtain the following $L^q - L^{q'}$ decay estimates:

$$\|e^{it\langle D\rangle}\varphi_j(D)f\|_{L^q} \leq C|t|^{-\frac{n}{2}+\frac{n}{q}}2^{j(\frac{n}{2}+1)(1-\frac{2}{p})}\|\widetilde{\varphi}_j(D)f\|_{L^{q'}} \tag{2.108}$$

for any $q \geq 2$ with $1/q + 1/q' = 1$.

Starting from estimates (2.108) and using the standard techniques of [**47**], [**66**], in particular the abstract Theorem 10.1 of [**66**], we obtain the full set of estimates including the endpoint case:

THEOREM 2.7. *The Klein-Gordon flow $u = e^{it\langle D\rangle}f$ satisfies the Strichartz estimates*

$$\|e^{it\langle D\rangle}f\|_{L^pH_q^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} \lesssim \|f\|_{L^2} \tag{2.109}$$

for any Schrödinger admissible couple (p,q).

Finally, the Dirac equation with mass can be handled in a similar way to Proposition 2.5:

PROPOSITION 2.6. *Let $n = 3$. The following Strichartz estimates hold:*

$$\|e^{it(\langle D\rangle+\beta)}f\|_{L^pH_q^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} \lesssim \|f\|_{L^2}, \tag{2.110}$$

for any Schrödinger admissible couple (p,q).

PROOF. As in the proof of Proposition 2.5, by the identity

$$(i\partial_t + (\mathcal{D} + \beta))(i\partial_t - (\mathcal{D} + \beta)) = (-\Box - 1)I_4$$

we obtain that each component of u solves a Klein-Gordon equation with initial data f and $(\mathcal{D} + \beta)f$. Thus estimate (2.110) follows immediately from the Strichartz estimates for the Klein-Gordon equation in space dimension $n = 3$. $\Box$

CHAPTER 3

Dispersion via wave operators

1. Introduction

To conclude the investigation on linear a priori estimates, we present in this chapter a slightly different approach to this kind of problems. It comes from the **Mathematical Scattering Theory**, and it is based on the study of the functional properties of the main object of this topics, which is called **Wave Operator**. As we see in the following, the boundedness of the wave operators in suitable functional spaces can be used to obtain dispersive-type estimates for perturbed equations as simple corollaries of the known estimates for the free equations. The main result of this Chapter (Theorem 3.1) has been proved in [**30**].

Let $H_0 = -d^2/dx^2$ be the one-dimensional Laplace operator on the line, and consider the perturbed operator $H = H_0 + V(x)$. For a potential $V(x) \in L^1(\mathbb{R})$, the operator H can be realized uniquely as a selfadjoint operator on $L^2(\mathbb{R})$ with form domain $H^1(\mathbb{R})$. The absolutely continuous spectrum of H is $[0, +\infty[$, the singular continuous spectrum is absent, and the possible eigenvalues are all strictly negative. Moreover, the *wave operators*

$$W_{\pm} f = L^2 - \lim_{s \to \pm\infty} e^{isH} e^{-isH_0} f \tag{3.1}$$

exist and are unitary from $L^2(\mathbb{R})$ to the absolutely continuous space $L^2_{ac}(\mathbb{R})$ of H. A very useful feature of $W_{\pm}$ is the *intertwining property*. If we denote by P_{ac} the projection of L^2 onto $L^2_{ac}(\mathbb{R})$, the property can be stated as follows: for any Borel function f,

$$W_{\pm} f(H_0) W^*_{\pm} = f(H) P_{ac} \tag{3.2}$$

(see e.g. [**36**], [**22**]).

Thanks to (3.2), one can reduce the study of an operator $f(H)$, or more generally $f(t, H)$, to the study of $f(t, H_0)$ which has a much simpler structure. When applied to the operators e^{itH}, $\frac{\sin(t\sqrt{H})}{\sqrt{H}}$, $\frac{\sin(t\sqrt{H+1})}{\sqrt{H+1}}$, this method can be used to prove decay estimates for the Schrödinger, wave and Klein-Gordon equations

$$iu_t - \Delta u + Vu = 0, \qquad u_{tt} - \Delta u + Vu = 0, \qquad u_{tt} - u_{xx} - \Delta u + u + Vu = 0,$$

provided one has some control on the L^p behaviour of $W_{\pm}$, $W^*_{\pm}$. Indeed, if the wave operators are bounded on L^p, the $L^q - L^{q'}$ estimates valid for the free operators extend immediately to the perturbed ones via the elementary argument

$$\|e^{itH} P_{ac} f\|_{L^q} \equiv \|W_+ e^{itH_0} W^*_+ f\|_{L^q} \leq C \|e^{itH_0} W^*_+ f\|_{L^q} \leq C t^{-\alpha} \|W^*_+ f\|_{L^{q'}} \leq C t^{-\alpha} \|f\|_{L^{q'}}$$

Such a program was developed systematically by K.Yajima in a series of papers [**115**], [**116**], [**117**] where he obtained the L^p boundedness for all p of $W_{\pm}$, under suitable assumptions on the potential V, for space dimension $n \geq 2$. The analysis was completed in the one dimensional case in Artbazar-Yajma [**5**] and Weder [**111**]. We remark that in high dimension $n \geq 4$ the decay estimates obtained by this method are the best available from the point of view of the assumptions on the potential; only in low dimension $n \leq 3$ more precise results have been proved (see [**49**], [**50**], [**90**], [**112**], [**118**] and [**33**]). We also mention [**51**] for an interesting class of related counterexamples.

In order to explain the results in more detail we recall a few notions. The relevant potential classes are the spaces

$$L^1_\gamma(\mathbb{R}) \equiv \{f\colon (1+|x|)^\gamma f \in L^1(\mathbb{R})\}. \tag{3.3}$$

Moreover, given a potential $V(x)$, the *Jost functions* are the solutions $f_\pm(\lambda, x)$ of the equation $-f'' + Vf = \lambda^2 f$ satisfying the asymptotic conditions $|f_\pm(\lambda, x) - e^{\pm i\lambda x}| \to 0$ as $x \to \pm\infty$. When $V(x) \in L^1_1$, the solutions $f_\pm$ are uniquely defined ([**36**]). Now consider the Wronskian

$$W(\lambda) = f_+(\lambda, 0)\partial_x f_-(\lambda, 0) - \partial_x f_+(\lambda, 0) f_-(\lambda, 0). \tag{3.4}$$

The function $W(\lambda)$ is always different from zero for $\lambda \in \mathbb{R} \setminus 0$, and hence for real λ it can only vanish at $\lambda = 0$. Then we say that 0 *is a resonance for H* when $W(0) = 0$, and that it is not a resonance when $W(0) \neq 0$. The first one is also called the *exceptional case.*

In [**111**] Weder proved that the wave operators are bounded on L^p for all $1 < p < \infty$, provided $V \in L^1_\gamma$ for $\gamma > 5/2$ (see also the following remark). The assumption can be relaxed to $\gamma > 3/2$ provided 0 is not a resonance. It is natural to conjecture that these conditions may be sharpened, also in view of the results Goldberg and Schlag [**49**] proved under the milder assumption $\gamma = 2$ in the general and $\gamma = 1$ in the nonresonant case.

Indeed, the main result of the present chapter is the following:

THEOREM 3.1. *Assume $V \in L^1_1$ and 0 is not a resonance, or $V \in L^1_2$ in the general case. Then the wave operators $W_\pm, W^*_\pm$ can be extended to bounded operators on L^p for all $1 < p < \infty$. Moreover, in the endpoint L^∞ case we have the estimate*

$$\|W_\pm g\|_{L^\infty} \leq C\|g\|_{L^\infty} + C\|\mathcal{H}g\|_{L^\infty}, \tag{3.5}$$

for all $g \in L^\infty \cap L^p$ for some $p < \infty$ such that $\mathcal{H}g \in L^\infty$, where $\mathcal{H}$ is the Hilbert transform on $\mathbb{R}$; the conjugate operators $W^*_\pm$ satisfy the same estimate.

REMARK 3.1. The appearence of the Hilbert trasform (see the beginning of Section 4 for a quick reminder) at the endpoint $p = \infty$ is not a surprise. Indeed, Weder proved that, under the assumptions $V \in L^1_\gamma$ for $\gamma > 5/2$ in the general case and $\gamma > 3/2$ in the nonresonant case, the wave operator involves explicitly the Hilbert transform. More precisely, let $\chi(x) \in C^\infty(\mathbb{R})$ be such that $\chi = 0$ for $x < 0$ and $\chi = 1$ for $x > 1$, then formula (1.12) in [**111**] states that

$$W_\pm = W_{\pm,r} \pm \chi(x) f_+(0, x)\mathcal{H}\Psi(D)(c_1 + c_2 P) \pm (1 - \chi(x)) f_-(0, x)\mathcal{H}\Psi(D)(c_3 + c_4 P)$$

where the operators $W_{\pm,r}$ are bounded on L^1 and L^∞, $Pf(x) := f(-x)$, $\Psi(\xi) \in C^\infty_0$ is a suitable cutoff, and the constants c_j can be expressed in terms of the transmission and reflection coefficients. From this decomposition it is clear that the wave operator in general can not be bounded on L^∞, but only from L^∞ to BMO. Notice also that the Hilbert transform terms vanish in the unperturbed case $V \equiv 0$.

At the opposite endpoint $p = 1$, we get an even weaker result by duality (see Remark 3.8). Weder's decomposition suggests that the stronger bound

$$\|W_{\pm}g\|_{L^1} \leq C\|g\|_{L^1} + C\|\mathcal{H}g\|_{L^1} \tag{3.6}$$

should be true (and is indeed true under his assumptions on the potential). Notice that (3.6) is equivalent to say that $W_{\pm}$ are bounded operators from the Hardy space $\mathcal{H}_1$ to L^1, and by duality this would also imply that $W_{\pm}$ are bounded operators from L^∞ to BMO.

Remark 3.2. Our proof is based on the improvement of some results of Deift and Trubowitz [**36**], combined with the stationary approach of Yajima [**115**], [**5**], and some precise Fourier analysis arguments. Quite inspirational have been the papers [**49**] and [**113**], both for showing there was room for improvement in the assumptions on the potential, and for the very effective harmonic analysis approach.

Remark 3.3. In the proof of Theorem 3.1 we split as usual the wave operator into high and low energy parts; the high energy part is known to be easier to handle since the resolvent is only singular at frequency $\lambda = 0$. Here we can prove that the high energy part is bounded on L^p for all p, including the cases $p = 1$ and $p = \infty$, under the weaker assumption $V \in L^1(\mathbb{R})$ (see Section 2 and Lemma 3.1).

Remark 3.4. An essential step in the low energy estimate is a study of the Fourier properties of the Jost functions; this kind of analysis is classical (see [**1**]) and the fundamental estimates were obtained by Deift and Trubowitz in [**36**]. In Section 3 we improve their results by showing that the L^1 norms of the Fourier transforms of the Jost functions satisfy a linear bound as $|x| \to +\infty$ instead of an exponential one as in [**36**]. In the resonant case we can prove a quadratic bound (see Lemmas 3.3, 3.4 and Corollary 3.1).

Remark 3.5. It is possible to continue the analysis and prove that the wave operators are bounded on Sobolev spaces $W^{k,p}$, under the additional assumption $V \in W^{k,1}$ (see also [**111**] where the boundedness from $W^{k,\infty}$ to BMO_k is proved under stronger assumptions on the potential), but we prefer not to pursue this question here.

Theorem 3.1 has several applications; here we shall focus on the dispersive estimates for the one dimensional Schrödinger and Klein-Gordon equations with variable rough coefficients.

Consider first the initial value problem

$$iu_t - a(x)u_{xx} + b(x)u_x + V(x)u = 0, \qquad u(0,x) = f(x). \tag{3.7}$$

Then we obtain the following decay result, where the notation $f \in L^2_1$ means $(1+|x|)f \in L^2$. Notice that the following result can also be obtained as a consequence of the dispersive $L^\infty - L^1$ estimate proved in [**49**] (and in [**112**] under stronger assumptions on the potential).

Proposition 3.1. *Assume $V \in L^1_2$, $a \in W^{2,1}(\mathbb{R})$ and $b \in W^{1,1}(\mathbb{R})$ with*

$$a(x) \geq c_0 > 0 \qquad a', b \in L^2_1, \qquad a'', b' \in L^1_2 \tag{3.8}$$

for some constant c_0. Then the solution of the initial value problem (3.7) *satisfies*

$$\|P_{ac}u(t,\cdot)\|_{L^q} \leq Ct^{\frac{1}{q}-\frac{1}{2}}\|f\|_{L^{q'}}, \qquad 2 \leq q < \infty, \quad \frac{1}{q} + \frac{1}{q'} = 1. \tag{3.9}$$

The same result holds if $a = 1$, $b = 0$ and $V \in L^1_1$, provided 0 is not a resonance for H.

PROOF. It is sufficient to perform the change of variables $u(t, x) = \sigma(x)w(t, c(x))$ with

$$c(x) = \int_0^x a(s)^{-1/2} ds, \qquad \sigma(x) = a(x)^{1/4} \exp\left(\int_0^x \frac{b(s)}{2a(s)} ds\right) \tag{3.10}$$

to reduce the problem to a Schrödinger equation with a potential perturbation $\widetilde{V}(y)$ defined by

$$\widetilde{V}(c(x)) = V(x) + \frac{1}{16a(x)}(2b(x) + a'(x))(2b(x) + 3a'(x)) - \frac{1}{4}(2b(x) + a''(x)); \tag{3.11}$$

notice that $\widetilde{V}$ satisfies the assumptions of Theorem 3.1 provided (3.8) hold. Hence the solution of the transformed problem satisfies a dispersive estimate like (3.9), and coming back to the original variables we conclude the proof. □

REMARK 3.6. The range of indices allowed in (3.9) is sufficient to deduce the full set of Strichartz estimates, as it is well known. It is interesting to compare this with the result of Burq and Planchon [**16**] who proved the Strichartz estimates for the variable coefficient equation

$$iu_t - \partial_x(a(x)\partial_x u) = 0$$

assuming only that $a(x)$ is of BV class and bounded from below.

REMARK 3.7. In view of the next application, we recall the definition of nonhomogeneous *Besov spaces*. Choose a Paley-Littlewood partition of unity, i.e., a sequence of smooth cutoffs $\phi_j \in C_0^\infty(\mathbb{R})$ with $\sum_{j\geq 0} \phi_j(\lambda) = 1$ and $\operatorname{supp}\phi_j = [2^{j-1}, 2^{j+1}]$ for $j \geq 1$, $\operatorname{supp}\phi_0 = [-2, 2]$. Then the $B^s_{p,r}$ Besov norm is defined by

$$\|g\|^r_{B^s_{p,r}} \equiv \sum_{j\geq 0} 2^{jsr} \|\phi_j(\sqrt{H_0})g\|^r_{L^p}$$

with obvious modification for $r = \infty$. It is then natural to define the *perturbed Besov norm* corresponding to the selfadjoint operator $H = H_0 + V$ as

$$\|g\|^r_{B^s_{p,r}(V)} \equiv \sum_{j\geq 0} 2^{jsr} \|\phi_j(\sqrt{H})g\|^r_{L^p}.$$

Now, from the L^p boundedness of the wave operators and the intertwining property in the form

$$\phi_j(\sqrt{H})W_\pm = W_\pm \phi_j(\sqrt{H_0})$$

we obtain immediately the Besov space bounds

$$\|W_\pm f\|_{B^s_{p,r}(V)} \leq C\|f\|_{B^s_{p,r}}, \qquad \|W^*_\pm f\|_{B^s_{p,r}} \leq C\|f\|_{B^s_{p,r}(V)} \tag{3.12}$$

(in the second one we used the inequality $\|P_{ac}\phi(H)f\|_{L^p} \leq C\|\phi(H)f\|_{L^p}$ which is true since the eigenfunctions belong to $L^1 \cap L^\infty$).

We now consider the initial value problem for the one dimensional Klein-Gordon equation

$$u_{tt} - a(x)u_{xx} + u + b(x)u_x + V(x)u = 0, \qquad u(0, x) = 0, \quad u_t(0, x) = g(x). \tag{3.13}$$

Our second application is the following, proved in an identical way as Proposition 3.1:

PROPOSITION 3.2. *Assume $a = 1$, $b = 0$ and $V \in L^1_2$, or $V \in L^1_1$ and 0 is not a resonance. Then the solution of the initial value problem* (3.13) *satisfies*

$$\|P_{ac}u(t,\cdot)\|_{L^q} \leq Ct^{\frac{1}{q}-\frac{1}{2}}\|g\|_{B^{\frac{1}{2}-\frac{3}{q}}_{q',q}(V)}, \qquad 2 \leq q < \infty, \quad \frac{1}{q}+\frac{1}{q'}=1. \tag{3.14}$$

The same decay rate is true for general coefficients a, b, V satisfying the assumptions of Proposition 3.1 (with the Besov norm replaced by a suitable norm of the initial data).

PROOF. In the unperturbed case, (3.14) can be obtained as usual by interpolating the dispersive $L^\infty - B^{1/2}_{1,1}$ estimate with the conservation of the H^1 norm i.e. the energy. The perturbed case is handled by the change of variables (3.10) and an application of Theorem 3.1 as in the proof of Proposition 3.1. In the general case the Besov norm in (3.14) must be replaced by $\|h\|_{B^{\frac{1}{2}-\frac{3}{q}}_{q',q}(\widetilde{V})}$ with $\widetilde{V}$ as in (3.11) and $h = (g/\sigma)|_{c^{-1}(y)}$. $\square$

The rest of the paper is devoted to the proof of Theorem 3.1. We first analyze the high energy part, in Section 2; Section 3 contains a detailed study of the Fourier properties of the Jost functions, necessary for the analysis of the low energy part studied in Section 4.

2. The high energy analysis

In the estimate of the high frequency part of the wave operator we shall use the standard representation as a distorted Fourier transform; considering e.g. the operator W_-, we have

$$W_-g(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\varphi(\lambda,x)e^{-i\lambda y}\,d\lambda\right)g(y)\,dy, \tag{3.15}$$

where the generalized eigenfunction $\varphi(\lambda, x)$ is defined as the solution to the Lippman-Schwinger equation (see e.g. [**5**], [**111**])

$$\varphi(\lambda,x) = e^{i\lambda x} - R_0(\lambda^2+i0)V\varphi(\lambda,x). \tag{3.16}$$

Here R_0 denotes the free resolvent $R_0(z) = (-\Delta - z)^{-1}$; we recall that the limits

$$R_0(\lambda \pm i0) = \lim_{\epsilon\to 0} R_0(\lambda \pm i\epsilon) = \frac{1}{2i}\int \frac{e^{\pm i\lambda|x-y|}}{\lambda} f(y)\,dy, \tag{3.17}$$

exist in the norm of bounded operators from the weighted $L^2_{1/2+\epsilon}$ to the weighted $L^2_{-1/2-\epsilon}$ spaces, for any $\lambda \in]0,\infty[$ (see e.g. [**2**]). The strong singularity at $\lambda = 0$ is the main source of difficulties in the study of the wave operator.

The perturbed resolvent $R_V(z) = (-\Delta + V - z)^{-1}$ is related to R_0 by the identity

$$R_V = R_0(I + VR_0)^{-1}. \tag{3.18}$$

We recall that under the assumption $V \in L^1_1$ the limiting absorption principle (3.17) holds also for R_V (see [**6**], [**31**]).

By the representation (3.17) it is clear that for $\lambda \geq \lambda_0 = \|V\|_{L^1}$ the operator R_0V is bounded on L^∞ with norm less than 1/2. In particular, for λ large enough, $I + R_0(\lambda^2 + i0)V$

can be inverted by a Neumann series, the solution $\phi(\lambda, x)$ of (3.16) is well defined and it can be represented by a uniformly convergent series

$$\varphi(\lambda,x)=\sum_{n\geq 0}(-1)^n\left(R_0(\lambda^2+i0)V\right)^n e^{ikx}, \qquad |\lambda|\geq \lambda_0:=\|V\|_{L^1}, \quad x\in\mathbb{R}. \tag{3.19}$$

Now take a smooth cutoff function $\Phi\in C^\infty(\mathbb{R}^+)$ such that

$$0\leq\Phi\leq 1, \qquad \Phi(\lambda^2)=0 \quad \text{for} \quad 0\leq\lambda\leq\lambda_0, \qquad \Phi(\lambda^2)=1 \quad \text{for} \quad \lambda\geq\lambda_0+1$$

and consider the high energy part of the wave operator

$$W_-\Phi(H_0)g(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\varphi(\lambda,x)e^{-i\lambda y}g(y)\Phi(\lambda^2)\,d\lambda dy.$$

We split this operator into positive and negative frequencies, i.e., writing

$$\chi(\lambda)=\begin{cases}\Phi(\lambda^2) & \text{for } \lambda>0,\\ 0 & \text{for } \lambda\leq 0\end{cases} \qquad \psi(\lambda)=\begin{cases}\Phi(\lambda^2) & \text{for } \lambda<0,\\ 0 & \text{for } \lambda\geq 0\end{cases}$$

we define the operators

$$Ag(x)=\frac{1}{2\pi}\int\int\varphi(\lambda,x)e^{-i\lambda y}g(y)\chi(\lambda)\,d\lambda dy, \qquad Bg(x)=\frac{1}{2\pi}\int\int\varphi(\lambda,x)e^{-i\lambda y}g(y)\psi(\lambda)\,d\lambda dy. \tag{3.20}$$

In the following we shall study the positive part Ag; clearly the estimate of the negative piece Bg is completely analogous. By (3.15) and (3.19), the integral kernel $K(x,y)$ of the operator A can be represented as

$$K(x,y)=\sum_{n\geq 0}(-1)^n\int\left\{\left(R_0(\lambda^2+i0)V\right)^n e^{i\lambda\cdot}\right\}(x)\;e^{-i\lambda y}\chi(\lambda)\,d\lambda. \tag{3.21}$$

We shall estimate the terms of the series (3.21) separately. Notice that for $n\geq 2$ we can write them explicitly as

$$K_n(x,y)=\left(\frac{i}{2}\right)^n\int\!\!\dots\!\!\int\frac{\chi(\lambda)}{\lambda^n}e^{i\lambda(|x-y_1|+|y_1-y_2|+\dots+|y_{n-1}-y_n|+y_n-y)}\prod_{j=1}^n V(y_j)\,dy_1\dots dy_n\,d\lambda. \tag{3.22}$$

On the other hand, for $n=0,1$ we have the formal expressions

$$K_0(x,y)=\int e^{i\lambda(x-y)}\chi(\lambda)\,d\lambda, \qquad K_1(x,y)=\frac{i}{2}\int\int\frac{\chi(\lambda)}{\lambda}e^{i\lambda(|x-y_1|+y_1-y)}V(y_1)\,dy_1\,d\lambda \tag{3.23}$$

which can defined precisely by adding a cutoff on $[0,L]$ and then sending $L\to+\infty$ (see below). Denoting by A_n the operator with kernel $K_n(x,y)$, we have

$$Ag(x)=(2\pi)^{-1}\Big(A_0g(x)-A_1g(x)+\sum_{n\geq 2}A_ng(x)\Big). \tag{3.24}$$

Then we have:

LEMMA 3.1. *Assume $V\in L^1(\mathbb{R})$ and let $0\leq\Phi\leq 1$ be a smooth function such that $\Phi(\lambda^2)=0$ for $\lambda<\|V\|_{L^1}$ and $\Phi(\lambda^2)=1$ for $\lambda>\|V\|_{L^1}+1$. Then the high energy parts of the wave operators $W_\pm$ are bounded on L^p for all $1\leq p\leq\infty$:*

$$\|W_\pm\Phi(H_0)g\|_{L^p}\leq C\|g\|_{L^p}. \tag{3.25}$$

The same holds for the conjugate operators $\Phi(H_0)W_\pm^*$.

PROOF. By standard duality arguments, it will be sufficient to prove the estimates for $p=\infty$; since the proof is completely analogous for any of the four operators $W_\pm$, $W_\pm^*$, we shall consider only W_-. By the discussion above, we see that it is sufficient to estimate the operator A defined in (3.24).

We shall estimate each term A_n in the series (3.24) separately. For the term A_0, we can write by (3.23)

$$A_0 g(x) = \int \Big(\int e^{i\lambda(x-y)} \chi(\lambda) g(y) \Big) = \int\int e^{i\lambda(x-y)}[1-(1-\chi(\lambda))]g(y) = g(x) - [\widehat{(1-\chi)} * g](x),$$

(recall the notations $\hat{h} = \mathcal{F}h$ for the Fourier transform of a function h) whence we obtain

$$\|A_0 g\|_{L^\infty} \le \Big(1 + \|\widehat{(1-\chi)}\|_{L^1}\Big) \|g\|_{L^\infty} \le C_0 \|g\|_{L^\infty}. \tag{3.26}$$

Consider now the term A_1, which by (3.23) can be written formally

$$A_1 g(x) = \frac{i}{2} \int \Big(\int \Big(\int \frac{\chi(\lambda)}{\lambda} e^{i\lambda(|x-z|+z-y)} V(z) g(y)\, dz \Big)\, d\lambda \Big)\, dy. \tag{3.27}$$

More precisely, fixed a function $\psi(\lambda) \in C_c^\infty$ equal to 1 on $[-1,1]$ and vanishing outside $[-2,2]$, we define the truncated operators

$$A_{1,L} g = \frac{i}{2} \int \Big(\int \Big(\gamma_L(\lambda) e^{i\lambda(|x-z|+z-y)} V(z) g(y)\, dz \Big)\, d\lambda \Big)\, dy, \qquad \gamma_L(\lambda) = \frac{1}{\lambda}\chi(\lambda)\psi_L(\lambda), \tag{3.28}$$

where $\psi_L(\lambda) \equiv \psi(\lambda/L)$. We claim that the operators $A_{1,L}$ are uniformly bounded on L^∞, and that for each $g \in L^\infty$ the sequence $A_{1,L}g$ converges to $A_1 g$ uniformly as $L \to +\infty$. To prove this, we notice that by Fubini's theorem (3.28) can be rewritten as

$$A_{1,L} g(x) = \frac{i}{2} \int\int \hat{\gamma}_L(|x-z| + z - y) V(z) g(y)\, dz\, dy, \tag{3.29}$$

It is clear that the claim follows as soon as we can prove that $\hat{\gamma}_L$ converges in $L^1(\mathbb{R})$ when $L \to +\infty$: indeed, we have

$$\|A_{1,L} g - A_{1,M} g\|_{L^\infty} \le \|V\|_{L^1} \|\hat{\gamma}_L - \hat{\gamma}_M\|_{L^1} \|g\|_{L^\infty}.$$

To prove the claim, decompose γ_L as follows:

$$\gamma_L(\lambda) = \psi_L(\lambda) \cdot \eta(\lambda), \qquad \eta(\lambda) = \frac{\lambda}{1+\lambda^2} + (\chi - 1)\frac{\lambda}{1+\lambda^2} + \frac{\chi(\lambda)}{\lambda(1+\lambda^2)}. \tag{3.30}$$

We notice that $\widehat{\eta}(\xi) \in L^1$; indeed, the Fourier transform of the first term is proportional to $\operatorname{sgn}(\xi)\exp(-|\xi|)$, while the remaining terms are smooth and decay faster than $|\lambda|^{-3}$. Since $\hat{\psi}_L$ is a δ-sequence, we conclude that $\hat{\gamma}_L = \hat{\psi}_L * \hat{\eta}$ converges to $\hat{\eta}$ in $L^1(\mathbb{R})$. As a consequence, $A_{1,L}g$ converge uniformly to

$$A_1 g \equiv \frac{i}{2} \int\int \hat{\eta}(|x-z| + z - y) V(z) g(y) dz dy$$

which is then a bounded operator on L^∞:

$$\|A_1 g\|_{L^\infty} \le \|V\|_{L^1} \|\hat{\eta}\|_{L^1} \|g\|_{L^\infty}. \tag{3.31}$$

To conclude the proof, it remains to estimate the operators A_n for $n \geq 2$. By the explicit formula (3.22) we obtain

$$A_n g(x) = \frac{i^n}{2^n} \int \psi_n(\lambda) e^{i\lambda(|x-y_1|+|y_1-y_2|+\cdots+|y_{n-1}-y_n|+y_n-y)} \prod_{j=1}^{n} V(y_j) g(y)\, dy_1 \dots dy_n\, d\lambda\, dy,$$

where $\psi_n(\lambda) := \chi(\lambda)/\lambda^n$. By Fubini's Theorem this can be written

$$A_n g(x) = \frac{i^n}{2^n} \int \hat{\psi}_n(|x-y_1| + |y_1-y_2| + \cdots + |y_{n-1}-y_n| + y_n - y) \prod_{j=1}^{n} V(y_j) g(y)\, dy_1 \dots dy_n\, dy,$$

and then we immediately get the inequality

$$\|A_n g(x)\|_{L^\infty} \leq \frac{1}{2^n} \|V\|_{L^1}^n \|\hat{\psi}_n\|_{L^1} \|g\|_{L^\infty}. \tag{3.32}$$

To compute the norm of $\hat{\psi}_n$, introduce the scaling operators S_h defined as $S_h g(x) = g(hx)$; then we can write

$$\psi_n(\lambda) = \lambda_0^{-n} \cdot S_{1/\lambda_0} \left(\frac{\chi_0(\lambda)}{\lambda^n} \right) \quad \text{where} \quad \chi_0(\lambda) := \chi(\lambda \cdot \lambda_0), \qquad \lambda_0 = \|V\|_{L^1}.$$

and hence

$$\begin{aligned} \|\hat{\psi}_n\|_{L^1} &= \lambda_0^{-n} \|\mathcal{F}(\chi_0/\lambda^n)\|_{L^1} \leq C\lambda_0^{-n} \|\langle\xi\rangle^2 \mathcal{F}(\chi_0/\lambda^n)\|_{L^\infty} \\ &\leq C\lambda_0^{-n} \|(1-\Delta)(\chi_0/\lambda^n)\|_{L^1} \leq C_0 n^2 \lambda_0^{-n} \equiv C_0 n^2 \|V\|_{L^1}^{-n} \end{aligned}$$

for some constant C_0 independent of n and λ_0 This inequality together with (3.32) gives

$$\|A_n g(x)\|_{L^\infty} \leq C_0 n^2 2^{-n} \|g\|_{L^\infty}. \tag{3.33}$$

By the estimates (3.26), (3.31), (3.33) and by formula (3.24) we conclude the proof of the Lemma. □

3. Fourier properties of the Jost Functions

Throughout this section we shall assume that $V \in L^1_1(\mathbb{R})$ (at least).

The *Jost functions* $f_\pm(z,x)$ are defined as the solutions of $-f''_\pm(z,x) + V(x) f_\pm(z,x) = z^2 f_\pm(z,x)$ satisfying the asymptotic conditions $\left| f_\pm(z,x) - e^{\pm izx} \right| \to 0$ as $x \to \pm\infty$. It is well known (see [**36**]) that $f_\pm(\lambda, x)$ are well defined for all $\lambda, x \in \mathbb{R}$. Using the Jost functions it is possible to write the following explicit representation of the integral kernel of the perturbed resolvent $R_V(\lambda^2 \pm i0)$:

$$K_\pm(x,y) = \frac{1}{2\pi i} \frac{f_+(\pm\lambda, y) f_-(\pm\lambda, x)}{W(\pm\lambda)} \qquad \text{for } x < y, \tag{3.34}$$

and x and y reversed for $x > y$; here $W(\lambda)$ denotes the Wronskian of f_+, f_- defined in (3.4). It is always true (see [**36**]) that $W(\lambda) \neq 0$ for any real $\lambda \neq 0$; thus the only possible real zero of the Wronskian is at $\lambda = 0$, and when $W(0) = 0$ we say that 0 is a *resonance* for $-\Delta + V$.

The modified Jost functions $m_\pm$ are given by the relation $f_\pm(\lambda, x) = e^{\pm i\lambda x} m_\pm(\lambda, x)$; equivalently, they can be characterized as the unique solutions of the equations $m''_\pm(\lambda, x) \pm 2i\lambda m'_\pm(\lambda, x) =$

$V(x)m_\pm(\lambda, x)$ satisfying the asymptotic conditions $m_\pm(\lambda, x) \to 1$ for $x \to \pm\infty$. Moreover, we can also obtain $m_\pm(\lambda, x)$ as the unique solutions of the Volterra integral equations

$$m_\pm(\lambda, x) = 1 \pm \int_x^{+\infty} D_\lambda(\pm(t-x))V(t)m_\pm(\lambda, t)\,dt, \qquad D_\lambda(x) := \frac{e^{2i\lambda x} - 1}{2i\lambda}. \tag{3.35}$$

The properties of the functions $m_\pm(\lambda, x)$ are well known, see e.g. [**36**]. Here we shall only need a few basic facts: in particular, when $V \in L^1_1$, then $m_\pm(\lambda, x) \in C(\mathbb{R}^2)$; and when $V \in L^1_2$, then $m_\pm(\lambda, x) \in C^1(\mathbb{R}^2)$ and $\frac{\lambda}{W(\lambda)} \in C(\mathbb{R})$.

As customary we shall denote by $B_\pm$ the Fourier transform w.r. to λ of the functions $m_\pm - 1$, and precisely

$$B_\pm(\xi, x) = \int_{\mathbb{R}} e^{-2i\lambda\xi}\,(m_\pm(\lambda, x) - 1)\,d\lambda. \tag{3.36}$$

(notice the factor 2 in the exponential). For each $x \in \mathbb{R}$ the function $B_+(\xi, x)$ is well defined, real valued, belongs to $L^2(\mathbb{R})$ and actually vanishes for $\xi < 0$; this means that $m_+(\cdot, x) - 1$ belongs to the *Hardy space* H^{2+} (see [**36**] for details). Analogously, $B_-(\xi, x)$ belongs to $L^2(\mathbb{R})$ and vanishes for $\xi > 0$, i.e., $m_-(\cdot, x) - 1 \in H^{2-}$.

If we take the Fourier transform of equation (3.35), we obtain that $B_+(\xi, x)$ satisfies the *Marchenko equation*

$$B_+(\xi, x) = \int_{x+\xi}^\infty V(t)\,dt + \int_0^\xi dz \int_{x+\xi-z}^\infty V(t)B_+(z, t)\,dt \tag{3.37}$$

($B_-(\xi, x)$ satisfies a symmetric equation).

The functions $B_\pm(\xi, x)$ have many additional properties of boundedness and regularity; however we shall only be concerned here with the properties of the L^1 norms $\|B_\pm(\cdot, x)\|_{L^1}$. Writing

$$\eta(x) = \int_x^\infty |V(t)|dt, \qquad \gamma(x) = \int_x^\infty (t-x)|V(t)|dt \equiv \int_x^\infty \int_y^\infty |V(t)|dtdy,$$

the well-known estimate of Deift and Trubowitz is the following:

Lemma 3.2. *Assume $V \in L^1_1$. Then, for all $\xi, x \in \mathbb{R}$, the solution $B_+(\xi, x)$ to* (3.37) *is well defined and satisfies the estimates*

$$|B_+(\xi, x)| \le e^{\gamma(x)}\eta(\xi + x), \qquad |\partial_x B_+(\xi, x) + V(x+\xi)| \le e^{\gamma(x)}\eta(x+\xi). \tag{3.38}$$

In particular, $B(\cdot, x)$ is in $L^1 \cap L^\infty$ for any x and

$$\|B_+(\cdot, x)\|_{L^1} \le e^{\gamma(x)}\gamma(x), \qquad \|\partial_x B_+(\cdot, x)\|_{L^1} \le \eta(x) + e^{\gamma(x)}\gamma(x). \tag{3.39}$$

The function B_- has similar properties, with the behaviours at $\pm\infty$ reversed. Notice that the behaviour of $\gamma(x)$ is the following:

$$\gamma(x) \le \|V\|_{L^1_1} \quad \text{for } x \ge 0, \qquad \gamma(x) \le \|V\|_{L^1_1} + |x| \cdot \|V\|_{L^1} \quad \text{for } x \le 0. \tag{3.40}$$

In other words, the estimate shows that $\|B_+(\cdot, x)\|_{L^1}$ is bounded by a constant depending on $\|V\|_{L^1_1}$ for $x > 0$, but it gives only an exponential bound for negative x. A similar estimate holds for the function B_-, exchanging the behaviours as $x \to +\infty$ and $x \to -\infty$.

A crucial tool in the study of the low energy case will be an essential improvement of the (3.39): indeed, we can prove that the norm of B_+ (resp. B_-) has at most a linear growth as $x \to -\infty$ (resp. $x \to +\infty$).

LEMMA 3.3. *Assume $V \in L^1_1$; then the functions $B_\pm(\xi, x)$ satisfy the estimates*

$$\|B_\pm(\cdot,x)\|_{L^1} \leq C \quad \text{for } \pm x \geq 0, \qquad \|B_\pm(\cdot,x)\|_{L^1} \leq C\langle x\rangle \quad \text{for } \pm x \leq 0 \tag{3.41}$$

for some constant C depending on $\|V\|_{L^1_1}$.

PROOF. We prove the result for B_+, the proof for B_- is identical. The behaviour for positive x is already contained in the Deift-Trubowitz estimate. Now, starting from the Marchenko equation (3.37), we integrate with respect to ξ from 0 to ∞ (recall that $B_\pm$ vanish for $\xi < 0$) and we have immediately

$$\|B_+(\cdot,x)\|_{L^1} \leq \sqrt{2}\langle x\rangle \cdot \|V\|_{L^1_1} + \int_0^\infty d\xi \int_{-\infty}^{\xi} dz \int_{x+\xi-z}^{\infty} |V(t)| \cdot |B_+(z,t)|\, dt.$$

Setting $z' := \xi - z$ and exchanging the order of integration we obtain

$$\|B_+(\cdot,x)\|_{L^1} \leq \sqrt{2}\langle x\rangle \cdot \|V\|_{L^1_1} + \int_x^\infty |V(t)| \cdot (t-x) \cdot \|B_+(\cdot,t)\|_{L^1}\, dt. \tag{3.42}$$

Now we remark that

$$\int_x^\infty t|V(t)| \cdot \|B_+(\cdot,t)\|_{L^1} dt \leq \int_0^\infty t|V(t)| \cdot \|B_+(\cdot,t)\|_{L^1} dt$$

which is obvious when $x > 0$ and is also evident for $x < 0$ since the integral from x to 0 is negative. Using the Deift-Trubowitz estimate (3.39) we see that $\|B_+(\cdot,t)\| \leq C_0 = C_0(\|V\|_{L^1_1})$ for $t > 0$, and hence in conclusion

$$\int_x^\infty t|V(t)| \cdot \|B_+(\cdot,t)\|_{L^1} dt \leq C_1 \equiv C_1(\|V\|_{L^1_1}) \quad \text{for all } x \in \mathbb{R}.$$

Thus inequality (3.42) gives

$$\|B_+(\cdot,x)\|_{L^1} \leq \sqrt{2}\langle x\rangle \cdot \|V\|_{L^1_1} + C_1(\|V\|_{L^1_1}) + |x| \int_x^\infty |V(t)| \cdot \|B_+(\cdot,t)\|_{L^1} dt$$

which implies

$$\frac{1}{\langle x\rangle}\|B_+(\cdot,x)\|_{L^1} \leq C_2(\|V\|_{L^1_1}) + \int_x^\infty \langle t\rangle |V(t)| \frac{\|B_+(\cdot,t)\|_{L^1}}{\langle t\rangle}\, dt. \tag{3.43}$$

Applying Gronwall's lemma for $x < 0$ we conclude the proof. □

In the resonant case $W(0) = 0$ it will be necessary to make the stronger assumption $V \in L^1_2$. In this case, we know that the Jost functions are C^1 in both variables and we shall study the behaviour of the functions

$$C_\pm(\xi, x) = \int_{\mathbb{R}} e^{-2i\lambda\xi} \partial_\lambda m_\pm(\lambda, x) d\xi \equiv 2i\xi B_\pm(\xi, x). \tag{3.44}$$

As above, a direct application of the Deift-Trubowitz estimate gives an optimal bound only on a half line. Indeed, if we multiply (3.38) by 2ξ and integrate in ξ we obtain

$$\|C_+(\cdot,x)\|_{L^1} \leq 2e^{\gamma(x)} \int_0^\infty \xi \int_{x+\xi}^\infty |V(t)| dt d\xi = e^{\gamma(x)} \int_x^\infty (t-x)^2 |V(t)| dt \tag{3.45}$$

after exchanging the order of integration. Recalling (3.40) we obtain that $\|C_+(\cdot,x)\|_{L^1} \leq C(\|V\|_{L^1_2})$ for $x \geq 0$, but we can only get an exponential growth for negative x (symmetric result for C_-).

We can improve this estimate by a different argument:

LEMMA 3.4. *Assume $V \in L^1_2$; then the functions $C_\pm(\xi,x) = 2i\xi B_\pm(\xi,x)$ satisfy*

$$\|C_\pm(\cdot,x)\|_{L^1} \leq C \quad \text{for } \pm x \geq 0, \qquad \|C_\pm(\cdot,x)\|_{L^1} \leq C\langle x\rangle^2 \quad \text{for } \pm x \leq 0 \tag{3.46}$$

for some constant C depending on $\|V\|_{L^1_2}$.

PROOF. We will consider only C_+; the proof for C_- is identical. We have already proved above the estimate of C_+ on the positive half-line. To prove the estimate for $x < 0$ we start again from Marchenko's equation (3.37); if we multiply both sides by 2ξ and integrate in ξ we obtain

$$\|C_+(\cdot,x)\|_{L^1} \leq 2\int_0^\infty \int_{x+\xi}^\infty |V(t)| \cdot \xi \, dt d\xi + 2\int_0^\infty d\xi \int_0^\xi d\sigma \int_{x+\xi-\sigma}^\infty |V(t)| \cdot |B_+(\sigma,t)| \cdot \xi \, dt$$

$$\equiv 2\int_x^\infty (t-x)^2|V|dt + \int_x^\infty |V|(t-x)^2\|B_+(\cdot,t)\|_{L^1}dt + \int_x^\infty |V|(t-x)\|C_+(\cdot,t)\|_{L^1}dt \tag{3.47}$$

after a suitable rearrangement of the order of integration. Call the three integrals on the right I, II, III respectively. For the first one we have obviously $I \leq 4\langle x\rangle^2\|V\|_{L^1_2}$. For the second one, we remark that $(t-x)^2 \leq x^2$ when $x < t < 0$, while $(t-x)^2 \leq 2t^2 + 2x^2$ when $t > 0$, so that

$$II \leq x^2 \int_x^0 |V(t)| \cdot \|B(\cdot,t)\|_{L^1}dt + 2\int_0^\infty (t^2+x^2)|V(t)| \cdot \|B(\cdot,t)\|_{L^1}dt;$$

recalling (3.41), this implies

$$II \leq x^2 \int_x^0 |V(t)|\langle t\rangle dt \cdot C(\|V\|_{L^1_1}) + 2\int_0^\infty (t^2+x^2)|V(t)| \cdot C(\|V\|_{L^1_1}) \leq \langle x\rangle^2 C(\|V\|_{L^1_2}). \tag{3.48}$$

For the last term we proceed as follows: we write

$$III = \int_x^\infty t|V(t)|\|C_+(\cdot,t)\|_{L^1}dt - x\int_x^\infty |V(t)|\|C_+(\cdot,t)\|_{L^1}dt$$

and, as above, we remark that the first integral increases if we replace the lower integration limit with 0:

$$\int_x^\infty t|V(t)|\|C_+(\cdot,t)\|_{L^1}dt \leq \int_0^\infty t|V(t)|\|C_+(\cdot,t)\|_{L^1}dt \leq C(\|V\|_{L^1_2})$$

where we have used the bound for $x > 0$ already proved. Thus we have

$$III \leq C(\|V\|_{L^1_2}) + \langle x\rangle \int_x^\infty \langle t\rangle^2|V(t)| \cdot \langle t\rangle^{-2}\|C_+(\cdot,t)\|_{L^1}dt. \tag{3.49}$$

In conclusion we have proved that

$$\langle x\rangle^{-2}\|C_+(\cdot,x)\|_{L^1} \leq C(\|V\|_{L^1_2}) + \int_x^\infty \langle t\rangle^2|V(t)| \cdot \langle t\rangle^{-2}\|C_+(\cdot,t)\|_{L^1}dt. \tag{3.50}$$

Applying as before Gronwall's lemma for $x < 0$, we conclude the proof of the Lemma. □

A useful consequence of (3.46) is an estimate of the Fourier transform of the functions

$$n_\pm(\lambda,x) = \frac{m_\pm(\lambda,x) - m_\pm(0,x)}{\lambda} \tag{3.51}$$

which are clearly related to the derivatives $\partial_\lambda m_\pm$; the usefulness of these quantities in the resonant case had already been remarked in [**5**].

COROLLARY 3.1. *Assume* $V \in L^1_2$*; then the functions* $\widetilde{C}_\pm(\xi,x) = \int_{\mathbb{R}} e^{-2i\lambda\xi} n_\pm(\lambda,x)d\lambda$ *satisfy*

$$\|\widetilde{C}_\pm(\cdot,x)\|_{L^1} \leq C \quad \text{for } \pm x \geq 0, \qquad \|\widetilde{C}_\pm(\cdot,x)\|_{L^1} \leq C\langle x\rangle^2 \quad \text{for } \pm x \leq 0 \tag{3.52}$$

for some constant C *depending on* $\|V\|_{L^1_2}$.

PROOF. Since $n_\pm(\lambda,x) = \int_0^1 \partial_\lambda m_\pm(\lambda s,x)ds$, we can write, using Fubini's theorem and the rescaling properties of the Fourier transform,

$$\widetilde{C}_\pm(\xi,x) = \int_0^1 \mathcal{F}_{\lambda\to\xi}\left(\partial_\lambda m_\pm(\lambda s,x)\right) ds = \int_0^1 s^{-1} C_\pm(\xi/s,x)ds.$$

The integral Minkowski inequality now gives

$$\|\widetilde{C}_\pm(\cdot,x)\|_{L^1} \leq \int_0^1 s^{-1}\|C_\pm(\cdot/s,x)\|_{L^1}ds \equiv \int_0^1 \|C_\pm(\cdot,x)\|_{L^1}ds \equiv \|C_\pm(\cdot,x)\|_{L^1}$$

and by (3.46) the proof is concluded. □

We conclude this section by studying the Fourier properties of the Wronskian $W(\lambda)$ defined in (3.4), which can be equivalently written

$$W(\lambda) = m_+(\lambda,0)\partial_x m_-(\lambda,0) - \partial_x m_+(\lambda,0)m_-(\lambda,0) - 2i\lambda m_+(\lambda,0)m_-(\lambda,0).$$

Notice that the following result is also proved in [**49**] by partly different arguments.

LEMMA 3.5. *Let* $\chi(\lambda) \in C_0^\infty(\mathbb{R})$ *be a smooth cutoff. If* $V \in L^1_1(\mathbb{R})$ *and* $W(0) \neq 0$ *then*

$$\mathcal{F}\left(\frac{\chi(\lambda)}{W(\lambda)}\right) \in L^1(\mathbb{R}). \tag{3.53}$$

On the other hand, if $V \in L^1_2(\mathbb{R})$ *and* $W(0) = 0$ *then*

$$\mathcal{F}\left(\frac{\chi(\lambda)\lambda}{W(\lambda)}\right) \in L^1(\mathbb{R}). \tag{3.54}$$

PROOF. Let $\chi_1 \in C_0^\infty(\mathbb{R})$ be a second cutoff such that $\chi_1 \equiv 1$ on the support of χ. By the Deift-Trubowitz estimates (see Lemma 3.2) we know that both $m_\pm(\lambda,0) - 1$ and $\partial_x m_\pm(\lambda,0)$ have Fourier transform in L^1; then writing

$$\chi_1 W(\lambda) \equiv \chi_1 m_+(\lambda,0)\partial_x m_-(\lambda,0) - \chi_1\partial_x m_+(\lambda,0)m_-(\lambda,0) - 2i\lambda\chi_1 m_+(\lambda,0)m_-(\lambda,0)$$

we see that $\chi_1 W$ can be written as a sum of products in which each factor has a Fourier transform in L^1, and we conclude that $\chi_1 W$ has Fourier transform in L^1.

Recall now that by Wiener's Lemma, if a function $a(\lambda)$ does not vanish on the support of $b(\lambda)$ and both $\hat{a}, \hat{b} \in L^1$, we have also $\mathcal{F}(b/a) \in L^1$. This implies that

$$\left\|\mathcal{F}_{\lambda\to\xi}\left(\frac{\chi(\lambda)}{W(\lambda)}\right)\right\|_{L^1_\xi} \equiv \left\|\mathcal{F}_{\lambda\to\xi}\left(\frac{\chi(\lambda)}{\chi_1(\lambda)W(\lambda)}\right)\right\|_{L^1_\xi} < \infty.$$

Consider now the resonant case with $V \in L^1_2$. Using the functions $n_\pm$ defined in (3.51) we can rewrite W as follows:

$$W(\lambda) = \lambda n_+(\lambda,0)\partial_x m_-(\lambda,0) + \lambda m_+(0,0)\partial_x n_-(\lambda,0) + m_+(0,0)\partial_x m_-(0,0)$$
$$- \lambda n_-(\lambda,0)\partial_x m_+(\lambda,0) - \lambda m_-(0,0)\partial_x n_+(\lambda,0) - m_-(0,0)\partial_x m_+(0,0) - 2\lambda m_+(\lambda,0)m_-(\lambda,0);$$

from this formula and the assumption $W(0) = 0$ we see that the term $m_+(0,0)\partial_x m_-(0,0) - m_-(0,0)\partial_x m_+(0,0)$ must vanish, hence we obtain

$$\begin{aligned}\frac{W(\lambda)}{\lambda} =& n_+(\lambda,0)\partial_x m_-(\lambda,0) + m_+(0,0)\partial_x n_-(\lambda,0) - \\ & - n_-(\lambda,0)\partial_x m_+(\lambda,0) - m_-(0,0)\partial_x n_+(\lambda,0) - 2m_+(\lambda,0)m_-(\lambda,0).\end{aligned} \tag{3.55}$$

We know already that the functions $m_\pm(\lambda,0) - 1$, $\partial_x m_\pm(\lambda,0)$ and $n_\pm(\lambda,0)$ have Fourier transform in L^1; this follows as above from the Deift-Trubowitz estimate and from our Corollary 3.1 (see (3.52)). We can show that also $\partial_x n_\pm(\lambda,0)$ have the same property. Indeed, write $\partial_x n_\pm(\lambda,x) = \int_0^1 \partial_x \partial_\lambda m_\pm(\lambda s, x) ds$; by Fubini's theorem and the rescaling properties of the Fourier transform and the integral Minkowski inequality we have

$$\|\mathcal{F}_{\lambda\to\xi}\left(\partial_x n_\pm(\lambda,0)\right)\|_{L^1_\xi} = \left\|\int_0^1 2i\xi s^{-1}\partial_x B_\pm(\xi/s,0)ds\right\|_{L^1_\xi} \leq 2\int_0^1 \|\xi s^{-1}\partial_x B_\pm(\xi/s,0)\|_{L^1_\xi} ds$$

whence

$$\|\mathcal{F}_{\lambda\to\xi}\left(\partial_x n_\pm(\lambda,0)\right)\|_{L^1_\xi} \leq \|\xi\partial_x B_\pm(\xi,0)\|_{L^1_\xi}. \tag{3.56}$$

Recalling now the Deift-Trubowitz estimate (3.38), we have immediately

$$|\xi\partial_x B_\pm(\xi,0)| \leq C|\xi|\cdot\left[\eta(\xi) + |V(\xi)|\right] \quad \Longrightarrow \quad \|\xi\partial_x B_\pm(\xi,0)\|_{L^1_\xi} \leq C\|V\|_{L^1_2}$$

and this proves that the Fourier transform of $\partial_x n_\pm(\lambda,0)$ belongs to $L^1(\mathbb{R})$.

Now, coming back to (3.55), and choosing a cutoff χ_1 as above, we see that $\chi_1(\lambda)W(\lambda)/\lambda$ can be written as a sum of products of functions with Fourier transform in L^1 and hence it also has Fourier transform in L^1; applying Wiener's Lemma exactly as before we conclude the proof. □

4. The low energy analysis

In this section we shall study the low energy part of the wave operator W_+; the estimate for W_- is completely analogous. By the stationary representation formula (see e.g. [**115**]), given a cutoff $\Phi(\lambda^2)$ supported near zero, we can represent the low energy part of W_+ as follows:

$$W_+\Phi(H_0)g = \Phi(H_0)g - \frac{2}{\pi}\int_0^{+\infty} R_V(\lambda^2 - i0)V\,\Im R_0(\lambda^2 + i0)\lambda\Phi(\lambda^2)g\,d\lambda. \tag{3.57}$$

Thus it is sufficient to study the boundedness in L^p of the operator

$$Ag := \int_0^{+\infty} R_V(\lambda^2 - i0)V\,\Im R_0(\lambda^2 + i0)\lambda\chi(\lambda)g\,d\lambda \tag{3.58}$$

for an even cutoff function $\chi(\lambda) = \Phi_0(\lambda^2) \in C_0^\infty(\mathbb{R})$.

As remarked in the Introduction, an $L^\infty - L^\infty$ estimate will be impossible in general, owing to the presence of a Hilbert transform term in the wave operator. We recall that the *Hilbert transform* on $\mathbb{R}$ is the operator

$$\mathcal{H}g(y) = \frac{1}{\pi} V.P. \int_{\mathbb{R}} \frac{g(s)}{y-s} ds \equiv \frac{1}{2\pi i} \int e^{iy\lambda} \frac{\lambda}{|\lambda|} \widehat{g}(\lambda) d\lambda.$$

We also recall that $\mathcal{H}^2 = -1$, and that $\mathcal{H}$ is a bounded operator on L^p for all $1 < p < \infty$, but not on L^1 and on L^∞.

In order to state a simple but useful interpolation lemma we introduce the space L^∞_0 of bounded functions vanishing at infinity (i.e., $g \to 0$ as $|x| \to \infty$), with the L^∞ norm, and

$$L^p_{\mathcal{H}} = \{g \in L^p \colon \mathcal{H}g \in L^p\}, \qquad \|g\|_{L^p_{\mathcal{H}}} = \|g\|_{L^p} + \|\mathcal{H}g\|_{L^p}. \tag{3.59}$$

Notice that the last definition is relevant only when $p = 1$ or $p = \infty$, since we have otherwise $L^p_{\mathcal{H}} \simeq L^p$ for $1 < p < \infty$. Our interpolation lemma is then the following:

LEMMA 3.6. *Let T be a bounded operator on L^2, and assume that*

$$\|Tg\|_{L^\infty} + \|T^*g\|_{L^\infty} \leq C\|g\|_{L^\infty} + C\|\mathcal{H}g\|_{L^\infty}, \qquad \forall g \in C^\infty_0. \tag{3.60}$$

Then T and T^* can be extended to bounded operators on L^p for all $1 < p < \infty$.

PROOF. The complex interpolate $X = [L^p, L^\infty_0 \cap L^\infty_{\mathcal{H}}]_\theta$ coincides with L^{p_θ} as expected:

$$X = [L^p, L^\infty_0 \cap L^\infty_{\mathcal{H}}]_\theta = L^{p_\theta}, \qquad \frac{1}{p_\theta} = \frac{1-\theta}{p}, \quad 0 < \theta < 1, \quad 1 < p < \infty. \tag{3.61}$$

To prove this, first of all notice that the inclusions $C^\infty_0 \subseteq L^p \subseteq L^p$ and $C^\infty_0 \subseteq L^\infty_0 \cap L^\infty_{\mathcal{H}} \subseteq L^\infty$ imply that $C^\infty_0 \subseteq X \subseteq L^{p_\theta}$ as sets. Moreover, the (bounded) injection operator $i : L^p \to L^p$ and $i : L^\infty_0 \cap L^\infty_{\mathcal{H}} \to L^\infty$ is also bounded from X to L^{p_θ} by complex interpolation with norm ≤ 1, i.e., $\|f\|_{L^{p_\theta}} \leq \|f\|_X$. Finally, given any compact set K, denote by $L^p(K)$ the subspace of L^p of functions with support contained in K; if we consider the injection operator

$$i : L^p(K) \cap L^p_{\mathcal{H}} \equiv L^p(K) \to L^p \cap L^p_{\mathcal{H}} \equiv L^p \quad \text{and} \quad i : L^\infty(K) \cap L^\infty_{\mathcal{H}} \to L^\infty_0 \cap L^\infty_{\mathcal{H}}$$

and we use again complex interpolation, we obtain that the injection $i : L^{p_\theta}(K) \to X$ is bounded with norm ≤ 1. Summing up, we have proved that $\|f\|_X = \|f\|_{L^{p_\theta}}$ for all functions $f \in L^{p_\theta}$ with compact support. Since X contains C^∞_0, this proves the claim that $X = L^{p_\theta}$ as Banach spaces.

Now, by a density argument we see that (3.60) implies that T, T^* can be extended to bounded operators from $L^\infty_0 \cap L^\infty_{\mathcal{H}}$ to L^∞, and on the other hand they are bounded on L^2 by assumption. Using (3.61), by interpolation we obtain that T, T^* are bounded on all L^p for $2 \leq p < \infty$, and by duality we conclude the proof. ☐

REMARK 3.8. In the endpoint case $p = \infty$ we can modestly improve (3.60) to

$$\|Tg\|_{L^\infty} \leq C\|g\|_{L^\infty} + C\|\mathcal{H}g\|_{L^\infty}, \qquad \forall g \in L^\infty \cap L^\infty_{\mathcal{H}} \cap L^p \tag{3.62}$$

for some $p < \infty$; this follows immediately by a density argument. Moreover, in the opposite endpoint $p = 1$, by duality, we obtain that $\|Tg\|_{L^1 + L^1_{\mathcal{H}}} \leq C\|g\|_{L^1}$ where $L^1 + L^1_{\mathcal{H}}$ is the Banach space with norm $\|g\| = \inf\{\|g_1\|_{L^1} + \|g_2\|_{L^1_{\mathcal{H}}}\}$, $g = g_1 + g_2$, $g_1 \in L^1$, $g_2 \in L^1_{\mathcal{H}}$.

We are now ready to prove our estimate of the low frequency part of the wave operator:

LEMMA 3.7. *Assume $V \in L^1_1$ and the nonresonant condition $W(0) \neq 0$ is satisfied. Let $\Phi(\lambda^2)$ be a smooth compactly supported cutoff function. Then the low energy parts of the wave operators $W_\pm$ satisfy the estimates*

$$\|W_\pm \Phi(H_0) g\|_{L^\infty} \leq C\left(\|g\|_{L^\infty} + \|\mathcal{H}g\|_{L^\infty}\right) \qquad \forall g \in L^1 \cap L^\infty \cap L^\infty_{\mathcal{H}} \tag{3.63}$$

and hence can be extended to bounded operatos on L^p, for all $1 < p < \infty$. The same properties hold for the conjugate operators $\Phi(H_0)W^*_\pm$.

PROOF. The proof for the operators $W_\pm$ and $W^*_\pm$ is completely analogous, hence we shall focus on the estimate for W_+. By Lemma 3.6, it is sufficient to prove that $W_+\Phi(H_0)$ satisfies (3.63); moreover, using the stationary representation formula (3.57), the problem is reduced to estimating the operator A defined by (3.58).

By the explicit expression of the kernel of R_V in terms of the Jost functions (3.34), we can split A as $A = A_1 + A_2$ where (forgetting constants)

$$A_1 g(x) = \int_0^{+\infty} d\lambda \int_{x<y} dy \frac{f_+(-\lambda, y) f_-(-\lambda, x)}{W(-\lambda)} V(y) \lambda \chi(\lambda) \Im R_0(\lambda^2 + i0) g(y) \tag{3.64}$$

while A_2 is given by a symmetric formula. In the following we shall estimate the operator A_1; the proof for A_2 is completely analogous.

By the relations $f_\pm(\lambda, x) = e^{\pm i\lambda x} m_\pm(\lambda, x)$ and

$$m_\pm(-\lambda, x) = \overline{m_\pm(\lambda, x)}, \qquad W(-\lambda) = \overline{W(\lambda)}$$

(see. e.g. [**36**]), we have

$$A_1 g(x) = \int_0^{+\infty} d\lambda \int_{x<y} dy \frac{\overline{m_+(\lambda, y) m_-(\lambda, x)}}{\overline{W(\lambda)}} V(y) e^{i\lambda(x-y)} \lambda \chi(\lambda) \Im R_0(\lambda^2 + i0) g(y). \tag{3.65}$$

By Fubini's Theorem we can exchange the order of integration and rewrite (3.65) as follows:

$$A_1 g(x) = \int_{x<y} \mathcal{F}_{\lambda \to \xi} \left(\frac{\overline{m_+(\lambda, y) m_-(\lambda, x)}}{\overline{W(\lambda)}} \chi(\lambda) \mathbf{1}_{(0,+\infty)}(\lambda) \Im R_0(\lambda^2 + i0) g(y) \right) \Bigg|_{\xi = x-y} V(y) dy \tag{3.66}$$

where $\mathcal{F}$ denotes the standard Fourier transform from the λ to the ξ variable and $\mathbf{1}_{(0,+\infty)}$ is the characteristic function of the half line $(0, +\infty)$.

Now choose a C_0^∞ cutoff function $\psi(\lambda)$ such that $\psi \equiv 1$ on $\operatorname{supp} \chi$; then the function

$$G(\lambda, x, y) = \frac{\overline{m_+(\lambda, y) m_-(\lambda, x)}}{\overline{W(\lambda)}} \chi(\lambda) \mathbf{1}_{(0,+\infty)}(\lambda) \Im R_0(\lambda^2 + i0) g(y)$$

can be written as a product

$$G(\lambda, x, y) = F_1(\lambda, y) F_2(\lambda, x) F_3(\lambda) F_4(\lambda, y) \tag{3.67}$$

where

$$F_1(\lambda, y) = \overline{m_+(\lambda, y)} \psi(\lambda), \qquad F_2(\lambda, x) = \overline{m_-(\lambda, x)} \psi(\lambda), \qquad F_3(\lambda) = \frac{\psi(\lambda)}{\overline{W(\lambda)}}$$

and

$$F_4(\lambda, y) = \chi(\lambda) \mathbf{1}_{(0,+\infty)}(\lambda) \Im R_0(\lambda^2 + i0) g(y).$$

We are interested in the Fourier transform of G with respect to λ; this can be written as the convolution of the transforms $\widehat{F}_j$, $j = 1, 2, 3, 4$.

By Lemma 3.5 (see (3.53)) we already know that

$$\|\widehat{F}_3(\xi)\|_{L^1_\xi} = C_0 < \infty. \tag{3.68}$$

Consider now $\widehat{F}_1(\xi, y)$, which can be written

$$\widehat{F}_1(\xi, y) = \mathcal{F}((\overline{m_+(\lambda, y)} - 1)\psi_1 + \psi_1) = \overline{B_+(-\xi/2, y)} * \widehat{\psi}_1 + \widehat{\psi}_1$$

(the inessential factor 1/2 comes from the nonstandard Fourier transform used in Definition (3.36), and the minus sign from the conjugation). Recalling Lemma 3.3, we get

$$\|\widehat{F}_1(\cdot, y)\|_{L^1} \leq \begin{cases} C & \text{for } y \geq 0 \\ C\langle y\rangle & \text{for } y \leq 0 \end{cases} \qquad \|\widehat{F}_2(\cdot, x)\|_{L^1} \leq \begin{cases} C & \text{for } x \leq 0 \\ C\langle x\rangle & \text{for } x \geq 0 \end{cases}$$

for some C depending on $\|V\|_{L^1_1}$. Recalling that in (3.66) we have $x < y$, we can write

$$\|\mathcal{F}(F_1F_2)\|_{L^1_\xi} \leq \|\widehat{F}_1(\cdot, y)\|_{L^1}\|\widehat{F}_2(\cdot, x)\|_{L^1} \leq \begin{cases} C\langle y\rangle & \text{for } x < y < 0 \\ C & \text{for } x < 0 < y \\ C\langle x\rangle \leq C\langle y\rangle & \text{for } 0 < x < y \end{cases}$$

and in conclusion

$$\|\mathcal{F}(F_1F_2)\|_{L^1_\xi} \leq C(\|V\|_{L^1_1}) \cdot \langle y\rangle. \tag{3.69}$$

Coming back to $G(\lambda, x, y)$, if we put together (3.68) and (3.69) and we use Young's inequality, we have proved that, for $x < y$,

$$\|\widehat{G}(\cdot, x, y)\|_{L^\infty} \leq C(\|V\|_{L^1_1}) \cdot \langle y\rangle \cdot \|\widehat{F}_4(\cdot, y)\|_{L^\infty}. \tag{3.70}$$

It remains to estimate

$$\|\widehat{F}_4(\cdot, y)\|_{L^\infty} \equiv \sup_\xi \left|\mathcal{F}_{\lambda\to\xi}\left(\chi(\lambda)\mathbf{1}_{(0,+\infty)}(\lambda)\lambda\Im R_0(\lambda^2 + i0)g(y)\right)\right|.$$

We have

$$\begin{aligned} \mathcal{F}\left(\chi(\lambda)\mathbf{1}_{(0,+\infty)}(\lambda)\lambda\Im R_0(\lambda^2 + i0)g(y)\right) &= \int_0^\infty e^{i\lambda\xi}\lambda\chi(\lambda)\Im R_0(\lambda^2 + i0)g(y)d\lambda \\ &\equiv Ce^{i\xi\sqrt{H_0}}\chi(\sqrt{H_0})g \end{aligned}$$

by the spectral theorem. Now we remark that the function

$$U(\xi, y) = e^{i\xi\sqrt{H_0}}\chi(\sqrt{H_0})g$$

is a solution of the one dimensional wave equation

$$U_{\xi\xi} + H_0U \equiv U_{\xi\xi} - U_{yy} = 0,$$

with initial data

$$U(0, y) = U_0(y) = \chi(\sqrt{H_0})g, \qquad U_\xi(0, y) = U_1(y) = i\sqrt{H_0}\chi(\sqrt{H_0})g.$$

By the explicit representation formula of the solution to the wave equation we have then

$$U(\xi, y) = \frac{U_0(\xi + y) + U_0(\xi - y)}{2} + \frac{1}{2}\int_{\xi-y}^{\xi+y} U_1(\sigma)d\sigma.$$

The first term is easy to bound:

$$\left|\frac{U_0(\xi+y)+U_0(\xi-y)}{2}\right| \le \|U_0\|_{L^\infty} = \|\chi(\sqrt{H_0})g\|_{L^\infty} \le C\|g\|_{L^\infty} \tag{3.71}$$

since $\chi(\sqrt{H_0})$ is bounded on L^∞ as it is well known. On the other hand, we can write

$$U_1(y) = i\sqrt{H_0}\chi(\sqrt{H_0})g(y) = \int e^{i\lambda y}|\lambda|\chi(\lambda)\widehat{g}(\lambda)d\lambda \equiv \int e^{i\lambda y}(-i\lambda)\chi(\lambda)\frac{i\lambda}{|\lambda|}\widehat{g}(\lambda)d\lambda$$

and this implies, apart from a constant,

$$U_1(y) = \frac{d}{dy}\chi(\sqrt{H_0})\mathcal{H}g \quad \Longrightarrow \quad \int_{\xi-y}^{\xi+y} U_1(\sigma)d\sigma = \chi(\sqrt{H_0})\mathcal{H}g(\xi+y) - \chi(\sqrt{H_0})\mathcal{H}g(\xi-y).$$

In conclusion

$$\left|\frac{1}{2}\int_{\xi-y}^{\xi+y} U_1(\sigma)d\sigma\right| \le \|\chi(\sqrt{H_0})\mathcal{H}g\|_{L^\infty} \le C\|\mathcal{H}g\|_{L^\infty}$$

and summing up we have proved that

$$\|\widehat{F}_4\|_{L^\infty_{\xi,y}} \le C\|U\|_{L^\infty_{\xi,y}} \le C\left(\|g\|_{L^\infty} + \|\mathcal{H}g\|_{L^\infty}\right). \tag{3.72}$$

By (3.66), (3.70) and (3.72) we finally obtain

$$\|A_1 g\|_{L^\infty} \le C(\|V\|_{L^1_1}) \cdot \left(\|g\|_{L^\infty} + \|\mathcal{H}g\|_{L^\infty}\right).$$

The operator A_2 can be estimated in a similar way, and this concludes the proof of the Lemma. □

We pass to the analysis of the resonant case $W(0) = 0$.

Lemma 3.8. *Assume $V \in L^1_2$ and we are in the resonant case $W(0) = 0$. Let $\Phi(\lambda^2)$ be a smooth compactly supported cutoff function. Then the following estimate holds:*

$$\|W_+\Phi(H_0)g\|_{L^\infty} \le C\left(\|g\|_{L^\infty} + \|\mathcal{H}g\|_{L^\infty}\right) \qquad \forall g \in L^1 \cap L^\infty \cap L^\infty_{\mathcal{H}} \tag{3.73}$$

where $\mathcal{H}$ is the Hilbert transform on $\mathbb{R}$, and hence can be extended to bounded operators on L^p, for all $1 < p < \infty$. The same estimate holds for the conjugate operators $\Phi(H_0)W_\pm^*$.

Proof. As in the proof of Lemma 3.7, the problem is reduced to estimating the L^∞ norm of $Ag = A_1 g + A_2 g$ where $A_j g$ are defined as above (see (3.65)). The new difficulty now is of course the denominator $W(\lambda)$ which vanishes at $\lambda = 0$. Thus we decompose Ag into several terms:

$$Ag = I_1 + I_2 + II_1 + II_2 + III_1 + III_2$$

where, recalling the notation (3.51).

$$I_1 = \int_0^{+\infty} d\lambda \int_{x>y} dy \frac{-\lambda}{W(-\lambda)} n_+(-\lambda, y) m_-(-\lambda, x) e^{i\lambda(y-x)} V(y)\lambda\chi(\lambda)\Im R_0(\lambda^2 + i0)g(y), \tag{3.74}$$

$$II_1 = \int_0^{+\infty} d\lambda \int_{x>y} dy \frac{-\lambda}{W(-\lambda)} m_+(0, y) n_-(-\lambda, x) e^{i\lambda(y-x)} V(y)\lambda\chi(\lambda)\Im R_0(\lambda^2 + i0)g(y), \tag{3.75}$$

$$III_1 = \int_0^{+\infty} d\lambda \int_{x>y} dy \frac{m_+(0, y) m_-(0, x)}{W(-\lambda)} V(y) e^{i\lambda(y-x)} \lambda\chi(\lambda)\Im R_0(\lambda^2 + i0)g(y). \tag{3.76}$$

while as usual, I_2, II_2 and III_2 have symmetric expressions with x and y interchanged. We notice the expression for III_2 which will be necessary in the following:

$$III_2 = \int_0^{+\infty} d\lambda \int_{x<y} dy \frac{m_+(0,x)m_-(0,y)}{W(-\lambda)} V(y) e^{i\lambda(x-y)} \lambda\chi(\lambda)\Im R_0(\lambda^2+i0)g(y) \tag{3.77}$$

Since $W(0) = 0$, we know that for $\lambda = 0$ the Jost functions $f_+(0,x) \equiv m_+(0,x)$ and $f_-(0,x) \equiv m_-(0,x)$ are linearly dependent, i.e.,

$$m_-(0,x) = c_0 \cdot m_+(0,x) \tag{3.78}$$

for some constant $c_0 \neq 0$. Moreover, by definition $m_\pm(0,x) \to 1$ as $\pm x \to \infty$, and together with (3.78) this implies that $m_\pm(0,x)$ are bounded on $\mathbb{R}$:

$$|m_\pm(0,x)| \leq c_1, \qquad x \in \mathbb{R}. \tag{3.79}$$

Finally, when $W(0) = 0$ we have (see e.g. [**5**])

$$\int_{-\infty}^{+\infty} V(y) m_\pm(0,y) dy = 0. \tag{3.80}$$

The terms of type I and II are handled in a way very similar to the proof of Lemma 3.7. In order to estimate the term I_1, we write it as

$$I_1 = \int_{x<y} \mathcal{F}_{\lambda\to\xi}\left(G(\lambda,x,y)\right)|_{\xi=y-x} V(y)dy, \qquad G(\lambda,x,y) = F_1(\lambda,y)F_2(\lambda,x)F_3(\lambda)F_4(\lambda,y),$$

where, choosing a $C_0^\infty(\mathbb{R})$ cutoff function ψ such that $\psi \equiv 1$ on $\operatorname{supp}\chi$,

$$F_1(\lambda,y) = n_+(-\lambda,y), \qquad F_2(\lambda,x) = m_-(-\lambda,x)\psi(\lambda), \qquad F_3(\lambda) = \frac{\psi(\lambda)\lambda}{W(-\lambda)}$$

and

$$F_4(\lambda,y) = \lambda\chi(\lambda)\mathbf{1}_{(0,+\infty)}(\lambda)\Im R_0(\lambda^2+i0)g(y).$$

Then we have

$$\sup_{x<y} \|\widehat{G}(\cdot,x,y)\|_{L^\infty} \leq \sup_{x<y}\left(\|\widehat{F}_1(\cdot,y)\|_{L^1} \cdot \|\widehat{F}_2(\cdot,x)\|_{L^1} \cdot \|\widehat{F}_3\|_{L^1} \cdot \|\widehat{F}_4(\cdot,y)\|_{L^\infty}\right)$$

Using Lemma 3.5 we see that $\|\widehat{F}_3\|_{L^1} = C_0 < \infty$, and by Lemma 3.2 and Corollary 3.1 we obtain as before (by considering the three cases $x<y<0$, $x<0<y$ and $0<x<y$)

$$\|\widehat{F}_1(\cdot,y)\|_{L^1} \cdot \|\widehat{F}_2(\cdot,x)\|_{L^1} \leq C(\|V\|_{L^1_2}) \cdot \langle y\rangle^2 \qquad \text{for } x<y.$$

Thus we arrive at

$$\|\widehat{G}(\cdot,x,y)\|_{L^\infty} \leq C(\|V\|_{L^1_2}) \cdot \langle y\rangle^2 \cdot \|\widehat{F}_4(\cdot,y)\|_{L^\infty}$$

and the remaining term $\|\widehat{F}_4(\cdot,y)\|_{L^\infty}$ has already been estimated in (3.72). Summing up we have proved that

$$|I_1| \leq C(\|V\|_{L^1_2}) \cdot \left(\|g\|_{L^\infty} + \|\mathcal{H}g\|_{L^\infty}\right).$$

The estimate of I_2 is completely analogous; the estimate of the terms II_1 and II_2 is even easier, keeping into account that the functions $m_\pm(0,x)$ are bounded on $\mathbb{R}$ (see (3.79)).

Consider now the more delicate terms III_1, III_2. Since $m_-(0,x) = c_0 \cdot m_+(0,x)$ we can put the two integrals back together as follows:

$$III_1 + III_2 = c_0 \int_0^{+\infty} d\lambda \int dy \frac{m_+(0,y)m_+(0,x)}{W(-\lambda)} V(y) e^{i\lambda|y-x|} \lambda\chi(\lambda) \Im R_0(\lambda^2 + i0) g(y).$$

We decompose this integral in a different way:

$$III_1 + III_2 = c_0 IV_1 + c_0 IV_2$$

where

$$IV_1 = \int_0^{+\infty} d\lambda \int dy \frac{m_+(0,y)m_+(0,x)}{W(-\lambda)} V(y) \big[e^{i\lambda|y-x|} - e^{i\lambda|x|}\big] \lambda\chi(\lambda) \Im R_0(\lambda^2 + i0) g(y) \tag{3.81}$$

and

$$IV_2 = \int_0^{+\infty} d\lambda \int dy \frac{m_+(0,y)m_+(0,x)}{W(-\lambda)} V(y) e^{i\lambda|x|} \lambda\chi(\lambda) \Im R_0(\lambda^2 + i0) g(y). \tag{3.82}$$

Using the identity

$$e^{i\lambda|y-x|} - e^{i\lambda|x|} = \int_0^1 e^{i\lambda(s|x-y|+(1-s)|x|)} ds \cdot i\lambda \cdot (|x-y| - |x|)$$

and Fubini's theorem we can rewrite IV_1 as follows:

$$IV_1 = \int_0^1 ds \int \mathcal{F}_{\lambda\to\xi} \left(\frac{\lambda}{W(-\lambda)} \lambda\chi(\lambda) \mathbf{1}_{(0,+\infty)}(\lambda) \Im R_0(\lambda^2 + i0) g(y) \right) \bigg|_{\xi = s|x-y|+(1-s)|x|} K \, dy$$

where

$$K = K(x,y) = i m_+(0,y) m_+(0,x) V(y) \left(|x-y| - |x|\right);$$

notice that

$$|K(x,y)| \leq C|y| \cdot |V(y)| \tag{3.83}$$

by (3.79). At this point, we can proceed as above using Lemma 3.5 and (3.72) to obtain

$$\left\| \mathcal{F}_{\lambda\to\xi} \left(\frac{\lambda}{W(-\lambda)} \lambda\chi(\lambda) \mathbf{1}_{(0,+\infty)}(\lambda) \Im R_0(\lambda^2 + i0) g(y) \right) \right\|_{L^\infty_\xi} \leq C(\|V\|_{L^1_2}) \cdot \left(\|g\|_{L^\infty} + \|\mathcal{H}g\|_{L^\infty}\right).$$

whence the estimate of IV_1 follows immediately.

To conclude the proof, it remains to estimate the term IV_2. By property (3.80) we have trivially

$$\int_0^{+\infty} d\lambda \int dy \frac{m_+(0,y)m_+(0,x)}{W(-\lambda)} V(y) e^{i\lambda|x|} \lambda\chi(\lambda) \Im R_0(\lambda^2 + i0) g(0) \equiv 0 \tag{3.84}$$

(indeed, in the inner integral only $V(y)$ and $m_+(0,y)$ depend on y). Thus we can subtract (3.84) from IV_2 and rewrite it in the form

$$IV_2 = \int_0^{+\infty} d\lambda \int dy \frac{m_+(0,y)m_+(0,x)}{W(-\lambda)} V(y) e^{i\lambda|x|} \lambda\chi(\lambda) \big[\Im R_0(\lambda^2 + i0) g(y) - \Im R_0(\lambda^2 + i0) g(0)\big].$$

We now use the elementary identity

$$\Im R_0(\lambda^2 + i0) g(y) - \Im R_0(\lambda^2 + i0) g(0) = \int_0^y \partial_s \left(\Im R_0(\lambda^2 + i0) g(s)\right) ds$$

and we obtain, after applying Fubini's theorem,

$$IV_2 = \int dy \int_0^y ds\, m_+(0,y)m_+(0,x)V(y) \int_0^{+\infty} e^{i\lambda|x|} \frac{\lambda\chi(\lambda)}{W(-\lambda)} \partial_s \left(\Im R_0(\lambda^2+i0)g(s)\right) d\lambda \tag{3.85}$$

Since

$$\Im R_0(\lambda^2+i0)g(x) = -\frac{1}{4\lambda}\int_{-\infty}^{\infty}(e^{i\lambda(x-y)}+e^{-i\lambda(x-y)})g(y)dy = -\frac{1}{4\lambda}(e^{i\lambda x}\widehat{g}(\lambda)+e^{-i\lambda x}\widehat{g}(-\lambda))$$

we can write for $\lambda > 0$

$$\partial_x \Im R_0(\lambda^2+i0)g(x) = -\frac{i}{4\lambda}\cdot\lambda\cdot\left(\frac{\lambda}{|\lambda|}e^{i\lambda x}\widehat{g}(\lambda)+\frac{-\lambda}{|-\lambda|}e^{-i\lambda x}\widehat{g}(-\lambda)\right)$$

and comparing the two identities and recalling the characterization of the Hilbert transform as a Fourier multiplier, we arrive at the formula

$$\partial_x \Im R_0(\lambda^2+i0)g(x) = C\lambda \Im R_0(\lambda^2+i0)\mathcal{H}g(x)$$

for a suitable constant C. Thus, by (3.85), we obtain (apart from a constant)

$$IV_2 = \int dy \int_0^y ds\, m_+(0,y)m_+(0,x)V(y) \int_0^{+\infty} e^{i\lambda|x|} \frac{\lambda}{W(-\lambda)}\lambda\chi(\lambda)\Im R_0(\lambda^2+i0)\mathcal{H}g(s)d\lambda$$

and this can be estimated exactly as the other terms considered above.

The proof is concluded. □

CHAPTER 4

Nonlinear Schrödinger equations

The final chapter of this thesis is devoted to some nonlinear applications of the dispersive-type estimates we introduced in the previous part of the work. A lot of physical models are usually described by nonlinear equations, and the first natural mathematical questions for these equations are related to existence and unicity of the solutions. Here we give two examples of nonlinear Schrödinger equations, and point out the attention on existence, unicity and blow-up of the solutions. The reference for the details of this chapter are [**38**] and [**40**].

1. NLS with time dependent coefficients

The interest for nonlinear Schrödinger equations finds motivations in the great number of physical models they describe. Here we consider the following problem:

$$\begin{cases} iu_t(t,x) + a(t)\Delta u(t,x) = \pm |u|^{\gamma-1}u(t,x) \\ u(0,x) = f(x), \end{cases} \tag{4.1}$$

where $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$, $f : \mathbb{R}^n \to \mathbb{C}$, $n \geq 2$, $\gamma \geq 1$ and $a(t) \geq 0$ is a real valued function. In analogy with the case $a(t) \equiv 1$, we frequently refer to *defocusing equation* when we take the positive sign at the right hand side of (4.1), otherwise we speak of *focusing equation*.

It is well known that the Cauchy problem (4.1), in the case $a(t) = 1$, is well posed in L^2 for $\gamma < 1 + \frac{4}{n}$ and in H^1 for $\gamma < 1 + \frac{4}{n-2}$; in the critical cases, global well-posedness holds, provided the initial data are small enough, while blow up phenomena can occur for the focusing equation when the initial datum is not assumed to be small (see [**20**], [**46**], [**63**]). If $a(t) > 0$ is strictly positive, the same techniques used for the constant case $a(t) = 1$ permit to obtain the same classical results.

On the other hand, the literature for the case in which $a(t) \geq 0$ can vanish is very small; this case can be regarded as a model for a nonrelativistic particle with variable speed of propagation (or equivalently with variable mass), and we are interested in the treatment of the case in which the speed can vanish (or equivalently the mass can explode). The motivation for our approach to this kind of problem is strictly mathematical: the question is how the degeneration of operator $a(t)\Delta$ can influence the critical exponents γ for the well-posedness of the Cauchy problem (4.1).

The analogous situation for the semilinear wave equation was treated in [**28**] first, and followed in [**29**], [**34**], [**39**], [**72**]: the main step is to reduce the problem to a wave equation with a time-dependent potential of order zero, with a suitable change in the time variable, and then find suitable a priori energy estimates for the solution. The algebraic structure of Schrödinger equation permits to regard equation (4.1) as a standard nonlinear Schrödinger equation, as it is shown in Section 3; this gives the possibility to use the linear technology for the free Schrödinger equation (namely Strichartz estimates), with some refinement, and it represents a strong difference with the case of the wave equation, in which the resulting potential does not allow to use classical linear estimates, because it is too singular.

In what follows, the coefficient a will satisfy the following assumptions:

$$a \in \mathcal{C}(\mathbb{R}), \qquad a \geq 0; \tag{4.2}$$

$$a \text{ vanish on a discrete set } Z = \{t_n\}_{n=0,1,,\dots} \subset \mathbb{R} \text{ with maximum order } \lambda > 0; \tag{4.3}$$

$$\exists \epsilon > 0 : a'(t) < 0, \text{ if } t_i - \epsilon < t < t_i, \text{ and } a'(t) > 0, \text{ if } t_i < t < t_i + \epsilon, i \in \mathbb{N}. \tag{4.4}$$

The small ϵ in (4.4) is uniform with respect to i; assumption (4.4) avoids the possibility that a could reach the zeroes with infinite oscillations. Moreover observe that the primitive function

$$A(t) = \int_0^t a(s)\, ds \tag{4.5}$$

is well defined at each $t \in \mathbb{R}$.

The simplest example is given by

$$a(t) = |t - t_0|^\lambda \tag{4.6}$$

with $\lambda > 0$.

The natural questions of well-posedness of the Cauchy problem (4.1) in L^2_x and H^1_x are solved in the main theorems of this paper. The sketch of the proofs is standard: a contraction argument based on Strichartz-type estimates permits to prove local (in time) well-posedness in the natural spaces, then the continuation to global solution is performed by means on the conservation laws of equation (4.1), that are investigated in Section 4. The lack of energy conservation for this equation (see Section 4) represent a strong difference with the case $a(t) \equiv 1$ and hence the proof of global well-posedness in $\mathcal{C}(\mathbb{R}; H^1)$ requires a more direct investigation on the lifespan of the local solutions (see Section 7).

In the special case when a is given by (4.6), the equation is invariant under the following scaling

$$v_\epsilon(t, x) = \epsilon^\alpha u(\epsilon^\beta t, \epsilon x), \tag{4.7}$$

with

$$\alpha = \frac{2}{(\lambda + 1)(\gamma - 1)}, \qquad \beta = \frac{2}{\lambda + 1}. \tag{4.8}$$

The mass is conserved by this scaling if

$$\gamma = 1 + \frac{4}{n(\lambda + 1)}, \tag{4.9}$$

while the H^1_x-norm is conserved for

$$\gamma = 1 + \frac{4}{(n - 2)(\lambda + 1)}. \tag{4.10}$$

This suggests that (4.9) and (4.10) should be the critical powers for the L^2 and H^1 well-posedness, respectively, and we can state our main results:

THEOREM 4.1. *Let $n \geq 2$, a satisfy assumptions* (4.2), (4.3), (4.4) *and let $\lambda > 0$ be the highest order of the zeroes of a. Assume that*

$$1 \leq \gamma < 1 + \frac{4}{n(\lambda+1)};$$

then, for each initial datum $f \in L^2$ there exists a unique global solution $u \in \mathcal{C}(\mathbb{R}; L^2)$ of the Cauchy problem (4.1).

In the critical case

$$\gamma = 1 + \frac{4}{n(\lambda+1)},$$

we have global well-posedness in $\mathcal{C}(\mathbb{R}; L^2)$, provided $\|f\|_{L^2} < \epsilon_0$, for some $\epsilon_0 > 0$ sufficiently small.

In the following Theorem we state the local H^1 well-posedness for equation (4.1):

THEOREM 4.2. *Let $n \geq 2$, a satisfy assumptions* (4.2), (4.3), (4.4) *and let $\lambda > 0$ be the highest order of the zeroes of a. Assume that*

$$1 \leq \gamma < 1 + \frac{4}{(n-2)(\lambda+1)};$$

then, for each initial datum $f \in H^1$ there exists a unique local solution $u \in \mathcal{C}(I; H^1)$ of the Cauchy problem (4.1), *where I is the greatest interval including the origin in which a does not vanish. In the critical case*

$$\gamma = 1 + \frac{4}{(n-2)(\lambda+1)}$$

we have local well-posedness in $\mathcal{C}(I; H^1)$, provided $\|f\|_{H^1} < \epsilon_0$, for some $\epsilon_0 > 0$ sufficiently small.

REMARK 4.1. The result of previous theorem includes also the case in which $a(0) = 0$; in that case the interval I contains the origin and does not contain any other zero point for the funcion a. We also remark that the subcritical result in Theorem 4.2 is already known (see for example Lemma 3.1 in [**21**]).

We conclude with the following global theorem (valid only in the sub-critical range):

THEOREM 4.3. *Let $n \geq 3$ and let a satisfy assumptions* (4.2), (4.3), (4.4) *and let $\lambda > 0$ be the highest order of the zeroes of a. Assume that*

$$1 \leq \gamma < 1 + \frac{4}{(n-2)(\lambda+1)}; \tag{4.11}$$

then, for each initial datum $f \in H^1$ there exists a unique global solution $u \in \mathcal{C}(\mathbb{R}; H^1)$ of the defocusing Cauchy problem (4.1).

In the case $n = 2$, the same result holds, under the more restrictive assumption

$$1 \leq \gamma \leq 1 + \frac{2(\lambda+1)}{\lambda}. \tag{4.12}$$

Let $n \geq 3$ and let us impose the above assumptions, with the more restrictive condition

$$1 \leq \gamma < 1 + \min\left\{\frac{4}{(n-2)(\lambda+1)}, 1 + \frac{4}{n}\right\}; \tag{4.13}$$

then, for each initial datum $f \in H^1$ *there exists a unique global solution* $u \in \mathcal{C}(\mathbb{R}; H^1)$ *of the focusing Cauchy problem* (4.1)*.*

In the case $n = 2$*, the same result holds, under the more restrictive assumption*

$$1 \leq \gamma < \min \left\{ 1 + \frac{2(\lambda+1)}{\lambda}, 1 + \frac{4}{n} \right\}. \tag{4.14}$$

REMARK 4.2. Observe that Theorems 4.1 and 4.3 (that are in fact valid also for $\lambda = 0$) cover the known results for $a \equiv 1$, also in the case $n = 2$ (in fact the critical exponent there is ∞).

The crucial role in the contraction argument for the local well-posedness is played by Strichartz estimates. We recall here some well known facts, starting with the following Definition:

DEFINITION 4.1. Let $n \geq 2$; a couple (p, q) is said to be *Schrödinger admissible* if the following conditions are satisfied:

$$\frac{2}{p} = \frac{n}{2} - \frac{n}{q}, \tag{4.15}$$

$$p > 2 \quad \text{if } n = 2, \qquad p \geq 2 \quad \text{if } n \geq 3. \tag{4.16}$$

The Schrödinger propagator $e^{it\Delta}$ is unitary on $L^2(\mathbb{R}^n)$ and satisfies the following Strichartz estimates

$$\|e^{it\Delta} f\|_{L^p(I;L^q)} \leq C\|f\|_{L^2}, \tag{4.17}$$

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s, \cdot)\, ds \right\|_{L^p(I;L^q)} \leq C\|F\|_{L^{\widetilde{p}'}(I;L^{\widetilde{q}'})}, \tag{4.18}$$

for all Schrödinger admissible couples (p, q), $(\widetilde{p}, \widetilde{q})$, with $\widetilde{p}'$ and $\widetilde{q}'$ the Hölder conjugates (see [**47**], [**66**]).

In order to use a contraction argument, we need some refinement of estimates (4.17) and (4.18). In the last years, several papers (see e.g. [**3**], [**79**]) treated the question of extending Strichartz estimates to the more general setting of Lorentz spaces $L^{p,r}$, that are usually also defined by real interpolation of L^p spaces (see [**10**]). In particular, it has been recently proved in [**3**] that, for any admissible pair $(p, q), (\widetilde{p}, \widetilde{q})$ and r with $2 < r \leq p$, we have

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s) ds \right\|_{L^{p,r}(I;L^q)} \leq C\|F\|_{L^{\widetilde{p}',2}(I;L^{\widetilde{q}'}) \cap L^2(I;L^{\frac{2n}{n+2},2})}. \tag{4.19}$$

The advantage of this estimate is that it reaches the endpoint, but the space $L^{\widetilde{p}',2}(I; L^{\widetilde{q}'}) \cap L^2(I; L^{\frac{2n}{n+2},2})$ is too "small" to be mapped in a generally bigger space and perform a contraction. Hence we need to prove directly a slightly different family of estimates, as stated in the following Theorem:

THEOREM 4.4. *For any Schrödinger admissible pair* $(p, q), (\widetilde{p}, \widetilde{q})$*, with* $2 < p, \widetilde{p} < \infty$*, and* r, s *satisfying*

$$\widetilde{p}' \leq s' < r \leq p, \qquad r \geq 2 \tag{4.20}$$

the following estimates hold:

$$\|e^{it\Delta}f\|_{L^{p,r}(I;L^{q,r})} \le C\|f\|_{L^2}, \tag{4.21}$$

$$\left\|\int_0^t e^{i(t-s)\Delta}F(s,\cdot)\,ds\right\|_{L^{p,r}(I;L^{q,r})} \le C\|F\|_{L^{\tilde{p}',s'}(I;L^{\tilde{q}',s'})}. \tag{4.22}$$

Remark 4.3. We find that estimates (4.21) and (4.22) are of independent interest. They involve Lorentz spaces, both t and x variables, and the nontrivial fact is that these spaces are not characterized as real interpolation of mixed $L^p_tL^q_x$ spaces, except for some very particular cases (see for example [**10**], [**26**]); our proof is based on direct dispersive methods. Moreover, we remark that the space at the right hand side in (4.22) is bigger than the space at the right hand side in (4.19), hence (4.22) is stronger in this sense. On the other hand, Theorem 4.4 does not comprehend the endpoint estimate, that actually is not needed in the proof of the previous well-posedness theorems.

The rest of the paper is organized as follows: Section 2 is devoted to the proof of Theorem 4.4; in Section 3, by algebraic computations, we transform equation (4.1) into a nonlinear Schroedinger equation with constant coefficients for the linear part; in Section 4 we investigate on the conservation laws of equation (4.1), and in the final part (Sections 5, 6 and 7) we prove Theorems 4.1, 4.2 and 4.3.

2. Strichartz estimates

This section is devoted to the proof of the generalized Strichartz estimates in Theorem 4.4. We prove (4.21) and (4.22) by a standard TT^* method based on dispersive estimates (see [**47**]). For simplicity of notations, we denote the Schrödinger group by $U(t) = e^{it\Delta}$. Let us start by the well known decay

$$\|U(t)f\|_{L^q} \le t^{-\frac{n}{2}+\frac{n}{q}}\|f\|_{L^{q'}}, \tag{4.23}$$

for $q \ge 2$, and $1/q + 1/q' = 1$. By interpolation of L^q-dispersive estimates, we immediately obtain their generalization to Lorentz spaces:

$$\|U(t)f\|_{L^{q,r}} \le t^{-\frac{n}{2}+\frac{n}{q}}\|f\|_{L^{q',r}}, \tag{4.24}$$

for all q, r such that

$$2 < q < \infty, \qquad r \ge 1.$$

Hence we can estimate

$$\begin{aligned}\left\|\int_0^t U(t-s)F(s,\cdot)ds\right\|_{L^{q,r}_x} &\le \int_0^t \|U(t-s)F(s,\cdot)\|_{L^{q,r}_x}\,ds\\ &\le \int_0^t |t-s|^{-\frac{n}{2}+\frac{n}{q}}\,\|F(s,\cdot)\|_{L^{q',r}_x}\,ds,\end{aligned} \tag{4.25}$$

by (4.24). Then, for the mixed norm we have:

$$\left\|\int_0^t U(t-s)F(s,\cdot)ds\right\|_{L^{p,r}(I;L^{q,r})} \le \left\||t|^{-\frac{n}{2}+\frac{n}{q}} * \|F(s,\cdot)\|_{L^{q',r}_x}\right\|_{L^{p,r}(I)}; \tag{4.26}$$

by Young inequality in Lorentz spaces (see [**80**]), observing that

$$|t|^{-\alpha} \in L^{\frac{1}{\alpha},\infty}, \qquad \alpha > 0,$$

we obtain

$$\left\| \int_0^t U(t-s)F(s,\cdot)ds \right\|_{L^{p,r}(I;L^{q,r})} \leq \|F\|_{L^{p',r}(I;L^{q',r})}, \tag{4.27}$$

under the admissibility conditions

$$\frac{2}{p} = \frac{n}{2} - \frac{n}{q}, \qquad 2 < p < \infty.$$

Observe that the space at the right hand side of (4.27) is not the dual of the space at the left hand side, in fact

$$(L^{p,r})' = L^{p',r'}.$$

We claim that estimate (4.27) can be improved, substituting r with r' at the right hand side. To this aim, let us consider the operator

$$TF = \int U(t)F(t,\cdot)dt,$$

and

$$\|TF\|_{L^2}^2 = \int \left(\int U(t)F(t,\cdot)dt \int U(-s)F(s,\cdot)ds \right) dx.$$

By Fubini's Theorem we integrate with respect to s first and we use Hölder inequality in Lorentz spaces in t and x to obtain

$$\|TF\|_{L^2}^2 \leq \|F\|_{L^{p',r'}(I;L^{q',r'})} \left\| \int U(t-s)F(s,\cdot)ds \right\|_{L^{p,r}(I;L^{q,r})}; \tag{4.28}$$

applying (4.27) we get

$$\|TF\|_{L^2}^2 \leq \|F\|_{L^{p',r'}(I;L^{q',r'})} \|F\|_{L^{p',r}(I;L^{q',r})}. \tag{4.29}$$

Now, since $r \geq 2$ (i.e. $r' \leq r$) by assumption (4.20), and since

$$L^{p,r} \subset L^{p,s}, \qquad r \leq s,$$

by (4.29) we conclude

$$\|TF\|_{L^2} \leq \|F\|_{L^{p',r'}(I;L^{q',r'})}, \tag{4.30}$$

for any Schrödinger admissible couple (p,q) with $2 < p < \infty$ and $r \geq 2$. Observe that $U(t)$ is the adjoint operator of T, hence the dual of estimate (4.30) reads

$$\|U(t)f\|_{L^{p,r}(I;L^{q,r})} \leq \|f\|_{L^2}, \tag{4.31}$$

and this proves (4.21). Mixing (4.21) and (4.30) we also obtain

$$\left\| \int U(t-s)F(s,\cdot)ds \right\|_{L^{p,r}(I;L^{q,r})} \leq \|F\|_{L^{\tilde{p}',s'}(I;L^{\tilde{q}',s'})}, \tag{4.32}$$

where the admissible pairs (p,q), $(\tilde{p},\tilde{q})$ and the Lorentz exponents $r, s \geq 2$ are unrelated. To conclude the proof of (4.22) it is sufficient to apply Proposition 2.1 in [**3**], that is the Lorentz generalization of the well known result by Christ and Kiselev in the L^p setting (see [**22**]).

3. Reduction to a nonlinear Schrödinger equation

As a first step we reduce equation (4.1) into a nonlinear Schrödinger equation with constant coefficients, by a suitable change of variables. Let us consider the function $A(t)$, defined in (4.5); by the assumptions (4.2), (4.3), (4.4), A is strictly increasing, then the inverse function is well-defined and continuous at each $t \in \mathbb{R}$; we will denote with $c(t)$ the inverse. Differentiating, we notice that c solves the Cauchy problem

$$\begin{cases} c'(t)a(c(t)) = 1 \\ c(0) = 0. \end{cases} \tag{4.33}$$

Then c is almost everywhere differentiable, except for the finite number of points whose images via c are the zeroes of a. In such points, the first derivative c' is singular, and we can easily estimate the order of the singularities, with the following Lemma, whose proof is an immediate consequence of assumptions (4.2), (4.3), (4.4):

LEMMA 4.1. *Let a satisfy assumptions* (4.2), (4.3) *and* (4.4) *and let λ be the highest order of the zeroes of a; let us consider the set*

$$Z := \{t \in \mathbb{R} : a(c(t)) = 0\}.$$

Then, c' has a pole in each $t_0 \in Z$, and the highest order of these singularities is $\lambda/(\lambda+1)$. In particular, for any real interval I, we have

$$c' \in L^{\frac{\lambda+1}{\lambda},\infty}(I). \tag{4.34}$$

Now, let u be a solution of (4.1), and let us define

$$w(t, x) := u(c(t), x); \tag{4.35}$$

we easily check that w solves

$$\begin{cases} iw_t + \Delta w = \pm c'(t)|w|^{\gamma-1}w \\ w(0, x) = f(x). \end{cases} \tag{4.36}$$

In what follows, we are going to look to solutions of (4.1) (or equivalently (4.36) of the integral form

$$u(t) = e^{iA(t)\Delta} f \pm i \int_0^t e^{iA(t-s)\Delta} |u|^{\gamma-1} u(s, \cdot)\, ds, \tag{4.37}$$

$$w(t) = e^{it\Delta} f \pm i \int_0^t e^{i(t-s)\Delta} c'(s)|w|^{\gamma-1} w(s)\, ds. \tag{4.38}$$

4. Conservation laws

The aim of this section is to present the main conservation properties of equation (4.1), to be applied in the passage from local to global well-posedness.

We start by the conservation of mass, i.e. the L^2_x-norm of the solution. By a formal computation, multiplying the equation in (4.1) by $\overline{u}$, integrating and taking the imaginary parts of the resulting identity we obtain

$$\|u(t)\|_{L^2} = \|u(0)\|_{L^2}. \tag{4.39}$$

Clearly, this computation makes sense if some regularity property is assumed for u: more precisely, if u is a, H^1 solution of (4.1), the above identity holds. In view to prove the well-posedness of (4.1) in $\mathcal{C}(\mathbb{R}; L^2)$, we need to extend the a priori conservation (4.39) to a class of solutions with less regularity. By the change of variables (4.35), it is sufficient to prove the property for any solution w of (4.36): we prove the following Lemma.

LEMMA 4.2. *Let $U(t) = e^{it\Delta}$, $f \in L^2$ and let w be a solution of the integral equation*

$$w(t) = U(t)f \pm i \int_0^t U(t-s)(c'(s)|w(s)|^{\gamma-1}w(s))ds, \tag{4.40}$$

with $w \in L^{p,r}(I; L^{q,r})$, for some Schrödinger admissible pair (p, q), and $r \geq 2$. Then the mass of w is conserved, i.e. (4.39) *holds.*

PROOF. The argument is identical to the one introduced by Ozawa in [**81**], Proposition 1; we write here the details for sake of completeness. Let us rewrite (4.40) as

$$U(-t)w(t) = f \pm i \int_0^t U(-s)(c'(s)|w(s)|^{\gamma-1}w(s))ds. \tag{4.41}$$

Since $U(t)$ is unitary on L^2 we have

$$\begin{aligned} \|w(t)\|_{L^2}^2 &= \|U(-t)w(t)\|_{L^2}^2 \\ &= \|f\|_{L^2}^2 \pm 2\Im\left(f, \int_0^t U(-s)c'(s)|w(s)|^{\gamma-1}w(s)ds\right) \\ &\quad + \left\|\int_0^t U(-s)c'(s)|w(s)|^{\gamma-1}w(s)ds\right\|_{L^2}^2. \end{aligned} \tag{4.42}$$

Following [**81**], we exchange the order of the time integral and the scalar product in the middle term at the right hand side of (4.42), that turns to be equal to

$$\pm 2\Im \int_0^t \left(U(s)f, c'(s)|w(s)|^{\gamma-1}w(s)\right) ds; \tag{4.43}$$

this has to be interpreted as the duality coupling on

$$\left(L^\infty(I; L^2) \cap L^{p,r}(I; L^{q+1,r})\right) \times \left(L^1(I; L^2) + L^{p',r'}(I; L^{\frac{q+1}{q},r'})\right),$$

with the choice

$$p = \frac{4(q+1)}{n(q-1)}.$$

Finally, a direct computation shows that the last term at the right hand side of (4.42) cancels out the quantity (4.43) (see [**81**] for details), hence the proof is complete. ☐

In view to prove global H^1-well-posedness for (4.1) we introduce the defocusing and focusing energies $E_d(t), E_f(t)$, defined by

$$E_d(t)[u] = \frac{1}{2}a(t)\|\nabla u(t)\|_{L^2}^2 + \frac{1}{\gamma+1}\|u(t)\|_{L^{\gamma+1}}^{\gamma+1}, \tag{4.44}$$

$$E_f(t)[u] = \frac{1}{2}a(t)\|\nabla u(t)\|_{L^2}^2 - \frac{1}{\gamma+1}\|u(t)\|_{L^{\gamma+1}}^{\gamma+1}. \tag{4.45}$$

As for the mass, a formal computation consisting in multiplying the equation (4.1) by $\overline{u}_t$, integrating by parts with respect to x and taking the resulting real parts, we see that

$$E_d'(t)[u] = \frac{1}{2}a'(t)\|\nabla u\|_{L^2}^2, \tag{4.46}$$

$$E_f'(t)[u] = \frac{1}{2}a'(t)\|\nabla u\|_{L^2}^2. \tag{4.47}$$

In terms of the solution w of (4.36), the energies are defined as

$$E_d(t)[w] = \frac{1}{2}\|\nabla w(t)\|_{L^2}^2 + \frac{c'(t)}{\gamma+1}\|w(t)\|_{L^{\gamma+1}}^{\gamma+1}, \tag{4.48}$$

$$E_f(t)[w] = \frac{1}{2}\|\nabla w(t)\|_{L^2}^2 - \frac{c'(t)}{\gamma+1}\|w(t)\|_{L^{\gamma+1}}^{\gamma+1}, \tag{4.49}$$

where c is defined by (4.33). The same formal computations as above lead to the identities

$$E_d'(t)[w] = \frac{c''(t)}{\gamma+1}\|w\|_{L^{\gamma+1}}^{\gamma+1}, \tag{4.50}$$

$$E_f'(t)[w] = \frac{c''(t)}{\gamma+1}\|w\|_{L^{\gamma+1}}^{\gamma+1}, \tag{4.51}$$

that are equivalent to (4.46) and (4.47). All these computations makes sense under the assumption of H^2-regularity for u or equivalently w; as for the mass conservation we need to show that identities (4.46), (4.47), (4.50), (4.51) are true for H^1-solutions of (4.1) or equivalently (4.36): hence we state the following Lemma:

LEMMA 4.3. *Let $f \in H^1$ and let w be a solution of the integral equation* (4.40)*, with $w \in L^{p,r}(I; W^1_{q'r})$, for some Schrödinger admissible pair p, q, and $r \geq 2$. Then* (4.50) *and* (4.51) *hold.*

We omit here the details of the proof, that follows the same lines of Proposition 2 in [**81**]; we only remark that the inverse change of (4.35) permits to obtain estimates (4.46) and (4.47) for solutions u of (4.1) satisfying the same assumptions of the previous Lemma.

It is clear from the above estimates that both in the defocusing and in the focusing case the energy is not conserved, and this is a difference with the case $a(t) \equiv 1$. By (4.46), $E_d'[u]$ has the sign of a', hence it is clear that, by the assumptions on a, the energy cannot blow up in any point t. This gives the following information about the gradient of u:

$$a(t)\|\nabla u\|_{L^2}^2 \leq E_d(t)[u] \leq C. \tag{4.52}$$

As a consequence, the only points in which $\|\nabla u\|_{L^2}$ can blow up are the zeroes of a, and the left hand side of (4.52) is bounded for all t.

The situation for the focusing equation is different: using the Gagliardo-Niremberg inequality

$$\|u\|_{L^{\gamma+1}}^{\gamma+1} \leq C\|u\|_{L^2}^{2+(\gamma-1)\frac{2-n}{2}}\|\nabla u\|_{L^2}^{(\gamma-1)\frac{n}{2}}, \tag{4.53}$$

we see by (4.47) that the energy is bounded from below by

$$E_f(t)[u] \geq \|\nabla u\|_{L^2}^2\left(\frac{1}{2}a(t) - \frac{C}{\gamma+1}\|u\|_{L^2}^{2+(\gamma-1)\frac{2-n}{2}}\|\nabla u\|_{L^2}^{(\gamma-1)\frac{n}{2}-2}\right). \tag{4.54}$$

Now, suppose by contradiction that $\|\nabla u(t)\|_{L^2}$ blows up at some t such that $a(t) \neq 0$. Then, if $\gamma < 1 + \frac{4}{n}$ (in other words $(\gamma - 1)\frac{n}{2} - 2 < 0$), as the mass is conserved, the right hand side in (4.54) is positive and we have the estimate

$$\|\nabla u(t)\|_{L^2}^2 \leq CE_f(t)[u], \tag{4.55}$$

for some $C > 0$; hence also the energy blows up in t. By (4.47), (4.55) and the Gronwall's Lemma we see that $E_f(t)[u] < \infty$, and this is a contradiction. As a consequence, the only points at which the energy can blow up are the zeroes of a. On the other hand, suppose that $a(t_i) = 0$ and

$$\|\nabla u(t)\|_{L^2} \to \infty,$$

as $t \to t_i^-$, in such a way that

$$\frac{1}{2}a(t) - \frac{C}{\gamma + 1}\|u\|_{L^2}^{2+(\gamma-1)\frac{2-n}{2}}\|\nabla u\|_{L^2}^{(\gamma-1)\frac{n}{2}-2} \geq 0, \tag{4.56}$$

if t is very close to t_i (in other words let us assume that the $\dot{H}^1$-norm blows up sufficiently fast). Then, by (4.54) we get

$$E_f(t)[u] \geq Ca(t)\|\nabla u\|_{L^2}^2; \tag{4.57}$$

hence the energy is positive and, by (4.47), we have $E_f(t_i) < \infty$ (because a' is negative at the left of t_i). This means that (4.52) holds also in this case, i.e.

$$a(t)\|\nabla u\|_{L^2}^2 \leq C, \tag{4.58}$$

for all t. We traduce (4.52) and (4.58) in terms of w, in fact by (4.35), (4.33) and (4.33) we immediately obtain the following a priori estimate:

$$\|\nabla w\|_{L^2}^2 \leq C(t_i - c(t))^{-\frac{\lambda}{\lambda+1}}, \tag{4.59}$$

for each t_i such that $a(t_i) = 0$ and $c(t)$ close to t_i. This estimate will be crucial in the proofs of the global results.

5. L^2 global well-posedness

This Section is devoted to the proof of Theorem 4.1. Let us consider a solution w of (4.36) and denote by $g(w) = \pm c'(t)|w|^{\gamma-1}$.

The solution of (4.36) is obtained as a fixed point of the map ϕ defined by:

$$\phi(w)(\cdot) = U(t)f(\cdot) - i\int_0^t U(t-s)g(w(s,\cdot))\,ds, \tag{4.60}$$

where as usual we denote $U(t) = e^{it\Delta}$. In what follows, we will prove that ϕ is a contraction in suitable Banach spaces, by means of Strichartz estimates; we start with the critical case, that is the simplest one.

5.1. The critical case. Here we put our attention on the critical power

$$\gamma = 1 + \frac{4}{n(\lambda+1)}. \tag{4.61}$$

Let $I = [-T, T]$ be a real interval, and let us denote

$$X = X(p,q) = L^\infty(I; L^2) \cap L^{p,2}(I; L^{q,2}), \tag{4.62}$$

for any Schrödinger admissible couple (p, q), with $2 < p < \infty$; X is a Banach space with the norm

$$\|F\|_X = \|F\|_{L^\infty(I;L^2)} + \|F\|_{L^{p,2}(I;L^{q,2})}.$$

Applying estimates (4.21) and (4.22), with the choice $s' = 2 - \epsilon$, for a small $\epsilon > 0$, we have

$$\|\phi(w)\|_X \leq C\left(\|f\|_{L^2} + \left\|c'|w|^\gamma\right\|_{L^{\widetilde{p}',s'}(I;L^{\widetilde{q}',s'})}\right). \tag{4.63}$$

By Hölder inequality in Lorentz spaces (see [**80**]), and recalling (4.34), we have

$$\|\phi(w)\|_X \leq C\left(\|f\|_{L^2} + \|\,|w|^\gamma\|_{L^{\widetilde{r}',s'}(I;L^{\widetilde{q}',s'})}\right), \tag{4.64}$$

where $\widetilde{r}'$ is defined by

$$\frac{1}{\widetilde{r}'} = \frac{1}{\widetilde{p}'} - \frac{\lambda}{\lambda+1}; \tag{4.65}$$

combining (4.65) with the admissibility condition

$$\frac{2}{\widetilde{p}} = \frac{n}{2} - \frac{n}{\widetilde{q}}, \qquad \widetilde{p} > 2,$$

it turns out that $\widetilde{r} = \widetilde{r}'/(\widetilde{r}' - 1)$ satisfies

$$\frac{2}{\widetilde{r}} = \frac{2\lambda}{\lambda+1} + \frac{n}{2} - \frac{n}{\widetilde{q}}, \tag{4.66}$$

in the range

$$\max\left\{\frac{2(\lambda+1)}{3\lambda+1}, 1\right\} < \widetilde{r} < \frac{\lambda+1}{\lambda}. \tag{4.67}$$

By the definition of the Lorentz norms, we easily see that

$$\|\,|h|^\alpha\|_{L^{p,r}(\mathbb{R}^n)} = \|h\|^\alpha_{L^{p\alpha,r\alpha}(\mathbb{R}^n)},$$

for any $\alpha \geq 0$; hence, coming back to (4.64), we have proved that

$$\|\phi(w)\|_X \leq C\left(\|f\|_{L^2} + \|w\|^\gamma_{L^{\widetilde{r}'\gamma,s'\gamma}(I;L^{\widetilde{q}'\gamma,s'\gamma})}\right). \tag{4.68}$$

Now, since the critical value of γ is bigger than 1, we can choice ϵ sufficiently small such as $s'\gamma > 2$; hence, by the embedding

$$L^{p,r} \hookrightarrow L^{p,s}, \qquad r \leq s,$$

we conclude that

$$\|\phi(w)\|_X \leq C\left(\|f\|_{L^2} + \|w\|^\gamma_{L^{\widetilde{r}'\gamma,2}(I;L^{\widetilde{q}'\gamma,2})}\right). \tag{4.69}$$

By simple algebraic computations, we see that the couple (p_0, q_0) defined by

$$p_0 = \widetilde{r}'\gamma, \qquad q_0 = \widetilde{q}'\gamma \tag{4.70}$$

satisfies the admissibility condition

$$\frac{2}{p_0} = \frac{n}{2} - \frac{n}{q_0}$$

if and only if γ is the critical value in (4.61).

REMARK 4.4. In Lemma 4.4, we will prove that the range condition $p_0 > 2$ can be satisfied, by a suitable choice of $\widetilde{r}$ in the interval (4.67).

At this point, we come back to (4.69); with the choice $(p, q) = (p_0, q_0)$, we have the following inequality:

$$\|\phi(w)\|_X \leq C \left(\|f\|_{L^2} + \|w\|_X^\gamma \right), \tag{4.71}$$

for some $C > 0$. It is clear from estimate (4.71) that the map ϕ has a convex closed invariant set, provided the initial datum f satisfies

$$\|f\|_{L^2} \leq \epsilon_0,$$

for some $\epsilon_0 > 0$ sufficiently small; with similar computations, we prove that ϕ is a contraction on X, then the local existence of a unique solution of (4.36). By the conservation of mass, we conclude that the solution is global in time.

The proof will be complete after the following Lemma:

LEMMA 4.4. *There exists a suitable choice of $\widetilde{r}$ in the range* (4.67), *such that the condition $p_0 = \widetilde{r}'\gamma > 2$ is satisfied.*

Proof. We distinguish the cases $\lambda \geq 1$ and $0 < \lambda < 1$.

In the case $\lambda \geq 1$, the condition (4.67) is

$$1 < \widetilde{r} < \frac{\lambda + 1}{\lambda}; \tag{4.72}$$

the function

$$\widetilde{r}' = \frac{\widetilde{r}}{\widetilde{r} - 1}$$

is strictly decreasing with respect to $\widetilde{r}$, and the lowest endpoint $\widetilde{r} = 1$ obviously satisfies the condition $2 < \widetilde{r}'\gamma < \infty$; hence we can choose $\widetilde{r}$ greatest as possible in the range (4.72), in order to obtain the lowest value of $p_0 > 2$. We omit here the precise computations.

In the case $0 < \lambda < 1$ the condition (4.67) becomes

$$\frac{2(\lambda + 1)}{3\lambda + 1} < \widetilde{r} < \frac{\lambda + 1}{\lambda}; \tag{4.73}$$

as in the previous case, it is sufficient to prove that the lowest endpoint $\widetilde{r} = 2(\lambda + 1)/(3\lambda + 1)$ largely satisfies the range condition $\widetilde{r}'\gamma > 2$. By a trivial computation we see that the inequality

$$\frac{2(\lambda + 1)}{3\lambda + 1}\gamma = \frac{2(\lambda + 1)}{3\lambda + 1} \left[1 + \frac{4}{n(\lambda + 1)} \right] > 2$$

is satisfied, for $\lambda \in (0, 1)$, in any dimension $n \geq 1$, hence we conclude as in the previous case.

5.2. The subcritical case. We pass to the study of the subcritical case

$$1 \leq \gamma < 1 + \frac{4}{n(\lambda+1)}. \tag{4.74}$$

Let us consider the map ϕ defined in (4.60); the computations in the previous Section work until estimate (4.68); to pass to (4.69) we need to exclude the case $\gamma = 1$, for which we are not able to choice ϵ in such a way that $s'\gamma > 2$. Anyway, the case $\gamma = 1$ can be easily recovered by an approximation argument that is identical to the one presented at the end of Section 6.2.

Once we have (4.69), we can apply Hölder inequality in Lorentz spaces, and we get

$$\|\phi(w)\|_X \leq C \left(\|f\|_{L^2} + \|w\|^{\gamma}_{L^{p,2}(I;L^{\widetilde{q}',2})} T^{\theta}, \right) \tag{4.75}$$

where

$$\theta = \frac{1}{\widetilde{r}'} - \frac{\gamma}{p} \tag{4.76}$$

and $T = \sup I$. Now we impose the conditions

$$\theta > 0, \qquad \widetilde{q}'\gamma = q,$$

that, in terms of $\widetilde{r}'$ and p are the following:

$$\widetilde{r}'\gamma < p, \qquad \widetilde{q}'\gamma = q. \tag{4.77}$$

Combining (4.77) with (4.66), (4.67) and the admissibility for the couple (p, q) we see that this are compatible with the subcritical range (4.74). Moreover, with the same computations of Lemma 4.4, we check that there always exists a choice of $\widetilde{r}$ such that $\widetilde{r}'\gamma > 2$.

In conclusion, we obtained the following estimate

$$\|\phi(w)\|_X \leq C \left(\|f\|_{L^2} + T^{\theta} \|w\|^{\gamma}_X \right); \tag{4.78}$$

it is clear that, if T is sufficiently small, ϕ has a closed convex invariant set, and with the same computations we prove that ϕ is a contraction in X; finally, by the conservation of mass we conclude that the solution is global in time.

6. H^1 local well-posedness

We pass to the proof of Theorem 4.2. At this level, the proof does not make any difference between the defocusing and the focusing cases. As in the previous Section, we only have to show the H^1 local well-posedness for (4.36). Differentiating the equation (4.36) with respect to x, we see that ∇w solves the system

$$\begin{cases} i\,(\nabla w)_t + \Delta(\nabla w) = \pm c'(t) \left(\frac{\gamma+1}{2} |w|^{\gamma-1} \nabla w + \frac{\gamma-1}{2} |w|^{\gamma-3} w^2 \nabla \overline{w} \right) \\ \nabla w(0,x) = \nabla f(x), \end{cases} \tag{4.79}$$

at each component. Hence, writing

$$G(w) = \pm \frac{\gamma+1}{2} |w|^{\gamma-1} \nabla w + \frac{\gamma-1}{2} |w|^{\gamma-3} w^2 \nabla \overline{w}, \tag{4.80}$$

if w is a solution of (4.36) then ∇w is a fixed point of the map

$$\phi(w) = U(t)\nabla f - i \int_0^t U(t-s) c'(s) G(w(s,\cdot))\, ds. \tag{4.81}$$

Observe that we have the estimate

$$|G(w)| \leq C|w|^{\gamma-1}|\nabla w|, \tag{4.82}$$

where the constant $C > 0$ depends on γ.

6.1. The critical case. We start with the critical case

$$\gamma = 1 + \frac{4}{(n-2)(\lambda+1)}. \tag{4.83}$$

As before, let us consider a real interval $I = [-T, T]$ and the spaces $X(p,q)$ introduced in (4.62). Let w be a solution of (4.36); by Strichartz estimates (4.21) and (4.22) applied to (4.81), we have

$$\|\phi(v)\|_{X(p,q)} \leq C\|\nabla f\|_{L^2} + C\left\|c'|w|^{\gamma-1}\nabla w\right\|_{L^{\widetilde{p}',s'}(I;L^{\widetilde{q}',s'})}, \tag{4.84}$$

with the choice $s' = 2-\epsilon$, for some small $\epsilon > 0$. After recalling (4.34), we apply Hölder inequality in Lorentz spaces and we obtain

$$\|\phi(v)\|_{X(p,q)} \leq C\|\nabla f\|_{L^2} + C\left\||w|^{\gamma-1}\nabla w\right\|_{L^{\widetilde{r}',s'}(I;L^{\widetilde{q}',s'})}, \tag{4.85}$$

where $\widetilde{r}'$ is defined in (4.65). Moreover, the conjugate $\widetilde{r}$ satisfies the admissibility condition (4.66) and the range condition (4.67).

We estimate the last term in inequality (4.85): by Hölder inequalities in Lorentz and L^p spaces, in both time and space variables, we have

$$\left\| |w|^{\gamma-1}\nabla w\right\|_{L^{\widetilde{r}',s'}(I;L^{\widetilde{q}',s'})} \leq \left\||w|^{\gamma-1}\right\|_{L^{r_1,\alpha}(I;L^{q_1,\alpha})} \|\nabla w\|_{L^{r_2,2}(I;L^{q_2,2})}, \tag{4.86}$$

under the following conditions:

$$\frac{1}{\widetilde{r}'} = \frac{1}{r_1} + \frac{1}{r_2}, \qquad \frac{1}{\widetilde{q}'} = \frac{1}{q_1} + \frac{1}{q_2}, \qquad \frac{1}{\alpha} + \frac{1}{2} \geq \frac{1}{s'}. \tag{4.87}$$

First, we impose the couple (r_2, q_2) to be Schrödinger admissible, i.e.

$$\frac{2}{r_2} = \frac{n}{2} - \frac{n}{q_2}, \qquad r_2 > 2 : \tag{4.88}$$

this permits to treat the term of order one in inequality (4.86). For the nonlinear term in (4.86), we need further conditions on the exponents r_1, s, q_1. If we impose that

$$(\gamma-1)r_1 \geq 1, \qquad (\gamma-1)q_1 \geq 1, \qquad (\gamma-1)\alpha \geq 1, \tag{4.89}$$

we get the identity

$$\left\| |w|^{\gamma-1}\right\|_{L^{r_1,\alpha}(I,L^{q_1,\alpha})} = \|w\|^{\gamma-1}_{L^{(\gamma-1)r_1,(\gamma-1)\alpha}(I;L^{(\gamma-1)q_1,(\gamma-1)\alpha})}. \tag{4.90}$$

A crucial comment on the exponent α: by the third condition in (4.87), as $s' = 2-\epsilon$ and ϵ is arbitrarily small, we can choose α arbitrarily big. In particular, let us take α such that $(\gamma-1)\alpha > 2$ (this is possible because the critical value of γ is strictly bigger than 1); by (4.90) and the Lorentz embeddings we can then estimate

$$\left\| |w|^{\gamma-1}\right\|_{L^{r_1,\alpha}(I,L^{q_1,\alpha})} \leq \|w\|^{\gamma-1}_{L^{(\gamma-1)r_1,2}(I;L^{(\gamma-1)q_1,2})}. \tag{4.91}$$

Remark 4.5. In Lemma 4.5 we will prove that all the conditions we are imposing are compatible, and hence we can choose the right exponents to have well-posedness in the suitable spaces.

Using (4.91), the inequality (4.86) gives

$$\| |w|^{\gamma-1}\nabla w\|_{L^{\widetilde{r}',s'}(I;L^{\widetilde{q}',s'})} \leq \|w\|^{\gamma-1}_{L^{(\gamma-1)r_1,2}(I;L^{(\gamma-1)q_1,2})}\|\nabla w\|_{L^{r_2,2}(I;L^{q_2,2})}. \tag{4.92}$$

By Sobolev embeddings in Lorentz spaces (see e.g. [**105**]), we estimate

$$\|w\|^{\gamma-1}_{L^{(\gamma-1)r_1,2}(I;L^{(\gamma-1)q_1,2})} \leq \|\nabla w\|^{\gamma-1}_{L^{(\gamma-1)r_1,2}(I;L^{y,2})}, \tag{4.93}$$

where

$$1-\frac{n}{y} = -\frac{n}{(\gamma-1)q_1}. \tag{4.94}$$

We are ready to impose the admissibility of the couple $((\gamma-1)r_1, y)$, i.e.

$$\frac{2}{(\gamma-1)r_1} = \frac{n}{2}-\frac{n}{y}, \qquad (\gamma-1)r_1 > 2. \tag{4.95}$$

A simple algebraic computation shows that (4.66), (4.87), (4.88), (4.94) and (4.95) are satisfied if and only if γ is the critical value in (4.83).

Finally, recollecting (4.85), (4.86) and (4.93), with the conditions (4.88) and (4.95), there exist a Schrödinger admissible couple (p,q) such that

$$\|\phi(v)\|_{X(p,q)} \leq C\|\nabla f\|_{L^2} + C\|\nabla w\|^{\gamma-1}_{X(p,q)}, \tag{4.96}$$

for some $C>0$. The last estimate, together with Sobolev embeddings, proves that the map ϕ has a closed convex invariant set in $L^\infty(I;H^1)\cap L^{p,1}(I;W^{1,q})$, provided $\|\nabla f\|_{L^2} < \epsilon_0$ is sufficiently small, for some Schrödinger admissible couple (p,q). With similar computations, we prove that ϕ is a contraction on the same space, then there exists a local solution of (4.36); finally, by the energy arguments discussed in the Introduction, we conclude that the solution exists until the first zero of a, in which the gradient can blow up.

The following Lemma concludes the proof, showing that all the conditions we imposed on the exponents are compatible.

LEMMA 4.5. *There exists a nonempty range for $\widetilde{r}$ in which the conditions* (4.66)*,* (4.87)*,* (4.88)*,* (4.89) *and* (4.95) *are compatible.*

Proof. By (4.87), (4.89) and (4.95), we have

$$\frac{1}{r_2} > \frac{1}{\widetilde{r}'} - \frac{\gamma+1}{2}, \qquad \frac{1}{q_2} \geq \frac{1}{\widetilde{q}'} - \gamma + 1;$$

then, summing the last two inequalities, we estimate

$$\frac{2}{r_2}+\frac{n}{q_2} > \frac{2}{\widetilde{r}'}+\frac{n}{\widetilde{q}'} - (1+n)(\gamma+1),$$

and using (4.88) and the admissibility condition (4.66) we obtain

$$\gamma \geq 1 + \frac{2}{(\lambda+1)(n+1)}. \tag{4.97}$$

The H^1-critical value of γ satisfies the last inequality, because

$$\gamma = 1+\frac{4}{(n-2)(\lambda+1)} \geq 1+\frac{2}{(\lambda+1)(n+1)},$$

for each n, λ. The last effort we have to do is to control that the condition $r_2 > 2$ in (4.88) can be satisfied. By (4.87), (4.89) and (4.95) we have

$$\frac{1}{r_2} > 1 - \frac{1}{\widetilde{r}} - \frac{\gamma - 1}{2}; \tag{4.98}$$

the condition (4.98) has to be consistent with $r_2 > 2$, then we observe that

$$1 - \frac{1}{\widetilde{r}} - \frac{\gamma - 1}{2} < \frac{1}{2} \Leftrightarrow \widetilde{r}(2 - \gamma) < 2. \tag{4.99}$$

In the case $\gamma > 2$, the condition (4.99) is obviously satisfied when $\widetilde{r}$ is in the range (4.67). On the other hand, in the case $1 \leq \gamma \leq 2$, we easily verify that

$$\max\left\{\frac{2(\lambda+1)}{3\lambda+1}, 1\right\} < \frac{2}{2-\gamma};$$

hence, the new range condition for $\widetilde{r}$ is

$$\max\left\{\frac{2(\lambda+1)}{3\lambda+1}, 1\right\} < \widetilde{r} < \min\left\{\frac{\lambda+1}{\lambda}, \frac{2}{2-\gamma}\right\}. \tag{4.100}$$

Hence the proof is complete.

6.2. The subcritical case. We pass to the study of the subcritical nonlinearity

$$1 + \frac{2}{(\lambda+1)(n+2)} \leq \gamma < 1 + \frac{4}{(n-2)(\lambda+1)}. \tag{4.101}$$

The computations made for the H^1-critical case in the previous Section work until estimate (4.93), provided we exclude the case $\gamma = 1$: then we have

$$\|\phi(v)\|_{X(p,q)} \leq C\|\nabla f\|_{L^2} + C\|\nabla w\|^{\gamma-1}_{L^{(\gamma-1)r_1,2}(I;L^{y,2})}\|\nabla w\|_{L^{r_2,2}(I;L^{q_2,2})}, \tag{4.102}$$

for some $C > 0$. The exponents r_1, r_2, y, q_2 satisfies conditions (4.66), (4.67), (4.87), (4.88), (4.89) and (4.94); the consistence of these conditions will be studied in Lemma 4.6.

To treat the nonlinear term in (4.102), we apply Hölder inequality for Lorentz spaces, with respect to time, and we get

$$\|\nabla w\|^{\gamma-1}_{L^{(\gamma-1)r_1,2}(I;L^{y,2})} \leq T^{\theta}\|\nabla w\|^{\gamma-1}_{L^{p,2}(I;L^{y,2})}, \tag{4.103}$$

where

$$\theta = \frac{1}{r_1} - \frac{\gamma-1}{p} \tag{4.104}$$

and $T = \sup I$. Here the exponent p is fixed in the space $X(p,q)$. We need to impose the condition $\theta \in (0,1]$, that means

$$\frac{p}{p+1} \leq (\gamma-1)r_1 < p; \tag{4.105}$$

the inequality on the left of (4.105) is weaker than the condition (4.89) for r_1, that will be discussed in Lemma 4.6. Hence, we impose the conditions

$$(\gamma-1)r_1 < p, \qquad y = q, \tag{4.106}$$

that with a simple algebraic computation turn out to be true for the subcritical values

$$1 \leq \gamma < 1 + \frac{4}{(n-2)(\lambda+1)}.$$

Finally, by (4.102) and (4.103), there exists a Schrödinger admissible couple (p,q) such that

$$\|\phi(v)\|_{X(p,q)} \le C\|\nabla f\|_{L^2} + CT^{\theta}\|\nabla w\|^{\gamma}_{X(p,q)}. \tag{4.107}$$

Now let us denote by

$$M = 2C\|\nabla f\|_{L^2} \tag{4.108}$$

and by B_M the ball of radius M in the Banach space $X(p,q)$; by (4.107) we see that ϕ maps B_M into itself if

$$T \le C_1 M^{\frac{1-\gamma}{\theta}}. \tag{4.109}$$

With the same computation we see that ϕ is a contraction on $X(p,q)$ provided (4.109) holds: consequently, ϕ has a (unique) fixed point in $X(p,q)$ and the local existence is proved. This local solution can be uniquely extended in time until the first possible blow up point, by means of the energy estimates in Section 4. Estimate (4.109) will be the crucial information to prove global existence in the following Section.

To complete the local theory we need the following Lemma, analogous to Lemma 4.5:

Lemma 4.6. *There exists a nonempty range for $\widetilde{r}$ in which the conditions* (4.66), (4.87), (4.88) *and* (4.89) *are compatible.*

Proof. The proof is identical to Lemma 4.5; here we don't have the condition (4.95). By (4.87) and (4.89) we obtain

$$\frac{1}{r_2} \ge \frac{1}{\widetilde{r}'} - \gamma + 1, \qquad \frac{1}{q_2} \ge \frac{1}{\widetilde{q}'} - \gamma + 1;$$

summing the last inequalities we get

$$\frac{2}{r_2} + \frac{n}{q_2} \ge \frac{2}{\widetilde{r}'} + \frac{n}{\widetilde{q}'} - (2+n)(\gamma - 1).$$

By (4.88) and the admissibility (4.66) the last inequality gives

$$\gamma \ge 1 + \frac{2}{(\lambda+1)(n+2)},$$

as required in the assumptions. Finally, as in Lemma 4.5 we prove that the condition $r_2 > 2$ can be satisfied, in a suitable range for $\widetilde{r}$. In fact, by (4.87) and (4.89) we have

$$\frac{1}{r_2} \ge 2 - \frac{1}{\widetilde{r}} - \gamma;$$

to be consistent with $r_2 > 2$ we need that

$$2 - \frac{1}{\widetilde{r}} - \gamma < \frac{1}{2} \Leftrightarrow \frac{1}{\widetilde{r}} > \frac{3-2\gamma}{2}.$$

In the case $\gamma \ge \frac{3}{2}$, the last condition is always satisfied in the range (4.67). On the other hand, we easily see that

$$\max\{\frac{2(\lambda+1)}{3\lambda+1}, 1\} < \frac{2}{3-2\gamma},$$

for any $\lambda \ge 0$ and $1 \le \gamma < \frac{3}{2}$. Hence, the required range for $\widetilde{r}$ is

$$\max\{\frac{2(\lambda+1)}{3\lambda+1}, 1\} < \widetilde{r} \le \min\{\frac{\lambda+1}{\lambda}, \frac{2}{3-2\gamma}\}, \tag{4.110}$$

and the proof is complete.

By means of Lemma 4.6 and the above computations, we have proved Theorem 4.2 in the range

$$1 + \frac{2}{(\lambda+1)(n+2)} \leq \gamma < 1 + \frac{4}{(\lambda+1)(n-2)}. \tag{4.111}$$

Now, with a simple approximation argument, we treat the lower powers

$$1 \leq \gamma < 1 + \frac{2}{(\lambda+1)(n+2)}. \tag{4.112}$$

First, let us observe that

$$1 + \frac{2}{(\lambda+1)(n+2)} < 1 + \frac{4}{(\lambda+1)(n)},$$

for each $\lambda \geq 0$, $n \geq 2$. Hence, by Theorem 4.1, for any initial datum $f \in H^1$, there exists a unique global solution $w \in \mathcal{C}(\mathbb{R}; L^2)$ of (4.36), when γ satisfies (4.112). Approximating the nonlinearity with a sequence of sufficiently regular functions, and applying the standard theory (see e.g. [**20**]), we produce a sequence of approximated local solutions $w_n \in \mathcal{C}(I; H^1)$, with the uniform bound

$$\|w_n\|_{\mathcal{C}(I;H^1)} \leq C, \tag{4.113}$$

or some $C > 0$. Moreover, the sequence w_n converges to w strongly in $\mathcal{C}(I; L^2)$, and by (4.113) has a subsequence that converges weakly in $\mathcal{C}(I; H^1)$. By the uniqueness of the limit we conclude the proof.

7. Global H^1 theory

This final Section is devoted to the proof of the global Theorem 4.3. As discussed in the Section 4, the main difficulty is that the energy does not control the gradient of the solution, hence we cannot say a priori that, starting from an initial datum in H^1, the solution remains in H^1 for all times. Moreover, as it is clear from (4.109), the lifespan of the local solution goes to 0 as the initial norm goes to ∞: hence we are not a priori able to say that, iterating the local existence, the series of the lifespans diverges, in such a way that the solution is global.

On the other hand, the energy investigation of Section 4 produced the information (4.59) about the blow up rate of the gradient: by mixing (4.59) and (4.109) we show that our solution is in fact global (under the assumptions of Theorem 4.3).

Let us denote by t_0 the first zero of a. Let $\epsilon > 0$ be small and t be such that $t_0 - c(t) < \epsilon$; by the local theory we know that the solution w is in H^1 at the time t. Starting with the initial datum in t, we iterate the proof of local solution: if we are able to cross the possible blow up point we have done. By (4.109) and (4.59) we obtain the lower bound

$$T \geq C(t_0 - c(t))^{-\frac{\lambda(1-\gamma)}{2\theta(\lambda+1)}}, \tag{4.114}$$

where T is the new lifespan. We claim that the new solution cross the point $\widetilde{t}_0$, whose image via c is t_0: this means that $c(t) + T > t_0$, i.e.

$$C(t_0 - c(t))^{-\frac{\lambda(1-\gamma)}{2\theta(\lambda+1)}} \geq t_0 - c(t). \tag{4.115}$$

If ϵ is sufficiently small, we can neglect the constants and (4.115) holds if

$$-\frac{\lambda(1-\gamma)}{2\theta(\lambda+1)} \leq 1,$$

or equivalently

$$\theta \geq \frac{\lambda(\gamma-1)}{2(\lambda+1)}. \tag{4.116}$$

The condition (4.116) has to be compatible with $\theta \in (0,1]$, then we impose

$$\frac{\lambda(\gamma-1)}{2(\lambda+1)} \leq 1 \Rightarrow \gamma < 1 + \frac{2(\lambda+1)}{\lambda}. \tag{4.117}$$

It is easy to verify that, if $n \geq 3$, the critical exponent satisfies

$$1 + \frac{4}{(n-2)(\lambda+1)} \leq 1 + \frac{2(\lambda+1)}{\lambda}, \tag{4.118}$$

for each $\lambda > 0$; hence, by (4.116), we have a final condition on θ that is

$$\frac{\lambda(\gamma-1)}{2(\lambda+1)} \leq \theta \leq 1. \tag{4.119}$$

The upper inequality in (4.119) was already discussed in Section 6. As for the lower condition, by the definition (4.104) we have

$$\frac{\gamma-1}{p} \leq \frac{1}{r_1} - \frac{\lambda(\gamma-1)}{2(\lambda+1)}. \tag{4.120}$$

We impose that

$$\frac{1}{r_1} - \frac{\lambda(\gamma-1)}{2(\lambda+1)} \geq \delta > 0 \Rightarrow (\gamma-1)r_1 < \frac{2(\lambda+1)}{\lambda} - \frac{\delta}{\gamma-1}, \tag{4.121}$$

for some small $\delta > 0$. If we assume that

$$p \geq \frac{2(\lambda+1)}{\lambda}, \tag{4.122}$$

inequality (4.121) is compatible with (4.89) and (4.106), and we have the final condition on r_1:

$$1 \leq (\gamma-1)r_1 < \frac{2(\lambda+1)}{\lambda} - \frac{\delta}{\gamma-1}. \tag{4.123}$$

We recall that, by (4.87), it is necessary that $r_1 \geq \widetilde{r}'$; moreover the Hölder conjugate $\widetilde{r}$ has to live in the range (4.110). A simple computation shows that these conditions are compatible with (4.123). Assumption (4.122) is not restrictive, in fact, once we performed a fixed point argument for one choice of $X(p_0, q_0)$, we can make the same, by interpolation, for all the choices $p > p_0$. At this point, by (4.120) we get

$$p \geq \frac{2r_1(\lambda+1)(\gamma-1)}{2(\lambda+1) - \lambda(\gamma-1)r_1}. \tag{4.124}$$

The right hand side of the last inequality is bounded by a great constant depending on δ, that is fixed. Hence there is a suitable choice of p (sufficiently large) such that (4.124) is satisfied.

In conclusion, all the above considerations hold in dimension $n = 2$, provided the more restrictive assumption (4.117) is satisfied by γ. This completes the proof.

8. Blow-up threshold for weakly coupled NLS

We can pass to another nonlinear example. Let us consider the following Cauchy problem for two coupled nonlinear Schrödinger equations

$$\begin{cases} i\phi_t + \Delta\phi + \left(|\phi|^{2p} + \beta|\psi|^{p+1}|\phi|^{p-1}\right)\phi = 0 \\ i\psi_t + \Delta\psi + \left(|\psi|^{2p} + \beta|\phi|^{p+1}|\psi|^{p-1}\right)\psi = 0 \\ \phi(0,x) = \phi_0(x) \qquad \psi(0,x) = \psi_0(x), \end{cases} \tag{4.125}$$

where $\phi, \psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$, $\phi_0, \psi_0 : \mathbb{R}^n \to \mathbb{C}$, $p \geq 0$ and β is a real positive constant.

This kind of problems arises as a model for propagation of polarized laser beams in birefringent Kerr medium in nonlinear optics (see, for example, [**76, 9, 41, 68**] and the references therein for a complete discussion of the physics of the problem). The two functions ϕ and ψ are the components of the slowly varying envelope of the electrical field, t is the distance in the direction of propagation, x are the orthogonal variables and Δ is the diffraction operator. The case $n = 1$ corresponds to propagation in a planar geometry, $n = 2$ is the propagation in a bulk medium and $n = 3$ is the propagation of pulses in a bulk medium with time dispersion (in this case x includes also the time variable).

The focusing nonlinear terms in (4.125) describe the dependence of the refraction index of the material on the electric field intensity and the birefringence effects. The parameter $\beta > 0$ has to be interpreted as the *birefringence* intensity and describes the coupling between the two components of the electric field envelope. The case $p = 1$ (i.e. cubic nonlinearities in (4.125)) is known as Kerr nonlinearity in the physical literature.

We are interested in a slightly more general model, in order to cover the physical cases and to discuss some results about Cauchy problem (4.125) from a more general point of view.

Our aim is to study the $H^1 \times H^1$ well-posedness of problem (4.125), with respect to the nonlinearity, in analogy with the case of the single focusing nonlinear Schrödinger equation

$$\begin{cases} i\psi_t + \Delta\psi + |\psi|^{2p}\psi = 0 \\ \psi(0,x) = f(x), \end{cases} \tag{4.126}$$

for $\psi : \mathbb{R}^{1+n} \to \mathbb{C}$ and $f : \mathbb{R}^n \to \mathbb{C}$.

The well known results for the single equation can be summarized as follows. By standard scaling arguments it possible to claim that the critical exponent for the H^1 local well-posedness of (4.126) is $p = 2/(n-2)$ (see [**20**]). Indeed, contraction techniques based on Strichartz estimates (see [**47**], [**66**]), permit to prove that (4.126) is locally well-posed in H^1 for $p < 2/(n-2)$ (see [**46**], [**20**]).

To pass from local to global well-posedness, it is natural to introduce the energy function given by

$$E(t) = \frac{1}{2}\|\nabla\psi\|_2^2 - \frac{1}{2p+2}\|\psi\|_{2p+2}^{2p+2},$$

that is conserved along any solution ψ of (4.126). For $p < 2/n$ the unique local H^1 solution can be extended globally in time by a continuation argument. In the *critical case* $p = 2/n$, we can also extend local solutions to global ones, provided the initial data are not too large in the L^2. Finally, for $2/n \leq p < 2/(n-2)$ without restriction on the data, it is possible to prove that the

L^2 norm of the gradient, in general, blows up in a finite time (see e.g. the original work [**48**] or [**20**]).

By a physical point of view it is very interesting to determine the threshold for the initial mass of the wave packet, that is the L^2 norm of the initial datum, which separates global existence and blow-up in the critical case. We recall that $\psi = e^{it}u(x) \in H^1$ is a *ground state solution* for (4.126) if u is a nonzero critical point of the action functional

$$A(u) = E(u) + \frac{1}{2}\|u\|_2^2 = \frac{1}{2}\left(\|\nabla u\|_2^2 + \|u\|_2^2\right) - \frac{1}{2+4/n}\|u\|_{2+4/n}^{2+4/n},$$

having the smallest action level; clearly u solves

$$-\Delta u + u = |u|^{4/n}u. \tag{4.127}$$

In [**114**], Weinstein proved that if the initial mass is smaller than a constant C_n, depending only on the space dimension n, than there exists a unique global H^1 solution; moreover C_n is the L^2 norm of any ground state solutions of (4.126) and can be numerically estimated. Moreover we want to point out that this kind of phenomena for the single equation present other kind of universality properties related, for example, to the blow-up profile (see [**9, 77, 78**] and the references therein).

Our main goal is to state the analogous result for the coupled system (4.125). The critical exponent for the local $H^1 \times H^1$ well-posedness has to be again $p = 2/(n-2)$; so for $p < 2/(n-2)$ it is possible to prove that (4.125) possesses a unique local solution (see Remark 4.2.13 in [**20**] and Section 9 below). The natural energy for (4.125) is the following:

$$E(t) = \frac{1}{2}\left(\|\nabla\phi\|_2^2 + \|\nabla\psi\|_2^2\right) - \frac{1}{2p+2}\left(\|\phi\|_{2p+2}^{2p+2} + 2\beta\|\phi\psi\|_{p+1}^{p+1} + \|\psi\|_{2p+2}^{2p+2}\right). \tag{4.128}$$

Also here it is possible to prove that $E(t)$ is conserved (see Section 9); hence the same techniques for the single equation can be applied to extend local solution to global ones. Now we can state our first result.

THEOREM 4.5. *Assume that $p < 2/n$. Then the Cauchy problem* (4.125) *is globally well posed in $H^1 \times H^1$, i.e. for any $(\phi_0, \psi_0) \in H^1 \times H^1$ there exists a unique solution $(\phi, \psi) \in C\left(\mathbb{R}; H^1 \times H^1\right)$.*

Also here ground state solutions of (4.125) play a crucial role in the dynamics of the system. In this case they are solutions of the form $(\phi, \psi) = e^{it}\left(u(x), v(x)\right)$, where the functions u and v have to be a least action solution of a elliptic system (see (4.136) below).

Since the birefringence tends to split a pulse into two pulses in two different polarization directions, the properties of the ground state solutions of (4.125) depend strongly on the coupling parameter. If β is sufficiently small, that is the interaction is weak, any ground state is a *scalar* solution, i.e. one of the two components is zero. On the other hand when the birefringence is strong, $\beta \gg 1$, we have *vector* ground states, i.e. all the components are distinct from zero (see [**4, 74**]). This suggest that the also the blow-up phenomena, in the critical case, should depend on the parameter β. It is natural to claim that in a weak interaction regime the behaviour has to be exactly the same of the single equation. Otherwise if $\beta \gg 1$, we expect that the analogous of the Weinstein threshold C_n should depend on β also. These claims are proved in the following main theorem.

THEOREM 4.6. *Assume that $p = 2/n$. Then there exists a constant $C = C_{n,\beta}$ such that the Cauchy problem* (4.125) *is globally well posed in $H^1 \times H^1$ if*

$$\|\phi_0\|_2^2 + \|\psi_0\|_2^2 < C.$$

Moreover there exists a pair (ϕ_0, ψ_0) such that $\|\phi_0\|_2^2 + \|\psi_0\|_2^2 = C_{n,\beta}$ and the corresponding solution blows up in a finite time. The constant $C_{n,\beta}$ has the following behaviour

$$\begin{cases} C_{n,\beta} = C_n & \text{if } \beta \leq 2^{2/n} - 1, \\ C_{n,\beta} \geq C_n \dfrac{(1+\beta)}{2^{2/n}} & \text{if } \beta \geq 2^{2/n} - 1, \end{cases} \tag{4.129}$$

where C_n is the blow-up threshold of a single equation.

REMARK 4.6. In the supercritical case the solution of the Cauchy problem for (4.125) exists locally in time, by the results in [**20**]. It is possible to prove that the solution exists globally in time if the assumption $\|\phi_0\|^2, \|\psi_0\|^2 \ll 1$ is satisfied (see Theorem 6.1.1 in [**20**]).

REMARK 4.7. As observed above, the Kerr nonlinearities (corresponding to $p = 1$) are physically relevant; in this case the system (4.125) becomes

$$\begin{cases} i\phi_t + \Delta\phi + \left(|\phi|^2 + \beta|\psi|^2\right)\phi = 0 \\ i\psi_t + \Delta\psi + \left(|\psi|^2 + \beta|\phi|^2\right)\psi = 0. \end{cases} \tag{4.130}$$

The above results (Theorems 4.5 and 4.6) can be summarized in the following way:

i) if $n = 1$ the Cauchy problem (4.130) is globally (in time) well posed in $H^1 \times H^1$,
ii) if $n = 2$ the cubic nonlinearity is critical, so the Cauchy problem (4.130) is globally well posed for small data; moreover the blow-up threshold $C_{2,\beta}$ is constant for any $\beta \leq 1$ and tends to infinity as $\beta \to +\infty$,
iii) if $n \geq 3$ a solution of the Cauchy problem (4.130) exists globally in time provided the initial datum is sufficiently small in $L^2 \times L^2$.

The single equation with a Kerr nonlinearity has been studied also in bounded domains or on compact manifolds (see, for example, [**14, 15**] and the references therein). The study of coupled nonlinear Schrödinger equations in bounded domains or on compact manifolds should be interesting in view to extend the results for a single equation.

The paper is organized in the following way: Section 9 is devoted to the proofs of the existence results above, in Section 10 it is proved a Gagliardo-Nirenberg inequality (see (4.133)) which is the fundamental tool to obtain Theorems 4.5 and 4.6. Section 11 deals with a blow-up result which shows the sharpness of constant $C_{n,\beta}$, while in Section 12 the proof of Theorem 4.6 is completed.

9. Global existence results

The first part of this work is devoted to the proof of Theorem 4.5. The theory for the single nonlinear Schrödinger equation (4.126) was developed in [**46**] and [**63**]; the proof of the local well-posedness is a contraction argument based on Strichartz estimates, and the conservation of both the mass and the energy allows to extend the local solution globally in time. The fixed point technique also works in the case of a system, hence problem (4.125) is locally well-posed in H^1 for $0 \le p \le 2/(n-2)$. We omit here the straightforward computations, see for example Remarks 4.2.13 and 4.3.4 in [**20**].

Let us now study the conservation laws for system (4.125). Multiplying the equations in (4.125) by $\overline{\phi}$ and $\overline{\psi}$ respectively, integrating in x and taking the resulting imaginary parts, we see that

$$\frac{d}{dt}\|\phi\|_2^2 = 0, \qquad \frac{d}{dt}\|\psi\|_2^2 = 0, \tag{4.131}$$

i.e. the conservation of the masses. Note that these computations make sense if ϕ, ψ are H^1 solutions (it is possible to prove (4.131) also in the case of L^2 solutions, following for example the techniques of [**81**]).

Now we consider the energy $E(t)$ defined in (4.128). Let (ϕ, ψ) be a solution to (4.125); multiplying the equations in (4.125) by $\overline{\phi}_t$ and $\overline{\psi}_t$ respectively, integrating by parts in x and taking the resulting real parts, we easily obtain the energy conservation

$$E'(t) = 0. \tag{4.132}$$

This formal computation needs H^2 regularity for ϕ, ψ, but (4.132) makes sense (and can be proved) also for H^1 solutions. To prove this, following exactly the same computations of Ozawa in [**81**], Proposition 2; we omit here the details.

In order to obtain an a priori control on the gradient of the solutions, we introduce a Gagliardo-Nirenberg inequality (see Section 10 below):

$$\left(\|u\|_{2p+2}^{2p+2} + 2\beta\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2}\right) \le C_{n,p,\beta}\left(\|u\|_2^2 + \|v\|_2^2\right)^{p+1-p\frac{n}{2}}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{p\frac{n}{2}}, \tag{4.133}$$

that gives the following bound from below:

$$E(t) \ge \frac{1}{2}(\|\nabla\phi\|_2^2 + \|\nabla\psi\|_2^2)\left[1 - \frac{C_{n,p,\beta}}{p+1}\left(\|\phi\|_2^2 + \|\psi\|_2^2\right)^{p+1-p\frac{n}{2}}\left(\|\nabla\phi\|_2^2 + \|\nabla\psi\|_2^2\right)^{p\frac{n}{2}-1}\right]. \tag{4.134}$$

If $p < 2/n$, we easily see by (4.134) that the norms $\|\nabla\phi\|_2, \|\nabla\psi\|_2$ cannot blow up in a finite time, because of the conservation of both the mass and the energy; as a consequence, global well-posedness in H^1 is proved in the subcritical range. The power $p = 2/n$ is critical, in the sense that this nonlinearity is sufficiently high to generate H^1 solutions blowing up in a finite time. On the other hand, also in this case, the smallness assumption

$$(\|\phi\|_2^2 + \|\psi\|_2^2)^{2/n} < \frac{p+1}{C_{n,p,\beta}} \tag{4.135}$$

allows by (4.134) to obtain the same a priori control for the gradient in terms of the energy, hence the global existence in the Theorems 4.5 and 4.6 is proved. The last part of Theorem 4.6 is proved in Section 11.

10. Gagliardo-Nirenberg inequality

Our next step is to discuss, following the approach of [**114**], the behavior of the best constant $C_{n,p,\beta}$ in the Gagliardo-Nirenberg inequality (4.133); this will allow us to understand which is the critical initial level defining the border line between global well-posedness and blow-up phenomena. This involves the existence of minimal energy stationary solutions of (4.125) and allows us to clarify the concept of *ground state*.

Consider the functional

$$J_{n,p,\beta}(u,v) = \frac{\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{pn/2}\left(\|u\|_2^2 + \|v\|_2^2\right)^{p+1-pn/2}}{\left(\|u\|_{2p+2}^{2p+2} + 2\beta\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2}\right)}, \quad u, v \in H^1;$$

the infimum of $J_{n,p,\beta}$ on $H^1 \times H^1$ is clearly the reciprocal of the best constant $C_{n,p,\beta}$ in (4.133). First of all we want to point out that, for any $u, v \in H^1$ and for any $\mu, \lambda > 0$, if we set $u_{\mu,\lambda}(x) = \mu u(\lambda x)$ and $v_{\mu,\lambda}(x) = \mu v(\lambda x)$ it follows

$$\|u_{\mu,\lambda}\|_2^2 = \mu^2\lambda^{-n}\|u\|_2^2, \quad \|\nabla u_{\mu,\lambda}\|_2^2 = \mu^2\lambda^{2-n}\|\nabla u\|_2^2,$$
$$\|v_{\mu,\lambda}\|_2^2 = \mu^2\lambda^{-n}\|v\|_2^2, \quad \|\nabla v_{\mu,\lambda}\|_2^2 = \mu^2\lambda^{2-n}\|\nabla v\|_2^2,$$
$$\|u_{\mu,\lambda}\|_{2p+2}^{2p+2} = \mu^{2p+2}\lambda^{-n}\|u\|_{2p+2}^{2p+2}, \quad \|v_{\mu,\lambda}\|_{2p+2}^{2p+2} = \mu^{2p+2}\lambda^{-n}\|v\|_{2p+2}^{2p+2},$$

so that

$$J_{n,p,\beta}(u_{\mu,\lambda}, v_{\mu,\lambda}) = J_{n,p,\beta}(u,v).$$

Assume that the infimum of $J_{n,p,\beta}$ is achieved by $(\tilde u, \tilde v)$: since the value of the functional is invariant with respect to the above scalings, we can assume that the best constant in (4.133) is achieved by the pair $(\tilde u, \tilde v)$ such that

$$\left(\|\tilde u\|_2^2 + \|\tilde v\|_2^2\right) = \left(\|\nabla \tilde u\|_2^2 + \|\nabla \tilde v\|_2^2\right) = 1.$$

Therefore $(\tilde u, \tilde v)$ is a weak solution of the following system of two weakly coupled elliptic equations

$$\begin{cases} -\dfrac{pn}{2}\Delta \tilde u + \dfrac{(2-n)p+2}{2}\tilde u = \dfrac{1}{C_{n,p,\beta}}\left(|\tilde u|^{2p} + \beta|\tilde u|^{p-1}|\tilde v|^{p+1}\right)\tilde u \\ -\dfrac{pn}{2}\Delta \tilde v + \dfrac{(2-n)p+2}{2}\tilde v = \dfrac{1}{C_{n,p,\beta}}\left(|\tilde v|^{2p} + \beta|\tilde v|^{p-1}|\tilde u|^{p+1}\right)\tilde v, \end{cases}$$

Now consider the pair $(\tilde u_{\mu,\lambda}, \tilde v_{\mu,\lambda})$, corresponding to the choice of parameters

$$\mu = \left(\frac{2}{C_{n,p,\beta}(2p+2-pn)}\right)^{1/2p}, \quad \lambda = \left(\frac{pn}{(2p+2-pn)}\right)^{1/2};$$

this pair solves the following elliptic system

$$\begin{cases} -\Delta \tilde u_{\mu,\lambda} + \tilde u_{\mu,\lambda} = \left(|\tilde u_{\mu,\lambda}|^{2p} + \beta|\tilde u_{\mu,\lambda}|^{p-1}|\tilde v_{\mu,\lambda}|^{p+1}\right)\tilde u_{\mu,\lambda} \\ -\Delta \tilde v_{\mu,\lambda} + \tilde v_{\mu,\lambda} = \left(|\tilde v_{\mu,\lambda}|^{2p} + \beta|\tilde v_{\mu,\lambda}|^{p-1}|\tilde u_{\mu,\lambda}|^{p+1}\right)\tilde v_{\mu,\lambda}. \end{cases} \tag{4.136}$$

Note that the preceding system is variational in nature, so that any (weak) solution is a critical point of the functional

$$I_{n,p,\beta}(u,v) = \frac{1}{2}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|u\|_2^2 + \|v\|_2^2\right) - \frac{1}{2p+2}\left(\|u\|_{2p+2}^{2p+2} + \beta\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2}\right).$$

Recently the problem of existence of positive solutions for elliptic systems of this kind has been studied by many authors (see, for example, [**4, 7, 35, 71, 74, 96, 120**]). In [**74**] particular attention is given to the existence and some qualitative properties of the ground state solutions of (4.136): a ground state solution is a nontrivial solution (i.e. distinct from the pair $(0,0)$) which has the least critical level. In particular it is possible to prove existence of ground state solutions for system (4.136) solving the following minimization problem

$$\inf_{(u,v)\in\mathcal{N}} I_{n,p,\beta}(u,v), \tag{4.137}$$

where $\mathcal{N} \subset H^1 \times H^1$ is the Nehari manifold, that is

$$\mathcal{N} = \left\{(u,v) \neq (0,0) : \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|u\|_2^2 + \|v\|_2^2 = \|u\|_{2p+2}^{2p+2} + 2\beta\|uv\|_{p+1}^{p+1}\|v\|_{2p+2}^{2p+2}\right\}.$$

Since $\mathcal{N}$ is a smooth (of class C^2) manifold containing all the nontrivial critical points of the functional, that is all the weak solutions of (4.136), clearly a ground state solution has to realize the minimum. We want to point out that $I_{n,p,\beta}$ is bounded from below on $\mathcal{N}$, so the minimization problem (4.137) is well-posed, moreover it is possible to prove that any minimizing sequences is compact (up to translations) and that the minimum is achieved.

Let

$$m_{n,p,\beta} = \inf_{\mathcal{N}} I_{n,p,\beta} = I_{n,p,\beta}(\tilde{u}_{\mu,\lambda}, \tilde{v}_{\mu,\lambda})$$

be the level of any ground state solution of (4.136); we want to prove that there is a direct relation between $C_{n,p,\beta}$ and $m_{n,p,\beta}$. Recalling that any critical point of $I_{n,p,\beta}$ is a weak solution of (4.136), multiplying (4.136) by $(\tilde{u}_{\mu,\lambda}, \tilde{v}_{\mu,\lambda})$ and integrating on $\mathbb{R}^n$ we obtain that

$$\begin{aligned} &\|\nabla\tilde{u}_{\mu,\lambda}\|_2^2 + \|\tilde{u}_{\mu,\lambda}\|_2^2 = \|\tilde{u}_{\mu,\lambda}\|_{2p+2}^{2p+2} + \beta\|\tilde{u}_{\mu,\lambda}\tilde{v}_{\mu,\lambda}\|_{p+1}^{p+1}, \\ &\|\nabla\tilde{v}_{\mu,\lambda}\|_2^2 + \|\tilde{v}_{\mu,\lambda}\|_2^2 = \|\tilde{v}_{\mu,\lambda}\|_{2p+2}^{2p+2} + \beta\|\tilde{u}_{\mu,\lambda}\tilde{v}_{\mu,\lambda}\|_{p+1}^{p+1}. \end{aligned} \tag{4.138}$$

Moreover in this case, Pohozaev identity reads

$$\begin{aligned} &\frac{n-2}{2}\left(\|\nabla\tilde{u}_{\mu,\lambda}\|_2^2 + \|\nabla\tilde{v}_{\mu,\lambda}\|_2^2\right) + \frac{n}{2}\left(\|\tilde{u}_{\mu,\lambda}\|_2^2 + \|\tilde{v}_{\mu,\lambda}\|_2^2\right) \\ &\qquad = \frac{n}{2p+2}\left(\|\tilde{u}_{\mu,\lambda}\|_{2p+2}^{2p+2} + 2\beta\|\tilde{u}_{\mu,\lambda}\tilde{v}_{\mu,\lambda}\|_{p+1}^{p+1} + \|\tilde{v}_{\mu,\lambda}\|_{2p+2}^{2p+2}\right). \end{aligned} \tag{4.139}$$

Putting together the above identities we have that

$$\begin{aligned} &\mu^2\lambda^{2-n}\left(\|\nabla\tilde{u}\|_2^2 + \|\nabla\tilde{v}\|_2^2\right) = \left(\|\nabla\tilde{u}_{\mu,\lambda}\|_2^2 + \|\nabla\tilde{v}_{\mu,\lambda}\|_2^2\right) = n m_{n,p,\beta}, \\ &\mu^2\lambda^{-n}\left(\|\tilde{u}\|_2^2 + \|\tilde{v}\|_2^2\right) = \left(\|\tilde{u}_{\mu,\lambda}\|_2^2 + \|\tilde{v}_{\mu,\lambda}\|_2^2\right) = \left(2 - n + \frac{2}{p}\right) m_{n,p,\beta}, \\ &\mu^{2p+2}\lambda^{-n}\left(\|\tilde{u}\|_{2p+2}^{2p+2} + 2\beta\|\tilde{u}\tilde{v}\|_{p+1}^{p+1} + \|\tilde{v}\|_{2p+2}^{2p+2}\right) \\ &\quad = \left(\|\tilde{u}_{\mu,\lambda}\|_{2p+2}^{2p+2} + 2\beta\|\tilde{u}_{\mu,\lambda}\tilde{v}_{\mu,\lambda}\|_{p+1}^{p+1} + \|\tilde{v}_{\mu,\lambda}\|_{2p+2}^{2p+2}\right) = \frac{2p+2}{p} m_{n,p,\beta}. \end{aligned}$$

All the above calculations imply that the following equalities hold

$$\frac{1}{C_{n,p,\beta}} = J_{n,p,\beta}(\tilde{u},\tilde{v}) = J_{n,p,\beta}(\tilde{u}_{\mu,\lambda}, \tilde{v}_{\mu,\lambda}) = m_{n,p,\beta}^p \frac{n^{pn/2}(2p+2-pn)^{p+1-pn/2}}{2(p+1)p^{p-pn/2}}. \tag{4.140}$$

Note that, in the critical case $p = 2/n$, (4.140) becomes

$$\frac{1}{C_{n,2/n,\beta}} = 2^{2/n}\frac{n}{n+2} m_{n,2/n,\beta}^{2/n} = \frac{n}{n+2}\left(\|\tilde{u}_{\mu,\lambda}\|_2^2 + \|\tilde{v}_{\mu,\lambda}\|_2^2\right)^{2/n}. \tag{4.141}$$

The arguments above, in particular (4.140), show that a minimum point of $J_{n,p,\beta}$, through a suitable scaling, has to correspond to a ground state solution of (4.136) (or to a least energy nontrivial critical point of $I_{n,p,\beta}$). Now, since in [**74**] it is proved the existence of ground state solutions to (4.136), we have obtained the existence of a minimum point for the functional $J_{n,p,\beta}$; this shows that inequality (4.133) is sharp and that there exists at least a pair of functions for which equality holds. More generally we have proved that the functionals $J_{n,p,\beta}$ and $I_{n,p,\beta}$ possess the same number of critical values.

The validity of inequality (4.133) follows by the above arguments.

11. Blow-up results

In view to prove the sharpness of the constant C inthe statement of Theorem 4.6, we introduce (following [**48**] and [**89**]) another physically relevant quantity, that plays a crucial role in the analysis of blow-up phenomena: the *variance* $V(t)$, which is defined by

$$V(t) = \int |x|^2|\phi(t,x)|^2\,dx + \int |x|^2|\psi(t,x)|^2\,dx. \tag{4.142}$$

As in the case of a single Schrödinger equation, we will prove a relation between the time behavior of V and that of the H^1-norm of the solutions: as we will see in the following, the precise calculation of the first and second derivatives of V in terms of the solutions of (4.125) is the main tool for the description of the blow-up (see for example [**20**] for a proof in the case of a single equation).

More precisely, we prove the following Lemma:

LEMMA 4.7. *Let (ϕ,ψ) be a solution of system* (4.125) *on an interval $I = (-t_1, t_1)$; then, for each $t \in I$, the variance satisfies the following identities:*

$$V'(t) = 4\Im \int \left[\left(x\cdot\nabla\phi\right)\overline{\phi} + \left(x\cdot\nabla\psi\right)\overline{\psi}\right]\,dx, \tag{4.143}$$

$$V''(t) = 8\int \left(|\nabla\phi|^2 + |\nabla\psi|^2\right)dx - \frac{4np}{p+1}\int\left(|\phi|^{2p+2} + 2\beta|\phi\psi|^{p+1} + |\psi|^{2p+2}\right)dx. \tag{4.144}$$

Proof. We introduce the following notations:

$$\begin{aligned} z =&(z^1,\dots,z^n)\in\mathbb{C}^n;\\ z\cdot w =&\sum_i z^i w^i, \qquad z,w\in\mathbb{C}^n;\\ u_i =&\frac{\partial u}{\partial x_i}, \qquad u:\mathbb{R}^n\to\mathbb{C}. \end{aligned}$$

Multiplying the equations in (4.125) by $2\overline{\phi}$ and $2\overline{\psi}$ respectively, and taking the resulting imaginary parts, we obtain

$$\frac{\partial}{\partial t}|\phi|^2 = -2\Im(\overline{\phi}\Delta\phi) = -2\nabla\cdot(\Im\overline{\phi}\nabla\phi), \tag{4.145}$$

$$\frac{\partial}{\partial t}|\psi|^2 = -2\Im(\overline{\psi}\Delta\psi) = -2\nabla\cdot(\Im\overline{\psi}\nabla\psi). \tag{4.146}$$

Now, multiplying (4.145) and (4.146) by $|x|^2$, and integrating by parts in x, we immediately obtain (4.143).

In order to prove (4.144), let us multiply the equations in (4.125) by $2(x\cdot\nabla\overline{\phi})$ and $2(x\cdot\nabla\overline{\psi})$ respectively, let us integrate in x and sum the equations for the real parts, to get:

$$\begin{aligned} 0 =&2\Re\int i\left[(x\cdot\nabla\overline{\phi})\phi_t(x\cdot\nabla\overline{\psi})\psi_t\right]dx + 2\Re\int\left[(x\cdot\nabla\overline{\phi})\Delta\phi + (x\cdot\nabla\overline{\psi})\Delta\psi\right]dx\\ &+ 2\Re\int\left[(x\cdot\nabla\overline{\phi})(|\phi|^{2p} + \beta|\psi|^{p+1}|\phi|^{p-1})\phi+\right.\\ &\left.+(x\cdot\nabla\overline{\psi})(|\psi|^{2p} + \beta|\phi|^{p+1}|\psi|^{p-1})\psi\right]dx. \end{aligned}$$

We rewrite the last identity in the form

$$\mathrm{I} = \mathrm{II} + \mathrm{III}, \tag{4.147}$$

where

$$\begin{aligned}
\mathrm{I} =& 2\Re \int i\big[(x\cdot\nabla\overline{\phi})\phi_t (x\cdot\nabla\overline{\psi})\psi_t\big]\,dx,\\
\mathrm{II} =& -2\Re \int \big[(x\cdot\nabla\overline{\phi})\Delta\phi + (x\cdot\nabla\overline{\psi})\Delta\psi\big]\,dx,\\
\mathrm{III} =& -2\Re \int \big[(x\cdot\nabla\overline{\phi})(|\phi|^{2p} + \beta|\psi|^{p+1}|\phi|^{p-1})\phi\\
&+(x\cdot\nabla\overline{\psi})(|\psi|^{2p} + \beta|\phi|^{p+1}|\psi|^{p-1})\psi\big]\,dx.
\end{aligned}$$

For the first term, we have

$$\mathrm{I} = -\Re\int i\sum_j \left(x^j\overline{\phi}_j\phi_t - x^j\phi_j\overline{\phi}_t + x^j\overline{\psi}_j\psi_t - x^j\psi_j\overline{\psi}_t\right)\,dx,$$

which can be written in the form

$$\begin{aligned}
\mathrm{I} =& \Re\int i\sum_j x^j\left[(\overline{\phi}_j\phi)_t - (\phi\overline{\phi}_t)_j + (\overline{\psi}_j\psi)_t - (\psi\overline{\psi}_t)_j\right]\,dx\\
=&\frac{d}{dt}\Re\int i\left[(x\cdot\nabla\overline{\phi})\phi + (x\cdot\nabla\overline{\psi})\psi\right]\,dx + n\Re\int i(\phi\overline{\phi}_t + \psi\overline{\psi}_t)\,dx.
\end{aligned}$$

Now we evaluate the last equality using the equations in (4.125), obtaining

$$\begin{aligned}
\mathrm{I} =& \frac{d}{dt}\Im\int\left[(x\cdot\nabla\phi)\overline{\phi} + (x\cdot\nabla\psi)\overline{\psi}\right]\,dx - n\int\left(|\nabla\phi|^2 + |\nabla\psi|^2\right)\,dx\\
&+ n\int\left[(|\phi|^{2p} + \beta|\psi|^{p+1}|\phi|^{p-1})|\phi|^2 + (|\psi|^{2p} + \beta|\phi|^{p+1}|\psi|^{p-1})|\psi|^2\right]\,dx\\
=& \frac{d}{dt}\Im\int\left[(x\cdot\nabla\phi)\overline{\phi} + (x\cdot\nabla\psi)\overline{\psi}\right]\,dx - n\int\left(|\nabla\phi|^2 + |\nabla\psi|^2\right)\,dx\\
&+ n\int\left(|\phi|^{2p+2} + 2\beta|\psi\phi|^{p+1} + |\psi|^{2p+2}\right)\,dx.
\end{aligned} \tag{4.148}$$

A multiple integration by parts in II gives the Pohozaev identity

$$\mathrm{II} = (2-n)\int\left(|\nabla\phi|^2 + |\nabla\psi|^2\right)\,dx. \tag{4.149}$$

As for the term III, we write it by components:

$$\begin{aligned}
\mathrm{III} = -\sum_j\int&\Big\{x^j\left[|\phi|^{2p}(2\Re\overline{\phi}_j\phi) + |\psi|^{2p}(2\Re\overline{\psi}_j\psi)\right]\\
&+\beta x^j\left[|\phi|^{p-1}|\psi|^{p+1}(2\Re\overline{\phi}_j\phi) + |\psi|^{p-1}|\phi|^{p+1}(2\Re\overline{\psi}_j\psi)\right]\Big\}\,dx.
\end{aligned} \tag{4.150}$$

Observe that

$$\begin{aligned}
|\phi|^{2p}(2\Re\overline{\phi}_j\phi) + |\psi|^{2p}(2\Re\overline{\psi}_j\psi) &= \frac{1}{p+1}\left(|\phi|_j^{2p+2} + |\psi|_j^{2p+2}\right),\\
|\phi|^{p-1}|\psi|^{p+1}(2\Re\overline{\phi}_j\phi) + |\psi|^{p-1}|\phi|^{p+1}(2\Re\overline{\psi}_j\psi) &= \frac{2\beta}{p+1}\left(|\phi|^{p+1}|\psi|^{p+1}\right)_j;
\end{aligned}$$

hence, integrating by parts in (4.150) we have

$$\text{III} = \frac{n}{p+1} \int \left(|\phi|^{2p+2} + |\psi|^{2p+2} + 2\beta |\phi|^{p+1} |\psi|^{p+1} \right) dx. \tag{4.151}$$

Finally, recollecting (4.147), (4.148), (4.149), (4.151) and (4.143), we complete the proof of (4.144). ∎

REMARK 4.8. Note that (4.144) can be rewritten, recalling the definition of E, in the following equivalent form

$$V''(t) = 16E(t) - 8\frac{np-2}{2p+2} \int \left(|\phi|^{2p+2} + 2\beta |\phi\psi|^{p+1} + |\psi|^{2p+2} \right) dx. \tag{4.152}$$

In the critical case $p = 2/n$ the equation above reduces to

$$V''(t) = 16E(t);$$

hence the variance V of any solution of (4.125) with negative initial energy vanish in a finite time. For each $h : \mathbb{R}^n \to \mathbb{C}$ we can estimate

$$\|h\|_{L^2}^2 = \||h|^2\|_{L^1} \leq \||x|h\|_{L^2} \left\| \frac{h}{|x|} \right\|_{L^2},$$

by Cauchy-Schwartz inequality. As a consequence of the standard Hardy's inequality we obtain

$$\|h\|_{L^2}^2 \leq \||x|h\|_{L^2} \|\nabla h\|_{L^2}.$$

Applying the last inequality to any solution of of (4.125) with negative initial energy, since the mass is conserved and the variance vanish in a finite time the L^2 norm of the gradient needs necessarely to blow up in a finite time.

REMARK 4.9. Consider the following pair

$$\frac{e^{-i\frac{|x|^2-4}{4(1-t)}}}{(1-t)^{n/2}} \left(U\left(\frac{x}{1-t}\right), V\left(\frac{x}{1-t}\right) \right),$$

where (U, V) is a ground state solution of (4.136). This is an explicit example of a blow-up solution, which shows that Theorem 4.6 is sharp. Indeed the pair solves (4.125) and has initial value $e^{-i(|x|^2-4)/4}(U(x), V(x))$ which attains the critical blow-up threshold.

12. On the blow-up threshold

If $p = 2/n$, we have obtained the following characterization of the blow-up threshold

$$\frac{1}{C_{n,2/n,\beta}} = \frac{1}{C_{n,\beta}} = \frac{n}{n+2} \left(\|U\|_2^2 + \|V\|_2^2 \right)^{2/n}, \tag{4.153}$$

where (U, V) is a ground state solution of (4.136). In order to prove Theorem 4.6 we have to estimate the quantities involved in (4.153).

In [**74**] (see Theorem 2.5) it is proved that, if $\beta < 2^{2/n} - 1$, then any ground state of the elliptic system (4.136) is a scalar function, that is one of the components of the ground state solution is zero. So we can assume, without loss of generality, that the ground state is $(z, 0)$, where $z \in H^1$ is the unique ground state solution (see [**114**]) of the equation

$$-\Delta z + z = |z|^{\frac{4}{n}} z. \tag{4.154}$$

This implies that the constant $C_{n,\beta} = C_n$ depends only on n for any $\beta \leq 2^{2/n} - 1$, since the coupling parameter β now does not play a role in the problem of selecting the ground state solution. Moreover C_n is exactly the blow-up threshold for a single nonlinear Schrödinger equation, introduced and numerically computed in [**114**].

If $\beta \geq 2^{2/n} - 1$, $C_{n,\beta}$ depends on n and β and its expression is unknown, but we can estimate it using a suitable test-pair. Let $\hat{z}$ be the unique positive ground state solution of

$$-\Delta \hat{z} + \hat{z} = (1+\beta)|\hat{z}|^{\frac{4}{n}} \hat{z},$$

it is easy to see that the pair $(\hat{z}, \hat{z})$ is a positive solution of (4.136) for any β; and the following inequality holds

$$\frac{1}{C_{n,\beta}} = \frac{n}{n+2} \left(\|U\|_2^2 + \|V\|_2^2\right)^{2/n} \leq \frac{n}{n+2} \left(2\|\hat{z}\|_2^2\right)^{2/n}.$$

Clearly we have an inequality since $(\hat{z}, \hat{z})$ could not be a ground state solution of (4.136).

Using the scaling as in Section 10 we can estimate $C_{n,\beta}$ with C_n. Recalling that z is the ground state solution of (4.154) and noticing that the L^2 norm of z is related to C_n (see (1.3) in [**114**] and also (4.141)) we obtain that

$$\|\hat{z}\|_{4/n}^2 = \frac{\|z\|_{4/n}^2}{1+\beta} = \frac{n+2}{n(1+\beta)C_n}.$$

Collecting the inequalities above we have

$$C_{n,\beta} \geq \left(\frac{1+\beta}{2^{2/n}}\right) C_n,$$

so that the claim is proved. ∎

Now we give some concluding remarks.

Remark 4.10. Note that it is possible to extend this argument to systems with more than two nonlinear Schrödinger equations, using some results about the elliptic counterpart contained in [**4**], Section 6.

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