Nicola Squartini

Trace process and global limit theorems for sums of i.i.d random variables using quasicumulants

Dottorato in Matematica, Roma «La Sapienza» (2007).

<http://www.bdim.eu/item?id=tesi_2007_SquartiniNicola_1>

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December 12, 2007

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Introduction

We present here general limit theorems for sums of i.i.d. random variables under the basic assumption that the logarithm of the characteristic function has the form:

$$(*)\qquad \begin{aligned} \log\varphi(t) &= |t|^{\alpha}\left(-c+\lambda(t)+R(t)\right) \\ \lambda(t) &= \sum_{\substack{k\geq 1, j\geq 0\\ k+j\alpha\leq r}} \lambda_{kj}^{+} t^{k}\,|t|^{j\alpha} \end{aligned}$$

for $t \geq 0$, with $c, \lambda_{kj}^{+} \in \mathbb{C}$, $0 < \alpha \leq 2$ and $R(t) = o(|t|^{r})$ $t \to 0$; for $t \leq 0$ with have the same expression in term of coefficients λ_{kj}^{-}. The coefficients λ_{kj}^{-} and λ_{kj}^{+} are called the *left* and *right quasicumulants* of the characteristic function, and they are extremely useful quantities when the usual cumulants or moments do not exist. In the case $\alpha = 1, 2$ (i.e. when the random variable is in the domain of attraction of the Cauchy and Gaussian law respectively) the quasicumulants correspond to the left and right derivative of $\log\varphi(t)$ at the point $t = 0$.

If (X_n) is a sequence of i.i.d. random variables with characteristic function satisfying (*) it is possible to derive in terms of the quasicumulants an asymptotic expansion in powers of $n^{-k/\alpha-j}$ for the probability $p_n(x) := \mathbb{P}(S_n = x)$, $S_n = X_1 + \cdots + X_n$, if X_n are lattice random variables, or for the density $p_n(x)$ of S_n, if X_n have density. This expansion generalize the classical Chebyshev-Edgeworth-Cramér expansion for for sums of i.i.d. random variables having finite variance and a certain number of higher order moments. If we also assume the differentiability C^{ν} of $\varphi(t)$ in all point except the origin (and $2k\pi$, $k \in \mathbb{Z}$, for lattice random variables) we can improve the estimate in x of the remainder term of the asymptotic expansion. Using this result it is easy to prove in the cases $0 < \alpha \leq 1$ and $\alpha = 2$ the *global limit theorem*, i.e. large deviation holding uniformly on the whole real line.

The collection of all characteristic functions satisfying (*) is called Δ_{α}; the name was an idea of Petrov (see [7]) who noticed that such characteristic functions, having a singularity at $t = 0$, resemble a "Δ".

The motivation of the assumption (*) came from the following problem. If $\{(X_n, Y_n)\}$ is a two dimensional random walk, let τ_n be the successive times when $Y_n = 0$, and define $\mathfrak{X}_n = X_{\tau_n}$: $\mathfrak{X}_n$ is the *trace* of the random walk $\{(X_n, Y_n)\}$ through the x-axis. In the continuous time case it was shown by Molchanov [6] that the trace of a two dimensional Brownian motion on the x-axis, parametrized by the local time on the axis, is a Cauchy process. In analogy we proved in [7] that the random variable $\mathfrak{X}_1$ belongs to the domain of attraction of the Cauchy law, and more the characteristic function satisfies:

$$\log \varphi(t) = \begin{cases} -ct + \sum_{k=2}^{r} c_k^+ (it)^k + R^+(t) & t \geq 0 \\ ct + \sum_{k=2}^{r} c_k^- (it)^k + R^-(t) & t < 0 \end{cases}$$

with $c > 0$, and R^+ and R^- are $o(|t|^r)$; so $\varphi(t) \in \Delta_1$. We then wanted to see if we could get example of (*) with general $0 < \alpha < 2$ using a procedure similar to the trace process, i.e. by subordinating a random walk with finite variance to a sequence of random times. The local times arising from the trace process all belong to the stable law of exponent $1/2$, so in order to construct random times with different exponent we considered the return times to 0 of the Markov chain on the natural numbers corresponding to the Jacobi polynomials: varying one of the parameters in the definition of the Jacobi polynomials we can get all the different exponents $0 < \alpha' < 1$ for the random times, and all $0 < \alpha < 2$ in (*). The equivalence between families of orthogonal polynomials and Markov chains on the natural numbers can be found in [14], while for a discussion on the subordinated processes in continuous time we suggest [6] or [11].

The idea of using some substitute of the moments in proving limit theorems seems to rise to Linnik ([4], Ch. 14): he introduced the "pseudomoments" defined as the left and right derivatives of the characteristic function at the point $t = 0$, and used them to in order to prove limit theorems holding on the whole real line for the distributions of sums of i.i.d. random variables with finite variance and regular-decaying density. Linnik assumptions are a particular case of (*) with $\alpha = 2$, and his result follows from the consideration in Chapter 4.

In Chapter 1 we introduce the trace process in a general setting and derive the formula for the characteristic function. We prove that in the case of the two dimensional random walk the characteristic function satisfies (*) with $\alpha = 1$.

In Chapter 2 we introduce the analogy between Markov chains on the natural numbers and families of orthogonal polynomials, which we took from [14], and we develop the particular case of the Jacobi polynomials, deducing the asymptotic for the generating function of the return time to 0.

In Chapter 3 we study the asymptotic expansion of $p_n(x)$ assuming (*), and we give non-uniform estimate on the remainder term when the differentiability of the characteristic function outside $t = 0$ is assumed.

In Chapter 4 we derive the global limit theorems for $0 < \alpha \leq 1$ and $\alpha = 2$ as a corollary of the results of Chapter 3. The global limit theorem for $\alpha = 2$ incorporates the result of Linnik ([4], Ch. 14) but the method for the proof is different.

Chapter 1

Trace process and Cauchy law

1.1 Introduction

We begin our discussion on the limit theorems with an example of random walk whose characteristic function can be analyzed in term of quantities which we will later call *quasicumulants*. Consider a 2-dimensional random walk $\{(X_n, Y_n)\}_{n\geq 0}$ where the random variables X_n and Y_n represent the position of a particle on the x-axis and the y-axis respectively; assume $X_0 = Y_0 = 0$. Let $A = \{(x, 0) \colon x \in \mathbb{Z}\}$ be the x-axis, we define τ_n as the successive times for which the particle is in A; let $\mathfrak{X}_n = X_{\tau_n}$. We call the new random walk $\{\mathfrak{X}_n\}_{n\geq 0}$ the *trace process*, as it is the trace that the particle leaves on the x-axis. In this chapter we study the characteristic function of the trace process.

1.2 The general case

Let us start from the following general case: let $\{X_n\}$ be a homogeneous Markov chain on the countable state space S, with transition matrix $P = (p_{x,y})_{x,y\in S}$. Let $A \subset S$ be a proper subset, and assume that the random time $\tau = \min\{n \geq 1 \colon X_n \in A\}$ is finite with probability 1. Let $\{\mathfrak{X}_n\}$ be the process defined by the trace that X_n leaves in its passages in A, i.e. $\mathfrak{X}_n = X_{\tau_n}$, where τ_n is the time when $\{X_n\}$ returns to A for the nth time. We are going to derive an expression for the transition matrix Q of the Markov chain $\{\mathfrak{X}_n\}$; the entries of Q are $q_{x,y} = \mathbb{P}_x(X_\tau = y)$, $x, y \in A$.

Given a bounded function f on A, we define, for $z \in \mathbb{C}$ and $|z| \leq 1$, the linear operator

$$(U_z f)(x) = \mathbb{E}_x[z^{\tau'} f(X_{\tau'})], \quad x \in S$$

where $\tau' := \min\{n \geq 0 \colon X_n \in A\}$; it is well defined since the argument of the expected value is bounded. It is well known from the discrete potential theory, that the function $u_z(x) := (U_z f)(x)$ satisfies the relation

$$\begin{cases} u_z(x) = z(Pu_z)(x) & \text{if } x \notin A, \\ u_z(x) = f(x) & \text{if } x \in A. \end{cases} \tag{1.1}$$

If we denote by $l^\infty(S)$ and $l^\infty(A)$ the spaces of bounded functions on S and A respectively, which are Banach spaces with respect to the supremum norm, it is clear that $U_z\colon l^\infty(A) \to l^\infty(S)$ has $\|U_z\| \le 1$. For $|z| \le 1$ we also define the operator $Q_z\colon l^\infty(A) \to l^\infty(A)$

$$(Q_z f)(x) = \mathbb{E}_x[z^\tau f(X_\tau)], \quad x \in A.$$

This operator is a strongly continuous function of the parameter $|z| \le 1$, this means that $Q := Q_1 = \lim_{z\to 1} Q_z$ and, as easy to see Q is the transition operator of the trace process on A.

In order to find an expression for Q_z, will will investigate the relation between Q_z and U_z. Namely

$$\begin{aligned}(Q_z f)(x) = \mathbb{E}_x[z^\tau f(X_\tau)] &= \sum_{n=1}^{+\infty} P(\tau = n)\mathbb{E}_x[z^n f(X_n)|\tau = n] \\ &= \sum_{y\in A} p(x,y) z f(y) + \sum_{n=2}^{+\infty}\sum_{y\notin A} z^n P_x(X_1 = y, \tau = n)\mathbb{E}_y[f(X_n)] \\ &= z\sum_{y\in A} p(x,y) f(y) + z\sum_{y\notin A} p(x,y)\mathbb{E}_y[z^{\tau'} f(X_{\tau'})] \\ &= z(RPU_z f)(x)\end{aligned}$$

where $R\colon l^\infty(S) \to l^\infty(A)$ is the restriction operator $(Rf)(x) = f(x)$, $x \in A$ (above we used the fact that if the initial point y is not in A then $\tau = \tau'$). Therefore $Q_z = zRPU_z$.

What about U_z? Introducing the extension operator $M_{\delta_A}\colon l^\infty(A) \to l^\infty(S)$[1] defined as

$$M_{\delta_A} f(x) = \begin{cases} f(x) & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

we can rewrite relation (1.1) in one line as

$$(U_z f)(x) = z(PU_z f)(x) + (M_{\delta_A}(1 - Q_z)f)(x),$$

which means

$$(1 - zP)U_z = M_{\delta_A}(1 - Q_z)$$

If $|z| < 1$, then $1 - zP$ is invertible, hence $U_z = (1 - zP)^{-1}M_{\delta_A}(1 - Q_z)$. So

$$\begin{aligned}Q_z &= zRP(1 - zP)^{-1}M_{\delta_A}(1 - Q_z) \\ &= zRP(1 - zP)^{-1}M_{\delta_A} - zRP(1 - zP)^{-1}M_{\delta_A}Q_z\end{aligned}$$

from which

$$[1 + zRP(1 - zP)^{-1}M_{\delta_A}]Q_z = zRP(1 - zP)^{-1}M_{\delta_A}$$
$$[1 + zRP(1 - zP)^{-1}M_{\delta_A}]Q_z = 1 + zRP(1 - zP)^{-1}M_{\delta_A} - 1$$

and finally

$$Q_z = 1 - [1 + zRP(1 - zP)^{-1}M_{\delta_A}]^{-1} \tag{1.2}$$

if $|z|$ is small enough.

[1]The notation for the extension operator comes from the fact that it "multiplies by the function δ_A".

1.3 The trace process

Now we specialize the formula in (1.2) to the case of an ordinary lattice random walk, i.e. $S = \mathbb{Z}^{d+1}$, $A = \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}\colon y = 0\}$ and the transition matrix P is translation invariant, that is

$$p_{(x_1,y_1),(x_2,y_2)} = p(x_2 - x_1, y_2 - y_1), \quad x_1, x_2 \in \mathbb{Z}^d \text{ and } y_1, y_2 \in \mathbb{Z}.$$

As we mention above, the basic assumption is that the return time τ of our random walk $\{(X_n, Y_n)\}$ is finite with probability 1. Condition for this can be found easily by looking at the projection of the random walk $\{(X_n, Y_n)\}$ on the y-axis, which correspond to the random walk $\{Y_n'\}$ with transition probability

$$p'(y) = \sum_{x \in \mathbb{Z}^d} p(x, y)$$

The random time τ is finite with probability 1 if and only if the projected random walk $\{Y_n'\}$ is recurrent. $\{Y_n'\}$ is a 1-dimensional random walk, and it is proved in [12] that a 1-dimensional random walk with finite first moment is recurrent if and only if the first moment is zero.

It is easy to see that the operators R, P and M_{δ_A} can be regarded as operators on l^2 rather then l^∞, and they are still bounded. Hence we get from (1.2) that also $Q_z : l^2(A) \to l^2(A)$ is bounded if $|z|$ is small enough.

Let $\mathcal{F}\colon l^2(S) \to L^2(\mathbb{T}^{d+1})$ and $\mathcal{F}_1 : l^2(A) \to L^2(\mathbb{T}^d)$ be Fourier transforms, we want to find the operator $\hat{Q}_z : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ such that $\hat{Q}_z \mathcal{F}_1 = \mathcal{F}_1 Q_z$. Direct calculation shows that the operator $\hat{M}_{\delta_A} : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^{d+1})$ such that $\hat{M}_{\delta_A} \mathcal{F}_1 = \mathcal{F} M_{\delta_A}$ is given by

$$(\hat{M}_{\delta_A} \hat{f})(s, t) = \hat{f}(s), \quad s \in \mathbb{T}^d \text{ and } t \in \mathbb{T},$$

and the operator $\hat{R}\colon L^2(\mathbb{T}^{d+1}) \to L^2(\mathbb{T}^d)$ such that $\hat{R}\mathcal{F} = \mathcal{F}_1 R$ is

$$(\hat{R}\hat{f})(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(s, t)dt, \quad s \in \mathbb{T}^d \text{ and } t \in \mathbb{T}.$$

Therefore it follows from (1.2) that

$$\hat{Q}_z \hat{f}(s) = \left[1 - \frac{1}{G_z(-s)}\right] \hat{f}(s), \quad s \in \mathbb{T}^d \tag{1.3}$$

where $G_z(s)$ is given by the expression

$$G_z(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dt}{1 - z\hat{p}(s, t)}. \tag{1.4}$$

Notice that the Fourier transform $\hat{p}'(t)$ of the transitions probability for the projected random walk $\{Y_n'\}$ is given by $\hat{p}(0, t)$, so the function $G_z(0)$ coincide with the Green function of the $\{Y_n'\}$, i.e. $\sum_{n \geq 0} p'^n(0) z^n$.

Theorem 1.3.1. *Let $\{(X_n, Y_n)\}$ be a random walk on the state space $\mathbb{Z}^d \times \mathbb{Z}$, with transition matrix $P = (p(x_2 - x_1, y_2 - y_1))$, $x_i \in \mathbb{Z}^d$, $y_j \in \mathbb{Z}$, $i, j = 1, 2$. If the first moment of the projected random walk $\{Y_n'\}$ is zero, then the trace random walk $\{\mathfrak{X}_n\}$ on the hyperplane $A = \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}\colon y = 0\}$ is well* *defined, and the function $q(x)$, $x \in \mathbb{Z}^d$, which define the transition matrix $Q = (q(x_2 - x_1))$, has the Fourier transform*

$$\hat{q}(s) = 1 - \frac{1}{G(-s)}, \quad s \in \mathbb{T}^d \tag{1.5}$$

with $G(s) = G_1(s)$ as defined in (1.4).

Proof. We know that $Q = \lim_{z \to 1} Q_z$, so passing to the limit in (1.3) gives us the desired result. □

Remark 1.3.1. If $\{X_n\}$ and $\{Y_n\}$ are independent, then $\hat{p}(s,t) = \hat{p}_1(s)\hat{p}_2(t)$ and $G(s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{dt}{1-\hat{p}_1(s)\hat{p}_2(t)}$. It is known (see, for example, [12]) that the generating function of τ, $F(z) = \mathbb{E}[z^\tau]$ can be expressed as

$$F(z) = 1 - \frac{1}{G_z}$$

where $G_z = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{dt}{1-z\hat{p}_2(t)}$. Therefore (1.5) reduces to $\hat{q}(s) = F(\hat{p}_1(s))$.

Remark 1.3.2. For the projected random walk $\{\overline{Y}_n\}$ the Fourier transform of $\bar{p}(y)$ is $\hat{\bar{p}}(t) = \hat{p}(0,t)$, which means that $G(0) = G_1$.

Example 1.3.1. Suppose $\{(X_n, Y_n)\}$ is the usual nearest neighbors random walk $p(\pm 1, \pm 1) = 1/4$, then substituting its Fourier transform $\hat{p}(s,t) = \frac{1}{2}(\cos s + \cos t)$ into (1.5) and integrating we get

$$\hat{q}(s) = 1 - \sqrt{1 - \cos s - \frac{1}{4}\sin^2 s}$$

Example 1.3.2. If $\{X_n\}$ and $\{Y_n\}$ are independent symmetric random walks on $\mathbb{Z}$, with the same transition probability $p(\pm 1) = 1/2$, then $F(z) = \mathbb{E}[z^\tau] = 1 - \sqrt{1-z^2}$ and, by Remark 1.3.1,

$$\hat{q}(s) = 1 - \sqrt{1 - \cos^2 s} = 1 - |\sin s|\,.$$

1.4 Asymptotic of $\hat{q}(s)$

For our consideration we further restrict to the case $d = 1$. We want to study the behaviour of the c.f. $\hat{q}(s)$ and show that, under mild regularity condition on the moments of the original random walk, it belongs to the domain of attraction of the Cauchy law.

Consider the integral

$$G(s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{dt}{1-\hat{p}(s,t)}. \tag{1.6}$$

Of course, as $\hat{p}(s,t)$ is the characteristic function of an aperiodic random walk, if s is away from 0 we have no trouble with integral (1.6), and $G(s)$ has the same regularity as $\hat{p}(s,t)$, which we assume C^{r+3}. We will therefore study $G(s)$ for small value of s.

By the Morse Lemma (see Appendix A) there exist neighborhoods $0 \in U \subset \mathbb{T}$, $0 \in V \subset \mathbb{T}$ and a function $u\colon U \times V \to \mathbb{R}$ of class C^{r+1} such that

$$\hat{p}(s,t) = -u(s,t)^2 + \hat{p}(s,\alpha(s))$$

where $\alpha(s)$ is the maximum of $\hat{p}(s,t)$ as a function of t, for fixed s (it is the maximum because we assume $\hat{p}''(s,t)$ positive definite). By the continuity of $u(s,t)$ we can find a smaller neighborhood $0 \in U_1 \subseteq U$ such that for $s \in U_1$, $u(s,V) \supset \,]-\delta,\delta[$, for some fixed δ independent on s. Hence for $s \in U_1$ we define $A(s) = \{t \in V\colon u(s,t)^2 < \delta^2\}$ and we split the integral

$$\begin{aligned} G(s) &= \frac{1}{2\pi} I_1(s) + \frac{1}{2\pi} I_2(s), \\ I_1(s) &= \int_{A(s)} \frac{dt}{1-\hat{p}(s,t)}, \\ I_2(s) &= \int_{\mathbb{T}\setminus A(s)} \frac{dt}{1-\hat{p}(s,t)}. \end{aligned}$$

The integral $I_2(s)$ is C^{r+3} because the integration variable t is uniformly bounded away from 0 when $s \in U_1$. As for $I_1(s)$, we can change variable and write

$$I_1(s) = \tilde{I}_1(\epsilon) = \int_{-\delta}^{\delta} \frac{f(s,u)}{\epsilon^2 + u^2} du \tag{1.7}$$

$$\epsilon^2 = \epsilon^2(s) = 1 - \hat{p}(s,\alpha(s)) \tag{1.8}$$

where $f(s,u)$ is C^r (notice that the integration variable u of I_2 runs exactly from $-\delta$ to δ by our assumption $s \in U_1$).

Lemma 1.4.1. *If $f(s,x)$ is of class C^{2r}, then the integral*

$$I(\alpha,s) = \alpha \int_{-\delta}^{\delta} \frac{f(s,x)}{\alpha^2 + x^2} dx$$

is of class C^{2r} in the variable (α,s), and $I(0,s) \neq 0$ if $f(s,0) \neq 0$.

Proof. Write the Taylor expansion

$$f(s,x) = \sum_{k=0}^{2r} a_k(s) x^k + \theta(s,x) x^{2r}$$

where $\theta(s,x) \to 0$ when $x \to 0$. Hence

$$\begin{aligned} I(\alpha,s) &= \sum_{k=0}^{r} 2a_{2k}(s)\gamma_k(\alpha) + \tilde{\gamma}_r(\alpha), \\ \gamma_k(\alpha) &= \alpha \int_0^{\delta} \frac{x^{2k}}{\alpha^2 + x^2} dx, \\ \tilde{\gamma}_r(\alpha,s) &= \alpha \int_{-\delta}^{\delta} \frac{\theta(s,x) x^{2k}}{\alpha^2 + x^2} dx. \end{aligned}$$

First notice that the $\gamma_k(\alpha)$ are analytic. Indeed

$$\begin{aligned}\gamma_k(\alpha) &= \alpha \int_0^\delta \frac{x^2 x^{2k-2} + \alpha^2 x^{2k-2}}{\alpha^2 + x^2} dx - \alpha^3 \int_0^\delta \frac{x^{2k-2}}{\alpha^2 + x^2} dx \\ &= \alpha \int_0^\delta x^{2k-2} dx - \alpha^2 \gamma_{k-1}(\alpha) = \sum_{j=1}^{k} \alpha^{2j-1} \int_0^\delta x^{2k-2j} dx + (-1)^k \alpha^{2k} \gamma_0(\alpha)\end{aligned}$$

and $\gamma_0(\alpha) = \arctan \frac{\delta}{\alpha}$ is analytic. As for $\tilde{\gamma}_r(\alpha, s)$, we repeat the same procedure:

$$\begin{aligned}\tilde{\gamma}_r(\alpha, s) &= \alpha \int_{-\delta}^{\delta} \theta(s, x) x^{2k-2} dx - \alpha^2 \tilde{\gamma}_{r-1}(\alpha, s) \\ &= \sum_{j=1}^{r} \alpha^{2j-1} \int_{-\delta}^{\delta} \theta(s, x) x^{2r-2j} dx + (-1)^r \alpha^{2r} \tilde{\gamma}_0(\alpha, s)\end{aligned}$$

where $\tilde{\gamma}_0(\alpha, s) = \alpha \int_{-\delta}^{\delta} \frac{\theta(s,x)}{\alpha^2 + x^2} dx \to 0$ when $\alpha \to 0$. Therefore $I(\alpha, s)$ is a sum of a C^{2r}function plus a remainder term $\alpha^{2r}\tilde{\gamma}_0(\alpha, s)$.

Moreover, since $\gamma_0(0) \neq 0$ and $\gamma_k(0) = 0 = \tilde{\gamma}_r(0, s)$ is k and r are $\neq 0$, then $I(0, s) \neq 0$ if $f(s, 0) \neq 0$. □

Considering equation (1.8), we notice that, since $\alpha(0) = 0$ and $\hat{p}'(0, 0) = 0$, $\epsilon^2(s)$ has a Taylor expansion

$$\epsilon^2(s) = \sum_{k=2}^{r+2} a_k s^k + R(s)$$

and $a_2 > 0$ by the non-degeneracy condition of the random walk. Therefore

$$\epsilon(s) = a\,|s|\,g(s), \quad a > 0$$

with $g(s)$ of the class C^r and $g(0) = 1$.

Now we have all the ingredients to prove the following

Theorem 1.4.2. *Suppose the characteristic function $\hat{p}\colon \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ of the random walk $\{(X_n, Y_n)\}$ on $\mathbb{Z}^2$ with 0 first moment belongs to the class C^{r+3}, and $\hat{p}''(0, 0)$ is positive definite. Then the characteristic function*

$$\hat{q}(s) = 1 - \frac{1}{G(-s)}$$

of the trace random walk, with $G(s)$ given by (1.6), admits the representation

$$\hat{q}(s) = 1 - a\,|s|\,F(s) \tag{1.9}$$

with $a > 0$, $F(s)$ of the class C^r, $F(0) = 1$ and $F(s) \neq 0$ if $s \neq 0$.

Chapter 2

Subordinated random walks and quasicumulants

2.1 Subordinated random walks

Before formulating and proving general limit theorems for sums of independent random variables we want to show how such theorems are important by showing how characteristic functions having quasicumulants appear naturally in the theory of random walks. In the previous chapter we analyzed the case of the trace of a r.w., which always belong to the domain of attraction of the Cauchy law. In order to have other limiting laws we need to use the idea of *subordinated random walk*.

Let $(T_n)_{n\geq 1}$ be a sequence of i.i.d.r.v. with values in the positive integers (random times). For $z \in \mathbb{C}$, $|z| \leq 1$, define the common generating function of the T_n's as $F(z) = \mathbb{E}[z^{T_1}]$. The sum $S_n = T_1 + \cdots + T_n$, $S_0 = 0$, has generating function $F(z)^n$. Let $(X_n)_{n\geq 0}$ be a homogeneous random walk with transition function $p(x)$, $x \in \mathbb{Z}$, and characteristic function $\hat{p}(t)$. The random walk $(\mathfrak{X}_n)_{n\geq 0}$ defined as $\mathfrak{X}_n = X_{S_n}$ is called the r.w *subordinated* to X_n; the Markov chain S_n is called the *subordinator*, and X_n and $\mathfrak{X}_n$ are the *direct* and *subordinated* r.w. respectively. Elementary calculations show that the c.f. of $\mathfrak{X}_1$ is given by $\hat{q}(t) = F(\hat{p}(t))$ (cfg. Remark 1.3.1).

In many cases $F(z)$ can be written as

$$F(z) = 1 - (1-z)^{\alpha} c(z) \tag{2.1}$$

where $0 < \alpha < 1$, and $c(z)$ is of class C^n in a neighborhood of the point $z = 1$, and $c(1) \neq 0$.

Example 2.1.1. The function $F(z) = 1 - c(1-z)^{\alpha}$, $0 < c \leq 1$ is the generating function of the distribution

$$\begin{aligned} p_0 &= 1 - c \\ p_1 &= \alpha c \\ p_2 &= \frac{\alpha(\alpha-1)}{2}(-c) \\ &\cdots \end{aligned}$$

Theorem 2.1.1. *If the generating function satisfies* (2.1) *and the c.f.* $\hat{p}(t)$ *can be expanded as*

$$\hat{p}(t) = 1 - \frac{m_2}{2}t^2 + \cdots + \frac{m_r}{r!}(it)^r + o(|t|^r)$$

with $m_2 > 0$*, then we have*

$$\hat{q}(t) = 1 - |t|^{2\alpha} g(t)$$

where $g(t)$ *is of class* C^m*,* $m = \min\{r-2, 2n\}$*, and* $g(0) = (m_2/2)^\alpha c(1)$.

In the next section we will show that using the relationship between orthogonal polynomials (OP) and Markov chains on the half axis, interesting examples of generating functions satisfying (2.1) can be produced.

2.2 Basics of orthogonal polynomials

Let μ be a probability measure on the real line with compact support, and which is not concentrated on a finite number of points. Then on the Hilbert space $L^2(\mathbb{R}, \mathcal{B}, \mu)$ we can apply the Graham-Schmidt orthogonalization process to the family of monomials $\{1, x, x^2, \dots\}$ (for the moment we do not choose any particular normalization). The result is a family of polynomials $\{R_n(x)\}_{n\geq 0}$ which are orthogonal with respect to μ, i.e. $(R_j|R_k) = 0$ if $j \neq k$, and $\deg R_n(x) = n$. The smallest interval $[a, b]$ which contains the support of μ is called the interval of orthogonality. The first important result in the theory of orthogonal polynomials is there exist coefficients a_n, b_n and c_n so that the following relation holds

$$xR_n(x) = a_n R_{n+1}(x) + b_n R_n(x) + c_n R_{n-1}(x). \tag{2.2}$$

The coefficients a_n and c_n depend on the particular choice of the normalization while b_n do not, in fact

$$a_n = \frac{(R_{n+1}|xR_n)}{\|R_{n+1}\|^2}, \quad b_n = \frac{(R_n|xR_n)}{\|R_n\|^2}, \quad c_n = \frac{(R_{n-1}|xR_n)}{\|R_{n-1}\|^2} \tag{2.3}$$

and $a_n c_{n+1} > 0$ for $n \geq 0$. We can choose the normalization which sets the leading coefficient equal to one, and denote the corresponding family of OP by the symbol $M_n(x)$. In this case we have $a_n = 1$ and $c_{n+1} > 0$. The following theorem, due to Favard, provides the converse to what we showed so far.

Theorem 2.2.1. *Let* $\{M_k(x)\}_{n\geq 0}$ *be a family of monic polynomials with degree* $\deg M_k(x) = k$*, which satisfies the recurrence relations*

$$\begin{aligned} M_1(x) &= x - b_0 \\ M_{n+1}(x) &= (x - b_n)M_n(x) - c_n M_{n-1} \quad n \geq 1 \end{aligned} \tag{2.4}$$

with $c_n > 0$*. Then there exists a probability distribution* μ *such that* $(M_j|M_k) = 0$ *if* $j \neq k$.

The structure of the zeros of a family of OP is described by the next classical result.

Theorem 2.2.2. *If $\{R_n(x)\}$ is a family of OP with interval of orthogonality $[a,b]$, then:*

(A) the roots of $R_n(x)$ are all real, simple and contained inside $]a,b[$;

(B) if $x_1 < \cdots < x_n$ are the roots of $R_n(x)$, then for every $m > n$ in the interval $]x_k, x_{k+1}[$ ($k = 0, \ldots, n$, $x_0 = a$, $x_{n+1} = b$) there is at least one root of $R_m(x)$.

For the proofs of the above Theorems and a more detailed discussion on orthogonal polynomials see, for example, [13].

2.3 Orthogonal polynomials and Markov chains

Consider the Markov chain on $(X_n)_{n\geq 0}$ on the non-negative integers with transition probabilities $\mathbb{P}(X_{k+1} = n+1|X_k = n) = p_n$, $\mathbb{P}(X_{k+1} = n|X_k = n) = r_n$, $\mathbb{P}(X_{k+1} = n|X_k = n+1) = q_{n+1}$, for $n \geq 0$, and $p_n + r_n + q_n = 1$ ($q_0 = 0$). We assume $p_n > 0$ and $q_{n+1} > 0$ for $n \geq 0$ so that the chain is irreducible. The measure on $\mathbb{N}$ given by $\pi(n) = p_0 \cdots p_{n-1}/q_1 \cdots q_n$ if $n \geq 1$, and $\pi(0) = 1$ is invariant under the action of the transition matrix P.

On the Hilbert space $\mathcal{H} = L^2(\mathbb{N}, \pi)$ of all complex valued functions $f(n)$ s.t. $\sum |f(n)|^2 \pi(n) < \infty$, with the scalar product $(f|g) = \sum \overline{f(n)} g(n)\pi(n)$, we define the operator P as

$$(Pf)(n) = p_n f(n+1) + r_n f(n) + q_n f(n-1).$$

This is the usual action of the transition matrix on L^2; P is selfadjoint since the chain is reversible, and $\|P\| = 1$, therefore the spectrum $\sigma(P)$ is contained in the interval $[-1,1]$. Using the assumption that the chain is irreducible it can be shown that the vector δ_0 is cyclic for P, i.e. the linear span of the set $\{P^n\delta_0\}$ is dense in $\mathcal{H}$.

Now define the family of polynomials recursively as

$$\begin{aligned} Q_0(x) &= 1 \\ xQ_n(x) &= p_n Q_{n+1}(x) + r_n Q_n(x) + q_n Q_{n-1}(x) \quad n \geq 1. \end{aligned} \tag{2.5}$$

By Theorem 2.2.1 there exists a probability measure μ on $\mathbb{R}$ such that the $Q_n(x)$'s are orthogonal to each other. In fact it can be shown (see [14]) that the measure μ coincide with the spectral measure of P relative to the vector δ_0, i.e. $\forall z \in \mathbb{C}$ with $\operatorname{Im} z > 0$ we can write

$$(\delta_0|(P - zI)^{-1}\delta_0) = \int_{-1}^{1} \frac{\mu(dx)}{x - z},$$

(the function $F(z) = \int \frac{\mu(dx)}{x-z}$ is called the *Borel transform* of μ).

Let us notice the polynomials $Q_n(x)$ satisfy the normalization $Q_n(1) = 1$. Viceversa if μ is a measure concentrated on $[-1,1]$ (and not concentrated on a finite set) we can construct the family of monic polynomials $M_n(x)$ orthogonal w.r.t. μ and by Theorem 2.2.2 we have $M_n(1) > 0$. The new OP's defined by $Q_n(x) = M_n(x)/M_n(1)$ will satisfy the recurrence relations (2.5) with $p_n > 0$, $q_{n+1} > 0$ ($n \geq 0$) and $r_n = b_n$; moreover $p_n + r_n + q_n = 1$. Therefore the measure μ is the spectral measure of a Markov chain on the natural numbers in all cases when the coefficients r_n (or b_n) are non-negative.

By the functional calculus we have the formula for the probability of returning to 0 after k steps

$$p_{00}(k) = \int_{-1}^{1} x^k \mu(dx)$$

so the generating function for these probabilities is given by

$$G(z) = \sum_{k=0}^{\infty} p_{00}(k) z^k = \int_{-1}^{1} \frac{\mu(dx)}{1 - zx}, \quad |z| \le 1 \tag{2.6}$$

and $G(z)$ is related to the Borel transform by the formula $G(z) = -F(1/z)/z$.

Let us introduce the probability measure on the interval $[-1, 1]$

$$\mu_{\alpha\beta}(dx) = n_{\alpha\beta}(1-x)^{\alpha}(1+x)^{\beta} dx$$

$\alpha, \beta > -1$ and $n_{\alpha\beta}$ is the normalizing factor. The family $\{P_n^{(\alpha,\beta)}(x)\}$ of OP generated by $\mu_{\alpha\beta}$ using the normalization

$$P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n+1+\alpha)}{n!\Gamma(1+\alpha)} \tag{2.7}$$

are called *Jacobi polynomials*. There are classical formulas for the coefficients a_n, b_n and c_n (see for example [13])

$$\begin{aligned} a_n &= \frac{2(n+1)(n+1+\alpha+\beta)}{(2n+1+\alpha+\beta)(2n+2+\alpha+\beta)} \\ b_n &= \frac{(\beta^2-\alpha^2)}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)} \\ c_n &= \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+1+\alpha+\beta)}. \end{aligned} \tag{2.8}$$

So, from (2.3) and (2.7),

$$\begin{aligned} p_n &= \frac{2(n+1+\alpha)(n+1+\alpha+\beta)}{(2n+1+\alpha+\beta)(2n+2+\alpha+\beta)} \\ r_n &= \frac{(\beta^2-\alpha^2)}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)} \\ q_n &= \frac{2n(n+\beta)}{(2n+\alpha+\beta)(2n+1+\alpha+\beta)}. \end{aligned} \tag{2.9}$$

As easy to see r_n is non-negative iff one of the three happens:

(A) $\alpha = \beta$ ($r_n = 0$ for $n \ge 0$);

(B) $\beta > 0$ and $-\beta < \alpha < \beta$ ($r_n > 0$ for $n \ge 0$);

(C) $\beta > 0$ and $\alpha = -\beta$ ($r_0 = \beta$ and $r_n = 0$ for $n \ge 1$).

If $\alpha = \beta \neq -1/2$ it is customary to write the measure as $\mu_\gamma(dx) = (1-x^2)^{\gamma-1/2}dx$ ($\gamma = \alpha + 1/2$); the family of OP $\{C_n^{(\gamma)}(x)\}$ generated by $\mu_\gamma(dx)$ using the normalization $C_n^{(\gamma)}(1) = \Gamma(n+2\gamma)/n!\Gamma(2\gamma)$ are called *Gegenbauer polynomials*. The case $\alpha = \beta = -1/2$ give rise to the *Chebyshev polynomials* $\{T_n(x)\}$ under the normalization $T_n(1) = 1$; they correspond to the Markov chain $p_0 = 1$ and $p_n = q_n = 1/2$ for $n \geq 1$. Another family of classical OP is that of the *Legendre polynomials* $\{P_n(x)\}$ which have $\alpha = \beta = 1$ and normalization $P_n(1) = 1$; they have transition probabilities $p_n = (n+1)/(2n+1)$ and $q_n = n/(2n+1)$.

2.4 Asymptotic of the generating function

The generating function (2.6) for the Jacobi Markov chain becomes

$$G_{\alpha\beta}(z) = n_{\alpha\beta}\int_{-1}^{1}\frac{(1-x)^\alpha(1+x)^\beta}{1-zx}dx. \tag{2.10}$$

The goal of this section is to study the asymptotic of this function in the variable z for different values of the parameters α and β.

An expression for (2.10) can be given using the *hypergeometric series*, which is defined as

$${}_2F_1(a,b;c;\zeta) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{\zeta^n}{n!} \tag{2.11}$$

where the symbol $(*)_n := (*)(*+1)\cdots(*+n-1)$ is called the *rising factorial.* The series (2.11) is convergent for $|\zeta| < 1$. The hypergeometric series has the representation

$${}_2F_1(a,b;c;\zeta) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1\frac{w^{b-1}(1-w)^{c-b-1}}{(1-\zeta w)^a}dw \tag{2.12}$$

when $0 < \operatorname{Re} b < \operatorname{Re} c$. The LHS of (2.12) is called the *Euler hypergeometric integral* (see [13]). It is easily seen that after the change of variable $w = (x+1)/2$ in (2.12), (2.10) can be written as

$$G_{\alpha\beta}(z) = n_{\alpha\beta}\frac{2^{\alpha+\beta+1}\Gamma(\beta+1)\Gamma(\alpha+1)}{(1+z)\Gamma(\alpha+\beta+2)}{}_2F_1\left(1,\beta+1;\alpha+\beta+2;\frac{2z}{1+z}\right)$$

for $|z| < 1$. However this formula doesn't give us information on the asymptotic when $z \to 1$; the case $\alpha > 0$ is not interesting for us since the integral (2.10) is finite for $z = 1$ (the Markov chain is transient).

Theorem 2.4.1. *For $\alpha < 0$ the integral* (2.10) *can be written, in a neighborhood of $z = 1$, as*

$$G_{\alpha\beta}(z) = \left(\frac{1-z}{z}\right)^\alpha h(z) + l(z)$$

where $h(z)$ and $l(z)$ are analytic and $h(1) \neq 0$, and the power α on the RHS is the branch which gives real values for real arguments.

From this we can derive an expression for the generating function of the time T_1 of first return to 0, $F_{\alpha\beta}(z) := \sum_{n\geq 1} f_n z^n$, $f_n := \mathbb{P}(T_1 = n)$:

$$F_{\alpha\beta}(z) = 1 - \frac{1}{G_{\alpha\beta}(z)} = 1 - (1-z)^{-\alpha} c(z)$$

with $c(z)$ analytic and $c(1) \neq 0$. For the particular case of the Chebyshev polynomials ($\alpha = -1/2$) we have $c(z) = 1$.

Chapter 3

Limit theorems using quasicumulants

3.1 The class Δ_α

Suppose $\varphi(t)$ is the c.f. corresponding to the probability distribution μ. For $0 < \alpha \le 2$ let's introduce the class Δ_α of all c.f. satisfying the condition:

$$\log \varphi(t) = |t|^\alpha \left(-c(t) + \lambda(t) + R(t)\right) \tag{3.1}$$

in some neighborhood of $t = 0$, where $c(t) = c \in \mathbb{C}$, $\operatorname{Re} c > 0$, for $t \ge 0$ and $c(t) = \overline{c}$ for $t < 0$, and

$$\lambda(t) = \sum_{\substack{k \ge 1, j \ge 0 \\ k + j\alpha \le r}} \lambda^+_{kj} t^k \, |t|^{j\alpha} \tag{3.2}$$

$$\lambda(-t) = \overline{\lambda(t)} \tag{3.3}$$

and $R(-t) = \overline{R(t)}$, $R(t) = o(|t|^r)$ $t \to 0$, and $r \in \mathbb{R}$. We call the coefficients λ^+_{kj} the *normalized (right) quasicumulants* of order (k, j) of the c.f. $\varphi(t)$; for $t < 0$ expansion (3.2) holds with $\lambda(t) = \sum \lambda^-_{kj} t^k \, |t|^{j\alpha}$, where $\lambda^-_{kj} = (-1)^k \overline{\lambda^+_{kj}}$ and λ^-_{kj}'s are the normalized left quasicumulants. It follows from the definition that if $\varphi_1, \varphi_2 \in \Delta_\alpha$ and have normalized quasicumulants $\lambda^{(1)+}_{kj}$ and $\lambda^{(2)+}_{kj}$ respectively, then the c.f. $\varphi_1(t)\varphi_2(t)$ belongs to Δ_α with $\lambda^+_{kj} = \lambda^{(1)+}_{kj} + \lambda^{(2)+}_{kj}$.

Proposition 3.1.1. *The following statements are true:*

(a) *if* $\alpha_1 \ne \alpha_2$ *then* $\Delta_{\alpha_1} \cap \Delta_{\alpha_2} = \emptyset$*;*

(b) *if* $\varphi(t)$ *is the strictly stable distribution of exponent* α*, then* $\varphi(t) \in \Delta_\alpha$*;*

(c) *if* $\varphi(t) \in \Delta_\alpha$ *then* $\varphi(t)$ *belongs to the domain of attraction of the strictly stable distribution of exponent* α *and the complex number* c *in* (3.1) *satisfies* $-\pi\alpha/2 \le \arg c \le \pi\alpha/2$ *if* $0 < \alpha < 1$*, and* $-\pi(2-\alpha)/2 \le \arg c \le \pi(2-\alpha)/2$ *if* $1 < \alpha \le 2$.

Example 3.1.1. Suppose $\varphi(t)$ satisfies:

$$\log \varphi(t) = \begin{cases} -\sigma^2 \frac{t^2}{2} + \sum_{k=3}^{r} \frac{c_k^+}{k!}(it)^k + R^+(t) & \text{if } t \geq 0 \\ -\sigma^2 \frac{t^2}{2} + \sum_{k=3}^{r} \frac{c_k^-}{k!}(it)^k + R^-(t) & \text{if } t < 0 \end{cases} \tag{3.4}$$

where both $R^+(t)$ and $R^-(t)$ are $o(|t|^r)$ and $\sigma \in \mathbb{R}$. In view of $\varphi(-t) = \overline{\varphi(t)}$ necessarily we have $c_k^- = \overline{c_k^+}$ and for $t \geq 0$ $R^-(-t) = \overline{R^+(t)}$. The coefficients c_k^- and c_k^+ are called the *(left and right) quasicumulants* (of order k) of the c.f. and they correspond to the left and right-sided derivatives of $\log \varphi(t)$ at the point $t = 0$. If we define:

$$R(t) := \begin{cases} R^+(t) & t \geq 0 \\ R^-(t) & t < 0 \end{cases}$$

$$\lambda_{k,0}^+ := -i^k \frac{c_{k+2}^+}{(k+2)!}$$

then $\varphi(t) \in \Delta_2$; the definition of $\lambda_{k,0}^+$ in terms of the quasicumulants c_{k+2} justifies the definition of "normalized quasicumulant". Notice that if $\varphi(t)$ has the cumulant c_k, then $c_j = c_j^+ = c_j^- \in \mathbb{R}$ for all $j \leq k$. Viceversa let $\varphi(t)$ satisfy (3.4) and let r_0 be the first index such that $c_{r_0}^+ \neq c_{r_0}^-$ (eq. $\operatorname{Im} c_{r_0}^+ \neq 0$), then by a classical result in probability theory (see for example [2]) the distribution corresponding to $\varphi(t)$ has $2m_0$ moments, where $m_0 := \max\{n \colon 2n \leq r_0\}$; therefore if $k \leq 2m_0$, then $c_k^+ = c_k^- = c_k$ where c_k is the usual cumulant.

Example 3.1.2. Let $\varphi(t)$ satisfy:

$$\log \varphi(t) = \begin{cases} -ct + \sum_{k=2}^{r} c_k^+ (it)^k + R^+(t) & t \geq 0 \\ \bar{c}t + \sum_{k=2}^{r} c_k^- (it)^k + R^-(t) & t < 0 \end{cases} \tag{3.5}$$

with $c = a - ib$, $a > 0$, and R^+ and R^- $o(|t|^r)$. Then $\varphi(t) \in \Delta_1$ with $\lambda_{k,0}^+ = i^{k+1} c_{k+1}^+ / (k+1)!$. As in Ex. 3.1.1 the coefficients c_k^- and c_k^+ are called left and right quasicumulants, since they coincide with the left and right derivatives of $\log \varphi(t)$ w.r.t the variable it.

3.2 Generalized Chebyshev-Edgeworth-Cramér expansion

Consider a sequence $(X_n)_{n=1}^{\infty}$ of i.i.d. random variables with common characteristic function $\varphi(t)$ belonging to Δ_α, and let $S_n = X_1 + \cdots + X_n$. We will consider two cases simultaneously.

Case A: X_n are lattice random variables, i.e. they take values in the set $a + h\mathbb{Z}$ where $h > 0$ is called the *span* of the distribution of X_n. The span is called *maximal* if there is no other a_1 and $h_1 > h$ such that X_n takes values in $a_1 + h_1\mathbb{Z}$ with probability 1. For lattice random variables we will always assume, without lost of generality, that the maximal span is equal to 1, which is equivalent to say that $|\varphi(t)| = 1$ iff $t = 2k\pi$ (iff $\varphi(t) = 1$).

Case B: the common c.f. $\varphi(t)$ satisfies the condition $\int_{-\infty}^{\infty} |\varphi(t)|^p \, dt < \infty$ for some $p \geq 1$. Since $|\varphi(t)| \leq 1$ $\forall t$, $\varphi \in L^q$ $\forall q \geq p$; in particular $\varphi^n \in L^1$ for all integers $n \geq p$. Therefore S_n has continuous and bounded density for all $n \geq p$.

With an abuse of notation we will denote by $p_n(x)$ either the probability $\mathbb{P}(S_n = x)$ $x \in \mathbb{Z}$, if we are in case A, or the density of the r.v. S_n, $n \geq p$, if we are in case B. By the Fourier Theory $p_n(x)$ has the integral expression

$$p_n(x) = \frac{1}{2\pi} \int_{-L}^{L} \varphi(t)^n e^{-itx} dt$$

with $L = \pi$ or ∞ depending whether we are in case A or B.

For an arbitrary $\delta > 0$ we can split the above integral as:

$$\begin{aligned} p_n(x) &= \frac{1}{2\pi} \big(I_1(n,x) + I_2(n,x)\big) \\ I_1(n,x) &= \int_{|t|\leq\delta} \varphi(t)^n e^{-itx} dt \\ I_2(n,x) &= \int_{\delta<|t|<L} \varphi(t)^n e^{-itx} dt. \end{aligned}$$

The asymptotic of $I_2(n,x)$ for large values of n is a consequence of the following well known results.

Lemma 3.2.1. *If $\varphi(t)$ is the c.f. of a lattice distribution with the maximal span h, then for every $\delta > 0$ there exists a positive number c such that $|\varphi(t)| \leq e^{-c}$ in the domain $\delta \leq |t| \leq \frac{2\pi}{h} - \delta$.*

Lemma 3.2.2. *If $\varphi(t)$ is a c.f. satisfying the Cramér condition*

$$\text{(C)} \qquad \limsup_{|t|\to\infty} |\varphi(t)| < 1,$$

then for every $\delta > 0$ there exists a positive number c such that $|\varphi(t)| \leq e^{-c}$ for $|t| \geq \delta$.

For the case A, using Lemma 3.2.1 and the hypothesis that the maximal span is 1 we get:

$$|I_2(n,x)| \leq \int_{\delta<|t|<\pi} |\varphi(t)|^n \, dt \leq 2\pi e^{-cn},$$

where the constant c is independent on $x \in \mathbb{Z}$. For the case B, let n_0 be the smallest integer greater then p; since $\varphi^{n_0}(t)$ is the c.f. of the density $p_{n_0}(x)$, it satisfies the Cramér condition (C), therefore using Lemma 3.2.2 we have:

$$|I_2(n,x)| \leq \int_{|t|>\delta} |\varphi(t)|^{n_0} \left(|\varphi(t)|^{n_0}\right)^{\frac{n-n_0}{n_0}} dt \leq e^{c\frac{n-n_0}{n_0}} \left\|\varphi^{n_0}\right\|_1.$$

This means that in both cases $I_2(n,x) = O(e^{-c_1 n})$ as $n \to \infty$ uniformly in $x \in \mathbb{Z}$ or $x \in \mathbb{R}$.

Using the hypothesis (3.2) we can rewrite the integral I_1 as:

$$\begin{aligned} I_1(n,x) &= \int_{|t|\leq\delta} e^{n\log\varphi(t) - itx} dt \\ &= \int_{|t|\leq\delta} e^{-nc(t)|t|^\alpha - itx} \exp\left[n\,|t|^\alpha \lambda(t) + n\,|t|^\alpha R(t)\right] dt \\ &= \int_0^\delta e^{-nc|t|^\alpha - itx} \exp\left[n\,|t|^\alpha \lambda(t) + n\,|t|^\alpha R(t)\right] dt \\ &\quad + \int_0^\delta e^{-n\bar{c}|t|^\alpha + itx} \exp\left[n\,|t|^\alpha \overline{\lambda(t)} + n\,|t|^\alpha \overline{R(t)}\right] dt \end{aligned}$$

The natural change of variable $s = n^{1/\alpha} t$ leads to the expression:

$$I_1(n,x) = \frac{1}{n^{1/\alpha}} \left[I_1^+(n,y) + I_1^-(n,y) \right], \quad y = \frac{x}{n^{1/\alpha}} \tag{3.6}$$

$$I_1^+(n,y) = \int_0^{n^{1/\alpha}\delta} e^{-c|s|^\alpha - isy} \exp\left[|s|^\alpha \lambda\left(\frac{s}{n^{1/\alpha}}\right) + |s|^\alpha R\left(\frac{s}{n^{1/\alpha}}\right) \right] ds \tag{3.7}$$

$$I_1^-(n,y) = \overline{I_1^+(n,y)}. \tag{3.8}$$

Let $u := n^{-1/\alpha}$, $w(s,u) := s^\alpha\big(\lambda(us) + R(us)\big)$ and $z(s,u) := s^\alpha \lambda(us)$. We have the Taylor expansion

(3.9)

$$\begin{aligned} e^{z(s,u)} &= \sum_{l=0}^{\infty} \frac{z^l}{l!} = \sum_{l=0}^{\infty} \frac{1}{l!} s^{l\alpha} \sum_{\sum h_{m,n} = l} \binom{l}{(h_{m,n})} \prod (\lambda_{mn}^+ u^{m+n\alpha} s^{m+n\alpha})^{h_{m,n}} \\ &= \sum_{l=0}^{\infty} s^{l\alpha} \sum_{\sum h_{m,n} = l} \prod \frac{(\lambda_{mn}^+ u^{m+n\alpha} s^{m+n\alpha})^{h_{m,n}}}{h_{m,n}!} \\ &= \sum_{k\ge 0, j \ge 0} u^{k+j\alpha} s^{k+j\alpha} \gamma_{k,j}(s) \end{aligned}$$

$$\gamma_{k,j}(s) = \begin{cases} 1 & k=0,\ j=0 \\ 0 & k=0,\ j\ge 1 \\ \sum_{l=1}^{k} s^{l\alpha} \sum_{\substack{\sum h_{m,n} = l \\ \sum m h_{m,n} = k \\ \sum n h_{m,n} = j}} \prod \frac{(\lambda_{mn}^+)^{h_{m,n}}}{h_{m,n}!} & \text{otherwise} \\ \end{cases} \tag{3.10}$$

where the symbol $\binom{l}{(h_{m,n})}$ denotes the multinomial coefficients with all the integers $h_{m,n}$ on the second line, and all the products above and the sums "$\sum h_{m,n}$" are extended over all pairs of integers $m \ge 1$ $n \ge 0$ with $m + n\alpha \le r$. In fact in place of (3.9) we consider $e^z = \sum_{l=0}^{r} z^l/l! + e^{z'} z^{r+1}/(r+1)!$ which gives the formula

$$e^{z(s,u)} = \sum_{\substack{k\ge 0, j\ge 0 \\ k+j\alpha \le r'}} u^{k+j\alpha} s^{k+j\alpha} \gamma_{k,j}(s) + e^{z'(s,u)} \frac{z(s,u)^{r+1}}{(r+1)!}$$

with $r \le r'$, and it follows from the calculations above that $z(s,u)^{r+1} = o(u^r)$. Let

$$\begin{aligned} J_1^+(n,y) &= \int_0^{n^{1/\alpha}\delta} e^{-cs^\alpha - isy} \left(\sum_{\substack{k\ge 0, j\ge 0 \\ k+j\alpha \le r}} u^{k+j\alpha} s^{k+j\alpha} \gamma_{k,j}(s) \right) ds, \\ J_1^-(n,y) &:= \overline{J_1^+(n,y)}; \end{aligned}$$

then

$$\begin{aligned} \left| I_1^+(n,y) - J_1^+(n,y) \right| \le \int_0^{n^{1/\alpha}\delta} e^{-\operatorname{Re} cs^\alpha} \Bigg[\left| e^{w(s,u)} - e^{z(s,u)} \right| \\ + \left| \sum_{r < k+j\alpha \le r'} u^{k+j\alpha} s^{k+j\alpha} \gamma_{k,j}(s) \right| + \left| e^{z'(s,u)} \frac{z(s,u)^{r+1}}{(r+1)!} \right| \Bigg] ds. \end{aligned} \tag{3.11}$$

If θ is a real number such that $|w(s,u)| \le \theta$ and $|z(s,u)| \le \theta$ then we have the inequality $|e^w - e^z| \le e^\theta |w - z|$. In the interval of integration in (3.11) θ can be chosen as as^α where $a > 0$ is arbitrary small (if δ is small enough). Also we have $|w - z| \le \epsilon s^\alpha (us)^r$ and $\left| e^{z'(s,u)} \frac{z(s,u)^{r+1}}{(r+1)!} \right| \le Ce^\theta s^{\alpha(r+1)}(us)^{r+1}$ for any $\epsilon > 0$ and some $C > 0$ if δ is sufficiently small. Therefore

(3.12)

$$\begin{aligned} \left| I_1^+(n,y) - J_1^+(n,y) \right| \le & \epsilon \int_0^{n^{1/\alpha}\delta} e^{-(\operatorname{Re} c - a)s^\alpha} s^\alpha u^r s^r ds \\ & + \sum_{r < k + j\alpha \le r'} u^{k+j\alpha} \int_0^{n^{1/\alpha}\delta} e^{-\operatorname{Re} c s^\alpha} s^{k+j\alpha} |\gamma_{k,j}(s)| \, ds \\ & + C \int_0^{n^{1/\alpha}\delta} e^{-(\operatorname{Re} c - a)s^\alpha} s^{\alpha(r+1)} (us)^{r+1} ds \\ \le & \epsilon M u^r \end{aligned}$$

which means $I_1^+(n,y) = J_1^+(n,y) + o(n^{-r/\alpha})$, $n \to \infty$, uniformly in y. Obviously the same estimate holds between I_1^- and J_1^-, and going back to (3.6) we derive

$$\begin{aligned} I_1(n,x) = & \sum \frac{1}{n^{\frac{k}{\alpha}+j}} \left[\int_0^{n^{1/\alpha}\delta} e^{-cs^\alpha - ity} q_{k,j}(s) ds + \int_0^{n^{1/\alpha}\delta} e^{-\bar{c}|s|^\alpha + ity} \overline{q_{k,j}(s)} ds \right] \\ & + o(n^{-r/\alpha}) \end{aligned}$$

with $q_{k,j}(s) := s^{k-1+j\alpha} \gamma_{k-1,j}(s)$, and the sum is over all pair of integers $k \ge 1$, $j \ge 0$ with $k + j\alpha \le r + 1$.

We conclude our estimate by noticing that for any $k \ge 1$ and $j \ge 0$

$$\left| \int_{n^{1/\alpha}\delta}^\infty e^{-c|s|^\alpha - ity} q_{k,j}(s) ds \right| \le \int_{n^{1/\alpha}\delta}^\infty e^{-\frac{\operatorname{Re} c}{2}|s|^\alpha} ds = O\left(\frac{e^{-\frac{\operatorname{Re} c}{2} n \delta^\alpha}}{n^{1-1/\alpha}} \right).$$

We therefore proved the following

Theorem 3.2.3. *Let $(X_n)_{n=1}^\infty$ be a sequence of i.i.d.r.v. with common c.f. $\varphi(t)$ belonging to Δ_α, and assume one of the following two cases:*

(A) the r.v. X_n have lattice distribution with maximal span equal to 1;

(B) $\int_{-\infty}^\infty |\varphi(t)|^p \, dt < \infty$ for some $p \ge 1$.

Then we have the asymptotic expansion

$$p_n(x) = \sum_{\substack{k \ge 1, j \ge 0 \\ k + j\alpha \le r+1}} \frac{1}{n^{k/\alpha + j}} U_{k,j}^{(\alpha,c)} \left(\frac{x}{n^{1/\alpha}} \right) + o\left(n^{-\frac{r+1}{\alpha}} \right), \quad n \to \infty \tag{3.13}$$

uniformly in $x \in \mathbb{Z}$ in the case A, or in $x \in \mathbb{R}$ in the case B. The functions $U_{k,j}^{(\alpha,c)}(y)$ have the expression:

$$U_{k,j}^{(\alpha,c)}(y) = \frac{1}{\pi} \operatorname{Re} \int_0^\infty e^{-cs^\alpha - isy} q_{k,j}(s) ds \tag{3.14}$$

$$q_{k,j}(s) = \begin{cases} 1 & k = 1, \ j = 0 \\ 0 & k = 1, \ j \ge 1 \\ \sum_{l=1}^{k-1} s^{(l+j)\alpha + k - 1} q_{k,j}^{(l)} & \textit{otherwise} \end{cases} \tag{3.15}$$

with the numbers $q_{k,j}^{(l)}$ *depending only on the* $\lambda_{m,n}^{+}$ *with* $m \leq k-1$ $n \leq j$ *according to the formula*

$$q_{k,j}^{(l)} = \sum \prod_{\substack{1 \leq m \leq k-1 \\ 0 \leq n \leq j}} \frac{(\lambda_{m,n}^{+})^{h_{m,n}}}{h_{m,n}!}$$

where the above sum is taken over all possible arrays of non-negative integers $h_{m,n}$ $1 \leq m \leq k-1$ $0 \leq n \leq j$ *satisfying the relations* $\sum h_{m,n} = l$, $\sum m h_{m,n} = k-1$ *and* $\sum n h_{m,n} = j$; *in particular* $q_{k,j}^{(1)} = \lambda_{k-1,j}^{+}$ *and*

$$q_{k,j}^{(k-1)} = \sum \prod_{n=0}^{j} \frac{(\lambda_{1,n}^{+})^{h_{1,n}}}{h_{1,n}!}.$$

From (3.14) and (3.15) we can see that once we know the coefficients $q_{k,j}^{(l)}$, the calculation of the functions $U_{k,j}^{(\alpha,c)}(y)$ reduces to the calculation of the integrals:

$$I_{k,l}^{(\alpha,c)}(y) = \int_0^{\infty} e^{-c|s|^{\alpha} - isy} s^{l\alpha + k} ds \tag{3.16}$$

and $U_{k,j}^{(\alpha,c)}(y)$ will be a linear combination with real coefficients of the functions

$$\mathrm{Re}(e^{i\theta} I_{k-1,l+j}^{(\alpha,c)}(y)) \tag{3.17}$$

for $l = 1, \ldots, k-1$ and $0 \leq \theta < \pi$. For $I_{k,l}^{(\alpha,c)}(y)$ we can write the generating function

$$G^{(\alpha,c)}(w,z) := \sum_{k,l=0}^{\infty} \frac{I_{k,l}^{(\alpha,c)}(y)}{k!l!} (iw)^k z^l \tag{3.18}$$

for $w \in \mathbb{R}$ and $z \in \mathbb{R}^{+}$, and if we consider

$$I_{0,0}^{(\alpha,c)}(y) = \int_0^{\infty} e^{-cs^{\alpha} - isy} ds$$

we find, summing up (3.18), that $G^{(\alpha,c)}(w,z) = p_{(\alpha, c - z^{1/\alpha})}^{+}(y-w)$. Therefore

$$I_{k,l}^{(\alpha,c)}(y) = (-i)^k \frac{d^{k+l}}{dw^k dz^l} I_{0,0}^{(\alpha,c)}(y-w) \Big|_{w=0,z=0}.$$

Remark 3.2.1. Consider a c.f. $\varphi(t)$ satisfying the hypothesis of Ex. 3.1.1 or Ex. 3.1.2. If the quasicumulants are all real (i.e. $c_k^{+} = c_k^{-}$) then the coefficients $q_{k,j}^{(l)}$ are real if k is odd and imaginary if k is even. Therefore in expression (3.17) we have $\theta = 0$ if k is odd and $\theta = \pi/2$ if k is even, and this allows us to consider, instead of $I_{0,0}^{(\alpha,c)}(y)$, the more convenient $p_{(\alpha,c)}(y) := \pi^{-1} \operatorname{Re} I_{0,0}^{(\alpha,c)}(y)$, $\alpha = 1, 2$, which is a strictly stable density.

3.3 Non-uniform expansions

In the previous Section we derived the asymptotic expansion for $p_n(x)$ assuming the regularity of the c.f. $\varphi(t)$ only near the origin. However if we further require $\varphi(t)$ to be sufficiently regular for all t, except $t = 0$ (or $t = 2k\pi$), we can get a sharper estimate of the remainder term when x is large; additionally we can get that the remainder term not only goes to zero when $n \to \infty$ but when $n + |x| \to \infty$. As in the previous Section we formulate our results for case A and B simultaneously.

Theorem 3.3.1. *Let $(X_n)_{n=1}^{\infty}$ be a sequence of i.i.d.r.v. with common c.f. $\varphi(t) \in \Delta_\alpha$, and assume one of the following two cases:*

(A) (1) the r.v.'s X_n have lattice distribution with maximal span equal to 1*,*
(2) $\varphi(t) \in C^\nu(]0, 2\pi[)$, $\nu \leq r + \alpha$;

(B) (1) $\int_{-\infty}^{\infty} |\varphi(t)|^p \, dt < \infty$ for some $p \geq 1$,
(2) $\varphi(t) \in C^\nu(\mathbb{R} \setminus \{0\})$, $\nu \leq r + \alpha$, and all derivatives are bounded outside a small neighborhood of 0*;*

Then $p_n(x)$ has the expansion:

$$p_n(x) = \sum_{\substack{k \geq 1, j \geq 0 \\ k + j\alpha \leq r+1}} \frac{1}{n^{k/\alpha + j}} U_{k,j}^{(\alpha,c)}\left(\frac{x}{n^{1/\alpha}}\right) + \frac{n^{(\nu - r - 1)/\alpha}}{n^{\nu/\alpha} + x^\nu} \gamma(n, x) \tag{3.19}$$

holding for $x \in \mathbb{Z}$ in the case A, or for $x \in \mathbb{R}$ in the case B, and $\gamma(n, x) \to 0$ as $n + |x| \to \infty$.

Remark 3.3.1. In the case of attraction to the Gaussian law ($\alpha = 2$) if the r.v.'s X_n have $r + 2$ moments and $\nu \leq r + 2$, then the boundedness of the derivatives in B2 yields.

For the proof of this theorem we will relay on the following elementary result, which provides a criterion to check whether a function $f(n, x)$ goes to zero when $n + |x| \to \infty$.

Lemma 3.3.2. *Given a function $f \colon \mathbb{N} \times \mathbb{R} \to \mathbb{C}$, then $f(n, x) \to 0$ when $n + |x| \to \infty$ if and only if the following two conditions are satisfied:*

(A) $\sup_x |f(n, x)| \to 0$ as $n \to \infty$;

(B) for all fixed n, $f(n, x) \to 0$ as $|x| \to \infty$.

Proof. The necessity is obvious. For the sufficiency, suppose that $\exists \epsilon > 0$ such that there exists $\{n_k\}$ and $\{x_k\}$ such that $n_k + |x_k| \to \infty$ and $|f(n_k, x_k)| \geq \epsilon$ $\forall k$. We have one of the following two cases: either $\{n_k\}$ is unbounded or it is bounded. If it is unbounded then we can extract a subsequence $\{n_{k_j}\}$ which tends to infinity and we contradict A. If $\{n_k\}$ is bounded then it has only a finite set of values and there exists a subsequence k_j such that $|x_{k_j}| \to \infty$ and $\{n_{k_j}\}$ is constant. Therefore we contradict B. $\square$

Consider the formula $p_n(x) = \frac{1}{2\pi}\int_{-L}^{L}\varphi^n(t)e^{-itx}dt$ with $L = \pi$ or ∞ depending whether we are in case A or B. Let δ_1 and δ_2 be arbitrary numbers satisfying $0 < \delta_1 < \delta_2 < L$, and introduce the *partition of the unity*, i.e. a function $\chi(t) \in C^\infty(\mathbb{R})$ such that $0 \le \chi(t) \le 1$ $\forall t \in \mathbb{R}$, $\chi(t) = 1$ if $|t| \le \delta_1$ and $\chi(t) = 0$ if $|t| \ge \delta_2$; for such function we introduce the complementary $\tilde{\chi}(t) := 1 - \chi(t)$[1]. Using $\chi(t)$ we can split the integral formula for $p_n(x)$ as:

$$p_n(x) = \frac{1}{2\pi}\big(I_1(n,x) + I_2(n,x)\big) \tag{3.20}$$

$$I_1(n,x) = \int_{-\infty}^{\infty}\varphi^n(t)\chi(t)e^{-itx}dt \tag{3.21}$$

$$I_2(n,x) = \int_{-L}^{L}\varphi^n(t)\tilde{\chi}(t)e^{-itx}dt. \tag{3.22}$$

The asymptotic for $I_2(n,x)$ is given by the following:

Lemma 3.3.3. *There exist functions $g_1(n,x)$ and $g_2(n,x)$ which go to zero when $n + |x| \to \infty$, such that*

(A) $I_2(n,x) = n^{-(r+1)/\alpha}g_1(n,x)$;

(B) $I_2(n,x) = \frac{n^{(\nu-r-1)/\alpha}}{x^\nu}g_2(n,x)$, if $x \neq 0$.

Therefore $|I_2(n,x)| \le \dfrac{n^{(\nu-r-1)/\alpha}}{n^{\nu/\alpha} + |x|^\nu}g(n,x)$, where $g(n,x)$ goes to zero when $n + |x| \to \infty$.

Proof. Let us start with the case $L = \pi$. For A it is enough to take $g_1(n,x) := n^{(r+1)/\alpha}I_2(n,x)$; then by Lemma 3.2.1 $g_1(n,x) \to 0$ as $n \to \infty$ uniformly in x exponentially fast. Also for fixed n $g_1(n,x) \to 0$ as $|x| \to \infty$ by the Riemann-Lebesgue Lemma, hence $g_1(n,x) \to 0$ when $n + |x| \to \infty$ by Lemma 3.3.2.

For B we integrate by parts:

$$\begin{aligned} I_2(n,x) &= \int_{\delta_1 \le |t| < \pi}\varphi^n(t)\tilde{\chi}(t)e^{-itx}dt \\ &= \frac{i}{x}\varphi^n(t)\tilde{\chi}(t)e^{-itx}\Big|_{t=\delta_1}^{t=\pi} + \frac{i}{x}\varphi^n(t)\tilde{\chi}(t)e^{-itx}\Big|_{t=-\pi}^{t=-\delta_1} \\ &\quad - \frac{i}{x}\int_{\delta_1 \le |t| < \pi}\frac{d}{dt}\left[\varphi^n(t)\tilde{\chi}(t)\right]e^{-itx}dt, \end{aligned}$$

and since $\varphi(-\pi) = \varphi(\pi)$ and $\tilde{\chi}(\delta_1) = 0$ we have:

$$I_2(n,x) = -\frac{i}{x}\int_{\delta_1 \le |t| < \pi}\frac{d}{dt}\left[\varphi^n(t)\tilde{\chi}(t)\right]e^{-itx}dt.$$

Therefore iterating the integration by parts:

$$\begin{aligned} I_2(n,x) = -\frac{1}{x^2}\int_{\delta_1 \le |t| < \pi}\frac{d^2}{dt^2}\left[\varphi^n(t)\tilde{\chi}(t)\right]e^{-itx}dt = \cdots \\ = \frac{(-i)^\nu}{x^\nu}\int_{\delta_1 \le |t| < \pi}\frac{d^\nu}{dt^\nu}\left[\varphi^n(t)\tilde{\chi}(t)\right]e^{-itx}dt. \end{aligned}$$

[1]For the construction of the partition of the unity see, for example, [5]

The function $g_2(n,x) := n^{(r+1-\nu)/\alpha}(-i)^\nu \int_{\delta_1 \le |t| < L} \frac{d^\nu}{dt^\nu}\left[\varphi^n(t)\tilde{\chi}(t)\right] e^{-itx} dt$ goes to zero uniformly in x exponentially fast when $n \to \infty$ since we can factor out from the expression $\frac{d^\nu}{dt^\nu}\left[\varphi^n(t)\tilde{\chi}(t)\right]$ a term of the shape $\varphi^{n-\nu}(t)$. Again B follows from Lemma 3.3.2 and the fact that $g_2(n,x) \to 0$ for fixed n when $|x| \to \infty$.

For the case $L = \infty$ we use the same method with the obvious adjustments. For A we define $g_1(n,x) := n^{(r+1)/\alpha} I_2(n,x)$, and use Lemma 3.2.2, the Riemann-Lebesgue Lemma and the Lemma 3.3.2. For B we use successive integration by parts, plus the fact that $\left.\frac{d^k}{dt^k}\left[\varphi^n(t)\tilde{\chi}(t)\right]\right|_{t=\infty} = \left.\frac{d^k}{dt^k}\left[\varphi^n(t)\tilde{\chi}(t)\right]\right|_{t=-\infty} = 0$ ($k = 0,\ldots,\nu-1$) since they are the sum of terms of the form $\varphi^{n-n_0}(t)$ (which tends to zero as $|t| \to \infty$) times some derivative of $\varphi(t)$, which are bounded by condition B2 of Theorem 3.3.1. So we get

$$I_2(n,x) = \frac{(-i)^\nu}{x^\nu} \int_{|t| \ge \delta_1} \frac{d^\nu}{dt^\nu}\left[\varphi^n(t)\tilde{\chi}(t)\right] e^{-itx} dt.$$

As in the case $L = \pi$, the function $g_2(n,x)$ defined above proves B.

Define $g(n,x) = 2\max\{|g_1(n,x)|, |g_2(n,x)|\}$. The inequalities

$$|I_2(n,x)| \le \frac{n^{-(r+1)/\alpha}}{2} g(n,x), \quad |I_2(n,x)| \le \frac{n^{(\nu-r-1)/\alpha}}{2|x|^\nu} g(n,x)$$

hold, therefore

$$|I_2(n,x)| \le \min\left\{n^{-(r+1)/\alpha}, \frac{n^{(\nu-r-1)/\alpha}}{|x|^\nu}\right\} \frac{g(n,x)}{2}$$

and since

$$\min\left\{n^{-(r+1)/\alpha}, \frac{n^{(\nu-r-1)/\alpha}}{|x|^\nu}\right\} = \frac{n^{-(r+1)/\alpha}}{\max\{1, |x|^\nu / n^{\nu/\alpha}\}} \le \frac{2n^{-(r+1)/\alpha}}{1 + |x|^\nu / n^{\nu/\alpha}} = \frac{2n^{(\nu-r-1)/\alpha}}{n^{\nu/\alpha} + |x|^\nu}$$

the last statement of the Theorem is also proved. □

We now restrict our attention to the integral $I_1(n,x)$. As in (3.6) $I_1(n,x) = n^{-1/\alpha}[I_1^+(n,y) + I_1^-(n,y)]$, $y = n^{-1/\alpha}x$. Using the notation of Section 3.2, let $\rho(s,u) = e^{w(s,u)} - \sum_{\substack{k\ge 0, j \ge 0\\ k+j\alpha \le r}} u^{k+j\alpha} s^{k+j\alpha} \gamma_{k,j}(s)$ ($u = n^{-1/\alpha}$, $s = n^{1/\alpha}t$). By condition A2 and B2, $\rho(s,u)$, $s > 0$, is ν-times differentiable with respect to s; moreover, by the consideration in Section 3.2, $\rho(s,u) = o(s^{r+\alpha})$, hence $\lim_{s\to 0} \frac{\partial^k \rho}{\partial s^k}(s,u) = 0$ $\forall k = 0,\ldots,\nu$.

Define the integrals:

$$
\begin{aligned}
I_1^{(1)+}(n,y) &= \int_0^{u^{-1}\delta_2} e^{-cs^\alpha - isy}\left(\sum_{\substack{k\geq 0, j\geq 0\\ k+j\alpha\leq r}} u^{k+j\alpha}s^{k+j\alpha}\gamma_{k,j}(s)\right)\chi(us)ds\\
I_1^{(1)-}(n,y) &= \overline{I_1^{(1)+}(n,y)}\\
I_1^{(2)+}(n,y) &= \int_0^{u^{-1}\delta_2} e^{-cs^\alpha - isy}\rho(s,u)\chi(us)ds\\
I_1^{(2)-}(n,y) &= \overline{I_1^{(2)+}(n,y)}
\end{aligned}
$$

and the identity $I_1(n,x) = u[I_1^{(1)+}(n,y) + I_1^{(1)-}(n,y) + I_1^{(2)+}(n,y) + I_1^{(2)-}(n,y)]$ holds.

Lemma 3.3.4.

$$
\left|I_1^{(2)+}(n,y)\right| \leq \frac{1}{n^{r/\alpha}}\frac{1}{1+y^\nu}h(n,x) \tag{3.23}
$$

where $h(n,x)\to 0$ *when* $n+|x|\to\infty$.

Proof. Using the same inequalities as in (3.11) and (3.12) we see that the integral $I_1^{(2)+}(n,y) = n^{-r/\alpha}h_1(n,x)$ where $h_1(n,x)\to 0$ when $n+|x|\to\infty$. On the other hand, integrating by parts when $y\neq 0$, we get

$$
\begin{aligned}
I_1^{(2)+}(n,y) &= \frac{i}{y}e^{-cs^\alpha}\rho(s,u)\chi(us)e^{-isy}\Big|_{s=0}^{s=u^{-1}\delta_2}\\
&\quad - \frac{i}{y}\int_0^{u^{-1}\delta_2}\frac{\partial}{\partial s}\left[e^{-cs^\alpha}\rho(s,u)\chi(us)\right]e^{-isy}ds\\
&= -\frac{i}{y}\int_0^{u^{-1}\delta_2}\frac{\partial}{\partial s}\left[e^{-cs^\alpha}\rho(s,u)\chi(us)\right]e^{-isy}ds = \cdots\\
&= \frac{(-i)^\nu}{y^\nu}\int_0^{u^{-1}\delta_2}\frac{\partial^\nu}{\partial s^\nu}\left[e^{-cs^\alpha}\rho(s,u)\chi(us)\right]e^{-isy}ds.
\end{aligned}
\tag{3.24}
$$

In the above we used the fact that all off-integral terms vanish due to the identity $\left.\dfrac{\partial^k\rho}{\partial s^k}\right|_{s=0} = 0$ $(k = 0,\ldots,\nu)$ and $\chi(\delta_2) = 0$. So $I_1^{(2)+}(n,y) = \dfrac{n^{-r/\alpha}}{y^\nu}h_2(n,y)$ where $h_2(n,y)\to 0$ as $n+|x|\to\infty$. Therefore (3.23) follows by the same argument as in Lemma 3.3.3. □

Trivially the estimate (3.23) holds also for $I_1^{(2)-}(n,y)$. In turn $I_1^{(1)+}(n,y)$ can be written as

$$
\begin{aligned}
I_1^{(1)+}(n,y) = &\sum_{\substack{k\geq 0, j\geq 0\\ k+j\alpha\leq r}} u^{k+j\alpha}\int_0^\infty e^{-cs^\alpha - isy}s^{k+j\alpha}\gamma_{k,j}(s)ds\\
&- \sum_{\substack{k\geq 0, j\geq 0\\ k+j\alpha\leq r}} u^{k+j\alpha}\int_{u^{-1}\delta_1}^\infty e^{-cs^\alpha - isy}s^{k+j\alpha}\gamma_{k,j}(s)\tilde{\chi}(us)ds
\end{aligned}
\tag{3.25}
$$

and the estimate for the second integral is given by

$$\left| \int_{u^{-1}\delta_1}^{\infty} e^{-cs^{\alpha}-isy} s^{k+j\alpha} \gamma_{k,j}(s)\tilde{\chi}(us)ds \right| \leq \frac{1}{n^{r/\alpha}} \frac{1}{1+y^{\nu}} l(n,x) \tag{3.26}$$

where $l(n,x) \to 0$ as $n + |x| \to \infty$ (the proof of (3.26) is a repetition of the proofs of Lemma 3.3.3 and Lemma 3.3.4).

Proof. (Theorem 3.3.1) It follows from (3.20) and (3.25) and the estimates in Lemma 3.3.3 and Lemma 3.3.4. □

Chapter 4

Global limit theorems: the Gaussian and Cauchy cases

4.1 Introduction

In this chapter we develop what we called *global limit theorems* as corollary of the results in Chapter 3; these are a kind of large deviation result holding uniformly on the full line. We got inspiration for this from Linnik [4], and we developed for the first time in [7]. We will consider here only the case when $0 < \alpha < 1$, the case $\alpha = 1$ (Cauchy case) and $\alpha = 2$ (Gaussian case) (see examples 3.1.1 and 3.1.2). Notice that in the last two cases it is irrelevant to consider the index j in the definition (3.2) of $\lambda(t)$, so we will omit and write simply λ_k^+ and λ_k^-. We also provide examples where these theorems apply. We begin with the Gaussian case since it is the more interesting.

4.2 Gaussian case

Assume the c.f. $\varphi(t)$ is as in Example 3.1.1 and satisfies the hypothesis of Theorem 3.3.1. Define $\lambda_k^+ = \lambda_{k,0}^+$ and $\lambda_k^- = \lambda_{k,0}^-$. The asymptotic expansion (3.19) can be rewritten as

$$p_n(x) = \frac{1}{\sqrt{2\pi n}\sigma} e^{-\frac{x^2}{2\sigma^2 n}} + \sum_{k=2}^{r-1} \frac{1}{n^{k/2}} U_k\left(\frac{x}{n^{1/2}}\right) + \frac{n^{(\nu-r+1)/2}}{n^{\nu/2}+x^\nu}\gamma(n,x) \tag{4.1}$$

with $U_k(y) := U_{k,0}^{(2,\sigma^2/2)}(y)$. As in the previous chapter, the functions U_k $k \geq 2$ are defined as

$$U_k(y) = \frac{1}{\pi}\sum_{l=1}^{k-1} \mathrm{Re}(q_k^{(l)} I_{k-1,2l}^{(2,\sigma^2/2)}(y)) \tag{4.2}$$

with

$$q_k^{(l)} = \sum \prod_{m=1}^{k-1} \frac{(\lambda_m^+)^{h_m}}{h_m!} \tag{4.3}$$

where the above sum is taken over all possible choices of non-negative integers $h_1, \dots, h_m$ satisfying the relations $\sum_{m=1}^{k-1} h_m = l$ and $\sum_{m=1}^{k-1} m h_m = k-1$. If $u_k^{(l)}$ and $v_k^{(l)}$ are the real and imaginary part respectively of $q_k^{(l)}$ then (4.2) can be rewritten as:

$$U_k(y) = \sum_{l=1}^{k-1} \frac{1}{\sigma^{k+2l}} \left(u_k^{(l)} \Lambda_{k-1+2l}^{(+)} \left(\frac{y}{\sigma}\right) + v_k^{(l)} \Lambda_{k-1+2l}^{(-)} \left(\frac{y}{\sigma}\right) \right) \tag{4.4}$$

$$\Lambda_k^{(+)}(x) = \frac{1}{\pi} \int_0^{+\infty} e^{-\frac{t^2}{2}} t^k \cos tx dt \tag{4.5}$$

$$\Lambda_k^{(-)}(x) = \frac{1}{\pi} \int_0^{+\infty} e^{-\frac{t^2}{2}} t^k \sin tx dt. \tag{4.6}$$

The coefficients $q_l^{(k)}$ are calculated in the same way as in the classical Chebyshev-Edgeworth-Cramér expansion, however there is a difference in the asymptotic of the functions $U_k(x)$ if this functions arise from quasicumulants instead of cumulants. If all c_k^+ are real for $k \le r_0 - 1$ (i.e. they are cumulants), then by (4.3) it follows that when k is even the $u_k^{(l)}$'s are 0 and when k is odd the $v_k^{(l)}$'s are 0, for all $k \le r_0 - 2$. Since for $\Lambda_k^{(+)}$ and $\Lambda_k^{(-)}$ we have the well known expressions

$$\Lambda_k^{(+)} = \frac{(-1)^{\frac{k}{2}}}{\sqrt{2\pi}} h_k(x) e^{-\frac{x^2}{2}}, \quad \text{for } k \text{ even} \tag{4.7}$$

$$\Lambda_k^{(-)} = \frac{(-1)^{\frac{k-1}{2}}}{\sqrt{2\pi}} h_k(x) e^{-\frac{x^2}{2}}, \quad \text{for } k \text{ odd} \tag{4.8}$$

where $h_k(x)$ is the kth Hermite polynomial, all $U_k(x)$'s have the exponential decay $e^{-\frac{x^2}{2}}$ when $|x| \to \infty$, for $k \le r_0 - 2$. But if in the expansion (3.4) there are some $c_k^+ \neq c_k^-$, then we will have some non-zero coefficient for $\Lambda_k^{(+)}$, k odd, or for $\Lambda_k^{(-)}$, k even, and in such cases the following Proposition shows that the asymptotic is only power-decay.

Proposition 4.2.1. *Let $h \ge 0$ be an integer. Then for $x \neq 0$ and $k = 2h+1$ we have:*

$$\int_0^{+\infty} e^{-\frac{t^2}{2}} t^k \cos tx dt = (-1)^{\frac{k+1}{2}} k! \frac{1}{x^{k+1}} + \sum_{j=h+1}^{n-1} (-1)^{j+1} \frac{d_{2j+1}}{x^{2j+2}} + \frac{(-1)^n}{x^{2n}} \int_0^{+\infty} D^{2n}(e^{-\frac{t^2}{2}} t^k) \cos tx dt, \tag{4.9}$$

while for $x \neq 0$ and $k = 2h$ we have:

$$\int_0^{+\infty} e^{-\frac{t^2}{2}} t^k \sin tx dt = (-1)^{\frac{k}{2}} k! \frac{1}{x^{k+1}} + \sum_{j=h+1}^{n-1} (-1)^{j} \frac{d_{2j}}{x^{2j+1}} + \frac{(-1)^n}{x^{2n}} \int_0^{+\infty} D^{2n}(e^{-\frac{t^2}{2}} t^k) \sin tx dt, \tag{4.10}$$

with $d_j = D^j(e^{-\frac{t^2}{2}} t^k)|_{t=0}$; moreover $\frac{(-1)^n}{x^{2n}} \int_0^{+\infty} D^{2n}(e^{-\frac{t^2}{2}} t^k) \cos tx dt = o(x^{-2n})$ and $\frac{(-1)^n}{x^{2n}} \int_0^{+\infty} D^{2n}(e^{-\frac{t^2}{2}} t^k) \sin tx dt = o(x^{-2n})$ by Lebesgue Lemma.

The following is the Global Limit Theorem for the Gaussian case.

Theorem 4.2.2. *Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables such that the common characteristic function $\varphi(t)$ belongs to Δ_2, and condition (3.4) is fulfilled with $r \geq r_0+1$ where r_0 is the first index such that $c^+_{r_0} \neq c^-_{r_0}$. Assume one of the following two cases:*

(A) *(1) the r.v.'s X_n take values in $\mathbb{Z}$ and have maximal span equal to 1;*
(2) $\varphi(t) \in C^{r_0+1}(]0, 2\pi[)$;

(B) *(1) $\int_{-\infty}^{\infty} |\varphi(t)|^p \, dt < \infty$ for some $p \geq 1$;*
(2) $\varphi(t) \in C^{r_0+1}(]0, \infty[)$ with bounded derivatives outside a small neighborhood of 0.

Then we have the formula:

$$p_n(x) = \left(\frac{1}{\sqrt{2\pi n}\sigma} e^{-\frac{x^2}{2\sigma^2 n}} + \frac{n \operatorname{Im} c^+_{r_0}}{\pi(\sqrt{n}^{r_0+1} + x^{r_0+1})} \right) (1 + \epsilon(n, x))$$

where $\epsilon(n, x) \to 0$ when $n + |x| \to 0$.

Proof. The first $U_k(x)$ with power decay at infinity is $U_{r_0-1}(x)$, and by the formulas (4.2)-(4.6),(4.9) and (4.10) we have

$$U_{r_0-1}(x) = \frac{r_0! M}{\pi x^{r_0+1}} + W_{r_0-1}(x)$$

$$M = \begin{cases} (-1)^{\frac{r_0+1}{2}} \operatorname{Re} q^{(1)}_{r_0-1} & \text{if } r_0 \text{ is odd} \\ (-1)^{\frac{r_0}{2}} \operatorname{Im} q^{(1)}_{r_0-1} & \text{if } r_0 \text{ is even} \end{cases}$$

with $W_{r_0-1}(x) = O(x^{-r_0-2})$ and $q^{(1)}_{r_0-1} = (r_0!)^{-1}(i)^{r_0} c^+_{r_0}$. All other $U_k(x)$'s, k even, have decay at infinity which is $O(x^{-r_0-2})$.

For $|x| \geq c > 0$ and $y := n^{-1/2}x$ we split formula (4.1) as:

$$p_n(x) = \rho(n, y) + \Gamma(n, y),$$

$$\rho(n, y) = \frac{1}{\sqrt{2\pi n}\sigma} e^{-\frac{y^2}{2}} + \frac{1}{\sqrt{n}^{r_0-1}} \frac{r_0! M}{\pi y^{r_0+1}},$$

$$\Gamma(n, y) = \sum_{k=2}^{r_0-2} \frac{1}{\sqrt{n}^k} U_k(y) + \frac{1}{\sqrt{n}^{r_0-1}} W_{r_0-1}(y) + \frac{1}{\sqrt{n}^{r_0}} U_{r_0}(y) + \frac{1}{(\sqrt{n})^{r_0}} \frac{1}{1 + y^{r_0+1}} \gamma(n, \sqrt{n} y),$$

and then divide both sides by $\rho(n, y)$. By analyzing the orders of n and y which appear in $\Gamma(n, y)$ and $\rho(n, y)$ it is clear that $\Gamma(n, y)/\rho(n, y) \to 0$ as $n + |x| \to \infty$. Therefore also $\epsilon(n, x) := \Gamma(n, n^{-1/2}x)/\rho(n, n^{-1/2}x) \to 0$ as $n + |x| \to \infty$. Since (4.9) and (4.10) are asymptotics for large x, we can safely replace $\frac{r_0! M}{\pi y^{r_0+1}}$ with $\frac{r_0! M}{\pi(1+y^{r_0+1})}$, and a slight modification of $\epsilon(n, x)$ yields the result. $\square$

We now show, with two examples, the wide range of applicability of the global limit theorem stated above.

4.2.1 Regular tails for lattice distributions.

Consider a symmetric random variable X with values in $\mathbb{Z}$, satisfying the property:

$$(4.11) \qquad P(X=x) = \frac{a_2}{x^4} + \frac{a_3}{x^6} + \cdots + \frac{a_m}{x^{2m}} + O\left(\frac{1}{x^{2m+1+\epsilon}}\right), \quad x \in \mathbb{Z}$$

when $|x| \to \infty$. Notice that X has $2s_0$ moments, where s_0 is the smallest index such that $a_{s_0+1} \neq 0$. We will show that X satisfies the condition of Theorems 3.3.1 and 4.2.2.

The condition (A1) for the maximal span is automatically guaranteed by (4.11). The characteristic function of X can be decomposed as:

$$\varphi(t) = a_2 f_4(t) + a_3 f_6(t) + \cdots + a_m f_{2m}(t) + R(t)$$

$$f_{2l}(t) = 2\sum_{x=1}^{\infty} \frac{\cos tx}{x^{2l}}$$

and $R(t)$ is the Fourier transform corresponding to the remainder term in (4.11) and belongs to $C^{2m}(\mathbb{R})$. For $f_{2l}(t)$ we have the formula

$$f_{2l}(t) = (-1)^{l-1}\frac{2(2\pi)^{2l}}{2(2l)!}\sum_{j=0}^{2l}\binom{2l}{j}B_{2l-j}\left(\frac{t}{2\pi}\right)^j, \quad t \in [0, 2\pi]$$

where B_s are the Bernoulli numbers (see [7] p.11 and [3]). Proceeding as in [7] it can be shown that, although $f_{2l}(t)$ is C^∞ in $]0, 2\pi[$, the derivative of order $2l-1$ has a jump at 0 (or 2π):

$$\frac{d^{2l-1}f_{2l}}{dt^{2l-1}}(0^-) = -\frac{d^{2l-1}f_{2l}}{dt^{2l-1}}(0^+) > 0.$$

Therefore $\varphi(t)$ belongs to Δ_2 and satisfies condition (A2) of Theorem 4.2.2 with $r = 2m$ and $r_0 = 2s_0 + 1$.

4.2.2 Regular tails for distributions with density

Let $p(x)$, $x \in \mathbb{R}$, be a symmetric bounded density satisfying

$$(4.12) \qquad p(x) = \frac{a_2}{x^4} + \frac{a_3}{x^6} + \cdots + \frac{a_m}{x^{2m}} + O\left(\frac{1}{x^{2m+1+\epsilon}}\right)$$

when $|x| \to \infty$. As in the previous example, $r_0 = 2s_0+1$ where s_0 is the smallest index such that $a_{s_0+1} \neq 0$. Let us prove that the c.f.

$$\varphi(t) = \int_{-\infty}^{+\infty} e^{itx}p(x)dx$$

satisfies the hypothesis of Theorem 4.2.2 with $r = 2m$.

Condition (B1) is satisfied since $p(x)$ is bounded. $\varphi(t)$ can be split as:

$$\varphi(t) = a_2 g_4(t) + a_3 g_6(t) + \cdots + a_m g_{2m}(t) + R(t)$$

$$g_{2l}(t) = 2\int_1^{+\infty} \frac{\cos tx}{x^{2l}}dx$$

and $R(t)$ corresponds to the remainder term in (4.12) and is $2m$ times differentiable.

For $t \geq 0$ consider the identity

$$\int_{C_R^+} \frac{e^{itz}}{z^{2l}} dz = 0$$

where C_R^+ is the contour in the complex plane defined by the semicircles $\Gamma_R^+ = \{Re^{it\theta} \colon 0 \leq \theta \leq \pi\}$ and $\Gamma_1^+ = \{e^{it\theta} \colon 0 \leq \theta \leq \pi\}$ and the segments $[-R, -1]$ and $[1, R]$. Then

$$\int_{\Gamma_R^+} \frac{e^{itz}}{z^{2l}} dz - \int_{\Gamma_1^+} \frac{e^{itz}}{z^{2l}} dz + 2 \int_1^R \frac{\cos tx}{x^{2l}} dx = 0.$$

Since the integral over Γ_R^+ in the above identity goes to 0 as $R \to \infty$ for $t \geq 0$, passing to the liming when $R \to \infty$ we get

$$g_{2l}(t) = \int_{\Gamma_1^+} \frac{e^{itz}}{z^{2l}} dz \tag{4.13}$$

which implies not only that $g_{2l}(t)$ is C^∞ on $]0, \infty[$, but also that it has any number of one-sided derivatives at 0^+. If instead of C_R^+ we consider $C_R^- = -C_R^+$, we obtain and expression for $g_{2l}(t)$ when $t \leq 0$, namely

$$g_{2l}(t) = -\int_{\Gamma_1^-} \frac{e^{itz}}{z^{2l}} dz \tag{4.14}$$

with $\Gamma_1^- = \{e^{it\theta} \colon -\pi \leq \theta \leq 0\}$. From (4.13) and (4.14) we derive a formula for the one-sided derivatives of $g_{2l}(t)$ at 0:

$$g_{2l}^{(h)}(0^\pm) = \pm(i)^h \int_{\Gamma_1^\pm} z^{h-2l} dz = \begin{cases} \frac{(i)^h}{h-2l+1}[(-1)^{h+1} - 1] & \text{if } h \neq 2l-1 \\ \pm(-1)^{l+1} i\pi & \text{if } h = 2l-1 \end{cases}$$

which shows that the derivative of order $2l-1$ at 0 has opposite sign from the left an from the right: $g_{2l}^{(2l-1)}(0^+) = -g_{2l}^{(2l-1)}(0^-)$.

The c.f. $\varphi(t)$ belongs to Δ_2 and the first half of condition (B2) of Theorem 4.2.2 is satisfied. It remains to show that the derivative of $\varphi(t)$ are bounded. $R(t)$ has $2m$ bounded derivatives since it is the Fourier transform of a function $O(x^{-2m-1-\epsilon})$. We know from (4.13) that the derivatives of $g_{2l}(t)$ are continuous on $[0, \infty[$ and have finite limit at 0^+; let us prove they tend to 0 when $t \to \infty$. For $t > 0$ use the substitution $y = tx$:

$$g_{2l}(t) = t^{2l-1} \int_t^{+\infty} \frac{\cos y}{y^{2k}} dy. \tag{4.15}$$

Since $g_{2l}(t) \to 0$ as $t \to \infty$, it follows that

$$\int_t^{+\infty} \frac{\cos y}{y^{2k}} dy = o(t^{1-2l}), \quad t \to \infty. \tag{4.16}$$

The derivative $g_{2l}^{(h)}(t)$ is a sum of terms of the shape

$$D^u(t^{2l-1}) D^v \left(\int_t^{+\infty} \frac{\cos y}{y^{2k}} dy \right), \quad u + v = h$$

which go to 0 when $t \to \infty$ because of (4.16).

Remark 4.2.1. Linnik [4] obtained a result analogue to Theorem 4.2.2, for sequence of i.i.d. symmetric random variables assuming that they have bounded density with the asymptotic (4.12).

4.3 The case $0 < \alpha < 1$

If the c.f. satisfies the hypothesis of Theorem 3.3.1, we have the asymptotic expansion

$$\begin{aligned} p_n(x) =& \frac{1}{n^{1/\alpha}} p_{(\alpha,c)}\left(\frac{x}{n^{1/\alpha}}\right) \\ &+ \sum_{\substack{k\geq 2, j\geq 0\\ k+j\alpha\leq r+1}} \frac{1}{n^{k/\alpha+j}} U_{k,j}^{(\alpha,c)}\left(\frac{x}{n^{1/\alpha}}\right) + \frac{n^{(\nu-r-1)/\alpha}}{n^{\nu/\alpha}+x^\nu}\gamma(n,x) \end{aligned}$$

$$U_{k,j}^{(\alpha,c)}(y) = \frac{1}{\pi}\sum_{l=1}^{k-1} \mathrm{Re}(q_{k,j}^{(l)} I_{k-1,l+j}^{(\alpha,c)}(y))$$

where $p_{(\alpha,c)}(y)$ is the strictly stable density of parameters (α, c). The integrals $I_{k,l}^{(\alpha,c)}(y)$ defined in (3.2) have the property

$$I_{k,l}^{(\alpha,c)}(-y) = I_{k,l}^{(\alpha,\bar{c})}(y)$$

and if $0 < \alpha < 1$ using integration on the complex plane we can write them as a convergent series of powers of $1/y$, for $y > 0$:

$$I_{k,l}^{(\alpha,c)}(y) = -\sum_{n=0}^{\infty} \frac{\Gamma((l+n)\alpha+k+1)}{n!}(-c)^n \frac{1}{(iy)^{(l+n)\alpha+k+1}}$$

(see, for example, Feller [2]). So when $y \to \infty$, $p_{(\alpha,c)}(y) = \pi^{-1}\,\mathrm{Re}\, I_{0,0}^{(\alpha,c)}(y) \sim \mathrm{Re}\, \frac{\Gamma(1+\alpha)c}{\pi(iy)^{1+\alpha}}$, and when $(k,l) \neq (0,0)$ $I_{k,l}^{(\alpha,c)}(y) \sim -\frac{\Gamma(l\alpha+k+1)}{(iy)^{l\alpha+k+1}}$. From the above it is easy to notice that the function $U_{k,j}^{(\alpha,c)}(y)$ has a power decay for $y \to \infty$ faster then $U_{k',j'}^{(\alpha,c)}(y)$ if $l\alpha + k > l'\alpha + k'$. So the in this case the global limit theorem is even easier than in the Gaussian case, as it doesn't depend on the presence of an actual quasicumulant ($c_k^+ \neq c_k^-$).

Theorem 4.3.1. *Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables such that the common characteristic function $\varphi(t)$ belongs to Δ_α with $0 < \alpha < 1$ and satisfies* (3.2) *for some $r \geq 2$. Assume one of the following two cases:*

- *(A)* *(1) the r.v.'s X_n take values in $\mathbb{Z}$ and have maximal span equal to 1;*
  (2) $\varphi(t) \in C^2(]0, 2\pi[)$;
- *(B)* *(1) $\int_{-\infty}^{\infty} |\varphi(t)|^p\, dt < \infty$ for some $p \geq 1$;*
  (2) $\varphi(t) \in C^2(]0, \infty[)$ with bounded derivatives outside a small neighborhood of 0.

Then we have for all x the formula:

$$p_n(x) = \frac{1}{n^{1/\alpha}} p_{(\alpha,c)}\left(\frac{x}{n^{1/\alpha}}\right)(1 + \epsilon(n,x))$$

where $p_{(\alpha,c)}(y)$ is the strictly stable density of parameters (α, c), and $\epsilon(n,x) \to 0$ when $n + |x| \to 0$.

Proof. The Theorem can be proved using the same argument as in the proof of Theorem 4.2.2, and the considerations above on the function $U_{k,j}^{(\alpha,c)}(y)$. $\square$

4.4 The Cauchy case

If the c.f. satisfies the condition of the Example 3.1.2 and the hypothesis of Theorem 3.3.1, the asymptotic expansion (3.13) can be rewritten as

$$p_n(x) = \frac{1}{\pi}\frac{na}{(na)^2 + (x-b)^2} + \sum_{k=2}^{r} \frac{1}{n^k} U_k\left(\frac{x}{n}\right) + \frac{n^{(\nu-r)}}{n^\nu + x^\nu}\gamma(n,x)$$

$$U_k(y) = \frac{1}{\pi}\sum_{l=1}^{k-1} \mathrm{Re}(q_k^{(l)} I_{k-1,l}^{(1,c)}(y))$$

with $c = a - ib$ and $q_k^{(l)}$ defined as in section 4.2. This times we have an explicit expression for $I_{k-1,l}^{(1,c)}(y)$ coming from (3.16):

$$I_{k-1,l}^{(1,c)}(y) = \frac{(-1)^l (k+l)!}{(c+ix)^{k+l+1}}.$$

As in the case $0 < \alpha < 1$ each function $U_k(y)$ has a power decay for $y \to \infty$ and $U_k(y)$ decays faster than $U_{k'}(y)$ if $k > k'$. So also in the case $\alpha = 1$ we have the global limit theorem even in absence of actual quasicumulants.

Theorem 4.4.1. *Let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables such that the common characteristic function $\varphi(t)$ belongs to Δ_1 and satisfies* (3.5) *for some integer $r \geq 2$. Assume one of the following two cases:*

(A) *(1) the r.v.'s X_n take values in $\mathbb{Z}$ and have maximal span equal to* 1*;*
(2) $\varphi(t) \in C^2(]0, 2\pi[)$;

(B) *(1) $\int_{-\infty}^{\infty} |\varphi(t)|^p \, dt < \infty$ for some $p \geq 1$;*
(2) $\varphi(t) \in C^2(]0, \infty[)$ with bounded derivatives outside a small neighborhood of 0*.*

Then we have for all x the formula:

$$p_n(x) = \frac{1}{\pi}\frac{na}{(na)^2 + (x-b)^2}(1 + \epsilon(n,x))$$

where $\epsilon(n,x) \to 0$ when $n + |x| \to 0$.

We formulate now the same two example on the regular tails as the previous section, and the proofs are just a repetition.

Example 4.4.1. Consider a symmetric random variable X with values in $\mathbb{Z}$, satisfying the property:

$$P(X = x) = \frac{a_1}{x^2} + \frac{a_2}{x^4} + \cdots + \frac{a_m}{x^{2m}} + O\left(\frac{1}{x^{2m+1+\epsilon}}\right), \quad x \in \mathbb{Z} \tag{4.17}$$

when $|x| \to \infty$, with $a_1 > 0$. Then the c.f. of X belongs to $C^{2m}([0, 2\pi])$, and the Theorem 4.4.1 applies.

Example 4.4.2. Let $p(x)$, $x \in \mathbb{R}$, be a symmetric bounded density satisfying

$$p(x) = \frac{a_1}{x^2} + \frac{a_2}{x^4} + \cdots + \frac{a_m}{x^{2m}} + O\left(\frac{1}{x^{2m+1+\epsilon}}\right) \tag{4.18}$$

when $|x| \to \infty$, with $a_1 > 0$. Then the c.f. of X belongs to $C^{2m}([0, 2\pi])$ and has bounded derivatives, and the Theorem 4.4.1 applies.

The following counterexample shows the importance of the differentiability conditions A2 and B2.

Example 4.4.3. Let

$$\varphi(t) = \begin{cases} 1 - |t| & \text{for } |t| \leq 1 \\ 0 & \text{for } |t| > 1. \end{cases}$$

This is the characteristic function of of a symmetric random variable with density

$$p(x) = \frac{1 - \cos x}{\pi x^2}.$$

The density $p_n(x)$ of the sum of i.i.d. random variables with the above density cannot obey Theorem 4.4.1 because it will always have oscillatory behaviour. Similar example can be formulated for the lattice case.

Appendix A

The Morse-Palais Lemma with parameters

In this appendix we will prove a generalized version of the well-known Morse-Palais Lemma, where we have dependency of the function on parameters. The scheme of the proof is taken from [5], and the generalization is quite straight forward. Also the reader should refer to [5] for the notion of "C^p-morphism", "C^p-isomorphism", and so on.

In what follows **E**, **F**, and **G** are Banach spaces, and $U \in \mathbf{E}$ and $V \in \mathbf{F}$ are open neighborhoods of the points $x_0 \in \mathbf{E}$ and $y_0 \in \mathbf{F}$ respectively. The following Lemma is a generalization of the Inverse Mapping Theorem with addition of an extra parameter.

Lemma A.0.2. *Let $f\colon U \times V \to \mathbf{G}$ be C^p-morphism with $p \geq 1$, and assume that $D_1 f(x_0, y_0)\colon \mathbf{E} \to \mathbf{G}$ is a linear isomorphism. Then there exists an open neighborhood $U_1 \times V_1 \subseteq U \times V$ of (x_0, y_0) such that*

$$f(\cdot, y)\colon U_1 \to \mathbf{G}$$

is a C^p-isomorphism on its image for every $y \in V_1$.

Proof. Since $D_1 f(x_0, y_0)$ is an isomorphism, we can assume that $\mathbf{E} = \mathbf{G}$. Further, using an affine change of coordinates, we can assume without lost of generality that that $x_0 = 0$, $y_0 = 0$, $f(x_0, y_0) = 0$ and $D_1 f(x_0, y_0) = id$.

If $g(x, y) := x - f(x, y) + f(0, y)$, then $g'(0, 0) = 0$, and by continuity there exist $0 \in \widetilde{U} \subseteq U$ and $0 \in \widetilde{V} \subseteq V$ such that $|D_1 g(x, y)| < 1/2$ and $|D_2 g(x, y)| < \lambda$ for $x \in \widetilde{U}$ and $y \in \widetilde{V}$. Introduce the following letters for the closed balls in **E**:

$$B_\delta = \overline{B_\delta(0; \mathbf{E})} \quad B_{\delta,y} = \overline{B(f(0, y), \delta; \mathbf{E})}$$

By the Mean Value Theorem it follows that $|g(x, y)| \leq \frac{1}{2}|x|$ if $x \in \widetilde{U}$ and $y \in \widetilde{V}$, therefore, if δ is so small that $B_\delta \subset \widetilde{U}$, $g(\cdot, y)$ maps B_δ into $B_{\delta/2}$.

We want to prove that, for fixed $y \in \widetilde{V}$, if $z \in \mathbf{E}$ satisfies $|z - f(0.y)| \leq \delta/2$ (i.e. $z \in B_{\delta/2,y}$) then there exists a unique $x \in B_\delta$ such that $f(x, y) = z$. Consider the map

$$g_z(x, y) = z - f(0, y) + g(x, y);$$

if $z \in B_{\delta/2,y}$ then $g_z(\cdot, y)$ maps B_δ into itself, and we have the contraction property

$$|g_z(x_1, y) - g_z(x_2, y)| = |g(x_1, y) - g(x_2, y)| \leq \frac{1}{2}|x_1 - x_2|$$

by the Mean Value Theorem. Therefore we constructed a map $\varphi(z, y)$ that for each $y \in \widetilde{V}$ and $z \in B_{\delta/2,y}$ gives the unique fixed point of $g_z(\cdot, y)$, but as easy to see the fixed point of $g_z(\cdot, y)$ is the one point x such that $f(x, y) = z$, hence $\varphi(\cdot, y)$ is the inverse of $f(\cdot, y)$. $\varphi(z, y)$ is continuous on $B_{\delta/2,y} \times \widetilde{V}$ because

$$\begin{aligned} \text{(A.1)} \quad |\varphi(z_1, y_1) - \varphi(z_2, y_2)| &= |g_{z_1}(\varphi(z_1, y_1), y_1) - g_{z_2}(\varphi(z_2, y_2), y_2)| \\ &\leq |z_1 - z_2| + |f(0, y_1) - f(0, y_2)| + |g(\varphi(z_1, y_1), y_1) - g(\varphi(z_2, y_2), y_2)| \end{aligned}$$

but

$$\begin{aligned} &|g(\varphi(z_1, y_1), y_1) - g(\varphi(z_2, y_2), y_2)| \\ &\quad = |g(\varphi(z_1, y_1), y_1) - g(\varphi(z_2, y_2), y_1) + g(\varphi(z_2, y_2), y_1) - g(\varphi(z_2, y_2), y_2)| \\ &\quad \leq \left| \int_0^1 D_1 g(\varphi(z_2, y_2) + t[\varphi(z_1, y_1) - \varphi(z_2, y_2)], y_1)[\varphi(z_1, y_1) - \varphi(z_2, y_2)]dt \right| \\ &\qquad\qquad + \left| \int_0^1 D_2 g(\varphi(z_2, y_2), y_2 + t(y_1 - y_2))(y_1 - y_2)dt \right| \\ &\qquad\qquad\qquad \leq \frac{1}{2}|\varphi(z_1, y_1) - \varphi(z_2, y_2)| + \lambda |y_1 - y_2| \end{aligned}$$

which together with (A.1) implies

$$|\varphi(z_1, y_1) - \varphi(z_2, y_2)| \leq 2\big[\,|z_1 - z_2| + |f(0, y_1) - f(0, y_2)| + \lambda |y_1 - y_2|\,\big].$$

We claim that the derivative of $\varphi(z, y)$ is given by

$$\text{(A.2)} \qquad \big(D_1 f(\varphi(z, y), y)^{-1}, -D_1 f(\varphi(z, y), y)^{-1} D_2 f(\varphi(z, y), y)\big)$$

In order to prove this consider the points $x_1 = \varphi(z_1, y_1)$ and $x_2 = \varphi(z_2, y_3)$ belonging to $B_\delta(0; \mathbf{E})$, and write the following estimate

$$\begin{aligned} &\big|\varphi(z_1, y_1) - \varphi(z_2, y_2) - D_1 f(\varphi(z_2, y_2), y_2)^{-1}(z_1 - z_2) + \\ &D_1 f(\varphi(z_2, y_2), y_2)^{-1} D_2 f(\varphi(z_2, y_2), y_2)(y_1 - y_2)\big| \leq \big|D_1 f(\varphi(z_2, y_2), y_2)^{-1}\big| \cdot \\ &|D_1 f(\varphi(z_2, y_2), y_2)(x_1 - x_2) - f(x_1, y_1) + f(x_2, y_2) + D_2 f(x_2, y_2)(y_1 - y_2)| \end{aligned}$$

where $\big|D_1 f(\varphi(z_2, y_2), y_2)^{-1}\big|$ is bounded and the second factor of the product is $o(|(x_1 - x_2, y_1 - y_2)|)$, i.e. $o(|(z_1 - z_2, y_1 - y_2)|)$ by the continuity of φ.

By (A.2), since $D_1 f$ and $D_2 f$ are C^{p-1}, we deduce that $\varphi(z, y)$ is C^p. $\square$

As corollaries of Lemma A.0.2 we have the following well known results:

Corollary A.0.3. (Inverse Mapping Theorem) Let $f\colon U \to \mathbf{G}$ be C^p-morphism with $p \geq 1$, such that $f'(x_0)\colon \mathbf{E} \to \mathbf{G}$ is a linear isomorphism. Then there exists an open neighborhood $x_0 \in U_1 \subseteq U$ such that $f(\cdot)$ is a C^p-isomorphism of U_1 on its image.

Corollary A.0.4. (Implicit Function Theorem) Let $f\colon U \times V \to \mathbf{G}$ be C^p-morphism with $p \geq 1$, such that $f(x_0, y_0) = 0$ and $D_1 f(x_0, y_0)\colon \mathbf{E} \to \mathbf{G}$ is a linear isomorphism. Then there exist open neighborhoods $x_0 \in U_1 \subseteq U$ and $y_0 \in V_1 \subseteq V$ and a C^p-map $g\colon V_1 \to U_1$ with $g(y_0) = x_0$, such that a point $(x, y) \in U_1 \times V_1$ satisfies

$$f(x, y) = 0$$

if and only if $x = g(y)$.

Proof. Consider $\varphi\colon U_1 \times V_1 \to \mathbf{G}$ as in the proof of Lemma A.0.2 and define

$$g(y) = \varphi(0, y).$$

□

Lemma A.0.5. *Let $A\colon U \times V \to \mathrm{Sym}(\mathbf{E})$ be a C^p-map taking values into the set of symmetric invertible operators of $\mathbf{E}$. Then there exists $0 \in U_1 \subseteq U$ and $0 \in V_1 \subseteq V$, and a C^p-map $\varphi\colon U_1 \times V_1 \to \mathbf{E}$ of the form*

$$\varphi(x, y) = C(x, y)x \text{ with } C\colon U_1 \times V_1 \to \mathrm{End}(\mathbf{E}) \tag{A.3}$$

such that $\varphi(\cdot, y)$ is a C^p-isomorphism of U_1 onto its image for every $y \in V_1$, and

$$(x|A(x, y)x) = (\varphi(x, y)|A(0, y)\varphi(x, y)) = (C(x, y)x|A(0, y)C(x, y)x).$$

Proof. Let us construct a map C such that

$$C^*(x, y)A(0, y)C(x, y) = A(x, y). \tag{A.4}$$

Put $B(x, y) = A(0, y)^{-1}A(x, y)$: since $B(0, y) = id$ there exists a neighborhood $\widetilde{U} \times \widetilde{V}$ where the square root power series expansion of $B(x, y)$ converges, therefore we can define $C(x, y) = B(x, y)^{1/2}$. Let us check relation (A.4) for $C(x, y)$ when $x \in \widetilde{U}$ and $y \in \widetilde{V}$.

Since $A(x, y)$ is self adjoint, we have

$$B^*(x, y) = A(x, y)A(0, y)^{-1}$$

and consequently

$$B^*(x, y)A(0, y) = A(0, y)B(x, y). \tag{A.5}$$

Since $C(x, y)$ is a uniform limit of polynomials in $B(x, y)$, and $C^*(x, y)$ is a uniform limit of the same polynomials in $B^*(x, y)$, we get that relation (A.5) holds also for $C(x, y)$, i.e.

$$C^*(x, y)A(0, y) = A(0, y)C(x, y)$$

which implies

$$C^*(x, y)A(0, y)C(x, y) = A(0, y)C(x, y)C(x, y) = A(0, y)B(x, y) = A(x, y).$$

It remains to prove that $\varphi(\cdot, y)$ defined as in (A.3) is a C^p-isomorphism of some $U_1 \subseteq \widetilde{U}$, for every $y \in V_1 \subseteq V$. This is a consequence of Lemma A.0.2, as $D_1\varphi(0, y) = C(0, y)\colon \mathbf{E} \to \mathbf{E}$ is an isomorphism. □

Next is what we called the *Morse-Palais Lemma with parameters*:

Theorem A.0.6. *Let $f: U \times V \to \mathbb{R}$ be a C^{p+2}-map, with $p \geq 1$. If $x_0 \in U$ is such that $D_1 f(x_0, y_0) = 0$ and $D_1^2 f(x_0, y_0)$ is a linear isomorphism for some $y_0 \in V$, then there exist neighborhoods $x_0 \in U_1 \subseteq U$ and $y_0 \in V_1 \subseteq V$, and a C^p-map $\varphi: U_1 \times V_1 \to \mathbf{E}$ such that $\varphi(\cdot, y)$ is a C^p-isomorphism of U_1 onto its image for every $y \in V_1$, and*

$$f(x, y) = (\varphi(x, y) | A(y) \varphi(x, y))$$

if $x \in U_1$ and $y \in V_1$.

Proof. We first "normalize" f, i.e. find $\tilde{f}$ such that

$$\tilde{f}(0, y) = 0 \text{ and } D_1 \tilde{f}(0, y) = 0 \tag{A.6}$$

in some small neighborhood of y_0. In order to do this, consider the equation $D_1 f(x, y) = 0$: since (x_0, y_0) satisfies the equation and $D_1^2 f(x_0, y_0)$ is a linear isomorphism, by the Implicit Function Theorem there exists a C^{p+1}-map $\alpha(y)$ such that $D_1 f(\alpha(y), y) = 0$ for every $y \in \widetilde{V} \subseteq V$.

The map

$$\Phi(x, y) = \begin{pmatrix} x + \alpha(y) \\ y \end{pmatrix} \tag{A.7}$$

is a C^{p+1}-isomorphism for $y \in \widetilde{V}$ and $x \in U$, therefore the function $\tilde{f}(x, y) = f(\Phi(x, y)) - f(\alpha(y), y)$ satisfies condition (A.6) in addition to the hypothesis of our Theorem.

Applying the Mean Value Theorem to the first variable of $\tilde{f}$ we get

$$\tilde{f}(x, y) = \int_0^1 D_1 \tilde{f}(tx, y) x dt$$

and applying it again, this time to $D_1 \tilde{f}$, we get

$$\tilde{f}(x, y) = \int_0^1 \int_0^1 (t D_1^2 \tilde{f}(stx, y) | x, x) ds dt.$$

Now the operator valued map

$$A(x, y) = \int_0^1 \int_0^1 t D_1^2 \tilde{f}(stx, y) ds dt$$

fulfills the hypothesis of Lemma A.0.5, therefore there exists a C^p-map $\varphi(x, y)$ such that

$$\tilde{f}(x, y) = (\varphi(x.y) | A(0, y) \varphi(x, y)).$$

Consider now $\psi(x, y) = (\varphi \circ \Phi^{-1})(x, y)$ and obtain

$$f(x, y) = (\psi(x, y) | A(\Phi^{-1}(0, y)) \psi(x, y)) + f(\Phi^{-1}(\alpha(y), y)).$$

□

Remark A.0.1. If in Theorem A.0.6 $f(x, y)$ is such that $D_1^2 f(x_0, y_0)$ is positive defined and invertible, then it is positive defined and invertible in a neighborhood of (x_0, y_0), so we can define $\psi'(x, y) = A(\Phi^{-1}(0, y))^{1/2} \psi(x, y)$, and get

$$f(x, y) = |\psi'(x, y)|^2 + f(\Phi^{-1}(\alpha(y), y)).$$

Bibliography

[1] Harald Cramér. On asymptotic expansions for sums of independent random variables with a limiting stable distribution. *Sankhyā Ser. A 25 (1963), 13-24; addendum, ibid.*, 25:216, 1963.

[2] William Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. Wiley, New York, 1971.

[3] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products.* Academic Press, New York, 1980.

[4] I. A. Ibragimov and Yu. V. Linnik. *Independent and stationary sequences of random variables.* Wolters-Noordhoff, Groningen, 1971.

[5] Serge Lang. *Fundamentals of differential geometry.* Springer-Verlag, New York, 1999.

[6] S. A. Molchanov and E. Ostrovskiĭ. Symmetric stable processes as traces of degenerate diffusion processes. *Teor. Verojatnost. i Primenen.*, 14:127–130, 1969.

[7] S. A. Molchanov, V. V. Petrov, and N. Squartini. Quasicumulant and limit theorems in case of the Cauchy limiting law. *Markov Process. Related Fields*, 13:597–624, 2007.

[8] L. V. Osipov. Asymptotic expansions of the distribution function of a sum of random variables with non uniform estimates for the remainder term. *Vestnik Leningrad. Univ.*, (1):51–59, 1972.

[9] Valentin V. Petrov. *Sums of independent random variables.* Springer-Verlag, New York, 1975.

[10] Valentin V. Petrov. *Limit theorems of probability theory.* Oxford University Press, New York, 1995.

[11] Ken-iti Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1999. Translated from the 1990 Japanese original, Revised by the author.

[12] Frank Spitzer. *Principles of random walks.* Springer-Verlag, New York, 1976.

[13] Gábor Szegő. *Orthogonal polynomials*. American Mathematical Society, Providence, R.I., fourth edition, 1975. American Mathematical Society, Colloquium Publications, Vol. XXIII.

[14] Thomas Whitehurst. An application of orthogonal polynomials to random walks. *Pacific J. Math.*, 99(1):205–213, 1982.