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Theory on foliated manifolds with cylindrical ends

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Index Theory on foliated manifolds with cylindrical ends

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Abstract

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1 Introduction

The aim of this PHD thesis is to prove an Atiyah Patodi Singer index formula for a Dirac operator on a manifold with cylindrical ends which is foliated by a foliation that respects the cylindrical structure........

Geometric setting
The whole geometric structure is introduced. We speak about cylindrical foliations and all the data needed to define the longitudinal Dirac operator associated to a Clifford bundle. Every cylindrical foliation arises from a gluing process starting from a foliated manifold with boundary and the foliation transverse to the boundary. The first geometrical invariant of a foliation is its holonomy. It enters into index theory in essential way providing a natural desingularization of the leaf space where doing analysis. Following Ramachandran we work at level of the equivalence relation of being on the same leaf. This is the most elementar level of desingularization.

Von Neumann algebras, foliations and index theory

Von Neumann algebras and Breuer Fredholm theory with traces. In this section generalities about Von Neumann algebras are given. These are particular $*$–subalgebras of all bounded operators acting on an Hilbert space. We specialize to Von Neumann algebras that can be equipped with a semi–finite normal faithful traces likewise Von Neumann algebras arising from foliations admitting a holonomy invariant transverse measure.

With a trace $\tau : M^+ \longrightarrow [0, \infty]$ one has a natural notion of dimension of a closed subspace affiliated to M, i.e. a subspace V whose projection Pr_V belongs to M. This is the relative dimension $\tau(\mathrm{Pr}_V)$. Relative dimension is the cornerstone of a theory of Fredholm operators inside M. This story goes back to the general seminal work of Breuer [14, 15]. For this reason relatively Fredholm operators are called Breuer–Fredholm. A Breuer–Fredholm operator has a finite real index with some stability properties as in the classical theory.

Transverse measures and Von Neumann algebras. In the spirit of Alain Connes non commutative geometry Von Neumann algebras stand for measure spaces while C^*–algebras describes topological spaces. In the seminal work [24] has shown that a foliation with a given transverse measure gives rise to a Von Neumann algebra whose properties reflect the properties of the measure. First we define transverse measures as measures on the sigma ring of all Borel transversals. This is acted by the holonomy pseudogroup. When the measure is invariant w.r.t. this action one has a holonomy invariant measures.

If a holonomy invariant measure exists then the associated W^*– algebra is type I or type II (the first type appears only in the ergodic case). In particular there's a natural trace whose definition is explicitly given as an integral of suitable objects living along leaves against the transverse measure.

Then transverse measures appear as some kind of measure on the space of the leaves.

In this section we define the Von Neumann algebra associated to the transverse measure and a square representation of the Borel equivalence relation $x\mathcal{R}y$ iff x and y are in the same leave. For a vector bundle E this is the algebra of uniformly bounded fields of operators $x \longmapsto A_x : L^2(L_x; E) \longrightarrow L^2(L_x; E)$ (L_x is the leave of x) acting on the Borel field of Hilbert spaces $x \longmapsto L^2(X; E)$ suitably identified using the transverse measure. Thinking of an operator as a family of leafwise operators the trace has a natural meaning, it is the integral against the transverse measure of a family of leafwise measures called local traces.

For self adjoint intertwining operators, using the spectral theorem and the trace on M (coming from a transverse measure Λ) one can define a measure on $\mathbb{R}$ called the spectral measure (depending on the trace). Breuer–Fredholm properties of the operator are easily described in terms of this spectral measure. In particular one can define some kind of essential spectrum called the Λ–essential spectrum. Belonging of zero to the essential spectrum is equivalent for the operator to be Breuer–Fredholm. We show also that for elliptic operators the essential spectrum is governed by the behavior of the operator outside compact subsets on the ambient manifold. Actually if one fix a compact set K on X every leave can intersect K infinite times then our notion of "lieing outside K" must be explained with care. We call this result the Splitting principle. It will be useful in the study of the Dirac operator.

Analysis of the Dirac operator. Consider the leafwise Dirac operator on X associated to the geometric datas of the first section. This is obtained from the collection of all Dirac operators $\{D_x\}_x$ one for each leave L_x. If the foliation is assumed even dimensional this is $\mathbb{Z}_2$–graded $D = D^+ \oplus D^-$ with respect to a natural involution on the bundle E. This is called the Chiral Dirac operator.
This leafwise family of operators gives an operator affiliated to the Von Neumann algebra M (the transverse measure gives the glue to join all the operators together). In particular each spectral projection of D defines a projection in M. If the foliated manifold is compact Connes shown this is Breuer–Fredholm and the index, the relative dimension of Kernel minus CoKernel is related to topological invariants of the foliation by the Connes index formula.

$$\mathrm{ind}_\Lambda(D^+) = \langle \mathrm{Ch}(D^+)\,\mathrm{Td}(X), [C_\Lambda] \rangle$$

At right handside one finds the coupling with longitudinal characteristic classes and the homology class of a closed current C_Λ associated to the transverse measure by the Ruelle–Sullivan method.

Finite dimensionality of the index problem. In our cylindrical case, the operator is in general non Breuer–Fredholm. As a general philosophical principle for manifolds with cylindrical ends and product–structure operators, Fredholm properties of the operator on the natural L^2 space are essentially captured by the spectrum at zero of the operator on the cross section (the base of the cylinder). Thanks to the splitting principle the Philosophy

$$\text{invertibility at boundary} \quad \Longleftrightarrow \quad \text{Freholm property}$$

carries on to the foliated case if one looks at the Λ–essential spectrum of the leafwise operator on the foliation induced on the transverse section of the cylinder (this is to be thinked the foliation at infinity).

Now it's a well known fact that lots of Dirac type operators of capital importance in Physics and Geometry are not invertible at the boundary. One example for all is the Signature operator, our main application here.

However some work on elliptic regularity and the use of the generalized eigenfunction expansion of Browder and Gårding shows that the Λ–dimension of the projection on the L^2 kernel of D^+ and D^- are finite projections of the V.N. algebra M. In particular we can define the L^2 chiral index of D^+ as

$$\operatorname{ind}_{L^2,\Lambda}(D^+) = \dim_\Lambda \operatorname{Ker}_{L^2}(D^+) - \dim_\Lambda \operatorname{Ker}_{L^2}(D^-).$$

On a compact foliated manifold, if a family of operators is implemented by a family of leafwise uniformly smoothing schwartz kernels the finite trace property follows immediately from the remarkable fact that integrating a longitudinal Radon measure against a transverse measure gives a finite mass measure on the ambient. Now the ambient is a manifold with a cylinder, hence Radon longitudinal measures do not give finite measures in general. Our strategy to prove the finite dimensionality of the L^2 index problem is to show that the field of L^2 projections on the kernel of D_x^+ enjoys the additional property to be locally traceable with respect to a bigger family of Borel sets. To be more precise we prove that for every compact set K on the boundary of the cylinder of a leave the operator $\chi_{K\times\mathbb{R}^+}\Pi_{\operatorname{Ker}_{L^2}(D^+)}\chi_{K\times\mathbb{R}^+}$ is trace class on $L^2(L_x)$. This is completely sufficient (by the integration process) to assure finite dimensionality.

Breuer–Fredholm perturbation. Once finite dimensionality of kernels is proven we perform a perturbation argument to change the Dirac operator into a Breuer–Fredholm one. This is done following very closely Boris Vaillant master thesis [?] where the same problem is studied for Galois coverings of manifolds with cylindrical ends. Since we are working with Von Neumann algebras the possibilty to use Borel functional calculus gives a great help in a way that we can define our two parameters perturbation essentially by subtracting, on the cylinder the boundary operator restricted to some small spectral interval near zero

$$D \rightsquigarrow D_{\epsilon,u}, \quad D_{\epsilon,0} := D_\epsilon$$

Next we prove (through the splitting principle) that $D_{\epsilon,u}$ is Breuer–Fredholm for small parameters and its index approximates the chiral index. Actually we have to consider separately the two parameters limits.

The analysis of the relation between the perturbed Fredholm index and the chiral L^2 index requires the introduction of weighted L^2 spaces along the leaves, $e^{u\theta}L^2$ for $u > 0$ (r is the cylindrical coordinate). Smooth solutions belonging to each weighted space are called Extended Solutions, $\operatorname{Ext}(D^\pm)$. They enter naturally into the A.P.S index formula naturally but do not form a closed subspace in L^2. Some care is needed in showing their finite Λ–dimensionality.

The remaining part of the section is devoted in the proof of the fundamental asymptotic relations

$$\lim_{\epsilon\to 0} \operatorname{ind}_{L^2,\Lambda}(D_\epsilon^+) = \operatorname{ind}_{L^2,\Lambda}(D^+), \qquad \lim_{\epsilon\to 0} \dim_\Lambda \operatorname{Ext}(D_\epsilon^\pm) = \operatorname{Ext}(D^\pm).$$

Cylindrical finite propagation speed and Cheeger Gromov Taylor type estimates. To prove the index formula we need some pointwise estimates on the Schwartz kernels of functions of the leafwise Dirac operator. Our perturbation on the cylinder has the shape $D+Q$ where Q is some selfadjoint order zero pseudodifferential operator on the base of the cylinder (actually Q is just a sum of a uniformly smoothing operator and $u\,\mathrm{Id}$) in particular one can repeat the proof of energy estimates as in the Book by John Roe for example [65] for the wave equation no more on a small geodesic ball but on a strip $\partial L_x \times (a,b)$ (∂L_x is the base of the cylinder) finding out that unitary cylindrical diffusion speed holds i.e. if ξ_0 is supported in $\partial L_x \times (a,b)$ then the solution of the wave equation $e^{iQ}\xi_0$ is supported in $\partial L_x \times (a-|t|, b+|t|)$. This is sufficient to extimate kernels of class schwartz spectral functions of D and Q following the method of Cheeger, Gromov and Taylor [21] obtaining decaying estimates as in the next example for the heat kernel where $[\cdot]$ is the Schwartz kernel,

$$|\nabla_{z_1}^l \nabla_{z_2}^k [T^m e^{-tT^2}](z_1, z_2)| \leq C(k,l,m,T) e^{(|s_1-s_2|-r_1)^2/6t}. \tag{1}$$

Here r_1 is some positive number and $z_i = (x_i, s_i)$ are two points on the cylinder with $|s_1 - s_2| > 2r_1$. It is clear why one calls these Chegeer Gromov Taylor estimates on the cylindrical direction. There is also an extremely useful relative version of estimate (1) where one can estimate the difference of the kernels of spectral functions of two operators that agree on some open subset of the cylinder.
In practice we shall collect all these estimates, one for each leaf. Thanks to the uniformly bounded geometry of the leaves the constants are independent. This is an extremely important fact.

The foliated eta invariant.
Since its first apparition in [4] the eta invariant of a Dirac operator as the difference between the local and global term on the Atiyah Patodi Singer index formula

$$1/2\eta(D_0) = \int_X \omega_D - \{\mathrm{ind}(D^+) + 1/2\,\mathrm{dimKer}(D_0)\}$$

or the spectral asimmetry defined as the regular value at zero of the meromorphic function (summation over eigenvalues)

$$\eta_{D_0}(s) := \sum_{\lambda \neq 0} \frac{\mathrm{sign}\,\lambda}{|\lambda|^s}, \quad \mathrm{Re}(s) > \dim \partial X \tag{2}$$

has becomed a central character of Spectral geometry and modern Physics.
The foliation eta invariant on a compact manifold (when a transverse invariant measure is fixed) was defined independently and essentially in the same way by Peric [58] and Ramachandran [62] and enters into our A.P.S index formula exactly in the way it enters classically. It should be strongly remarked that Peric and Ramachandran numbers are not the same. The reason is simple. Peric uses the holonomy groupoid to desingularize the space of the leaves while Ramachandran works directly on the Borel equivalence relation. Due to their global nature the eta invariants obtained are not the same. As a striking consequence one get the awareness that on a cylindrical foliated manifold every choice of desingularization from the equivalence relation to the holonomy (or the monodromy groupoid) leads to different index formulas with different eta invariants. This is a genuine feature of the boundary (cylindrical) case.
Since we work with the Borel equivalence relation our eta–invariant is that of Ramachandran. So consider the base Dirac operator $D^{\mathcal{F}_\partial}$ the eta function[1] of $D^{\mathcal{F}_\partial}$ is defined for $\mathrm{Re}(s) \leq 0$ by

$$\eta(D^{\mathcal{F}_\partial}, s) := \frac{1}{\Gamma((s+1)/2)} \int_0^\infty t^{\frac{s-1}{2}}\, \mathrm{tr}_\Lambda(D^{\mathcal{F}_\partial} e^{-D^{\mathcal{F}_\partial}}) dt, \quad |\lambda| > 0, \quad s > -1.$$

It can be shown that this is meromorphic for $\mathrm{Re}(\lambda) \leq 0$ with eventually simple poles at $(\dim \mathcal{F}_\partial - k)/2, \quad k = 0, 1, ...$ and a regular value at zero.

[1]the relation with (2) comes from the identity $\mathrm{sign}(\lambda)|\lambda|^{-1} = \Gamma\big(\frac{s+1}{2}\big)^{-1} \int_0^\infty t^{\frac{s-1}{2}} \lambda e^{-t\lambda^2} dt$

In this section we describe this result extending the result to some classes of perturbations of the operator needed in the proof of the index formula. We shall consider perturbations of the form $Q = D^{\mathcal{F}_\partial} + K$ with K some uniformly smoothing spectral function $K = f(D^{\mathcal{F}_\partial})$, $f : (-a, a) \longrightarrow \mathbb{R}$. For $f = \chi_{(-\epsilon,\epsilon)}$ more can be said about the family $Q_u := D^{\mathcal{F}_\partial} + D^{\mathcal{F}_\partial} f(D^{\mathcal{F}_\partial}) + u$ in fact we can define

$$\eta_\Lambda(Q_u) = \mathrm{LIM}_{\delta\to 0} \int_\delta^k \frac{t^{-1/2}}{\Gamma(1/2)} \operatorname{tr}_\Lambda(Q_u e^{-tQ_u^2})dt + \int_k^\infty \frac{t^{-1/2}}{\Gamma(1/2)} \operatorname{tr}_\Lambda(Q_u e^{-tQ_u^2})dt$$

where LIM is the constant term in the asymptotic development in powers of δ near zero of the function $\delta \longmapsto \int_\delta^k$. Moreover two important formulas hold true

- $\eta_\Lambda(Q_u) - \eta_\Lambda(Q_0) = \operatorname{sign}(u) \operatorname{tr}_\Lambda(f(D^{\mathcal{F}_\partial})$
-
$$\eta_\Lambda(Q_0) = 1/2(\eta_\Lambda(Q_u) + \eta_\Lambda(Q_{-u})). \tag{3}$$

This only requires a minimal modification of Vaillant proof.

The index formula.
Finally we prove the index formula

$$\operatorname{ind}_{L^2,\Lambda}(D^+) = \langle \hat{A}(X) \operatorname{Ch}(E/S), [C_\Lambda] \rangle + 1/2[\eta_\Lambda(D_0^{\mathcal{F}}) - h_\Lambda^+ + h_\Lambda^+]$$

where $h_\Lambda^\pm := \dim_\Lambda(\operatorname{Ext}(D^\pm) - \dim_\Lambda(\operatorname{Ker}_{L^2}(D^\pm)$. Our proof is a modification of Vaillant proof that in turn is inspired by Müller proof of the L^2–index formula on manifolds with corners of codimension two [54]. This is a (of course) a proof based on the heat equation.
The starting point is the identity

$$\operatorname{ind}_{L^2,\Lambda}(D_\epsilon^+) = \lim_{u\downarrow 0} 1/2\{\operatorname{ind}_\Lambda(D_{\epsilon,u}^+) + \operatorname{ind}_\Lambda(D_{\epsilon,-u}^+) + h_{\Lambda,\epsilon}^- - h_{\Lambda,\epsilon}^+\} \tag{4}$$

where

$$h_{\Lambda,\epsilon}^\pm := \dim_\Lambda \operatorname{Ext}(D_\epsilon^\pm) - \dim_\Lambda \operatorname{Ker}_{L^2}(D_\epsilon^\pm)$$

definition also valid for $\epsilon = 0$.
Next we prove

$$\operatorname{ind}_\Lambda(D_{\epsilon,u}^+) = \langle \hat{A}(X) \operatorname{Ch}(E/S), [C_\Lambda] \rangle + 1/2\eta_\Lambda(D_{\epsilon,u}^{\mathcal{F}_\partial}) + g(u) \tag{5}$$

with $g(u) \longrightarrow 0$.
Equation (5) combined with (4) and (3) becomes, after the u–limit

$$\operatorname{ind}_{L^2,\Lambda}(D_\epsilon^+) = \langle \hat{A}(X) \operatorname{Ch}(E/S), [C_\Lambda] \rangle + 1/2\eta_\Lambda(D^{\mathcal{F}_0}) + h_\epsilon^- - h_\epsilon^-.$$

The last step is to assure that under $\epsilon \to 0$ each ϵ–depending object in the above equation goes to the corresponding value for $\epsilon = 0$.

Some words about the proof of (5). This is inspired from the work of Müller [54]. We start from the convergence into the space of leafwise smoothing kernels of $[\exp(-tD_{\epsilon,u}^2)]$ to $[\operatorname{Ker}_{L^2}(D_{\epsilon,u})]$. The choice of cut off functions ϕ_k supported in X_{k+1} (X_m is the manifold truncated at $r = m$) gives an exaustion of X into compact pieces. Consider the equation

$$\begin{aligned}\operatorname{ind}_\Lambda(D_{\epsilon,u}^+) = \operatorname{str}_\Lambda \chi_{\{0\}}(D_{\epsilon,u}) = \lim_{k\to+\infty} \lim_{t\to+\infty} \operatorname{str}_\Lambda(\phi_k e^{-tD_{\epsilon,u}^2} \phi_k) = \\ \lim_{k\to\infty} \operatorname{str}_\Lambda(\phi_k e^{-sD_{\epsilon,u}^2} \phi_k) - \int_s^\infty \operatorname{str}_\Lambda(\phi_k D_{\epsilon,u}^2 e^{-tD_{\epsilon,u}^2} \phi_k)dt.\end{aligned} \tag{6}$$

The t–integral is splitted into $\int_s^{\sqrt{k}} + \int_{\sqrt{k}}^{\infty}$ the second one going to zero thanks to the Breuer–Fredholm property of $D_{\epsilon,u}$. More work is needed in the study of the first one, the responsible of the presence of the eta invariant in the formula. Using heavily the relative version of the Cheeger–Gromov–Taylor estimate (1) one shows that

$$\lim_{k\to\infty} \mathrm{LIM}_{s\to 0} \int_s^{\sqrt{k}} = 1/2\eta_\Lambda(D_{\epsilon,u}^{\mathcal{F}_0}).$$

The first addendum in (6) will lead to the well known local term

$$\lim_{k\to\infty} \mathrm{LIM}_{s\to 0} \operatorname{str}_\Lambda(\phi_k e^{-sD_{\epsilon,u}^2}\phi_k) = \langle \hat{A}(X)\operatorname{Ch}(E/S), [C_\Lambda]\rangle.$$

This requires some work in developing the asymptotic expansion, we have to consider three pieces of X separately again making use of relative kernel estimates.

Comparison with Ramachandran index formula.
In this section.......

The signature formula.
The main application....

2 Geometric Setting

Definition 2.1 — A p–dimensional foliation $\mathcal{F}$ on a manifold with boundary X_0 is transverse to the boundary if it is given by a foliated atlas $\{U_\alpha\}$ with homeomorphisms $\phi_\alpha : U_\alpha \longrightarrow V_\alpha \times W_\alpha$ with V_α open in $\mathbb{H}^p := \{(x_1, ..., x_p) \in \mathbb{R}^p : x_1 \geq 0\}$ and W^q open in $\mathbb{R}^q$ with change of coordinated $\phi_\alpha(u, v)$ in the shape

$$v' = \phi(v, w), \quad w' = \psi(w) \tag{7}$$

(ψ is a local diffeomorphism). Such an atlas is supposed maximal among all collections of this type. The integer p is the dimension of the foliation, q its codimension and $p + q = \dim(X_0)$.

In each foliated chart, connected components of subsets as $\phi_\alpha^{-1}(V_\alpha \times \{w\})$ are called plaques. The plaques coalesce (thanks to the change of coordinate condition (7)) to give maximal connected injectively immersed (not embedded !) submanifolds called leaves. One uses the notation $\mathcal{F}$ for the set of leaves. Note that in general each leaf passes infinitely times trough a foliated chart so a foliation is only locally a fibration. Taking the tangent spaces to the leaves one gets an integrable subbundle $T\mathcal{F} \subset TX_0$ that's transverse to the boundary i.e $T\partial X_0 + T\mathcal{F} = TX_0$ in other words the boundary is a submanifold that's transverse to the foliation.

2.1 Holonomy

We skip the definition of a foliation on a manifold without boundary recall only that is defined by foliated charts as in the definition 2.1 above with local models $U \times V$ where U is an open set in $\mathbb{R}^p$ Let X a manifold equipped with a (p, q)–foliation. If X has boundary the foliation is assumed transverse to the boundary according to definition 2.1.

Definition 2.2 — A function $f : X \longrightarrow \mathbb{R}$ is called distinguished if each point x is in the domain of a foliated chart $U \xrightarrow{\phi} V \times W_\alpha$ such that $f_{|U} = \phi \circ \mathrm{Pr}_V$ where $\mathrm{Pr}_V : U \times V \longrightarrow V$ is the projection on the second factor.

Let $\mathcal{D}$ the collection of all the germs of distinguished maps with the obvious projection $\sigma : \mathcal{D} \longrightarrow X$ sending the germ of f at x onto x. Consider a foliated chart (U, ϕ) and P a plaque of U, then P individuates the set $\tilde{P} \subset \mathcal{D}$ of the distinguished germs $\{[\phi \circ \mathrm{Pr}_V]_x\}_{x \in P}$. When P varies over all the possible foliated charts these sets form the base of a topology of a p–dimensional manifold on $\mathcal{D}$ called the leaf topology. The mapping $\sigma : \mathcal{D} \longrightarrow \mathcal{F}$ is a covering ([36])where $\mathcal{F}$ is the non paracompact manifold of the disjoint union of all the leaves (equivalently use the plaques to give X a topology where the connected components are exactly the leaves with their natural topology). Let $\gamma : x \longrightarrow y$ a continuous leafwise path. Since σ is a covering map there's a holonomy map $h_\gamma : \sigma^{-1}(x) \longrightarrow \sigma^{-1}(y)$ sending the point $\pi \in \sigma^{-1}(x)$ into the endpoint of the unique lifting $\tilde{\gamma}$ of γ starting from π.

Definition 2.3 — A q–dimensional submanifold $Z \subset X$ is a transversal if for every $z \in Z$ there exists a distinguished map $\pi : U \longrightarrow \mathbb{R}^q$ such that $\pi_{|Z \cap U}$ is an homeomorphism.

There are many equivalent definitions of transverse submanifold for example at infinitesimal level, one can ask, $T_z Z \oplus T_z \mathcal{F} = T_z X$. The definition given here makes possible to realize that holonomy acts in a natural way on the disjoint union of all transversals [59].

First we give a slight different version of holonomy. For a continuous leafwise path $\gamma : x \longrightarrow y$ we can choose a path of foliated charts $(U_0, \phi_1), ..., (U_k, \phi_m)$ associated to a decomposition $0 = s_0, ..., 1 = s_m$ of $[0, 1]$ such that $\gamma_{|[s_l, s_{l+1}]} \subset U_l$ and each plaque of U_l meets at only a plaque of U_{l+1}. Following the plaques along γ one obtain a mapping of the plaques of U_0 to the plaques U_m hence, composing with the distinguished maps associated a germ of diffeomorphism of $\mathbb{R}^q$. Since the inclusion of a transversal compose with a distinguished mapping to give coordinates on the transversal this is also a germ of diffeomorphism $H_{T_0 T_1}(\gamma)$ of transversals T_0 around x and T_1 around y.
The connection with the holonomy map given before in terms of the holonomy covering is given as follows. Let $\pi \in \sigma^{-1}(x)$ and f a distinguished map defined around x. The diffeomorphism $H_{T_0 T_1}(\gamma)$ allows to define a local coordinate system on T_1 defined around y and in turn a distinguished map $f_1 : V \longrightarrow \mathbb{R}^q$ defined around y. Then the germ of f_1 at y coincides with $h_\gamma(\pi) \in \sigma^{-1}(y)$.
It is clear that the relation

$$\gamma \sim \tau \quad \text{iff} \quad h_\gamma = h_\gamma(\tau) \tag{8}$$

is weaker than homotopy (obvious by the definition in terms of lifting).

Definition 2.4 — The holonomy groupoid G of the foliation is the quotient of the homotopy groupoid (the set of all equivalence fixed points homotopy classes of leafwise continuous paths) under the relation (8).

One can show that this procedure gives a finite dimensional reduction of the homotopy groupoid. In fact in the case $\partial X = \emptyset$ G is a smooth, in general non–Hausdorff $2p + q$–dimensional manifold where the local coordinates are given by mappings in the form of $(U \times V) \times_{h_\gamma} (U' \times V')$ where $x \in U \times V$, $y \in U' \times V'$, $\gamma : x \longrightarrow y$ is a leafwise path and one uses the graph of the holonomy $h_\gamma : V \longrightarrow V'$ ([77, 24, 53]). Finally

Definition 2.5 — A pseudogroup of a manifold X is a family Γ of diffeomorphisms defined on open subsets of X such that

1. if $\Phi \in \Gamma$ then $\Phi^{-1} \in \Gamma$
2. Γ is closed under composition when possible (depending on domains and ranges).
3. If $\Phi : U \longrightarrow W$ is in Γ then every restriction of Φ to open subsets $V \subset U$ is in Γ.
4. If $\Phi : U \longrightarrow W$ is a diffeomorphism such that every point in U has a neighborhood on which Φ restricts to an alement of Γ then $\Phi \in \Gamma$.
5. The identity belongs to Γ.

The holonomy pseudogroup of a foliation is the pseudogroup Γ acting on the disjoint union of all (regular) whose germs at every point are germs of holonomy mappings defined by some leafwise path.

2.2 Longitudinal Dirac operator

Let $X = X_0 \cup Z$ be a connected manifold with cylindrical end meaning that X_0 is a compact manifold with boundary and $Z = \partial X_0 \times [0, \infty)_r$ is the cylindrical end. Suppose that X has a Riemannian metric g that is product type on the cylinder $g_{|Z} = g_{\partial X_0} + dr \otimes dr$.
Let given on X a smooth oriented foliation $\mathcal{F}$ with leaves of dimension $2p$ respecting the cylindrical structure i.e.

1. The submanifold ∂X_0 is transversal to the foliation and inherits a $(2p-1,q)$ foliation $\mathcal{F}_\partial = \mathcal{F}_{|\partial X_0}$ with foliated atlas given by $\phi_\alpha : U_\alpha \cap \partial X_0 \longrightarrow \partial V_\alpha \times W_\alpha$. Note that the codimension is the same.

2. The restriction of the foliation on the cylinder is product type $\mathcal{F}_{|Z} = \mathcal{F}_\partial \times [0,\infty)$.

The orientation we choose is the one given by $(e_1,..,e_{2p-1},\partial_r)$ where $(e_1,..,e_{2p-1})$ is a positive leafwise frame for the induced boundary foliation. Let $E \longrightarrow X$ be a leafwise Clifford bundle with leafwise Clifford connection ∇^E and Hermitian metric h^E. Suppose each geometric structure is of product type on the cylinder meaning that if $\rho : \partial X_0 \times [0,\infty) \longrightarrow \partial X_0$ is the base projection

$$E_{|Z} \simeq \rho^*(E_{|\partial X_0}), \quad h^E_{|\partial X_0} = \rho^*(h^E_{|\partial X_0}), \quad \nabla^E_{|Z} = \rho^*(\nabla^E_{|\partial X_0}).$$

Each geometric object restricts to the leaves to give a longitudinal Clifford module that's canonically $\mathbb{Z}_2$ graded by the leafwise chirality element. One can check immediately that the positive and negative boundary eigenbundles $E^+_{\partial X_0}$ and $E^-_{\partial X_0}$ are both modules for the Clifford structure of the boundary foliation (see Appendix A.2 for more informations). Leafwise Clifford multiplication by ∂_r induces an isomorphism of leafwise Clifford modules between the positive and negative eigenbundles

$$c(\partial_r) : E^+_{\partial X_0} \longrightarrow E^-_{\partial X_0}.$$

Put $F = E^+_{|\partial X_0}$ the whole Clifford module on the cylinder $E_{|Z}$ can be identified with the pullback $\rho^*(F \oplus F)$ with the following action: tangent vectors to the boundary foliation $v \in T\mathcal{F}_\partial$ acts as $c^E(v) \simeq c^F(v)\Omega$ with $\Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ while in the cylindrical direction $c^F(\partial_r) \simeq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In particular one can form the longitudinal Dirac operator assuming under the above identification the shape[2]

$$D = c(\partial_r)\partial_r + c_{|\mathcal{F}_0}\nabla^{E_{|\mathcal{F}_\partial}} = c(\partial_r)\partial_r + \Omega D^{\mathcal{F}_\partial} = c(-\partial_r)[-\partial_r - c(-\partial_r)\Omega D^{\mathcal{F}_\partial}]. \tag{9}$$

Here $D^{\mathcal{F}_\partial}$ is the leafwise Dirac operator on the boundary foliation. In the following these identifications will be omitted letting D act directly on $F \oplus F$ according to

$$\begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} : F \oplus F \longrightarrow F \oplus F$$

$$\begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\partial_r + D^{\mathcal{F}_\partial} \\ \partial_r + D^{\mathcal{F}_\partial} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \partial_u + D^{\mathcal{F}_\partial} \\ -\partial_u + D^{\mathcal{F}_\partial} & 0 \end{pmatrix}$$

where $u = -r$, $\partial_u = -\partial_r$ (interior unit normal) note this is the opposite of A.P.S. notation. We are using the notation $X = X_k \cup Z_k$ with $Z_k = \partial X_0 \times [k,\infty)$ and $X_k = X_0 \cup (\partial X_0 \times [0,k])$ also $Z^b_a := \partial X_0 \times [a,b]$ and where there's no danger of confusion Z_x is the cylinder of the leaf passing trough x, $Z_x = L_x \cap Z_0$.

3 The Atiyah Patodi Singer index theorem

We are going to recall the classical Atiyah–Patodi–Singer index theorem in [4] So let X_0 a compact $2p$ dimensional manifold with boundary ∂X_0 and a Clifford bundle E with all the geometric structure as in the previous section. We take, here the opposite orientation of A.P.S

[2]we choose to insert $-\partial_r$ the inward pointing normal to help the comparison with the orientation of A.P.S

i.e. we use the exterior unit normal to induce the boundary operator instead of the interior one as pointed out by A.P.S this is a way to declare what is the positive eigenbundle for the natural splitting. In other words

$$D^+_{\text{here}} = D^-_{\text{APS}}.$$

The operator writes in a collar around the boundary

$$\begin{pmatrix} 0 & D^- \\ D_+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\partial_r + D_0 \\ \partial_r + D_0 & 0 \end{pmatrix}$$

where ∂_r is the exterior unit normal and D_0 is a Dirac operator on the boundary. It is shown in [3] that the K–theory of the boundary manifold contains topological obstructions to the existence of elliptic boundary value conditions of local type (for the signature operator they are always non zero). If one enlarges the point of view to admit global boundary conditions a Fredholm problem can properly set up. More precisely consider the boundary operator D_0 acting on the boundary manifold ∂X_0. This is a first order elliptic differential operator with discrete spectrum on $L^2(\partial X_0; F)$. Let $P = \chi_{[0,\infty)}(D_0)$ be the spectral projection on the non negative part of the spectrum. This is a pseudo–differential operator ([4]). Atiyah Patodi and Singer prove the following facts

- The (unbounded) operator $D^+ : C^\infty(X; E^+, P) \longrightarrow C^\infty(X, E^-)$ with domain

  $$C^\infty(X; E^+, P) := \{s \in C^\infty(X; E^+) : P(s_{|\partial X_0}) = 0\}$$

  is Fredholm and the index is given by the formula

  $$\text{ind}_{\text{APS}}(D^+) = \int_{X_0} \hat{A}(X)\,\text{Ch}(E) - h/2 + \eta(0)/2$$

  with the standard Atiyah–Singer $\hat{A}$ integrand (exactly as in the closed case) and two correcting terms:

  1. $h := \text{Ker}(D_0)$ is the dimension of the kernel of the boundary operator
  2. $\eta(0)$, the eta invariant of D_0 gives a measure of the asymmetry of the spectrum of the boundary operator D_0. This is extensively explained in section 7.

- The index formula can be interpreted as a natural L^2 problem on the manifold with a cylinder attached, $X = X_0 \cup_{\partial X_0} (\partial X_0 \times [0, \infty))$ with every structure pulled back. More precisely the kernel of $D^+ : C^\infty(X; E^+, P) \longrightarrow C^\infty(X, E^-)$ is naturally isomorphic to the kernel of D^+ extended to an ubounded operator on $L^2(X)$ while to describe the kernel of its Hilbert space adjoint i.e. the closure of D^- with the adjoint boundary condition $D^- : C^\infty(X; E^-, 1 - P) \longrightarrow C^\infty(X, E^+)$ we have to introduce the space of extended L^2 solutions.

  A locally square integrable solution s of the equation $D^- s = 0$ on X is called an extended solution if for large positive r

  $$s(y, r) = g(y, r) + s_\infty(y) \tag{10}$$

  where y is the coordinate on the base ∂X_0 and $g \in L^2$ while s_∞ solves $D_0 s_\infty = 0$ and is called the limiting value of s.

  APS prove that the kernel of $(D^+)^*$ (Hilbert space adjoint of D^+ with domain given by the APS boundary condition) is naturally isomorphic to the space of L^2 extended solution of D^- on X. Moreover

  $$\text{ind}_{\text{APS}}(D^+) = \dim_{L^2}(D^+) - \dim_{L^2}(D^-) - h_\infty(D^-) = \text{ind}_{L^2}(D^+) - h_\infty(D^-) \tag{11}$$

where $\mathrm{ind}_{L^2}(D^+) := \dim_{L^2}(D^+) - \dim_{L^2}(D^-)$ and the number $h_\infty(D^-)$ is the dimension of the space of limiting values of the extended solutions of D^-. In this sense the APS index can be interpreted as an L^2–index. The number at right in (11) is called often the L^2 extended index. Along the proof of (11) the authors prove that

$$h = h_\infty(D^+) + h_\infty(D^-) \tag{12}$$

and conjecture that it must be true at level of the kernel of D_0 i.e.

every section in $\mathrm{Ker}(D_0)$ *is uniquely expressible as a sum of limiting values coming from* D^+ *and* D^-.

The conjecture was solved by Melrose with the invention of the b–calculus, a pseudo–differential calculus on a compactification of X that furnished a totally new point of view on the APS problem [50].

With (11) and (12) the index formula is

$$\mathrm{ind}_{L^2}(D^+) = \int_{X_0} \hat{A}(X)\,\mathrm{Ch}(E) + \frac{\eta(0)}{2} + \frac{h_\infty(D^-) - h_\infty(D^+)}{2}.$$

Finally a naive remark about the introduction of extended solutions in order to motivate our definition of $h_\infty(D^\pm)$ (equation (34) and (75)) in our Von Neumann setting. For a real parameter u say that a distributional section s on the cylinder is in the weighted L^2–space $e^{ur}L^2(\partial X_0 \times [0,\infty); E^\pm)$ if $e^{-ur}s \in L^2$. The operator $D^\pm$ trivially esxtends to act on each weighted space. Now it is evident from (10) that an L^2–extended solution of the equation $D^+s = 0$ is in each $e^{ur}L^2$ for positive u. Viceversa let $s \in \bigcap_{u>0} \mathrm{Ker}_{e^{ur}L^2}(D^+)$. Keep u fixed, then $e^{-ur}s \in L^2$ can be represented in terms of a complete eigenfunction expansion for the boundary operator D_0,

$$e^{-ur}s = \sum_\lambda \phi_\lambda(y) g(r).$$

Solving $D^+s = 0$ together with the condition $e^{-ur}s \in L^2$ leads to the representation (on the cylinder) $s(y,r) = \sum_{\lambda > -u} \phi_\lambda(y) g_{0\lambda}(y) e^{-\lambda r}$. Since u is arbitrary we see that s should have a representation as a sum

$$s(y,r) = \sum_{\lambda \geq 0} \phi_\lambda(y) g_{0\lambda} e^{-\lambda r}$$

over the non negative eigenvalues of D_0, i.e. s is an extended solution with limiting value $\sum_{\lambda=0} \phi_0(y) g_{00}$. We have proved that

$$\mathrm{Ext}(D^\pm) = \bigcap_{u>0} \mathrm{Ker}_{e^u rL^2}(D^\pm).$$

4 Von Neumann algebras, foliations and index theory

4.1 Non–commutative integration theory.

The measure–theoretical framework of non–commutative integration theory is particular fruitful when applied to measured foliations. The non–commutative integration theory of Alain Connes [25] provides us a measure theory on every measurable groupoid $(G, \mathcal{B})$ with $G^{(0)}$ the space of unities. In our applications G will be the mostly the equivalence relation $\mathcal{R}$ or sometimes the holonomy groupoid of a foliation. Transverse measures will be defined from holonomy invariant transverse measures. Below a list of fundamental objects and facts. This very contained and simplified survey in fact the general theory admits the existence of a modular function that says, in the case of foliations how transverse measure of sets changes under holonomy(under flows generated by fields tangent to the foliation). Hereafter our modular function is everywhere 1, corresponding to the geometrical case of a foliation equipped with a holonomy invariant transverse measure (this is a definition we give below).

Measurable groupoids . A groupoid is a small cathegory G where every arrow is invertible. The set of objects is denoted by $G^{(0)}$ and there are two maps $s, r : G \longrightarrow G^{(0)}$ where $\gamma : s(\gamma) \longrightarrow r(\gamma)$. Two arrows γ_1, γ_2 can be composed if $r(\gamma_2) = s(\gamma_1)$ and the result is $\gamma_1 \cdot \gamma_2$. The set of composable arrows is $G^{(2)} = \{(\gamma_1, \gamma_2) : r(\gamma_2) = s(\gamma_1)\}$. As a notation $G_x = r^{-1}(x)$, $G^x = s^{-1}(x)$ for $x \in G^{(0)}$. An equivalence relation $\mathcal{R} \subset X \times X$ is a groupoid with $r(x, y) = x$ and $s(x, y) = y$, in this manner $(z, x) \cdot (x, y) = (z, y)$. The range of the map $(r, s) : G \longrightarrow G^{(0)} \times G^{(0)}$ is an equivalence relation called the principal groupoid associated to G. In this sense groupoids desingularize equivalence relations. A measurable groupoid is a pair $(G, \mathcal{B})$ where G is a groupoid and $\mathcal{B}$ is a σ–field on G making measurable the structure maps r, s, composition $\circ : G^{(2)} \longrightarrow G$ and the inversion $\gamma \longmapsto \gamma^{-1}$.

Kernels are mappings $x \longmapsto \lambda^x$ where λ^x is a positive measure on G, supported on the r–fiber $G^x = r^{-1}(x)$ with a measurability property i.e. for every set $A \in \mathcal{B}$ the function $y \longmapsto \lambda^y(A) \in [0, +\infty]$ must be measurable.

A kernel λ is called proper if there exists an increasing family of measurable sets $(A_n)_{n\in\mathbb{N}}$ with $G = \cup_n A_n$ making the functions $\gamma \longmapsto \lambda^{s(\gamma)}(\gamma^{-1}(A))$ bounded for every $n \in \mathbb{N}$. The point here is that every element $\gamma : x \longrightarrow y$ in G defines by left traslation a measure space isomorphism $G^x \longrightarrow G^y$ and calling $R(\lambda)_\gamma := \gamma\lambda^x$ (push–forward measure) one has a kernel in the usual sense i.e. a mapping with value measures. The definition of properness is in fact properness for $R(\lambda)$.

The space of proper kernels is denoted by $\mathcal{C}^+$.

Transverse functions are kernels $(\nu^x)_{x\in X}$ with the left invariance property $\gamma\nu^{s(\gamma)} = \nu^{r(\gamma)}$ for every $\gamma \in G$. One checks at once that properness is equivalent to the existence of an increasing family of measurable sets $(A_n)_n$ with $G = \cup_n A_n$ such that the functions $x \longmapsto \nu^x(A_n)$ are bounded for every $n \in \mathbb{N}$. The space of proper transverse functions is denoted $\mathcal{E}^+$.

The support of a transverse function ν is the measurable set $\mathrm{supp}(\nu) = \{x \in G^{(0)} : \nu^x \neq 0\}$. This is *saturated* w.r.t. the equivalence relation induced by G on $G^{(0)}$, $x\mathcal{R}y$ iff there exists $\gamma : x \longrightarrow y$. If $\mathrm{supp}(\nu) = G^{(0)}$ we say that ν is faithful.

When $G = \mathcal{R}$ or the holonomy groupoid these gives families of positive measures one for each leaf in fact in the first case the invariance property is trivial, in the second case we are giving a measure ν^x on each holonomy cover G^x with base point x but the invariance property says that these are invariant under the deck trasformations together with the change of base points then push forward on the leaf under $r : G^x \longrightarrow L_x$.

Convolution. The groupoid structure provides an operation on kernels. For fixed kernels λ_1 and, λ_2 on G their convolution product is the kernel $\lambda_1 * \lambda_2$ defined by

$$(\lambda_1 * \lambda_2)^y = \int (\gamma\lambda_2^x) d\lambda_1^y(\gamma), \quad y \in X.$$

It is a fact that if λ is a kernel and ν is a transverse function then $\nu * \lambda$ is a transverse function. Clearly $R(\lambda_1 * \lambda_2) = R(\lambda_1) \circ R(\lambda)$ the standard composition of kernels on a measure space.

Transverse invariant measures (actually are transverse measures of modulo $\delta = 1$). These are linear mappings $\Lambda : \mathcal{E}^+ \longrightarrow [0, +\infty]$ such that

1. Λ is normal i.e $\Lambda(\sup \nu_n) = \sup \Lambda(\nu_n)$ for every increasing sequence ν_n in $\mathcal{E}^+$ bounded by a transverse function. Since the sequence is bounded by an element of $\mathcal{E}^+$ the expression $\sup \nu_n$ makes sense in $\mathcal{E}^+$.
2. Λ is invariant under the right traslation of G on $\mathcal{E}^+$. This means that
$$\Lambda(\nu) = \Lambda(\nu * \lambda)$$
for every $\nu \in \mathcal{E}^+$ and kernel λ such that $\lambda^y(1) = 1$ for every $y \in G^{(0)}$.

A transverse measure is called semi–finite if it is determined by its finite values i.e $\Lambda(\nu) = \sup\{\Lambda(\nu'),\ \nu' \leq \nu,\ \Lambda(\nu') < \infty\}$. We shall consider only semi–finite measures.

A transverse measure is σ–finite if there exists a faithful transverse function ν of kind $\nu = \sup \nu_n$ with $\lambda(\nu_n) < \infty$.

The coupling of a transverse function $\nu \in \mathcal{E}^+$ and a transverse measure Λ produces a positive measure Λ_ν on $G^{(0)}$ through the equation $\Lambda_\nu(f) := \Lambda((f \circ s)\nu$ the invariance property reflects downstairs in the property $\Lambda_\nu(\lambda) = \Lambda(\nu * \lambda)$ for $\nu \in \mathcal{E}^+$ and $\lambda \in \mathcal{C}^+$.

Measures on the base $G^{(0)}$ that can be represented as Λ_ν are characterized by a theorem of disintegration of measures.

THEOREM 4.5 — (Connes [24]) Let ν a transverse proper function with support A. The mapping $\Lambda \longmapsto \Lambda_\nu$ is a bijection between the set of transverse measures on $G_A^A = r^{-1}(A) \cup s^{-1}(A)$ and the set of positive measures μ on $G^{(0)}$ satisfying the following equivalent relations

1. $(\mu \circ \nu)\tilde{} = \mu \circ \nu$
2. $\lambda, \lambda' \in \mathcal{C}^+, \nu * \lambda = \nu * \lambda' \in \epsilon^+ \Longrightarrow \mu(\lambda(1)) = \mu(\lambda'(1))$.

Nex we shall explain this procedure of disintegration in a geometrical way for foliations.

We shall see that what is important here is the class of null–measure subsets of $G^{(0)}$. A saturated set $A \subset G^{(0)}$ is called Λ–trascurable if $\Lambda_\nu(A) = 0$ for every $\nu \in \mathcal{E}^+$.

Representations. Let H be a measurable field of Hilbert spaces by definition this is a mapping $x \longmapsto H_x$ from $G^{(0)}$ with values Hilbert spaces. The measurability structure is assigned by a linear subspace of the free product vector space of the whole family $\mathcal{M} \subset \Pi_{x \in G^{(0)}} H_x$ meaning that

1. For every $\xi \in \mathcal{M}$ the function $x \longmapsto \|\xi(x)\|$ is measurable.
2. A section $\eta \in \Pi_{x \in G^{(0)}} H_x$ belongs to $\mathcal{M}$ if and only if the function $\langle \eta(x), \xi(x) \rangle$ is measurable for every $\xi \in \mathcal{M}$.
3. There exists a sequence $\{\xi_i\}_{i \in \mathbb{N}} \subset \mathcal{M}$ such that $\{\xi_i(x)\}_{i \in \mathbb{N}} \subset \mathcal{M}$ is dense in H_x for every x.

Elements of $\mathcal{M}$ are called measurable sections of H.

Suppose a measure μ on $G^{(0)}$ has been chosen. One can keep together the Hilbert spaces H_x taking their direct integral
$$\int H_x d\mu(x).$$

This is defined as follows, first select the set of square integrable sections in $\mathcal{M}$. This is the set of sections s such that the integral $\int_{G^{(0)}} \|s(x)\|^2_{H_x} d\mu(x) < \infty$ then identitify two square integrable sections if they are equal outside a μ–null set. The direct integral comes equipped with a natural hilbert space structure with product

$$\langle s, t\rangle := \int_{G^{(0)}} \langle s(x), t(x)\rangle_{H_x} d\mu(x).$$

The notation $s = \int_{G^{(0)}} s(x)d\mu(x)$ for an element of the direct integral is clear. A field of bounded operators $x \longmapsto B_x \in B(H_x)$ is called measurable if sends measurable sections to measurable sections. A mesurable family of operators with operator norms uniformely bounded esssup $\|B_x\| < \infty$ defines a bounded operator called decomposable $B := \int_{G^{(0)}} B_x d\mu(x)$ on the direct integral in the simplest way

$$Bs := \int_{G^{(0)}} B_x d\mu(x)\, s = \int_{G^{(0)}} B_x s(x) d\mu(x).$$

For example each element of the abelian Von Neumann algebra $L^\infty_\mu(G^{(0)})$ defines a decomposable operator acting by pointwise multiplication. One gets an involutive algebraic isomorphism of $L^\infty_\mu(G^{(0)})$ onto its image in $B(\int H_x d\mu(x))$ called the algebra of diagonal operators. One can ask when a bounded operator $T \in B(\int H_x d\mu(x))$ is decomposable i.e. when $T = \int T_x d\mu(x)$ for a family of uniformely bounded operators $(T_x)_x$. The answer is precisely when it belongs to the commutant of the diagonal algebra.

A representation of G on H is the datum of an Hilbert space isomorphism $U(\gamma) : H_{s(\gamma)} \longrightarrow H_{r(\gamma)}$ for every $\gamma \in G$ with

1. $U(\gamma_1^{-1}\gamma_2) = U(\gamma_1)^{-1}U(\gamma_2), \quad \forall \gamma_1, \gamma_2 \in G, \quad r(\gamma_1) = r(\gamma_2).$
2. For every couple ξ, η of measurable section the function defined on G according to $\gamma \longmapsto \langle \eta_{r(\gamma)}, U(\gamma)\eta_{s(\gamma)}\rangle$, is measurable.

A fundamental example is given by the left regular representation of G defined by a proper transverse function $\nu \in \mathcal{E}^+$ in the following way. The measurable field of Hilbert space is $L^2(G, \nu)$ defined by $x \longmapsto L^2(G^x, \nu^x)$ with the unique measurable structure making measurable the family of sections of kind $y \longmapsto f_{|G^x}$ obtained from every measurable f on G such that each $\int |f|^2 d\nu^x$ is finite. For every $\gamma : x \longrightarrow y$ in G one has the left traslation $L(\gamma) : L^2(G^x, \nu^x) \longrightarrow L^2(G^y, \nu^y)$, $\quad (L(\gamma)f)(\gamma') = f(\gamma^{-1}\gamma')$, $\gamma' \in G^y$.

Intertwining operators are morphisms between representations. If (H, U), (H', U') are representations of G an intertwining operator is a measurable family of operators $(T_x)_{x\in G^{(0)}}$ of bounded operators $T_x : H_x \longrightarrow H'_x$ such that

1. Uniform boundedness, $\sup \|T_x\| < \infty$.
2. For every $\gamma \in G$ there follows $U'(\gamma)T_{s(\gamma)} = T_{r(\gamma)}U(\gamma)$.

Looking at a representation as a measurable functor, an intertwining operator gives a natural transformation between representations. The vector space of intertwing operators from H to H' is denoted by $\mathrm{Hom}_G(H, H')$.

Square integrable representations. Fix some transverse function $\nu \in \mathcal{E}^+$. For a representation of G the property of being equivalent to some sub–representation of the infinite sum of the regular left representation L^ν is independent from ν and is the definition of square integrability for representations. Actually, due to measurability issues much care is needed here to define sub representations (see section 4 in [24]) but the next fundamental remark assures that square integrable representations are very commons in applications.

Measurable functors and representations. Let $\tilde{\mathcal{R}}_+$ be the cathegory of (standard) measure spaces without atoms i.e. objects are triples $(\mathcal{Z}, \mathcal{A}, \alpha)$ where $(\mathcal{Z}, \mathcal{A})$ is a standard measure space and α is a σ–finite positive measure.

Measurability of a functor $F : G \longrightarrow \tilde{\mathcal{R}}_+$ is a measure structure on the disjoint union $Y = \bigcup_{x\in G^{(0)}} F(x)$ making the following structural mappings measurable

1. The projection $\pi : Y \longrightarrow G^{(0)}$.
2. The natural bijection $\pi^{-1}(x) \longrightarrow F(x)$.
3. The map $x \longmapsto \alpha^x$, a σ–finite measure on $F(x)$.
4. The map sending $(\gamma, z) \in G \times X$ with $s(\gamma) = \pi(z)$ into $F(\gamma)z \in Y$.

Usually one assumes that Y is union of a denumerable collection $(Y_n)_n$ making every function $\alpha^x(Y_n)$ bounded. With a measurable functor F one has an associated representation of G denoted by $L^2 \bullet F$ defined in the following way: the field of Hilbert space is $x \longmapsto L^2(F(x), \alpha^x)$ and if $\gamma : x \longrightarrow y$ define $U(\gamma) : L^2(F(x), \alpha^x) \longrightarrow L^2(F(y), \alpha^y)$ by $f \longmapsto F(\gamma^{-1}) \circ f$. Proposition 20 in [24] shows that this is a square–integrable representation.

Random hilbert spaces and Von Neumann algebras. We have seen that every fixed transverse measure Λ defines a notion of Λ–null measure sets (for saturated sets) hence an equivalence relation on $\mathrm{End}_G(H_1, H_2)$ the vector space of all intertwining operators $T, S : H_1 \longrightarrow H_2$ between two square integrable representations H_i. Each equivalence class is called a random operator and the set of random operators is denoted by $\mathrm{End}_\Lambda(H_1, H_2)$. Also square integrable representations can be identified modulo Λ–null sets. An equivalence class of square integrable representations is by definition a random hilbert space.

Theorem 2 in [24] says that $\mathrm{End}_\Lambda(H)$ is a Von Neumann algebra for every random Hilbert space.

More precisely choose some $\nu \in \mathcal{E}^+$ and put $\mu = \Lambda_\nu$ and $m := \mu \bullet \nu$ to form the Hilbert space $\mathcal{H} = L^2(G, m)$. For a function f on G denote $Jf = f^\sharp(\gamma) = \overline{f(\gamma^{-1})}$, consider the space $\mathcal{A}$ of measurable functions f on G such that $f, f^\sharp \in L^2(G, m)$ and $\sup(\nu|f^\sharp|) < \infty$. Equip $\mathcal{A}$ with the product $f *_\nu g = f\nu * g$. The structure $\mathcal{A}$ has is that of an Hilbert algebra (a left–Hilbert algebra in the modular case) i.e $\mathcal{A}$ is a $*$–algebra with positive definite (separable) pre–Hilbert structure such that

1. $\langle x, y\rangle = \langle y^*, x^*\rangle, \quad \forall x, y \in \mathcal{A}$.
2. The representation of $\mathcal{A}$ on $\mathcal{A}$ by left multiplication is bounded, involutive and faithful.

With such structure one can speak about the left regular representation λ of $\mathcal{A}$ on the Hilbert space completion $\mathcal{H}$ of $\mathcal{A}$ itself. The double commutant $\lambda''(\mathcal{A})$ of this representation is the Von Neumann algebra $W(\mathcal{A})$ associated to the Hilbert algebra $\mathcal{A}$. It is a remarkable fact that $W(\mathcal{A})$ comes equipped with a semifinite faithful normal trace τ such that

$$\tau(\lambda(y^*)\tau(x)) = \langle x, y\rangle \quad \forall x, y \in \mathcal{A}.$$

Furthermore one knows that the commutant of $\lambda(\mathcal{A})$ in $\mathcal{H}$ is generated by the algebra of right multiplications $\lambda'(\mathcal{A}) = J\lambda(\mathcal{A})J$ for the conjugate–linear isometry $J : \mathcal{H} \longrightarrow \mathcal{H}$ defined by the involution in $\mathcal{A}$. For every Λ–random Hilbert space H one can use the measure Λ_ν on $G^{(0)}$ to form the direct integral $\nu(H) = \int H_x d\Lambda_\nu(x)$. Remember that the direct integral is the set of equivalence classes modulo Λ_ν zero measure of square integrable measurable sections. Now, directly from the definition, an intertwining operator $T \in \mathrm{Hom}_\Lambda(H_1, H_2)$ is a decomposable operator defining a bounded operator $\nu(T) : \nu(H_1) \longrightarrow \nu(H_2)$.

Put $W(\nu)$ for the Von Neumann algebra associated to the Hilbert algebra $L^2(G, m)$, $m = \Lambda_\nu \bullet \nu$, $\nu \in \mathcal{E}^+$.

Theorem 4.5 — (Connes) Fix some transverse function $\nu \in \mathcal{E}^+$

1. For every Λ–random Hilbert space H there exists a unique normal representation of $W(\nu)$ in $\nu(H)$ such that $U_\nu(f) = U(f\nu)$ $f \in \mathcal{A}_\nu$. Here $U(f\nu)$ is defined by $(U(f\nu)\xi)_y = \int U(\gamma)\xi_x d(f\nu^y)(\gamma)$.
2. The correspondence $H \longmapsto \nu(H)$, $T \longmapsto \nu(T)$ is a functor from the (W^*)–cathegory $\mathcal{C}_\Lambda$ of random Hilbert spaces and intertwining operators to the cathegory of $W(\nu)$ modules.
3. If the transverse measure ν is faithful the functor above is an equivalence of cathegories.

Then in the case of faithful transverse measures one gets an isometry of $\mathrm{End}_\Lambda(H)$ on the commutant of $W(\nu)$ on the direct integral $\nu(H)$. In particular $\mathrm{End}_\Lambda(H)$ is a Von Neumann algebra.

Transverse integrals. The most important notion of non commutative integration theory is the integral of a random variable against a transverse measure. A positive random variable on $(G, \mathcal{B}, \Lambda)$ is nothing but a measurable functor F as defined above. Let $X := \bigcup_{x \in G^{(0)}} F(x)$ disjoint union measure space and $\bar{\mathcal{F}}^+$ the space of measurable functions with values in $[0, +\infty]$ while $\mathcal{F}^+$ is for functions with values on $(0, +\infty]$. Kernels λ on G acts as convolution kernels on $\bar{\mathcal{F}}^+$ according to $(\lambda * f)(z) = \int f(\gamma^{-1}z)d\lambda^y(\gamma)$, $y = \pi(z) \in G^{(0)}$. This is an associative operation $(\lambda_1 * \lambda_2) * f = \lambda_1 * (\lambda_2 * f)$.

Now to define the integral $\int F d\lambda$ choose some ν faithful and put

$$\int F d\lambda = \sup\{\Lambda_\nu(\alpha(f)),\ f \in \mathcal{F}^+,\ \nu * f \leq 1\},$$

this is independent from ν and enjoys the following properties

1. there exist random variables F_1, F_2 with $F = F_1 \oplus F_2$ such that $\int F_1 d\Lambda = 0$ and a function $f_2 \in \mathcal{F}^+(X_2)$ with $X_2 = \bigcup_{x \in G^{(0)}} F_2(x)$ with $\nu * f_2 = 1$.
2. Monotony. If $f, f' \in \mathcal{F}(X)$ satisfy $\nu * f \leq \nu * f' \leq 1$ then

   $$\Lambda_\nu((\alpha(f)) \leq \Lambda_\nu((\alpha(f'))$$

   in particular for F_2 as in 1.

   $$\int F_2 d\Lambda = \Lambda_\nu((\alpha(f')).$$

Traces. Let A be a Von Neumann algebra with the cone of positive elements A^+.

A weight on a A is a functional $\phi : A^+ \longrightarrow [0, \infty]$ such that

1. $\phi(a + b) = \phi(a) + \phi(b)$, $a, b \in A^+$
2. $\phi(\alpha a) = \alpha\phi(a)$, $\alpha \in \mathbb{R}^+$, $a \in A^+$.

a weight is a trace if $\phi(a^*a) = \phi(aa^*)$, $a \in A^+$. A weight is called

- faithful if $\phi(a) = 0 \Rightarrow a = 0$, $a \in A^+$.
- normal if for every increasing net a_{ii} of positive elements with least upper bound a then
$$\phi(a) = \sup\{\phi(a_i)\}.$$
- Semifinite if the linear span of a the set of ϕ–finite elements, $\{a \in A^+ : \phi(a) < \infty\}$ is σ–weak dense.

Every V.N algebra has a semifinite normal faithful weight.

The Von Neumann algebra $\mathrm{End}_\Lambda(H)$ associated to a square integrable representation comes equipped with a bijection $T \longmapsto \Phi_T$ between positive operators and semifinite normal weights $\Phi_T : \mathrm{End}_\Lambda(H) \longrightarrow [0, +\infty]$ where Φ_T is faithful if and only if T_x is not singular Λ–a.e. The construction of this correspondence uses the fact, for a faithful transverse function ν the direct integral $\nu(H) = \int H_x d\Lambda_\nu(x)$ is a module over the Von Neumann algebra $W(\nu)$ associated to the Hilbert algebra $\mathcal{A}$ above described.

The notation of Connes is

$$\Phi_T(1) := \int \mathrm{Trace}(T_x) d\Lambda(x)$$

i.e. the mapping $T \longmapsto \Phi_T(1)$ is the canonical trace on $\mathrm{End}_\Lambda(H)$. In fact this is related to the type I Von Neumann algebra P of classes modulo equality Λ_ν almost everywhere of measurable fields $(B_x)_{x \in G^{(0)}}$, $B_x \in B(H_x)$ of bounded operators. Remember that P has a canonical trace $\rho(B) = \int \mathrm{Trace}(B_x) d\Lambda_\nu(x)$ hence we can define

$$\rho_T(B) := \int \mathrm{Trace}(T_x B_x) d\Lambda_\nu(x).$$

The next lemma will be important in our applications

Lemma 4.6 — For a faithful transverse function ν there's a unique operator valued weight[3] E_ν from P to $\mathrm{End}_\Lambda(H)$ such that the diagram

$$\begin{tikzcd} P^+ \arrow[d, "E_\nu"'] \arrow[dr, "\rho_T(\cdot) = \int \mathrm{Trace}(T_x \cdot) d\Lambda_\nu(x)"] & \\ \mathrm{End}_\Lambda(H) \arrow[r, "\Phi_T"'] & \mathbb{C} \end{tikzcd}$$

is commutative. Moreover E_ν is such that if $B = (B_x)_{x \in G^{(0)}}$, $B \in P^+$ if an operator making bounded the corresponding family

$$C_y := \int U(\gamma) B_x U(\gamma)^{-1} d\nu^y$$

then $E_\nu(B) = C$.

Let F be a random variable and put $H = L^2 \bullet F$. The integration process above defines a semi–finite faithful trace on the Von Neumann algebra $\mathrm{End}_\Lambda(H)$. In fact, for $T \in \mathrm{End}_\Lambda^+(H)$ let F_T the new random variable defined by $x \mapsto (F(x), \alpha_T(x))$ where

[3] see [74] for the definition

$\alpha_T(x)$ is the measure on $F(x)$ such that $\alpha_T(x)(f) = \mathrm{Trace}_{\mathrm{L}^2}(T_x^{1/2}M(f)T_x^{1/2})$ where f is a bounded measurable function on $F(x)$ and $M(f)$ the corresponding multiplication operator on $L^2(F(x))$. The trace is

$$\Phi_T(1) = \int F_T d\Lambda.$$

In the following we shall use often the notation $\mathrm{tr}_\Lambda(T) = \Phi_T(1)$ to emphasize the dependence on Λ.

With a trace one can develop a dimesion theory for square integrable representation i.e. a dimension theory for random Hilbert spaces that's very similar to the dimension theory of Γ–Hilbert modules.

The formal dimension of the random Hilbert space H is

$$\dim_\Lambda(H) = \int \mathrm{Trace}(1_{H_x}) d\Lambda(x)$$

here some fundamental properties

Lemma 4.7 —

1. If $\mathrm{Hom}_\Lambda(H_1, H_2)$ contains an invertible element then $\dim_\Lambda(H_1) = \dim_\Lambda(H_2)$.
2. $\dim_\Lambda(\oplus H_i) = \sum \dim_\Lambda(H_i)$.
3. $\dim_{\sum \Lambda_i}(\oplus H) = \sum \dim_{\Lambda_i}(H)$.

4.2 Holonomy invariant transverse measures

The main example of a non–commutative measure space is the space of leaves of a foliation. It is, in general impossible to look at the space of leaves as the quotient measure space. A famous example is the Cronecker foliation on the thorus $\mathbb{T}^2$ given by irrational flows ([25]). The foliation is ergodic i.e. a function almost everywhere constant along the leaves must be constant on the ambient. In particular every Lebesgue space of classical analysis is one dimensional. A central concept is that of holonomy invariant transverse measure introduced by Plante [59] and Ruelle and Sullivan [68]. According to Connes [24] a transverse measure provides a measure on the space of leaves. Actually there's a most general modular theory. Holonomy invariant measures correspond to the simplest case.

4.2.1 Measures and currents

Let X be a manifold equipped with a foliation of dimension p and codimension q. We suppose always that the foliation is *oriented* i.e. the bundle of p–dimensional leafwise forms $\wedge^p_{\mathbb{C}} T\mathcal{F}$ is trivial. This is not truly a restrictive assumption, in fact in the non–orientable case one can make use of densities instead of forms to define currents. Currents are directly related to holonomy invariant transverse measures by the Ruelle–Sullivan isomorphism. The goal of this section is to introduce all these notions and prove the relations between them.
There is a weak version of the concept of a transversal

DEFINITION 4.8 — A Borel subset $T \subset X$ is called a Borel transversal if the intersection of T with each leaf is (finite) denumerable.

The set of all Borel transversals $\mathcal{T}$ is a σ–ring i.e it is closed under the operation of relative complementation and denumerable union. Recall that a σ–ring is a σ–algebra if contains the entire space. This is in general clearly not the case of the set of all Borel transversals hence holonomy measures will be defined only on σ–rings.

DEFINITION 4.9 — A holonomy invariant transverse measure is a σ–additive map $\mu : \mathcal{T} \longrightarrow [0, +\infty]$ such that

1. For a Borel bijection $\psi : B_1 \longrightarrow B_2$ with $\psi(x) \sim x$ (the relation of being on the same leaf) then $\mu(B_1) = \mu(B_2)$.
2. μ is Radon i.e. for every compact $K \subset B$ then $\mu(K) < \infty$.

DEFINITION 4.10 — A holonomy invariant transverse distribution is the datum for every transverse submanifold T of a linear and continuous[4] map $\delta_T : C_c^\infty(T) \longrightarrow \mathbb{C}$ such that if $\psi : T_1 \longrightarrow T_2$ is the holonomy of a path γ on X,

$$\langle \delta_{T_1}, f \rangle = \langle \delta_{T_2}, f \circ \psi \rangle.$$

Now let $\mathrm{Hom}_{\mathrm{cont}}(C_c^\infty(\wedge^d T_{\mathbb{C}}^* X), \mathbb{C})$ the space of d–dimensional currents on X. This is the dual space of the t.v.s. given by the compactly supported d–forms equipped with the topology of the direct limit of Frechet spaces. The operations of Lie derivative L_V and contraction i_V w.r.t. a vector field V and the De Rham exterior derivative d extends to distribution just by duality [25].
Note that a d–differential form ω can be restricted to a subbundle S of the tangent bundle just by evaluation of ω to the d–vectors belonging to $\wedge^d S_{\mathbb{C}}^* \subset \wedge^p T^* X_{\mathbb{C}}$.

DEFINITION 4.11 — A d–dimensional current (d is the dimension of the leaves) C is said a foliated current if it is invariant under the operation of restriction i.e $\langle C, \omega \rangle = 0$ for every p–form ω such that $\omega_{|T\mathcal{F}} = 0$.

Notice that for a d–dimensional foliated current C the condition of being closed is equivalent to require $\partial_X C = 0$ for every section $X \in C^\infty(X; T\mathcal{F})$.

PROPOSITION 4.12 — For a manifold X equipped with a d–dimensional foliation is equivalent to give

1. A holonomy invariant transverse distribution.
2. A closed foliated d–current.

PROOF— We define first holonomy invariant transverse distributions relative to regular

[4] w.r.t. the usual topology of the direct limit i.e. a distribution in the usual sense

atlas and show they define closed foliated d–currents. Since the definition of current does not depend on the atlas and every h.i.t. distribution restricts to a h.i.t. distribution relative to each regular atlas the proof will be complete. For a foliated chart $\Omega \longrightarrow V \subset \mathbb{R}^{n-d} \times \mathbb{R}^d$ the local transversal associated is the quotient space defined by the relation $x \sim y$ if x, y belongs to the same plaque of Ω. In particular a local transversal is the space of plaques in Ω. We say that the inclusion $\Omega \hookrightarrow \Omega'$ of distinct open sets is regular and write $\Omega \lhd \Omega'$ if the inclusion mapping $i : \Omega \hookrightarrow \Omega'$ passes to the quotient to define a smooth mapping on the transversals. In particular each plaque of Ω meets only a plaque of Ω'.
We say that a foliated atlas $\{(\Omega_i, \phi_i)\}_i$ of $(X, \mathcal{F})$ of foliated charts Ω_i is a good cover if

1. $\{\Omega_i\}_i$ is locally finite
2. for every i, j such that $\overline{\Omega}_i \cap \overline{\Omega}_j \neq \emptyset$ there exist a distinct open set Ω such that $\Omega_i \lhd \Omega$ and $\Omega_j \lhd \Omega$.

Standard methods show that a regular atlas always exists.
Now define a transverse distribution related to a regular cover to be a distribution on every local transversal T_Ω of each finite intersection $\Omega = \Omega_1 \cap ... \cap \Omega_k$ with the property of (relative) holonomy invariance i.e the distribution associated to $T_{\Omega \cap \Omega'}$ is equal to the restriction of the distribution associated to T_Ω and the distribution associated to $T_{\Omega'}$.
So let C be a closed foliated current and $\{\Omega_i\}_i$ a regular atlas for $\mathcal{F}$. For every i choose a differential d–form ω_i compactly supported in some neighborhood of $\Omega_i \simeq L_i \times T_i$ such that $\int_{L(t)} \omega_i = 1$ for every $t \in T_i$. A transverse distribution δ_i on the local transversal T_i is now defined by

$$\langle \delta_i, f \rangle := \langle C, f\omega_i \rangle \quad f \in C_c^\infty(T_i).$$

This definition is independent from the choice of the forms ω_i in fact if $\int_{L(t)} \omega_i = \int_{L(t)} \omega_i' = 1$ there must be some family $d+1$–forms $t \longmapsto \sigma(t)$ such that $d_{L(t)}\sigma(t) = \omega(t) - \omega'(t)$. This family can be extended to a form σ on Ω_i using the trivial connection. But C is foliated and closed then,

$$\langle C, \omega_i - \omega_i' \rangle = \langle C, d f \sigma \rangle = 0.$$

The independence from the choice of ω_i also proves the relative holonomy invariance in fact, for two distinct sets $\Omega_i \cup \Omega_j$ one can choose ω_{ij} such that $\int_{L_i(t)} \omega_{ij} = \int_{L_j(t)} \omega_{ij} = 1$ for $t \in T_i \cap T_j$.
Viceversa let δ a holonomy invariant transverse distribution relative to a good cover. Define first a closed foliated d–current C_Ω on Ω for every $\Omega_i \simeq L_i \times T_i$ of the cover then patch together using a smooth partition of the unity.
If ω is a compactly supported d–form on Ω define

$$\langle C_\Omega, \omega \rangle := \langle \delta, \int_L \omega_{|F} \rangle,$$

in other words we let δ act on the function on T defined by $t \longmapsto \int_{L(t)} \omega_F(l, t)$. This collection of local currents is coherent with intersections by means of the holonomy invariance in fact $C_\Omega = C_{\Omega'}$ on $\Omega \cap \Omega'$. Furthermore every C_Ω is closed since $\langle C_\Omega, d\omega \rangle = \langle \delta_T, \int_L d\omega_{|F} \rangle = \langle \delta_T, 0 \rangle$ The property of being foliated is immediate since by costruction they depend only on the values of the forms on the foliation. □

Remark — Actually there is also another interesting geometric definition of a holonomy invariant measure as a (Radon) measure on X that is invariant in the direction of the leaves i.e. a measures on the ambient manifold that is invariant under flows generated by vector fields tangent to the foliation. Also a notion of distribution invariant in the direction of the leaves can be defined (see [24]).

To complete the picture one has to speak about positivity. Recall that our foliation is oriented.

DEFINITION 4.13 — A closed d–current C is positive in the direction of the leaves if $\langle C, \omega \rangle \geq 0$ for every d–form that restricts to a positive form on the leaves.

THEOREM 4.13 — Is equivalent to give on an manifold X with an oriented foliation

1. A holonomy invariant transverse measure i.e. a (Radon) measure on the σ–ring of all transversals invariant under the action of the holonomy pseudogroup Γ.
2. An measure on X invariant in the direction of the leaves.
3. A closed foliated current positive in the direction of the leaves.

PROOF — Apart for the case of invariant measures on X that are positive in the direction of the leaves for whose we make reference to [24] the only observation to do here is that a foliated current that is positive in the direction of the leaves defines a positive transverse distribution. $\square$

4.2.2 Tangential cohomology

Let $\wedge^k T^*\mathcal{F}$ the bundle of exterior forms of the foliation. In the terminology of Moore and Schochet this is a tangential vector bundle i.e. it has a canonical foliation compatible with the vector bundle structure. In a local trivialization over a foliated chart

$$\begin{tikzcd}
& \wedge^k T^*\mathcal{F} \arrow[r] \arrow[d, "\pi"'] & U \times \mathbb{R}^{\binom{p}{k}} \arrow[dl] \\
L \times T \arrow[r, "\simeq"] & U &
\end{tikzcd}$$

this foliation is given by the product foliation $\left(L \times \mathbb{R}^{\binom{p}{q}}\right) \times T$, in particular the bundle projection maps leaves into leaves.

DEFINITION 4.14 — A continuous section of $\wedge^k T^*\mathcal{F}$ is called a tangential k differential form if in every trivialization as above it restricts to be a smooth section on every plaque $L \times \{t\}$. The space of tangential k–differential forms is denoted with $\Omega^k_\tau(X)$ and $\Omega^k_{\tau,c}(X)$ is the subspace of compactly supported ones.

In a foliated chart with leafwise cordinates $x_1, ..., x_p$ and transversal coordinate t, a tangentially smooth differential form can be written

$$\omega = \sum_{i_l} a_{i_l}(x_1, ..., x_p, t) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \tag{13}$$

with a_{i_l} and all of its derivatives w.r.t. $x_1, ..., x_p$ continuous in all its variables. One can hence form the tangential De Rham operator $d_\tau : \Omega^k_{\tau c}(X) \longrightarrow \Omega^k_{\tau c}(X)$ just applying the standard De Rham operator plaque by plaque. We have defined the complex $(\Omega^*_{\tau c}(X), d_\tau)$ of tangential forms with compact support (d_τ is an example of leafwise differential operator, it decrease supports).

Definition 4.15 — The homology of the complex $(\Omega^*_{\tau c}(X), d_\tau)$ is called the tangential cohomology with compact support and denoted by $H^*_{\tau c}(X)$.

We can naturally define also tangential cohomology starting with forms without the condition of compactness of the support. In general the tangential cohomology has infinite dimension this is due to the fact that the continuous transverse control is much more relaxing than smoothness in every direction. In fact there is an interesting question on how the dimension of these spaces changes passing from tangential continuity (also measurability) to smoothness. In Chapter III of [53] there are examples. In the case the foliation is given by the fibers of a trivially local fiber bundle $F \hookrightarrow M \longrightarrow X$ the tangential cohomology turns out to be naturally isomorphic to the space of continuous sections of the bundle $H \longrightarrow X$ where the fiber $H_x = H^*_{\mathrm{dR}}(M_x)$ is the De Rham cohomology of the fiber above x.
Let's topologize each space $\Omega^\bullet_{\tau c}(X)$ by requiring uniform convergence of every coefficient function a_{i_l} in (13) with its tangential derivatives in every compact subset of each foliated chart. It often happens that the topological vector space $H^\bullet_{\tau c}(X)$ is not Hausdorff, this is the reason why it is convenient to take its maximal Haudorff quotient to define the closed tangential cohomology[5]

$$\overline{H}^k_\tau(X) := H^k_\tau(X)/\overline{\{0\}} = \mathrm{Ker}(d_\tau : \Omega^k_{\tau c} \longrightarrow \Omega^{k+1}_{\tau c})/\overline{\mathrm{Range}(d_\tau : \Omega^{k-1}_{\tau c} \longrightarrow \Omega^k_{\tau c}})}.$$

In general this leads to different spaces, for the irrational flow on the torus $\overline{H}^1_\tau(\mathbb{T}, \mathbb{R}) \cong \mathbb{R}$ while $H^1_\tau(\mathbb{T}, \mathbb{R})$ is infinite dimensional ([53]).

Definition 4.16 — Elements of the topological dual of $\Omega^\bullet_{\tau c}(X)$ i.e. continuous linear functionals $C : \Omega^\bullet_{\tau c}(X) \longrightarrow \mathbb{C}$ are called tangential currents. The space of tangential currents is denoted by

$$\Omega^\tau_k := \mathrm{Hom}_{\mathrm{con.}}(\Omega^k_{\tau c}(X); \mathbb{C}).$$

Note that a foliated current of definition 4.11 is a current in the ordinary sense that passes to define a tangential current under the restriction morphism $(\cdot)_{|\mathcal{F}} : \Omega^k(X) \longrightarrow \Omega^k_\tau(X)$. The differential $d_\tau : \Omega^\bullet_\tau(X) \longrightarrow \Omega^{\bullet+1}_\tau(X)$ (omit the subscript τ by simplicity of notation) is continuous and extends by duality to currents, $d_* : \Omega^\tau_\bullet(X) \longrightarrow \Omega^\tau_{\bullet-1}(X)$ according to the sign convention $\langle \omega, d_* \rangle = (-1)^{k-1}\langle d_\tau \omega, c\rangle$. There is an isomorphism

$$\mathrm{Hom}_{\mathrm{con.}}(H^k_{\tau c}(X); \mathbb{R}) \cong H^\tau_k(X; \mathbb{R})$$

and theorem 4.3, page 22 is essentially the Ruelle–Sullivan isomorphism[6]

$$\mathrm{MT}(X) \longrightarrow \mathrm{Hom}_{\mathrm{con.}}(H^p_{\tau c}, \mathbb{R})$$

between the vector space of signed holonomy invariant transverse Radon measures and the topological dual space of the top degree tangential homology. The tangential current defined by a measure Λ is called the Ruelle Sullivan current C_Λ.

[5] sometimes is called the tangential reduced cohomology

[6] at this level this is only a vector space iso. but one can consider the $*$–weak topology on the space of measures to force this to be a topological iso. However we don't need continuity.

4.2.3 Transverse measures and non commutative integration theory

At this point we have used the name *transverse measure* for at least two objects, measures on the union of all transversals and transverse measures in the equivalence relation $\mathcal{R}$ (or the holonomy groupoid, G is the same) according to definition 4.1. In the rest of the section we clarify the relationship between them. First we need a couple of definitions

DEFINITION 4.17 — A transverse measure Λ in the sense of non commutative integration theory for the equivalence relation $\mathcal{R}$ (or the holonomy groupoid G) is called locally finite if $\Lambda(\nu) < \infty$ for every $\nu \in \mathcal{E}^+$ with

1. ν is locally bounded i.e. $\sup \nu^x(K) < \infty$ for every K compact in $\mathcal{R}$
2. ν is compactly supported i.e. ν^x is supported in $s^{-1}(K)$ for a compact $K \subset X$.

DEFINITION 4.18 — The characteristic function ν_A of a subset $A \subset X$ is the transverse function defined by $\nu_A^x(B) = |s^{-1}(A) \cap G^x \cap B|$ or equivalently $\nu(f)(y) := \sum_{\gamma \in G^y,\, s(\gamma) \in A} f(\gamma)$ for a Borel function f on G.

Note that the characteristic function is nothing but the lift $s^{-1}(\mu_A)$ of the counting measure concentrated in A. This actually shows that $\gamma \nu_A^x = \nu_A^y$, $\gamma \in G_x^y$.

THEOREM 4.18 — (Connes [24]) Let Λ be a locally finite transverse measure for $\mathcal{R}$ (G). Let Z a transverse submanifold, for a compact set $K \subset Z$ define $\tau(K) := \Lambda(\nu_K)$. This is the definition of a Positive Radon measure on Z that is holonomy invariant.
In other words the correspondence $\Lambda \longmapsto \tau$ is a bijection

$$\{\text{Locally finite transverse measures on } \mathcal{R}\} \longrightarrow \{\text{Holonomy invariant transverse measures on } X\}.$$

Remember that there is a coupling between transverse measures Λ on $\mathcal{R}$ and transverse functions ν to produce a measure on X defined by $\Lambda_\nu(f) = \Lambda((s \circ f)\nu)$ then $\Lambda_{\nu_K}(1) = \Lambda(\nu_k) = \tau(K)$.

DEFINITION 4.19 — Choose some Radon measure α on the ambient X call the lift of α is the transverse measure $\nu^x := s^*(\alpha)$ where $s : G^x \longrightarrow X$. We say that a lift is transversally measurable if for every foliated chart $\Omega \cong U \times T$ it is represented as a weakly measurable mapping $T \longrightarrow \mathrm{Ra}(U)$ from T to the space of Radon measures on U, bounded if Ω is relatively compact.

PROPOSITION 4.20 — (Connes [24]) The map $\alpha \longmapsto s^*(\alpha)$ is a bijection transversally measurable Radon measures on X and transverse functions ν suc that $\sup \nu(K) < \infty$ for every compact $K \subset G$.

PROPOSITION 4.21 — Choose some Radon measure α on X with support X. Let $\nu = s^*(\alpha)$. The mapping $\Lambda \longmapsto \Lambda_\nu$ is a bijection between locally finite transverse measures on G and Radon measures μ on X with the property:

for every disintegration of μ on a foliated chart along the fibers of the distinct mapping $\Omega \cong U \times T \longrightarrow T$ the conditional measures satisfy

$$d\mu_t = d\alpha_t.$$

In practice the above propositions furnishes a geometrical recipe to recognize the measure Λ_ν on the base X if Λ is a transverse measure on the foliation i.e. a measure on the σ–ring of all Borel transversals. In fact choose some foliated atlas $\Omega_i \simeq U_i \times T_i$ with the set of coordinates (x,t) and a subordinate smooth partition of the unit φ_i. Then for a function f

$$\Lambda_\nu(f) = \sum_i \int_{T_i} \int_{U_i} \varphi_i(x,t) f(x,t) d\nu_t(x) d\Lambda_{T_i}(t)$$

where $\nu_t(x)$ is the longitudinal measure ν restricted to the plaque $U_i \times \{t\}$. We shall refer to this Fubini type decomposition as to the integration process according to the terminology of the book by Moore and Schochet [53].

4.3 Von Neumann algebras and Breuer Fredholm theory for foliations

Let $\mathcal{R}$ the equivalence relation of the foliation. For square integrable representations on the measurable fields of Hilbert spaces H_i let $\mathrm{Hom}_{\mathcal{R}}(H_1, H_2)$ the vector space of all intertwining operators. The choice of a holonomy invariant measure Λ on the foliation gives rise to a transverse measure on $\mathcal{R}$ in the sence of non commutative integration theory hence a quotient projection

$$\mathrm{Hom}_{\mathcal{R}}(H_1, H_2) \longrightarrow \mathrm{Hom}_{\Lambda}(H_1, H_2)$$

given by identification modulo Λ-a.e. equality. Elements of $\mathrm{Hom}_{\Lambda}(H_1, H_2)$ are called Random operators. If $H_1 = H_2 = H$, then $\mathrm{Hom}_{\mathcal{R}}(H, H) = \mathrm{End}_{\mathcal{R}}(H)$ is an involutive algebra, the quotient via Λ is a Von Neumann algebra[7]

$$\mathrm{Hom}_{\mathcal{R}}(H) \longrightarrow \mathrm{End}_{\Lambda}(H).$$

For a vector bundle $E \longrightarrow X$ let $L^2(E)$ be the Borel field of Hilbert spaces on X, of leafwise square integrable sections $\{L^2(L_x, E_{|L_x})\}_{x \in X}$. There is a natural square integrable representation of $\mathcal{R}$ on $L^2(E)$ the one given by $(x,y) \longmapsto \mathrm{Id} : L^2(L_x, E) \longrightarrow L^2(L_y, E)$. Denote $\mathrm{End}_{\mathcal{R}}(E)$ the vectorspace of all intertwining operators and $\mathrm{Hom}_{\Lambda}(E)$ the corresponding Von Neumann algebra.

Since we need unbounded operators we have to define measurability for fields of closed unbounded operators. Remember that the polar decomposition $T = u|T|$ is determined by the couple of bounded operators u and $(1 + T^*T)^{-1}$.

Definition 4.22 — We say that a field of unbounded closed operators T_x is measurable if are measurable the fields of bounded operators u_x and $|T_x|$.

Remark — .In the paper [56] about unbounded reduction theory. An unbounded field of

[7] to be precise this is a W^* algebra in fact it is not naturally represented on some Hilbert space. The choice of a longitudinal measure ν gives however a representation $\mathrm{End}_{\mathcal{R}}(H) \longrightarrow B(\int_X H_x d\Lambda_\nu(x))$ on the direct integral of the field H_x

closed operators A is said measurable if the family corresponding to the projection on the graph is measurable on $H \oplus H$ with the direct sum measure structure. Writing the projection on the graph as

$$(\xi, \eta) \longmapsto ((1 + A^*A)^{-1}(\xi + A^*\eta), A(1 + A^*A)^{-1}(\xi + A^*\eta))$$

we can see that these definition is equivalent to the one given here

Next, we review some ingredients from Breuer theory of Fredholm operators on Von Neumann algebras, adapted to our weight–theory case with some notions translated in the language of the essential Λ–spectrum, a straightforward generalization of the essential spectrum of a self–adjoint operator. Main references are [14, 15] and [18] and [19].

Remember that the set of projections $\mathcal{P} := \{A \in \mathrm{End}_\Lambda(E), A^* = A, A^2 = A\}$ of a Von Neumann algebra, has the structure of a complete lattice i.e. for every family $\{A_i\}_i$ of projections one can form their join $\vee A_i$ and their meet $\wedge A_i$. Then for a random operator $A \in \mathrm{End}_\Lambda(E)$ we can define its projection on the range $R(A) \in \mathcal{P}(\mathrm{End}_\Lambda(E))$ and the projection on its kernel $N(A) \in \mathcal{P}(\mathrm{End}_\Lambda(E))$ according to $R(A) := \vee\{P \in \mathcal{P}(\mathrm{End}_\Lambda(E)) : PA = A\}$ and $N(A) := \wedge\{P \in \mathcal{P}(\mathrm{End}_\Lambda(E)) : PA = P\}$. If A is the class of the measurable field of operators A_x, it is clear that $R(A)$ and $N(A)$ are the classes of $R(A)_x$ and $N(A)_x$.

DEFINITION 4.23 — Let H_i, $i = 1, .., 3$ be square integrable representations of $\mathcal{R}$ define

1. Λ–finite rank random operators. $B^f_\Lambda(H_1, H_2) := \{A \in \mathrm{Hom}_\Lambda(H_1, H_2) : \mathrm{tr}_\Lambda R(A) < \infty\}$
2. Λ–compact random operators. $B^\infty_\Lambda(H_1, H_2)$ is the norm closure of finite rank operators.
3. Λ–Hilbert–Schmidt random operators
$$B^2_\Lambda(H_1, H_2) := \{A \in \mathrm{Hom}_\Lambda(H_1, H_2) : \mathrm{tr}_\Lambda(A^*A) < \infty\}.$$
4. Λ–trace class operators. $B^1_\Lambda(H) = B^2_\Lambda(H)B^2_\Lambda(H)^* = \{\sum_{i=1}^n S_iT_i^* : S_i, T_i \in B^2_\Lambda(H)\}$.

LEMMA 4.24 — $B^*_\Lambda(H)$ is a $*$–ideal in $\mathrm{End}_\Lambda(E)$. An element $A \in B^*_\Lambda(H)$ iff $|A| \in B^*_\Lambda(H)$, $* = f, 1, 2, \infty$. The following inclusion holds

$$B^f_\Lambda(E) \subset B^1_\Lambda(E) \subset B^2_\Lambda(E) \subset B^\infty_\Lambda(E).$$

Furthermore

$$B^1_\Lambda(E) = \{A \in \mathrm{End}_\Lambda(E) : \mathrm{tr}_\Lambda |A| < \infty\}.$$

PROOF— The proof is very similar to the standard case. □

An important inequality is the following, take $A \in B^1_\Lambda(E)$ and $C \in \mathrm{End}_\Lambda(H)$. We have polar decompositions $A = U|A|$, $C = V|C|$ then $|A| = U^*A \in B^1_\Lambda(E)$, $|A|^{1/2} \in B^2_\Lambda(E)$ and

$$|\mathrm{tr}_\Lambda(CA)| \leq \|C\| \,\mathrm{tr}_\Lambda |A|. \tag{14}$$

For the proof, being a very standard calculation in Von Neumann algebras can be found in chapter V of [74].

Definition 4.25 — A random operator $F \in \mathrm{Hom}_\Lambda(E_1, E_2)$ is Λ–Fredholm (Breuer–Fredholm) if there exist $G \in \mathrm{Hom}_\Lambda(E_2, E_1)$ such that $FG - \mathrm{Id} \in B^\infty_\Lambda(E_2)$ and $GF - \mathrm{Id} \in B^\infty_\Lambda(E_1)$.

Definition 4.26 — For an unbounded field of closed operators $T_x : H_1 \longrightarrow H_2$ between two measurable fields of Hilbert spaces H_i the field of bounded operators

$$T_x : (\mathrm{Domain}(T_x), \|\cdot\|_{T_x}) \longrightarrow H_2$$

where $\|\cdot\|_{T_x}$ is the graph norm is measurable by Remark 4.3. We say that T is Λ–Breuer–Fredholm when this field of bounded operators is Λ–Breuer–Fredholm.

Proposition 4.27 — A random operator $F \in \mathrm{Hom}_\Lambda(H_1, H_2)$ is Λ–Fredholm if and only if $N(F)$ is Λ–finite rank and there exist some finite rank projection $S \in \mathrm{End}_\Lambda(H_2)$ such that $R(\mathrm{Id} - S) \subset R(F)$.

Hence from the proposition above Λ–Fredholm operators F have a finite Λ–index. In fact $\mathrm{tr}_\Lambda(N(F)) < \infty$ and

$$\mathrm{tr}_\Lambda(1 - R(F)) \leq \mathrm{tr}_\Lambda(S) < \infty,$$

making clear the next definition.

Definition 4.28 — Let $F \in \mathrm{Hom}_\Lambda(H_1, H_2)$ be Λ–Fredholm. The Λ index of F is defined by

$$\mathrm{ind}_\Lambda(F) := \mathrm{tr}_\Lambda(N(F)) - \mathrm{tr}_\Lambda(1 - R(F)).$$

The next result contained in The Shubin book [69] motivates the definition of an useful instrument called the Λ–essential spectrum

Lemma 4.29 — Let M be a Von Neumann algebra endowed with a semi–finite faithful trace τ, $S = S^* \in M$. Then S is τ–Breuer–Fredholm if and only if there exists $\epsilon > 0$ such that $\tau(E(-\epsilon, \epsilon)) < \infty$, where $E(\Delta)$ is the spectral projection of S corresponding to a Borel set Δ. Besides if $S = S^*$ is τ–Breuer–Fredholm then $\mathrm{ind}_\tau S = 0$.

So consider a measurable field T of unbounded intertwining operators. If T is selfadjoint (every T_x is self–adjoint a.e.) the parametrized (measurable) spectral Theorem (cf. Theorem XIII.85 in [63]) shows that for every bounded Borel function f the family $x \longmapsto f(T_x)$ is a measurable field of uniformely bounded intertwining operators defining a unique random operator. In other words

$$\{f(T_x)\}_x \in \mathrm{End}_\Lambda(H).$$

For a Borel set $U \subset \mathbb{R}$ let $\chi_T(U)$ be the family of spectral projections $x \longmapsto \chi_U(T_x)$. Denote $H_T(U)$ the measurable field of Hilbert spaces corresponding to the family of the images $(H_T(U))_x = \chi_U(T)H_x$. Let $\mathrm{tr}_\Lambda : \mathrm{End}^+_\Lambda(H) \longrightarrow [0, +\infty]$ the semifinite normal faithful trace defined by Λ. The formula

$$\mu_{\Lambda,T}(U) := \mathrm{tr}_\Lambda(\chi_T(U)) = \dim_\Lambda(H_U(T))$$

defines a Borel measure on $\mathbb{R}$.

DEFINITION 4.30 — We call the Borel measure defined above the Λ–spectral measure of T.

REMARK — Clearly this is not in general a Radon measure (i.e. finite on compact sets). In fact due to the non–compactness of the ambient manifold a spectral projection of a relatively compact set of an (even elliptic) operator is not trace class. In the case of elliptic self adjoint operators with spectrum bounded by below this is the Lebesgue–Stiltijes measure associated with the spectrum distribution function relative to the Λ–trace. This is the (not decreasing) function $\lambda \longmapsto \mathrm{tr}_\Lambda\, \chi_{(-\infty,\lambda)}(T)$. A good reference on this subject is the work of Kordyukov [38].

Notice the formula
$$\int f d\mu_{\Lambda,T} = \mathrm{tr}_\Lambda(f(T))$$
for each bounded Borel function $f : \mathbb{R} \longrightarrow [0, \infty)$. The proof of this fact easily follows starting from characteristic functions. Here the normality property of the trace plays a fundamental role. A detailed argument can be found in [58]. Next we introduce, inspired by [76] the *hero* of this section.

DEFINITION 4.31 — The essential Λ–spectrum of the measurable field of unbounded self–adjoint operators T is
$$\mathrm{spec}_{\Lambda,e}(T) := \{\lambda \in \mathbb{R} : \mu_{\Lambda,T}(\lambda - \epsilon, \lambda + \epsilon) = \infty, \forall \epsilon > 0\}.$$

LEMMA 4.32 — For Random operators the Λ–essential spectrum is stable under compact perturbation. If $A \in \mathrm{End}_\Lambda(E)$ is selfadjoint $A = A^*$ and $S = S^* \in B^\infty_\Lambda(E)$ then
$$\mathrm{spec}_{\Lambda,e}(A + S) = \mathrm{spec}_{\Lambda,e}(A).$$
Then if tr_Λ is infinite i.e. $\mathrm{tr}_\Lambda(1) = \infty$ we have $\mathrm{spec}_{\Lambda,e}(A) = \{0\}$ for every $A = A^* \in B^\infty_\Lambda(E)$.

PROOF — Let $\lambda \in \mathrm{spec}_{\Lambda,e}(A)$, by definition $\dim_\Lambda H_A(\lambda - \epsilon, \lambda + \epsilon) = \infty$. Then consider the field of Hilbert spaces
$$G_{\epsilon,x} := \big\{t \in \chi_{(-\lambda-\epsilon,\lambda+\epsilon)}(A_x)H_x;\ \|S_x t\| < \epsilon\|t\|\big\} = H_{S_x}(-\epsilon, \epsilon) \cap H_{A_x}(-\lambda - \epsilon, \lambda + \epsilon).$$
This actually shows that G_ϵ is Λ–finite dimensional infact $H_{A_x}(-\lambda - \epsilon, \lambda + \epsilon)$ is Λ–infinite dimensional while $H_{S_x}(-\epsilon, \epsilon)$ is Λ–finite codimensional. This showing that $\lambda \in \mathrm{spec}_{\Lambda,e}(A+S)$. The second statement is immediate. □

There is a spectral characterization of Λ–Fredholm random operators as expected after Lemma [?].

PROPOSITION 4.33 — For a random operator $F \in \mathrm{Hom}_\Lambda(H_1, H_2)$ the following are equivalent

1. F is Λ–Fredholm.
2. $0 \notin \mathrm{spec}_{\Lambda,e}(F^*F)$ and $0 \notin \mathrm{spec}_{\Lambda,e}(FF^*)$.

3. $0 \notin \mathrm{spec}_{\Lambda,e} \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix}$

4. $N(F)$ is Λ–finite rank and there exist some finite rank projection $S \in \mathrm{End}_\Lambda(H_2)$ such that $R(\mathrm{Id} - S) \subset R(F)$.

4.3.1 Splitting principle

Let $E \longrightarrow X$ be a vector bundle. For every $x \in X$ and integer k consider the Sobolev space $H^k(L_x, E)$ of sections of E, obtained by completion of $C^\infty_c(L_x, E)$ with respect to the k Sobolev norm

$$\|s\|^2_{H^k(L_x;E)} := \sum_{i=0}^{k} \|\nabla^k s\|^2_{L^2(\otimes^k T^* L_x;E)},$$

here the longitudinal Riemannian connection has been used. This is the definition of a Borel field of Hilbert spaces with natural Borel structure given by the inclusion into L^2. In fact, by Proposition 4 of Dixmier [27] p.167 to prescribe a measure structure on a field of hilbert spaces H it is enough to give a countable sequence $\{s_j\}$ of sections with the property that for $x \in X$ the countable set $\{s_j(x)\}$ is complete orthonormal. In the appendix of Heitsch and Lazarov paper [33] is shown, making use of holonomy that a family with the property that each s_j is smooth and compactly supported on each leaf can be choosen.

Definition 4.34 — Consider a field $T = \{T_x\}_{x\in X}$ (not necessarily Borel by now) of continuous intertwining operators $T_x : C^\infty_c(L_x; E_{|L_x}) \longrightarrow C^\infty_c(L_x; E_{|L_x})$.

- We say that T is of order $k \in \mathbb{Z}$ if T_x extends to a bounded operator

$$H^m(L_x, E_{|Lx}) \longrightarrow H^{m-k}(L_x, E_{|Lx})$$

  for each $m \in \mathbb{Z}$ and for x a.e.

- We say that the T is elliptic if each T_x satisfies a Garding inequality

$$\|s\|_{H^{m+k}_x} \leq C(L_x, m, k)[\|s\|_{H^m_x} + \|T_x s\|_{H^m_x}],$$

  and the family $\{C(L_x, m, k)\}_{x\in X}$ is bounded outside a null set in X.

Since each leaf L_x is a manifold with bounded geometry for a family of elliptic selfadjoint intertwining operators $\{T_x\}_{x\in X}$ every T_x is essentially selfadjoint with domain $H^k(L_x; E_{|L_x})$. It makes sense again to speak of measurability of such a family.

Definition 4.35 — For two fields of operators P and P' say that $P = P'$ outside a compact $K \subset X$ if for every leaf L_x and every section $s \in C^\infty_c(L_x \setminus K; E)$ then $Ps = P's$. This property holding x a.e in X with respect to the standard Lebesgue measure class.

Theorem 4.35 — The splitting principle. Let P and P' two Borel fields of (unbounded) selfadjoint order 1 elliptic intertwining operators. If $P = P'$ outside a compact set $K \subset X$ then

$$\mathrm{spec}_{\Lambda,e}(P) = \mathrm{spec}_{\Lambda,e}(P').$$

Proof — Let $\lambda \in \mathrm{spec}_{\Lambda,e}(P)$, for each $\epsilon > 0$ put $\chi_\epsilon^\lambda := \chi_{(\lambda-\epsilon,\lambda+\epsilon)}$ put $G_\epsilon := \chi_\epsilon^\lambda(P)$, then $\mathrm{tr}_\Lambda(G_\epsilon) = \infty$. The projection G_ϵ amounts to the Borel field of projections $\{\chi_\epsilon^\lambda(P_x)\}_{x\in X}$. By elliptic regularity on each Hilbert space $G_{\epsilon,x}$ every Sobolev norm is equivalent in fact the spectral theorem and Gårding inequality show that for $s \in G_{\epsilon,x}$ and $k \in \mathbb{N}$

$$\|s\|_{H_x^{k+2}} \leq C(P_1, k+2)\{\|s\|_{L_x^2} + \|(P_1-\lambda)^k s\|_{L_x^2}\} \leq (C+\epsilon^k)\|s\|_{L_x^2}$$

where $C(P_1, k+2)$ is a constant bigger than each leafwise Gårding constant.
Now choose two cut–off functions $\phi, \psi \in C_c^\infty(X)$ with $\phi_K = 1$ and $\psi_{|\mathrm{supp}\,\phi} = 1$. Consider the following fields of operators

$$B_\phi : L_x^2 \xrightarrow{\chi_\lambda^\epsilon} G_{\epsilon,x} \xrightarrow{\phi} L_x^2, \tag{15}$$

$$C_\psi : L_x^2 \xrightarrow{\chi_\lambda^\epsilon} (G_{\epsilon,x}, \|\cdot\|_{L^2}) \longrightarrow (G_{\epsilon,x}, \|\cdot\|_{H^k}) \xrightarrow{\psi} H_x^1 \tag{16}$$

for a k sufficiently big in order to have the Sobolev embedding theorem. We declare that $C_\psi^* C_\psi \in \mathrm{End}_\Lambda(E)$ is Λ–compact. In fact consider by simplicity the case in which ψ is supported in a foliation chart $U \times T$. The integration process shows that the trace of $C_\psi^* C_\psi$ is given by integration on T of the local trace on each plaque $U_t = U \times \{t\}$. Now the operator $C_{\psi,x}^* C_{\psi,x}$ is locally traceable by Theorem 1.10 in Moore and Schochet [53] since by Sobolev embedding the range of C_Ψ is made of continuous sections (the fact that each sobolev norm is equivalent on G_ϵ makes the teorem appliable i.e don't care in forming the adjoint w.r.t. H^1 norm or L^2). These local traces are uniformly bounded in $U \times T$ from the uniformity of the Gårding constants for the family since we are multiplying by a compactly supported function ψ. Actually we have shown that $C_\psi^* C_\psi$ is Λ–trace class. There follows from Lemma 4.32 about Λ–compact operators that the projection $\tilde{G}_\epsilon := \chi_{(-\epsilon^2,\epsilon^2)}(C_\psi^* C_\psi)$ is Λ–infinite dimensional in fact $\mathrm{spec}_{\Lambda,e}(C_\psi^* C_\psi) = \{0\}$.
Now $1 - B_\phi$ is Λ–Fredholm (also B_ϕ is Λ–compact) then its kernel has finite Λ–dimension. Also since $C_\psi^* C_\psi \chi_\epsilon^\lambda = C_\psi^* C_\psi$ then $\tilde{G}_\epsilon \chi_\epsilon^\lambda = \tilde{G}_\epsilon$ hence $(1 - B_\phi)\tilde{G}_\epsilon = (1-\phi)\tilde{G}_\epsilon \subset \mathrm{domain}(P')$ is Λ–infinite dimensional.
Take $s \in \tilde{G}_\epsilon$, from the definition

$$\|\psi s\|_{H^1}^2 = \langle C_\psi s, C_\psi s\rangle_{H^1} = \langle C_\psi^* C_\psi s, s\rangle_{L^2} \leq \epsilon^2 \|s\|_{L^2}^2$$

then

$$\begin{aligned}\|(P'-\lambda)(1-\phi)s\|_{L^2} \leq \|[P,\phi]s\|_{L^2} + \|(1-\phi)(P-\lambda)s\|_{L^2} \leq C\|\psi s\|_{H^1} + \\ \|(P-\lambda)s\|_{L^2} \leq \epsilon(1+C)\|s\|_{L^2}.\end{aligned}$$

The second chain of inequalities follows from

$$\begin{aligned}(P'-\lambda)(1-\phi)s = (P-\lambda)(1-\phi)s = ([P-\lambda, 1-\phi] - (1-\phi)(P-\lambda))s \\ = -([P,\phi] + (1-\phi)(P-\lambda))s.\end{aligned}$$

Finally the spectral theorem for (unbounded) self adjoint operators shows that $(1-\phi)\tilde{G}_\epsilon \subset \chi_{(\sigma,\tau)}(P')$ with $\sigma = \lambda - \epsilon(1+C), \tau = \lambda + \epsilon(1+C)$. In particular $\lambda \in \mathrm{spec}_{\Lambda,e}(P')$. □

Corollary 4.36 — Consider two foliated manifolds X and Y (with cylindrical ends or bounded geometry) with holonomy invariant measures Λ_1, Λ_2 and bounded geometry vector bundles $E_1 \longrightarrow X$ and $E_2 \longrightarrow Y$. Suppose there exist compact sets $K_1 \subset X$ and $K_2 \subset Y$ such that outside $X \setminus K_1$ and $Y \setminus K_2$ are isometric with an isometry that identifyies every geometric structure as the bundles and the foliation with the transverse measure. If P and P' are operators as in Theorem 4.5, page 29 with $P = P'$ on $X \setminus K_1 \simeq Y \setminus K_2$ in the sense of definition 4.35 then

$$\operatorname{spec}_{\Lambda_1,e}(P) = \operatorname{spec}_{\Lambda_2,e}(P').$$

PROOF— The proof of 4.5, page 29 can be repeated word by word till the introduction of the element $(1-\phi)\tilde{G}_\epsilon$ that can be considered as an element of $\operatorname{End}_{\Lambda_2}(E_2)$ through the fixed isometry. □

5 Analysis of the Dirac operator

5.1 Finite dimensionality of the index problem

Consider the leafwise Dirac operator D. This is a measurable field of unbounded first order differential operators $\{D_x\}_{x\in X}$. Its measurability property is easily checked observing that is equivalent to prove the measurability of the field of bounded operators

$$(D_x + i)^{-1} : L^2(L_x; E) \longrightarrow H^1(L_x; E).$$

Here the field of natural Sobolev spaces has the canonical structure given by inclusion into L^2. Now, the self–adjointness of D_x with domain $H^1(L_x; E)$ shows that

$$(D_x + i) : H^1(L_x; E) \longrightarrow L^2(L_x; E)$$

is a Hilbert space isomorphism. Choose two sections s, t of the domain and range respectively with the additional property that are smooth when restricted to each leaf then

$$\langle (D_x + i)s(x), t(x)\rangle_{L^2(L_x;E)} = \langle s(x), (D_x - i)t(x)\rangle_{L^2(L_x;E)}$$

and the measurability of the right–hand side is clear. Now it remains to apply the Example 2. in Dixmier [27] p-180 to have that the leafwise inverse family is measurable (Borel).

Since the foliation is even dimensional there is a canonical involution $\tau = i^p c(e_1 \cdots e_{2p})$ giving a parallel hortonormal ± 1 eigenbundles splitting $E = E^+ \oplus E^-$. Moreover the Dirac operator is odd with respect to this splitting. That's to say that D anticommutes with τ giving a pair of first order leafwise differential elliptic operators $D^\pm_x : C^\infty_c(L_x; E^\pm) \longrightarrow C^\infty_c(L_x; E^\mp)$. We continue to use the same notation for their unique L^2–closure and we have $D = D^+ \oplus D^-$ with $D^+ = (D^-)^*$.

The operator D^+ is called the chiral longitudinal Dirac operator, in general this is not a Breuer–Fredholm operator. In fact Fredholm properties are governed by its behavior at the boundary i.e its restriction to the base of the cylinder ∂X_0. Just in the one leaf situation D^+ is Fredholm in the usual sense if and only if 0 is not in the continuous spectrum of D^-D^+ or equivalently if the continuous spectrum has a positive lower bound. However the L^2 kernels of D^+ and D^- are finite dimensional and made of smooth sections. The difference

$$\dim_\Lambda \operatorname{Ker}_{L^2}(D^+) - \dim_\Lambda \operatorname{Ker}_{L^2}(D^-)$$

is the definition of the L^2–chiral index of D^+ giving the usual fredholm index when the operator is Fredholm. Notice that in the non Fredholm case the L^2 index is not stable under compactly supported perturbations making difficult its computation.

We are going to show that in our foliation case the chiral index problem is Λ–finite dimensional in the following sense exlpained in four steps.

- By an application of the parametrized measurable spectral theorem the projections on the L^2–kernels of $D^{\pm}$ belong to the Von Neumann algebras of the corresponding bundles, $\chi_{\{0\}}(D^+) \in \mathrm{End}_\Lambda(\mathrm{E}^{\pm})$ and decompose as a Borel family of bounded operators $\{\chi_{\{0\}}(D^{\pm})_x\}_x$ corresponding to the projections on the L^2 kernels of $D^{\pm}_x$. Furthermore they are implemented by a Borel family of uniformly smoothing Schwartz kernels.

- The family of projections above give rise to a longitudinal measure on the foliation. These measure are the local traces $U \longmapsto \mathrm{tr}_{L^2(L_x)}[\chi_U \cdot \chi_{\{0\}}(D^{\pm})_x \cdot \chi_U]$ where for a Borel $U \subset L_x$ the operator χ_U acts on $L^2(L_x)$ by multiplication. In terms of the smooth longitudinal Riemannian density these measures are represented by the pointwise traces of the leafwise Schwartz kernels. We prove that these local traces has the following finiteness property completely analog to the Radon property for compact foliated spaces.

  Finiteness property for local traces of projections on the kernel.

  Consider a leaf L_x. This is a bounded geometry manifold with a cylindrical end $\partial L_x \times \mathbb{R}^+$. We claim that for every compact $K \subset \partial L_x$

  $$\mathrm{tr}_{L^2(L_x)}[\chi_{K\times\mathbb{R}^+} \cdot \chi_{\{0\}}(D^{\pm})_x \cdot \chi_{K\times\mathbb{R}^+}].$$

  Since this list is aimed to the definition of the index the (rather long) proof of this statement is postponed immediately after.

- The integration process of a longitudinal measure against a transverse holonomy invariant measure immediately shows that the integrability condition above is sufficient to assure finite Λ–dimensionality of the L^2 kernels of $D^{\pm}$. Here the proof.

  First one has to choose a complete compact transversal S and a Borel map $f : X \longrightarrow S$ that respects the leaf equivalence relation displaying X as measure–theoretically fibering over S. Thanks to our assuptions on the foliation we can choose S composed by two pieces S_1 and S_2 where $S_1 = \partial X_0 \times \{0\}$ on the cylinder while S_2 is an *interior* transversal. Since we are working in the Borel world we can surely think that f restricts to U with values on S_1 and outside U with values on S_2. Now the integral ha two terms. The first integral, on S_1 is finite thanks to the finiteness property above in fact the situation here is a fibered integral of a standard Radon measure on the base times a finite measure. The interior term is finite thanks to proposition 4.22 in [53].

- **DEFINITION 5.37** — Define the chiral Λ–L^2–index

  $$\mathrm{Ind}_{L^2,\Lambda}(D^+) := \mathrm{tr}_\Lambda(\chi_{\{0\}}(D^+)) - \mathrm{tr}_\Lambda(\chi_{\{0\}}(D^-)) \in \mathbb{R}.$$

Proof of finiteness property of the local trace of kernel projections

PROOF — It is clear that it suffices to prove the property for each operator $(\cdot)_{|\partial_x\times\mathbb{R}^+}\chi_{\{0\}}(D^+_x)$.

Let us consider the operator D^+ on a fixed leaf L_x. This is a bounded geometry manifold with a cylindrical end $\partial L_x \times \mathbb{R}^+ = \{y \in L_x : r(y) \geq 1\}$ where the operator can be written in the form $B + \partial/\partial t$ acting on sections of $F \longrightarrow \partial L_x \times \mathbb{R}^+$. The boundary operator B is essentially selfadjoint on $L^2(\partial L_x; F)$ on the complete manifold ∂L_x (see [22] and [21] for a proof of self–adjointness using finite speed tecniques).
We are going to remind the Browder–Gårding type generalized eigenfunction expansion for B (see [26] 11, 300–307, [28] and [62] for an application to a A.P.S foliated and Galois covering index problems).
According to Browder–Gårding there exist

1. A sequence of smooth sectional maps $e_j : \mathbb{R} \times \partial L_x \longrightarrow F$ i.e. e_j is measurable and for every $\lambda \in \mathbb{R}$, $e_j(\lambda, \cdot)$ is a smooth section of F over ∂L_x such that $Be_j(\lambda, x) = \lambda e_j(\lambda, x)$.

2. A sequence of measures μ_j on $\mathbb{R}$ such that the map $V : C_c^\infty(\partial L_x; F) \longrightarrow \bigoplus_j L^2(\mathbb{R}, \mu_j)$ defined by $(Vs)_j(\lambda) = \langle s, e_j(\lambda, \cdot)\rangle_{L^2(\partial L_x)}$ (integration w.r.t Riemannian density) extends to an Hilbert space isometry

$$V : L^2(\partial L_x; F) \longrightarrow \bigoplus_j L^2(\mathbb{R}, \mu_j) =: \mathcal{H}_B$$

sending Borel spectral functions $f(B)$ into multiplication by $f(\lambda)$ with domain given by $\operatorname{dom} f(B) = \Big\{ s : \sum_j \int_{\mathbb{R}} |f(\lambda)|^2 |(Vs)_j(\lambda)|^2 d\mu_j(\lambda) < \infty \Big\}$. In particular beying an isometry means $\displaystyle\int_{\partial L_x} |s(x)|^2 dg = \sum_j \int_{\mathbb{R}} |(Vs)_j|^2 d\mu_j(\lambda)$.

Notice that $e_j(\lambda, \cdot)$ need not be square integrable on L_x. Taking tensor product with $L^2(\mathbb{R})$ we have the isomorphism

$$L^2(\partial L_x \times \mathbb{R}^+, F) \simeq L^2(\partial L_x, F) \otimes L^2(\mathbb{R}) \xrightarrow{\sim} [\oplus_j L^2(\mathbb{R}, \mu_j)] \otimes L^2(\mathbb{R}^+) = \mathcal{H}_B \otimes L^2(\mathbb{R}^+) \quad (17)$$

where $R^+ = (0, \infty)_r$. Under the identification $W := V \otimes \mathrm{Id}$ the operator D^+ is sent into $\lambda + \partial_r$ acting on the space $\mathcal{H}_B \otimes L^2(\mathbb{R}^+)$. Now let s be an L^2–solution of $D_x s = 0$. By elliptic regularity it restricts to the cylinder as an element $s(x, r) \in C^\infty(\mathbb{R}^+, H^\infty(\partial L_x; F)) \cap L^2(\mathbb{R}^+; L^2(\partial L_x, F))$ solution of $(\partial_r + B)s = 0$ then

$$\begin{aligned} \partial_r (Vs)_j(\lambda, t) &= \partial_r \int_{\partial L_x} \langle s(x, r), e_j(x, r)\rangle dg = \int_{\partial L_x} \langle dr\, s(x, r), e_j(\lambda, x)\rangle dg \\ &= -\int_{\partial L_x} \langle Bs(x, r), e_j(\lambda, x)\rangle dg = \int_{\partial L_x} \langle s(x, r), Be_j(x, r)\rangle dg \\ &= -\lambda \int_{\partial L_x} \langle Bs(x, r), e_j(\lambda, x)\rangle dg = -\lambda (Vs)_j(\lambda, r). \end{aligned} \quad (18)$$

Equation (18) says that all L^2 solutions of $D^+ = 0$ under the representation V on the cylinder are zero $\mu_j(\lambda)$–a.e. for $\lambda \leq 0$ for every j. Decompose, for fixed $a > 0$

$$L^2(\partial L_x \times \mathbb{R}^+; F) = L^2(\mathbb{R}^+; \mathcal{H}_B([-a, a])) \oplus L^2(\mathbb{R}^+, \mathcal{H}_B(\mathbb{R} \setminus [-a, a])) \quad (19)$$

where the notation is $\mathcal{H}_B(\Delta)$ for the spectral projection associated to χ_Δ. Let $\Pi_{\leq a}$ and $\Pi_{>a}$ respectively be the hortogonal projections corresponding to (19). Let $\chi_{\{0\}}(D_x^+)$ be the L^2 projection on the kernel, there's a composition

$$\Pi^a := \Pi_{\leq a} \circ (\cdot)_{|\partial L_x \times \mathbb{R}^+} \circ \chi_{\{0\}}(D_x^+)$$

defined trough

$$L^2(L_x) \longrightarrow \mathrm{Ker}_{L^2}(D_x^+) \longrightarrow L^2(\partial L_x \times \mathbb{R}^+) \longrightarrow L^2(\mathbb{R}^+; \mathcal{H}_B([-a,a])). \tag{20}$$

Thanks to the Browder–Garding expansion and equation (18) we can see that elements ξ belonging to the space $\Pi^a L^2(L_x)$ are in the form

$$\xi = \chi_{(0,\infty)}(\lambda) e^{-\lambda t} \zeta_0 \tag{21}$$

with $\zeta_0 = \zeta_{0j} \in H^\infty(\partial L_x; F)$ to be univoquely determined using boundary conditions. Formula (21) allows to define[8] the "boundary datas" mapping

$$\begin{gathered} \mathrm{BD} : \Pi^a L^2(L_x; F) \longrightarrow \mathcal{H}_B((0,a]) \\ W^{-1}(\chi_{(0,a]}(\lambda)\zeta_0 e^{-\lambda t}) \longmapsto W^{-1}(\chi_{(0,a]}(\lambda)\zeta_0) \end{gathered}$$

This is continuous and injective in fact injectivity is obvious while continuity follows at once from

$$\begin{aligned} \|\xi\|_{L^2(\partial L_x \times \mathbb{R}^+)} &= \sum_j \int_{\mathbb{R}} \int_0^\infty e^{-2\lambda t} |\zeta_{0j}(\lambda)|^2 dt d\mu_j(\lambda) \geq \sum_j \int_{[-a,a]} \int_0^\infty e^{-2\lambda t} |\zeta_{0j}(\lambda)|^2 dt d\mu_j(\lambda) \\ &\geq \sum_j \int_{[-a,a]} \int_0^\infty e^{-2at} |\zeta_{0j}(\lambda)|^2 dt d\mu_j(\lambda) = 1/(2a) \sum_j \int_{\mathbb{R}} |\chi_{[-a,a]} \zeta_{0j}(\lambda)|^2 d\mu_j(\lambda) \\ &= 1/(2a) \|\chi_{[-a,a]} \zeta_0\|_{\mathcal{H}_B}. \end{aligned}$$

Now choose an orthonormal basis $s_m = f_m \otimes g_m \in L^2(\partial L_x \times \mathbb{R}^+, F)$ and a compact set of the boundary $A \subset \partial L_x$, then put $\chi_{A\square} = \chi_{A\times(0,\infty)}(x,r)$. Consider the operator $\chi_{A\square}\Pi^a\chi_{A\square}$ acting on $L^2(L_x; F)$, now notice that Π^a acts on s_m via the natural embedding $L^2(\partial L_x) \subset L^2(L_x)$ then

$$\mathrm{tr}(\chi_{A\square}\Pi^a\chi_{A\square}) = \sum_m \langle \chi_{A\square}\Pi^a\chi_{A\square} s_m, s_m \rangle_{L^2(\partial L_x \times \mathbb{R}^+)}. \tag{22}$$

Write $\mathrm{BD}[\Pi^a\chi_{A\square}s_m] = W^{-1}[\chi_{(0,a]}(\lambda)\zeta_0^{(m)}]$ hence $[\Pi^a\chi_{A\square}s_m] = \chi_{(0,a]}(\lambda)\zeta_0^{(m)}e^{-\lambda t}$. By continuity of BD the sequence $\chi_{(0,a]}\zeta_0^{(m)}$ is bounded. Then (22) becomes

$$\begin{aligned} \mathrm{tr}(\chi_{A\square}\Pi^a\chi_{A\square}) &= \sum_m \langle W^{-1}[\chi_{(0,a]}(\lambda)\zeta_0^{(m)}e^{-\lambda t}], \chi_{A\square}s_m \rangle \\ &= \sum_m \langle \chi_{(0,a]}(\lambda)\zeta_0^{(m)}e^{-\lambda t}, W(\chi_{A\square}s_m) \rangle \end{aligned} \tag{23}$$

$$= \sum_m \int_{\mathbb{R}^+} \int_{\mathbb{R}\times\mathbb{N}} \chi_{(0,a]}(\lambda)\zeta_0^{(m)}e^{-\lambda t} \overline{\Big\{W(\chi_{A\square}s_m)\Big\}} d\mu(\lambda) dt \tag{24}$$

where μ is the direct sum of the μ_j's.

Last term of (23) can be estimated using Cauchy–Schwartz inequality and the trivial identity

[8] this is clearly inspired by Melrose definition [50] Chapter 6

$W(\chi_{A^\square} s_m) W(\chi_{A^\square} f_m \otimes g_m) = V(\chi_A(x) f_m(x)) g_m(t),$

$$\sum_m \int_{\mathbb{R}^+} \int_{\mathbb{R}\times\mathbb{N}} \chi_{(0,a]}(\lambda) \zeta_0^{(m)} e^{-\lambda r} \overline{\Big\{ w(\chi_{A^\square} s_m) \Big\}} d\mu(\lambda) dr$$
$$\leq \sum_m \Big\{ \int_{\mathbb{R}^+} \int_{\mathbb{R}\times\mathbb{N}} |g_m(r)|^2 |\zeta_0^{(m)}|^2 d\mu(\lambda) dr \Big\}^{1/2} \Big\{ \int_{\mathbb{R}^+} e^{-2ar} \int_{\mathbb{R}\times\mathbb{N}} \chi_{(0,a]} e^{-2(\lambda-a)} |V(\chi_A f_m)|^2 d\mu(\lambda) dr \Big\}^{1/2}$$
$$\leq \sum_m C \Big\{ \int_{\mathbb{R}\times\mathbb{N}} \chi_{(0,a]} |V(\chi_A f_m)|^2 d\mu(\lambda) dr \Big\}^{1/2} = C \sum_m \|\chi_A \mathcal{H}_B((0,a]) \chi_A f_m\|_{L^2(\partial L_x)}$$
$$\leq C \sum_m \langle \chi_A \mathcal{H}_B((0,a]) \chi_A f_m, f_m \rangle = C \mathrm{tr}(\chi_A \mathcal{H}_B((0,a]) \chi_A) < \infty. \quad (25)$$

In the last step we used the fact that for a projection on a closed subspace K one can compute its trace as, $\mathrm{tr}(K) = \sum_m \langle K f_m, f_m \rangle = \sum_m \|K f_m\|$ together with the fact that $\mathcal{H}_B((0,a])$ is a spectral projection of B hence uniformely smoothing. Let us now pass to examine the operator

$$\Pi_a := \Pi_{\geq a} \circ (\cdot)_{|\partial L_x \times \mathbb{R}^+} \circ \chi_{\{0\}}(D_x^+)$$

defined by

$$L^2(L_x) \longrightarrow \mathrm{Ker}_{L^2}(D_x^+) \longrightarrow L^2(\partial L_x \times \mathbb{R}^+) \longrightarrow L^2(\mathbb{R}^+; \mathcal{H}_B(\mathbb{R} \setminus [-a,a])). \quad (26)$$

arising from the second addendum of the splitting (19). Let φ_k be the characteristic function of $r \leq k$ and

$$\Lambda_k := \Pi_{\geq a} \circ \varphi_k \circ (\cdot)_{|\partial_x \times \mathbb{R}^+} \circ \chi_{\{0\}}(D_x^+).$$

Now

$$\|(\Pi_a - \Lambda_k)\xi\| = \|\Pi_{\geq a}(\varphi_k - 1)(\cdot)_{|\partial L_x \times \mathbb{R}^+} \chi_{\{0\}}(D_x^+)\xi\|_{L^2(\partial L_x \times \mathbb{R}^+)} \quad (27)$$
$$= \int_k^\infty \int_{(a,\infty)\times\mathbb{N}} e^{-2\lambda r} |\zeta_0|^2 d\mu(\lambda) dt \leq e^{-2ak} \int_{(a,\infty)\times\mathbb{N}} \int_0^\infty e^{-2\lambda r} |\zeta_0|^2 d\mu(\lambda) dr$$
$$\leq e^{-2ak} \|\xi\|_{L^2(\partial L_x \times \mathbb{R}^+)}.$$

Finally choose a compact $A \subset \partial L_x$, estimate (27) shows that $S_k := \chi_{A^\square} \Lambda_k \chi_{A^\square}$ converges uniformly to $\chi_{A^\square} \Pi_a \chi_{A^\square}$. Observe that S_k is compact by Rellich theorem and regularity theory in fact $\Pi_{\mathrm{Ker}(T^+)}$ is obtained by functional calculus from a rapid Borel function hence has a uniformly smoothing Schwartz–kernel (see the appendix for more informations). Since $\chi_{A^\times} \Lambda_k \Pi_{>a} \Pi_{\mathrm{Ker}(T^+)} \chi_{A^\times}$ is norm–limit of compact operators is compact but a compact projection is finite rank. □

5.2 Breuer–Fredholm perturbation

Our main application of the splitting principle is the construction of a Λ–Breuer Fredholm perturbation of the leafwise Dirac operator. Let θ be a smooth function satisfying $\theta = \theta(r) = r$ on Z_1 while $\theta(r) = 0$ on $X_{1/2}$, put $\dot{\theta} = d\theta/dr$. Let $\Pi_\epsilon := \chi_{I_\epsilon}(D^{\mathcal{F}_\partial})$ for $I_\epsilon := (-\epsilon, 0) \cup (0, \epsilon)$. Our perturbation will be the leafwise operator

$$D_{\epsilon,u} := D + \dot{\theta}\Omega(u - D^{\mathcal{F}_\partial}\Pi_\epsilon) \text{ for } \epsilon > 0, \quad u \in \mathbb{R} \quad (28)$$

that is $\mathbb{Z}_2$ odd as D. We write $D_{\epsilon,u} = D_{\epsilon,u}^+ \oplus D_{\epsilon,u}^-$ and $D_{\epsilon,u,x}$ for its restriction to L_x, also for brevity $D_{\epsilon,0} := D_\epsilon$.

Notice that the perturbed boundary operator is $D^{\mathcal{F}_\partial}_{\epsilon,u} = D^{\mathcal{F}_\partial}(1-\Pi_\epsilon) + u = D^{\mathcal{F}_\partial}_{\epsilon,0} + u$. Since for $\epsilon > 0$, 0 is an isolated point in the spectrum of $D^{\mathcal{F}_\partial}_{\epsilon,0}$ then $D^{\mathcal{F}_\partial}_{\epsilon,u}$ is invertible for $0 < |u| < \epsilon$. For further application let us compute the essential spectrum of $B^2_{\epsilon,u}$ where

$$B_{\epsilon,u} = D + \Omega(u - D^{\mathcal{F}_\partial}\Pi_\epsilon)$$

on the foliated cylinder $X_0 \times \mathbb{R}$ with product foliation $\mathcal{F}_\partial \times \mathbb{R}$. The Von Neumann algebra becomes $\mathrm{End}_{\Lambda_0}(E) \otimes B(L^2(\mathbb{R}))$ where $\mathrm{End}_{\Lambda_0}(E)$ is the Von Neumann algebra of the base i.e. the foliation induced on the transversal $X_0 \times \{0\}$. The integration process shows that the trace is nothing but $\mathrm{tr}_\Lambda = \mathrm{tr}_{\Lambda_0} \otimes \mathrm{tr}$ where the second factor is the canonical trace on $B(L^2(\mathbb{R}))$. We can write $B^2_{\epsilon,u} = \begin{pmatrix} 0 & -\partial_r + u + D^{\mathcal{F}_\partial}(1-\Pi_\epsilon) \\ \partial_r + u + D^{\mathcal{F}_\partial}(1-\Pi_\epsilon) & 0 \end{pmatrix}^2 = \begin{pmatrix} -\partial_r^2 & 0 \\ 0 & -\partial_r^2 \end{pmatrix} + \begin{pmatrix} 0 & u + D^{\mathcal{F}_\partial}(1-\Pi_\epsilon) \\ u + D^{\mathcal{F}_\partial}(1-\Pi_\epsilon) & 0 \end{pmatrix}^2 = -\partial_r^2 \,\mathrm{Id} + V^2$.
Consider the spectral measure μ_{Λ_0,V^2} of V^2 on the tranversal section $X_0 \times \{0\}$. We claim the following facts

1. $\omega := \inf \mathrm{supp}(\mu_{\Lambda_0,V^2}) > 0$
2. $\mu_{\Lambda,B^2_{\epsilon,u}}(a,b) = \infty, \quad 0 \le a < b, \quad \omega < b$
3. $\mu_{\Lambda,B^2_{\epsilon,u}}(a,b) = 0, \quad 0 \le a < b \le \omega.$

First of all 1. is obvious since $\mathrm{spec}(D^{\mathcal{F}_\partial}_{\epsilon,u}) \subset [(\epsilon+u)^2, \infty)$. To prove the second one observe first that we can use the Fourier transform in the cylindrical direction. This gives a spectral representation of $-\partial_r^2$ as the multiplication by y^2 on $L^2(\mathbb{R})$. Choose some $\gamma < (b-\omega)/2$. We can prove the following inclusion for the spectral projections

$$\chi_{(a,\gamma+\omega)}(V^2) \otimes \chi_{(0,\gamma)}(-\partial_r^2) \subset \chi_{(a,b)}(B^2_{\epsilon,u}). \tag{29}$$

In fact (29) follows from a (leafwise) spectral representation for V as the multiplication operator by x together with the implication $a < x^2 < \gamma + \omega, \quad 0 < y^2 < \gamma \Rightarrow a < x^2 + y^2 < b$. From (29) follows

$$\mu_{\Lambda,B^2_{\epsilon,u}}(a,b) \ge \mu_{\Lambda_0,V^2}(a,\gamma+\omega) \cdot \mathrm{tr}_{B(L^2(\mathbb{R}))}\, \chi_{(0,\gamma)}(-\partial_r^2) = \infty$$

in fact the first factor is non zero and the second is clearly infinite. Finally the third statement is very similar in the proof. We have shown that

$$\mathrm{spec}_{\Lambda,e}(B^2_{\epsilon,u}) = [\omega, \infty).$$

The perturbed boundary operator is $D^{\mathcal{F}_\partial}_{\epsilon,u} = D^{\mathcal{F}_\partial}(1-\Pi_\epsilon) + u = D^{\mathcal{F}_\partial}_{\epsilon,0} + u$. Since for $\epsilon > 0$, 0 is an isolated point in the spectrum of $D^{\mathcal{F}_\partial}_{\epsilon,0}$ then $D^{\mathcal{F}_\partial}_{\epsilon,u}$ is invertible for $0 < |u| < \epsilon$.

PROPOSITION 5.38 — The operator $D_{\epsilon,u}$ is Λ–Breuer–Fredholm if $0 < |u| < \epsilon$.

PROOF— The splitting principle (actually for order 2 operators but it makes no difference) says that the essential spectrum is determined by the operator on the cylinder for $r > 1$. The above calculation ends up the proof. □

In the next we shall investigate the relations between the index of the perturbed operator and the Dirac operator. At this aim the use of weighted L^2–spaces is fruitful as Melrose shows in [50].

Definition 5.39 — For $u \in \mathbb{R}$, denote $e^{u\theta}L^2$ the Borel field of Hilbert spaces (with obvious Borel structure given by $e^{u\theta} \cdot L^2 - \text{Borel structure}$. $\{e^{u\theta}L^2(L_x;E)\}_x$ where, for $x \in X$, $e^{u\theta}L^2(L_x;E)$ is the space of section–distributions w such that $e^{-u\theta}\omega \in L^2(L_x;E)$. Analog definition for weighted Sobolev spaces $e^{u\theta}H^k$ can be written.

Notice that $e^{u\theta}L^2(L_x;E) = L^2(L_x;E,e^{-2u\theta}dg_{|L_x})$ where dg is the leafwise Riemannian density so these Hilbert fields correspond to the representation of $\mathcal{R}$ with the longitudinal measure $x \in X \longmapsto e^{-2u\theta}dg_{|L_x} = r^*(e^{-2u\theta}dg)$ (transverse function, in the language of the non commutative integration theory [24]).
The operators D and its perturbation $D_{\epsilon,u}$ extend to a field of unbounded operators $e^{u\theta}L^2 \longrightarrow e^{u\theta}L^2$ with domain $e^{u\theta}H^1$. Put

$$e^{\infty\theta}L^2_x := \cup_{\delta>0}e^{\delta\theta}L^2_x.$$

In the next we will use, for brevity the following notation: $\partial L_x := L_x \cap (\partial X_0 \times \{0\})$ and

$$Z_x := \partial L_x \times [0,\infty)$$

for the cylindrical end of the leaf L_x.

For a smooth section $s^{\pm}$ such that $D^{\pm}_{\epsilon,u,x}s^{\pm} = 0$ we have $(D^{\pm}_{\epsilon,u,x})_{|\partial L_x\times\mathbb{R}^+}(s^{\pm})_{|\partial L_x\times\mathbb{R}^+} = 0$ that can be easily seen choosing smooth r–functions ϕ,ψ with $\phi_{X_0} = 1$, $\psi_{Z_{1/4}} = 1$, $\operatorname{supp}(\psi \subset Z_{1/8})$ and evaluating $[D^{\pm}_{\epsilon,u,x}(\phi(1-\psi)s + \phi\psi s) = 0]_{|\partial L_x\times\mathbb{R}^+}$.
The isomorphism W defined in (17) used in the proof of finiteness property for the kernel projection, can be defined also as an isomorphism $e^{u\theta}L^2(\partial L_x \times \mathbb{R}^+,F) \simeq \mathcal{H}_B \otimes e^{u\theta}L^2(\mathbb{R}^+)$ in a way that solutions of $D^{\pm}_{\epsilon,u,x}s^{\pm} = 0$ with conditions $s^{\pm} \in e^{\infty\theta} \cap L^2_x$ can be represented as solutions of $[\pm\partial_r + \lambda + \dot{\theta}(r)(u - \chi_\epsilon(\lambda)\lambda)]Ws^{\pm} = 0$ with $\chi_\epsilon(\lambda) = \chi_{(-\epsilon,0)\cup(\epsilon,0)}(\lambda)$ acting as a multiplier on $\bigoplus_j L^2(\mathbb{R},\mu_j)$. In particular (forgetting for brevity the restriction symbol)

$$Ws^{\pm} = \zeta^{\pm}_j(\lambda)\exp\{\mp u\theta(r) \mp \lambda[r - \theta(r)\chi_\epsilon(\lambda)]\} \tag{30}$$

with suitable choosen $\zeta^{\pm}_j(\lambda) \in L^2(\mu_j)$.

Proposition 5.40 — Let $\epsilon > \delta > 0$ and $\delta' \in \mathbb{R}$ then

1. $\xi \in \operatorname{Ker}_{e^{\delta'\theta}L^2}(D^+_x) \Longrightarrow \xi_{Z_x} = e^{-rD^{\mathcal{F}_\partial}_x}h$ with $h \in \chi(D^{\mathcal{F}_\partial}_x)_{(-\delta',\infty)}L^2_x$.
2. $\xi \in \operatorname{Ker}_{L^2}(D^+_{\epsilon,x}) \Longrightarrow \xi_{Z_x} = e^{-rD^{\mathcal{F}_\partial}_x + \theta(r)D^{\mathcal{F}_\partial}_x\Pi_{\epsilon,x}}h$, with $h \in \chi(D^{\mathcal{F}_\partial}_x)_{(\epsilon,\infty)}L^2_x$
3. $\xi \in \operatorname{Ker}_{e^{\delta\theta}L^2}(D^+_{\epsilon,x}) \Longrightarrow \xi_{|Z_x} = e^{-rD^{\mathcal{F}_\partial}_x + \theta(r)D^{\mathcal{F}_\partial}_x\Pi_{\epsilon,x}}h$, $h \in \chi(D^{\mathcal{F}_\partial}_x)_{(-\epsilon,\infty)}L^2_x$,

recall that $\Pi_{\epsilon,x} = \chi_{(-\epsilon,\epsilon)-\{0\}}(D^{\mathcal{F}_\partial}_x)$. Moreover the following identity (as fields of operators) holds true

$$D^{\pm}e^{\mp\theta(r)D^{\mathcal{F}_\partial}\Pi_\epsilon} = e^{\mp\theta(r)D^{\mathcal{F}_\partial}\Pi_\epsilon}D^{\pm}_\epsilon.$$

Proof —

1. From the representation formula (30) of formal solutions for $u = 0$, $\epsilon = 0$ it remains $\xi = \xi_j(\lambda)e^{-\lambda r}$. Then $e^{-\delta'\theta}\xi$ must be square integrable hence clearly $\xi_j(\lambda) = h_j(\lambda) \in \chi_{(-\delta',\infty)}(D^{\mathcal{F}_\partial}_x)$.

The remaining are proved in a very similar way. The last statement is merely a computation.
□

Solutions of $D^{\pm}_{\epsilon,x}s^{\pm}=0$ belonging to the space $\bigcap_{u>0}e^{u\theta}L^2(L_x;E^{\pm})$ are called L^2*–extended solutions*, in symbols $\mathrm{Ext}(D^{\pm}_{\epsilon,x})$. Next we study this space of solutions as x varies.

PROPOSITION 5.41 — For every $x\in X$ and $0<u<\epsilon$

$$1.\ \mathrm{Ker}_{L^2}(D^{\pm}_{\epsilon,x})=\mathrm{Ker}_{e^{-u\theta}L^2}(D^{\pm}_{\epsilon,x})=\mathrm{Ker}_{L^2}(D^{\pm}_{\epsilon,\mp u,x}) \tag{31}$$

$$2.\ \mathrm{Ext}(D^{\pm}_{\epsilon,x})=\mathrm{Ker}_{e^{u\theta}L^2}(D^{\pm}_{\epsilon,x})=\mathrm{Ker}_{L^2}(D^{\pm}_{\epsilon,\pm u,x}). \tag{32}$$

$$3.\ \mathrm{Ker}_{L^2}(D^{\pm}_{\epsilon,x})\subset\mathrm{Ext}(D^{\pm}_{\epsilon,x}) \tag{33}$$

PROOF— We show only the first equality of (31) the others being very similar. This is a simple application of equation (30). In fact, for $u=0$, $Ws^{\pm}=\zeta_j^{\pm}(\lambda)\exp\{\mp\lambda[r-\theta(r)\chi_\epsilon(\lambda)]\}$. The condition of being square integrable in $(\mathbb{R},\mu_j)\otimes(\mathbb{R}^+,dr)$ is easily seen to be equivalent to $\zeta_j^{+}(\lambda)=0$ $\lambda<\epsilon$, λ–a.e and $\zeta_j^{-}(\lambda)=0$ $\lambda>-\epsilon$ in particular, for $r\geq 1$ $Ws^{\pm}=\zeta_j^{\pm}(\lambda)e^{\mp\lambda r}\chi_{\pm\lambda\geq\epsilon}(\lambda)$ then $e^{u\theta}s^{\pm}\in L^2$ if $u<\epsilon$. For the reverse inclusion the proof is the same. For the third stament note that $e^{u\theta}L^2\subset e^{v\theta}L^2$ for every $u,v\in\mathbb{R}$ with $u\leq v$ then $\mathrm{Ker}_{L^2}\subset\mathrm{Ext}$. □

Proposition 5.41 shows at a single time that the mapping $x\longmapsto\mathrm{Ext}(D^{\pm}_{\epsilon,x})$ gives a Borel field of closed subspaces of L^2. No difference in notation between the space Ext and Ker and the corresponding projection in the Von Neumann algebra will be done in future. Inclusion (33) together with 5.38 and the finiteness property of the L^2–kernel projection says that the difference

$$h^{\pm}_{\Lambda,\epsilon}=\dim_\Lambda(\mathrm{Ext}(D^{\pm}_{\epsilon}))-\dim_\Lambda(\mathrm{Ker}_{L^2}(D^{\pm}_{\epsilon}))=\mathrm{tr}_\Lambda(\mathrm{Ext}(D^{\pm}_{\epsilon}))-\mathrm{tr}_\Lambda(\mathrm{Ker}_{L^2}(D^{\pm}_{\epsilon}))\in\mathbb{R} \tag{34}$$

is a finite number.

LEMMA 5.42 — For $\epsilon>0$

1. $\dim_\Lambda Ker_{L^2}(D^{\pm}_{\epsilon})=\lim_{u\downarrow 0}\dim_\Lambda Ker_{L^2}(D^{\pm}_{\epsilon,\mp u})=\lim_{u\downarrow 0}\dim_\Lambda Ker_{L^2}(D^{\pm}_{\epsilon,\pm u})-h^{\pm}_{\Lambda,\epsilon}$,
2. $\mathrm{Ind}_{L^2,\Lambda}(D^{+}_{\epsilon})=\lim_{u\downarrow 0}\mathrm{Ind}_\Lambda(D^{+}_{\epsilon,u})-h^{+}_{\Lambda,\epsilon}=\lim_{u\downarrow 0}\mathrm{Ind}_\Lambda(D^{+}_{\epsilon,-u})+h^{-}_{\Lambda,\epsilon}$

PROOF— Nothing to prove here, proposition 5.41 says that the limit is constant for u sufficiently small, the second one in the statement follows from the first by summation. □

Now define the extended solutions $\mathrm{Ext}(D^{\pm}_x)$ in the same way i.e. distributional solution of the differential operator $D^{\pm}_x:C^{\infty}_c(L_x;E^{\pm})\longrightarrow C^{\infty}_c(E^{\mp};E)$ belonging to each weighted L^2–space with positive weights,

$$\mathrm{Ext}(D^{\pm}_x)=\bigcap_{u>0}\mathrm{Ker}_{e^{u\theta}L^2}(D^{\pm})=\{s\in C^{-\infty}(L_x;E^{\pm});\ D^{\pm}s=0;\ e^{-u\theta}s\in L^2\ \forall u>0\}.$$

Here we have made use of the longitudinal Riemannian density to to identify sections with sections with values on density and the Hermitian metric on E, in a way that one has the

isomorphism $C^{-\infty}(L_x; E^\pm) \simeq C_c^\infty(L_x; (E^\pm)^* \otimes \Omega(L_x))$ to simplify the notation with distributional sections of the bundle E.
It is clear by standard elliptic regularity that extended solutions of $D^\pm$ are smooth on each leaf. In fact $D^\pm$ a first order differential elliptic operator and one can construct a parametrix i.e. an inverse of $D^\pm$ modulo a smoothing operator i.e. an operator sending each Sobolev space onto each Sobolev space (of the new, weighted metric).

Remark — By definition $\operatorname{Ext}(D^\pm) \subset e^{u\theta}L^2$ for every $u > 0$, define $\dim_\Lambda^{(\mathrm{u})}(\operatorname{Ext})$ as the trace in $\operatorname{End}_\Lambda(e^{u\theta}L^2)$ of the projection on the closure of Ext, now we must check that under the natural inclusion $e^{u\theta}L^2 \subset e^{u'\theta}L^2$ if $u < u'$, these dimensions are preserved. This is done at once in fact the inclusion $\operatorname{Ext}(D^\pm) \subset e^{u\theta}L^2 \hookrightarrow \operatorname{Ext}(D^\pm) \subset e^{u'\theta}L^2$ is bounded and extends to a bounded mapping

$$\overline{\operatorname{Ext}(D^\pm)}^{e^{u\theta}L^2} \longrightarrow \overline{\operatorname{Ext}(D^\pm)}^{e^{u'\theta}L^2}$$

with dense range. Now the unitary part of its polar decomposition is an unitary isomorphism then the Λ dimensions are the same by 1. in 4.7.

Definition 5.43 — The Λ–dimension of the space of extended solution is

$$\dim_\Lambda \operatorname{Ext}(D^\pm) := \dim_\Lambda \overline{\operatorname{Ext}(D^\pm)}^{e^{u\theta}L^2}$$

for some $u > 0$.

Proposition 5.44 —

1. $\lim_{\epsilon\downarrow 0} \dim_\Lambda \operatorname{Ker}_{L^2}(D^\pm_\epsilon) = \dim_\Lambda \operatorname{Ker}_{L^2}(D^\pm)$
2. $\lim_{\epsilon\downarrow 0} \operatorname{Ind}_{L^2,\Lambda} D^+_\epsilon = \operatorname{Ind}_{L^2,\Lambda} D^+$
3. $\lim_{\epsilon\downarrow 0} \dim_\Lambda \operatorname{Ext}(D^\pm_\epsilon) = \dim_\Lambda \operatorname{Ext}(D^\pm)$

Proof —

1. Let $\xi \in \operatorname{Ker}_{L^2}(D^+_{\epsilon,x})$ thanks to Proposition 5.40

$$\xi_{Z_x} = e^{-rD_x^{\mathcal{F}_\partial} + \theta(r)D_x^{\mathcal{F}_\partial}\Pi - \epsilon,x} h, \ \ h \in \chi_{(\epsilon,\infty)}(D_x^{\mathcal{F}_\partial})$$

   from $\Pi_{\epsilon,x}h = 0$ we get

$$\begin{aligned} D_x^+ \xi_{|Z_x} &= (D^+_{\epsilon,x} + \theta(r)D_x^{\mathcal{F}_\partial}\Pi_{\epsilon,x})\xi_{|Z_x} = \theta(r)D_x^{\mathcal{F}_\partial}\Pi_{\epsilon,x}(\xi_{|Z_x}) \\ &= \theta(r)D_x^{\mathcal{F}_\partial}\Pi_{\epsilon,x}(e^{-rD_x^{\mathcal{F}_\partial} + \theta(r)D_x^{\mathcal{F}_\partial}\Pi_{\epsilon,x}} h) = 0 \end{aligned}$$

   meaning that $\operatorname{Ker}_{L^2}(D^+_{\epsilon,x}) \subset \operatorname{Ker}_{L^2}(D^+)$. Moreover

$$\begin{aligned} &D^+_\epsilon(\operatorname{Ker}_{L^2}(D^+)) \\ &= \dot\theta D_x^{\mathcal{F}_\partial}\Pi_{\epsilon,x}(\operatorname{Ker}_{L^2}(D^+) \subset -\dot\theta D_x^{\mathcal{F}_\partial} e^{-rD_x^{\mathcal{F}_\partial}} \chi_{(-\epsilon,\epsilon)}(D_x^{\mathcal{F}_\partial})(L^2(\partial L_x \otimes L^2(\mathbb{R}^+)). \end{aligned}$$

Note that clearly $\dim_\Lambda \left[\dot{\theta} D_x^{\mathcal{F}_\partial} e^{-rD_x^{\mathcal{F}_\partial}} \chi_{(-\epsilon,\epsilon)}(D_x^{\mathcal{F}_\partial})(L^2(\partial L_x \otimes L^2(\mathbb{R}^+))\right] \longrightarrow_{\epsilon\to 0} 0$ by the normality of the trace. Then the family of operators

$$D^+_{\epsilon \,|\, \mathrm{Ker}_{L^2}(D^+)} : \mathrm{Ker}_{L^2}(D^+) \longrightarrow L^2$$

has kernel $\mathrm{Ker}_{L^2}(D^+_{\epsilon,x})$ and range with Λ dimension going to zero, 1. follows looking at an hortogonal decomposition $\mathrm{Ker}_{L^2}(D^+) = \mathrm{Ker}_{L^2}(D^+_\epsilon) \oplus \mathrm{Ker}_{L^2}(D^+)/\,\mathrm{Ker}_{L^2}(D^+_\epsilon)$.

2. Follows immediately from 1.

3. Consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Ker}_{e^{\delta\theta}L^2}(D^+) \arrow[r] \arrow[dr, "\Psi^+_\epsilon"] & \mathrm{Ker}_{e^{(\delta+\epsilon)\theta}L^2}(D^+) \\
 & \mathrm{Ker}_{e^{\delta\theta}L^2}(D^+_\epsilon) \arrow[u, "\Psi^-_\epsilon"']
\end{tikzcd}$$

where $\Psi^\pm_\epsilon = e^{\pm\theta\Pi_\epsilon D^{\mathcal{F}_\partial}}$. It is easily seen thanks to the representation of solutions in proposition 5.40 that each arrow is injective and bounded with respect to the inclusions

$$\begin{tikzcd}
e^{\delta\theta}L^2 \arrow[r] \arrow[dr] & e^{(\delta+2\epsilon)\theta}L^2 \\
 & e^{(\delta+\epsilon)\theta}L^2 \arrow[u]
\end{tikzcd} .$$

Then joining together the two diagrams,

$$\begin{tikzcd}
\mathrm{Ker}_{e^{\delta\theta}L^2}(D^+) \arrow[r] \arrow[dr, "\Psi^+_\epsilon"] & \mathrm{Ker}_{e^{(\delta+\epsilon)\theta}L^2}(D^+) \arrow[r] & e^{(\delta+2\epsilon)\theta}L^2 \arrow[d] \\
 & \mathrm{Ker}_{e^{\delta\theta}L^2}(D^+_\epsilon) \arrow[u, "\Psi^-_\epsilon"'] \arrow[r] & e^{(\delta+\epsilon)\theta}L^2
\end{tikzcd}$$

and using the last column to measure dimensions one gets the inequality

$$\dim_\Lambda \mathrm{Ker}_{e^{\delta\theta}L^2}(D^+) \leq \dim_\Lambda \mathrm{Ker}_{e^{\delta\theta}L^2}(D^+_\epsilon) \leq \dim_\Lambda \mathrm{Ker}_{e^{(\delta+\epsilon)\theta}L^2}(D^+)$$

from which 3. immediately follows.

□

6 Cylindrical finite propagation speed and Cheeger Gromov Taylor type estimates.

6.1 The standard case

A very important property of the Dirac operator on a manifold of bounded geometry X is finite propagation speed for the associated wave equation. Let $P \in \mathrm{UDiff}^1(X,E)$ uniformly elliptic first order (formally) self–adjoint operator.

DEFINITION 6.45 — The diffusion speed of P in x is the norm of the principal symbol

$$\sup_{v\in S^*_x} |\sigma_{\mathrm{pr}}(P)(x)|$$

(S_x^*) is the fibre of cosphere bundle at x). Taking the supremum on x in M one gets the maximal diffusion speed $c = c(P)$.
We say that an operator has finite propagation speed if its maximal diffusion speed is finite.

Remark — A (generalized) Dirac operator associated to bounded geometry datas (manifold and clifford structure) has finite propagation speed in fact its principal symbol is Clifford multiplication.

The starting point is an application of the spectral theorem to show that for every initial data $\xi_0 \in C_c^\infty(X, E)$ there is a unique solution $t \mapsto \xi(t)$ of the Cauchy problem for the wave equation associated with P,

$$\begin{cases} \partial\xi/\partial t - iP\xi = 0, \\ \xi(0) = \xi_0, \end{cases} \tag{35}$$

this solution is given by the application of the one parameter group of unitaries $\xi(t) = e^{itP}\xi_0$. By the Stone theorem the domain of P is invariant under each unitary e^{itP} and $e^{itP}$is bounded from each Sobolev space H^s into itself. In particular the domain of P is invariant under each unitary e^{itP}.

Lemma 6.46 — For θ suitably small and $x \in M$, $\|\xi(t)\|_{L^2 B(x,\theta - ct)}$ is decreasing in t. In particular $\mathrm{supp}(\xi_0) \subset B(x, r) \Longmapsto \mathrm{supp}(e^{itP}\xi_0) \subset B(x, r + ct)$.

Proof — The proof is in J. Roe's book [65] Prop. 5.5 and lemma 5.1. Next we shall prove something similar in the cylindrical end. First one proves that for a small geodesic ball of radius r the function $\|e^{itP}\xi_0\|_{L^2(B(x,r-ct))}$ is decreasing, this is called energy estimate then the second step follows easily. □

Finite propagation speed tecniques provides us with the construction of a functional calculus, a morphism of algebras $\mathcal{S}(\mathbb{R}) \longrightarrow B(L^2(X, E))$, $f \longmapsto f(P)$ with properties

- Continuity, $\|f(P)\| \leq \sup|f|$ hence it can be extended to $C_0(\mathbb{R})$, the space of continuous functions vanishing at infinity.
- If $f(x) = xg(x)$ then $f(P) = Pg(P)$.
- We have the representation formula in terms of the inverse Fourier transform

$$f(P) = \int_{\mathbb{R}} \hat{f}(t) e^{itP} dt/2\pi, \tag{36}$$

here $\hat{\cdot}$ is Fourier transform and the integral converges in the weak operator topology, namely $\langle f(P)x, y\rangle = \int \hat{f}(t)\langle e^{itP}x, y\rangle dt/2\pi$, for every $x, y \in L^2(X; E)$. If $X = S^1$ this is just Poisson summation formula.

Representation (36) leads further, as an example we recount how John Roe, using ideas contained in [21] used to build a pseudodifferential calculus.
Let $S^m(\mathbb{R})$ be the space of symbols of order $\leq m$ on the real line i.e. smooth functions such that $|f^\lambda(k)| \leq C_k(1 + |\lambda|)^{m-k}$. This is a Fréchet space with best constants C_k as seminorms and $\mathcal{S}(R) = \bigcap S^m(\mathbb{R})$.
Roe proves in [66] that for a bounded geometry Dirac operator D every spectral function $f(D)$ with f a symbol of order $\leq m$ is a uniform pseudodifferential operator of order m. The proof of this fact uses formula (36) together with a convolution smoothing technique.

Now formula (36) leads us to an easy method to obtain pointwise extimates of the Schwatz kernel $[f(P)]$ for a class Schwartz function f. In fact due to the ellipticity of P, $f(P)$ is a uniformly smoothing operator and $[f(P)] \in UC^\infty(X \times X; \mathrm{End}(E))$ (see the appendix A) here we have used the Riemannian density to remove the density coefficient in the Schwartz kernels.

PROPOSITION 6.47 — Take some section $\xi \in L^2(X;E)$ supported into a geodesic ball $B(x,r)$ then the following estimate holds true

$$\|f(P)\xi\|_{L^2(X-B(x,R))} \leq (2\pi)^{-1/2}\|\xi\|_{L^2(X)} \int_{\mathbb{R}-I_R} |\hat{f}(s)|ds, \tag{37}$$

where $I_R := (-\frac{r-R}{c}, \frac{r-R}{c})$ with the convention that $I_R = \emptyset$ if $R \leq r$.

PROOF — From the finite propagation speed

$$\mathrm{supp}(e^{itP}\xi) \subset B(x, r + c|t|). \tag{38}$$

From the identity (36),

$$\begin{aligned}
\|f(P)\xi\|_{L^2(X-B(x,R))} &= \left\| (2\pi)^{-1/2} \int_{\mathbb{R}} \hat{f}(s) e^{isP}\xi ds \right\|_{L^2(X-B(x,R))} \\
&\leq \left\| (2\pi)^{-1/2} \int_{\mathbb{R}-I_R} \hat{f}(s) e^{isP}\xi ds \right\|_{L^2(X)} \\
&\leq (2\pi)^{-1/2}\|\xi\|_{L^2(X)} \int_{\mathbb{R}-I_R} |\hat{f}(s)|ds
\end{aligned}$$

where $I_R := (-\frac{r-R}{c}, \frac{r-R}{c})$ with the convention that $I_R = \emptyset$ if $R \leq r$. In fact

$$\|f(P)\xi\|^2_{L^2(X-B(x,R))} = (2\pi)^{-1} \int_R |\hat{f}(s)|^2 \|e^{isP}\xi\|^2_{L^2(X-B(x,R))} ds$$

and the function $s \longmapsto \|e^{isP}\xi\|^2_{L^2(X-B(x,R))}$ is zero if $|s| < \frac{r-R}{c}$ from (38). □

So the point of view is that more far from the support of the section we want the L^2 norm of the image, larger pieces of the Fourier transform around zero can we remove. The extreme point of view is that spectral functions made by functions with compactly supported Fourier transforms will produce properly supported operators i.e. operators whose kernel lies within a δ–neighborhood of the diagonal. Estimate (37) is the starting point. The following proposition shows how to work out pointwise estimates on the kernel from this mapping properties. This is a very rough version of the ideas contained in [21]

PROPOSITION 6.48 — Let $r_1 > 0$ sufficiently small, $x, y \in X$ put

$$R(x,y) := \max\{0, d(x,y) - r_1\}$$

and $\bar{n} := [n/2+1]$, $n = \dim X$, $I(x,y) := (-R(x,y)/c, R(x,y)/c)$. For a class Schwartz function $f \in \mathcal{S}(\mathbb{R})$

$$\left|\nabla^l_x \nabla^k_y [f(P)]_{(x,y)}\right| \leq \mathcal{C}(P,l,k,r_1) \sum_{j=0}^{2\bar{n}+l+k} \int_{\mathbb{R}-I(x,y)} \left|\hat{f}^{(j)}(s)\right| ds. \tag{39}$$

PROOF— $|\nabla_x^l \nabla_y^k [f(P)]_{(x_0,y_0)}| \leq C_0 \|\nabla_x^l [f(P)]_{(x_0,\bullet)}\|_{H^{\bar{n}+k}(B(y_0,r_1/3))}$ where C_0 is the constant[9] of the Sobolev embedding $H^{\bar{n}+k}(B(y_0,r_1/3)) \longrightarrow UC^k(B(y_0,r_1/3))$ applied to the function $\nabla_x^l [f(P)]_{(x_0,\bullet)}$.
Then we have to apply the Gårding inequality of P

$$\begin{aligned}\|\nabla_x^l [f(P)]_{(x_0,\bullet)}\|_{H^{\bar{n}+k}(B(y_0,r_1/3))} &\leq C_1 \sum_{j=0}^{\bar{n}+k} \|\nabla_x^l P_y [f(P)]_{(x_0,\bullet)}\|_{L^2(B(y_0,r_1/2))} \\ &= C_1 \sum_{j=0}^{\bar{n}+k} \|\nabla_x^l [f(P)P]_{(x_0,\bullet)}\|_{L^2(B(x_0,r_1/2))}\end{aligned}$$

in fact by self adjointness $P_y[f(P)]_{(x_0,\bullet)} = [f(P)P]_{(x_0,\bullet)}$. No problem here in *localizing* the Gårding inequality we can choose in fact for each y_0 a function χ supported in $B(y_0,r_1)$ with $\chi_{|B(y_0,r_1/2)} = 1$. Then since the coefficients of P in normal coordinates are uniformly bounded, each $[P,\chi]$ is uniformly bounded. Let $\xi_j(y) := \chi_{B(y_0,r_1/2)}(y)\nabla_x^l [P^j f(P)]_{(x_0,y)}$ the inequality becomes $|\nabla_x^l \nabla_y^k [f(P)]_{(x_0,y_0)}| \leq C_0 C_1 \sum_{j=0}^{\bar{n}+k} \|\xi_j\|_{L^2(X)}$.
Now

$$\begin{aligned}\|\xi_j\|^2_{L^2(X)} = \int \chi_{B(y_0,r_1/2)} \nabla [P^j f(P)]_{(x_0,\bullet)} \xi_j(y) dy = |(\nabla_x^l P^j f(P) \xi_j)(x_0)| \\ \leq C_2 \|P^j f(P)\xi_j\|_{H^{\bar{n}+l}(B(x_0,r_1/3))} \leq C_2 C_3 \sum_{i=0}^{\bar{n}+l} \|P^{j+i} f(P)\xi_j\|_{L^2(B(x_0,r_1/2))}\end{aligned}$$

again by Sobolev embedding and Gårding inequality. The choice to keep every constant is motivated to control their dependence in order to apply these extimates leaf by leaf.
Finally putting everything together

$$\begin{aligned}|\nabla_x^l \nabla_y^k [f(P)]_{(x_0,y_0)}| &\leq \mathcal{C} \sum_{j=0}^{\bar{n}+k} \sum_{i=0}^{\bar{n}+l} \|P^{j+i} f(P)\|_{L^2(B(x_0,r_1/2)),L^2(B(y_0,r_{1/2}))} \\ &\underbrace{\leq \mathcal{C}}_{(37)} \sum_{j=0}^{2\bar{n}+l+k} \int_{\mathbb{R}-I(x_0,y_0)} |\hat{f}^{(j)}(s)| ds\end{aligned}$$

□

For the <u>heat</u> <u>kernel</u> $[f(P)] = [e^{-tP^2}]$ when $f(x) = e^{-tx^2}$, $\hat{f}(s) = (2t)^{-1/2} e^{-s^2/4t}$,

$$\begin{aligned}\hat{f}(s)^{(k)} &= \frac{1}{(2t)^{1/2}(4t)^{k/2}} ((4t)^{1/2}\partial_s)^k e^{-\left(\frac{s}{(4t)^{1/2}}\right)^2} \\ &= \frac{C(k)}{t^{(k+1)/2}} H_k\Big(\frac{s}{(4t)^{1/2}}\Big) e^{-\left(\frac{s}{(4t)^{1/2}}\right)^2},\end{aligned}$$

where H_k is the k–th Hermite polynomial. Then using the simple inequalities

$$\int_u^\infty e^{-x} dx \leq e^{-u^2}, \; y^s e^{ay^2} \leq \Big(\frac{s}{2ae}\Big)^{s/2}, \; a,s,u,y \in \mathbb{R}^+,$$
$$\int_u^\infty y^s e^{-y^2} dy = \int_u^\infty y^s e^{-\epsilon y^2} e^{-(1-\epsilon)y^2} dy \leq C(s,\epsilon) e^{-(1-\epsilon)u^2}$$

[9] if preferable one can suppose $B(y_0,r_1)$ a geodesic ball and multiply a multiply a cut off φ supported within distance $r_1/3$ from y_0 and use the global Sobolev embedding. In that case the constant depends on φ but using normal coordinates φ can be used well for each y_0

with $R = R(x,y)$ and $\eta = 2\bar{n} + l + k$

$$
\begin{aligned}
|\nabla_x^l \nabla_y^k [P^m e^{-tP^2}]_{(x,y)}| &\leq \mathcal{C} \sum_{j=m}^{\eta+n} t^{-j/2} \int_{R/c}^{\infty} \Big| H_j\Big(\frac{s}{(4t)^{1/2}}\Big)\Big| e^{-\Big(\frac{s}{(4t)^{1/2}}\Big)^2} (4t)^{-1/2} ds \qquad (40) \\
&\leq \mathcal{C} \sum_{j=m}^{\eta+m} t^{-j/2} \int_{R/2c\sqrt{t}}^{\infty} |H_j(x)| e^{-x^2} dx \\
&\leq \mathcal{C} \sum_{j=m}^{\eta+m} t^{-j/2} \int_{R/2c\sqrt{t}}^{\infty} (1 + x^j) e^{-x^2} dx \\
&\leq \mathcal{C} e^{-R^2/5c^2 t} \sum_{j=m}^{\lambda+m} t^{-j/2} \\
&\leq \begin{cases} \mathcal{C}(k,l,m,P) t^{-m/2} e^{-R^2/6c^2 t}, & t > T \\ \mathcal{C}(k,l,m,P) e^{R^2/6c^2 t}, & d(x,y) > 2r_1 \end{cases} \quad t \in \mathbb{R}^+.
\end{aligned}
$$

There's also a relative version of Proposition 6.48 in which two differential, formally self–adjoint uniformly elliptic operators P_1 and P_2 are considered. More precisely relative means that P_1 acts on $E_1 \longrightarrow X_1$ and P_2 acts on $E_2 \longrightarrow X_2$ with open sets $U_1 \subset X_1$, $U_2 \subset X_2$ and isometries φ, Φ

$$
\begin{array}{ccc}
E_{1|U_1} & \xrightarrow{\Phi} & E_{2|U_2} \\
\downarrow & & \downarrow \\
U_1 & \xrightarrow{\varphi} & U_2
\end{array}
$$

making possible to identify P_1 with P_2 upon $U = U_1 = U_2$ i.e.

$$\Phi(P_1 s) = P_2(\Phi s), \quad s \in C_c^\infty(U_1; E_1)$$

where Φ is again used to denote the mapping induced on sections

$$\Phi : C_c^\infty(U_1; E_1) \longrightarrow C_c^\infty(U_2; E_2),\ (\Phi s)(y) := \Phi_{\varphi^{-1}(y)} s(\varphi^{-1}(y)).$$

Thanks to the identification one calls $P = P_1 = P_2$ over U. Then the relative version of the estimate (39) is contained in the following proposition.

PROPOSITION 6.49 — Choose $r_2 > 0$ and let x, y be in U. Set

$$Q(x,y) := \max\{\min\{d(x,\partial U); d(y,\partial U)\} - r_2; 0\},\ J(x,y) := \Big(\frac{-Q(x,y)}{c}, \frac{Q(x,y)}{c}\Big).$$

For $f \in \mathcal{S}(\mathbb{R})$,

$$|\nabla_x^l \nabla_y^k ([f(P_1)] - [f(P_2)])_{(x,y)}| \leq \mathcal{C}(P_1, k, l, r_2) \sum_{j=0}^{2\bar{n}+l+k} \int_{\mathbb{R} - J(x,y)} |\hat{f}^{(j)}(s)| ds.$$

More precisely the reason of the dependence of the constant only to P_1 is that it depends upon $P_{1|U}$ where the operators coincide.

PROOF — This is very similar to the proof of 37. Choose $x_0, y_0 \in U$ then

$$\begin{aligned}|\nabla_x^l \nabla_y^k([f(P_1)] - [f(P_2)])_{(x_0,y_0)}| &\leq C\|\nabla_x^l([f(P_1)] - [f(P_2)]_{(x_0,\bullet)}\|_{H^{\bar{n}+k}(B(y_0,r_2/3))} \\ &\leq C\sum_{j=0}^{\bar{n}+k}\|\nabla_x^l([P_1^j f(P_1)] - [P_2^j f(P_2)])_{(x_0,\bullet)}\|_{L^2(B(y_0,r_2/2))}.\end{aligned} \tag{41}$$

Where the first step is Sobolev embedding $H^{\bar{n}+k} \longrightarrow UC^k$, again no problem in reducing the Sobolev norm to be computed on the ball $B(y_0, r_2/3)$ in fact one can suppose r_2 is smaller than the injectivity radius and build a cut off function χ. The Sobolev embedding is applied then to the section $\chi\nabla_x^k[f(P_1) - f(P_2)]$ and the resulting constant C will be depending also on χ but uniform geometry assumption makes χ universal in that can be used on each normal coordinate. For example in order one the argument one applies is

$$\|\nabla_y \chi t\|_{H^1} \leq \|(\nabla_y\chi)t\|_{L^2} + \|\chi\nabla_y t\|_{L^2} \leq D(\chi,1)\|t\|_{H^1(B(y_0,r_2/3))}$$

if χ is supported in $B(y_0, r_2/3)$.

The second step is Gårding inequality of P_1 and P_2 together with the fact that they coincide on U_1. The same argument with a cut off function χ_2 also works well with Gårding inequality. Let $\xi_j(y) := \chi_{B(y_0,r_2/2)}(y)\nabla_x^l\{[P_1^j f(P_1)]_{(x_0,y)} - [P_2^j f(P_2)]_{(x_0,y)}\}$ then

$$\begin{aligned}\|\xi_j\|^2_{L^2(B(y_0,r_2/2))} =& |(\nabla_x^l(P_1^j f(P_1) - P_2^j f(P_2))\xi_j)(x_0)| \\ &\leq C\|P_1^j f(P_1) - P_2^j f(P_2)\xi_j\|_{H^{\bar{n}+l}(B(x_0,r_2/3))} \\ &\leq C\sum_{i=0}^{\bar{n}+l}\|P_1^{j+i}f(P_1) - P_2^{j+i}f(P_2)\|_{L^2(B(x_0,r_2/2))} \\ &\leq C\|\xi_j\|_{L^2(U)}\sum_{i=0}^{\bar{n}+l}\int_{\mathbb{R}-J(x,y)}|\hat{f}^i(s)|ds\end{aligned} \tag{42}$$

in fact for a class Schwartz function g,

$$\begin{aligned}\|(g(P_1) - g(P_2))\xi_j\|_{L^2(B(x_0,r_2/2))} =& \left\|(2\pi)^{-1/2}\int_{\mathbb{R}}\hat{g}(s)(e^{isP_1} - e^{isP_2})\xi_j ds\right\|_{L^2(B(x_0,r_2/2))} \\ =& \left\|(2\pi)^{-1/2}\int_{\mathbb{R}-J(x_0,y_0)}\hat{g}(s)(e^{isP_1} - e^{isP_2})\xi_j ds\right\|_{L^2(B(x_0,r_2/2))}\end{aligned}$$

since $\mathrm{supp}(e^{iP_i s}\xi_j) \subset B(y_0, r_2/2 + c|s|)$ then $e^{isP_1\xi_j}$ and $e^{isP_2\xi_j}$ remain supported in U then $e^{isP_1}\xi_j = e^{isP_2}\xi_j$ by the uniqueness of the solution of the Cauchy problem for the wave equation. □

PROPOSITION 6.50 — The relative version of (40) is

$$|\nabla_x^l\nabla_y^k([P_1^m e^{-tP_1^2}] - [P_2^m e^{-tP_2^2}])_{(x,y)}| \leq \left\{\begin{array}{c}\mathcal{C}(k,l,m,P_1)t^{-m/2}e^{-Q(x,y)^2/6c^2t},\ t > T \\ \mathcal{C}(k,l,m,P_1)e^{-Q(x,y)^2/6c^2t},\end{array}\right. .$$

for $x, y \in U$, $d(x,\partial U)$, $d(y,\partial U) > r_2$ and $t \in \mathbb{R}^+$.

6.2 The cylindrical case

In this section our manifold L will be the generic leaf of the foliation i.e. start with a manifold with bounded geometry L_0 with boundary ∂L_0 composed of possibly infinite components and a product type Riemannian metric near the boundary. Glue an infinite cylinder $Z_0 = \partial L_0 \times [0, \infty)$ with product metric and denote $L := L_0 \cup_{\partial L_0} Z_0$. Let $E \longrightarrow L$ an Hermitian Clifford bundle. Every notation of section 2 is keeped on with the slight abuse that Z_0 is the cylinder here and in X. Recall that $E_{|Z_0} = F \oplus F$.

DEFINITION 6.51 — We say that a first order uniformly elliptic (formally) selfadjoint operator $T \in \mathrm{Op}^1(L; E)$ has product structure if

1. T restricts to L_0 and Z_0 i.e. $\mathrm{supp}(Ts) \subset L_0(Z_0)$ if s is supported on L_0 (Z_0).
2. $T_{|L_0}$ is a uniformly elliptic differential operator.
3. T restricts to the cylinder to have the form

$$T_{|Z_0} = c(\partial_r)\partial_r + \Omega B(r) = \begin{pmatrix} 0 & B(r) - \partial_r \\ B(r) + \partial(r) & 0 \end{pmatrix}$$

for a smooth[10] mapping $B : \mathbb{R}^+ \longrightarrow \mathrm{Op}^1(\partial L_0; E)$ with values on the subspace of uniformly elliptic and selfadjoint operators. Furthermore suppose that $B(r) \cong B$ is constant for $r \geq 2$.

However this is only a model embracing our Breuer–Fredholm perturbation of the Dirac operator in fact

$$(D_{\epsilon,u,x})_{|\partial_x \times \mathbb{R}^+} = c(\partial_r)\partial_r + \Omega \underbrace{(\dot{\theta}u - \dot{\theta}D^{\mathcal{F}_\partial}\Pi_\epsilon + D^{\mathcal{F}_\partial})}_{B(r)}. \tag{43}$$

In this sense every result from here to the end of the section has to be thought applied to $D_{\epsilon,u}$.
Again the spectral theorem shows that for a compactly supported section $\xi_0 \in C_c^\infty(L; E)$ there is a unique solution $t \mapsto \xi(t)$ of the Cauchy problem (35) for the wave equation associated with T. This solution is given by the application of the wave one parameter group e^{itT} with the same properties written above in the standard case.

PROPOSITION 6.52 — Cylindrical finite propagation speed. Let $U = \partial L_0 \times (a, b)$ $0 < a < b$ and $B(U, l) = \{x \in L : d(x, U) < l\}$. For $\xi_0 \in C_c^\infty(L; E)$ let $\xi(t) = e^{itT}\xi_0$ the solution of the wave equation. If $\alpha < a$ the function $\|\xi(t)\|_{L^2(B(U,\alpha - t))}$ is not increasing in t. In particular

$$\mathrm{supp}(\xi_0) \subset U \Longmapsto \mathrm{supp}(\xi(t)) \subset B(U, t).$$

PROOF — The product structure of the operator makes us possible to repeat the standard

[10] Some words about the smoothness condition on the mapping B. Here we shall make use only of pseudodifferential operators with uniformly bounded symbols, (almost everywhere they will be smoothing operators) hence the smoothness condition of the family is the usual one. In particular this is the smoothness of the family of operators acting on the fibers of $\partial L_0 \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $B(t) \in \mathrm{Op}^1(\partial L_0 \times \{t\}; E)$. If U is a coordinate set for ∂L_0 such a family is determined by a smooth mapping $p : \mathbb{R}^+ \longrightarrow \mathrm{S}^1_{\mathrm{hom}}(U)$ in the space of polihomogeneous symbols. Here smooth means that each derivative $t \longmapsto d^k\sigma/dt^k$ is continuous as a mapping with values in the space of symbols (with the symbols topology, see [77])

proof of the energy estimates and finite propagation speed that can be found in John Roe's book [65]. So let us consider

$$\begin{aligned}\frac{d}{dt}\|\xi(t)\|^2{}_{L^2(B(U,\alpha-t))} &= \frac{d}{dt}\int_{B(U,\alpha-t)} |\xi(t)|^2(z)dz \\ &\leq \left|\int_{B(U,\alpha-t)} \Big(\langle \xi(t), iT\xi(t)\rangle + \langle iT\xi(t), \xi(t)\rangle\Big)(z)dz\right| - \int_{\partial B(U,\alpha-t)} |\xi(t)|^2(z)dz.\end{aligned}$$

Since the operator T has product structure, the integration domain is a product and the operator $B(t)$ is selfadjoint on the base

$$\int_{B(U,\alpha-t)} \langle \xi(t), iT\xi(t)\rangle + \langle iT\xi(t), \xi(t)\rangle dz = \int_{B(U,\alpha-t)} \langle \xi(t), ic(\partial_r)\partial_r\xi(t)\rangle + \langle ic(\partial_r)\partial_r\xi(t), \xi(t)\rangle dz.$$

Here the fact that the function

$$r \longmapsto \int_{\partial L_0} \Big(\langle i\Omega B(r)\xi(t)_{|\partial L_0\times\{r\}}, \xi(t)_{|\partial L_0\times\{r\}}\rangle + \langle \xi(t)_{|\partial L_0\times\{r\}}, i\Omega B(r)\xi(t)_{|\partial L_0\times\{r\}}\rangle\Big)(x)dx$$

is identically zero by the self–adjointness of B has been used. Note that $\xi(t)_{|\partial L_0\times\{r\}}$ is in the domain of $B(r)$ by the Stone theorem (however it is certainly true for operators in the form of our perturbation (43)). Finally

$$\begin{aligned}\frac{d}{dt}\|\xi(t)\|^2_{L^2(B(U,\alpha-t))} &\leq \left|\int_{B(U,\alpha-t)} \Big(\langle \xi(t), ic(\partial_r)\partial_r\xi(t)\rangle + \langle ic(\partial_r)\partial_r\xi(t), ic(\partial_r)\partial_r\xi(t)\rangle(z)\Big)dz\right| \\ - \int_{\partial B(U,\alpha-t)} |\xi(t)|^2(z) &= \left|\int_{B(U,\alpha-t)} \partial_r\langle \xi(t), c(\partial_r)\xi(t)\rangle(z)dz\right| - \int_{\partial B(U,\alpha-t)} |\xi(t)|^2(z) \leq 0\end{aligned}$$

□

As a notation for a subset $H \in L$ and $t \geq 0$ put $H * t := B(H,t) \cup \partial L_0 \times (\alpha - t, \beta + t)$ where $\alpha := \inf\{r(z) : z \in H \cap Z_0\}$ and $\beta := \max\{r(z) : z \in H \cap Z_0\}$ in other words $H * t$ is the set of points at distance t from H in the cylindrical direction.
It is clear from (6.52) that the support of the solution of the wave problem satisfies

$$\mathrm{supp}(e^{itT}\xi) \subset \mathrm{supp}(\xi) * |t|.$$

Then the cylindrical basic Cheeger–Gromov–Taylor estimate similar to (37) is obtained in the following way:
first note that proposition 6.52 is certainly true if the propagation speed is c, for a section ξ supported into a ball $B(x, r_0)$ and $f \in \mathcal{S}(\mathbb{R})$ let $I_R := (-(R-r_0)/c, (R-r_0)/c)$ if $R > r_0$ and $I_R = \emptyset$ if $r \leq R$ then,

$$\begin{aligned}\|f(P)\xi\|_{L^2(L-B(x,r_0)*R)} &= \left\|(2\pi)^{-1/2}\int_{\mathbb{R}} \hat{f}(s)e^{isP}\xi ds\right\|_{L^2(L-B(x,r_0)*R)} \qquad (44)\\ &\leq \left\|(2\pi)^{-1/2}\int_{\mathbb{R}-I_R} \hat{f}(s)e^{isP}\xi ds\right\|_{L^2(L)} \\ &\leq (2\pi)^{-1/2}\|\xi\|_{L^2(L)}\int_{\mathbb{R}-I_R} |\hat{f}|ds, \qquad (45)\end{aligned}$$

since $\mathrm{supp}\, e^{isP}\xi \cap (L - B * R) = \emptyset$ for $|t| < (R - r_0)/c$.

PROPOSITION 6.53 — Choose two points on the cylinder $z_1 = (x_1, s_1)$ and $z_2 = (x_2, s_2)$ with $s_i > r_1$, $|s_1 - s_2| > 2r_1$, put $I(z_1, z_2) := \Big(-\frac{|s_1 - s_2| + r_1}{c}, \frac{|s_1 - s_2| - r_1}{c} \Big)$ then for $f \in \mathcal{S}(\mathbb{R})$,

$$|\nabla^l_{z_1} \nabla^k_{z_2} [f(P)]_{(z_1.z_2)}| \le \mathcal{C}(P, l, k) \sum_{j=0}^{2\bar{n}+l+k} \int_{R - I(z_1, z_2)} |\hat{f}(s)^{(j)}| ds$$

with $\bar{n} := [n/2 + 1]$

PROOF —

Imitate the proof of 6.48 till the estimate

$$|\nabla^l_x \nabla^k_y [f(P)]_{(x,y)}| \le C \sum_{j=0}^{\bar{n}+k} \|\xi_j\|_{L^2(L)}$$

where $\xi_j := \chi_{B(y, r_1 = 2)} \nabla^l_x [P^j f(P)]_{(x, \bullet)}$ and $x, y \in L$.
There is a subtle point to concentrate, it is when one let P^j act on $[f(P)]_{(x,\bullet)}$. This is perfectly granted by the smoothing properties of $f(P)$ in fact, let the bundle be $L \times \mathbb{R}$ and identify distributions with functions through the Riemannian density. The operator $f(P)$ extends to and operator from compactly supported distributions to distributions (actually takes values on smooth functions). Consider the family of Dirac masses $\delta_y(\cdot)$ concentrated at y, first note that

$$[f(P)]_{(x,y)} = (f(P)\delta_y(\cdot))(x) \tag{46}$$

in fact by selfadjointness

$$\langle f(P)\delta_y, s\rangle = \langle \delta_y, f(P)s\rangle = \int [f(P)]_{(z,y)} t(z) dz,$$

that's to say (46). Now the Sobolev embedding theorem says that $\delta_y \in H^k(X)$ with $k < -n/2$ with norms uniformly (in y) bounded. Since $f(P)$ maps every Sobolev space into each other Sobolev space, every section $[f(P)]_{(x,\bullet)}$ (and the symmetric one by selfadjointness) is in the domain of P^j.
Again

$$\begin{aligned} \|\xi_j\|^2_{L^2(L)} &= \|\chi_{B(y, r_1/2)} \nabla^l_x [P^j f(P)]_{(x,\bullet)}\|^2_{L^2(B(y, r_1/2))} \\ &= |\nabla^l_x P^j f(P) \xi_j(x)| \le C \|P^j f(P) \xi_j\|_{H^{\bar{n}+l} B(x, r_1/3)} \\ &\le C \sum_{i=0}^{\bar{n}+l} \|P^j f(P) \xi_j\|_{L^2(B(x, r_1/2))}. \end{aligned} \tag{47}$$

It's time to move on the cylindrical end, so let $x = (x_2, s_2)$, $y = (x_1, s_1)$ with $s_i > r_1$ and $|s_1 - s_2| > 2r_1$, then last term in (47) can be estimated by

$$\sum_{i=0}^{\bar{n}+l} \|P^{j+i} f(P) \xi_j\|_{L^2(V)}$$

with $V = L - B(y, r_1/2) * c(|s_1 - s_2| - r_1)/2$ so we can conclude by application of (45). □

COROLLARY 6.54 — With the notations of the proposition above

1. If $|s_1 - s_2| > 2r_1,\ s_i > r_1$

$$|\nabla^l_{z_1}\nabla^k_{z_2}[P^m e^{-tP^2}]_{(z_1,z_2)}| \leq \mathcal{C}(k,l,m,P)e^{-\dfrac{(|s_1-s_2|-r_1)^2}{6t}} \tag{48}$$

2. Let ψ_1, ψ_2 compactly supported with supports at r–distance d on the cylinder, then for the operator norm and $t > 0$

$$\|\psi_1 P^m e^{-tP^2}\psi_2\| \leq \mathcal{C}(m,\psi_1,\psi_2)e^{-d^2/6t}. \tag{49}$$

3. The relative version of (48) is

$$|\nabla^l_{z_1}\nabla^k_{z_2}[P^m e^{-tP^2} - T^m e^{-tT^2}]_{(z_1,z_2)}| \leq \mathcal{C}(k,l,m,P)e^{\{-(\min\{s_1,s_2\}-r_2)^2/6t\}}. \tag{50}$$

Proof — The second statement follows immediately from the first one while the third can be proven exactly in the way proposition 86 is proven. □

7 The eta invariant

7.1 The classical eta invariant

The eta invariant of Atiyah Patodi and Singer appears for the first time in the following theorem that we write in the cylindrical case.

Theorem 7.54 — Let X a compact manifold with boundary Y and product type metric on a collar $Y \times [0,1]$, attach an infinite cylinder $Y \times [-\infty, 0]$ to get the elongated manifold $\hat{X} := X \cup Y \times [-\infty, 0]$. Let $D : C^\infty(X;E) \longrightarrow C^\infty(X;F)$ a first order differential elliptic operator with product structure near the boundary i.e.

$$D = \sigma(\partial_u + A)$$

where $\sigma E_{|Y} \longrightarrow F_{|Y}E$ is a bundle isomorphism, ∂_u is the normal interior coordinate and A is the boundary self–adjoint elliptic operator. Then the operator D extends to sections of the bundles extended to $\hat{X}$ and has a finite L^2 index i.e the space of L^2 solutions of the equations $Ds = 0$ and $D^*s = 0$ and

$$\mathrm{ind}(D) = \dim_{L^2(\hat{X},E)}(D) - \dim_{L^2(\hat{X},E)}(D^*) = \int_X \alpha_0(x)dx - \eta(0)/2 - \frac{h_\infty(E) - h_\infty(F)}{2}$$

where

1. $h_\infty(E)$ is the dimension of the space of limiting values of the extended L^2 solutions. More precisely one says that s is an L^2 extended solution of the equation $Ds = 0$ with limiting value s_∞ if s is locally square integrable and for large $u < 0$

$$s(y,u) = g(y,u) + s_\infty(y),\ s_\infty(y) \in \mathrm{Ker}(A).$$

Analog definition for $h_\infty(F)$.

2. $\alpha_0(x)$ is the constant term in the asymptotic expansion as $t \to 0$ of

$$\sum e^{-t\mu'}|\phi'_\mu(x)|^2 - \sum e^{-t\mu''}|\phi''_\mu(x)|^2 \tag{51}$$

where μ', ϕ'_μ are the eigenvalues and eigenfunctions of D^*D on the double of X and μ'', ϕ''_μ are the corresponding objects for DD^*.

3. The number $\eta(0)$, is called the spectral asymmetry or the eta invariant of A is obtained as follows:

the summation on the non negative eigenvalues of A,

$$\eta(s) := \sum_{\lambda \neq 0} \mathrm{sign}(\lambda)|\lambda|^{-s}$$

converges absolutely for $\mathrm{Re}(s) >> 0$ extends to a meromorphic function on the whole s–plane with regular value at $s = 0$. Moreover if the asymptotic expansion at (51) has no negative powers of t then $\eta(s)$ is holomorphic for $\mathrm{Re}(s) > -1/2$. That's the case of the Dirac operator of a Riemannian manifold.

7.2 The foliation case

The existence of the eta invariant for the leafwise Dirac operator on a closed foliated manifold was shown by Peric [58] and Ramachandran [62]. In fact they build different invariants, Peric works with the holonomy groupoid of the foliation and Ramachandran with the equivalence relation but the methods are essentially the same. So consider a compact manifold Y with a foliation and a longitudinal Dirac structure i.e. every geometrical structure needed to form a longitudinal Dirac–type operator acting on the tangentially smooth sections of the bundle S, $D : C^\infty_\tau(Y;S) \longrightarrow (Y;S)$. In our index formula Y will be a transverse section of the cylinder sufficiently far from the compact piece and D is the operator at infinity. Suppose also a transverse holonomy invariant measure Λ is fixed.
Here the first issue to solve is to pass to the summation $\eta(s) = \sum_\lambda \mathrm{sign}(\lambda)|\lambda|^{-s}$ which deals with the discrete spectrum to a continuous spectrum and family version. The link is offered by the definition of Euler gamma function

$$\mathrm{sign}(\lambda)|\lambda|^{-s} = \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty t^{\frac{s-1}{2}}\lambda e^{-t\lambda^2}dt.$$

Each bounded spectral function of D belongs to the Von Neumann algebra of the foliation arising from the regular representation of the equivalence relation on the Borel field of L^2 spaces of sections of S. Replace the summation by integration w.r.t. the spectral measure of D (definition 4.30) and (formally) change the integration to define the eta function of D as

$$\eta_\Lambda(D;s) := \int_{-\infty}^\infty \mathrm{sign}(\lambda)|\lambda|^{-s}d\mu_D(\lambda) = \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty t^{\frac{s-1}{2}}\,\mathrm{tr}_\Lambda(De^{-tD^2})dt. \tag{52}$$

We shall use also the notation

$$\eta_\Lambda(D;s)_k := \int_k^\infty t^{\frac{s-1}{2}}\,\mathrm{tr}_\Lambda(De^{-tD^2})dt, \quad \eta_\Lambda(D;s)^k := \int_0^k t^{\frac{s-1}{2}}\,\mathrm{tr}_\Lambda(De^{-tD^2})dt$$

THEOREM 7.54 — (Ramachandran) The eta function (52) is a well defined meromorphic function for $\mathrm{Re}(s) \leq 0$ with eventually simple poles at $(\dim \mathcal{F} - k)/2$, $k = 0, 1, 2, \ldots.$, $\eta_\Lambda(D; s)$ is regular at 0 and its value $\eta_\Lambda(D; 0)$ is called the foliated eta invariant of D.

PROOF — Here a trace of the proof.

First step. For every $s \in \mathbb{C}$ with $\mathrm{Re}(s) \leq 0$ the integral

$$\int_1^\infty t^{\frac{s-1}{2}} \operatorname{tr}_\Lambda(De^{-tD^2})dt \tag{53}$$

is convergent then in some sense the most important piece of the eta function is the integral $\int_0^1$.

This is reminescent of the remark in the paper of Atiyah Patodi and Singer [4] where they define the function $K(t)$ to be the integral on the cylinder of the difference of the heat kernels $e^{-t\Delta_1} - e^{-t\Delta_2}$ of D and D^*,

$$K(t) = \int_0^\infty \int_{\partial Y} K(t, y, u)dydu = -\sum_\lambda \operatorname{sign}(\lambda)/2 \operatorname{erf}(|\lambda|\sqrt{t}) \sim_{t\to 0} \sum_{k \geq -n} a_k t^{k/2}$$

where ∂Y is the boundary manifold of dimension n. The remark they do is that the asymptotic expansion is the same replacing the integral with an integral on $\int_{[0,\delta]}$.

The convergence of (53) is proven by simple estimates and the use of the spectral measure. In particular here, by compactness the spectral measure $\mu_{\Lambda,D}$ is tempered i.e.

$$\int \frac{1}{(1 + |x|^l)} d\mu_{\Lambda,D} < \infty$$

for some positive l. In fact this measure corresponds to a positive functional [62]

$$I : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{R},\ I(f) := \operatorname{tr}_\Lambda(f(D)).$$

The same is obviously true for the square $D^2 = |D|^2$. Here the estimate. Start with $|t^{(s-1)/2}| \leq t^{(\mathrm{Re(s)}-1)/2} \leq t^{-1/2}$, $0 \leq t \leq \infty$ then

$$\begin{aligned}\int_1^\infty |t^{(s-1)/2} \operatorname{tr}_\Lambda(De^{-tD^2})dt| &\leq \int_1^\infty |t^{-1/2}| \operatorname{tr}_\Lambda(De^{-tD^2})dt| \\ &\leq \int_1^\infty t^{(s-1)/2} \operatorname{tr}_\Lambda(|D|e^{-tD^2})dt.\end{aligned}$$

The last integral is equal to

$$\int_1^\infty t^{-1/2}dt \int_0^\infty \lambda^{1/2} e^{-t\lambda} d\mu_{D^2}(\lambda)$$

hence

$$\begin{aligned}\int_0^\infty \lambda^{1/2} d\mu_{D^2}(\lambda) \int_1^\infty t^{-1/2} e^{-t\lambda} dt &= \int_0^\infty \lambda^{1/2} e^{-\lambda} d\mu_{D^2}(\lambda) \int_1^\infty t^{-1/2} e^{-\lambda(t-1)} dt \qquad (54) \\ &= \int_0^\infty e^{-\lambda} d\mu_{D^2}(\lambda) \int_0^\infty (u + \lambda)^{-1/2} e^{-u} du \\ &\leq \int_0^\infty e^{-\lambda} d\mu_{D^2}(\lambda) \int_0^\infty u^{-1/2} e^{-u} du \\ &= \pi^{1/2} \int_0^\infty e^{-\lambda} d\mu_{D^2}(\lambda) = \pi^{1/2} \operatorname{tr}_\Lambda(e^{-D^2}) < \infty\end{aligned}$$

Second step. The examination of the finite piece

$$\int_0^1 t^{\frac{s-1}{2}} \operatorname{tr}_\Lambda(De^{-tD^2})dt \tag{55}$$

is done using the expansion of the Schwartz kernel of the leafwise operator De^{-tD^2} in fact one can prove that there exists a family of tangentially smooth and locally computable functions $\{\Psi_m\}_{m\geq 0}$ [11] so that the kernel $K_t(x,y,n)$ (n the transverse parameter) of the leafwise bounded operator De^{-tD^2} has the asymptotic expansion

$$K_t(x,x,n) \sim \sum_{m\geq 0} t^{(m-\dim\mathcal{F}-1)/2}\Psi_m(x,n). \tag{56}$$

Moreover $\Psi_m = 0$ for m even. The proof is an adaptation of the classical situation, for example can be found in [66] and [24]. Now, thanks to the expansion (56), since the operator De^{-tD^2} is Λ trace class and the trace is the integral of the Schwartz kernel against the transverse measure we get the corresponding expansion for the trace

$$\frac{1}{\Gamma(\frac{s+1}{2})}\int_0^1 t^{\frac{s-1}{2}} \operatorname{tr}_\Lambda(De^{-tD^2})dt \sim \sum_{m\geq 0}\frac{2}{s+m-\dim\mathcal{F}}\int_Y \Psi_m d\lambda \tag{57}$$

where $\int \Psi_m d\lambda = \Lambda(\Psi_m dg)$ i.e. is the effect of the integration of the tangential measures $x \longmapsto \Psi_{m|l_x} \times dg_{|l_x}$. From (57) we see that the eta function has a meromorphic continuation to the whole plane with simple (at most) poles at $(\dim\mathcal{F}-k)/2,\ k=0,1,2,....$

Third step, regularity at the origin.

If $P=\dim\mathcal{F}$ is even we have said that the coefficients Ψ_m of the development (56) are zero for m even, then the eta function is regular at 0. If p is odd the regularity at zero follows from a very deep result of Bismut and Fried [12]. In fact they showed that the ordinary Dirac operator satisfies a remarkable cancellation property,

$$\operatorname{tr}(De^{-tD^2}) = O(t^{1/2}).$$

Since the Λ–trace can be, as pointed out by Connes ([24]), locally approximated by the regular trace their result applies to our setting to give

$$K_t(x,x,n) \sim \sum_{m\geq p+2} \underbrace{t^{(m-p-1)/2}\Psi(x,n)}_{\text{almost everywhere}},$$

and the regularity at the origin follows immediately.

□

7.3 Eta invariant for perturbations of the Dirac operator

Let Let us consider slightly more general operators

1. $P = D + K$ where $K \in \mathrm{Op}^{-\infty}$ is leafwise uniformly smoothing obtained by functional calculus $K = f(D)$ where f is bounded Borel function supported in $(-a,a)$.

[11] in the case of the holonomy groupoid the Ψ_m are locally bounded i.e. bounded on every set in the form of $r^{-1}K$ for K compact in Y

Start with the computation

$$Qe^{-tQ^2} - De^{-tD^2} = De^{-t(D+K)^2} + Ke^{-t(D+K)^2} - De^{-tD^2} \tag{58}$$
$$= D\int_0^1 \partial_s e^{-s(D+K)^2-(t-s)D^2} dt + Ke^{-t(D+K)^2}$$
$$= Ke^{-t(D+K)^2} - D\int_0^1 e^{-s(D+K)^2}(KD+DK+K^2)e^{(t-s)D^2} ds.$$

The family (58) converges to 0 as $t \to 0$ in the Frechet topology of kernels in $\mathrm{Op}^{-\infty}$ with uniform transverse control i.e. for kernels $K(x,y,n)$ (n is the transverse parameter) one uses foliated charts to define seminorms that involve derivatives w.r.t. x, y. From (58) one gets the development

$$\mathrm{tr}_\Lambda(Qe^{-tQ^2}) \sim_{t\to 0} \sum_{m=0} t^{\frac{m-\dim\mathcal{F}-1}{2}} \int_Y \Psi_j d\Lambda + \mathrm{tr}_\Lambda(K) + g(t)$$

where $g \in C[0,\infty)$ with $g(0)=0$. Then an asymptotic development for $\eta_\Lambda(Q)(0)_1$ as (57) follows. For the non finite integral $\eta_\Lambda(Q,0)^1$ no problem in carrying on the estimate (54).

2. The smooth family $u \longmapsto Q_u := D+K+u$. The function $\mathrm{tr}_\Lambda(Q_u e^{-tQ_u^2})$ is smooth then

$$\partial_u \,\mathrm{tr}_\Lambda(Q_u e^{-tQ_u^2}) = \mathrm{tr}_\Lambda(Q_u^{'} e^{-tQ_u^2} - tQ_u(Q_u^{'}Q_u + Q_uQ_u^{'})e^{-tQ_u^2}) \tag{59}$$
$$= (1+2t\partial_t)\,\mathrm{tr}_\Lambda(Q_u^{'}e^{-tQ_u^2})$$

in fact $Q_u^{'} = \mathrm{I}$. By integration

$$\partial_u \eta_\Lambda(Q_u,s)_1 = \partial_u \int_0^1 \frac{t^{(s-1)/2}}{\Gamma(\frac{s+1}{2})}\,\mathrm{tr}_\Lambda(Q_u e^{-tQ_u^2})dt = \int_0^1 \frac{t^{(s-1)/2}}{\Gamma(\frac{s+1}{2})}(1+2t\partial_t)\,\mathrm{tr}_\Lambda(Q_u^{'}e^{-tQ_u^2})dt$$
$$= \int_0^1 \frac{t^{(s-1)/2}}{\Gamma(\frac{s+1}{2})}\,\mathrm{tr}_\Lambda(Q_u^{'}e^{-Q_u^2}) - \frac{s}{\Gamma(\frac{s+1}{2})}\int_0^1 t^{(s-1)/2}\,\mathrm{tr}_\Lambda(Q_u^{'}e^{-tQ_u^2})dt. \tag{60}$$

Now, from $Q_u^{'} = \mathrm{I}$ proceed as before using the asymptotic development of the heat kernel for $D+u$ [12]

$$\mathrm{tr}_\Lambda(Q_u^{'}e^{-tQ_u^2} = \mathrm{tr}_\Lambda(Q_u^{'}e^{-tQ_u^2} \sim \sum_{m\geq 0} a_m(D+u)t^{(m-\dim\mathcal{F})/2} + g(t)$$

where $g \in C[0,\infty)$, $g(0)=0$. We see that the integral in (60) admits a meromorphic expansion around zero in $\mathbb{C}$ with zero as a pole of almost first order. Then the derivative $\partial_u\eta_\Lambda(Q_u,s)_1$ is holomorphic around zero. The identity

$$\partial_u \,\mathrm{Res}_{|s=0}\,\eta_\Lambda(Q_u,s)_1 = \mathrm{Res}_{|s=0}\,\partial_u\eta_\Lambda(Q_u,s)_1 = 0$$

says that $\mathrm{Res}_{|s=0}\,\eta_\Lambda(Q_u,s)_1$ is constant in u then the function $\eta_\Lambda(Q_u,s)_1$ is holomorphic at zero since $\eta_\Lambda(Q_0,s)_1$ is holomorphic in 0.

3. Families in the form $Q_u = D+u+\Pi D$ for a spectral projection $\Pi = \chi_{(-a,a)}(D)$.

[12] $(D+u)^2$ is a generalized Laplacian

Proposition 7.55 — The eta invariant for Q_u exists and satisfies

$$\eta_\Lambda(Q_u) = \mathrm{LIM}_{\delta\to 0}\int_\delta^1 \frac{t^{-1/2}}{\gamma(1/2)}\,\mathrm{tr}_\Lambda(Q_u e^{-tQ_u^2})dt + \int_1^\infty \frac{t^{-1/2}}{\gamma(1/2)}\,\mathrm{tr}_\Lambda(Q_u e^{-tQ_u^2})dt$$

where LIM is the constant term in the asimptotic development in powers of δ as $t\to 0$. Moreover for every $u\in\mathbb{R}$ and $a>0$,

a. $\eta_\Lambda(Q_u) - \eta_\Lambda(Q_0) = \mathrm{sign}(u)\,\mathrm{tr}_\Lambda(\Pi)$

b. $\eta_\Lambda(Q_0) = 1/2\eta_\Lambda(Q_u) + 1/2\eta_\Lambda(Q_{-u})$

c. $|\eta_\Lambda(D) - \eta_\Lambda(Q_0)| = |\eta_\Lambda(\Pi D)| \le \mu_{\Lambda,D}((-a,a))$.

Proof — The first statement can be proved as above. a. using the spectral measure we have to compute the difference

$$\int_0^\infty t^{-1/2}\int_{\mathbb{R}}(x+u-\chi x)e^{-t(x+u-\chi x)^2}d\mu_{\Lambda,D}(x)\frac{dt}{\Gamma(1/2)} \\ -\int_0^\infty t^{-1/2}\int_{\mathbb{R}}(x-\chi x)e^{-t(x-\chi x)^2}d\mu_{\Lambda,D}(x)\frac{dt}{\Gamma(1/2)}$$

where $\chi=\chi_{(-a,a)}(x)$. Split the integral on $\mathbb{R}$ into two pieces, $|x|>a$ and $|x|\le a$.
First case $|x|>a$ changing the integration order the first integral is

$$\Gamma(1/2)^{-1}\int_{|x|>a}\int_0^\infty (x+u)t^{-1/2}e^{-t(x+u)^2}dt d\mu_{\Lambda,D}(x)$$

and performing the substitution $\sigma := t(x+u)^2$ in the second we see that the difference is zero.
Second case $|x|<a$, the second integral is zero, the first

$$\begin{aligned}&\int_0^\infty t^{-1/2}\int_{-a}^{a} ue^{-tu^2}d\mu_{\Lambda,D}(x)\frac{dt}{\Gamma(1/2)} = \int_0^\infty t^{-1/2}ue^{-tu^2}\frac{dt}{\Gamma(1/2)}\,\mathrm{tr}_\Lambda(\Pi)\\ &= \underbrace{\int_0^\infty u|u|\sigma^{-1/2}e^{-\sigma}\frac{d\sigma}{|u|^2}}_{tu^2=\sigma}\frac{\mathrm{tr}_\Lambda(\Pi)}{\Gamma(1/2)} = \mathrm{sign}(u)\frac{\mathrm{tr}_\Lambda(\Pi)}{\Gamma(1/2)}\int_0^\infty \sigma^{-1/2}e^{-\sigma^2}d\sigma \qquad (61)\\ &= \mathrm{sign}(u)\,\mathrm{tr}_\Lambda(\Pi).\end{aligned}$$

b. and c. follows easily from a. □

8 The index formula

First we introduce the supertrace notation. Since the bundle $E=E^+\oplus E^-$ is $\mathbb{Z}_2$–graded, there is a canonical Random operator τ obtained by passing to the Λ–class of the family of involutions $\tau_x : L^2(L_x;E)\longrightarrow L^2(L_x;E)$ represented w.r.t. the splitting by matrices

$$\tau_x := \begin{pmatrix} \mathrm{Id}_{L^2(L_x;E^+)} & 0 \\ 0 & -\mathrm{Id}_{L^2(L_x;E^-)}\end{pmatrix}.$$

Definition 8.56 — The Λ–supertrace of $B \in \mathrm{End}_\Lambda(E)$ is $\mathrm{str}_\Lambda(B) := \mathrm{tr}_\Lambda(\tau B)$.

Now according to proposition 5.38 for $0 < |u| < \epsilon$ the perturbed operator $D_{\epsilon,u}$ is Λ–Breuer–Fredholm. Consider the heat operator $e^{-tD^2_{\epsilon,u,x}}$ on the leaf L_x. This is a uniformly smoothing operator with a Schwartz kernel (remember that metric trivializes densities and $[\bullet]$ means Schwartz kernel)

$$[e^{-tD^2_{\epsilon,u,x}}] \in UC^\infty(L_x \times L_x; \mathrm{End}(E)).$$

It is a well know fact the convergence for $t \longrightarrow \infty$ in the Frechet space of UC^∞ sections of the heat kernel to the kernel of the projection on the L^2–Kernel,

$$\lim_{t\to\infty}[e^{-tD^2_{\epsilon,u,x}}] = [\chi_{\{0\}}(D_{\epsilon,u,x})].$$

This is explained in proposition A.16, page 89 and is a consequence of continuity of the functional calculus $RB(\mathbb{R}) \longrightarrow UC^\infty(\mathrm{End(E)})$ applied to the sequence of functions $e^{-t\lambda^2} \longrightarrow \chi_{\{0\}}$ in $RB(\mathbb{R})$. Choose cut–off functions $\phi_k \in C^\infty_c(X)$ such that $\phi_{k|X_k} = 1$, $\phi_{k|Z_{k+1}} = 0$. The measurable family of bounded operators $\{\phi_k e^{-tD^2_{\epsilon,u,x}}\phi_k\}_{x\in X}$ gives an intertwining operator $\phi_k e^{-tD^2_{\epsilon,u}}\phi_k \in \mathrm{End}_{\mathcal{R}}(L^2(E))$ hence a random operator $\phi_k e^{-tD^2_{\epsilon,u}}\phi_k \in \mathrm{End}_\Lambda(L^2(E))$.

Lemma 8.57 — The random operator $\phi_k e^{-tD^2_{\epsilon,u}}\phi_k \in \mathrm{End}_\Lambda(L^2(E))$ is Λ–trace class. The following formula (iterated limit) holds true

$$\mathrm{ind}_\Lambda(D^+_{\epsilon,u}) = \mathrm{str}_\Lambda(\chi_{\{0\}}(D_{\epsilon,u})) = \lim_{k\to\infty}\lim_{t\to\infty}\mathrm{str}_\Lambda(\phi_k e^{-tD^2_{\epsilon,u}}\phi_k). \tag{62}$$

Proof— For the first statement there's nothing to proof, it is essentially the closed foliated manifold case. The local traces define a tangential measure that are C^∞ in the leaves direction while Borel and uniformely bounded (by the uniform ellipticity of the operator) and we are integrating against the transverse measure on a compact set. More precisely we are evaluating the mass of a compact set through the measure Λ_h where h is the longitudinal measure that on the leaf L_x is given by

$$A \longmapsto \int_A \mathrm{str}_{\mathrm{End}(E)}[e^{-tD^2_{\epsilon,u}}]_{\mathrm{diag}}dg_{|L_x},$$

with $\mathrm{str}_{\mathrm{End}(E)}$ the pointwise supertrace defined on the space of sections of $\mathrm{End}(E) \to X$ by $(\mathrm{str}_{\mathrm{End}(E)}\,\gamma)(x) := \mathrm{tr}_{\mathrm{end}(E_x)}(\tau(x)\gamma(x))$.
The limit formula (62) is nothing that the Lebesgue dominated convergence theorem applied two times, first $\mathrm{str}_\Lambda(\chi_{\{0\}}(D_{\epsilon,u})) = \lim_{k\to\infty}\mathrm{str}_\Lambda(\phi_k\chi_{\{0\}}(D_{\epsilon,u})\phi_k)$ but for fixed k one finds $\mathrm{str}_\Lambda(\phi_k\chi_{\{0\}}(D_{\epsilon,u}))\phi_k) = \lim_{t\to\infty}\mathrm{str}_\Lambda(\phi_k e^{-tD^2_{\epsilon,u}}\phi_k)$. The possibility to apply the dominated convergence theorem is given again by the integration process in fact as written above every tangential measure has smooth density w.r.t to the Riemannian metric and convergence is within the Frechet topology of C^∞ functions.

$\square$

Now, Duhamel formula $d/dt\,\mathrm{str}_\Lambda(\phi_k e^{-tD^2_{\epsilon,u}}\phi_k) = -\mathrm{str}_\Lambda(\phi_k D^2_{\epsilon,u}e^{-tD^2_{\epsilon,u}}\phi_k)$ integrated between s and ∞ leads to the identity

$$\lim_{t\to\infty}\mathrm{str}_\Lambda(\phi_k e^{-tD^2_{\epsilon,u}}\phi_k) = \mathrm{str}_\Lambda(\phi_k e^{-sD^2_{\epsilon,u}}\phi_k) - \int_s^\infty \mathrm{str}_\Lambda(\phi_k D^2_{\epsilon,u}e^{-tD^2_{\epsilon,u}}\phi_k)dt.$$

Note that the right–hand side is independent from $s > 0$. Then

$$\mathrm{ind}_\Lambda(D^+_{\epsilon,u}) = \lim_{k\to\infty}\left[\mathrm{str}_\Lambda(\phi_k e^{-sD^2_{\epsilon,u}}\phi_k) - \int_s^\infty \mathrm{str}_\Lambda(\phi_k D^2_{\epsilon,u} e^{-tD^2_{\epsilon,u}}\phi_k)dt\right]. \tag{63}$$

Split the integral into

$$\int_s^\infty \mathrm{str}_\Lambda(\phi_k D^2_{\epsilon,u} e^{-tD^2_{\epsilon,u}}\phi_k)dt = \int_s^{\sqrt{k}} \mathrm{str}_\Lambda(\phi_k D^2_{\epsilon,u} e^{-tD^2_{\epsilon,u}}\phi_k)dt + \int_{\sqrt{k}}^\infty \mathrm{str}_\Lambda(\phi_k D^2_{\epsilon,u} e^{-tD^2_{\epsilon,u}}\phi_k)dt.$$

For future ease of reading make the following definitions

$$\alpha_0(k,s) = \mathrm{str}_\Lambda(\phi_k e^{-sD^2_{\epsilon,u}}\phi_k), \qquad \beta_0(k,s) = \textstyle\int_s^\infty \mathrm{str}_\Lambda(\phi_k D^2_{\epsilon,u} e^{-tD^2_{\epsilon,u}}\phi_k)dt$$

$$\beta_{01}(k,s) = \textstyle\int_s^{\sqrt{k}} \mathrm{str}_\Lambda(\phi_k D^2_{\epsilon,u} e^{-tD^2_{\epsilon,u}}\phi_k)dt, \qquad \beta_{02}(k,s) = \textstyle\int_{\sqrt{k}}^\infty \mathrm{str}_\Lambda(\phi_k D^2_{\epsilon,u} e^{-tD^2_{\epsilon,u}}\phi_k)dt$$

Then $\beta_0(k,s) = \beta_{01}(k,s) + \beta_{02}(k,s)$ and

$$\mathrm{ind}_\Lambda(D^+_{\epsilon,u}) = \lim_{k\to\infty}[\alpha_0(k,s) - \beta_0(k,s)] = [\alpha_0(k,s) - \beta_{01}(k,s) - \beta_{02}(k,s)]. \tag{64}$$

Let us start with β_{01}.

LEMMA 8.58 — Let $\eta_\Lambda(D^{\mathcal{F}_\partial}_{\epsilon,u})$ be the Ramachandran eta–invariant for the perturbed operator $D^{\mathcal{F}_\partial}_{\epsilon,u}$ on the foliation at the infinity. Then the following limit formula is true

$$\lim_{k\to\infty} \mathrm{LIM}_{s\to 0}\, \beta_{01}(k,s) = \lim_{k\to\infty} \mathrm{LIM}_{s\to 0} \int_s^{\sqrt{k}} \mathrm{str}_\Lambda(\phi_k D^2_{\epsilon,u} e^{-tD^2_{\epsilon,u}}\phi_k)ds, = 1/2\eta_\Lambda(D^{\mathcal{F}_\partial}_{\epsilon,u})$$

where as usual $\mathrm{LIM}_{s\to 0}\, g(s)$ is the constant term in the expansion of $g(s)$ in powers of s near zero.

PROOF — The integrand can be written as follows

$$\begin{aligned}
\mathrm{str}_\Lambda(\phi_k D^2_{\epsilon,u} e^{-tD^2_{\epsilon,u}}\phi_k) =& 1/2\,\mathrm{str}_\Lambda(\phi_k[D_{\epsilon,u}, D_{\epsilon,u} e^{-tD^2_{\epsilon,u}}]\phi_k) &(65)\\
=& 1/2\,\mathrm{str}_\Lambda([D_{\epsilon,u}, \phi_k D_{\epsilon,u} e^{-tD^2_{\epsilon,u}}\phi_k] - [D_{\epsilon,u}, \phi_k^2] D_{\epsilon,u} e^{-tD^2_{\epsilon,u}}) &(66)\\
=& 1/2\,\mathrm{str}_\Lambda(-[D_{\epsilon,u}, \phi_k^2] D_{\epsilon,u} e^{-tD^2_{\epsilon,u}})\\
=& -1/2\,\mathrm{str}_\Lambda(c(\partial_r)\partial_r(\phi_k^2) D_{\epsilon,u} e^{-tD^2_{\epsilon,u}}).
\end{aligned}$$

In the next we shall use the notation $[a,b] := ab - (-1)^{|a|\cdot|b|}ba$ for the Lie–superbracket[13] on the Lie–superalgebra of $\mathbb{C}$–linear endomorphisms of $L^2(X, E^+ \oplus E^-)$ while, when the standard bracket is needed we write $[a,b]_\circ := ab - ba$. notice that

$$[\alpha, ab] = [\alpha, a]b + (-1)^{|\alpha|\cdot|a|} a[\alpha, b].$$

Remember the definition of $D_{\epsilon,u}$, in the cylinder it can be written

$$D_{\epsilon,u} = D + \dot{\theta}\Omega(u - D^{\mathcal{F}_\partial}_{\epsilon,u}) = c(\partial_r)\partial_r + Q$$

[13] everything we say about super–algebras can be found in [9]

with the Clifford multiplication $c(\partial_r)=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and Q is $\mathbb{R}^+$–invariant in fact acts on the transverse section. The next identities are also useful

$$D_{\epsilon,u}=\begin{pmatrix} 0 & D^-_{\epsilon,u} \\ D^+_{\epsilon,u} & 0 \end{pmatrix},\quad e^{-tD^2_{\epsilon,u}}=\begin{pmatrix} e^{-tD^-_{\epsilon,u}D^+_{\epsilon,u}} & 0 \\ 0 & e^{-tD^+_{\epsilon,u}D^-_{\epsilon,u}} \end{pmatrix},$$

$$D^-_{\epsilon,u}e^{-tD^+_{\epsilon,u}D^-_{\epsilon,u}}=e^{-tD^-_{\epsilon,u}D^+_{\epsilon,u}}D^-_{\epsilon,u},\quad D^+_{\epsilon,u}e^{-tD^-_{\epsilon,u}D^+_{\epsilon,u}}=e^{-tD^+_{\epsilon,u}D^-_{\epsilon,u}}D^+_{\epsilon,u}.$$

These are nothing but a rephrasing of the identity

$$D_{\epsilon,u}e^{-tD^2_{\epsilon,u}}=e^{-tD^2_{\epsilon,u}}D_{\epsilon,u}$$

granted by the spectral theorem. Now it's time to use the Cheeger–Gromov–Taylor relative estimates. Consider the leafwise operator

$$S_{\epsilon,u}:=c(\partial_r)\partial_r+\Omega(u-D^{\mathcal{F}_\partial}_{\epsilon,u}) \tag{67}$$

on the infinite foliated cylinder (in both directions) $Y=\partial X_0\times\mathbb{R}$ with the product foliation $\mathcal{F}_\partial\times\mathbb{R}$. Choose some point $z_0=(x_0,r)$ on the cylinder. Estimate (50) says that we can compare the two kernels at the diagonal leaf by leaf for large r and this estimate is uniform on the leaves,

$$\|[D_{\epsilon,u,z_0}e^{-tD^2_{\epsilon,u,x_0}}]-[S_{\epsilon,u,z_0}e^{-tS^2_{\epsilon,u,z_0}}]\|_{(z,z)}\leq Ce^{-(r-r_2)^2/(6t)} \tag{68}$$

for $\underline{z=(x,r)\in L_{z_0}}$. From (68), since the derivatives of ϕ_k are supported on the cylindrical portion $Z_k^{k+1}=\partial X_0\times[k,k+1]$,

$$\int_s^{\sqrt{k}}|\operatorname{str}_\Lambda(c(\partial_r)\partial_r\phi_k^2D_{\epsilon,u}e^{-tD^2_{\epsilon,u}})-\operatorname{str}_\Lambda(c(\partial_r)\partial_r\phi_k^2S_{\epsilon,u}e^{-tS^2_{\epsilon,u}})|dt=\int_s^{\sqrt{k}}\int_{Z_k^{k+1}}\Theta(z,t)d\Lambda_g dt$$

where Λ_g is the coupling of Λ with the tangential Riemannian measure and $\Theta(z,r)$ is the function

$$\Theta(z,r):=\|c(\partial_r)\partial_r\phi_k^2[D_{\epsilon,u,z}e^{-tD^2_{\epsilon,u,z}}-S_{\epsilon,u,z}e^{-tS^2_{\epsilon,u,z}}]\|_{(z,z)}.$$

Let $\mathcal{T}_k$ be a transversal of the foliation $\mathcal{F}_k$ induced on the slice $\{r=k\}$ then $\mathcal{T}_k$ is also transversal for $\mathcal{F}$ (since the boundary foliation has the same codimension of $\mathcal{F}$). The transverse measure Λ defines also a transverse measure on the boundary foliation. Then the foliation $\mathcal{F}_{|Z_k^{k+1}}$ is fibering on $\mathcal{T}_k$ as in the diagram $\partial\mathcal{F}\times[k,k+1]\longrightarrow\mathcal{T}_k$. Use this fibration to disintegrate the measure Λ_g. This is splitted into $d\Lambda_\partial\times dr$ where Λ_∂ is the measure obtained applying the integration process of Λ (restricted to $\mathcal{F}_k$) to the $g_{|\partial}$. In local coordinates $(r,x_1,...,x_{2p-1})\times(x_{2p},...,x_n)$ the transversal is decomposed into pieces $\mathcal{T}_k=\{(k,x_1^0,...,x^0_{2p-1})\}\times\{(x_{2p},...,x_n)\}$ and we are taking integrals

$$\int_{\mathcal{T}_k\times\{x_1,...,x_{2p-1}\}}\int_{[k,k+1]}\Theta(r,x_1,...,x_{2p-1},x_{2p},...,x_n)dr\,\underbrace{dx_1\cdots dx_{2p-1}d\Lambda(x_{2p},..,x_n)}_{\text{this is }d\Lambda_\partial} \tag{69}$$

$$=:\int_{\mathcal{F}_k}\int_{[k,k+1]}\Theta(x,r)d\Lambda_\partial dr.$$

Equation (69) can be taken as a definition of a notation that will be used next. Notice that $\int_{\mathcal{F}_k}$ contains a slight abuse of notation, in fact to follow rigorously the integration receipt one should write $\int_{\partial X_0\times\{k\}}$. We prefer the first to stress the fact that we are splitting w.r.t the foliation induced on the transversal. With this notation in mind,

$$\begin{aligned}
&\int_s^{\sqrt{k}} |\operatorname{str}_\Lambda(c(\partial_r)\partial_r\phi_k^2 D_{\epsilon,u}e^{-tD^2_{\epsilon,u}}) - \operatorname{str}_\Lambda(c(\partial_r)\partial_r\phi_k^2 S_{\epsilon,u}e^{-tS^2_{\epsilon,u}})|dt \\
&\qquad = \int_s^{\sqrt{k}}\int_{\mathcal{F}_z}\int_{[k,k+1]} \|c(\partial_r)\partial_r\phi_k^2[D_{\epsilon,u}e^{-tD^2_{\epsilon,u}} - S_{\epsilon,u}e^{-tS^2_{\epsilon,u}}]\|_{((x,r),(x,r))}drd\Lambda_\partial dt \qquad (70)\\
&\qquad \leq C\int_s^{\sqrt{k}}\int_k^{k+1} e^{-(r-3)^2/6t}drdt \leq C\int_s^{\sqrt{k}} e^{-(k-3)^2/6t}dt \\
&\qquad \leq C\int_{1/\sqrt{k}}^{1/s} y^{-2}e^{-(k-3)^2y/6}dy \leq C(e^{-k^{3/2}/c_1} + e^{-c_2/s})
\end{aligned}$$

for sufficiently small[14] s and large k. This estimate says that

$$\lim_{k\to+\infty}\mathrm{LIM}_{s\to 0}\,\beta_{01}(k,s) = \lim_{k\to+\infty}\mathrm{LIM}_{s\to 0}\int_s^{\sqrt{k}} \operatorname{str}_\Lambda(c(\partial_r)\partial_r\phi_k^2 S_{\epsilon,u}e^{-tS^2_{\epsilon,u}})dt.$$

Now the second integral (on the cylinder) is explicitly computable in fact the Schwartz kernel of the operator $S_{\epsilon,u,z_0}e^{-tS^2_{\epsilon,u,z_0}}$ on the diagonal is easily checked to be

$$\begin{aligned}
&\big[S_{\epsilon,u,z_0}e^{-tS^2_{\epsilon,u,z_0}}\big]_{(z,z)} \\
=&\big(D^{\mathcal{F}_\partial}_{\epsilon,u,x_0}\Omega + c(\partial_r)\partial_r\big)\left(\big[e^{-t(D^{\mathcal{F}_\partial}_{\epsilon,u,x_0}\Omega)^2}\big]_{(x,y)}\frac{e^{-(r-s)^2/(4t)}}{\sqrt{4\pi t}}\right)\Bigg|_{y=x,\, s=r} \\
=&\frac{1}{\sqrt{4\pi t}}\Omega\big[D^{\mathcal{F}_\partial}_{\epsilon,u,x_0}e^{-tD^{\mathcal{F}_\partial}_{\epsilon,u,x_0}}\big]_{(x,x)},\quad z=(x,r)
\end{aligned}$$

i.e. it does not depend on the cylindrical coordinate r. Now the pointwise supertrace on $\mathrm{End}(E)$ is related to the trace on the positive boundary eigenbundle F via the identity (see the appendix on Clifford algebras)

$$\operatorname{str}^E(c(\partial_r)\Omega\bullet) = -2\operatorname{tr}^F(\bullet),$$

then

$$\begin{aligned}
&\int_s^{\sqrt{k}} \operatorname{str}_\Lambda(c(\partial_r)\partial_r\phi_k^2 S_{\epsilon,u}e^{-tS^2_{\epsilon,u}})dt \\
&\qquad\qquad = -2\int_s^{\sqrt{k}}\int_k^{k+1}\partial_r\phi_k^2 dr\int_{\mathcal{F}_0}\frac{1}{\sqrt{4\pi t}}\operatorname{tr}^F[D^{\mathcal{F}_\partial}_{\epsilon,u,x}e^{-t(D^{\mathcal{F}_\partial}_{\epsilon,u,x})^2}]_{(x,x)}\cdot d\Lambda_\partial dt \\
&\qquad\qquad = 2\int_s^{\sqrt{k}}\int_{\mathcal{F}_0}\frac{1}{\sqrt{4\pi t}}\operatorname{tr}^F[D^{\mathcal{F}_\partial}_{\epsilon,u,x}e^{-t(D^{\mathcal{F}_\partial}_{\epsilon,u,x})^2}]_{(x,x)}\cdot d\Lambda_\partial dt \\
&\qquad\qquad = \int_s^{\sqrt{k}}\int_{\mathcal{F}_0}\frac{1}{\sqrt{\pi t}}\operatorname{tr}^F[D^{\mathcal{F}_\partial}_{\epsilon,u,x}e^{-t(D^{\mathcal{F}_\partial}_{\epsilon,u,x})^2}]_{(x,x)}\cdot d\Lambda_\partial dt,
\end{aligned}$$

[14] $y^s e^{-ay^2} \leq (\frac{s}{2ae})^{s/2}$ for $s,u,y,a>0$

with the same argument on the splitting of measures as above. Finally it is clear from our discussion on the η–invariant (exactly proposition 7.55)

$$\begin{aligned}
&\lim_{k\to\infty} \mathrm{LIM}_{s\to 0}\, \beta_{01}(k,s) \\
&\quad = \lim_{k\to\infty} \mathrm{LIM}_{s\to 0} \int_s^{\sqrt{k}} \int_{\mathcal{F}_0} \frac{1}{\sqrt{\pi t}} \mathrm{tr}^F [D^{\mathcal{F}_\partial}_{\epsilon,u,x} e^{-t(D^{\mathcal{F}_\partial}_{\epsilon,u,x})^2}]_{(x,x)} \cdot d\Lambda_\partial dt = 1/2\eta_\Lambda(D^{\mathcal{F}_\partial}_{\epsilon,u}).
\end{aligned}$$

□

LEMMA 8.59 — Since $D_{\epsilon,u}$ is Λ–Breuer–Fredholm for $0<|u|<\epsilon$ then

$$\lim_{k\to\infty} \beta_{02}(k,s) = \lim_{k\to\infty} \int_{\sqrt{k}}^\infty \mathrm{str}_\Lambda(\phi_k D^2_{\epsilon,u} e^{-tD^2_{\epsilon,u}} \phi_k) dt = 0.$$

PROOF— From the very definition of the Λ–essential spectrum (see also lemma 4.29) there exists some $\sigma = \sigma(u) > 0$ such that the projection $\Pi_\sigma = \chi_{[-\sigma,\sigma]}(D_{\epsilon,u})$ has finite Λ–trace. Then

$$\begin{aligned}
|\beta_{02}(k,s)| &= \Big| \int_{\sqrt{k}}^\infty \mathrm{str}_\Lambda(\phi_k D^2_{\epsilon,u} e^{-tD^2_{\epsilon,u}} \phi_k) dt \Big| \\
&\le \int_{\sqrt{k}}^\infty |\,\mathrm{str}_\Lambda[\phi_k D_{\epsilon,u} e^{-D^2_{\epsilon,u}/2}(1-\Pi_\sigma) e^{-(t-1)D^2_{\epsilon,u}} e^{-D^2_{\epsilon,u}/2} D_{\epsilon,u}\phi_k]| dt \\
&+ \int_{\sqrt{k}}^\infty |\,\mathrm{str}_\Lambda[e^{-tD^2_{\epsilon,u}/2}\Pi_\sigma D_{\epsilon,u}\phi_k^2 D_{\epsilon,u}\Pi_\sigma e^{-tD^2_{\epsilon,u}/2}]| dt \\
&\le \underbrace{\int_{\sqrt{k}}^\infty e^{-(t-1)\sigma} |\,\mathrm{str}_\Lambda(\phi_k D^2_{\epsilon,u} e^{-D^2_{\epsilon,u}} \phi_k)| dt}_{\beta_{021}(k,s)} + \underbrace{\int_{\sqrt{k}}^\infty |\,\mathrm{str}_\Lambda(D^2_{\epsilon,u} e^{-tD^2_{\epsilon,u}}\Pi_\sigma)| dt}_{\beta_{022}(k,s)}.
\end{aligned}$$

Now the Schwartz kernel of $(D^2_{\epsilon,u} e^{-D^2_{\epsilon,u}})_x$ is uniformely bounded in x and varies in a Borel fashion transversally. When forming the Λ–supertrace we are integrating a longitudinal measure with C^∞–density w.r.t. the longitudinal measure given by the Riemannian density. Let as usual Λ_g the measure given by the integration of the Riemannian longitudinal measure with the transverse measure Λ. If A is a uniform bound on the leafwise Schwartz kernels of $(D^2_{\epsilon,u} e^{-D^2_{\epsilon,u}})$, and $\mathcal{T}_0$ is a complete transversal contained in the normal section of the cylinder (the same in lemma 8.58), we can extimate

$$\beta_{021}(k,s) \le \int_{\sqrt{k}}^\infty A(\Lambda_g(X_0) + \Lambda(\mathcal{T}_t)k) e^{-(t-1)\sigma} dt \longrightarrow_{k\to\infty} 0.$$

For the second addendum,

$$\begin{aligned}
\beta_{022}(k,s) &= \int_{\sqrt{k}}^\infty |\,\mathrm{str}_\Lambda(D^2_{\epsilon,u} e^{-tD^2_{\epsilon,u}}\Pi_\sigma)| dt \le \int_{\sqrt{k}}^\infty \int_{-\sigma}^\sigma x^2 e^{-tx^2} d\mu_{\Lambda,D_{\epsilon,u}}(x) dt \\
&= \int_{-\sigma}^\sigma e^{-\sqrt{k}x^2} \int_0^\infty x^2 e^{-tx^2} dt d\mu_{\Lambda,D_{\epsilon,u}}(x) \\
&\le C \int_{-\sigma}^\sigma e^{-\sqrt{k}x^2} d\mu_{\Lambda,D_{\epsilon,u}}(x) \le C\mu_{\Lambda,D_{\epsilon,u}}(x)([-\sigma,\sigma]) \longrightarrow_{k\to\infty} 0
\end{aligned}$$

since the Λ–essential spectrum of $D_{\epsilon,u}$ has a gap around zero and the normality property of the trace. $\square$

It is time to update equation (64),

$$\begin{aligned}\operatorname{ind}_\Lambda(D^+_{\epsilon,u}) &= \lim_{k\to\infty}[\alpha_0(k,s)-\beta_0(k,s)] = \lim_{k\to\infty}[\alpha_0(k,s)-\beta_{01}(k,s)-\beta_{02}(k,s)] \\ &= \lim_{k\to\infty}\mathrm{LIM}_{s\to 0}\,\alpha_0(k,s) - 1/2\eta_\Lambda(D^{\mathcal{F}_\partial}_{\epsilon,u}). \end{aligned} \tag{71}$$

LEMMA 8.60 — There exists a function $g(u)$ with $\lim_{u\to 0} g(u) = 0$ such that for $0<\epsilon<u$,

$$\lim_{k\to\infty}\mathrm{LIM}_{s\to 0}\,\alpha_0(k,s) = \lim_{k\to\infty}\mathrm{LIM}_{s\to 0}\,\operatorname{str}_\Lambda(\phi_k e^{-sD^2_{\epsilon,u}}\phi_k) = \langle \hat{A}(X)\,\mathrm{Ch(E/S)}, C_\Lambda\rangle + g(u).$$

Here the leafwise characteristic form $\hat{A}(X)\,\mathrm{Ch(E/S)}$ is supported on X_0, in particular it belongs to the domain of the Ruelle–Sullivan current C_Λ associated to the transverse measure Λ.

PROOF — This is the investigation of the behavior of the local supertrace of the family of the leafwise heat kernels

$$\operatorname{str}^E[e^{-sD^2_{\epsilon,u}}]_{|\,\mathrm{diag}}$$

on the leafwise diagonals. We can do it dividing into three separate cases

1. For $z\in X_0$ everything goes as in the classical computation by Atiyah Bott and Patodi [2],
$$\mathrm{LIM}_{s\to 0}\operatorname{str}^E[e^{-sD^2_{\epsilon,u,z}}]_{(x,x)}dg_z = \hat{A}(X)\,\mathrm{Ch}(E/S)(x),$$
where dg_z is the Riemannian density on the leaf L_z.
2. In the middle, $z\in\partial X_0\times[0,4]$ there's the cause of the presence of the defect function $g(u)$, more precisely we show that the asymptotic development of the local supertrace is the same for the comparison operator $S_{0,u}$ defined above
$$\operatorname{str}^E([e^{-sD^2_{\epsilon,u,z}}])_{(z,z)} \simeq \sum_{j\in\mathbb{N}} a_j(S_{0,u})_{(z)}s^{(j-\dim\mathcal{F})/2}$$
with coefficients $a_j(S_{0,u})$ smoothly depending on u satisfying $a_j(S_{0,u})=0$ for $j\le \dim\mathcal{F}/2$
3. Away from the base of the cylinder $z=(y,r)\in Z$ $r>4$ we find
$$[e^{-D^2_{\epsilon,u,z}}]_{(y,r)} = 0.$$

Below the proves of these facts.

1. We can consider the doubled manifold $2X_0$ so that we can apply the relative estimate of type Cheeger–Gromov–Taylor in the non–cylindrical case (the perturbation starts from the cylinder) i.e. proposition 6.50 shows that the two Schwartz kernels of the Dirac operator and the perturbed operator $D_{\epsilon,u}$ have the same development as $t\to 0$,
$$\|[e^{-tD^2_{\epsilon,u}} - e^{-tD^2}]_{(x,x)}\| \le Ke^{-\alpha/(6t)}.$$
And the local computation of Atiyah Bott and Patodi, or the Getzler rescaling ([50],[30]) can be performed as in the classical situation.

2. We are going to use an argument of comparison with the leafwise operator

$$S_{\epsilon,u} := c(\partial_r)\partial_r + \Omega(D^{\mathcal{F}_\partial} + \dot\theta(u - \Pi_\epsilon D^{\mathcal{F}_\partial}))$$

on the infinite cylinder $\partial X_0 \times \mathbb{R}$ equipped with the product foliation $\mathcal{F}_\partial \times \mathbb{R}$. Notice that, due to the presence of $\dot\theta$ this is a slightly different form of the operator (67). Choose some function ψ_1 supported in $\partial X_0 \times [-1, 5]$ and $\psi_{1|\partial X_0 \times [0,4]} = 1$. The first fact we show is

$$\lim_{s\to 0} \mathrm{str}_\Lambda(\psi_1(e^{-sS^2_{\epsilon,u}} - e^{-sS^2_{0,u}})\psi_1) = 0.$$

Now, $S_{\epsilon,u} = S_{0,u} - \Omega\Pi_\epsilon D^{\mathcal{F}_\partial} = c(\partial_r)\partial_r + H$ with $H = \Omega D^{\mathcal{F}_\partial} + \Omega\dot\theta u$ hence

$$\begin{aligned} S^2_{\epsilon,u} - S^2_{0,u} &= -[S_{0,u}, \Omega\dot\theta\Pi_\epsilon D^{\mathcal{F}_\partial}] + (\Omega\dot\theta\Pi_\epsilon D^{\mathcal{F}_\partial})^2 \\ &= -[c(\partial_r)\partial_r, \Omega\dot\theta\Pi_\epsilon D^{\mathcal{F}_\partial}] - [H, \Omega\dot\theta\Pi_\epsilon D^{\mathcal{F}_\partial}] + (\Omega\dot\theta\Pi_\epsilon D^{\mathcal{F}_\partial})^2 \\ &= -\Phi\dot\theta\Pi_\epsilon D^{\mathcal{F}_\partial} - 2(D^{\mathcal{F}_\partial} + \dot\theta u)(\dot\theta\Pi_\epsilon D^{\mathcal{F}_\partial}) + (\Omega\dot\theta\Pi_\epsilon D^{\mathcal{F}_\partial})^2. \end{aligned} \tag{72}$$

Apply the Duhamel formula

$$\begin{aligned} &|\,\mathrm{str}_\Lambda(\psi_1(S^2_{\epsilon,u} - S^2_{0,u})\psi_1| \\ &\quad = \big|\,\mathrm{str}_\Lambda(\psi_1 e^{-\delta S^2_{0,u}} e^{-(s-\delta)S^2_{\epsilon,u}}\psi_1)_{(\delta=s)} - \mathrm{str}_\Lambda(\psi_1 e^{-\delta S^2_{0,u}} e^{-(s-\delta)S^2_{\epsilon,u}}\psi_1)_{(\delta=0)}\big| \\ &\quad = \Big|\int_0^s \mathrm{str}_\Lambda(\psi_1^2\Pi_\epsilon)e^{-\delta S^2_{0,u}}(S^2_{\epsilon,u} - S^2_{0,u})\Pi_\epsilon e^{-(s-\delta)S^2_{\epsilon,u}} d\delta\Big|. \end{aligned}$$

Again from the Cheeger–Gromov relative estimates (49)

$$|\,\mathrm{tr}_\Lambda(\psi_1 e^{-\delta S^2_u}\Pi_\epsilon\psi_1)| \leq C\delta^{-1/2}$$

$$\|(S^2_{\epsilon,u} - S^2_{0,u})\Pi_\epsilon e^{-(s-\delta)S^2_{\epsilon,u}}\| \leq C(s-\delta)^{-1/2}$$

with the constants independent from $|u| < \epsilon$. Then the integral of the supertrace (72) can be estimated by the function of s, $h(s) = C\int_0^s (s-\delta)^{-1/2}\delta^{-1/2} d\delta \longrightarrow_{s\to 0} 0.$. To see this first split the integral into $\int_0^{s/2} + \int_{s/2}^s$ to prove finiteness then use the absolutely continuity of the integral for convergence to zero. Now from the limit $\lim_{s\to 0}\mathrm{str}_\Lambda(\psi_1(e^{-sS^2_{\epsilon,u}} - e^{-sS^2_{0,u}})\psi_1) = 0$ and the comparison argument we get that the asymptotic expansion for $s \to 0$ of $\mathrm{str}_\Lambda(\phi_k e^{-sD^2_{\epsilon,u}\phi_k})$ is the same of the comparison operator

$$S_{0,u} = \underbrace{c(\partial_r)\partial_r + \Omega D^{\mathcal{F}_\partial}}_{D} + \underbrace{\dot\vartheta u\Omega}_{\text{bounded perturbation}}$$

on the infinite cylinder. This is a very simple u–family of generalized laplacians (see [9] Chapter 2.7) and the Duhamel formula

$$e^{-tS^2_{0,u}} - e^{-tS^2_{0,0}} = -\int_0^u t\dot\vartheta\Omega e^{-tS_{0,v}} dvds$$

shows what is written in the statement i.e.

$$\mathrm{str}^E([e^{-sD^2_{\epsilon,u,z}}])_{(z,z)} \simeq \sum_{j\in\mathbb{N}} a_j(S_{0,u})_{(z)} s^{(j-\dim\mathcal{F})/2}$$

where the coefficients $a_j(S_{0,u})$ depend smoothly on u and satisfy $a_j(S_{0,u}) = 0$ for $j \leq \dim\mathcal{F}/2$ since $S_{0,0}$ is the Cylindrical Dirac operator. One can take for the definition of g,

$$g(u) := \sum_{j=0}^{\dim\mathcal{F}/2} \int_{\partial X_0\times[0,4]} a_j(S_{0,u})_{(z)} s^{(j-\dim\mathcal{F})/2} d\Lambda_g.$$

3. This is done again by comparison with $S_{\epsilon,u}$ consider the r–depending family of tangential tangential measures $(y,r) \in \partial X_0 \times [a,b] \longmapsto \mathrm{str}^E\, e^{-sD^2_{\epsilon,u,(x,r)}} dxdr$ where $x \in L_{(y,r)}$, once coupled with $d\Lambda$ gives a measure on X $\mu := \mathrm{str}^E\, e^{-sD^2_{\epsilon,u,(x,r)}} dxdr \cdot d\Lambda$. The Fubini theorem can certainly used during the integration process to find out that the mass of μ can be computed integrating first the r–depending tangential measures $y \longmapsto \mathrm{str}^E\, e^{-sD^2_{\epsilon,u,(y,r)}} dy$ against Λ on the foliation at infinity $(\partial X_0, \mathcal{F}_\partial)$ then the resulting function of r on $[a,b]$,

$$\begin{aligned}\mathrm{LIM}_{s\to 0}\int_{\partial X_0\times[a,b]} d\mu &= \mathrm{LIM}_{s\to 0}\int_a^b\int_{\partial X_0}\mathrm{str}^E([e^{-sS^2_{\epsilon,u}}])_{(y,r),(y,r)})dy\cdot d\Lambda dx\\ &= \mathrm{LIM}_{s\to 0}\frac{b-a}{\sqrt{4\pi s}}\,\mathrm{str}_\Lambda(e^{-s(D^{\mathcal{F}_\partial}_{\epsilon,u})^2}) = 0\end{aligned}$$

in fact the boundary operator $D^{\mathcal{F}_\partial}_{\epsilon,u}$ is invertible and the well–known Mc–Kean–Singer formula for foliations on compact ambient manifolds (formula (7.39) in [53]) says that $\mathrm{ind}_\Lambda(D^{\mathcal{F}_\partial}_{\epsilon,u}) = \mathrm{str}_\Lambda\, e^{-s(D^{\mathcal{F}_\partial}_{\epsilon,u})^2}$ independently from s.

□

Finally (71) becomes

$$\mathrm{ind}_\Lambda(D^+_{\epsilon,u}) = \langle \hat{A}(X)\,\mathrm{Ch}(E/S), C_\Lambda\rangle - 1/2\eta_\Lambda(D^{\mathcal{F}_\partial}_{\epsilon,u}) + g(u). \tag{73}$$

THEOREM 8.60 — The Dirac operator has finite dimensional $L^2 - \Lambda$–index and the following formula holds

$$\mathrm{ind}_{L^2,\Lambda}(D^+) = \langle \hat{A}(X)\,\mathrm{Ch}(E/S), [C_\Lambda]\rangle + 1/2[\eta_\Lambda(D^{\mathcal{F}_\partial}) - h^+_\Lambda + h^-_\Lambda] \tag{74}$$

where

$$h^\pm_\Lambda := \dim_\Lambda(\mathrm{Ext}(D^\pm) - \dim_\Lambda(\mathrm{Ker}_{L^2}(D^\pm) \tag{75}$$

with the dimension of the space of extended solutions as defined in the definition ?? after the remark i.e.

$$\dim_\Lambda \mathrm{Ext}(D^\pm) := \dim_\Lambda \overline{\mathrm{Ext}(D^\pm)}^{e^{u\theta}L^2}$$

independently from small $u > 0$.

PROOF — Start from

$$\mathrm{ind}_{L^2,\Lambda}(D^+_\epsilon) = \lim_{u\downarrow 0} 1/2\{\mathrm{ind}_\Lambda(D^+_{\epsilon,u}) + \mathrm{ind}_\Lambda(D^+_{\epsilon,-u}) + h^-_{\Lambda,\epsilon} - h^+_{\Lambda,\epsilon}\}, \tag{76}$$

here $h^\pm_{\Lambda,\epsilon} = \dim_\Lambda(\mathrm{Ext}(D^\pm_\epsilon)) - \dim_\Lambda(\mathrm{Ker}_{L^2}(D^\pm_\epsilon))$ for now proposition 5.41 says that

$$\mathrm{Ext}(D^\pm_\epsilon) = \mathrm{Ker}_{L^2}(D^\pm_{\epsilon,\pm}) = \mathrm{Ker}_{e^{u\theta}L^2}(D^\pm_\epsilon).$$

Use the identity $\mathrm{ind}_\Lambda(D^+_{\epsilon,u}) = \langle \hat{A(X)}\,\mathrm{Ch}(E/S), [C_\Lambda]\rangle + 1/2\eta_\Lambda(D^{\mathcal{F}_\partial}_{\epsilon,u}) + g(u)$ into (76),

$$\begin{aligned}\mathrm{ind}_{L^2,\Lambda}(D^+_\epsilon) = &\lim_{u\downarrow 0} 1/2\Big\{2\langle \hat{A}(X)\,\mathrm{Ch}(E/S), [C_\Lambda]\rangle + h^-_{\Lambda,\epsilon} - h^+_{\Lambda,\epsilon} + g(u) + g(-u) \\ &\underbrace{+1/2\eta_\Lambda(D^{\mathcal{F}_\partial}_{\epsilon,u}) + 1/2\eta_\Lambda(D^{\mathcal{F}_\partial}_{\epsilon,-u})}_{\eta_\Lambda(D^{\mathcal{F}_\partial}_\epsilon)\text{ by proposition 7.55}}\Big\} \\ =&\langle \hat{A}(X)\,\mathrm{Ch}(E/S), [C_\Lambda]\rangle + \frac{h^-_{\Lambda,\epsilon} - h^+_{\Lambda,\epsilon}}{2} + \frac{\eta_\Lambda(D^{\mathcal{F}_\partial}_\epsilon)}{2}.\end{aligned} \tag{77}$$

It remains to pass to the ϵ–limit remembering that:

- $\lim_{\epsilon\downarrow 0}\mathrm{ind}_{L^2,\Lambda}(D^+_\epsilon) = \mathrm{ind}_{L^2,\Lambda}(D^+)$ (Proposition 5.44),
- $\lim_{\epsilon\downarrow 0} h^-_{\Lambda,\epsilon} - h^+_{\Lambda,\epsilon} = h^- - h^+$ (again proposition 5.44)
- $\lim_{\epsilon\downarrow 0}\eta_\Lambda(D^{\mathcal{F}_\partial}_\epsilon) = \eta_\Lambda(D^{\mathcal{F}_\partial})$ (proposition 7.55).

□

9 Comparison with Ramachandran index formula

The Ramachandran index formula [62] stands into index theory for foliations exactly as the Atiyah–Patodi–Singer formula in the boundary value problem form stays classically. Our formula is in some sense the cylindrical point of view of this formula. In this section we prove that the two formulas are compatible and we do it exactly in the way it is done for the single leaf case by APS. First we recall the Ramachandran Theorem

9.1 The Ramachandran index

Since we have chosen an opposite orientation for the boundary foliation the Ramachandran index formula here written differs from the original in [62] exactly for its sign (as in section 3 for the APS formula). So let us consider the Dirac operator builded in section 2 but acting only on the foliation restricted to the compact manifold with boundary X_0. To be precise with notation let us call $\mathcal{F}_0$ the foliation restricted to X_0 with leaves $\{L^0_x\}_x$, equivalence relation $\mathcal{R}_0$ and $D^{\mathcal{F}_0}$ the Dirac operator acting on the field of Hilbert spaces $\{L^2(L^0_x;E)\}_{x\in X_0}$. Near the boundary

$$D^{\mathcal{F}_0} = \begin{pmatrix} 0 & D^{\mathcal{F}_0^-} \\ D^{\mathcal{F}_0^+} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\partial_r + D^{\mathcal{F}_\partial} \\ \partial_r + D^{\mathcal{F}_\partial} & 0 \end{pmatrix}$$

with the boundary operator $D^{\mathcal{F}_\partial}$. Let us consider the field of APS boundary conditions

$$B = \begin{pmatrix} \chi_{[0,\infty)}(D^{\mathcal{F}_\partial}) & 0 \\ 0 & \chi_{(-\infty,0)}(D^{\mathcal{F}_\partial}) \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & \mathrm{I}-P \end{pmatrix}$$

acting on the boundary foliation. In the order of ideas of Ramachandran paper (coming back from an idea of John Roe) this is a <u>self adjoint boundary condition</u> i.e. its interacts with the Dirac operator in the following way:

1. B is a field of bounded self–adjoint operators with $\sigma B + B\sigma = \sigma$ where σ is Clifford multiplication by the unit (interior) normal.

2. If b is the operator of restriction to the boundary acting on smooth sections then $(s_1, D^{\mathcal{F}_0} s_2) = (D^{\mathcal{F}_0} s_1, s_2)$ for every couple of smooth sections s_1 and s_2 such that $Bbs_1 = 0$ and $Bbs_2 = 0$.

Next Ramachandran proves using the generalized eigenfunction expansion of Browder and Gårding, that there's a field of restriction operators

$$H^k(X_0; E) \longrightarrow H^{k-1/2}(X_0; E)$$

extending b where the Sobolev spaces are defined taking into account the boundary i.e. for a leaf L^0_x, the space $H^k(L^0_x; E)$ is the completion of $C^\infty_c(L^0_x; E)$ (compact support possibly meeting the boundary) w.r.t. the usual L^2–based Sobolev norms. It follows from the restriction theorem that one can define the domain of D with boundary condition B as $H^\infty(X_0; E, B) := \{s \in H^\infty(X_0; E) : Bbs = 0\}$.

THEOREM 9.60 — (Ramachandran [62]) The family of unbounded operators D with domain $H^\infty(X_0; E, B)$ is essentially self–adjoint and Breuer–Fredholm in the Von Neumann algebra of the foliation with finite Λ–index in the sense of $\mathrm{ind}_\Lambda(D^{\mathcal{F}_0}) = \dim_\Lambda(\mathrm{Ker}(D^{\mathcal{F}_0^+})) - \dim_\Lambda(\mathrm{Ker}(D^{\mathcal{F}_0^-}))$ given by the formula

$$\mathrm{ind}_\Lambda(D^{\mathcal{F}_0}) = \langle \hat{A}(X)\,\mathrm{Ch}(E/S), [C_\Lambda]\rangle + 1/2[\eta_\Lambda(D^{\mathcal{F}}_0) - h] \tag{78}$$

Now we are going to prove compatibility between formula (78) and (74). First of all we have to relate the two Von Neumann algebras in play. Denote (according to our notation) with $\mathrm{End}_{\mathcal{R}_0}(E)$ the space of intertwining operators of the representation of $\mathcal{R}_0$ on $L^2(E)$ and, only in this section $\mathrm{End}_{\mathcal{R}_0,\Lambda}(E)$ the resulting Von Neumann algebra with trace $\mathrm{tr}_{\mathcal{R}_0,\Lambda}$ in order to make distinction from $\mathrm{End}_{\mathcal{R},\Lambda}(E)$ the Von Neumann algebra of random operators associated with the representation of $\mathcal{R}$. Start with a measurable fields of bounded operators $X_0 \ni B_x \longmapsto B_x : L^2(L^0_x; E) \longrightarrow L^2(L^0_x; E)$ with $B_x = B_y$ a.e. if $(x, y) \in \mathcal{R}_0$. There's a natural way to extend B to a field of operators in $\mathrm{End}_{\mathcal{R}}(E)$.

1. If $x \in X_0$ simply let $\imath B_x$ act to $L^2(L_x; E)$ to be zero on the cylinder

$$\imath B_x : L^2(L^0_x; E) \oplus L^2(\partial L^0_x \times (0, \infty); E) \longrightarrow L^2(L^0_x; E) \oplus L^2(\partial L^0_x \times (0, \infty); E)$$

$\imath B_x(s, t) := (B_x s, 0)$.

2. If $x \in \partial X_0 \times (0, \infty)$ define $\imath B_x := \imath B_{p(x)}$ where $p : \partial X_0 \times (0, \infty) \longrightarrow \partial X_0$ is the base projection and $\imath B_{p(x)}$ is defined by point 1.

PROPOSITION 9.61 — The map $\imath : \mathrm{End}_{\mathcal{R}_0}(E) \longrightarrow \mathrm{End}_{\mathcal{R}}(E)$ as defined above passes to the quotient to an injection

$$\imath : \mathrm{End}_{\mathcal{R}_0,\Lambda}(E) \longrightarrow \mathrm{End}_{\mathcal{R},\Lambda}(E)$$

between the Von Neumannn algebras of Random operators preserving the two natural traces

$$\mathrm{tr}_{\mathcal{R},\Lambda}(\imath B) = \mathrm{tr}_{\mathcal{R}_0,\Lambda}(B).$$

PROOF— The first part is clear. An intertwining operator $B = \{B_x\}_{x \in X_0}$ is zero Λ–a.e. in $X_=0$ then also does $\imath B$ in X for any transversal T contained in the cylinder can slide by holonomy to a transversal contained in X_0. About the identity on traces remember the link between the direct integral algebras and the algberas of random operators i.e. Lemma 4.6. Choose ν to be the longitudinal Riemannnian metric then Λ_ν is the integration of ν against Λ. Let P_0 be the Von Neumann algebra of Λ_ν–a.e. classes of measurable fields of operators $X_0 \ni x \longmapsto B_x \in B(L^2(L^0_x; E))$ and P the corresponding algebra builded replacing X_0 with X and $B(L^2(L^0_x; E))$ with $B(L^2(L^0_x; E))$. Notice that the family

$$X \ni y \longmapsto \int \imath B_x d\nu^y \tag{79}$$

is bounded for B in the domain of $\imath$ then Lemma 4.6 says that

$$\operatorname{tr}_{\mathcal{R},\Lambda}(\imath B) = \int_X \operatorname{Trace}(B_x) d\Lambda_\nu(x) = \int_{X_0} \operatorname{Trace}(B_x) d\Lambda_\nu(x) = \operatorname{tr}_{\mathcal{R}_0,\Lambda}(B).$$

□

Theorem 9.61 — Let $\operatorname{Pr}\operatorname{Ker}(D^{\mathcal{F}_0^\pm}) \in \operatorname{End}_{\mathcal{R}_0,\Lambda}(E)$ the projection on the Kernel of $D^{\mathcal{F}_0^\pm}$ with domain given by the boundary condition $Px = 0$, $(\mathrm{I} - P = 0)$ as in Ramachandran formula. Let also $\operatorname{Pr}\operatorname{Ker}_{L^2}(D^\pm) \in \operatorname{End}_{\mathcal{R},\Lambda}(E)$ be the projection on the L^2–kernel of the leafwise operator on the foliation with the cylinder attached and $\operatorname{Pr}\overline{\operatorname{Ext}(D^\pm)} \in \operatorname{End}_{\mathcal{R},\Lambda}(e^{u\theta}L^2E)$ be the projection on the closure of the space of extended solution seen in $e^{u\theta}$ for sufficiently small positive u.

1. $\imath \operatorname{Pr}\operatorname{Ker}(D^{\mathcal{F}_0^+})$ is equivalent to $\operatorname{Pr}\operatorname{Ker}_{L^2}(D^+)$ in $\operatorname{End}_{\mathcal{R},\Lambda}(E)$ i.e. there exists a partial isometry $u \in \operatorname{End}_{\mathcal{R},\Lambda}(E)$ such that

$$u^*u = \imath \operatorname{Pr}\operatorname{Ker}(D^{\mathcal{F}_0^+}), \quad uu^* = \operatorname{Pr}\operatorname{Ker}_{L^2}(D^+)$$

. In particular

$$\dim_{\mathcal{R}_0,\Lambda}\operatorname{Ker}(D^{\mathcal{F}_0^+}) = \dim_{\mathcal{R},\Lambda}\operatorname{Ker}_{L^2}(D^+).$$

2.

$$\imath \operatorname{Pr}\operatorname{Ker}_{L^2}(D^{\mathcal{F}_0^-}) \sim \operatorname{Pr}\overline{\operatorname{Ext}(D^-)}^{e^{u\theta}L^2},$$

for sufficiently small u and equivalence in $\operatorname{End}_\Lambda(e^{u\theta}L^2(E))$ with the inclusion

$$\imath : \operatorname{End}_{\mathcal{R}_0,\Lambda}(E) \longrightarrow \operatorname{End}_\Lambda(e^{u\theta}L^2(E))$$

defined as in proposition 9.61.

As a consequence

$$\dim_\Lambda \operatorname{Ker}(D^{\mathcal{F}_0^-}) = \dim_\Lambda \operatorname{Ext}(D^-).$$

Proof— The idea is contained in A.P.S. [4] when they prove the equivalence between the boundary value problem and the L^2 cylindrical problem. Their main instrument is the eigenfunction expansion of the operator at the boundary, now we use the Browder–Garding generalized expansion to see that any solution of the boundary value problems extends to a solution of the operator on the cylinder.

1. Use the Browder–Gårding expansion as in the proof of the finiteness of the projection on the kernel 5.1. For a single leaf, the isomorphism

$$L^2(\partial L_x^0 \times (-1,0]) \longrightarrow \bigoplus_{j\in\mathbb{N}} L^2(\mathbb{R},\mu_j) \otimes L^2((-1,0])$$

represents a solution of the boundary value problem as $h_j(r,\lambda) = \chi_{(-\infty,0)}(\lambda)e^{-\lambda r}h_{j0}(r)$ hence the solution can be extended to the cylinder of the leaf $\partial L_x^0 \times (0,\infty)$. This clearly gives a field of linear isomorphisms $T_x : \operatorname{Ker}(D_x^{\mathcal{F}_0^+}) \longrightarrow \operatorname{Ker}_{L^2}(D_x^+)$ for $x \in X_0$, first extend T_x to all $L^2(L_x^0; E)$ to be zero on $\operatorname{Ker}(D^{\mathcal{F}_0^+})^\perp$ then let x take values also in X according to the method explained before i.e. put $T_x := T_{p(x)}$ for x in the cylinder. Take the polar decomposition $T_x = u_x|T_x|$, then u_x is a partial isometry with initial space $\operatorname{Ker}(D_x^{\mathcal{F}_0^+})$ and range $\operatorname{Ker}(D_x^+)$, i.e

$$u_x^*u_x = \Pr\operatorname{Ker}(D_x^{\mathcal{F}_0^+}), \quad u_xu_x^* = \Pr\operatorname{Ker}(D_x^+).$$

We have to look at this relation into the Von Neumann algebra of the foliation on X. Split every L^2 space of the leaves as $L^2(L_{p(x)}^0;E)\oplus L^2(\partial L_{p(x)}^0 \times (0,\infty);E)$. With respect to the splitting, forgetting the indexes x downstairs, we have $u = \begin{pmatrix} u_{11} & 0 \\ u_{21} & 0 \end{pmatrix}$ acting on the field of $L^2(X;E)$ spaces of the leaves. Then $u^* = \begin{pmatrix} u_{11}^* & u_{21}^* \\ 0 & 0 \end{pmatrix}$ with conditions $u_{11}u_{21}^* = 0$ and $u_{21}u_{11}^* = 0$. Finally

$$uu^* = \begin{pmatrix} u_{11}u_{11}^* + u_{21}u_{21}^* & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Pr(D^{\mathcal{F}_0^+}) & 0 \\ 0 & 0 \end{pmatrix} = \imath\Pr(D^{\mathcal{F}_0^+})$$

and similarly $u^*u = \Pr(D^+)$.

2. It is very similar to statement 1. in fact writing the Browder–Gårding expansion and imposing the adjoint boundary condition one ends directly into the space of extended solutions.

□

To conclude now we can compare Ramachandran index with our index, let's compare formula (78) with (74) keeping in mind that, the index of Ramachandran is now our extended index (see section 3)

$$\operatorname{ind}_\Lambda(D^{\mathcal{F}_0}) = \operatorname{ind}_{\Lambda,\tilde{L^2}}(D^+) = \dim_\Lambda \operatorname{Ker}_{L^2}(D^+) - \dim_\Lambda \operatorname{Ext}(D^-)$$

to obtain the equation

$$\dim_\Lambda \operatorname{Ext}(D^-) - \dim_\Lambda \operatorname{Ker}_{L^2}(D^-) = (h_\Lambda^- - h_\Lambda^+)/2 + h/2.$$

The same argument applied to the (formal) adjoint of D^+ leads to the equation

$$\dim_\Lambda \operatorname{Ext}(D^+) - \dim_\Lambda \operatorname{Ker}_{L^2}(D^+) = (h_\Lambda^+ - h_\Lambda^-)/2 + h/2,$$

then

$$h = h_\Lambda^+ + h_\Lambda^-$$

as in A.P.S.

10 The signature formula

10.1 The classical signature formula

The reference for the notation about the signature operator is the book bt Berline Getzler and Vergne [9]. Let X be an oriented Riemannian manifold and $|dvol|$ the volume the unique volume form compatible with the metric i.e. the one assuming the value 1 on each positive oriented orthonormal frame. In other words $|\,\mathrm{dvol} = |\sqrt{g}dx|$. One can define the Hodge $*$ operator in the usual way

$$*e^{i_1} \wedge \cdots \wedge e^{i_k} = \mathrm{sign}(\sigma) e_{j_1} \wedge \cdots \wedge e_{i_{n-k}}$$

where $(e_1, ..., e_n)$ is an oriented orthonormal basis, $(i_1, ..., i_k)$ and $(j_i, ..., j_k)$ are complementary multindices and σ is the permutation $\sigma := \begin{pmatrix} 1 & . & . & . & . & n \\ i_1 & . & i_k & j_1 & . & j_{n-k} \end{pmatrix}$.
Since $*^2 = (-1)^{|\cdot|(n-|\cdot|)}$ this is an involution on even dimensional manifolds.
The bundle ΛT^*X of exterior algebras of X is a natural Clifford module under the action defined by

$$c(e^i) := \epsilon(e_i) - \iota(e^i) \tag{80}$$

where $\epsilon(e^i)\omega = e^i \wedge \omega$ is the exterior multiplication by e^i and $\iota(e_i)$ is the contraction by the tangent vector e_i. In other words it is the metric adjoint of exterior multiplication, $\epsilon(e^i)^* = \iota(e_i)$. The chirality involution

$$\tau := i^{[(n+1)/2]} c(e_1) \cdots c(e_n)$$

is related to the Hodge duality operator by

$$\tau = i^{[(n+1)/2]} * (-1)^{n|\cdot| + \frac{|\cdot|(|\cdot|-1)}{2}},$$

following from the identity (same deegree forms)

$$\int_X \alpha \wedge \tau\beta = (-1)^{n|\cdot|+|\cdot|(|\cdot|-1)/2} i^{[2n+1]/2} \int_X (\alpha, \beta)|dx|$$

while $\int_X \alpha \wedge *\beta = \int_X (\alpha, \beta)|dx|$. As a consequence one can write the adjoint of d in two different ways,

$$d^* = - * d * (-1)^{n|\cdot|+n} = -(-1)^n \tau d \tau.$$

Sections of the positive and negative eigenbundles of τ are called the self–dual and anti self–dual differential forms respectively and denoted by $\Omega^{\pm}(X)$.
Now suppose n is even, and X is compact. The bilinear form on the middle cohomolgy $H^{n/2}(X;\mathbb{R})$ defined by $(\alpha, \beta) \longmapsto \int_X \alpha \wedge \beta$ satisfies the identity

$$\int_X \alpha \wedge \beta = (-1)^{n/2} \int_X \beta \wedge \alpha.$$

In particular if n is divisible by four this is symmetric and has a signature $\sigma(X)$ i.e. the number $p - q$ related to the representation

$$Q(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_q^2$$

of the associated quadratic form (this is independent by the choosen basis). In this situation the chiral Dirac operator $d + d^*$ acting on the space of differential forms is called the Signature operator[15]

$$(d + d^*) = D^{\mathrm{sign}} = \begin{pmatrix} 0 & D^{\mathrm{sign},-} \\ D^{\mathrm{sign},+} & 0 \end{pmatrix} : \Omega^+(X) \oplus \Omega^-(X) \longrightarrow \Omega^+(X) \oplus \Omega^-(X)$$

[15] it differs from the Gauss–Bonnet operator $d + d^*$ only for the choice of the involution

The Atiyah–Singer index theorem in this case becomes the Hirzebruch signature theorem

$$\operatorname{ind}(D^{\mathrm{sign},+}) = \sigma(X) = \int_X \mathrm{L}(X)$$

where $\mathrm{L}(X)$ is the L–genus, $\mathrm{L}(X) = (\pi i)^{-n/2} \det^{1/2}\left(\frac{R}{\tanh(\mathrm{R}/2)}\right)$ for the Riemannian curvature R. The proof uses the Hodge theorem stating a natural isomorphism between the space of harmonic forms $\mathcal{H}^q(X)$ i.e. the kernel of the forms laplacian $\Delta = (d + d^*)^2$ and the cohomology $H^q(X)$ together with Poincaré duality.

Now on a ∂–manifold with product structure near the bounday the situation is much more complicated. The signature formula is the most important application of the index theorem in the A.P.S. paper. The operator writes on a collar around the boundary as

$$D^{\mathrm{sign},+} = \sigma(\partial_u + B)$$

where the isomorphism $\sigma : \Omega(\partial X) \longrightarrow \Omega^+(X)$ and B is the self–adjoint operator on $\Omega(\partial X)$ given by $B\alpha = (-1)^{k+p+1}(*_\partial d - d*_\partial)\alpha$ where here and in the next $\dim(X) = 4k$, $\epsilon(\alpha) = \pm 1$ according to α is even or odd degree and $*_\partial$ is the Hodge duality operator on ∂X. Since B commutes with $\alpha \longmapsto (-1)^{|\alpha|} *_\partial \alpha$ and preserves the parity of forms, $B = B^{\mathrm{ev}} \oplus B^{\mathrm{odd}}$ and the dimension of the kernel at the boundary as the η invariant are twice that of B^{ev}. The A.P.S index theorem says

$$\operatorname{ind}(D^{\mathrm{sign},+}) = h^+ - h^- - h^-_\infty = \int_X L - h(B^{\mathrm{ev}}) - \eta(B^{\mathrm{ev}})$$

or

$$\operatorname{ind}_{L^2}(D^{\mathrm{sign},+}) - h^-_\infty = \int_X L - h(B^{\mathrm{ev}}) - \eta(B^{\mathrm{ev}})$$

where $h^\pm$ are the dimensions of the L^2–harmonic forms on the manifold $\hat{X}$ with a cylinder attached and h^-_∞ is the dimension of the limiting values of extended L^2 harmonic forms in $\Omega^-(X)$.

The identifications of all these numbers with topological quantities require some work.

1. The space $\mathcal{H}(\hat{X})$ of L^2 harmonic forms on $\hat{X}$ is naturally isomorphic to the image $\hat{H}(X)$ of
$$H^*_0(\hat{X}) \longrightarrow H^*(\hat{X}).$$
Equivalently one can use the relative De Rham cohomology $H^*(X, \partial X) \longrightarrow H^*(X)$ defined imposing boundary conditions $\omega_{|\partial X} = 0$ on the De Rham complex. This statement plays in the ∂–case the role played by Hodge theory.

2. The signature $\sigma(X)$ of a ∂–manifold is defined to be the signature of the non–degenerate quadratic form on the middle–cohomology $\hat{H}^{2k}(X)$. This is induced by the degenerate quadratic form given by the cup–product on the relative cohomology $H^{2k}(X, \partial X)$. By Lefshetz duality the radical of this quadratic form is exactly the kernel of the mapping $H^{2k}(X, \partial X) \longrightarrow H^{2k}(X)$ then
$$\sigma(X) = h^+ - h^- = \operatorname{ind}_{L^2}(A).$$

3. Then A.P.S get rid of the third number h^-_∞ proving that $h^-_\infty = h^+_\infty = h(B^{\mathrm{ev}})$ that together with $h^+_\infty + h^-_\infty = 2h(B^{\mathrm{ev}})$ gives the final signature formula
$$\sigma(X) = \int_X L - \eta(B^{\mathrm{ev}}).$$

10.2 The foliation signature

Now we pass to our foliated case following the paper of Luck and Schick [48] signature are given for a Galois covering of a compact manifold with boundary and proven to be equivalent. So let X_0 be a compact manifold with boundary equipped with an oriented $4k$–dimensional foliation transverse to the boundary and every geometric structure of product type near the boundary. As usual attach an infinite cylinder $Z_0 = \partial X_0 \times [0,\infty)_r$ and extend every The leafwise signature operator corresponds to the leafwise Clifford action (80) on the leafwise exterior bundle $\Lambda T^*\mathcal{F}$. If $(e_1, ..., e_{4k-1}, \partial_r)$ is a leafwise positive orthonormal frame near the boundary, the leafwise chirality element [16] satisfies

$$\begin{aligned}\tau := i^{2k} c(e^1)\cdots c(e^{4k-1})c(dr) = i^{2k} * (-1)^{|\cdot|(|\cdot|-1)/2} \\ = -i^{2k} c(dr) c_\partial = -i^{2k} c(dr) *_\partial (-1)^{|\cdot|+|\cdot|(|\cdot|-1)/2}\end{aligned}$$

where $*$ is leafwise Hodge duality operator, $c_\partial = c(e^1)\cdots c(e^{4k-1})$ is, a part for the i^{2k} factor the leafwise boundary chirality operator and $*_\partial$ is the leafwise boundary Hodge operator. On the cylinder the leafwise bundle $\Lambda T^*\mathcal{F}$ is isomorphic to the pulled back bundle $\rho^*(\wedge T^*\mathcal{F}_{\partial X_0})$ (the projection on the base ρ will be omitted throughout) while separating the dr component on leafwise forms $\alpha = \omega + \beta \wedge dr$ yields an isomorphism

$$(\Lambda T^*\mathcal{F})_{\partial X_0} \longrightarrow (\Lambda T^*\partial\mathcal{F}) \oplus (\Lambda T^*\partial\mathcal{F}), \tag{81}$$

sometimes we shall write $(\Lambda T^*\partial\mathcal{F}) \wedge dr$ for the second addendum in (81) to remember this isomorphism. An easy computation involving rules as

$$d\omega = d_\partial \omega + (-1)^{|\omega|}\partial_r \omega \wedge dr$$

for $\omega \in C^\infty([0,\infty); \Lambda T^*\partial\mathcal{F})$ and $c(dr)(\omega + \alpha\wedge dr) = (-1)^{|\omega|}\omega \wedge dr - (-1)^{|\alpha|}\alpha$ shows that the operator can be written on the direct sum $(\Lambda T^*\partial\mathcal{F}) \oplus (\Lambda T^*\partial\mathcal{F})$ as the matrix

$$D^{\text{sign}} = \begin{pmatrix} d_\partial + c_\partial d_\partial c_\partial & -(-1)^{|\cdot|}\partial_r \\ (-1)^{|\cdot|}\partial_r & c_\partial d_\partial c_\partial \end{pmatrix} = c(dr)\partial_r + (d_\partial + c_\partial d_\partial c_\partial) \oplus (d_\partial + c_\partial d_\partial c_\partial) \tag{82}$$

and

$$\tau = i^{2k} \begin{pmatrix} 0 & c_\partial(-1)^{|\cdot|} \\ -c_\partial(-1)^{|\cdot|} & 0 \end{pmatrix}. \tag{83}$$

Since $d_\partial^* = \tau_\partial d_\partial \tau_\partial = c_\partial d_\partial c_\partial$ formula (82) is equivalent to

$$D^{\text{sign}} = c(dr)\partial_r + (d_\partial + d_\partial^*) \oplus (d_\partial + d_\partial^*).$$

There's also another important formula corresponding to the fact that $d + d^*$ anticommutes with τ. Denote $\Omega^\pm(\mathcal{F})$ the positive (negative) eigenbundles i.e. the bundles of leafwise auto–dual (anti auto–dual) forms. We can write the operator on the cylinder as an operator on sections of the direct sum $\rho^*(\Omega^+(\mathcal{F})_{\partial X_0} \oplus \Omega^+(\mathcal{F})_{\partial X_0})$ as the matrix

$$\begin{aligned}&\begin{pmatrix} 0 & -(-1)^{|\cdot|}\partial_r + (*_\partial d_\partial - d_\partial *_\partial) i^{2k}(-1)^{|\cdot|(|\cdot|-1)/2} \\ (-1)^{|\cdot|}\partial_r + (*_\partial d_\partial - d_\partial *_\partial) i^{2k}(-1)^{|\cdot|(|\cdot|-1)/2} & 0 \end{pmatrix} \\ &= c(dr)\partial_r + (*_\partial d_\partial - d_\partial *_\partial) i^{2k}(-1)^{|\cdot|(|\cdot|-1)/2}\Omega.\end{aligned} \tag{84}$$

To pass from one representation to another we have to consider the following compositions

$$\Lambda T^*\partial\mathcal{F} \xrightarrow{i_1} (\Lambda T^*\partial\mathcal{F}) \bigoplus (\Lambda T^*\partial\mathcal{F}) \wedge dr \xrightarrow{1+\tau} \Omega^+(\mathcal{F}) \xrightarrow{d+d^*} \Omega^-(\mathcal{F}) \xrightarrow{\text{Pr}_2} \Lambda T^*\partial\mathcal{F}\ .$$

[16] we omit simbols denoting leafwise action for ease of reading

and

$$\Lambda T^*\partial\mathcal{F} \xrightarrow{i_2} \Lambda(T^*\partial\mathcal{F}) \bigoplus (\Lambda T^*\partial\mathcal{F}) \wedge dr \xrightarrow{1-\tau} \Omega^-(\mathcal{F}) \xrightarrow{d+d^*} \Omega^+(\mathcal{F}) \xrightarrow{\mathrm{Pr}_1} \Lambda T^*\partial\mathcal{F}.$$

where i_j is the inclusion on the j–th factor and Pr_j is the corresponding projection.
The first definition we give is the most simple. It is merely the L^2 index of the signature operator on the foliated manifold with a cylinder attached.

DEFINITION 10.62 — The Λ–analytic signature of the foliated manifold with boundary X_0 is the measured L^2 index of the signature operator on the foliated manifold with a cylinder attached,

$$\sigma_{\Lambda,\mathrm{an}}(X_0, \partial X_0) := \mathrm{ind}_{L^2,\Lambda}(D^{\mathrm{sign},+}).$$

Now, by the standard identification of the Atiyah–Singer integrand for the signature operator [9], formula (74) becomes

$$\sigma_{\nu,\mathrm{an}}(X_0, \partial X_0) = \langle L(X), [C_\Lambda]\rangle + 1/2[\eta_\Lambda(D^{\mathcal{F}_\partial}) - h_\Lambda^+ + h_\Lambda^-]$$

where $L(X)$ is the tangential L–characteristic class and the numbers $h_\Lambda^\pm$ and the foliation eta–invariant are referred to the boundary signature operator.
As in [4] first we have to identify these numbers. Minor modifications of the proof of Vaillant [76] are needed in order to prove the following.

PROPOSITION 10.63 — For the foliated signature operator

$$h_\Lambda^+ = h_\Lambda^-. \tag{85}$$

Consequently the formula for the analytical signature is

$$\sigma_{\nu,\,\mathrm{an}}(X_0, \partial X_0) = \langle L(X), [C_\Lambda]\rangle + 1/2[\eta_\Lambda(D^{\mathcal{F}_\partial})].$$

PROOF — Use the representation (82) of the operator on the cylinder on the bundle $(\Lambda T^*\partial\mathcal{F}) \oplus (\Lambda T^*\partial\mathcal{F})$, here we can easily write the one parameter perturbation

$$D_\epsilon^{sign} = c(dr)\partial_r + (d_\partial + d_\partial^*) \oplus (d_\partial + d_\partial^*) - \dot{\theta}\Pi_\epsilon[(d_\partial + d_\partial^*) \oplus (d_\partial + d_\partial^*)]$$

where Π_ϵ the spectral projection $\Pi_\epsilon = \chi_{(-\epsilon,\epsilon)}((d_\partial + d_\partial^*) \oplus (d_\partial + d_\partial^*))$ of the leafwise boundary (signature) operator and θ is the function considered above in (28). For much clarity we make the position

$$d_\partial + d_\partial^* = D_\partial^{\mathrm{sign}} = \mathrm{S}_\partial$$

for the boundary signature operator. Now pass to the antidiagonal form

$$c(dr)\partial_r + (*_\partial d_\partial - d_\partial *_\partial) i^{2k}(-1)^{|\cdot|(|\cdot|-1)/2}\Omega. \tag{86}$$

It is a well known fact that only the middle dimension forms contribute to form the index in fact the leafwise kernel of the signature operator is the space of leafwise harmonic forms and decompose

$$\ker \Delta_x = \oplus_{i=0}^p \ker \Delta_x^{(i)}$$

where $\Delta_x^{(i)} : \Omega^i(L_x) \longrightarrow \Omega^i(L_x)$. The subspace $\ker \Delta_x^{(r)} \oplus \Delta_x^{(n-r)}$ is τ–invariant for each $0 \leq r \leq n$ and there is a field of unitary equivalences

$$[\ker \Delta_x^{(r)} \oplus \Delta_x^{(n-r)}]^+ \longrightarrow [\ker \Delta_x^{(r)} \oplus \Delta_x^{(n-r)}]^-$$

given by $\omega + \tau\omega \longmapsto \omega - \tau\omega$. Now choose a leaf and apply the Browder–Gårding expansion exactly as in section 5 to the boundary operator in (86). We forget the subscript indicating we are on a single leaf and the isomorphisms coming from the eigenfunction expansion. A section $\xi \in \mathrm{Ext}(D^{\mathrm{sign},\pm}_{\epsilon,x})$ can be written on the cylinder $r \geq 3$,

$$\xi^{\pm}(\lambda, r) = \zeta^{\pm}(\lambda, i)[\chi_{(-\epsilon,\epsilon)}(\lambda) + (1 - \chi_{(-\epsilon,\epsilon)}(\lambda))e^{\mp\lambda r}]$$

with the fundamental fact that the boundary datas $\zeta^{\pm}(\lambda, i) \in L^2(\pm[0,\infty) \times \mathbb{N}, \mu)$ are univoquely determined by $r = 0$. Now there's a coefficient that's constant in r. It is precisely $\zeta^{\pm}(\lambda, i)\chi_{(-\epsilon,\epsilon)}(\lambda)$ and can be seen (under the spectral isomorphism) to belong to the image of the spectral projection $\chi_{(-\epsilon,\epsilon)}(\mathrm{S}_\partial \oplus \mathrm{S}_\partial)$. This subspace of $L^2(\partial L_x; \Lambda T^*\partial L_x)$ is naturally $\mathbb{Z}_2$ graded in fact the chirality operator τ commutes with the boundary operator.

In particular

$$\zeta^{\pm}(\lambda, i)\chi_{(-\epsilon,\epsilon)}(\lambda) \in [\chi_{(-\epsilon,\epsilon)}(\mathrm{S}_\partial \oplus \mathrm{S}_\partial)L^2]^{\pm}$$

The splitting becomes more evident looking at the decomposition (82)

$$\chi_{(-\epsilon,\epsilon)}(\mathrm{S}_\partial \oplus \mathrm{S}_\partial) = \chi_{(-\epsilon,\epsilon)}(\mathrm{S}_\partial) \oplus \chi_{(-\epsilon,\epsilon)}(\mathrm{S}_\partial)$$

with τ acting on the right–hand side according to

$$\tau = \begin{pmatrix} 0 & -\tau_\partial(-1)^{|\cdot|} \\ \tau_\partial(-1)^{|\cdot|} & 0 \end{pmatrix},$$

exactly formula (83). So we have defined a measurable family of maps

$$\mathcal{J}_x^{\pm} : \mathrm{Ext}(D^{\mathrm{sign},\pm}_{\epsilon,x}) \longrightarrow [\chi_{(-\epsilon,\epsilon)}(\mathrm{S}_\partial)L^2 \oplus \chi_{(-\epsilon,\epsilon)}(\mathrm{S}_\partial)L^2]^{\pm}, \quad \xi^{\pm} \longmapsto \zeta^{\pm}(\lambda, i)\chi_{(-\epsilon,\epsilon)}(\lambda).$$

Now proposition 5.41 says that if we choose δ small, say $0 < \delta < \epsilon$ then $\mathrm{Ker}_{e^{-\delta\theta}L^2}(D^{\mathrm{sign},\pm}_{\epsilon,x})$ is closed in each $e^{-\delta\theta}L^2$ and $\mathrm{Ext}(D^{\mathrm{sign},\pm}_{\epsilon,x})$ is closed into each $e^{\delta\theta}L^2$. It follows that

- We have a Borel family of continuous and middle–exact sequences

$$\begin{aligned}(\mathrm{Ker}_{L^2}(D^{\mathrm{sign},\pm}_{\epsilon,x}), \|\cdot\|_{e^{-\delta\theta}L^2}) &\longrightarrow (\mathrm{Ext}(D^{\mathrm{sign},\pm}_{\epsilon,x}), \|\cdot\|_{e^{\delta\theta}L^2}) \\ &\longrightarrow [\chi_{(-\epsilon,\epsilon)}(\mathrm{S}_\partial)L^2 \oplus \chi_{(-\epsilon,\epsilon)}(\mathrm{S}_\partial)L^2]^{\pm}\end{aligned} \tag{87}$$

  where the last arrow is $\mathcal{J}_x^{\pm}$.

- $h^{\pm}_{\Lambda,\epsilon} = \dim_\Lambda(\mathrm{range}(J^{\pm}))$.

Now join togheter $\mathcal{J}_x := \mathcal{J}_x^+ + \mathcal{J}_x^-$ assume that

$$\mathrm{range}(\mathcal{J}_x) \subset \chi_{(-\epsilon,\epsilon)}(\mathrm{S}_\partial)L^2_x \oplus \chi_{(-\epsilon,\epsilon)}(\mathrm{S}_\partial)L^2_x$$

splits into a direct sum

$$\mathrm{range}(\mathcal{J}_x) = \mathcal{V}_x \oplus \mathcal{W}_x. \tag{88}$$

Then in this case the proof ends because the chirality element acts on $\mathrm{range}(\mathcal{J}_x)$ sending $\mathcal{V}_x$ into $\mathcal{W}_x$ and vice–versa then the $\pm$ eigenspaces must be isomorphic. □

So it remains to prove (88). First we need a lemma,

LEMMA 10.64 — If $0 < \delta < \epsilon$ The family of spaces $\mathrm{range}_{e^{\delta\theta}L^2}(D_\epsilon^{\mathrm{sign}})$ is <u>Λ–closed</u> this property meaning that for every $\gamma > 0$ there exists a Borel family of closed subspaces $M \subset \mathrm{range}_{e^{\delta\theta}L^2}(D_\epsilon^{\mathrm{sign}})$ such that

$$\dim_\Lambda \overline{\mathrm{range}(D_\epsilon^{\mathrm{sign}})}^{e^{\delta\theta}L^2} - \dim_\Lambda(M) < \gamma.$$

PROOF— The first is a direct consequence of the Λ–Fredholm of the perturbed operator $D^{\mathrm{sign}_\epsilon}$ on the field $e^{\delta\theta}L^2$ in fact the commutative diagram

$$\begin{tikzcd} e^{\delta\theta}L^2 \arrow[r, "D_\epsilon^{\mathrm{sign},\pm}"] & e^{\delta\theta}L^2 \\ L^2 \arrow[u, "e^{\delta\theta}"] \arrow[r, "D_\epsilon^{\mathrm{sign},\pm}+\delta\theta"] & D \arrow[u, "e^{\delta\theta}"'] \end{tikzcd} \tag{89}$$

and lemma 5.38 show that the operator on field of weighted spaces $e^{\delta\theta}L^2$ is Breuer–Fredholm than 0 is not contained in the Λ–essential spectrum of TT^* where $T = D_\epsilon^{\mathrm{sign},\pm}$ and T^* is the adjoint w.r.t the $e^{\delta\theta}$ norm and the spaces $M_\eta := \chi_{(-\infty,\eta)}(TT^*) \cup (\chi_{(\eta,+\infty)}(TT^*)$ are Λ–finite codimensional in the closure of the image of T in L^2 (L^2 because the vertical arrows in (89) are isomorphisms that preserve the Λ–dimension). □

PROPOSITION 10.65 — For every x the image of $\mathcal{J}$ splits,

$$\mathrm{range}(\mathcal{J}_x) = \mathcal{V}_x \oplus \mathcal{W}_x.$$

PROOF— Consider the first row of (87) i.e

$$(\mathrm{Ker}_{L^2}(D_{\epsilon,x}^{\mathrm{sign},\pm}), \|\cdot\|_{e^{-\delta\theta}L^2}) \longrightarrow (\mathrm{Ext}(D_{\epsilon,x}^{\mathrm{sign},\pm}), \|\cdot\|_{e^{\delta\theta}L^2})$$

with the non–degenerate pairing $e^{-\delta\theta} \times e^{\delta\theta} \longrightarrow \mathbb{C}$ on each leaf,

$$(\mathrm{Ker}_{L^2}(D_{\epsilon,x}^{\mathrm{sign},\pm}), \|\cdot\|_{e^{-\delta\theta}L^2})^\perp = (\mathrm{Ker}_{e^{-\delta\theta}L^2}(D_{\epsilon,x}^{\mathrm{sign},\pm}))^\perp = \overline{\mathrm{range}(D_{\epsilon,x}^{\mathrm{sign},\pm})}^{e^{\delta\theta}L^2}$$

then extend $\mathcal{J}$ to be zero on the $e^{\delta\theta}$–hortocomplement of $\mathrm{Ext}(D_{\epsilon,x}^{\mathrm{sign},\pm})$ then

$$\mathrm{range}(\mathcal{J}_x) = \tilde{\mathcal{J}}_x\Big(\overline{\mathrm{range}(D_{\epsilon,x}^{\mathrm{sign},\pm})}^{e^{\delta\theta}L^2}\Big).$$

Hence by continuity we can restric ourselves

range($\mathcal{J}$) = $\mathcal{J}(\mathcal{K})$ by the continuity of $\mathcal{J}$ we can restrict our attention to elements in

$$\mathcal{K}_x^0 := \mathrm{range}_{e^{\delta\theta}L^2}(D_{\epsilon,x}^{\mathrm{sign}}) \cap \underbrace{\mathrm{Ext}(D_{\epsilon,x}^{\mathrm{sign}})}_{e^{\delta\theta}-\mathrm{closed}}$$

for each x. So let $\xi \in \mathcal{K}^0$, by definition there exist $\alpha \in e^{\delta\theta}L^2(\Lambda T^*L_x)$ such that $\xi = D_\epsilon^{\mathrm{sign}}$ and $(D_\epsilon^{\mathrm{sign}})^2\xi = 0$. On the cylinder we can write $\alpha = \alpha_0 + \alpha_1 \wedge dr$ with $\alpha_i \in H^\infty(\partial L_x \times [0,\infty); \Lambda T^*L_x)$. Using again Browder–Gårding (or a spectral resulution, it's the same) of the boundary operator S_∂ we can see that in the region $r \geq 3$ these section satisfy the differential equation

$$-(\partial_r)^2\alpha_l + (1 - \chi_{(-\epsilon,\epsilon)}(\lambda))\lambda^2\alpha_l = 0$$

with solutions in the general form

$$\alpha_l(x, r) = r\beta_{l,1}(x) + \beta_{l,2} + O(e^{-\epsilon r})$$

and $\beta_{l,i} \in \chi_{(-\epsilon,\epsilon)}(\mathrm{S}_\partial)$. Keeping in mind the identities $d + d^* = d_\epsilon + d_\epsilon^*$ with $d_\epsilon := d - d\theta\Pi_\epsilon$ and $d_\epsilon^* := d^* - d^*\theta\Pi_\epsilon$, using the identity $(1 - \Pi_\epsilon)\beta_{0,j} = 0$

$$\begin{aligned} d_\epsilon\alpha_0(x, r) &= (\epsilon(dr)\partial_r + d(1 - \Pi_\epsilon))(r\beta_{0,1}(x) + \beta_{(0,2)}(x) + O(e^{-\epsilon r}) \\ &= dr \wedge \beta_{0,1}(x) + O(e^{-\epsilon r}). \end{aligned}$$

The calculation to show that the second piece $d_\epsilon\alpha_1(x, r) \wedge dr = O(e^{-\epsilon r})$ can be performed in the same way.
For the second piece of the signature operator

$$\begin{aligned} d_\epsilon\alpha_1(x, r) \wedge dr = (-\iota(dr)\partial_r + d^*(1 - \Pi_\epsilon))(r\beta_{1,1} \wedge dr + \beta_{1,2}(x) + O(e^{-\epsilon r})) \\ = -(-1)^{|\beta_{1,1}|}\beta_{1,1}(x) + O(e^{-\epsilon r}) \end{aligned}$$

with $d_\epsilon\alpha_0(x, r) = e^{-\epsilon r}$. This shows that

$$\mathcal{J}(\xi) = \mathcal{J}(d_\epsilon\alpha + d_\epsilon^*\alpha) = 0 \oplus (-1)^{|\beta_{0,1}|}\beta_{0,1} + (-1)^{|\beta_{1,1}|}\beta_{1,1}(x, r) \oplus 0$$

and concludes the proof. □

It remains to apply 10.64 to prove (85).

Remark — Everythig works with coeficients on a rank m leafwise flat bundle, the signature formula in this case becames

$$\sigma_{\nu,\,\mathrm{an}}(X_0, \partial X_0) = m\langle L(X), [C_\Lambda]\rangle + 1/2[\eta_\Lambda(D^{\mathcal{F}_\partial})].$$

Consider the measurable field of Hilbert spaces of L^2–harmonic forms

$$x \longmapsto \mathcal{H}_x := \ker\{\Delta_x^q : L^2(\Lambda^q T^* L_x) \longrightarrow L^2(\Lambda^q T^* L_x)\}$$

where L_x is a leaf of the foliation on the manifold X with cylindrical ends. Since leafwise harmonic forms are closed this is a field of subspaces of the fields of De Rham cohomologies $H^*(L_x)$ hence inherits the structure of a measurable field of Hilbert spaces futhermore it makes sence to speak about the space of tangentially continuous sections $\mathcal{H}_\tau^q$.
So if the dimension of the foliation is $\underline{\dim(\mathcal{F}) = 4k}$ as above, we have a well defined bilinear form on the middle–degree leafwise transversally continuous (transversally measurable also goes well)

$$\mathrm{s}_\Lambda^\infty : \mathcal{H}_\tau^{2k} \times \mathcal{H}_\tau^{2k} \longrightarrow \mathbb{C},\ (\alpha, \beta) \longmapsto \int_X \alpha \wedge \beta d\Lambda = \int_X (\alpha, *\beta) d\Lambda. \tag{90}$$

given by the wedge product followed by integration against the transverse measure. This bilinear form is defined on forms (and here is simmetric) with real coefficients and extended to be sesquilinear ($\mathbb{C}$–antilinear in the second variable) on forms with complex coefficients in the usual way, $\mathrm{s}_\Lambda^\infty(\alpha, \beta \otimes \gamma) := \bar{\gamma}\, \mathrm{s}_\Lambda^\infty(\alpha, \beta \otimes \gamma)$. For sesquilinear forms to be simmetric means

$$\overline{\mathrm{s}_\Lambda^\infty(\alpha, \beta)} = \mathrm{s}_\Lambda^\infty(\beta, \alpha).$$

This field of bilinear forms corresponds, by Riesz Lemma to a continuous (measurable) field of self–adjoint bounded operators $A_x : \mathcal{H}^{2k}_{\tau,x} \longrightarrow \mathcal{H}^{2k}_{\tau,x}$ univoquely defined by the property

$$\mathrm{s}^{\infty}_{\Lambda}(\alpha, \beta) = (\alpha, A\beta)$$

where at the right–hand side the scalar product of the field of Hilbert spaces i.e., the L^2 scalar product on forms. Now A determines a field of hortogonal splittings $\mathcal{H}^{2k}_{\tau,x} = V^+_x \oplus V^0_x \oplus V^-_x$ of Hilbert spaces where $V^{\pm}_x$ is the image of the spectral projection $\chi(0,\infty)(A_x)$ $(\chi(-\infty,0)(A_x))$ and V^0_x is the kernel of A_x. The pairing on the leaf passing trough x is non degenerate if and only if $A^0_x = 0$ but we are interested in the general behaviour using the transverse measure to integrate.

DEFINITION 10.66 — The signature on harmonic forms on the foliated elongated manifold is

$$\sigma^{\infty}_{\Lambda}(X) := \dim_{\Lambda} V^+ - \dim_{\Lambda} V^-.$$

THEOREM 10.66 — The analytical signature of the compact ∂–manifold and the signature on harmonic forms on the manifold with cylinder attached coincide,

$$\sigma_{\Lambda,\mathrm{an}}(X_0, \partial X_0) = \sigma^{\infty}_{\Lambda}(X) = \langle L(X), [C_{\Lambda}] \rangle + 1/2[\eta_{\Lambda}(D^{\mathcal{F}_{\partial}})]. \tag{91}$$

PROOF — Just the definition (90) says that $B = *_{|\Omega^{2k}}$ but since the dimension of the foliation is $4k$ we have $\tau_{|\Omega^{2k}} = *_{|\Omega^{2k}}$. It follows that

$$V^{\pm} = \ker_{L^2}(D^{\mathrm{sign},\pm}).$$

□

11 Random Hilber complexes

We prove some results we shall need next in the Chapter about the signature.

11.1 ∂–manifolds with bounded geometry

The generic leaf of $(X_0, \mathcal{F})$ is a Riemannian manifold with boundary with bounded geometry as those examined by Schick [70, 71, 72].

DEFINITION 11.67 — We say that a ∂–manifold with a Riemannian metric has bounded geometry if the following holds

Normal collar : there exists $r_C > 0$ so that the geodesic collar

$$N := [0, r_C) \times \partial M : (t, x) \longmapsto \exp_x(t\nu_x)$$

is a diffeomorphism onto its image, where ν_x is the unit inward normal vector at $x \in \partial M$. Equip N with the induced metric. In the sequel N and its image will be identified. Denote $\mathrm{im}[0, r_C/3) \times \partial M$ by $N_{1/3}$ and similarly $N_{2/3}$.

Injectivity radius of ∂M : the injectivity radius of ∂M is positive, $r_{\text{inj}}(\partial M) > 0$

Injectivity radius of M : there is $r_i > 0$ so that for $x \in M - N_{1/3}$ the exponential mapping is a diffeomorphism on $B(0, r_1) \subset T_x M$. In particular if we identify $T_x M$ with $\mathbb{R}^m$ via an orthonormal frame we have Gaussian coordinates $\mathbb{R}^m \supset B(0, r_i) \longrightarrow M$ around any point in $M - N_{1/3}$

Curvature bounds : for every $K \in \mathbb{N}$ there is some $C_K > 0$ so that $|\nabla^i R| \leq C_K$ and $|\nabla^\partial l| \leq C_K$, $0 \leq i \leq K$. Here ∇ is the Levi–Civita connection on M, ∇^∂ is the Levi–Civita connection on ∂M and l is the second fundamental form tensor with respect to ν.

Choose some $0 < r_1^C < r_{\text{inj}}(\partial M)$, near points $x' \in \partial M$ on the boundary one can define **normal collar coordinates** by iteration of the exponential mapping of ∂M and that of M,

$$k_{x'} : \underbrace{B(0, r_i^C)}_{\subset \mathbb{R}^{m-1}} \times [0, r_C) \longrightarrow M, (v, t) \longmapsto \exp^M_{\exp^{\partial M}_{x'}(v)}(t\nu).$$

For points $x \in M - N_{1/3}$ standard **Gaussian coordinates** are defined via the exponential mapping. In the following we shall call both **normal coordinates**. It is a non trivial fact that the condition on curvature bounds in definition 11.67 can be substituted by uniform control of each derivative of the metric tensor g_{ij} and its inverse g^{ij} on normal coordinates. The definition extends to bounded geometry vector bundles on δ–manifolds with bounded geometry and each object of uniform analysis like i.e. uniformly bounded differential operators can be defined [72]. In particular, using a suitable partition of the unity adapted to normal coordinates one can define uniform Sobolev spaces (different coordinates give equivalent norms so we get hilbertable spaces) and every basic result continues to hold.

Proposition 11.68 — Let $E \longrightarrow M$ a bundle of bounded geometry over M. Suppose F is bounded vector bundle over ∂M. Then the following hold for the Sobolev spaces $H^s(E), H^t(F)$, $s, t \in \mathbb{R}$ of sections.

1. $H^s(E), H^t(F)$ is an Hilbert space (inner product depending on the choices).

2. The usual (bounded) Sobolev embedding theorem holds with values on the Banach space $C^k_b(E)$ of all sections with the first k derivatives uniformly bounded,

$$H^s(E) \hookrightarrow C^k_b(E), \quad \text{whenever} \quad s > m/2 + k.$$

3. For the bundle of differential forms one can use as Sobolev norm the one coming from the integral of the norm of covariant differentials

$$\|\omega\|_k^2 := \sum_{i=0}^{k} \int_M \|\nabla^i \omega(x)\|^2_{T_x^* M \otimes \Lambda T^* M} |dx|.$$

4. For $s < t$ we have a bounded embedding with dense image $H^t(E) \subset H^s(E)$. The map is compact if and only if M is compact. We define

$$H^\infty(E) := \bigcap_s H^s(E), \quad H^{-\infty}(E) := \bigcup_s H^s(E).$$

5. Let $p : C^\infty(E) \longrightarrow C^\infty(F)$ a k–bounded boundary differential operator i.e the composition of an order k bounded differential operator on E with the morphism of restriction to the boundary. Then p extends to be a bounded operator

$$p : H^s(E) \longrightarrow H^{s-k-1/2}(F), \quad s > k + 1/2.$$

In particular we have the bounded restriction map $H^s(E) \longrightarrow H^{s-1/2}(E_{|\partial M}), s > 1/2$.

6. $H^s(E)$ and $H^{-s}(E)$ are dual to each other by extension of the pairing

$$(f, g) = \int_M g(f(x))|dx|;\ f \in C_0^\infty(E),\ g \in C_0^\infty(E^*)$$

where E^* is the dual bundle of E. If E is a bounded Hermitian or Riemannian bundle, then the norm on $L^2(E)$ defined by charts is equivalent to the usual L^2–norm

$$|f|^2 := \int_M (f, f)_x |dx|,\ f \in C_0^\infty(E).$$

Moreover $H^s(E)$ and $H^{-s}(E)$ are dual to each other by extension of $(f, g) = \int_M (f, g)_x |dx|$.

11.2 Random Hilbert complexes

Now we define the De Rham L^2 complexes along the leaves. These are particular examples of Hilbert complexes studied in complete generality in [16].
So let $x \in X_0$, consider the unbounded operator with Dirichlet boundary conditions

$$d_{L_x^0} : \Omega^k_{d,x} = \{\omega \in C_0^\infty(\Lambda T^k L_x^0); \omega_{|\partial M} = 0\} \subset L^2_x(\Lambda T^k L_x^0) \longrightarrow L^2_x(\Lambda T^k L_x^0).$$

Being a differential operator it is closable, let $A^k_x(L^0_x, \partial L^0_x)$ the domain of its closure i.e the set of L^2 limits ω of sequences ω_n such that also the $d\omega_n$ converges in L^2 to some $\eta =: d\omega$. The graph norm $\| \cdot \|^2_A := \| \cdot \|^2_{L^2} + \|d \cdot \|^2_{L^2}$ gives $\| \cdot \|^2_{L^2}$ the structure of an Hilbert space making d bounded. It is easily checked that $d(A^k_x) \subset \ker(d : A^{k+1}_x) \longrightarrow L^2_x)$ then we have a Hilbert cochain complex

$$\cdots \longrightarrow A^{k-1}_x \longrightarrow A^k_x \longrightarrow A^{k+1}_x \longrightarrow \cdots$$

with cycles $Z^k_x(L^0_x, \partial L^0_x) := \ker(d : A^k_x \longrightarrow A^{k+1}_x)$ and boundaries $B^k_x(L^0_x, \partial L^0_x) := \mathrm{range}(d : A^{k-1}_x \longrightarrow A^k_x)$.

Definition 11.69 — The L^2 (reduced)[17] relative De Rham cohomology of the leaf L^0_x is defined by the quotients

$$H^{k,x}_{dR,(2)}(L^0_x, \partial L^0_x) := \frac{Z^k_x(L^0_x, \partial L^0_x)}{B^k_x(L^0_x, \partial L^0_x)}.$$

Clearly the closure is to assure the quotient to be an Hilbert space. Similar L^2–De Rham cohomologies of the whole leaf, $H^{k,x}_{dR,(2)}(L^0_x)$ and of the boundary $H^{k,x}_{dR,(2)}(\partial L^0_x)$ are defined using no (Dirichlet) boundary conditions. In particular $A^k_x(L^0_x)$ will be used to denote the domain of the closure of the differential as unbounded operator on $L^2(L^0_x)$ defined on compactly

[17] the word reduced stands for the fact we use the closure to make the quotient, also the non reduced cohomology can be defined. For a Γ covering of a compact manifold the examination of the difference reduced/unreduced cohomology leads to the definition of interesting invariants

supported sections (the support possibly meeting the boundary). The subscript dR helps to make distinction with Sobolev spaces. Each one of this spaces is naturally isomorphic to a corresponding space of harmonic forms. More precisely

Definition 11.70 — The space of k–L^2 harmonic forms which fulfill Dirichlet boundary conditions on ∂L^0_x is

$$\mathcal{H}^k_{(2)}(L^0_x, \partial L^0_x) := \{\omega \in C^\infty \cap L^2,\ \omega_{|\partial L^0_x} = 0,\ (\delta\omega)_{|\partial L^0_x} = 0,\ \underbrace{(d\omega)_{|\partial L^0_x} = 0}_{\text{gratis}}\}$$

We shall see that the boundary conditions are exactly the square of the Dirichlet boundary condition on the Dirac operator $d+\delta$. Since each leaf is complete a generalization of an idea of Gromov shows that these forms are closed and co–closed, [70, 71]

$$\mathcal{H}^k_{(2)}(L^0_x, \partial L^0_x) = \{\omega \in C^\infty \cap L^2(\Lambda^k L^0_x), d\omega = 0,\ \delta\omega = 0,\ \omega_{|\partial L^0_x} = 0\}.$$

Furthermore there's the L^2–orthogonal Hodge decomposition [70, 71]

$$L^2(\Lambda^k T^* L^0_x) = \mathcal{H}^k_{(2)}(L^0_x, \partial L^0_x) \oplus \overline{d^{k-1}\Omega^{k-1}_{d,x}(L^0_x, \partial L^0_x)}^{L^2} \oplus \overline{\delta^{k+1}\Omega^{k+1}_{\delta,x}(L^0_x, \partial L^0_x)}^{L^2}$$

where $\Omega^{k-1}_{d,x} := \{\omega \in C^\infty_0(\Lambda^{k-1}T^*L^0_x),\ \omega_{|\partial L^0_x} = 0\}$ and the corresponding one for δ with no boundary conditions $\Omega^{k+1}_{\delta,x} := \{\omega \in C^\infty_0(\Lambda^{k+1}T^*L^0_x)\}$. These decompositions shows with a little work that the inclusion $\mathcal{H}^k(L^0_x, \partial L^0_x) \hookrightarrow A^k_x$ induces isomorphism in cohomology

$$\mathcal{H}^k(L^0_x, \partial L^0_x) \cong H^k_{dR,(2)}(L^0_x, \partial L^0_x).$$

This is a consequence of the fact that the graph norm (of d) and the L^2 norm coincide on the space of cycles Z^k_x.

For further use we mention also the specular Hodge decomposition where one imposes Neumann boundary conditions on L^2 harmonic forms and Dirichlet conditions on the domain of δ,

$$\begin{aligned} L^2(\Lambda^k T^* L^0_x) = &\ker(\Delta_k | \{\omega : (*\omega)_{|\partial} = 0 = (\delta\omega)_{|\partial}\}) \\ &\overline{d^{k-1}C^\infty_0(\Lambda^{k-1}T^*L^0_x)}^{L^2} \oplus \overline{\delta^{k+1}\{\omega \in C^\infty_0(\Lambda^{k-1}T^*L^0_x) : \omega_{|\partial} = 0\}}^{L^2} \end{aligned} \tag{92}$$

Then we can write the sequence of cochain complexes

$$\begin{tikzcd}
 & {} \arrow[d, "d"] & {} \arrow[d, "d"] & {} \arrow[d, "d"] & \\
0 \arrow[r] & A^{k-1}_x(L^0_x, \partial L^0_x) \arrow[r, "i"] \arrow[d, "d"] & A^{k-1}_x(L^0_x) \arrow[r, "r"] \arrow[d, "d"] & A^{k-1}_x(\partial L^0_x) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A^k_x(L^0_x, \partial L^0_x) \arrow[r, "i"] \arrow[d, "d"] & A^k_x(L^0_x) \arrow[r, "r"] \arrow[d, "d"] & A^k_x(\partial L^0_x) \arrow[r] \arrow[d, "d"] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

where each morphism must be considered as an unbounded operator on the corresponding L^2, i is bounded since is merely the restriction of the identity mapping on $L^2(L^0_x, \Lambda T^* L^0_x)$ and r is restriction to the boundary.

PROPOSITION 11.71 —

1. For every k the domain $A^k_x(L^0_x)$ is contained in the Sobolev space of forms $H^1(L^0_x, \Lambda T^* L^0_x)$ then the composition with r makes sense.

2. The rows are (weakly) exact i.e. one has to take closure of the images of i and r in the L^2 topology in the A^k_x's.

PROOF— 1. An element ω in $A^k_x(L^0_x)$ is an L^2–limit of smooth compactly supported forms ω_n with differential also convergent in L^2. Then since the Hodge $\star$ is an isometry on L^2 also $\delta\omega_n = \pm * \omega *$ converges. In particular we can control the L^2–norm of $d\omega$ and $\delta\omega$ this means we have control of the first covariant derivative, in fact $d + \delta = c \circ \nabla$ where c is the (unitary) Clifford action then the second term can made less that the norm of ∇ by bounded geometry. In particular we have control on the order one Sobolev norm by proposition 11.68. The remaining part follows from the fact that the restriction morphism is bounded from H^1 to $H^{1/2} \hookrightarrow L^2$. 2. The only non–trivial point is exactness in the middle but as a consequence of the bounded geometry the boundary condition on the first space extends to H^1 (see proposition 5.4 in the thesis of Thomas Schick [70] that together with point 1. is exactness. □

REMARK — Note that the proof of the proposition above says also that the induced morphisms i_* and r_* are bounded.

Every arrow induces morphisms on the reduced L^2 cohomology. Miming the algebraic construction of the connecting morphism (everything works thanks to the remark above) we have, for every x the long sequence of square integrable representations of the equivalence relation $\mathcal{R}$

$$\cdot \longrightarrow H^{k,x}_{dR,(2)}(L^0_x, \partial L^0_x) \xrightarrow{i_*} H^{k,x}_{dR,(2)}(L^0_x) \xrightarrow{r_*} H^{k,x}_{dR,(2)}(\partial L^0_x) \xrightarrow{\delta} H^{k-1,x}_{dR,(2)}(L^0_x, \partial L^0_x) \longrightarrow \cdot \, .$$

As x varies in X_0 they form measurable fields of Hilbert spaces. We discuss this aspect in a slightly more general way applicable to other situations. Remember that a measurable structure on a field of Hilbert spaces over X_0 is given by a fundamental sequence of sections, $(s_x)_{x\in X_0}$, $s_n(x) \in H_x$ such that $x \longmapsto \|s_n(x)\|_{H_x}$ is measurable and $\{s_(x)\}_n$ is total in H_x (see chapter IV in [74]).

PROPOSITION 11.72 — If for a family of closed densely defined operators (P_x) with minimal domain $\mathcal{D}(P_x)$ a fundamental sequence $s_n(x) \in \mathcal{D}(P_x)$ is a core for P_x and $P_x s_n(x)$ is measurable for every x and n then the family P_x is measurable in the sense of closed unbounded operators (definition 4.22 and the remark below) i.e. the family of projections Π^g_x on the graph is measurable in the square field $H_x \oplus H_x$ with product measurable structure.

PROOF— It is trivial in fact the graph is generated by vectors $(s_n(x), P_x s_n(x))$ then the projections is measurable. □

The lemma above can be applied to the $(A^k_x(L^0_x \partial L^0_x))_x$ in fact in the appendix of [33] a fundamental sequence φ_n of sections with the property that each $(\varphi_n(\cdot))_{|L^0_x}$ is smooth and compactly supported. Now the same proof works for manifold with boundary and, since the boundary has zero measure one can certainly require to each φ_n to be zero on the boundary.

In particular we have defined complexes of square integrable representations. Reduction modulo Λ–a.e. gives complexes of random Hilbert spaces (with unbounded differentials) for which we introduce the following notations,

- $(L^2(\Omega^\bullet X_0), d)$ is the complex of Random Hilbert spaces obtained by Λ a.e. reduction from the field of Hilbert complexes

$$\cdots \longrightarrow L^2(\Lambda^k T^* L_x^0) \xrightarrow{d} L^2(\Lambda^{k+1} T^* L_x^0) \longrightarrow \cdots \tag{93}$$

- $(H^\bullet_{dR,(2)}(X_0), d)$ is the complex of Random Hilbert spaces obtained by Λ a.e. reduction from the reduced L^2 cohomology of (93)
- $(L^2(\Omega^\bullet X_0, \partial X_0), d)$ is the complex of Random Hilbert spaces obtained by Λ a.e. reduction from the field of Hilbert complexes with Dirichlet boundary condition

$$\cdots \longrightarrow L^2(\Lambda^k T^* L_x^0) \xrightarrow{d} L^2(\Lambda^{k+1} T^* L_x^0) \longrightarrow \cdots \tag{94}$$

  with differentials considered as unbounded operators with domains $A^k_x(L^0_x, \partial L^0_x)$.
- $(H^\bullet_{dR,(2)}(X_0, \partial X_0), d)$ is the complex of Random Hilbert spaces of the cohomologies of the above complex.
- For the boundaries we have the corresponding complexes of Random Hilbert spaces $(L^2(\Omega^\bullet \partial X_0), d)$ and $(H^\bullet_{dR,(2)}(\partial X_0), d)$
- The Borel field of weakly exact sequences

$$0 \longrightarrow A^{k-1}_x(L^0_x, \partial L^0_x) \xrightarrow{i} A^{k-1}_x(L^0_x) \xrightarrow{r} A^{k-1}_x(\partial L^0_x) \longrightarrow 0$$

gives rise to a long sequence of Random Hilbert spaces

$$\begin{tikzcd} H^\bullet_{dR,(2)}(X_0, \partial X_0) \arrow[r, "i^*"] & H^\bullet_{dR,(2)}(X_0) \arrow[d, "r_*"] \\ & H^\bullet_{dR,(2)}(\partial X_0). \arrow[ul, "\delta"] \end{tikzcd} \tag{95}$$

The meaning of this construction is clear first one builds the long L^2 leafwise sequence of the pairs $(L^0_x, \partial L^0_x)$ then uses the transverse measure to collect the informations together.

Remark — The notation $H_{dR,(2)}(\partial X_0)$ must not be confusing. This is not obtained by the De Rham square integrable representation of the boundary foliation. This is deduced from Λ–equivalence from the square integrable representation of the equivalence relation $\mathcal{R}$ of the whole foliation.

Now we follow the paper by Cheeger and Gromov to a notion of exactness for the long sequence (95) and the right assumption assuring it.

Definition 11.73 — We say that a sequence of Random Hilbert spaces as (95) is Λ–weakly exact at a point if in the correspondig Von Neumann algebra of Endomorphisms the projection on the closure of the range of coming arrow coincide with the projection on the kernel of the starting one. These means i.e at point $\xrightarrow{i^*} H^k_{dR,(2)}(X_0) \xrightarrow{r^*}$,

$$\overline{\operatorname{range} i^*} = \ker i^* \in \operatorname{End}_\Lambda(H^k_{dR,(2)}(X_0)).$$

11.3 Spectral density functions and Fredholm complexes.

Let U,V two Random Hilbert spaces on $\mathcal{R}_0$ (for these consideration also the holonomy groupoid or, more generally a Borel groupoid should work) and an unbounded Random operator $f : \mathcal{D}(f) \subset U \longrightarrow V$ i.e start with a Borel family of closed densely defined operators $f_x : U_x \longrightarrow V_x$ intertwining the representation of $\mathcal{R}_0$. Since f is closable, the question of measurability is addressed in definition4.22. For every $\mu \geq 0$ put $\mathcal{L}(f, \lambda)$ as the set of measurable fields of subspaces $L_x \subset \mathcal{D}(f_x) \subset U_x$ (measurability is measurability of the family of the closures) such that for every $x \in X_0$ and $\phi \in L_x$, $\|f_x(\phi)\| \leq \mu\|\phi\|$. After reduction modulo Λ a.e. this becomes a set of Random Pre–Hilbert spaces we call $\mathcal{L}_\Lambda(f, \mu)$

DEFINITION 11.74 — The Λ–spectral density function of f is the monotone increasing function

$$\mu \longmapsto F_\Lambda(f, \mu) := \sup\{\dim_\Lambda : L \in \mathcal{L}_\Lambda(f, \mu)\}.$$

Where of course one has to pass to the closure to apply the Λ–dimension. We say f Fredholm if for some $\epsilon > 0$, $F_\Lambda(f, \epsilon) < \infty$

We want to show that this definition actually coincides with the definition given in term of the spectral measure of the positive self–adjoint operator f^*f.

LEMMA 11.75 — In the situation above

$$F_\Lambda(f, \mu) = \operatorname{tr}_\Lambda \chi_{[0,\mu^2]}(f^*f) = \dim_\Lambda \operatorname{range}(\chi_{[0,\mu^2]}(f^*f))$$

as a projection in $\operatorname{End}_\Lambda(U)$.
Notice that since f^*f is a positive operator $\chi_{[0,\mu^2]}(f^*f) = \chi_{(-\infty,\mu^2]}(f^*f)$ is the spectral projection associated to the spectral resolution $f^*f = \int_{-\infty}^{\infty} \mu d\chi_{(-\infty,\mu]}$.

PROOF — The spectral Theorem (a parametrized measurable version) shows that the ranges of the family of projections $\chi_{[0,\mu^2]}(f^*f)$ belong to the class $\mathcal{L}(f, \mu)$, then

$$\dim_\Lambda(\operatorname{range}(\chi_{[0,\mu^2)}(f^*f))) \leq F_\Lambda(f, \mu).$$

In fact it's clear that $\chi_{[0,\mu^2)}(f_x^*f_x)\omega = \omega \Rightarrow \|f\omega\| \leq \mu\|\omega\|$. But now for every $L \in \mathcal{L}(f, \mu)$ we get a family of injections $\chi_{\mu^2}(f_x^*f_x)_{|L_x} \longrightarrow \operatorname{range}(\chi_{\mu^2}(f_x^*f_x))$ that after reduction modulo Λ and with the crucial property of the formal dimension 3 in lemma 4.7 says

$$\dim_\Lambda(L) \leq \dim_\Lambda(\operatorname{range}(\chi_{[0,\mu^2]}(f^*f)).$$

□

DEFINITION 11.76 — A complex of random Hilbert cochains as $(L^2(\Omega^\bullet X_0), d)$ and its relative and boundary versions is said Λ–Fredholm at point k if the differential induced on the quotient

$$\frac{\mathcal{D}(d^k)}{\mathrm{range}(d^{k-1})} \xrightarrow{d} L^2(\Omega^{k+1}X_0)$$

gives by Λ a.e. reduction a left Fredholm unbounded operator in the sence of definition 11.74. In particular the condition involving the spectrum distribution function is

$$F_\Lambda\big(d| : \mathcal{D}(d^k) \cap \mathrm{range}(d^{k-1})^\perp \longrightarrow L^2(\Omega^{k+1}X_0), \mu\big) < \infty \tag{96}$$

for some positive number μ.
For this reasin one calls the left hand–side of (96)

$$F_\Lambda\Big(L^2(\Omega^k X_0, \partial X_0), \mu\Big) := F_\Lambda\big(d| : \mathcal{D}(d^k) \cap \mathrm{range}(d^{k-1})^\perp \longrightarrow L^2(\Omega^{k+1}X_0), \mu\big)$$

the spectral density function of the complex at point k.

REMARK — Definition above combined with lemma 11.75 says that we have to compute the formal dimension of $\chi_{[0,\mu^2]}(f^*f)$ where $f = d_{|\mathcal{D}(d)\cap\overline{\mathrm{range}(d^{k-1})}^\perp}$ but f is an injective restriction of d^k then every spectral projection $\chi_B(f^*f)$ projects onto a subspace that's orthogonal to $\ker(d^k)$ then

$$F_\Lambda\big(d| : \mathcal{D}(d^k) \cap \mathrm{range}(d^{k-1})^\perp \longrightarrow L^2(\Omega^{k+1}X_0), \mu\big) = \sup \mathcal{L}_\Lambda^\perp(f, \mu) \tag{97}$$

where $\mathcal{L}_\Lambda^\perp(f, \mu)$ is the set of Random fields of subspaces of $\mathcal{D}(d) \cap \ker(d)^\perp$ where d is bounded by μ (see Definition 11.74)

THEOREM 11.76 — All the three complexes of Random Hilbert spaces $(L^2(\Omega^\bullet X_0), d)$, $(L^2(\Omega^\bullet \partial X_0), d)$ and $(L^2(\Omega^\bullet X_0, \partial X_0), d)$ considered above are Λ–Fredholm.

PROOF— The proof follows by an accurate inspection of the relation between the differentials (with or without boundary conditions) and the Laplace operator trough the theory of selfadjoint boundary differential problems developed in [70]. To make the notation lighter let $M = L_x^0$ with $\partial M = \partial L_x^0$ the generic leaf. We concentrate on the relative sequence at point $d : A^k(M, \partial M) \longrightarrow A^{k+1}(M, \partial M)$ where the differential is an unbounded operator on L^2 with Dirichlet boundary conditions. Let $\mathcal{D}(d) = A^{k+1}(M, \partial M)$. The following Lemma is inspired by Lemma 5.11 in [47] where Neumann boudary conditions are imposed.

LEMMA 11.77 — Let $\ker(d)$ the kernel of d as unbounded operator with Dirichlet boundary conditions, then

$$\mathcal{D}(d) \cap \ker(d)^\perp = H^1_{\mathrm{Dir}} \cap \overline{\delta^{k+1} C_0^\infty(\Lambda^{k+1} T^* M)}^{L^2}$$

where H^1_{Dir} is the space of order 1 Sobolev k–forms ω such that $\omega_{|\partial M} = 0$.

PROOF— First of all remember that the differential operator $d + \delta : C^\infty(\Lambda^\bullet T^* M) \longrightarrow$

$C^\infty(\Lambda^\bullet T^*M)$ with either Dirichlet or Neumann boundary conditions is formally self–adjoint with respect to the greenian formula

$$(d^r\omega,\eta)-(\omega,\delta^{p+1}\eta)=\int_{\partial M}(\omega\wedge *^{p+1}\eta)_{|\eta}$$

and uniformly elliptic [70]. This means that this is an elliptic boundary value problem in the classical sense according to the definition of Lopatinski and Shapiro [57], Appendix I, together with a uniform condition on the local fundamental solutions. Now let $\omega\in C_0^\infty$ and $\eta\in\ker(d)$ i.e. $\eta_n\in C_0^\infty$, $(\eta_n)_{|\partial M}=0$, $\eta_n\xrightarrow{L^2}\eta$, $d\eta_n\xrightarrow{L^2}0$ then

$$(\eta,\delta\omega)=\lim_n(\eta_n,\delta\omega)=\underbrace{\lim_n(d\eta_n,\omega)}_{0}\pm\underbrace{\int_{\partial M}(\eta_n\wedge *\omega)_{|\partial M}}_{\eta_{|\partial M}=0}=0,$$

showing that $\overline{\delta C_0^\infty}\subset\mathcal{D}(d)\cap\ker(d)^\perp$. For the reverse inclusion take $\omega\in\mathcal{D}(d)\cap\ker(d)^\perp$ i.e. $\omega_n\in C_0^\infty$, $\omega_n\xrightarrow{L^2}\omega$, $d\omega_n\xrightarrow{L^2}0$. For fixed $\eta\in C_0^\infty$,

$$\underbrace{((d+\delta)\eta,\omega)}_{d\eta\in\ker(d),\omega\in\ker(d)^\perp}=(\delta\eta,\omega)=\lim_n(\delta\eta,\omega_n)\underbrace{=}_{\omega_{n\,|\partial M}=0}=\lim_n(\eta,d\omega).$$

Then we can apply the adjoint regularity theorem of Hörmander [70] Lemma 4.19, cor 4.22 saying that $\omega\in H^1_{\mathrm{loc}}$ then $(\delta\omega,\eta)=(\omega,d\eta)$ holds because for every $\eta\in C_0^\infty(M-\partial M)$, $d\eta\in\ker(d)$ then $\delta\omega=0$. It follows that for every $\sigma\in C_0^\infty$

$$0\underbrace{=}_{d\sigma\in\ker(d)}(d\sigma,\omega)=\underbrace{(\sigma,\delta\omega)}_{0}\pm\int_{\partial M}(\sigma\wedge *\omega)_{|\partial M}=\pm\int_{\partial M}(\overline{\omega}\wedge\overline{*\sigma})_{|\partial M}.$$

The last passage coming from the definition of the Hodge $*$ operator, $\sigma\wedge *\omega=(\sigma,\omega)dvol=(\overline{\omega},\overline{\sigma})dvol=\overline{\omega}\wedge\overline{*\sigma}$, where $\bar{\cdot}$ is the complex conjugate in $\Lambda T^*M\otimes\mathbb{C}$. Now from the density of $\{i^*(\overline{*\sigma})\}_{\sigma\in C_0^\infty}$ in $L^2(\partial M)$, $i:\partial M\hookrightarrow M$ the boundary condition $\omega_{|\partial M}=0$ follows in particular $\omega\in H^1_{\mathrm{Dir}}$. Now it remains to apply the Hodge decomposition

$$L^2(\Lambda^kT^*M)=\mathcal{H}^k_{(2)}(M,\partial M)\oplus\overline{d^{k-1}\Omega^{k-1}_d(M,\partial M)}^{L^2}\oplus\overline{\delta^{k+1}\underbrace{\Omega^{k+1}_\delta(M,\partial M)}_{\text{no }\partial-\text{conditions}}}^{L^2}$$

to deduce $\omega\in\overline{\delta^{k+1}C_0^\infty(\Lambda^{k+1}T^*M)}^{L^2}$. □

Consider again the formally selfadjoint boundary value problem $d+\delta$ with Dirichlet boundary conditions i.e $\mathcal{D}(d+\delta)=H^1_{\mathrm{Dir}}$. Its square in the sense of unbounded operators on L^2 is the laplacian Δ with domain

$$H^2_{\mathrm{Dir}}:=\{\omega\in H^2:\omega_{|\partial M}=0,\ ((d+\delta)\omega)_{|\partial M}=(\delta\omega)_{|\partial M}=0\}.$$

Let $\Delta_k^\perp$ the operator obtained from Δ on k–forms restricted to the orthogonal complement of its kernel, it is easy to see that the splitting

$$L^2(\Lambda^kT^*M)=\mathcal{H}^k_{(2)}(M,\partial M)\oplus\overline{d^{k-1}\Omega^{k-1}_d(M,\partial M)}^{L^2}\oplus\overline{\delta^{k+1}\underbrace{\Omega^{k+1}_\delta(M,\partial M)}_{\text{no }\partial-\text{conditions}}}^{L^2}$$

induces the following splitting on Δ_k,

$$\Delta_k^\perp = (\delta^{k+1} d^p)_{|\overline{\delta^{k+1}\Omega_\delta^{k+1}}} \oplus (d^{k-1}\delta^k)_{|\overline{d^{k-1}\Omega_d^{k-1}}}.$$

Lemma 11.78 — The following identies of unbounded operators hold

$$\begin{aligned}(\delta^{k+1} d^p)_{|\overline{\delta^{k+1}\Omega_\delta^{k+1}}} &= (d^k_{|\overline{\delta^{k+1}\Omega_\delta^{k+1}}})^* (d^k_{|\overline{\delta^{k+1}\Omega_\delta^{k+1}}}),\\ (d^{k-1}\delta^k)_{|\overline{d^{k-1}\Omega_d^{k-1}}} &= (d^{k-1}_{|\overline{\delta^k\Omega_\delta^k}})(d^{k-1}_{|\overline{\delta^k\Omega_\delta^k}})^*\end{aligned}$$

where the $d^k_{|\overline{\delta^{k+1}\Omega_\delta^{k+1}}}$ is the unbounded operator on the subspace $\overline{\delta^{k+1}\Omega_\delta^{k+1}}$ of L^2 with domain $H^1_{\mathrm{Dir}} \cap \overline{\delta^{k+1}\Omega_\delta^{k+1}}$ and range $\overline{d^{k+1}\Omega_d^{k+1}}$.

Proof — This is again the *dual* (in the sense of boundary conditions) statement of Lemma 5.16 in [47]. We first state that the Hilbert space adjoint of the operator d^k with domain $H^1_{\mathrm{Dir}} \cap \overline{\delta^{k+1}\Omega_\delta^{k+1}}$ and range $\overline{d^{k+1}\Omega_d^{k+1}}$ is exactly δ^{k+1} with domain $H^1_{\mathrm{Dir}} \cap \overline{d^k\Omega_d^k}$. We shall omit grades of forms and call d this restricted operator. Thanks to the intersection with H^1 this is also the restriction of $d+\delta$ to the same subspace, in particular $\omega \in \mathcal{D}(d^*) \subset \overline{dC_0^\infty}$ implies $\omega \in \mathcal{D}(d)$ and $d\omega = 0$. Take arbitrary $\eta \in H^1_{\mathrm{Dir}} \cap \overline{\delta C_0^\infty}$, then since $\delta\eta = 0$, $((d+\delta)\eta, \omega) = (d\eta, \omega) = (\eta, d^*\omega)$ and if $\eta \in H^1_{\mathrm{Dir}} \cap \overline{d\Omega_d}$, $((d+\delta)\eta,\omega) = (\delta\eta,\omega) = 0$. Since $\delta H^1_{\mathrm{Dir}} \perp \overline{d\Omega_d}$ this is immediately checked,
$\sigma \in d\Omega_d$, $\sigma = d\lambda$, $\lambda_{|\partial M} = 0$, $(\sigma, \delta\gamma) = \underbrace{(d\sigma,\gamma)}_{=0} + \int_{|\partial M} \underbrace{(\sigma \wedge *\gamma)_{|\partial M}}_{=0}$.
Also $(\eta, d^*\omega) = 0$ since $d^*\omega \in \overline{\delta C_0^\infty}$ and $d\Omega_{\mathrm{Dir}} \perp \delta C_0^\infty$. Then we can apply again the adjoint regularity theorem [70], Lemma 4.19 to deduce $\omega \in H^1_{\mathrm{loc}}$. The next goal is to show $\omega \in H^1_{\mathrm{Dir}}$ i.e. $d\omega, \delta\omega \in L^2$, $\omega_{|\partial M} = 0$ but $dx = 0 \in L^2$, $\delta\omega = (d+\delta)\omega = d^*\omega \in L^2$ and
$(\omega, d\delta\eta) = (d^*\omega, \delta\eta) = (\delta\omega, \delta\eta) = (\omega, d\delta\eta) \pm \overline{\int_{\partial M}(\delta\eta \wedge *\omega)_{|\partial M}}$ for every $\eta \in C_0^\infty$. Then $0 = \int_{\partial M}(\delta\eta \wedge *\omega)_{|\partial M} = \int_{\partial M}(\bar\omega \wedge \overline{*\delta\eta})_{|\partial M} \underbrace{=}_{=0} \int_{\partial M}(\omega \wedge *\delta\eta)_{|\partial M}$ for every η. The boundary condition follows by density. Finally it is clear that $\delta d_{|\mathcal{D}(d^*d)} = \Delta = \Delta^\perp$ but we have to prove the coincidence of the domains

$$\mathcal{D}(\Delta) \cap \overline{\delta C_0^\infty} = \mathcal{D}(d^*(d_{|\overline{\delta C_0^\infty}})),$$

now $\mathcal{D}(\Delta) = H^2_{\mathrm{Dir}} = \{\omega \in H^2,\ \omega_{|\partial M},\ (\delta\omega)_{|\partial M} = 0\} \subset \mathcal{D}(d^* d_{|\overline{\delta C_0^\infty}})$. Clearly

$$\omega \in \mathcal{D}(d^* d_{|\overline{\delta C_0^\infty}}) \Rightarrow \omega \in H^1_{\mathrm{Dir}} \cap \overline{\delta C_0^\infty},$$

$d\omega \in H^1_{\mathrm{Dir}}$ then $(d+\delta)\omega \in H^1$ and since $\omega_{|\partial M} = 0$ by elliptic regularity (for the boundary value problem $(d+\delta)$ with Dirichlet conditions [70]) $\omega \in H^2$. We have just checked the boundary conditions, finally $\omega \in H^2_{\mathrm{Dir}} = \mathcal{D}(\Delta)$. The second equality in the statement is proven in a very similar way. □

Now that the relation of d with Dirichlet boundary condition restricted to the complement of its kernel with the Laplacian ($\Delta^\perp$) is clear we can use elliptic regularity to deduce that the relative Random Hilbert complex is Λ–Fredholm. This has to be done in two steps, the first

is to show that the spectral function of the Laplacian controls the spectral function of the complex
$$F_\Lambda(\Delta_k^\perp,\sqrt{\mu})=F_\Lambda(L^2(\Omega^k X_0,\partial X_0),\mu)+F_\Lambda(L^2(\Omega^{k-1}X_0,\partial X_0),\mu) \tag{98}$$
in fact
$$\begin{aligned}F_\Lambda(\Delta_k^\perp,\sqrt{\mu}) &= F_\Lambda\Big((\delta^{k+1}d^k)_{|\overline{\delta^{k+1}\Omega_\delta^{k+1}}}),\sqrt{\mu}\Big)+F_\Lambda\Big((d^{k-1}\delta^k)_{|\overline{d^{k-1}\Omega_d^{k-1}}}),\sqrt{\mu}\Big)\\ &= F_\Lambda\Big((d^k_{|\overline{\delta^{k+1}\Omega_\delta^{k+1}}})^*(d^k_{|\overline{\delta^{k+1}\Omega_\delta^{k+1}}}),\sqrt{\mu}\Big)+F_\Lambda\Big((d^{k-1}_{|\overline{\delta^k\Omega_\delta^k}})(d^{k-1}_{|\overline{\delta^k\Omega_\delta^k}})^*,\sqrt{\mu}\Big)\\ &= F_\Lambda\Big(d^k_{|\overline{\delta^{k+1}\Omega_\delta^{k+1}}},\mu\Big)+F_\Lambda(d^{k-1}_{|\overline{\delta^k\Omega_\delta^k}},\mu\Big)\end{aligned}$$

where, at first step we have used the obvious fact that the spectral functions behave additively under direct sum of operators togheter with the remark after (11.76) , at the second step there are lemmas 11.77 and 11.78 together with the following properties of the spectral functions

- $F_\Lambda\Big(f^*f,\sqrt{\lambda}\Big)=F_\Lambda(f,\lambda)$
- $F_\Lambda(\phi,\lambda)=F_\lambda(\phi^*,\lambda)$

that can be adapted to hold in our situation with unbounded operators. Good references are the paper of Lott and Lück [44] and the paper of Lück and Schick [47] that inspired completely this treatment.
Thanks to (98) it remains to show that $\Delta_k^\perp$ is left Λ–Fredholm. We can use the heat kernel, in fact by elliptic regularity for each leaf the heat kernel $e^{-t\Delta_{k},x^\perp}(z,z')$ is smooth and uniformly bounded along the leaf on intervals $[t_0,\infty)$ [70] Theorem 2.35. As x varies in X_0 these bounds can made uniform by the uniform geometry (in fact the constants depend on the metric tensor, its inverse and a finite number of their derivatives in normal coordinates) and we get a family of smooth kernels that varies transversally in a measurable fashion since it is obtained by functional calculus from a measurable family of operators. Then they give a Λ–trace class element in the Von neumann algebra. Now the projections $\chi_{[0,\mu]}(f^*f)$ in definition 11.76 where f is the differential restricted to the complement of its kernel are obtained from the heat kernel as
$$\chi_{[0,\mu]}(f^*f)=\underbrace{\chi_{[0,\mu]}(\Delta_k^\perp)e^{\Delta_k^\perp}}_{\text{bounded}}\underbrace{\chi_{[0,\mu]}(\Delta_k^\perp)e^{-\Delta_k^\perp}}_{\Lambda-\text{trace class}}.$$

□

Remark — The same argument of elliptic regularity for b.v. problems togheter with the various Hodge decompositions shows that each term of the long sequence (95) <u>is a finite Random Hilbert space.</u>

12 L^2– De Rham signature

Let $\dim(\mathcal{F})=4k$ Consider the measurable field of Hilbert spaces $A_x^k(L_x^0,\partial L_x^0)$ of the minimal domains of the De Rham leafwise differential with Dirichlet boundary conditions $\omega_{|\partial L_x^0}=0$ as in section 11.2 with the graph Hilbert structure and the induce Borel structure. This square integrable representation of $\mathcal{R}_0$ carries a field of bounded symmetric sesquilinear forms defined by
$$s_x^0:A_x^{2k}(L_x^0,\partial L_x^0)\times A_x^{2k}(L_x^0,\partial L_x^0)\longrightarrow\mathbb{C},(\omega,\eta)\longmapsto\int_{L_x^0}\omega\wedge\overline{\eta}=\int_{L_x^0}(\omega,*\eta)d\nu^x$$

i.e. the $\mathbb{C}$–antilinear in the second variable extension of the wedge product on forms , $\overline{\sigma \otimes \gamma} = \sigma \otimes \bar{\gamma}$ is the complex conjugate and ν^x is the Leafwise Riemannian metric. Note that also the scalar product $(\cdot, \cdot)$ on forms is extended to be sesquilinear.

Lemma 12.79 — The sesquilinear form s_x^0 passes to the L^2 relative cohomology of the leaf $H^{2k}_{dR,(2)}(L_x^0, \partial L_x^0)$ factorizing through the image of the map $H^{2k}_{dR,(2)}(L_x^0, \partial L_x^0) \longrightarrow H^{2k}_{dR,(2)}(L_x^0)$ of the L^2 relative de Rham cohomology to the L^2 de Rham cohomology exactly as in the compact (one leaf) case.

Proof — The first assertion is simply Stokes theorem, in fact let $\omega \in A_x^{2k}(L_x^0, \partial L_x^0)$ i.e. $\omega_n \xrightarrow{L^2} \omega$, $d\omega_n \xrightarrow{L^2} 0$ and $\theta_m \in C_0^\infty(\Lambda T^{2k-1} L_x^0)$, $d\theta_m \xrightarrow{L^2} \varphi$ then

$$s_x^0(\omega, \varphi) = \lim_{n,m} \int_{L_x^0} \omega_n \wedge \overline{d\theta_m} = \lim_{n,m} \int_{L_x^0} d(\omega_n \wedge \overline{\theta_m}) = \lim_{n,m} \int_{\partial L_x^0} (\omega_n \wedge \theta_m)_{|\partial L_x^0} = 0.$$

The second one is clear and follows exactly from the classical case i.e. if $\beta_1 = \beta_2 + \lim_n d\rho_n$ with ρ_n compactly supported with no boundary conditions write

$$s_x^0([\alpha], [\beta]) = s_x^0([\alpha], [\beta_2]) + \lim_n \int \alpha \wedge \rho_n,$$

represent α as a L^2 limit of forms with Dirichlet boundary conditions than apply Stokes theorem again. □

For every x the sesquilinear form s_x^0 on the cohomology corresponds to a bounded selfadjoint operator $B_x \in B(H^{2k}_{dR,(2)}(L_x^0, \partial L_x^0))$ (a proof in [63]) univoquely determined by the condition $s_x^0(\alpha, \beta) = (\alpha, B_x \beta)$. Measurability properties of $(s_x^0)_{x \in X_0}$ are by definition (for us) measurability properties of the family $(B_x)_x$. It is clear that everything varies in a Borel fashion (use again a smooth fundamental sequence of vector fields as in [33]) then the B_x's define a self–adjoint random operator $B \in \mathrm{End}_\Lambda(H^{2k}_{dR,(2)}(X_0, \partial X_0))$.

Definition 12.80 — The Λ–L^2 De Rham signature of the foliated manifold X_0 with boundary ∂X_0 is

$$\sigma_{\Lambda, dR}(X_0, \partial X_0) := \mathrm{tr}_\Lambda\, \chi_{(0,\infty)}(B) - \mathrm{tr}_\Lambda\, \chi_{(-\infty,0)}(B)$$

as random operators in $\mathrm{End}_\Lambda(H^{2k}_{dR,(2)}(X_0, \partial X_0))$.

Theorem 12.80 — We have

$$\sigma_{\Lambda, dR}(X_0, \partial X_0) = \sigma_{\Lambda, \mathrm{an}}(X, \partial X_0)$$

then together with formula (91) w.rt. the manifold with cylinder attached X all the three signatures we have defined agree

$$\sigma_{\Lambda, dR}(X_0, \partial X_0) = \sigma_{\Lambda, \mathrm{an}}(X_0, \partial X_0) = \sigma_\Lambda^\infty(X) = \langle L(X), [C_\Lambda] \rangle + 1/2[\eta_\Lambda(D^{\mathcal{F}_\partial})]$$

Proof —

First step.This is done. We have just proved, following the method of Vaillant the equality $\sigma_{\Lambda,\mathrm{an}}(X_0, \partial X_0) = \sigma_\Lambda^\infty(X)$ where at right the signature on harmonic leafwise L^2–forms on the elonged manifold with elonged foliation i.e. the Λ signature of the Poincarè product on leafwise harmonic forms. Our reference is then the harmonic signature.

Second step. We shall prove $\sigma_{\Lambda,dR}(X_0, \partial X_0) = \sigma_\Lambda^\infty(X)$. Remember the notation $x \in X_0$, L_x^0 is the leaf of the compact foliated manifold with boundary, L_x is the leaf of the foliation on the manifold X with a cylinder attached. Consider the random Hilbert space $H^{2k}_{dr,(2)}(X_0)$ obtained from the various L^2 cohomologies of the leaves with no boundary conditions (this is called in [48] the L^2–homology since it naturally pairies with forms with Dirichlet boundary conditions). We have a family of restriction maps $X_0 \ni x \longmapsto r_x^p : \mathcal{H}^{2k}(L_x) \longrightarrow H^{2k}_{dR,(2)}(L_x^0)$ where we stress the fundamental fact that the variable x is the compact piece X_0 in order to obtain an intertwining operator $(\mathcal{H}^{2k}(L_x))_{x \in X_0} :\longmapsto H^{2k}_{dR,(2)}(L_x^0)$ where the first is seen as a square integrable representation of $\mathcal{R}_0$. There are also natural mappings $i_x^{2k} : H^{2k}_{dR,(2)}(L_x^0, \partial L_x^0) \longrightarrow H^{2k}_{dR,(2)}(L_x)$. The program of Lück and Schick fits well here and is:

1. Λ a.e. $\overline{\mathrm{range}(r_x^{2k})} = \mathrm{range}(i_x^{2k})$ and the signature can be computed looking the fields of sesquilinear Poincarè products on the images of i_x^{2k} as square integrable representations of $\mathcal{R}_0$,

$$\begin{tikzcd} H^{2k}_{dR,(2)}(L_x^0, \partial L_x^0) \arrow[r, "i_x^{2k}"] & H^{2k}_{dR,(2)}(L_x^0) \; . \\ \mathcal{H}^{2k}(L_x) \arrow[ur, "r_x^{2k}"'] & \end{tikzcd} \tag{99}$$

2. The signature of the field of products on the image of i_x^{2k} concides with the signature of the fields of Poincaré products on $(\mathcal{H}_x)_{x \in X_0}$ as square integrable representations of $\mathcal{R}_0$.

1.

□

A Analysis on Manifolds with bounded geometry

Hereafter we review some essential results about differential operators, and the Dirac one in particular, on manifolds with bounded geometry. This theory was developed by J. Roe [65, 66, 67], M. Shubin [69] and J. Lott [?] among others.

Let M be an oriented Riemannian manifold of bounded geometry, by definition,

1. the injectivity radius of M, $\mathrm{inj}(M)$, defined as the infimum on M of radii of regular geodesic balls is finite.
2. The Riemann curvature tensor is uniformly bounded with every covariant derivative.

DEFINITION A.81 — For an vector bundle to be of bounded geometry will mean that it is given a connection with uniformly bounded curvature together with every covariant derivative.

Natural examples are, compact manifolds, Galois covering of compact manifolds, the interior of a compact manifold with boundary equipped with a b–metric and finally leaves of a compact foliated manifold. An obvious but important property is that compact perturbations, i.e. connected sum preserve bounded geometry.

Note that a non–compact manifold with bounded geometry has infinite volume. Directly from

the definition one finds that if $\dim(M) = n$ there exists a positive number r such that the eclidean ball $B(0, r) \subset \mathbb{R}^n$ is the domain of exponential coordinates for every point in M. The Christoffel symbols of M regarded as a family of smooth functions depending on i, j, k and points m in B are a bounded subset of the Fréchet space $C^\infty(B)$. These geodetic balls can be used also to trivialize bundles by parallel traslation along geodesic rays of a fixed orthonormal basis at the center. Such frames are called *synchronous*. With a "*good coordinate ball*" one refers to this situation.

We shall consider till the end of this section Clifford modules of bounded geometry with $\mathbb{Z}_2$ graduated structure denoted generally by S and call D the associated Dirac operator.

Definition A.82 —

1. For $k \in \mathbb{N}$ the Sobolev space of sections of $H^k(S)$ is the completion of $C_c^\infty(S)$ under the norm
$$\|s\|_k = (\|s\|_{L^2}^2 + \|\nabla s\|_{L^2}^2 + \cdots \|\nabla^k s\|_{L^2}^2)^{1/2}.$$
2. For negative k, $H^k(S)$ is the dual space of $H^{-k}(S)$ regarded as a distributional sections space.
3. Put $H^\infty(S) = \bigcap_k H^k(S)$ equipped with its natural Fréchet topology, $H^\infty(S) = \bigcup H^k(S)$ with the weak topology that it inherits as as the dual of $H^\infty(S)$.

Definition A.83 —

1. Let $r \in \mathbb{N}$, the uniform C^r space is the Banach space of all C^r sections s of S such that the norm
$$|||s|||_r = \sup \left\{ |\nabla_{v_1} \cdots \nabla_{v_r} s(m)| \right\}$$
is finite, supremum taken over points $m \in M$ and collections $\{v_1, ..., v_q, 0 \leq q \leq r\}$ of unit vectors at m.
2. Also, $UC^\infty(S)$ is the Fréchet space $\bigcap_r UC^r(S)$.

The algebra of differential operators $\mathrm{Diff}^*(M, S)$ acting on S contains the subalgebra $\mathrm{UDiff}^*(M, S)$ of *uniform differential operators* generated by the uniform space $UC^\infty(\mathrm{End}(S))$ together with covariant derivatives ∇_X^S (as differential operators) along uniform vector fields $X \in UC^\infty(TM)$.

It turns out that for a differential operator to be uniformly elliptic is necessary and sufficient to have every derivative (also 0 order of course) of its symbol uniformly bounded on every good coordinate ball. A k–order uniform differential operator naturally defines continuous mappings, $H^r(M, S) \longrightarrow H^{r-k}(M, S)$ and $UC^l(M, S) \longrightarrow UC^{l-k}(M, S)$.

Definition A.84 — An uniform differential operator $P \in \mathrm{UDiff}^*(M, S)$ is uniformly elliptic if its principal symbol
$$\sigma_{\mathrm{pr}}(P) \in \mathrm{UC}(T^*M, \pi^*(\mathrm{End}(S))$$
has an uniform inverse in an ϵ–neightborhood of the zero section in T^*M.

THEOREM A.84 — (uniform Gårding's inequality) For an uniformly elliptic operator $T \in \mathrm{UDiff}^k(M,S)$, for every l there exists a positive constant $C(l)$ such that

$$\|s\|_{H^{s+k}} \leq C(l)\{\|s\|_{H^s} + \|Ps\|_{H^s}\}, \tag{100}$$

for every $s \in C_c^\infty(M,S)$.

PROOF — A straightforward generalization of compact case. □

Here a list of properties

In this framework the Sobolev embedding theorem reads as follows,

THEOREM A.84 — For $k, s \in \mathbb{N}$, $s > k + (\dim(M))/2$ There is a continuous inclusion $H^s(M,S) \longrightarrow UC^k(M,S)$ hence also a continuous inclusion of Fréchet spaces

$$H^\infty(S) \longrightarrow UC^\infty(S)$$

.

PROOF — As observed by J. Roe, this is an adaption of the standard compact case, in fact thanks to bounded geometry assumption the family of local Sobolev constant on good balls is bounded. □

Now by Schwartz kernel theorem a continuous linear operator[18] $T : C_c^\infty(M,S) \longrightarrow C^{-\infty}(M,s)$ is univoquely represented by its Schwartz kernel, the unique distribution–section $K_T \in C^{-\infty}(M \times M, \mathrm{END}(S) \otimes \mathrm{Pr}_1^*\Omega(M))$ satisfying the distributional equation

$$\langle K_T u, v\rangle = \langle K_T, v \boxtimes u\rangle$$

for every $u, v \in C_c^\infty(M,S)$. Here the *big endomorphism bundle* $\mathrm{END}(S) \longrightarrow M \times M$ has fiber $\mathrm{Hom}(S_x, S_y)$ over (x,y). the following is a group of definitions.

DEFINITIONS A.85 —

1. We say that T has order $k \in \mathbb{Z}$ if it extends to an operator in $B(H^s(M,S), H^{s-k}(M,s))$ for every s.
2. The space of k–order operators is denoted by $\mathrm{Op}^k(M,S)$. with seminorms given by $B(H^s(M,S), H^{s-k}(M,s))$.
3. The space $\mathrm{Op}^{-\infty}(M,S) = \bigcap_{k<0} \mathrm{Op}^k(M,S)$ is called the space of uniformly smoothig operators. In fact we shall see it is the space of operators with uniformly smooth kernels.
4. An element $T \in \mathrm{Op}^k(M,S)$, $k \geq 1$ is called elliptic if it satisfies the uniform Gårding inequality (100).

Below a list of properties that can be found in the papers cited at the beginning.

PROPOSITION A.86 —

[18] If T is not a pseudo–differential operator it is customary to require that it respects all the connected components of M.

- Ellipticity is stable under order 0 perturbations, if $T \in \mathrm{Op}^k(M,S)$ elliptic and $Q \in \mathrm{Op}^0(M,S)$ then $T+Q$ is elliptic.
- If $\in \mathrm{Op}^k(M,S)$ is elliptic and formally self–adjoint then every its spectral projection belongs to $\mathrm{Op}^0(M,s)$.
- It follows from the completeness of M that an elliptic and formally self–adjoint element $T \in \mathrm{Op}^k(M,S)$ ($k \geq 1$ as required by the definition of elliptic element) is essentially selfadjoint on $L^2(M,S)$.

  If T denotes its closure also one finds that $\mathrm{dom}(T) = H^k(M,S)$. In particular this is true for the Dirac operator D.

A.1 Spectral functions of elliptic operators

Last theorem says that an uniformly elliptic operator on a manifold with bounded geometry is essentially self–adjoint. We need some considerations about spectral functions of T. Let

$$RB(\mathbb{R}) := \{f : \mathbb{R} \longrightarrow \mathbb{C},\ \text{Borel}; \quad |(1+x^2)^{k/2} f(x)|_\infty < \infty \quad \forall k \in \mathbb{N}\}$$

be the space of rapidly decreasing Borel functions with Fréchet structure induce by the seminorms $|(1+x^2)^{k/2} \cdot |_\infty$
Let $RC(\mathbb{R})$ denote the closed subspace of continuous functions.

Proposition A.87 — For an elliptic element T and $l \in \mathbb{N}$ and rapid Borel functions f, $T^l f(T)$ is bounded in L^2 and the following Gårding inequality holds true,

$$\|f(T)\psi\|_{H^l} \leq C(l) \sum_{i=0}^{l} \|T^i f(T)\psi\|_{L^2} \leq C(l)\|\psi\|_{L^2} \sum_{i=0}^{l} |x^i f|_\infty \tag{101}$$

for every $\psi \in C_c^\infty(M,S)$. Suppose now, by simplicity of writing that T has order 1, making use of the duality

$$(H^s)^* = H^{-s}$$

one finds, for $k,l \in \mathbb{Z}$, $l \geq k$,

$$\|f(T)\psi\|_{H^l} \leq C(l,k) \sum_{i=0}^{l-k} \|T^i f(T)\psi\|_{H^k} \leq C(l,k)\|\psi\|_{H^k} \sum_{i=0}^{l-k} |x^i f|_\infty. \tag{102}$$

Proof — Observe first that the operator $T^l f(T)$ is the spectral function of T corresponding to the function $x^l f(x)$ on $\mathbb{R}$ hence is bounded. Again, since f is bounded no problem here in commuting relations, in particular $T^l f(T) = f(T) T^l$ (equality in the sense of unbounded operators) in particular $f(T) : L^2 \longrightarrow H^{l+k}$. Now from Gårding's inequality for T,

$$\|f(T)\psi\|_{H^l} \leq C(l) \sum_{i=0}^{l} \|T^i f(T)\psi\|_{L^2} \leq C(l)\|\psi\|_{L^2} \sum_{i=0}^{l} |x^i f|_\infty.$$

Inequality (102) follows at once from the first one (101) in fact the first step is to consider the transpose of $T^l f(T) : H^{-l} \longrightarrow H^{-k}$ while the second step is based on our very dual definition of Sobolev space of order negative. $\square$

Hence, we get continuity of the functional calculus $RB(\mathbb{R}) \longrightarrow B(H^l(M,S), H^k(M,S))$ for each l,k then continuity of $RB(\mathbb{R}) \longrightarrow \mathrm{Op}^{-\infty}(M,S)$. With a little work, using Sobolev embedding one can prove the following theorem.

THEOREM A.87 — Let $T \in \mathrm{Op}^k(M,S)$ uniformly elliptic and formally selfadjoint.

- If $L = [n/2+1]$, $n = \mathrm{dim}M$ and $l \in \mathbb{N}$ then the *kernel mapping*

$$\mathrm{Op}^{-2L-l}(M,S) \longrightarrow UC^l(M \times M, \mathrm{END}(S) \otimes \mathrm{Pr}_1^*\Omega(M)), T \longmapsto K_T,$$

  is continuous.

- For $f \in RB(\mathbb{R})$ the kernel of $f(T)$ is uniformly smoothing,

$$K_T \in UC^\infty(M \times M, \mathrm{END}(S) \otimes \mathrm{Pr}_1^*\Omega(M)).$$

  and the kernel mapping $RB(\mathbb{R}) \longrightarrow UC^\infty(M \times M, \mathrm{END}(S) \otimes \mathrm{Pr}_1^*\Omega(M))$ is continuous.

REMARK — Combining A.16, page 89 and **??** we see that every spectral projection Π_A of the Dirac operator obtained by a bounded Borel set $A \subset \mathbb{R}$ is represented by a uniformly smoothing kernel hence is locally traceable (in the usual sense on $L^2(M,S)$ w.r.t the Abelian Von Neumann algebra $L^\infty(M)$). This means that for every Borel set $B \subset M$ with compact closure the operator $\chi_B \Pi_A \chi_B$ is trace class, one gets a Radon measure $B \longmapsto \mathrm{trace}\,\chi_B \Pi_A \chi_B$ called the local trace of Π_A.

A.2 Some computations on Clifford algebras

Let $\mathbb{C}l(k)$ the (complex) Clifford algebra over the euclidean space $\mathbb{R}^k$, with generators $\mathbf{c}_1, \ldots, \mathbf{c}_k$ and relations ($\mathbf{c}_j$ orthonormal basis)

$$\mathbf{c}_i\mathbf{c}_j + \mathbf{c}_j\mathbf{c}_i = -2\delta_{ij}.$$

The algebra $\mathbb{C}l(k)$ is $\mathbb{Z}_2$-graded: $\mathbb{C}l(k) = \mathbb{C}l^+(k) \oplus \mathbb{C}l^-(k)$, being $\mathbb{C}l^+(k)$ the subalgebra spanned by products of even sets of generators.
The map $\mathbf{c}_i \longmapsto \mathbf{c}_i\mathbf{c}_{k+1}$ defines an isomorphism $\mathbb{C}l(k) \xrightarrow{\sim} \mathbb{C}l^+(k+1)$.
The volume element $\tau_k := i^{[(k+1)/2]}\mathbf{c}_1 \ldots \mathbf{c}_k \in \mathbb{C}l(k)$ satisfies $\tau_k^2 = 1$ and thus induces a $\mathbb{Z}_2$-grading on each representation of $\mathbb{C}l(k)$. Due to the fact

$$\tau_k\mathbf{c} = -(-1)^k\mathbf{c}\tau_k$$

for $\mathbf{c} \in \mathbb{R}^k \subset \mathbb{C}l(k)$ this induced grading is trivial if k is odd. $\mathbb{C}l(2l)$ has a unique irreducible representation, called its spinor space and we denote it by $S(2l)$. Its dimension is $\dim S(2l) = 2^l$. Decomposing into the ± 1-Eigenspaces of τ_{2l} we write $S(2l) = S^+(2l) \oplus S^-(2l)$. Via the identification $\mathbb{C}l(2l-1) \cong \mathbb{C}l^+(2l)$ the spaces $S^+(2l)$, $S^-(2l)$ are non-equivalent irreducible representations of $\mathbb{C}l(2l-1)$, which can be considered as being isomorphic representations of $\mathbb{C}l(2l-2) \cong \mathbb{C}l^+(2l-1)$ via the map $S^+(2l) \overset{c_{2l}}{\to} S^-(2l)$. This of course is then just the representation $S(2l-2)$ of $\mathbb{C}l(2l-2)$.
Notation: for $S^\pm(2l)$ we also write $S^\pm(2l-1)$ when these spaces are seen as representations of $\mathbb{C}l(2l-1)$.

$$\mathbb{C}l(2l-1) \longleftrightarrow \mathbb{C}l^+(2l) \longleftrightarrow \mathrm{End}^+(S^+(2l) \bigoplus S^-(2l)) \Longrightarrow \mathrm{End}(S^\pm(2l)) =: \mathrm{End}(S^\pm(2l-1)) .$$

It is easily seen that $\mathbb{C}l(2l)$ acts injectively on $S(2l)$. Comparison of dimensions then yields $\mathbb{C}(2l) \cong \mathrm{End}(S(2l))$, and, using $\mathbb{C}l(2l-1) \cong \mathbb{C}l^+(2l)$ also

$$\mathbb{C}l(2l-1) \cong \mathbb{C}l^+(2l) \cong \mathrm{End}^+(S(2l)).$$

The identification $\mathbb{C}l(2l-1) \longrightarrow \mathrm{End}(S^\pm(2l-1))$ maps τ_{2l-1} to ± 1 and one can show that the null space is $(1 \mp \tau_{2l-1})\mathbb{C}(2l-1)$.

$$\begin{tikzcd}
\mathrm{End}(S^+(2l) \bigoplus S^-(2l)) \arrow[r, leftrightarrow] & \mathbb{C}l(2l) = & \underbrace{\mathbb{C}l^+(2l)} \arrow[dll, leftrightarrow] & \bigoplus & \mathbb{C}l^-(2l) \\
\overbrace{\mathbb{C}l(2l-1)} = & \underbrace{\mathbb{C}l^+(2l-1)} \arrow[dl, leftrightarrow] & \bigoplus & \mathbb{C}l^-(2l-1) \\
\overbrace{\mathbb{C}l(2l-2)}
\end{tikzcd}$$

The traces $\mathrm{tr}^\pm$ on $\mathrm{End}(S^\pm(2l-1))$ and the graded trace str on $\mathrm{End}(S(2l))$ then induce traces on $\mathbb{C}(2l-1)$ and $\mathbb{C}(2l)$. On elements of the form $\mathbf{c}_I := \mathbf{c}_{i1} \ldots \mathbf{c}_{i|I|}$ where $I = \{i_1 \leq \ldots \leq i_{|I|}\} \subset \{1, \ldots, k\}$ these can be computed as follows

Lemma A.88 —

(a) In $\mathbb{C}l(2l)$ we have $\mathrm{str}(\tau_{2l}) = 2^l$ and $\mathrm{str}(1) = \mathrm{str}(\mathbf{c}_I) = 0$ for $I \neq \{1, \ldots, k\}$.

(b) In $\mathbb{C}l(2l-1)$ we have $\mathrm{str}(\tau_{2l-1}) = -\mathrm{tr}^-(\tau_{2l-1}) = \mathrm{tr}^\pm(1) = 2^{l-1}$ and for $I \neq \{1, \ldots, k\}$ we have $\mathrm{tr}^\pm(\mathbf{c}_1) = 0$.

On $(\mathbb{C}l(2l-1) - \mathbb{C}) \subset \mathbb{C}l(2l)$ therefore $\mathrm{tr}^\pm(\bullet) = \mp\frac{1}{2}\,\mathrm{str}(\mathbf{c}_{2l}\bullet)$ and on $\mathbb{C}l(2l) \subset \mathbb{C}l(2l+1)$ we have $\mathrm{str}(\bullet) = \pm i\,\mathrm{tr}^\pm(\mathbf{c}_{2l+1}\bullet)$

Proof — Cf. [9], Proposition 3.21 □

The map $S^+(2l) \xrightarrow{c_{2l}} S^-(2l)$ gives an identification $S(2l) \cong S^\pm(2l-1) \oplus S^\pm(2l-1)$. In this representation, $\mathbb{C}l(2l)$ acts on $S(2l)$ as follows

$$\mathbf{c}_i \in \mathbb{C}l(2l-1) \simeq \begin{pmatrix} 0 & \pm\mathbf{c}_i \\ \pm\mathbf{c}_i & 0 \end{pmatrix} \ \mathbf{c}_{2l} \simeq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\text{and str} \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{pmatrix} = \mathrm{tr}^\pm(\phi_1) - \mathrm{tr}^\pm(\phi_4)$$

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