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Branched Covers between Surfaces

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to my family

Abstract

Let $\widetilde{\Sigma}$ and Σ be closed, connected, and orientable surfaces, and let $f : \widetilde{\Sigma} \to \Sigma$ be a branched cover. For each branching point $x \in \Sigma$ the set of local degrees of f at $f^{-1}(x)$ is a partition of the total degree d. The total length of the various partitions is determined by $\chi(\widetilde{\Sigma})$, $\chi(\Sigma)$, d and the number of branching points via the Riemann-Hurwitz formula. A very old problem asks whether a collection of partitions of d having the appropriate total length (that we call a *candidate cover*) always comes from some branched cover. The answer is known to be in the affirmative whenever Σ is not the 2-sphere S, while for $\Sigma = S$ exceptions do occur. A long-standing conjecture however asserts that when the degree d is a prime number a candidate cover is always realizable. In the main result of this thesis, proved in Chapter 3, we provide strong supporting evidence for the conjecture. In particular, we exhibit three different sequences of candidate covers, indexed by their degree, such that for each sequence:

- The degrees giving realizable covers have asymptotically zero density in the naturals;
- Each prime degree gives a realizable cover.

Actually this result is a by-product of an extensive study of realizability of branched covers between surfaces. using the viewpoint of geometric 2-orbifolds, we split branched covers in three families, and, fully explointing the mentioned geometry,we completely solve the problem for two of them. Even if it seems to be very hard to determine realizability for the last family of covers, using different approaches we prove other partial results analysing two sub-families (chosen because relevant in order to understand more about the prime degree conjecture). At the end of this thesis we also partially extend the geometric approach to the last family.

Contents

Introduction

This thesis deals with the realizability problem of ramified covers between surfaces. To each such cover one naturally associates a set of data; on the other hand, for a given set of data, there exist easy necessary conditions it should satisfy to be associated to an existing ramified cover. A classical problem, first proposed by Hurwitz in 1891, asks whether these conditions are also sufficient for existence.

Chapter 1 gives the reader all the definitions and classical results about 2-orbifolds, needed to understand the methods used in the sequel.

Chapter 2 is a syntethic overview of the many different methods used in the past to face the Hurwitz existence problem. If one has a branched cover between orientable surfaces, one can assign to it a sort of passport, called its 'branch datum'. Branch data can also be defined abstractly, without starting from a branched cover, as symbols $\widetilde{\Sigma} \xrightarrow[\Pi]{d:1} \Sigma$, where $\widetilde{\Sigma}$ and Σ are closed surfaces, $d \in \mathbb{N}$ and Π is a finite set of partitions of d. However the branch data coming from covers satisfy some necessary conditions, notably the Riemann-Hurwitz formula. The question now is: if a branch datum satisfies the Riemann-Hurwitz condition, does there exist a surface branched cover having it as a branch datum? We address the reader to Hurwitz [10] himself, who produced an elegant answer to his problem for the special case of $S \to S$ (with S denoting the 2-sphere) when all ramification points but one are simple, and also to the classical [11, 25, 24, 5, 7, 6, 13, 4, 8, 14, 12], and the more recent [2, 15, 16, 19, 20, 17, 28]. When a branch datum satisfies the necessary conditions to come from a branched cover, we call it a *candidate branched cover*; if indeed it comes from a branched cover we call it *realizable*, and otherwise *exceptional*. Considerable energy has been devoted over the time to a general understanding of the exceptional candidate surface branched covers, and quite some progress has been made (in particular, it has been shown that exceptions can occur only if Σ is S or the projective plane; see for instance the survey of known results contained in [19], together with the later papers [20, 17, 28]), but the global pattern remains elusive. In particular the following conjecture proposed in [4] appears to be still open:

Conjecture 0.0.1. *If* $\widetilde{\Sigma} \xrightarrow[\Pi]{d:1} S$ *is a candidate surface branched cover and the degree* d *is a prime number then the candidate is realizable.*

We mention in particular that all exceptional candidate surface branched covers over S with $n = 3$ and $d \leqslant 20$ have been determined by computer in [28]. There are many of them, but none occurs for prime d.

In Chapter 2 we show that each candidate branched cover has an associated candidate cover between 2-orbifolds, and in Chapter 3 we determine the realizability of all candidate surface branched covers having associated candidate covers between 2-orbifold with non-negative Euler characteristic χ^{orb}; moreover, in Chapter 4 we perform a partial analysis of the remaining case of negative χ^{orb}. Our approach for $\chi^{\mathrm{orb}} \geqslant 0$ strongly depends on the geometric properties of 2-orbifolds, whereas for $\chi^{\mathrm{orb}} < 0$ we employ more classical techniques. At the end of our work however we also partially apply the geometric viewpoint to the case of negative χ^{orb}.

Main new results Using the geometry of Euclidean 2-orbifolds we will establish in Chapter 3 (among others) the next three theorems.

Theorem 0.0.2. *Suppose $d = 4k + 1$ for $k \in \mathbb{N}$. Then*

$$S \xrightarrow[(\underbrace{2,\ldots,2}_{2k},1),(\underbrace{4,\ldots,4}_{k},1),(\underbrace{4,\ldots,4}_{k},1)]{d:1} S$$

is a candidate surface branched cover, and it is realizable if and only if d can be expressed as $x^2 + y^2$ for some $x, y \in \mathbb{N}$.

Theorem 0.0.3. *Suppose $d = 6k + 1$ for $k \in \mathbb{N}$. Then*

$$S \xrightarrow[(\underbrace{2,\ldots,2}_{3k},1),(\underbrace{3,\ldots,3}_{2k},1),(\underbrace{6,\ldots,6}_{k},1)]{d:1} S$$

is a candidate surface branched cover and it is realizable if and only if d can be expressed as $x^2 + xy + y^2$ for some $x, y \in \mathbb{N}$.

Theorem 0.0.4. *Suppose $d = 3k + 1$ for $k \in \mathbb{N}$. Then*

$$S \xrightarrow[(\underbrace{3,\ldots,3}_{k},1),(\underbrace{3,\ldots,3}_{k},1),(\underbrace{3,\ldots,3}_{k},1)]{d:1} S$$

is a candidate surface branched cover and it is realizable if and only if d can be expressed as $x^2 + xy + y^2$ for some $x, y \in \mathbb{N}$.

What makes these results remarkable is the link with Conjecture 0.0.1. To make it clear we recall now the following arithmetic facts:

- A prime number of the form $4k + 1$ can always be expressed as $x^2 + y^2$ for $x, y \in \mathbb{N}$ (Fermat);
- A prime number of the form $6k + 1$ (or equivalently $3k + 1$) can always be expressed as $x^2 + xy + y^2$ for $x, y \in \mathbb{N}$ (Gauss);
- The integers that can be expressed as $x^2 + y^2$ or as $x^2 + xy + y^2$ with $x, y \in \mathbb{N}$ have asymptotically zero density in $\mathbb{N}$.

Moreover we have:

$$\begin{aligned}
&\{d\in\mathbb{N}:\ d=x^2+y^2 \text{ for } x,y\in\mathbb{N},\ x\not\equiv y \pmod 2\}\\
&=\{d\in\mathbb{N}:\ d\equiv 1 \pmod 4,\ d=x^2+y^2 \text{ for } x,y\in\mathbb{N}\},\\
&\{d\in\mathbb{N}:\ d=x^2+xy+y^2 \text{ for } x,y\in\mathbb{N} \text{ not both even},\ x\not\equiv y \pmod 3\}\\
&=\{d\in\mathbb{N}:\ d\equiv 1 \pmod 6,\ d=x^2+xy+y^2 \text{ for } x,y\in\mathbb{N}\},\\
&\{d\in\mathbb{N}:\ d=x^2+xy+y^2 \text{ for } x,y\in\mathbb{N},\ x\not\equiv y \pmod 3\}\\
&=\{d\in\mathbb{N}:\ d\equiv 1 \pmod 3,\ d=x^2+xy+y^2 \text{ for } x,y\in\mathbb{N}\}.
\end{aligned}$$

Now we can conclude that a candidate cover in any of our three statements is "exceptional with probability 1," because

$$\begin{aligned}
&\lim_{n\to\infty}\tfrac{1}{n}\cdot\#\{d\in\mathbb{N}:\ d\leqslant n,\ d=x^2+y^2 \text{ for } x,y\in\mathbb{N}\}=0,\\
&\lim_{n\to\infty}\tfrac{1}{n}\cdot\#\{d\in\mathbb{N}:\ d\leqslant n,\ d=x^2+xy+y^2 \text{ for } x,y\in\mathbb{N}\}=0
\end{aligned}$$

even though it is realizable when its degree is prime.

More new results The main idea underlying our results in Chapter 3 is to analyze the realizability of a given candidate surface branched cover $\widetilde{\Sigma} \overset{d:1}{\underset{\Pi}{\dashrightarrow}} \Sigma$ using the associated candidate 2-orbifold cover $\widetilde{X} \overset{d:1}{\dashrightarrow} X$ and the geometries of $\widetilde{X}$ and X. Note that $\chi^{\mathrm{orb}}(\widetilde{X})$, a generalization of the classical Euler characteristic, has the same sign as $\chi^{\mathrm{orb}}(X)$, being d times it. Moreover every orbifold Y with $\chi^{\mathrm{orb}}(Y)>0$ is either 'bad' (a technical notion defined below) or it has a spherical geometric structure, while it has an Euclidean structure if $\chi^{\mathrm{orb}}(Y)=0$, and a hyperbolic one if $\chi^{\mathrm{orb}}(Y)<0$. We will fully carry out the geometric approach in the case $\chi^{\mathrm{orb}}(X)\geqslant 0$, but we will also discuss part of the case $\chi^{\mathrm{orb}}(X)<0$. In the bad/spherical case the statement is quite expressive:

Theorem 0.0.5. *Let a candidate surface branched cover* $\widetilde{\Sigma} \overset{d:1}{\underset{\Pi}{\dashrightarrow}} \Sigma$ *have an associated candidate 2-orbifold cover* $\widetilde{X} \overset{d:1}{\dashrightarrow} X$ *with* $\chi^{\mathrm{orb}}(X)>0$*. Then* $\widetilde{\Sigma} \overset{d:1}{\underset{\Pi}{\dashrightarrow}} \Sigma$ *is exceptional if and only if* $\widetilde{X}$ *is bad and* X *is spherical. All exceptions occur with non-prime degree.*

Turning to the Euclidean case (which leads in particular to Theorems 0.0.2 to 0.0.4) we confine ourselves here to the following informal:

Theorem 0.0.6. *Let a candidate surface branched cover* $\widetilde{\Sigma} \overset{d:1}{\underset{\Pi}{\dashrightarrow}} \Sigma$ *have an associated candidate 2-orbifold cover* $\widetilde{X} \overset{d:1}{\dashrightarrow} X$ *with Euclidean* X*. Then its realizability can be decided explicitly in terms of* d *and* Π*. More precisely, as in Theorems 0.0.2 to 0.0.4, given* Π *the condition on* d *depends on a congruence and/or an integral quadratic form. No exceptions occur when* d *is a prime number.*

We conclude with our statements for the hyperbolic case, to which Chapter 4 is devoted. A 2-orbifold is called *triangular* if it has the form $S(p,q,r)$.

Theorem 0.0.7. *There exist* 9 *candidate surface branched covers having an associated candidate* 2*-orbifold cover* $\widetilde{X} \dashrightarrow X$ *with* $\widetilde{X}$ *and* X *being hyperbolic triangular orbifolds. All of them but two are realizable. Exceptions occur in degrees* 8 *and* 16 *(which are not prime).*

Theorem 0.0.8. *There exist* 141 *candidate surface branched covers having an associated candidate* 2*-orbifold cover* $\widetilde{X} \dashrightarrow X$ *with* $\widetilde{X}$ *being the sphere with four cone points, and* X *being a hyperbolic triangular orbifold. Among them there are* 29 *exceptional candidates, and they do not occur in prime degree.*

For the case of non-negative χ^{orb}, within the proofs of Theorems 0.0.5 and 0.0.6 we will describe explicit geometric constructions of all the realizable covers. The proofs of Theorems 0.0.7 and 0.0.8 have instead a more combinatorial flavor.

Chapter 1

Geometric 2-orbifolds

This chapter is devoted to the introduction of the most important objects in our work: geometric orbifolds and orbifold covers. These objects are a rather natural extension of differentiable manifolds; for this reason they were defined many times and studied by several mathematicians.

1.1 Orbifolds

We first meet orbifolds in Satake's work [23], with the name of V-manifolds. Orbifolds were reintroduced with more success by William P. Thurston during a course he gave in Princeton in 1978-79. Thurston's big improvement over Satake's earlier version was to show that the theory of cover spaces and fundamental groups works for orbifolds. One of the many aims of that course was to describe the very strong connection between geometry and low-dimensional topology. In the last chapter of the electronic version of the notes Thurston introduces the notion of orbifold: the idea is to study quotient spaces of $\mathbb{R}^n$ under the action of a group that acts properly discontinuously but not necessarily freely. Orbifolds were defined in great generality, but in this thesis we use only smooth locally orientable 2-dimensional orbifolds.

Definition 1.1.1. *Let X be a Hausdorff space, and let $\{U_i\}$ be an open covering of X closed under finite intersections. For each U_i there is an associated finite group Γ_i, an action of Γ_i on an open subset $\widetilde{U_i}$ of $\mathbb{R}^n$, and a homeomorphism φ_i between U_i and $\widetilde{U_i}/\Gamma_i$. Moreover if $U_i \subset U_j$ there should be an injective homomorphism $f_{ij} : \Gamma_i \hookrightarrow \Gamma_j$ and an embedding $\widetilde{\varphi}_{ij} : \widetilde{U_i} \hookrightarrow \widetilde{U_j}$ equivariant with respect to f_{ij} such that the following diagram commutes:*

$$
\begin{array}{ccc}
\widetilde{U}_i & \xrightarrow{\ \widetilde{\varphi}_{ij}\ } & \widetilde{U}_j \\
\downarrow & & \downarrow \\
\widetilde{U}_i/\Gamma_i & \xrightarrow{\ \varphi_{ij} = \widetilde{\varphi}_{ij}/\Gamma_i\ } & \widetilde{U}_j/\Gamma_i \\
\Big\uparrow \varphi_i & & \downarrow \\
 & & \widetilde{U}_j/f_{ij}(\Gamma_i) \\
 & & \uparrow \varphi_j \\
U_i & \subset & U_j
\end{array}
$$

A maximal covering of open sets as above is called an orbifold atlas*.*

Hence, an n-orbifold X is a space that locally looks like $\mathbb{R}^n/\Gamma$: it is locally homeomorphic to a quotient of $\mathbb{R}^n$ under the action of a finite group of homeomorphism. Note that each point $x \in X$ has an associated group Γ_x, well-defined up to isomorphism: in a local coordinate system $U = \widetilde{U}/\Gamma$; the group Γ_x is the isotropy group of any point in $\widetilde{U}$ correspondig to x. The *singular locus* of X is the set $\{x \in X \,|\, \Gamma_x \neq \{1\}\}$. Moreover, an orbifold is *smooth*, or *differentiable*, if each Γ_i acts smoothly on $\widetilde{U}_i$ and if the inclusion maps $\widetilde{\varphi}_{ij}$ are differentiable. In the smooth case, taking the average of the Euclidean metric of $\mathbb{R}^n$ over the action of the local group, one can assume without loss of generality that each local group Γ is a subgroup of the orthogonal group $O(n)$. Hence, for smooth 2-orbifolds, Γ is a finite subgroup of $O(2)$ and we know that there are only three types of subgroups in $O(2)$; therefore we can describe all types of singular points. We list the local model for each of them:

"mirror": $\mathbb{R}^2/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by a reflection;

"elliptic points of order n": $\mathbb{R}^2/\mathbb{Z}_n$, where $\mathbb{Z}_n$ acts a rotation of angle $2\pi/n$;

"corner reflectors of order n": $\mathbb{R}^2/D_n$, where D_n is the dihedral group of order $2n$ with presentation $\langle a, b : a^2 = b^2 = (ab)^n = 1\rangle$, and a and b are reflections with respect to lines making an angle of π/n.

Note that if we require a smooth 2-orbifold to be closed and locally orientable (i.e. for any x, the group Γ_x consists of preserving orientation homeomorphisms), we can only have elliptic points.

We say that an orbifold is *Riemannian* if on each $\widetilde{U_i}$ there is a Riemannian metric, if each Γ_i acts isometrically, and if the inclusion maps are isometries.

From now on, when we say 2-orbifold we mean closed, orientable, and smooth 2-dimensional orbifold. Let us state for them an equivalent and simpler definition, and the notion of orbifold cover:

Definition 1.1.2. *A closed, orientable, locally orientable 2-orbifold* $X = \Sigma(p_1, \ldots, p_n)$ *is a closed orientable surface* Σ *with* n *cone points of orders* $p_i \geqslant 2$*, at which* X *has the singular differentiable structure given by the quotient* $\mathbb{C}/_{\langle \mathrm{rot}(2\pi/p_i)\rangle}$.

Definition 1.1.3. *A degree* d *cover* $f : \widetilde{X} \to X$ *between closed orientable smooth* 2*-orbifolds is a map such that* $f^{-1}(x)$ *generically consists of* d *points and locally making a diagram of the following form commutative:*

$$\begin{array}{ccc} (\mathbb{C},0) & \stackrel{\mathrm{id}}{\longrightarrow} & (\mathbb{C},0) \\ \downarrow & & \downarrow \\ (\widetilde{X},\widetilde{x}) & \stackrel{f}{\longrightarrow} & (X,x) \end{array}$$

where $\widetilde{x}$ *and* x *have cone orders* $\widetilde{p}$ *and* $p = k \cdot \widetilde{p}$ *respectively, and the vertical arrows are the projections corresponding to the actions of* $\langle \mathrm{rot}(2\pi/\widetilde{p})\rangle$ *and* $\langle \mathrm{rot}(2\pi/p)\rangle$*, namely the maps defining the (possibly singular) local differentiable structures at* $\widetilde{x}$ *and* x.

The notion of orbifold cover was introduced by Thurston [26], together with the orbifold Euler characteristic (see the next paragraph for the definition) designed so that if $f : \widetilde{X} \xrightarrow{d:1} X$ is an orbifold cover then $\chi^{\mathrm{orb}}(\widetilde{X}) = d \cdot \chi^{\mathrm{orb}}(X)$. He also gave the notion of *orbifold universal cover* and established its existence.

Orbifold Euler Characteristic Here we define the generalization of the classic Euler characteristic introduced by Thurston for a smooth 2-orbifold X. To this end we take a cell deconposition of the underlying surface of X such that the group associated to the interior points of any cell is constant. Then we define the orbifold Euler characteristic of X as:

$$\chi^{\mathrm{orb}}(X) := \sum_c (-1)^{\dim(c)} \frac{1}{|\Gamma(c)|}$$

where c ranges over cells, and $\Gamma(c)$ is the group associated to each point of the cell c. It is also called the 'Euler number' of X, and it will be useful in the geometric classification of 2-orbifolds. Now we can give the general formula for the Euler number of a smooth 2-orbifold X based on Σ with m corner reflectors of orders $k_1, \ldots, k_m$ and n elliptic points of orders $p_1, \ldots, p_n$:

$$\chi^{\mathrm{orb}}(X) := \chi(\Sigma) - \sum_{i=1}^{n} \left(1 - \frac{1}{p_i}\right) - \frac{1}{2} \sum_{j=1}^{m} \left(1 - \frac{1}{k_j}\right) \tag{1.1}$$

Let us now consider an orbifold cover $f : \widetilde{X} \xrightarrow{d:1} X$ and show that $\chi^{\mathrm{orb}}(\widetilde{X}) = d \cdot \chi^{\mathrm{orb}}(X)$. Note that at any point of $x \in X$ we have:

$$d = \sum_{\tilde{x}: f(\tilde{x}) = x} \left(\frac{|\Gamma_x|}{|\Gamma_{\tilde{x}}|} \right). \tag{1.2}$$

Now we choose a cell decomposition τ of the surface underlying X, suitable for the computation of the Euler number; then we pull it back to the cover producing a cell decomposition $\tilde{\tau}$ of the surface underlying $\widetilde{X}$. Finally, we compute the Euler number for $\widetilde{X}$ grouping together those cells c_j of $\tilde{\tau}$ mapped by f to c, and using(1.2):

$$\chi^{\mathrm{orb}}(\widetilde{X}) = \sum_{c_j}(-1)^{\dim(\mathrm{c_j})}\frac{1}{|\Gamma(c_j)|} = \sum_{c}(-1)^{\dim(\mathrm{c})}\frac{d}{|\Gamma(c)|} = d\cdot\chi^{\mathrm{orb}}(X).$$

The geometrization theorem Let X be a closed Riemannian 2-orbifold, and denote by $|X|$ the surface underlying X. Then the curvature K of X is defined on $|X|$, except at the cone points and a version of the classical Gauss-Bonnet formula is valid:

$$\int_{|X|} K dA = 2\pi\chi^{\mathrm{orb}}(X).$$

If X has an elliptic, a Euclidean or a hyperbolic structure, this implies that $\chi^{\mathrm{orb}}(X)$ must be respectively positive, zero or negative; (and the area is $A(X) = 2\pi|\chi^{\mathrm{orb}}(X)|$ when X is elliptic or hyperbolic.) Moreover, in the case of connected differentiable 2-orbifolds Thurston showed that:

- If $\chi^{\mathrm{orb}}(X) > 0$ then X is either *bad* (not covered by a surface in the sense of orbifolds) or *spherical*, namely the quotient of the metric 2-sphere $\mathbb{S}$ under a finite isometric action;
- If $\chi^{\mathrm{orb}}(X) = 0$ (respectively, $\chi^{\mathrm{orb}}(X) < 0$) then X is *Euclidean* (respectively, *hyperbolic*), namely the quotient of the Euclidean plane $\mathbb{E}$ (respectively, the hyperbolic plane $\mathbb{H}$) under a discrete isometric action.

Using this fact we easily get the following result that we will need below (where *good* means "not bad"):

Lemma 1.1.4. *If $\widetilde{X}$ is bad and X is good then there cannot exist any orbifold cover $\widetilde{X} \to X$.*

The results just stated allow us to give the *topological classification* of 2-orbifolds according to theri geometry. We tabulate below all bad, elliptic, and Euclidean orbifolds:

- **Bad**: $S^2(n),\ S^2(n,m)$, $(2 \leqslant n < m)$;
- **Elliptic**: $S^2,\ S^2(n,n),\ S^2(2,2,n),\ S^2(2,3,3),\ S^2(2,3,4),\ S^2(2,3,5)$, $(n \geqslant 2)$;
- **Euclidean**: $T^2,\ S^2(2,3,6),\ S^2(2,4,4),\ S^2(3,3,3),\ S^2(2,2,2,2)$.

Orbifolds not listed here are all hyperbolic. Keeping in mind that $\chi^{\mathrm{orb}}(\Sigma(p_1,\ldots,p_n)) \leqslant \chi(\Sigma)$, we immediately deduce that the set of hyperbolic 2-orbifolds is infinite.

1.2 Geometric structures

There is one crucial geometric fact underlying the proofs of our main results for the case of positive and zero Euler characteristic. Namely, in these cases the geometry (if any) of an orbifold with cone points is *rigid* (up to rescaling), with the single exception of $S(2,2,2,2)$, where the space of moduli is easy to compute anyway. The case of negative Euler characteristic is however quite different: one knows that a hyperbolic 2-orbifold is rigid if and only if it is *triangular*, namely if it is based on the sphere and it has precisely three cone points. The dimension of the Teichmüller space of all other hyperbolic orbifolds tells us how difficult their geometric analysis is.

Teichmüller space of 2-orbifolds Let X be a closed, orientable 2-orbifold, with negative orbifold Euler characteristic, and let H the set of all hyperbolic metrics on X. Then the Theichmüller space of X, denoted by $\tau(X)$, is defined as $H/_\sim$, where $h_1 \sim h_2$ means that there exists an isometric orbifold diffeomorphism $(X, h_1) \to (X, h_2)$ that is isotopic to identity.

Thurston proves the following proposition in [26]:

Proposition 1.2.1. *The Teichmüller space $\tau(X)$ of an orbifold $X = (\Sigma(p_1, \ldots, p_n))$ with $\chi^{\rm orb}(X) < 0$ is homeomorphic to a Euclidean space of dimension $-3\chi(\Sigma) + 2n$.*

Exactly as for the Teichmüller space of surfaces, this proposition is a consequence of the construction of $\tau(X)$ as a parameter space, coming from the classical pants decomposition, suitably modified. As already announced, the only rigid hyperbolic 2-orbifolds based on S^2 are those with exactly three cone points; moreover any hyperbolic orbifold based on S^2 with more than three cone points has a n-dimensional Teichmüller space with $n \geqslant 1$.

Spherical structures As already mentioned, any closed good 2-orbifold X with $\chi^{\rm orb}(X) > 0$ has a spherical structure, given by the action of some finite group Γ of isometries on the metric sphere $\mathbb{S}$. We will now explicitly describe each relevant Γ, thus identifying X with the quotient $\mathbb{S}/\Gamma$. To this end we will always regard $\mathbb{S}$ as the unit sphere of $\mathbb{C} \times \mathbb{R}$.

The football The geometry of $S(p, p)$ is very easy, even if for consistency with what follows we will not give the easiest description. Consider in $\mathbb{S}$ a wedge with vertices at the poles $(0, \pm 1)$ and edges passing through $(1, 0)$ and $({\rm e}^{i\pi/p}, 0)$, so the width is π/p. Now define $\widetilde{\Gamma}_{(p,p)}$ as the group of isometries of $\mathbb{S}$ generated by the reflections in the edges of the wedge, and $\Gamma_{(p,p)}$ as the its subgroup of orientation-preserving isometries. Then $\Gamma_{(p,p)}$ is generated by the rotation of angle $2\pi/p$ around the poles $(0, \pm 1)$, a fundamental domain for $\Gamma_{(p,p)}$ is the union of any two wedges sharing an edge, and $S(p, p) = \mathbb{S}/\Gamma_{(p,p)}$. See Fig. 1.1-left.

Triangular orbifolds The remaining spherical 2-orbifolds $S(2, q, p)$ with either $q = 2$ or $q = 3$ and $p = 3, 4, 5$ are called *triangular*. The corresponding group $\Gamma_{(2,q,p)}$ is the subgroup of orientation-preserving elements of the group $\widetilde{\Gamma}_{(2,q,p)}$ generated by the reflections in the edges of a triangle $\Delta_{(2,q,p)}$ with angles $\pi/2, \pi/q, \pi/p$. A fundamental domain of $\Gamma_{(2,q,p)}$ is then the union of $\Delta_{(2,q,p)}$ with its image under any of the reflections in its edges, and $\Gamma_{(2,q,p)} = \langle \alpha, \beta, \gamma | \ \alpha^2 = \beta^q = \gamma^p = \alpha \cdot \beta \cdot \gamma = 1 \rangle$ where α, β, γ are the rotations centered at the vertices of $\Delta_{(2,q,p)}$.

The main point here is of course that the triangles $\Delta_{(2,q,p)}$ exist in $\mathbb{S}$. The choice for $q = 2$ and arbitrary p is easy: $A = (1, 0)$, $B = ({\rm e}^{i\pi/p}, 0)$ and $C = (0, 1)$, see Fig. 1.1-right. For $q = 3$ see Fig. 1.2.

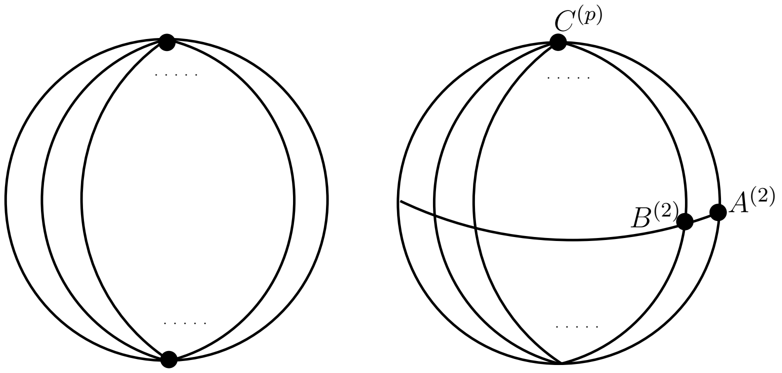

Figure 1.1. Tessellations of $\mathbb{S}$ by fundamental domains of $\widetilde{\Gamma}_{(p,p)}$ and $\widetilde{\Gamma}_{(2,2,p)}$

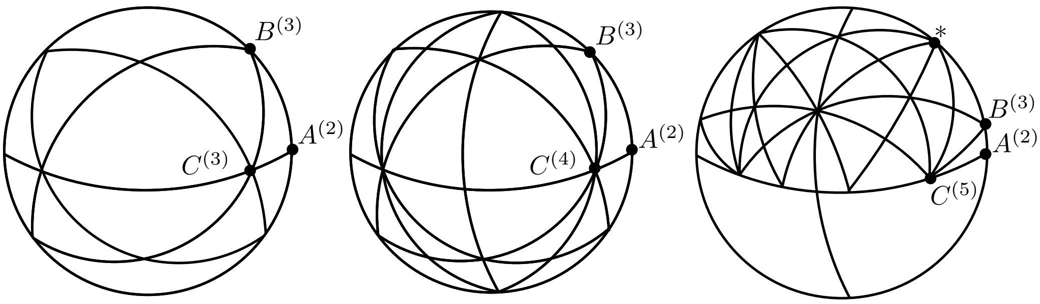

Figure 1.2. Tessellations of $\mathbb{S}$ by triangular fundamental domains of $\widetilde{\Gamma}_{(2,3,3)}$, $\widetilde{\Gamma}_{(2,3,4)}$, and (partial) $\widetilde{\Gamma}_{(2,3,5)}$

In each case the pictures also show the images of $\Delta(2,q,p)$ under the action of $\widetilde{\Gamma}_{(2,q,p)}$. A fundamental domain for the group $\Gamma_{(2,q,p)}$ giving $S(2,q,p)$ is always the union of any two triangles sharing an edge. In all the pictures for $q=3$ we have $A=(1,0)$. Moreover for $p=3$ we have $B=\left(\sqrt{\frac{1}{3}},\sqrt{\frac{2}{3}}\right)$ and $C=\left(\sqrt{\frac{1}{3}}+i\sqrt{\frac{2}{3}},0\right)$, while for $p=4$ we have $B=\left(\sqrt{\frac{2}{3}},\sqrt{\frac{1}{3}}\right)$ and $C=\left(\sqrt{\frac{1}{2}}+i\sqrt{\frac{1}{2}},0\right)$; for $p=5$ the exact values of the coordinates of B and C are more complicated.

Remark 1.2.2. Each group $\widetilde{\Gamma}_{(2,3,p)}$ is the symmetry group of a Platonic solid inscribed in $\mathbb{S}$: the tetrahedron for $p=3$, the octahedron (or its dual cube) for $p=4$, and the dodecahedron (or its dual icosahedron) for $p=5$.

Euclidean structures On each of the three triangular orbifolds $S(p,q,r)$ with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$ the Euclidean structure (unique up to rescaling) is constructed essentially as in the spherical case. We take a triangle $\Delta(p,q,r)$ in $\mathbb{E}$ with angles $\pi/p,\pi/q,\pi/r$, the group $\widetilde{\Gamma}_{(p,q,r)}$ generated by the reflections in the edges of $\Delta(p,q,r)$, and its subgroup $\Gamma_{(p,q,r)}$ of orientation-preserving isometries. Then $S(p,q,r)$ is the quotient of $\mathbb{E}$ under the action of $\Gamma_{(p,q,r)}$, which is generated by the rotations of angles $2\pi/p,2\pi/q,2\pi/r$ around the vertices of $\Delta(p,q,r)$, and a fundamental domain is given by the union of $\Delta(p,q,r)$ with any of its reflected copies in one of the edges. In each case we now make a precise choice for the vertices $\widetilde{A}^{(p)},\widetilde{B}^{(q)},\widetilde{C}^{(r)}$ of $\Delta(p,q,r)$ and determine the resulting area $\mathcal{A}$ of $S(p,q,r)$. See also Fig. 1.3.

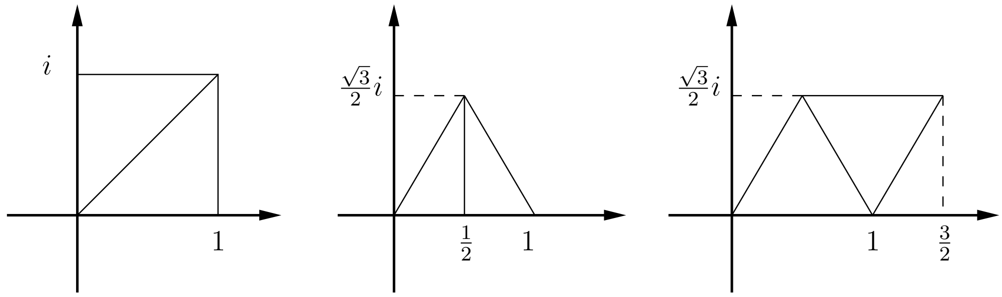

Figure 1.3. The fixed fundamental domains for $S(2,4,4)$, $S(2,3,6)$ and $S(3,3,3)$

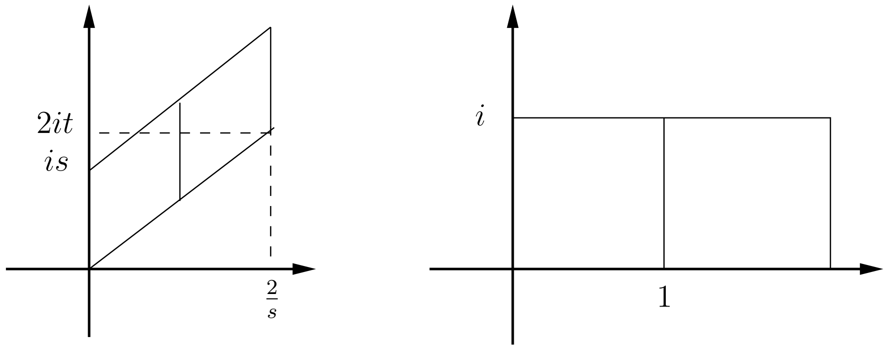

Figure 1.4. The fundamental domains for $S(2,2,2,2)$ for general s,t and for $s=t=1$

$$\begin{array}{lllll}\Delta(2,4,4): & \widetilde{A}^{(2)}=1, & \widetilde{B}^{(4)}=0, & \widetilde{C}^{(4)}=1+i, & \mathcal{A}(S(2,4,4))=1,\\ \Delta(2,3,6): & \widetilde{A}^{(2)}=\frac{1}{2}, & \widetilde{B}^{(3)}=0, & \widetilde{C}^{(6)}=\frac{1+i\sqrt{3}}{2}, & \mathcal{A}(S(2,3,6))=\frac{\sqrt{3}}{4},\\ \Delta(3,3,3): & \widetilde{A}^{(3)}=0, & \widetilde{B}^{(3)}=1, & \widetilde{C}^{(3)}=\frac{1+i\sqrt{3}}{2}, & \mathcal{A}(S(3,3,3))=\frac{\sqrt{3}}{2}.\end{array}$$

The situation for $S(2,2,2,2)$ is slightly different, since there is flexibility besides rescaling. For $s,t\in\mathbb{R}$ with $s>0$ we consider in $\mathbb{E}$ the quadrilateral $Q_{s,t}$ with corners

$$\widetilde{A}^{(2)}=0,\qquad \widetilde{B}^{(2)}=\frac{1}{s}+it,\qquad \widetilde{C}^{(2)}=\frac{1}{s}+i(s+t),\qquad \widetilde{D}^{(2)}=is$$

and we define $\Gamma^{s,t}_{(2,2,2,2)}$ as the group generated by the rotations of angle π around these points. Then the action of $\Gamma^{s,t}_{(2,2,2,2)}$ on $\mathbb{E}$ defines on $S(2,2,2,2)$ a Euclidean structure of area 2, and a fundamental domain is given by the union of $Q_{s,t}$ with any translate of itself having an edge in common with itself, as shown in Fig. 1.4-left. When $S(2,2,2,2)$ plays the rôle of X in $\widetilde{X} \dashrightarrow X$ we will endow it with the structure given by $s=t=1$, as shown in Fig. 1.4-right. It is an easy exercise to check that any other structure with area 2 is defined by $\Gamma^{s,t}_{(2,2,2,2)}$ for some s,t.

Chapter 2

The Hurwitz Problem

In this chapter we describe the Hurwitz existence problem, to which this thesis is devoted. We start with some preliminary definitions and facts, then we state the problem, and we conclude by describing diverse viewpoints and different methods used to attack it, giving also an overview of known results.

Branched covers Let $\widetilde{\Sigma}$ and Σ be closed, connected, and orientable surfaces, and $f : \widetilde{\Sigma} \to \Sigma$ be a branched cover, *i.e.* a map locally modeled on functions of the form $(\mathbb{C}, 0) \xrightarrow{z \mapsto z^k} (\mathbb{C}, 0)$ with $k \geqslant 1$. If $k > 1$ then 0 in the target $\mathbb{C}$ is a branching point, and k is the local degree at 0 in the source $\mathbb{C}$. There is a finite number n of branching points, and, removing all of them from Σ and their preimages from $\widetilde{\Sigma}$, we see that f induces a genuine cover of some degree d. The collection $(d_{ij})_{j=1}^{m_i}$ of the local degrees at the preimages of the i-th branching point is a partition Π_i of d. We now define:

- $\ell(\Pi_i)$ to be the length m_i of Π_i;
- Π as the set $\{\Pi_1, \ldots, \Pi_n\}$ of all partitions of d associated to f;
- $\ell(\Pi)$ to be the total length $\ell(\Pi_1) + \ldots + \ell(\Pi_n)$ of Π.

The Riemann-Hurwitz formula Once we have defined branched covers, it is useful to introduce the famous Riemann-Hurwitz formula: it links the number of branching points, the local degrees at their preimages, and the Euler characteristic of the surfaces involved in the cover. Its version for unramified covers between compact orientable surfaces is quite easy. Consider a topological cover between surfaces (possibly with boundary) $M_1 \to M_2$ of degree d. Take a sufficient small triangulation of the base surface, suitable for computing the Euler characteristic, and lift to the covering surface; it is evident that for each triangle, edge or vertex in the base, there are d copies in the cover. Hence:

$$\chi(M_1) = d \cdot \chi(M_2).$$

Now consider a degree d branched cover $\widetilde{\Sigma} \to \Sigma$, with Π its set of partitions of d; the Riemann-Hurwitz formula reads:

$$\chi(\widetilde{\Sigma}) - \ell(\Pi) = d \cdot \big(\chi(\Sigma) - n\big). \tag{2.1}$$

In order to show (2.1), we simply combine the multiplicativity of χ under genuine covers for surfaces with boundary, with the data given by local degrees at the singularities: in fact, outside an open neighborhood of the singularities, the branched cover is a topological cover of degree d; while a little disc around the i-th branching point has only $\ell(Pi_i)$ preimages, (and not d, as for nonsingular points). In the end, we sum up all terms, taking care of these missing preimages: $\chi(\widetilde{\Sigma}) = d \cdot \chi(\Sigma) + (\ell(\Pi_1) - d) + \ldots + (\ell(\Pi_n) - d)$. The announced formula descends immediately.

Candidate branched covers Consider again two closed, connected, and orientable surfaces $\widetilde{\Sigma}$ and Σ, integers $d \geqslant 2$ and $n \geqslant 1$, and a set of partitions $\Pi = \{\Pi_1, \ldots, \Pi_n\}$ of d, with $\Pi_i = (d_{ij})_{j=1}^{m_i}$, such that condition (2.1) is satisfied. We associate to these data the symbol

$$\widetilde{\Sigma} \xrightarrow[(d_{11},\ldots,d_{1m_1}),\ldots,(d_{n1},\ldots,d_{nm_n})]{d:1} \Sigma$$

that we will call a *candidate surface branched cover.* We remark that for candidate covers of the form $S \xrightarrow[\Pi]{d:1} S$, where S is the 2-sphere, with three branching points, as most of ours will be, the Riemann-Hurwitz formula (2.1) reads

$$\ell(\Pi) = d + 2. \tag{2.2}$$

The Hurwitz existence problem As we already mentioned in the Introduction, the *Hurwitz existence problem* asks which candidate surface branched covers are actually *realizable*, namely induced by some existent branched cover $f : \widetilde{\Sigma} \to \Sigma$. A non-realizable candidate surface branched cover will be called *exceptional.*

It is still an open problem, but the combined efforts of several mathematicians led in particular to the following results [11, 4]:

- If $\chi(\Sigma) \leqslant 0$ then any candidate surface branched cover is realizable, *i.e.* the Hurwitz existence problem has a positive solution in this case;
- If $\chi(\Sigma) > 0$, *i.e.* if Σ is the 2-sphere S, there exist exceptional candidate surface branched covers.

We give some more details and references in Section 2 of this chapter.

Generalization A version of the Hurwitz existence problem exists also for possibly non-orientable $\widetilde{\Sigma}$ and Σ. In this case other conditions must be required, for a branch datum, to be compatible, see [19], and below. We define a branch datum to be compatible if the following conditions hold:

1. $\chi(\widetilde{\Sigma}) - \ell(\Pi) = d(\chi(\Sigma) - n)$;
2. $n \cdot d - \ell(\Pi)$ is even;
3. If Σ is orientable then $\widetilde{\Sigma}$ is also orientable;
4. If Σ is non-orientable and d is odd then $\widetilde{\Sigma}$ is non-orientable;

5. If Σ is non-orientable and $\widetilde{\Sigma}$ is orientable, then each partition $(d_{ij})_{j=1,\dots,m_i}$ of d refines the partition $(d/2, d/2)$.

However it has been shown [6, 4] that again this generalized problem always has a positive solution if $\chi(\Sigma) \leqslant 0$, and that the case where Σ is the projective plane reduces to the case where Σ is the S.

According to these facts, in order to face the Hurwitz existence problem, it is not restrictive to *assume the candidate covered surface* Σ *is the* 2*-sphere.* Quite some progress has been made (see for instance Section 2 for known results, and the papers [19, 20, 17, 28]), but the global pattern remains elusive.

2.1 Diverse viewpoints

In its long history, the Hurwitz problem has been studied by several researchers. As a result, the original topological question has been reformulated in many different ways. Here we review the most famous and fruitful approaches.

Riemann surfaces Let Σ be a compact and connected Riemann surface, and $M(\Sigma)$ be the space of the meromorphic functions on it. It is well known that since Σ is compact, the zero set and the set of poles are finite. Let us regard anon constant $f \in M(\Sigma)$ as a map to $\mathbb{P}^1 \cong \hat{\mathbb{C}}$, mapping poles to ∞. Hence f becomes an analytic function from Σ to $\hat{\mathbb{C}}$. Actually, it is a ramified cover: one can always choose local coordinates in such a way that f is expressed as $z \mapsto z^d$ near a nonsingular point, and $z \mapsto z^m$, with $m = -\nu_p(f)$, ($\nu_p(f)$ is the order of f in p) near p pole; moreover, it is open and surjective.

It is a very classical fact in Riemann surface theory that if one fixes a finite set $B \subset \hat{\mathbb{C}}$, and denote by R_f the branching set in Σ, the following three sets are in 1-1 correspondence with each other:

- Equivalence classes of analytic maps of degree-d of some $f : \Sigma \to \hat{\mathbb{C}}$ such that $f(R_f) \subset B$
- Equivalence classes of degree d topological connected covers, $f_0 : \Sigma \backslash f^{-1}(B) \to \hat{\mathbb{C}} \backslash B$ which extend to some $f \in M(\Sigma)$ such that $f(R_f) \subset B$.
- Equivalence classes of $|B|$-tuples $(\sigma_1, \dots, \sigma_{|B|})$, of elements of $\mathfrak{S}_d$ generating a transitive subgroup, and such that $\sigma_1 \cdot \ldots \cdot \sigma_{|B|} = 1$.

(For the classical theory of Riemann surfaces see [1].)

Permutations It is evident, from the previous paragraph, that the set equivalences described there give us a truly algebraic description of the Hurwitz existence problem: in fact it establishes a correspondence between transitive subgroups of permutations in $\mathfrak{S}_d$ and degree-d branched covers of the sphere. To make it axplicit we recall here a fundamental fact, proved by Hurwitz, Husemoller, Ezell and Singerman, also reviewed in [19]: a realization of a candidate surface branched cover

$$\widetilde{\Sigma} \xrightarrow[(d_{11},\dots,d_{1m_1}),\dots,(d_{n1},\dots,d_{nm_n})]{d:1} \Sigma$$

corresponds to the choice of permutations $\sigma_1, \ldots, \sigma_n \in \mathfrak{S}_d$ such that:

- σ_i has cycles of lengths $(d_{ij})_{j=1}^{m_i}$;
- the product $\sigma_1 \cdots \sigma_n$ is the identity;
- the subgroup of $\mathfrak{S}_d$ generated by $\sigma_1, \ldots, \sigma_n$ acts transitively on $\{1, \ldots, d\}$.

Let us also give an example of how to find a realization through permutations. Consider the case of $\widetilde{\Sigma} \underset{(d_{11},\ldots,d_{1m_1}),\ldots,(d_{31},\ldots,d_{3m_3})}{\overset{d:1}{\dashrightarrow}} S$, that is a branched surface cover of the sphere with three branching points; fix a certain σ_1 with cycle lengths $(d_{1j})_{j=1}^{m_1}$, and let σ_2 vary in the conjugacy class of permutations having cycles of lengths $(d_{2j})_{j=1}^{m_2}$, checking that $\langle \sigma_1, \sigma_2 \rangle$ is a transitive subgroup, and that $\sigma_1 \cdot \sigma_2$ has cycle lengths d_{3j}. Taking $\sigma_3 = (\sigma_1 \cdot \sigma_2)^{-1}$, we obtain the three permutations, satisfying the required conditions, and hence realizing the given branch datum. It can be easily checked, for istance that the permutations

$$\begin{aligned}
\sigma_1 &= (1{,}12)(2{,}8)(9{,}10)(6{,}7)(3{,}4) \\
\sigma_2 &= (3{,}2{,}1)(6{,}5{,}4)(9{,}8{,}7)(12{,}11{,}10) \\
\sigma_3 &= (1{,}10{,}7{,}4)(2{,}9{,}11{,}12)(3{,}5{,}6{,}8)
\end{aligned}$$

realize the compatible branch datum $S \underset{(2,\ldots,2,1,1),(3,3,3,3),(4,4,4)}{\overset{12:1}{\dashrightarrow}} S$ (with the convention that $(1,2) \cdot (2,3) = (1,2,3)$).

Note that using permutations could take a long time to decide about realizability of covers with large degree like $S \underset{(2,\ldots,2),(3,\ldots,3),(5,\ldots,5)}{\overset{60:1}{\dashrightarrow}} S$: in the following chapter we prove that also this candidate branch datum is realizable, but using a different technique.

Dessins d'enfant This is a classical technique, introduced by Grothendieck in [9] for studying algebraic maps between Riemann surfaces. The main idea is to establish an association between ramified covers and bipartite graphs. Recall now that a bipartite graph is a finite 1-complex such that its set of vertices splits as $V_1 \sqcup V_2$ and each edge has one endpoint in V_1 and one in V_2.

We introduce Grothendieck's dessins in their original form, that is valid only for covers of the sphere with three branching points. We recommend [19] for further reading, where this classical technique is generalized to an arbitrary number of branching points.

Definition 2.1.1. *A* dessins d'enfant *on a surface* $\widetilde{\Sigma}$ *is a* bipartite *graph* $D \subset \widetilde{\Sigma}$ *such that* $\widetilde{\Sigma} \backslash D$ *consists of open discs. The* length *of one of these discs is the number of edges of* D *along which its boundary passes, counted with multiplicity.*

Here we state a proposition that makes it clear how to pass from branched covers to dessins. (See [19] for the proof.)

Proposition 2.1.2. *The realizations of a branch datum* $\widetilde{\Sigma} \underset{(d_{11},\ldots,d_{1m_1}),\ldots,(d_{31},\ldots,d_{3m_3})}{\overset{d:1}{\dashrightarrow}} S$ *correspond to the dessins d'enfant* $D \subset \widetilde{\Sigma}$ *with set of vertices* $V_1 \sqcup V_2$ *such that for* $i = 1, 2$ *the vertices in* V_i *have valences* $(d_{ij})_{j=1}^{m_i}$*, and the discs in* $\widetilde{\Sigma} \backslash D$ *have lengths* $(2d_{3j})_{j=1}^{m_3}$.

Permutations and dessins Consider a candidate branched cover $\widetilde{\Sigma} \underset{(d_{11},\ldots,d_{1m_1}),\ldots,(d_{31},\ldots,d_{3m_3})}{\overset{d:1}{\dashrightarrow}} S$; we will now describe for the reader's convenience a correspondence between a dessin d'enfant D realizing the cover, and a suitable choice of permutations in $\mathfrak{S}_d$. Let D have vertices $V_1 \sqcup V_2$ as usual; to produce a choice of $\sigma_1,\ \sigma_2$ corresponding to the same realization we proceed as follows:

(i) Assign labels from 1 to d to the edges of D;

(ii) Then consider the edge k and the vertex of V_i belonging to it: $\sigma_i(k)$ has one cycle for each vertex v of V_i,the cycle consisting of the labels of the edges incident to v, arranged as they appear around $v$in a counter-clockwise order.

On the other hand, given two permutations $\sigma_1,\ \sigma_2 \in \mathfrak{S}_d$ realizing the given branch datum, we construct a dessins $D(\sigma_1, \sigma_2)$ in the following way:

(i) Take the set of cycles of σ_i as the set of vertices V_i;

(ii) Draw an edge labeled k ($k = 1, \ldots, d$) if there are two cycles, one in σ_1 and one in σ_2, containing k.

We address the reader to [19] for further details.

Covers between 2-orbifolds vs surface branched covers As pointed out in [19] and spelled out below, any candidate surface branched cover $\widetilde{\Sigma} \underset{\Pi}{\overset{d:1}{\dashrightarrow}} \Sigma$ has a *preferred associated candidate* 2*-orbifold cover* $\widetilde{X} \overset{d:1}{\dashrightarrow} X$ satisfying $\chi^{\text{orb}}(\widetilde{X}) = d \cdot \chi^{\text{orb}}(X)$. Moreover $\widetilde{\Sigma} \underset{\Pi}{\overset{d:1}{\dashrightarrow}} \Sigma$ can be reconstructed from $\widetilde{X} \overset{d:1}{\dashrightarrow} X$ if some additional *cover instructions* are provided.

As one easily sees, distinct orbifold covers can induce the same surface branched cover (in the local model, the two cone orders can be multiplied by one and the same integer). However a surface branched cover has an "easiest" associated orbifold cover, *i.e.*, that with the smallest possible cone orders. This carries over to *candidate* covers, as we will now spell out. Consider a candidate surface branched cover

$$\widetilde{\Sigma} \underset{(d_{11},\ldots,d_{1m_1}),\ldots,(d_{n1},\ldots,d_{nm_n})}{\overset{d:1}{\dashrightarrow}} \Sigma$$

and define

$$\begin{aligned} p_i &= \text{l.c.m.}\{d_{ij} :\ j = 1, \ldots, m_i\}, & p_{ij} &= p_i / d_{ij}, \\ X &= \Sigma(p_1, \ldots, p_n), & \widetilde{X} &= \widetilde{\Sigma}\big((p_{ij})_{i=1,\ldots,n}^{j=1,\ldots,m_i}\big) \end{aligned}$$

where "l.c.m." stands for "least common multiple." Then we have a preferred associated candidate 2-orbifold cover $\widetilde{X} \overset{d:1}{\dashrightarrow} X$ satisfying $\chi^{\text{orb}}(\widetilde{X}) = d \cdot \chi^{\text{orb}}(X)$. Note that the original candidate surface branched cover cannot be reconstructed from $\widetilde{X}, X, d$ alone, but it can if $\widetilde{X} \overset{d:1}{\dashrightarrow} X$ is complemented with the *cover instructions*

$$(p_{11}, \ldots, p_{1m_1}) \dashrightarrow p_1, \qquad \ldots \qquad (p_{n1}, \ldots, p_{nm_n}) \dashrightarrow p_n$$

that we will sometimes include in the symbol $\widetilde{X} \overset{d:1}{\dashrightarrow} X$ itself, omitting the p_{ij}'s equal to 1. Of course a candidate surface branched cover is realizable if and only if the associated candidate 2-orbifold cover with appropriate cover instructions is realizable.

Remark 2.1.3. We must emphasize this passage: going to and coming from 2-orbifold covers is the key point of our study of surface branched covers.

2.2 Known results

In this section we briefly collect the main partial solutions to the Hurwitz problem obtained over the time.

Known results for $\Sigma \neq \mathbb{S}$ We outline some results which reduce the general Hurwitz problem to the case where the base surface is the sphere. The first theorem first appeared in [6, p. 125], and it is attributed to Shephardson. A proof can be found in [11, theorem 4] and [4, Prop. 3.3]:

Theorem 2.2.1. *A compatible branch datum with orientable base surface Σ and $\chi(\Sigma) \leqslant 0$ is realizable.*

The next result is proved in [6, Theorem 3.4] and [4, Prop. 3.3]:

Theorem 2.2.2. *A compatible branch datum with non-orientable Σ and $\widetilde{\Sigma}$ and $\chi(\Sigma) \leqslant 0$ is realizable.*

The following easy fact (stated in [4, Prop. 2.7]) together with the previous results let us complete the case $\chi(\Sigma) \leqslant 0$, in the affirmative: each compatible branch datum is realizable.

Proposition 2.2.3. *A compatible branch datum with Σ non-orientable and $\widetilde{\Sigma}$ orientable is realizable if and only if it is possible to decompose for all i the partition Π_i into a pair of partitions Π_i' and Π_i'' of $d/2$ in such a way that the branch datum*

$$\widetilde{\Sigma} \xrightarrow[\Pi_1', \Pi_1'', \ldots, \Pi_n', \Pi_n'']{d/2:1} \Sigma'$$

is realizable, where Σ' is the orientable double cover of Σ.

Now the aim is to understand what happens when the base has positive Euler characteristic. The following theorem is due to Edmonds, Kulkarni and Stong, [4, Theorem 5.1], and deals with the case $\Sigma = \mathbb{P}$, the projective plane.

Theorem 2.2.4. *A compatible branch datum with $\Sigma = \mathbb{P}$ and non-orientable $\widetilde{\Sigma}$ is realizable.*

Hence, the Hurwitz existence problem remains open when

- $\Sigma = \mathbb{S}$;
- $\Sigma = \mathbb{P}$ and orientable $\widetilde{\Sigma}$.

However, Proposition 2.2.3 tells us that the realizability issue of some $\widetilde{\Sigma} \dashrightarrow \mathbb{P}$ reduces to the realizability of a suitable $\widetilde{\Sigma} \dashrightarrow \mathbb{S}$. Then, from now on, we will discuss only branched covers of the sphere.

Known results for $\Sigma = \mathbb{S}$ When the base surface is the sphere, we do have exceptional branched covers. The easiest example is $\mathbb{S} \overset{4:1}{\underset{(3,1),(2,2),(2,2)}{\dashrightarrow}} \mathbb{S}$. Here we review only the main existence and non-existence results. One of the most interesting among non-existence results is the following, in the case of $\widetilde{\Sigma} = \mathbb{S}$ and $n = 3$, [4, Prop. 5.7]:

Lemma 2.2.5. *If $d = ab$, with $a, b > 1$, then the partitions*

$$(a, \ldots, a), (b + 1, 1, \ldots, 1), (a, a(b - 1))$$

give a compatible but non-realizable branch datum.

The following result proved in [19] extends the previous one:

Theorem 2.2.6. *Let $d = h \cdot k$, with $k,\ h \geqslant 2$; let $(h_j)_{j=1,\ldots,p}$ be a partition of h with $p \geqslant 2$. Then the branch datum*

$$\mathbb{S} \overset{d:1}{\underset{(k,\ldots,k),(h+p-1,1,\ldots,1),(kh_1,\ldots,kh_p)}{\dashrightarrow}} \mathbb{S}$$

is non-realizable.

In the same paper Pervova and Petronio also establish a theorem which implies a very efficient criterion to recognize exceptional data:

Theorem 2.2.7. *Suppose d and all d_{ij} for $i = 1, 2$ are even. If the branch datum*

$$\mathbb{S} \overset{d:1}{\underset{\Pi_1,\Pi_2,\Pi_3}{\dashrightarrow}} \mathbb{S}$$

is realizable, then (d_{3j}) refines the partition $(d/2, d/2)$.

On the other hand, one of the most general existence result was first stated by Thom (in [25]) in the case of $\widetilde{\Sigma} = \mathbb{S}$, then reproved in [2], and finally generalized to arbitrary cover by Edmonds, Kulkarni and Stongs in [4];

Theorem 2.2.8. *A compatible branch cover is realizable if one of the partitions of the degree is (d).*

A variation on this result is given in [4]: it classifies realizable branch data with one partition of the form $(d - 1, 1)$.

Because of the prime degree conjecture, discussed in the next section, we are most interested in covers with exactly three branching points. However there are also some results relevant to the case where the number of branching points is "large" compared to degree d. One of the most significant one is due to Edmonds, Kulkarni and Stong:

Theorem 2.2.9. *A branch datum with $d \neq 4$ and $n \cdot d - \ell(\Pi) \geqslant 3(d - 1)$ is realizable. The exceptional data with $d = 4$ are precisely those with partitions $(2, 2), \ldots, (2, 2), (3, 1)$.*

A consequence of this result is that the number of exceptional branch data for any fixed $d \neq 4$ is finite, [4, Corollary 4.4].

Moreover Pervova and Petronio, in [20], studied the branch data in which one partition is $(d-2,2)$, and proved the following three theorems, using an extension of a geometric criterion for the existence of a branched cover previously introduced by Baránski in [2], (i.e. the existence of certain families of graphs on the covering surface, called minimal checkerboard graphs).

Theorem 2.2.10. *Let* $\mathbb{S} \xrightarrow[(d-2,2),\Pi_2,\Pi_3]{d:1} \mathbb{S}$ *be a compatible branch datum. If d is odd then it is realizable. If $d = 2k$ is even then the compatible and non-realizable branch data are precisely those of the following types:*

- $\mathbb{S} \xrightarrow[(2k-2,2),(2,\ldots,2),(2,\ldots,2)]{2k:1} \mathbb{S}$ *with* $k > 2$*;*
- $\mathbb{S} \xrightarrow[(2k-2,2),(2,\ldots,2),(k+1,1,\ldots,1)]{2k:1} \mathbb{S}$.

Theorem 2.2.11. *With the single exception of* $\mathbb{T} \xrightarrow[(4,2),(3,3),(3,3)]{6:1} \mathbb{S}$*, every compatible datum of the form* $\mathbb{T} \xrightarrow[(d-2,2),\Pi_2,\Pi_3]{d:1} \mathbb{S}$ *is realizable.*

Theorem 2.2.12. *If* $g \geqslant 2$*, then every compatible branch datum of the form* $g\mathbb{T} \xrightarrow[(d-2,2),\Pi_2,\Pi_3]{d:1} \mathbb{S}$ *is realizable.*

It is worth mentioning that in [17] Pakovich gives an interesting generalization of Theorem 2.2.10, using an entirely different approach. Pakovich extends the result to the branch data of the form $\mathbb{S} \xrightarrow[(d-k,k),\Pi_2,\ldots,\Pi_n]{d:1} \mathbb{S}$: he completely answers the problem in the case of branched covers between spheres with an arbitrary number of branching points and when Π has at least a "small partition" (it is called small when $\ell(\Pi_i) \leqslant 2$).

Here is one more result from [19] about the realizability of covers with three branchings, the proof of which is based on the idea of composing covers:

Theorem 2.2.13. *Let* $\widetilde{\Sigma} \xrightarrow[\Pi_1,\Pi_2,\Pi_3]{d:1} \mathbb{S}$ *be a compatible branch datum. Let* $p \geqslant 3$ *be odd and suppose that all* d_{ij} *are divisible by* p*. Then the datum is realizable.*

2.3 The prime degree conjecture

In [4] the Hurwitz existence problem is reduced to the case of branched covers of the 2-sphere; moreover the paper contains in addition the following interesting:

Conjecture 2.3.1. *If* $\widetilde{\Sigma} \xrightarrow[\Pi]{d:1} S$ *is candidate surface branched cover and the degree d is a prime number then the candidate is realizable.*

Sure it has been partially motivated by a lemma they showed in the same paper, about exceptionality in non-prime degree, already mentioned (Theorem 2.2.5); and in [19] and [20] there are many other examples of exceptionality of covers in non-prime degree.

In order to approach this conjecture, it is important to notice that in [4] the authors stated that establishing Conjecture 0.0.1 in the special case of three branching points would imply the general case.

We find in [28] some results supporting this conjecture: in fact Zheng lists all exceptional candidate surface branched covers with $n = 3$ and $d \leqslant 20$ (he has been determined them by computer) and none of them occurs for prime d. We address the reader to the end of Chapter 3, where the relevance of our results with respect to this conjecture is explained.

Chapter 3

Orbifold Covers in $\chi^{\text{orb}} \geqslant 0$

This chapter is devoted to the complete analysis of orbifold covers with $\chi^{\text{orb}} \geqslant 0$. It uses the geometric description of orbifolds given in the Chapter 1, and it ends with remarks about the prime degree conjecture.

3.1 The geometric approach

To analyze the realizability of a candidate surface branched cover we will switch to the associated candidate 2-orbifold cover $\widetilde{X} \dashrightarrow X$ and we will use geometry either to explicitly construct a map $f : \widetilde{X} \to X$ realizing it, or to show that such an f cannot exist.

To explain how this works we first note that any 2-orbifold X with a fixed geometric structure of type $\mathbb{X} \in \{\mathbb{S}, \mathbb{E}, \mathbb{H}\}$ has a well-defined distance function. This is because the structure is given by a quotient map $\mathbb{X} \to X$, that for obvious reasons we will call *geometric universal cover of* X, defined by an isometric and discrete (even if not free) action. Therefore a piecewise smooth path α in X has a well-defined length obtained by lifting it to a path $\widetilde{\alpha}$ in $\mathbb{X}$, even if $\widetilde{\alpha}$ itself is not unique (even up to automorphisms of $\mathbb{X}$) when α goes through some cone point of X. Now we have the following:

Proposition 3.1.1. *Let $f : \widetilde{X} \to X$ be a 2-orbifold cover. Suppose that X has a fixed geometry with geometric universal cover $\pi : \mathbb{X} \to X$. Then there exists a geometric structure on $\widetilde{X}$ with geometric universal cover $\widetilde{\pi} : \mathbb{X} \to \widetilde{X}$ and an isometry $\widetilde{f} : \mathbb{X} \to \mathbb{X}$ such that $\pi \circ \widetilde{f} = f \circ \widetilde{\pi}$.*

Proof. We define the length of a path in $\widetilde{X}$ as the length of its image in X under f, and we consider the corresponding distance. Analyzing the local model of f, one sees that this distance is compatible with a local orbifold geometric structure also of type $\mathbb{X}$, so there is one global such structure on $\widetilde{X}$, with geometric universal cover $\widetilde{\pi} : \mathbb{X} \to \widetilde{X}$. The properties of the universal cover imply that there is a map $\widetilde{f} : \mathbb{X} \to \mathbb{X}$ such that $\pi \circ \widetilde{f} = f \circ \widetilde{\pi}$. By construction $\widetilde{f}$ preserves the length of paths, but $\mathbb{X}$ is a manifold, not an orbifold, so $\widetilde{f}$ is a local isometry. In particular it is a cover, but $\mathbb{X}$ is simply connected, so $\widetilde{f}$ is a homeomorphism and hence an isometry. $\square$

Any spherical 2-orbifold X is *rigid*, namely the geometric universal cover $\mathbb{S} \to X$ is unique up to automorphisms of $\mathbb{S}$ and X, so in the spherical case one is not faced with any choice while applying Proposition 3.1.1. On the contrary Euclidean 2-orbifolds are never rigid, since the metric can always be rescaled (and it can also be changed in more essential ways on the torus T and on $S(2,2,2,2)$, see below). In this case we will slightly modify the content of Proposition 3.1.1 by rescaling $\widetilde{X}$ so that its area equals that of X, in which case $\widetilde{f}$ is no more an isometry but merely a complex-affine map $\mathbb{C} \to \mathbb{C}$, with $\mathbb{C}$ identified to $\mathbb{E}$. More precisely:

Proposition 3.1.2. *Let $f : \widetilde{X} \xrightarrow{d:1} X$ be a 2-orbifold cover. Suppose that X has a fixed Euclidean structure with geometric universal cover $\pi : \mathbb{E} \to X$. Then there exists a Euclidean structure on $\widetilde{X}$ with geometric universal cover $\widetilde{\pi} : \mathbb{E} \to \widetilde{X}$ such that X and $\widetilde{X}$ have the same area, and a map $\widetilde{f} : \mathbb{E} \to \mathbb{E}$ of the form $\widetilde{f}(z) = \lambda \cdot z + \mu$, with λ, $\mu \in \mathbb{C}$, such that $\pi \circ \widetilde{f} = f \circ \widetilde{\pi}$. This implies that $d = |\lambda|^2$.*

Proof. With respect to the structure on $\widetilde{X}$ given by Proposition 3.1.1 the area of $\widetilde{X}$ is d times that of X, so the scaling factor is $1/\sqrt{d}$. After rescaling $\widetilde{f}$ is therefore $\sqrt{d}$ times an isometry, and the conclusion follows. □

Remark 3.1.3. The structure of this chapter is the consequence of splitting orbifold covers with respect to the sign of the Euler characteristic. Despite our efforts the hyperbolic case is far from been solved, because of the difficulty of improving a substantial extension to the geometric tool. Then, in the next chapter we restrict ourselves to the study of hyperbolic 2-orbifold covers relevant for the prime degree conjecture.

3.2 Positive Euler characteristic

In this section we will establish Theorem 0.0.5. More precisely we will show:

Theorem 3.2.1. *A candidate surface branched cover $\widetilde{\Sigma} \dashrightarrow \Sigma$ having an associated candidate 2-orbifold cover $\widetilde{X} \dashrightarrow X$ with $\chi^{\mathrm{orb}}(X) > 0$ is exceptional if and only if $\widetilde{X}$ is bad and X is spherical. This occurs precisely for the following candidate covers, in none of which the degree is prime:*

$$\begin{array}{lll}
S \xrightarrow[(2,\ldots,2,1),(3,3,3),(3,3,3)]{9:1} S & S \xrightarrow[(2,\ldots,2,1),(3,3,3),(4,4,1)]{9:1} S & S \xrightarrow[(2,\ldots,2),(3,3,3,1),(4,4,2)]{10:1} S \\
S \xrightarrow[(2,\ldots,2),(3,\ldots,3,1),(4,\ldots,4)]{16:1} S & S \xrightarrow[(2,\ldots,2),(3,\ldots,3,1),(5,5,5,1)]{16:1} S & S \xrightarrow[(2,\ldots,2),(3,\ldots,3),(4,\ldots,4,2)]{18:1} S \\
S \xrightarrow[(2,\ldots,2,1),(3,\ldots,3),(5,\ldots,5,1)]{21:1} S & S \xrightarrow[(2,\ldots,2,1),(3,\ldots,3,1),(5,\ldots,5)]{25:1} S & S \xrightarrow[(2,\ldots,2),(3,\ldots,3),(5,\ldots,5,1)]{36:1} S \\
S \xrightarrow[(2,\ldots,2),(3,\ldots,3,1),(5,\ldots,5)]{40:1} S & S \xrightarrow[(2,\ldots,2,1),(3,\ldots,3),(5,\ldots,5)]{45:1} S & S \xrightarrow[(2,\ldots,2),(2,\ldots,2),(h,2k-h)]{2k:1} S
\end{array} \tag{3.1}$$

with $k > h \geqslant 1$ in the last item.

In addition to proving this result we will describe all $\widetilde{\Sigma} \dashrightarrow \Sigma$ having associated $\widetilde{X} \dashrightarrow X$ with $\chi^{\mathrm{orb}}(X) > 0$ not listed in the statement, and we will explicitly construct a geometric realization of each such $\widetilde{X} \dashrightarrow X$. To outline our argument, we first recall that the 2-orbifolds X with $\chi^{\mathrm{orb}}(X) > 0$ are

$$S, \quad S(p), \quad S(p,q), \quad S(2,2,p), \quad S(2,3,3), \quad S(2,3,4), \quad S(2,3,5).$$

In particular for any relevant $\widetilde{\Sigma} \xrightarrow[\Pi]{d:1} \Sigma$ we have $\widetilde{\Sigma} = \Sigma = S$. Moreover X is bad if and only if it is $S(p)$ for $p > 1$ or $S(p,q)$ for $p \neq q > 1$, and in all other cases it has a rigid spherical structure. Our main steps will be as follows:

- We will determine all the candidate surface branched covers having an associated candidate $\widetilde{X} \dashrightarrow X$ with positive $\chi^{\mathrm{orb}}(\widetilde{X})$ and $\chi^{\mathrm{orb}}(X)$, and the corresponding cover instructions for $\widetilde{X} \dashrightarrow X$;
- For each spherical X with $\chi^{\mathrm{orb}}(X) > 0$ we will explicitly describe (and fix) the geometric universal cover $\pi : \mathbb{S} \to X$;
- For each $\widetilde{X} \dashrightarrow X$ with $\chi^{\mathrm{orb}}(X) > 0$ (complemented with its cover instructions) associated to some candidate surface branched cover, except when $\widetilde{X}$ is bad and X is spherical, we will explicitly describe an isometry $\widetilde{f} : \mathbb{S} \to \mathbb{S}$ such that there exists $f : \widetilde{X} \to X$ realizing $\widetilde{X} \dashrightarrow X$ with $\pi \circ \widetilde{f} = f \circ \widetilde{\pi}$, where π and $\widetilde{\pi}$ are the geometric universal covers of $\widetilde{X}$ and X described in the previous step.

To list all candidate surface branched covers having an associated candidate $\widetilde{X} \dashrightarrow X$ with positive $\chi^{\mathrm{orb}}(X)$ our steps will be as follows:

- We consider all possible pairs $(\widetilde{X}, X)$ such that $\chi^{\mathrm{orb}}(\widetilde{X})/\chi^{\mathrm{orb}}(X)$ is an integer $d > 1$;
- Supposing X has n cone points of orders $p_1, \ldots, p_n$, we consider all possible ways of grouping the orders of the cone points of $\widetilde{X}$ as
$$(q_{11}, \ldots, q_{1\mu_1}), \ldots, (q_{n1}, \ldots, q_{n\mu_n})$$
so that q_{ij} divides p_i for all i and j;
- We determine $m_i \geqslant \mu_i$ so that, setting $q_{ij} = 1$ for $j > \mu_i$ and $d_{ij} = \frac{p_i}{q_{ij}}$, we have that $\sum_{j=1}^{m_i} d_{ij}$ is equal to d for all i;
- We check that p_i is the least common multiple of $(d_{ij})_{j=1}^{m_i}$.

This leads to the candidate surface branched cover $S \xrightarrow[\Pi]{d:1} S$ with $\Pi_i = (d_{ij})_{j=1}^{m_i}$ and $\Pi = (\Pi_i)_{i=1}^{n}$, having associated candidate $\widetilde{X} \dashrightarrow X$ with cover instructions
$$(q_{11}, \ldots, q_{1m_1}) \dashrightarrow p_1, \ldots, (q_{n1}, \ldots, q_{nm_n}) \dashrightarrow p_n.$$
For the sake of brevity we will group together our statements depending on the type of $\widetilde{X}$. In the proofs it will sometimes be convenient to carry out the steps outlined above in a different order. In particular, it is often not easy to determine beforehand when $\chi^{\mathrm{orb}}(\widetilde{X})/\chi^{\mathrm{orb}}(X)$ is an integer, so this condition is imposed at the end. Moreover, whenever X has three cone points, instead of $\chi^{\mathrm{orb}}(\widetilde{X}) = d \cdot \chi^{\mathrm{orb}}(X)$ we will use the equivalent formula (2.2), expressed in terms of the data of the would-be candidate surface branched cover.

Remark 3.2.2. Suppose some $\widetilde{X} \stackrel{d:1}{\dashrightarrow} X$ where X has $n \geqslant 1$ cone points is associated to some $\widetilde{\Sigma} \xrightarrow[\Pi_1,\ldots,\Pi_n]{d:1} \Sigma$. Then each of the n partitions of d in Π has at least one entry larger than 1, otherwise $\widetilde{\Sigma} \xrightarrow[\Pi]{d:1} \Sigma$ has an "easier" associated candidate $\widetilde{Y} \dashrightarrow Y$, where Y has less than n cone points. In particular $d > 1$.

Proposition 3.2.3. *The candidate surface branched covers having associated candidate $\widetilde{X} \dashrightarrow X$ with bad $\widetilde{X}$ are precisely those listed in (3.1).*

Proof. We start with $\widetilde{X} = S(\widetilde{p})$ for $\widetilde{p} \geqslant 2$. If $X = S$ or $X = S(p)$ then $\chi^{\rm orb}(\widetilde{X})/\chi^{\rm orb}(X) < 2$, so there is no relevant candidate.

Now suppose $X = S(p,q)$ and $\widetilde{p} \dashrightarrow p$, so $p = k\cdot\widetilde{p}$ for some k, whence $\mu_1 = 1$, $\mu_2 = 0$, and $d = k + (m_1 - 1)\cdot p = m_2\cdot q$. Combining these relations with $1 + \frac{1}{\widetilde{p}} = d\cdot\left(\frac{1}{p} + \frac{1}{q}\right)$ we get $m_1 + m_2 = 2$, so $m_1 = m_2 = 1$ and $k = q = d$, but p is not l.c.m.(k), so again there is no relevant candidate.

Turning to $X = S(2,2,p)$ we can have either $\widetilde{p} \dashrightarrow 2$ or $\widetilde{p} \dashrightarrow p$. In the first case we should have $\Pi_1 = (2,\ldots,2,1)$ and $\Pi_2 = (2,\ldots,2)$, which is impossible because d should be both even and odd. In the second case we have $d = 2k$ and $m_1 = m_2 = k$, whence $m_3 = 2$, so we get item 12 in (3.1) in the special case where h divides $2k - h$ or conversely. Other instances of item 12 will be found below.

Now let $X = S(2,3,3)$. If $\widetilde{p} \dashrightarrow 2$ (or $\widetilde{p} \dashrightarrow 3$) then $\widetilde{p} = 2$ (or $\widetilde{p} = 3$) and computing $\chi^{\rm orb}$ we get $d = 9$ (or $d = 8$). In the first case we get item 1 in (3.1), in the second case we get nothing because $\Pi_2 = (3,\ldots,3,1)$ which is incompatible with $d = 8$.

The discussion for $X = S(2,3,p)$ with $p = 4,5$ is similar. We examine where $\widetilde{p}$ can be mapped to, we deduce what it is (except that both 2 and 4 are possible when $\widetilde{p} \dashrightarrow p = 4$), in each case we determine d using $\chi^{\rm orb}$ and we check that there exist appropriate partitions of d. For $X = S(2,3,4)$ we get items 4 and 6 in (3.1), with $\widetilde{p} = 2 \dashrightarrow 4$ in 6, while for $X = S(2,3,5)$ we get items 9 to 11 in (3.1).

Let us now consider $\widetilde{X} = S(\widetilde{p},\widetilde{q})$ with $\widetilde{p} \neq \widetilde{q} > 1$ and again examine the various X's, noting first that X cannot be S or $S(p)$ since $d \geqslant 2$. For $X = S(p,q)$ suppose first $\widetilde{p},\widetilde{q} \dashrightarrow p$. Then $p = k\cdot\widetilde{p} = h\cdot\widetilde{q}$ and $d = k + h + (m_1 - 2)\cdot p = m_2\cdot q$, which we can combine with $\frac{1}{\widetilde{p}} + \frac{1}{\widetilde{q}} = d\cdot\left(\frac{1}{p} + \frac{1}{q}\right)$ easily getting $m_1 = 2$ and $m_2 = 0$, which is impossible. Now suppose $\widetilde{p} \dashrightarrow p$ and $\widetilde{q} \dashrightarrow q$, so $p = k\cdot\widetilde{p}$ and $q = h\cdot\widetilde{q}$ whence $d = k + (m_1 - 1)\cdot p = h + (m_2 - 1)\cdot q$, which leads to $m_1 = m_2 = 1$, but then we cannot have $p = $ l.c.m.(k) or $q = $ l.c.m.(h), so we get nothing.

If $X = S(2,2,p)$ then we cannot have $(\widetilde{p},\widetilde{q}) \dashrightarrow 2$ or $\widetilde{p} \dashrightarrow 2, \widetilde{q} \dashrightarrow 2$, otherwise $\widetilde{X}$ would be good. If $\widetilde{p} \dashrightarrow 2$ and $\widetilde{q} \dashrightarrow p$ then d should be both even and odd, which is impossible. So $(\widetilde{p},\widetilde{q}) \dashrightarrow p$, $d = 2k$, $m_1 = m_2 = k$, $m_3 = 2$ and we get the last item in (3.1) with h and $2k - h$ not multiple of each other.

For $X = S(2,3,p)$, considering where $\widetilde{p}$ and $\widetilde{q}$ can be mapped, we again see what they can be, we determine d using $\chi^{\rm orb}$, and we check that three appropriate partitions exist, getting nothing for $p = 3$, items 2 and 3 in (3.1) for $p = 4$, and items 5, 7, and 8 for $p = 5$, which completes the proof. □

We omit the straight-forward proof of the next result:

Proposition 3.2.4. *The candidate surface branched covers having associated candidate $S \dashrightarrow X$ are*

$$\begin{array}{lll} S \overset{p:1}{\underset{(p),(p)}{\dashrightarrow}} S & S \overset{2p:1}{\underset{(2,\ldots,2),(2,\ldots,2),(p,p)}{\dashrightarrow}} S & S \overset{12:1}{\underset{(2,\ldots,2),(3,\ldots,3),(3,\ldots,3)}{\dashrightarrow}} S \\ S \overset{24:1}{\underset{(2,\ldots,2),(3,\ldots,3),(4,\ldots,4)}{\dashrightarrow}} S & S \overset{60:1}{\underset{(2,\ldots,2),(3,\ldots,3),(5,\ldots,5)}{\dashrightarrow}} S & \end{array} \tag{3.2}$$

Proposition 3.2.5. *The candidate surface branched covers having associated candidate $S(\widetilde{p},\widetilde{p}) \dashrightarrow X$ with $\widetilde{p} > 1$ are*

$$\begin{array}{lll} S \overset{4:1}{\underset{(2,2),(3,1),(3,1)}{\dashrightarrow}} S & S \overset{6:1}{\underset{(2,2,1,1),(3,3),(3,3)}{\dashrightarrow}} S & S \overset{6:1}{\underset{(2,2,2),(3,3),(4,1,1)}{\dashrightarrow}} S \\ S \overset{8:1}{\underset{(2,\ldots,2),(3,3,1,1),(4,4)}{\dashrightarrow}} S & S \overset{12:1}{\underset{(2,\ldots,2,1,1),(3,3,3,3),(4,4,4)}{\dashrightarrow}} S & S \overset{12:1}{\underset{(2,\ldots,2),(3,\ldots,3),(4,4,2,2)}{\dashrightarrow}} S \\ S \overset{12:1}{\underset{(2,\ldots,2),(3,\ldots,3),(5,5,1,1)}{\dashrightarrow}} S & S \overset{20:1}{\underset{(2,\ldots,2),(3,\ldots,3,1,1),(5,\ldots,5)}{\dashrightarrow}} S & S \overset{30:1}{\underset{(2,\ldots,2,1,1),(3,\ldots,3),(5,\ldots,5)}{\dashrightarrow}} S \\ S \overset{2k+1:1}{\underset{(2,\ldots,2,1),(2,\ldots,2,1),(2k+1)}{\dashrightarrow}} S & S \overset{2k+2:1}{\underset{(2,\ldots,2,1,1),(2,\ldots,2),(2k+2)}{\dashrightarrow}} S & \end{array} \tag{3.3}$$

with arbitrary $k \geqslant 1$ in the last two items.

Proof. Since $\chi^{\mathrm{orb}}(S(\widetilde{p},\widetilde{p})) = \frac{2}{\widetilde{p}} \leqslant 1$ and $d \geqslant 2$ we cannot have $X = S$ or $X = S(p)$. Suppose then $X = S(p,q)$, so $\frac{2}{\widetilde{p}} = \frac{d}{p} + \frac{d}{q}$. If $\widetilde{p},\widetilde{p} \dashrightarrow p$ then $p = k\cdot\widetilde{p}$ and $d = 2k + (m_1 - 2)\cdot p = m_2\cdot q$, whence $m_1 = 2$ and $m_2 = 0$, which is absurd. If $\widetilde{p} \dashrightarrow p$ and $\widetilde{p} \dashrightarrow q$ then $p = k\cdot\widetilde{p}$ and $q = h\cdot\widetilde{p}$ whence $d = k + (m_1 - 1)\cdot p = h + (m_2 - 1)\cdot q$ which gives $m_1 = m_2 = 1$, but then we cannot have $p = \mathrm{l.c.m.}(k)$ or $q = \mathrm{l.c.m.}(h)$, so we get nothing.

Assume now $X = S(2,2,p)$. If $(\widetilde{p},\widetilde{p}) \dashrightarrow 2$ then $\widetilde{p} = 2$ and $p = d = 2k+2$, which leads to the last item in (3.3). If $\widetilde{p} \dashrightarrow 2$ and $\widetilde{p} \dashrightarrow 2$ then $\widetilde{p} = 2$ and $p = d = 2k+1$, so we get the penultimate item in (3.3). Of course we cannot have $\widetilde{p} \dashrightarrow 2$ and $\widetilde{p} \dashrightarrow p$ otherwise d should be both even and odd. If $(\widetilde{p},\widetilde{p}) \dashrightarrow p$ then $p = k\cdot\widetilde{p}$ and $d = 2k$, so $\Pi_3 = (k,k)$, but then $p \neq \mathrm{l.c.m.}(k,k)$.

If $X = S(2,3,3)$ then $d\cdot\widetilde{p} = 12$. Of course we cannot have $\widetilde{p} \dashrightarrow 2$ and $\widetilde{p} \dashrightarrow 3$. If $(\widetilde{p},\widetilde{p}) \dashrightarrow 2$ then $\widetilde{p} = 2$ and $d = 6$, so we get item 2 in (3.3), while if $(\widetilde{p},\widetilde{p}) \dashrightarrow 3$ then $\widetilde{p} = 3$ and $d = 4$, which is impossible since there would be a partition of 4 consisting of 3's only. For $\widetilde{p} \dashrightarrow 3$ and $\widetilde{p} \dashrightarrow 3$ again $\widetilde{p} = 3$ and $d = 4$, whence item 1 in (3.3).

The discussion for $X = S(2,3,p)$ with $p = 4,5$ is similar. We get items 3 to 6 in (3.3) for $p = 4$ and items 7 to 9 for $p = 5$. □

Proposition 3.2.6. *The candidate surface branched covers having associated candidate $S(2,2,\widetilde{p}) \dashrightarrow X$ with $\widetilde{p} > 1$ are*

$$\begin{array}{ll} S \overset{4:1}{\underset{(2,1,1),(3,1),(4)}{\dashrightarrow}} S & S \overset{6:1}{\underset{(2,2,1,1),(3,3),(4,2)}{\dashrightarrow}} S \\ S \overset{6:1}{\underset{(2,2,1,1),(3,3),(5,1)}{\dashrightarrow}} S & S \overset{10:1}{\underset{(2,\ldots,2,1,1),(3,3,3,1),(5,5)}{\dashrightarrow}} S \\ S \overset{15:1}{\underset{(2,\ldots,2,1,1,1),(3,\ldots,3),(5,5,5)}{\dashrightarrow}} S. & \end{array} \tag{3.4}$$

Proof. Note first that X cannot be S or $S(p)$ since $d \geqslant 2$. If $X = S(p,q)$ we have the following possibilities:

- $(2,2,\widetilde{p}) \dashrightarrow p$. Then $p = 2k$, so $d = k + k + 2k/\widetilde{p} + (m_1 - 3)2k = m_2 q$;
- $\widetilde{p} \dashrightarrow p$ and $(2,2) \dashrightarrow q$. Then $p = k\widetilde{p}$ and $q = 2h$, so $d = k + (m_1 - 1)k\widetilde{p} = h + h + (m_2 - 2)2h$;
- $(2,\widetilde{p}) \dashrightarrow p$ and $2 \dashrightarrow q$. Then $p = 2k$ and $q = 2h$, so $d = k + 2k/\widetilde{p} + (m_1 - 2)2k = h + (m_2 - 1)2h$.

Since $\frac{1}{\widetilde{p}} = \frac{d}{p} + \frac{d}{q}$, in all cases we deduce that $m_1 + m_2 = 2$, which is absurd, so we do not get any candidate cover.

Now suppose $X = S(2,2,p)$. We have the following possibilities:

$(2,2,\widetilde{p}) \dashrightarrow 2$	$(2,2,\widetilde{p}) \dashrightarrow p$	
$(2,2) \dashrightarrow 2,\ \widetilde{p} \dashrightarrow 2$	$(2,2) \dashrightarrow 2,\ \widetilde{p} \dashrightarrow p$	$\widetilde{p} \dashrightarrow 2,\ (2,2) \dashrightarrow p$
$(2,\widetilde{p}) \dashrightarrow 2,\ 2 \dashrightarrow 2$	$(2,\widetilde{p}) \dashrightarrow 2,\ 2 \dashrightarrow p$	$2 \dashrightarrow 2,\ (2,\widetilde{p}) \dashrightarrow p,$
$2 \dashrightarrow 2,\ 2 \dashrightarrow 2,\ \widetilde{p} \dashrightarrow p$	$2 \dashrightarrow 2,\ \widetilde{p} \dashrightarrow 2,\ 2 \dashrightarrow p$	

and $\widetilde{p}$ must actually be 2 in items 1, 3, 5, 6, 7 and 10. Items 1, 3, 5, 6 and 8 are then impossible because d should be both even and odd. In item 2 we have $d = 2k$ and $m_1 = m_2 = k$, whence $m_3 = 2$, which is impossible. In items 4 and 7 we have $d = 2k$, whence $m_1 = k + 1$ and $m_2 = k$, so $m_3 = 1$, which is absurd. In items 9 and 10 we have $d = 2k + 1$ and $m_1 = m_2 = k + 1$, whence $m_3 = 1$, which is absurd. This shows that there is no candidate cover for $X = S(2,2,p)$.

The cases where $X = S(2,3,p)$ for $p = 3,4,5$ are easier to discuss and hence left to the reader. For $p = 3$ there is nothing, for $p = 4$ there are items 1 and 2 in (3.4), and for $p = 5$ items 3 to 5. □

The candidate surface branched covers having associated candidate covers $S(2,3,\widetilde{p}) \dashrightarrow X$ for $\widetilde{p} = 3,4,5$ are not hard to analyze using the same methods employed above, so we will not spell out the proofs of the next two results. The most delicate point is always to exclude the cases $X = S(p,q)$ and $X = S(2,2,p)$.

Proposition 3.2.7. *The only candidate surface branched cover having associated candidate* $S(2,3,3) \dashrightarrow X$ *is* $S \underset{(2,2,1),(3,1,1),(5)}{\overset{5:1}{\dashrightarrow}} S$.

Proposition 3.2.8. *There are no candidate surface branched covers having associated candidate of the form* $S(2,3,4) \dashrightarrow X$ *or* $S(2,3,5) \dashrightarrow X$.

Proposition 3.2.9. *All the candidate surface branched covers of Proposition 3.2.5 are realizable.*

Proof. The following list describes the candidate orbifold covers $\widetilde{X} \dashrightarrow X$ and the cover instructions associated to the items in (3.3), together with the corresponding isometry $\widetilde{f} : \mathbb{S} \to \mathbb{S}$ inducing the desired f via the geometric universal covers $\widetilde{\pi} : \mathbb{S} \to \widetilde{X}$ and $\pi : \mathbb{S} \to X$ fixed above.

- $S(3,3) \underset{3 \dashrightarrow 3,\ 3 \dashrightarrow 3}{\overset{4:1}{\dashrightarrow}} S(2,3,3)$ and $\widetilde{f}$ is the rotation around $(\pm i, 0)$ sending $(0,1)$ to point B of Fig. 1.2-left;
- $S(2,2) \underset{(2,2) \dashrightarrow 2}{\overset{6:1}{\dashrightarrow}} S(2,3,3)$ and $\widetilde{f} = \mathrm{rot}_{(\pm i,0)}(\pi/2)$;

- $S(4,4) \xrightarrow[(4,4) \dashrightarrow 4]{6:1} S(2,3,4)$ and $\widetilde{f}$ is the identity;
- $S(3,3) \xrightarrow[(3,3) \dashrightarrow 3]{8:1} S(2,3,4)$ and $\widetilde{f}$ is the rotation around $(\pm i, 0)$ sending $(0,1)$ to point B of Fig. 1.2-center;
- $S(2,2) \xrightarrow[(2,2) \dashrightarrow 2]{12:1} S(2,3,4)$ and $\widetilde{f} = \mathrm{rot}_{(\pm i,0)}(\pi/2)$;
- $S(2,2) \xrightarrow[(2,2) \dashrightarrow 4]{12:1} S(2,3,4)$ and $\widetilde{f}$ is the identity;
- $S(5,5) \xrightarrow[(5,5) \dashrightarrow 5]{12:1} S(2,3,5)$ and $\widetilde{f}$ is the rotation around $(\pm i, 0)$ mapping $(0,1)$ to the point labelled $*$ in Fig. 1.2;
- $S(3,3) \xrightarrow[(3,3) \dashrightarrow 3]{20:1} S(2,3,5)$ and $\widetilde{f}$ is the rotation around $(\pm i, 0)$ mapping $(0,1)$ to point B in Fig. 1.2-right;
- $S(2,2) \xrightarrow[(2,2) \dashrightarrow 2]{30:1} S(2,3,5)$ and $\widetilde{f}$ is the identity;
- $S(2,2) \xrightarrow[2 \dashrightarrow 2,\ 2 \dashrightarrow 2]{2k+1:1} S(2,2,2k+1)$ and $\widetilde{f} = \mathrm{rot}_{(\pm i,0)}(\pi/2)$;
- $S(2,2) \xrightarrow[(2,2) \dashrightarrow 2]{2k+2:1} S(2,2,2k+2)$ and $\widetilde{f} = \mathrm{rot}_{(\pm i,0)}(\pi/2)$.

The proof is complete. □

Proposition 3.2.10. *All the candidate surface branched covers of Proposition 3.2.6 are realizable.*

Proof. The candidate orbifold covers with cover instructions associated to the items in (3.4), and the corresponding isometries $\widetilde{f} : \mathbb{S} \to \mathbb{S}$, are as follows:

- $S(2,2,3) \xrightarrow[(2,2) \dashrightarrow 2,\ 3 \dashrightarrow 3]{4:1} S(2,3,4)$ and $\widetilde{f}$ is the rotation around $(\pm 1, 0)$ mapping $(0,1)$ to point B in Fig. 1.2-center followed by a rotation of angle $\pi/6$ around $\pm B$;
- $S(2,2,2) \xrightarrow[(2,2) \dashrightarrow 2,\ 2 \dashrightarrow 4]{6:1} S(2,3,4)$ and $\widetilde{f}$ is the identity;
- $S(2,2,5) \xrightarrow[(2,2) \dashrightarrow 2,\ 5 \dashrightarrow 5]{6:1} S(2,3,5)$ and $\widetilde{f}$ is the rotation around $(\pm 1, 0)$, sending $(0,1)$ to the point of order 5 best visible in the upper hemisphere in Fig. 1.2-right;
- $S(2,2,3) \xrightarrow[(2,2) \dashrightarrow 2,\ 3 \dashrightarrow 3]{10:1} S(2,3,5)$ and $\widetilde{f}$ is the rotation around $(\pm i, 0)$ that maps the pole $(0,1)$ to point B in Fig. 1.2-right followed by a rotation of angle $\pi/6$ around $\pm B$;
- $S(2,2,2) \xrightarrow[(2,2,2) \dashrightarrow 2]{15:1} S(2,3,5)$ and $\widetilde{f}$ is the identity.

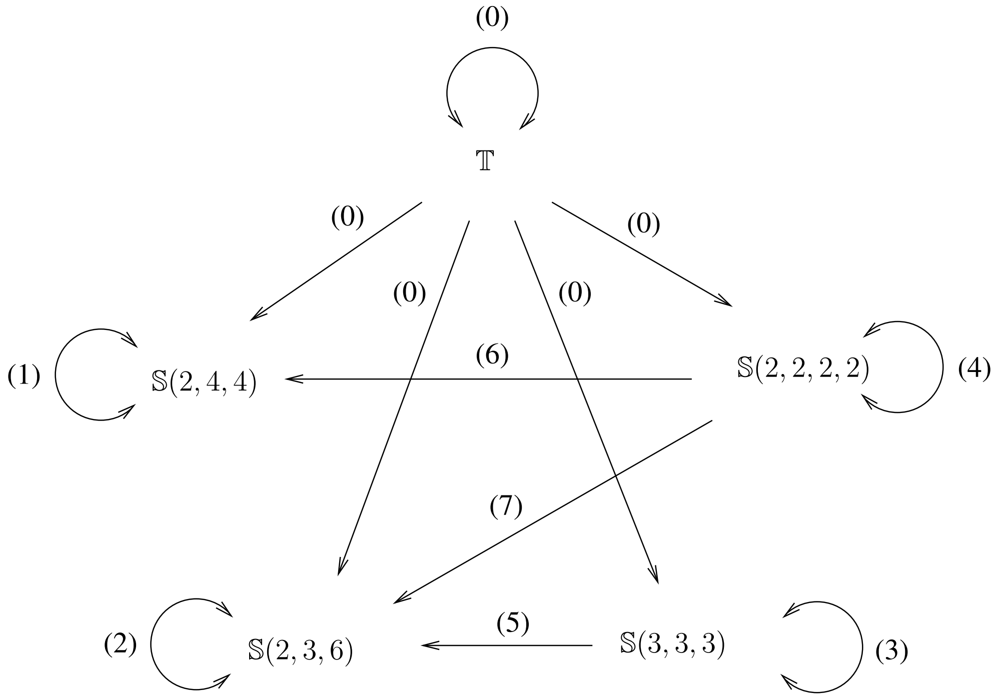

Figure 3.1. Possible covers between Euclidean orbifolds

The proof is complete. □

Proposition 3.2.11. *The candidate surface branched cover of Proposition 3.2.7 is realizable.*

Proof. The candidate orbifold cover is in this case

$$S(2,3,3) \underset{2 \dashrightarrow 2,\ (3,3)\dashrightarrow 3}{\overset{5:1}{\dashrightarrow}} S(2,3,5)$$

and the isometry $\widetilde{f} : \mathbb{S} \to \mathbb{S}$ inducing its realization is the rotation of angle $\pi/4$ around $(\pm 1, 0)$. □

3.3 The Euclidean case

In this section we investigate realizability of candidate surface branched covers having associated candidate orbifold covers $\widetilde{X} \dashrightarrow X$ with $\chi^{\rm orb}(\widetilde{X}) = \chi^{\rm orb}(X) = 0$. This means that $\widetilde{X}$ and X must belong to the list

$$T, \quad S(2,4,4), \quad S(2,3,6), \quad S(3,3,3), \quad S(2,2,2,2)$$

where T is the torus. Recalling that the orders of the cone points of $\widetilde{X}$ must divide those of X, we see that the only possibilities are the cases (0) to (7) shown in Fig. 3.1, that we will analyze using Euclidean geometry. Namely:

- We will fix on X a Euclidean structure given by some $\pi : \mathbb{E} \to X$;

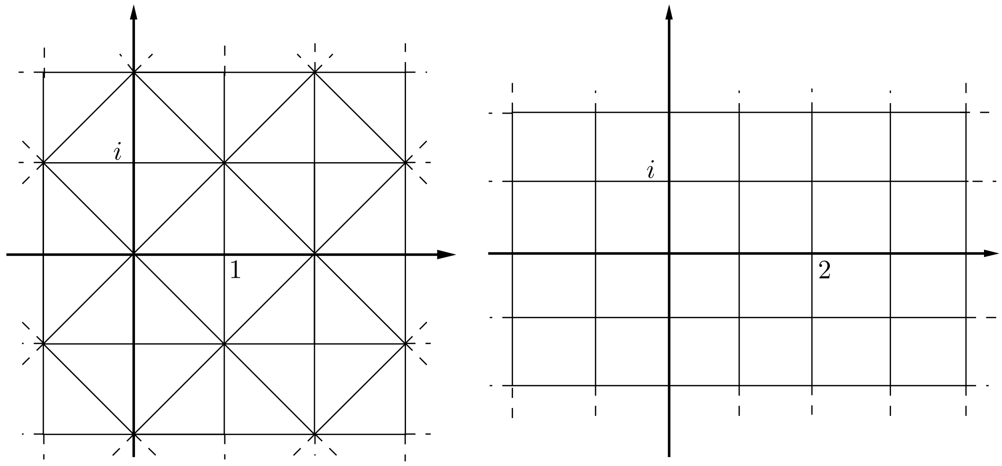

Figure 3.2. Tessellations of $\mathbb{E}$ induced by the geometric structures fixed on $S(2,4,4)$ and $S(2,2,2,2)$.

- We will assume that $\widetilde{X} \stackrel{d:1}{\dashrightarrow} X$ is realized by some map f, we will use Lemma 3.1.2 to deduce there is a corresponding affine map $\widetilde{f} : \mathbb{E} \to \mathbb{E}$, and we will analyze $\widetilde{f}$ to show that d must satisfy certain conditions;

- We will employ the calculations of the previous point to show that if d satisfies the conditions then $\widetilde{f}$, whence f, exists.

General geometric tools Our analysis of the candidate covers (1)-(7) of Fig. 3.1 relies on certain facts that we will use repeatedly. The first is the exact determination of the lifts of the cone points, that we now describe. For any of our four Euclidean X's, with the structure $\pi : \mathbb{E} \to X = \mathbb{E}/\Gamma$ we have fixed, and any vertex $\widetilde{V}^{(p)}$ of the fundamental domain for Γ described above, we set $V^{(p)} = \pi(\widetilde{V}^{(p)})$, so that its cone order is p. Then $\pi^{-1}(V^{(p)})$ will be some set $\{\widetilde{V}_j^{(p)}\} \subset \mathbb{E}$, with j varying in a suitable set of indices. The exact lists of lifts are as follows:

$$\begin{array}{ll} S(2,4,4): & \text{(see Fig. 3.2-left)} \\ \widetilde{A}^{(2)}_{a,b} = a+ib & a,b \in \mathbb{Z},\ a \not\equiv b \ (\text{mod } 2) \\ \widetilde{B}^{(4)}_{a,b} = a+ib & a,b \in \mathbb{Z},\ a \equiv b \equiv 0 \ (\text{mod } 2) \\ \widetilde{C}^{(4)}_{a,b} = a+ib & a,b \in \mathbb{Z},\ a \equiv b \equiv 1 \ (\text{mod } 2); \end{array} \qquad (3.5)$$

$$\begin{array}{ll} S(2,2,2,2): & \text{(see Fig. 3.2-right)} \\ \widetilde{A}^{(2)}_{a,b} = a+ib & a,b \in \mathbb{Z},\ a \equiv b \equiv 0 \ (\text{mod } 2) \\ \widetilde{B}^{(2)}_{a,b} = a+ib & a,b \in \mathbb{Z},\ a \equiv 1,\ b \equiv 0 \ (\text{mod } 2) \\ \widetilde{C}^{(2)}_{a,b} = a+ib & a,b \in \mathbb{Z},\ a \equiv b \equiv 1 \ (\text{mod } 2) \\ \widetilde{D}^{(2)}_{a,b} = a+ib & a,b \in \mathbb{Z},\ a \equiv 0,\ b \equiv 1 \ (\text{mod } 2); \end{array} \qquad (3.6)$$

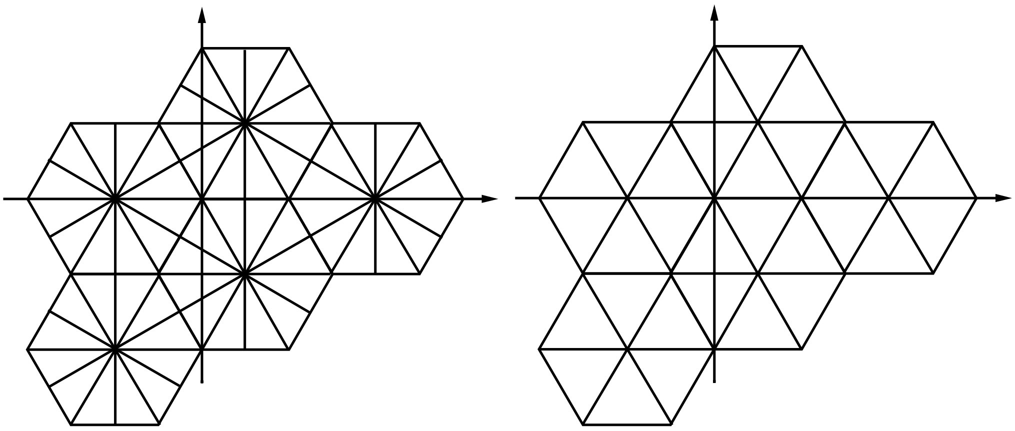

Figure 3.3. Tessellations of $\mathbb{E}$ induced by the geometric structure fixed on $S(2,3,6)$ and $S(3,3,3)$

$$\begin{array}{lll} S(2,3,6): & \text{with } \omega = \frac{1+i\sqrt{3}}{2} \text{ (see Fig. 3.3-left)} \\ \widetilde{A}^{(2)}_{a,b} = \frac{1}{2}(a+\omega b) & a,b \in \mathbb{Z} \text{ not both even, } a-b \equiv 1 \pmod 3 \\ \widetilde{B}^{(3)}_{a,b} = a+\omega b & a,b \in \mathbb{Z},\ a-b \not\equiv 2 \pmod 3 \\ \widetilde{C}^{(6)}_{a,b} = a+\omega b & a,b \in \mathbb{Z},\ a-b \equiv 2 \pmod 3; \end{array} \tag{3.7}$$

$$\begin{array}{lll} S(3,3,3): & \text{with } \omega = \frac{1+i\sqrt{3}}{2} \text{ (see Fig. 3.3-right)} \\ \widetilde{A}^{(3)}_{a,b} = a+\omega b, & a,b \in \mathbb{Z},\ a-b \equiv 0 \pmod 3 \\ \widetilde{B}^{(3)}_{a,b} = a+\omega b, & a,b \in \mathbb{Z},\ a-b \equiv 1 \pmod 3 \\ \widetilde{C}^{(3)}_{a,b} = a+\omega b, & a,b \in \mathbb{Z},\ a-b \equiv 2 \pmod 3. \end{array} \tag{3.8}$$

Also important for us will be the symmetries of our Euclidean X's. More precisely we will use the following facts:

- There exists a symmetry of $S(2,4,4)$ switching $B^{(4)}$ and $C^{(4)}$;
- Any permutation of $\{A^{(3)}, B^{(3)}, C^{(3)}\}$ is induced by a symmetry of $S(3,3,3)$;
- Any permutation of $\{A^{(2)}, B^{(2)}, C^{(2)}, D^{(2)}\}$ is induced by a symmetry of $S(2,2,2,2)$.

Here is another tool we will use very often. As one sees, if $\Gamma < \mathrm{Isom}^+(\mathbb{E})$ defines a Euclidean structure on a 2-orbifold $X = \mathbb{E}/\Gamma$ then Γ has a maximal torsion-free subgroup, a rank-2 lattice $\Lambda(\Gamma)$. For $u \in \mathbb{C}$, let τ_u denote the translation $z \mapsto z+u$. Then the lattices for the groups we have fixed are as follows:

$$\Lambda_{(2,4,4)} = \langle \tau_2, \tau_{2i} \rangle, \qquad \Lambda^{s,t}_{(2,2,2,2)} = \left\langle \tau_{2is}, \tau_{2\left(\frac{1}{s}+it\right)} \right\rangle,$$
$$\Lambda_{(2,3,6)} = \Lambda_{(3,3,3)} = \left\langle \tau_{i\sqrt{3}}, \tau_{\frac{3+i\sqrt{3}}{2}} \right\rangle.$$

Moreover the following holds:

Lemma 3.3.1. *Let* $\widetilde{\Gamma},\Gamma<\mathrm{Isom}^+(\mathbb{E})$ *define Euclidean orbifolds* $\widetilde{X}=\mathbb{E}/\widetilde{\Gamma}$ *and* $X=\mathbb{E}/\Gamma$. *Suppose that* $\Lambda(\widetilde{\Gamma})=\langle\tau_{\widetilde{u}_1},\tau_{\widetilde{u}_2}\rangle$ *and* $\Lambda(\Gamma)=\langle\tau_{u_1},\tau_{u_2}\rangle$. *Let* $\widetilde{f}:\mathbb{E}\to\mathbb{E}$ *given by* $\widetilde{f}(z)=\lambda\cdot z+\mu$ *have an associated orbifold cover* $\widetilde{X}\to X$. *Then* $\lambda\cdot\widetilde{u}_1$ *and* $\lambda\cdot\widetilde{u}_2$ *are integer linear combinations of* u_1 *and* u_2.

Proof. The map $\widetilde{f}$ induces a homomorphism $\widetilde{f}_*:\widetilde{\Gamma}\to\Gamma$ given by $\widetilde{f}_*(\tau_u)=\tau_{\lambda\cdot u}$ which maps $\Lambda(\widetilde{\Gamma})$ to $\Lambda(\Gamma)$, and the conclusion easily follows. □

Exceptions and geometric realizations We will now state and prove 8 theorems corresponding to the cases (0)-(7) of Fig. 3.1, thus establishing Theorem 0.0.6. Cases (1)-(3) imply in particular Theorems 0.0.2, 0.0.3, and 0.0.4.

Theorem 3.3.2 (case (0) in Fig. 3.1)**.** *The candidate surface branched covers having associated candidate* $T \dashrightarrow X$ *are*

$$T\overset{k:1}{\dashrightarrow}T \qquad T\underset{(2,\ldots,2),(2,\ldots,2),(2,\ldots,2),(2,\ldots,2)}{\overset{2k:1}{\dashrightarrow}}S \qquad T\underset{(3,\ldots,3),(3,\ldots,3),(3,\ldots,3)}{\overset{3k:1}{\dashrightarrow}}S$$
$$T\underset{(2,\ldots,2),(4,\ldots,4),(4,\ldots,4)}{\overset{4k:1}{\dashrightarrow}}S \qquad T\underset{(2,\ldots,2),(3,\ldots,3),(6,\ldots,6)}{\overset{6k:1}{\dashrightarrow}}S$$

with arbitrary $k\geqslant 1$, *and they are all realizable.*

Proof. The first assertion and realizability of any $T\overset{k:1}{\dashrightarrow}T$ are easy. For any $X\neq T$ let $X=\mathbb{E}/\Gamma$ and identify T to $\mathbb{E}/\Lambda(\Gamma)$. Since $\Lambda(\Gamma)<\Gamma$ we have an associated orbifold cover $T\to X$, which realizes the relevant $T\dashrightarrow X$ in the special case $k=1$. The conclusion follows by taking compositions. □

Theorem 3.3.3 (case (1) in Fig. 3.1)**.** *The candidate surface branched covers having associated candidate* $S(2,4,4)\overset{d:1}{\dashrightarrow}S(2,4,4)$ *are*

$$S\underset{(2,\ldots,2,1),(4,\ldots,4,1),(4,\ldots,4,1)}{\overset{4k+1:1}{\dashrightarrow}}S \quad S\underset{(2,\ldots,2),(4,\ldots,4,2),(4,\ldots,4,1,1)}{\overset{4k+2:1}{\dashrightarrow}}S \quad S\underset{(2,\ldots,2),(4,\ldots,4),(4,\ldots,4,2,1,1)}{\overset{4k+4:1}{\dashrightarrow}}S$$

for $k\geqslant 1$, *and they are realizable if and only if, respectively:*

- $d=x^2+y^2$ *for some* $x,y\in\mathbb{N}$ *of different parity;*
- $d=2(x^2+y^2)$ *for some* $x,y\in\mathbb{N}$ *of different parity;*
- $d=4(x^2+y^2)$ *for some* $x,y\in\mathbb{N}$ *not both zero.*

Proof. A candidate $S(2,4,4)\overset{d:1}{\dashrightarrow}S(2,4,4)$ can be complemented with any one of the following cover instructions

$$\begin{array}{lll} 2\dashrightarrow 2,\ 4\dashrightarrow 4,\ 4\dashrightarrow 4, & 2\dashrightarrow 4,\ (4,4)\dashrightarrow 4, & (2,4,4)\dashrightarrow 4,\\ 2\dashrightarrow 2,\ (4,4)\dashrightarrow 4, & (2,4)\dashrightarrow 4,\ 4\dashrightarrow 4 & \end{array}$$

and it is very easy to see that the first three come from the candidate surface branched covers of the statement, while the last two do not come from any candidate cover (recall that the simplified version $\ell(\Pi)=d+2$ (2.2) of the Riemann-Hurwitz formula must be satisfied).

Now suppose there is a realization $f : S(2,4,4) \xrightarrow{d:1} S(2,4,4)$ of one of the three relevant candidate covers. Proposition 3.1.2 implies that there is a geometric universal cover $\widetilde{\pi} : \mathbb{E} \to S(2,4,4)$ and an affine map $\widetilde{f} : \mathbb{E} \to \mathbb{E}$ with $\pi \circ \widetilde{f} = f \circ \widetilde{\pi}$. But the Euclidean structure of $S(2,4,4)$ is unique up to scaling, so $\widetilde{\pi} = \pi$. If $\widetilde{f}(z) = \lambda \cdot z + \mu$, since $\Lambda_{(2,4,4)} = \langle \tau_2, \tau_{2i} \rangle$, Lemma 3.3.1 implies that $\lambda = n + im$ for some $n, m \in \mathbb{Z}$, whence $d = n^2 + m^2$.

We now employ the notation of (3.5) and note that for all three candidates we can assume, by the symmetry of $S(2,4,4)$, that $f\left(B^{(4)}\right) = B^{(4)}$, whence that $\widetilde{f}\left(\widetilde{B}^{(4)}_{0,0}\right) = \widetilde{B}^{(4)}_{0,0}$, namely $\mu = 0$. We then proceed separately for the three candidates. For the first one we have $f\left(A^{(2)}\right) = A^{(2)}$ so $\widetilde{f}\left(\widetilde{A}^{(2)}_{1,0}\right) = \lambda$ is some $\widetilde{A}^{(2)}_{*}$. Therefore n and m have different parity, and we can set $x = |n|$ and $y = |m|$ getting that $d = x^2 + y^2$ for $x, y \in \mathbb{N}$ of different parity. Conversely if d has this form we define $\widetilde{f}(z) = (x + iy) \cdot z$. Then

$$\widetilde{f}\left(\widetilde{A}^{(2)}_{1,0}\right) = \widetilde{A}^{(2)}_{x,y}, \quad \widetilde{f}\left(\widetilde{B}^{(4)}_{0,0}\right) = \widetilde{B}^{(4)}_{0,0}, \quad \widetilde{f}\left(\widetilde{C}^{(4)}_{1,1}\right) = (x-y) + i(x+y) = \widetilde{C}^{(4)}_{x-y,x+y}$$

where the last equality depends on the fact that $x - y \equiv x + y \equiv 1 \pmod 2$. It easily follows that $\widetilde{f}$ induces a realization of the candidate.

For the second candidate $f\left(A^{(2)}\right) = C^{(4)}$, hence $\widetilde{f}\left(\widetilde{A}^{(2)}_{1,0}\right) = \lambda$ is some $\widetilde{C}^{(4)}_{*}$, namely n and m are odd. Setting $x = \frac{1}{2}|n+m|$ and $y = \frac{1}{2}|n-m|$ we see that $x, y \in \mathbb{N}$ have different parity and $d = 2(x^2 + y^2)$. Conversely if $d = 2(x^2 + y^2)$ with x, y of different parity, we define $\widetilde{f}(z) = ((x+y) + i(x-y)) \cdot z$. Then

$$\widetilde{f}\left(\widetilde{A}^{(2)}_{1,0}\right) = \widetilde{C}^{(4)}_{x+y,x-y}, \quad \widetilde{f}\left(\widetilde{B}^{(4)}_{0,0}\right) = \widetilde{B}^{(4)}_{0,0}, \quad \widetilde{f}\left(\widetilde{C}^{(4)}_{1,1}\right) = \widetilde{B}^{(4)}_{2y,2x}$$

from which it is easy to see that $\widetilde{f}$ induces a realization of the candidate.

For the last candidate $f\left(A^{(2)}\right) = B^{(4)}$ hence $\widetilde{f}\left(\widetilde{A}^{(2)}_{1,0}\right)$ is some $\widetilde{B}^{(4)}_{*}$, so n and m are even. Setting $x = \frac{1}{2}|n|$ and $y = \frac{1}{2}|m|$ we see that $d = 4(x^2 + y^2)$ for $x, y \in \mathbb{N}$. Conversely if $d = 4(x^2 + y^2)$ and we define $\widetilde{f}(z) = (2x + 2iy) \cdot z$ then

$$\widetilde{f}\left(\widetilde{A}^{(2)}_{1,0}\right) = \widetilde{B}^{(4)}_{2x,2y}, \quad \widetilde{f}\left(\widetilde{B}^{(4)}_{0,0}\right) = \widetilde{B}^{(4)}_{0,0}, \quad \widetilde{f}\left(\widetilde{C}^{(4)}_{1,1}\right) = \widetilde{B}^{(4)}_{2(x-y),2(x+y)}$$

therefore $\widetilde{f}$ induces a realization of the candidate. □

Remark 3.3.4. One feature of the above proof is worth pointing out. After assuming that some degree-d candidate $S(2,4,4) \dashrightarrow S(2,4,4)$ is realized by some map, we have always used only "two thirds" of the branching instruction to show that d has the appropriate form. The same phenomenon will occur in all the next proofs, except that of Theorem 3.3.8. Note however that to check that some $\widetilde{f}$ defined starting from a degree d with appropriate form induces a realization of the corresponding candidate, we had to check all three conditions.

As an illustration of the previous proof, we provide in Fig. 3.4 a description of the map $\widetilde{f} : \mathbb{E} \to \mathbb{E}$ inducing a realization of $S \overset{5:1}{\underset{(2,2,1),(4,1),(4,1)}{\dashrightarrow}} S$.

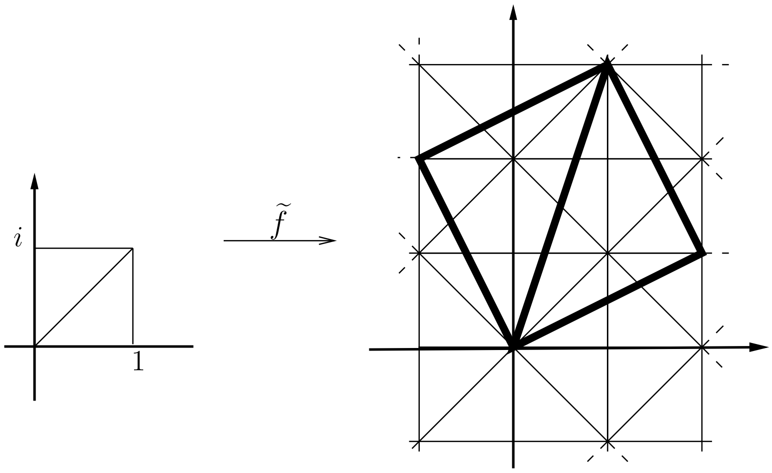

Figure 3.4. A map $\widetilde{f}:\mathbb{E}\to\mathbb{E}$ inducing a degree-5 cover of $S(2,4,4)$ onto itself

Theorem 3.3.5 (case (2) in Fig. 3.1)**.** *The candidate surface branched covers having associated candidate* $S(2,3,6)\overset{d:1}{\dashrightarrow}S(2,3,6)$ *are*

$$S \xrightarrow[(2,\ldots,2,1),(3,\ldots,3,1),(6,\ldots,6,1)]{6k+1:1} S \qquad S \xrightarrow[(2,\ldots,2,1),(3,\ldots,3),(6,\ldots,6,2,1)]{6k+3:1} S$$
$$S \xrightarrow[(2,\ldots,2),(3,\ldots,3,1),(6,\ldots,6,3,1)]{6k+4:1} S \qquad S \xrightarrow[(2,\ldots,2),(3,\ldots,3),(6,\ldots,6,3,2,1)]{6k+6:1} S$$

with $k\geqslant 1$*, and they are realizable if and only if, respectively:*

- $d=x^2+xy+y^2$ *with* $x,y\in\mathbb{N}$ *not both even and* $x\not\equiv y\pmod 3$*;*
- $d=3(x^2+3xy+3y^2)$ *with* $x,y\in\mathbb{N}$ *not both even;*
- $d=12(x^2+3xy+3y^2)+16$ *with* $x,y\in\mathbb{N}$*;*
- $d=12(x^2+3xy+3y^2)$ *with* $x,y\in\mathbb{N}$*.*

Proof. A candidate orbifold cover $S(2,3,6)\dashrightarrow S(2,3,6)$ can be complemented with the cover instructions

$$\begin{array}{ll} 2\dashrightarrow 2,\ 3\dashrightarrow 3,\ 6\dashrightarrow 6 & \quad 2\dashrightarrow 2,\ (3,6)\dashrightarrow 6, \\ 3\dashrightarrow 3,\ (2,6)\dashrightarrow 6, & \quad (2,3,6)\dashrightarrow 6 \end{array}$$

which are associated to the candidate surface branched covers of the statement (formula (2.2) is always satisfied in this case).

The scheme of the proof is now as for case (1): we consider the universal cover $\pi:\mathbb{E}\to S(2,3,6)$ we have fixed, we assume that a map $f:S(2,3,6)\xrightarrow{d:1}S(2,3,6)$ realizing some candidate cover exists, we use Proposition 3.1.2 to find $\widetilde{f}:\mathbb{E}\to\mathbb{E}$ with $\widetilde{f}(z)=\lambda\cdot z+\mu$ and $\pi\circ\widetilde{f}=f\circ\pi$, and we show that $d=|\lambda|^2$ has the appropriate form. Moreover essentially the same calculations will also allow us to prove the converse. We will always use the notation fixed in (3.7), in particular $\omega=\frac{1+i\sqrt{3}}{2}$.

We first apply Lemma 3.3.1. Since $\Lambda_{(2,3,6)}=\left\langle \tau_{i\sqrt{3}},\tau_{\frac{3+i\sqrt{3}}{2}}\right\rangle$ we must have

$$\lambda\cdot i\sqrt{3}=n\cdot i\sqrt{3}+m\cdot\frac{3+i\sqrt{3}}{2}$$

for some $n, m \in \mathbb{Z}$, which easily implies that

$$\lambda = n + \frac{m}{2} - i\sqrt{3}\frac{m}{2} = (n+m) - \omega m$$

therefore $d = n^2 + nm + m^2$. Notice that Lemma 3.3.1 does not give any other condition, because $\lambda \cdot \frac{3+i\sqrt{3}}{2} = (-m)\cdot i\sqrt{3} + (n+m)\cdot \frac{3+i\sqrt{3}}{2}$.

We now analyze our four candidates, starting from the first one. Since

$$f\left(A^{(2)}\right) = A^{(2)}, \quad f\left(B^{(3)}\right) = B^{(3)}, \quad f\left(C^{(6)}\right) = C^{(6)}$$

we can assume $\widetilde{f}\left(\widetilde{B}^{(3)}_{0,0}\right) = \widetilde{B}^{(3)}_{0,0}$, so $\mu = 0$, and $\widetilde{f}\left(\widetilde{A}^{(2)}_{1,0}\right) = \widetilde{A}^{(2)}_{a,b}$ for some $a, b \in \mathbb{Z}$ not both even with $a - b \equiv 1 \pmod 3$. Since $a = n + m$ and $b = -m$ we deduce that n, m are not both even and $n - m \equiv 1 \pmod 3$. If $n, m \geqslant 0$ or $n, m \leqslant 0$ we set $x = |n|$ and $y = |m|$, getting $d = x^2 + xy + y^2$ with $x, y \in \mathbb{N}$ not both even and $x \not\equiv y \pmod 3$. Otherwise we have $n > 0 > m$ up to permutation. If $n \geqslant -m$ we set $x = n + m$ and $y = -m$, otherwise $x = n$ and $y = -n - m$, and again we have $d = x^2 + xy + y^2$ with $x, y \in \mathbb{N}$ not both even and $x \not\equiv y \pmod 3$.

Conversely, assume $d = x^2 + xy + y^2$ with $x, y \in \mathbb{N}$ not both even and $x \not\equiv y \pmod 3$. Up to changing sign to both x and y we can suppose that $x - y \equiv 1 \pmod 3$ and define $\widetilde{f}(z) = ((x+y) - \omega y)\cdot z$. Of course $\widetilde{f}\left(\widetilde{B}^{(3)}_{0,0}\right) = \widetilde{B}^{(3)}_{0,0}$, the above calculations show that $\widetilde{f}\left(\widetilde{A}^{(2)}_{1,0}\right) = \widetilde{A}^{(2)}_{x+y,-y}$, and using the identity $\omega^2 = \omega - 1$ we have $\widetilde{f}\left(\widetilde{C}^{(6)}_{0,1}\right) = (x + y - \omega y)\cdot\omega = y + \omega x = \widetilde{C}^{(6)}_{y,x}$ where the last equality depends on the fact that $y - x \equiv -(x - y) \equiv 2 \pmod 3$. This easily implies that $\widetilde{f}$ induces a realization of the first candidate.

Let us turn to the second candidate. Since

$$f\left(A^{(2)}\right) = A^{(2)}, \quad f\left(B^{(3)}\right) = C^{(6)}, \quad f\left(C^{(6)}\right) = C^{(6)}$$

we can assume $\widetilde{f}\left(\widetilde{B}^{(3)}_{0,0}\right) = \widetilde{C}^{(6)}_{0,1}$, namely $\mu = \omega$. In addition $\widetilde{f}\left(\widetilde{A}^{(2)}_{1,0}\right) = \widetilde{A}^{(2)}_{a,b}$ for some $a, b \in \mathbb{Z}$ not both even with $a - b \equiv 1 \pmod 3$. Since $a = n + m$ and $b = 2 - m$ we deduce that n, m are not both even and $n \equiv m \pmod 3$. Setting $x = n$ and $y = (m - n)/3$ we then get $d = 3(x^2 + 3xy + 3y^2)$ for $x, y \in \mathbb{Z}$ not both even. Reducing to the case $x, y \in \mathbb{N}$ is a routine matter that we leave to the reader.

Conversely, assume $d = 3(x^2 + 3xy + 3y^2)$ for $x, y \in \mathbb{N}$ not both even, set $n = x$ and $m = x + 3y$ and define

$$\widetilde{f}(z) = ((n+m) - \omega m)\cdot z + \omega.$$

Then $\widetilde{f}\left(\widetilde{A}^{(2)}_{1,0}\right)$ is some $\widetilde{A}^{(2)}_{*}$ by the above calculations, while

$$\widetilde{f}\left(\widetilde{C}^{(6)}_{0,1}\right) = \widetilde{f}(\omega) = (x + 3y) + \omega(x+1)$$

which is some $\widetilde{C}^{(6)}_{*}$ because $(x+3y) - (x+1) \equiv 2 \pmod 3$. It follows that $\widetilde{f}$ induces a realization of the candidate.

For the third candidate we have

$$f\left(A^{(2)}\right) = C^{(6)}, \quad f\left(B^{(3)}\right) = B^{(3)}, \quad f\left(C^{(6)}\right) = C^{(6)}$$

so we can take $\mu = 0$ and $\widetilde{f}\left(\widetilde{A}^{(2)}_{1,0}\right) = \widetilde{C}^{(6)}_{a,b}$ for some $a, b \in \mathbb{Z}$ with $a - b \equiv 2 \pmod 3$. Since $a = (n+m)/2$ and $b = -m/2$ we have that n and m are even and $n - m \equiv 1 \pmod 3$. So we can define $x = \frac{n+4}{2}$ and $y = \frac{m-n-8}{6}$ and we have $d = 12(x^2 + 3xy + 3y^2) + 16$ with $x, y \in \mathbb{Z}$. Again it is easy to reduce to the case $x, y \in \mathbb{N}$.

Conversely, suppose $d = 12(x^2 + 3xy + 3y^2) + 16$ with $x, y \in \mathbb{N}$, set $n = 2(x-2)$ and $m = 2(x + 3y + 2)$, and define $\widetilde{f}(z) = ((n+m) - \omega m) \cdot z$. Then of course $\widetilde{f}\left(\widetilde{B}^{(3)}_{0,0}\right) = \widetilde{B}^{(3)}_{0,0}$, the above calculations show that $\widetilde{f}\left(\widetilde{A}^{(2)}_{1,0}\right)$ is some $\widetilde{C}^{(6)}_{*}$, while

$$\widetilde{f}\left(\widetilde{C}^{(6)}_{0,1}\right) = \widetilde{f}(\omega) = 2(x + 3y + 2) + \omega 2(x - 2)$$

which is some $\widetilde{C}^{(6)}_{*}$ because $2(x + 3y + 2) - 2(x - 2) \equiv 2 \pmod 3$, hence $\widetilde{f}$ induces a realization of the candidate.

For the last candidate

$$f\left(A^{(2)}\right) = C^{(6)}, \quad f\left(B^{(3)}\right) = C^{(6)}, \quad f\left(C^{(6)}\right) = C^{(6)}$$

so we can assume $\widetilde{f}\left(\widetilde{B}^{(3)}_{0,0}\right) = \widetilde{C}^{(6)}_{0,1}$, namely $\mu = \omega$. In addition $\widetilde{f}\left(\widetilde{A}^{(2)}_{1,0}\right) = \widetilde{C}^{(6)}_{a,b}$ for some $a, b \in \mathbb{Z}$ with $a - b \equiv 2 \pmod 3$. Since $a = (n+m)/2$ and $b = (2-m)/2$ we deduce that n, m are both even and $n \equiv m \pmod 3$. Setting $x = n/2$ and $y = (m-n)/6$ we then get $d = 12(x^2 + 3xy + 3y^2)$ for $x, y \in \mathbb{Z}$, and again we can reduce to $x, y \in \mathbb{N}$.

Conversely suppose $d = 12(x^2 + 3xy + 3y^2)$ for $x, y \in \mathbb{N}$, set $n = 2x$ and $m = 2(x + 3y)$ and define $\widetilde{f}(z) = ((n+m) - \omega m) \cdot z + \omega$. Then $\widetilde{f}\left(\widetilde{B}^{(3)}_{0,0}\right) = \widetilde{C}^{(6)}_{1,0}$ and $\widetilde{f}\left(\widetilde{A}^{(2)}_{1,0}\right)$ is some $\widetilde{C}^{(6)}_{*}$ by the above calculations, while

$$\widetilde{f}\left(\widetilde{C}^{(6)}_{0,1}\right) = \widetilde{f}(\omega) = (2x + 3y) + (2x + 1)\omega$$

which is some $\widetilde{C}^{(6)}_{*}$ because $(2x + 3y) - (2x + 1) \equiv 2 \pmod 3$, hence $\widetilde{f}$ induces a realization of the candidate. $\square$

Theorem 3.3.6 (case (3) in Fig. 3.1)**.** *The candidate surface branched covers inducing* $S(3,3,3) \stackrel{d:1}{\dashrightarrow} S(3,3,3)$ *are*

$$S \xrightarrow[(3,\ldots,3,1),(3,\ldots,3,1),(3,\ldots,3,1)]{3k+1:1} S \qquad S \xrightarrow[(3,\ldots,3),(3,\ldots,3),(3,\ldots,3,1,1,1)]{3k+3:1} S$$

for $k \geqslant 1$*, and they are realizable if and only if, respectively*

- $d = x^2 + xy + y^2$ *with* $x, y \in \mathbb{N}$ *and* $x \not\equiv y \pmod 3$*;*
- $d = 3(x^2 + 3xy + 3y^2)$ *with* $x, y \in \mathbb{N}$*.*

Proof. The possible cover instructions are

$$3 \dashrightarrow 3,\ 3 \dashrightarrow 3,\ 3 \dashrightarrow 3, \qquad (3,3) \dashrightarrow 3,\ 3 \dashrightarrow 3, \qquad (3,3,3) \dashrightarrow 3.$$

The second one is not associated to any candidate surface branched cover, and the other two are associated to the candidates in the statement.

We follow again the same scheme, using the notation of (3.8). If $\widetilde{f}(z) = \lambda\cdot z + \mu$ realizes a candidate then, as in the previous proof, Lemma 3.3.1 implies that $\lambda = (n+m) - \omega m$ for $n, m \in \mathbb{Z}$, and $d = n^2 + nm + m^2$. Moreover from the symmetry of $S(3,3,3)$ we can assume $f\left(A^{(3)}\right) = A^{(3)}$, whence $\widetilde{f}\left(\widetilde{A}^{(3)}_{0,0}\right) = \widetilde{A}^{(3)}_{0,0}$, namely $\mu = 0$.

For the first candidate we have $f\left(B^{(3)}\right) = B^{(3)}$ up to symmetry, whence $\widetilde{f}\left(\widetilde{B}^{(3)}_{1,0}\right) = \lambda$ is some $\widetilde{B}^{(3)}_{a,b}$ with $a - b \equiv 1 \pmod 3$, therefore $n - m \equiv 1 \pmod 3$. Exactly as in the previous proof we deduce that $d = x^2 + xy + y^2$ with $x, y \in \mathbb{N}$ and $x \not\equiv y \pmod 3$. The converse is proved as above. Switching signs if necessary we assume $x - y \equiv 1 \pmod 3$, we set $\widetilde{f}(z) = ((x+y) - \omega y)\cdot z$ and note that $\widetilde{f}\left(\widetilde{C}^{(3)}_{0,1}\right) = \widetilde{f}(\omega) = y + \omega x$ is $\widetilde{C}^{(6)}_{y,x}$ because $y - x \equiv 2 \pmod 3$.

For the second candidate $f\left(B^{(3)}\right) = A^{(3)}$, whence $\widetilde{f}\left(\widetilde{B}^{(3)1,0}\right) = \lambda$ is some $\widetilde{A}^{(3)}_{*}$, which implies that $n \equiv m \pmod 3$. Setting $x = n$ and $y = (m-n)/3$ we see that $x, y \in \mathbb{Z}$ and $d = 3(x^2 + 3xy + y^2)$, and the conclusion is as usual. □

Theorem 3.3.7 (case (4) in Fig. 3.1)**.** *The candidate surface branched covers inducing some* $S(2,2,2,2) \overset{d:1}{\dashrightarrow} S(2,2,2,2)$ *are*

$$S \xrightarrow[(2,\ldots,2,1),(2,\ldots,2,1),(2,\ldots,2,1),(2,\ldots,2,1)]{2k+1:1} S$$
$$S \xrightarrow[(2,\ldots,2,1,1),(2,\ldots,2,1,1),(2,\ldots,2),(2,\ldots,2)]{2k+2:1} S$$
$$S \xrightarrow[(2,\ldots,2,1,1,1,1),(2,\ldots,2),(2,\ldots,2),(2,\ldots,2)]{2k+4:1} S$$

for $k \geqslant 1$*. The first two are always realizable, the last one is if and only if* d *is a multiple of* 4.

Proof. The first assertion is easy (but note that now the Riemann-Hurwitz formula cannot be used in its simplified form (2.2), it reads $\ell(\Pi) = 2d + 2$). The second assertion is proved as usual, except that we have to deal with the flexibility of $S(2,2,2,2)$. We assume that a map $f : S(2,2,2,2) \xrightarrow{d:1} S(2,2,2,2)$ realizing some candidate exists and we put on the target $S(2,2,2,2)$ the structure π defined by $\Gamma^{1,1}_{(2,2,2,2)}$. Then we deduce from Lemma 3.1.2 that there exists a structure $\widetilde{\pi}$ on the source $S(2,2,2,2)$ also with area 2, and $\widetilde{f} : \mathbb{E} \to \mathbb{E}$, such that $\widetilde{f}(z) = \lambda\cdot z + \mu$ with $\pi \circ \widetilde{f} = f \circ \widetilde{\pi}$ and $d = |\lambda|^2$. Then $\widetilde{\pi}$ is defined by some $\Gamma^{s,t}_{(2,2,2,2)}$.

We first note that by the symmetry of $S(2,2,2,2)$ we can assume $\mu = 0$. Then we apply Lemma 3.3.1. Since

$$\Lambda^{1,1}_{(2,2,2,2)} = \langle \tau_2, \tau_{2i}\rangle, \qquad \left\langle \tau_{2is}, \tau_{2\left(\frac{1}{s}+it\right)}\right\rangle$$

there exist $m, n, p, q \in \mathbb{Z}$ such that

$$\begin{cases} \lambda\cdot is = n + im \\ \lambda\cdot\left(\frac{1}{s} + it\right) = p + iq \end{cases} \tag{3.9}$$

and some easy computations show that all the relevant quantities can be determined explicitly in terms of n, m, p, q, namely:

$$s = \sqrt{\frac{n^2+m^2}{pm - qn}}, \qquad t = \frac{sp}{n} - \frac{m}{ns}, \qquad \lambda = \frac{1}{s}(m - in)$$

so in particular $d = |\lambda|^2 = pm - qn$. Note also that equations (3.9) already give us also the images of the lifts of the cone points.

For the first candidate we have $d = 4a \pm 1$ for some $a \geqslant 1$, we set $n = 2a,\ m = 1,\ p = \pm 1,\ q = -2$, we compute s, t, λ as above and we see that the corresponding map $\widetilde{f}$ induces a realization of the candidate, because

$$\widetilde{f}\left(\widetilde{A}^{(2)}\right) = \widetilde{A}^{(2)}_{0,0} \qquad \widetilde{f}\left(\widetilde{B}^{(2)}\right) = \widetilde{B}^{(2)}_{\pm 1,-2}$$
$$\widetilde{f}\left(\widetilde{C}^{(2)}\right) = \widetilde{C}^{(2)}_{2a\pm 1,-1} \quad \widetilde{f}\left(\widetilde{D}^{(2)}\right) = \widetilde{D}^{(2)}_{2a,1}.$$

For the second candidate we have $d = 4a + 1 \pm 1$ for some $a \geqslant 1$, we set $n = 2a,\ m = 1 \pm 1,\ p = 1,\ q = -2$ we compute s, t, λ as above and we see that the corresponding map $\widetilde{f}$ induces a realization of the candidate, because

$$\widetilde{f}\left(\widetilde{A}^{(2)}\right) = \widetilde{A}^{(2)}_{0,0} \qquad \widetilde{f}\left(\widetilde{B}^{(2)}\right) = \widetilde{B}^{(2)}_{1,-2}$$
$$\widetilde{f}\left(\widetilde{C}^{(2)}\right) = \widetilde{B}^{(2)}_{2a+1,\pm 1-1} \quad \widetilde{f}\left(\widetilde{D}^{(2)}\right) = \widetilde{A}^{(2)}_{2a,1\pm 1}.$$

For the last candidate each lift of a cone point has some $\widetilde{A}^{(2)}_{*}$ as its image, therefore n, m, p, q must all be even, which implies that d is a multiple of 4, as prescribed in the statement. Conversely if $d = 4a$ for $a > 1$ we set $n = m = q = 2,\ p = 2(a+1)$ we compute s, t, λ as above and we see that the corresponding map $\widetilde{f}$ induces a realization of the candidate, because

$$\widetilde{f}\left(\widetilde{A}^{(2)}\right) = \widetilde{A}^{(2)}_{0,0} \qquad \widetilde{f}\left(\widetilde{B}^{(2)}\right) = \widetilde{A}^{(2)}_{2(a+1),2}$$
$$\widetilde{f}\left(\widetilde{C}^{(2)}\right) = \widetilde{A}^{(2)}_{2(a+2),4} \quad \widetilde{f}\left(\widetilde{D}^{(2)}\right) = \widetilde{A}^{(2)}_{2,2}.$$

The proof is complete. □

Theorem 3.3.8 (case (5) in Fig. 3.1)**.** *The candidate surface branched covers having associated candidate* $S(3,3,3) \overset{d:1}{\dashrightarrow} S(2,3,6)$ *are*

$$S \xrightarrow[(2,\ldots,2),(3,\ldots,3,1,1,1),(6,\ldots,6)]{6k:1} S \qquad S \xrightarrow[(2,\ldots,2),(3,\ldots,3,1,1),(6,\ldots,6,2)]{6k+2:1} S$$
$$S \xrightarrow[(2,\ldots,2),(3,\ldots,3,1),(6,\ldots,6,2,2)]{6k+4:1} S \qquad S \xrightarrow[(2,\ldots,2),(3,\ldots,3),(6,\ldots,6,2,2,2)]{6k+6:1} S$$

for $k \geqslant 1$*, and they are realizable, respectively:*

- *if and only if* $d = 6(x^2 + 3xy + 3y^2)$ *for* $x, y \in \mathbb{N}$*;*
- *if and only if* $d = 2(x^2 + xy + y^2)$ *for* $x, y \in \mathbb{N}$ *and* $x \not\equiv y \pmod 3$*;*
- *never;*
- *if and only if* $d = 6(x^2 + 3xy + 3y^2)$ *for* $x, y \in \mathbb{N}$*.*

Proof. The first assertion is easy. For the second one we proceed as above, except that now the Euclidean structure $\widetilde{\pi}$ on $S(3,3,3)$ is not that we have fixed above, because its area should be $\frac{\sqrt{3}}{4}$ rather than $\frac{\sqrt{3}}{2}$, so the triangle $\Delta(3,3,3)$ must be rescaled by a factor $1/\sqrt{2}$. The lattices to which we can apply Lemma 3.3.1 are therefore

$$\frac{1}{\sqrt{2}}\cdot\Lambda_{(3,3,3)} = \left\langle \tau_{i\sqrt{\frac{3}{2}}}, \tau_{\frac{3+i\sqrt{3}}{2\sqrt{2}}} \right\rangle, \qquad \Lambda_{(2,3,6)} = \left\langle \tau_{i\sqrt{3}}, \tau_{\frac{3+i\sqrt{3}}{2}} \right\rangle.$$

As in the proof of Theorem 3.3.5 (except for the new factor) we deduce that

$$\lambda = \sqrt{2}\cdot((n+m)-\omega m), \qquad d = |\lambda|^2 = 2(n^2+nm+m^2).$$

Therefore $\widetilde{f}$ maps the lifts of the cone points of $S(3,3,3)$ to

$$\widetilde{f}(0)=\mu, \quad \widetilde{f}\left(\frac{1}{\sqrt{2}}\right) = (n+m)-\omega m+\mu, \quad \widetilde{f}\left(\frac{\omega}{\sqrt{2}}\right) = m+n\omega+\mu.$$

For the first candidate all these points should be some $\widetilde{B}_*^{(3)}$ from (3.7), so we can assume $\mu = 0$ and

$$n+m-(-m)\not\equiv 2 \ (\mathrm{mod}\ 3), \quad m-n\not\equiv 2\ (\mathrm{mod}\ 3) \quad \Rightarrow \qquad n\equiv m\ (\mathrm{mod}\ 3).$$

Setting $x = n$ and $y = (m-n)/3$ we then see that $d = 6(x^2+3xy+3y^2)$ for $x,y\in\mathbb{Z}$, and as above we can reduce to $x,y\in\mathbb{N}$, so d has the appropriate form. The converse follows from the same computations: if $d = 6(x^2+3xy+3y^2)$ we set $n = x$ and $m = x+3y$ and we see that the corresponding $\widetilde{f}$ realizes the candidate.

For the second candidate again $\mu = 0$ and, by the symmetry of $S(3,3,3)$, we can assume $1/\sqrt{2}$ is mapped to some $C_*^{(6)}$, namely $n-m\equiv 2$ (mod 3), so in particular $n\not\equiv m$ (mod 3). Therefore $d = 2(x^2+xy+y^2)$ for some $x,y\in\mathbb{Z}$ with $x\not\equiv y$ (mod 3), and once again restricting to $x,y\in\mathbb{N}$ makes no difference, so d has the prescribed form. The construction is easily reversible because if $n-m\equiv 2$ (mod 3) then $m-n\not\equiv 2$ (mod 3), which also proves that the third candidate is never realizable.

For the last candidate we can assume $\mu=\omega$, and

$$\begin{matrix} (n+m)-(1-m)\equiv 2\ (\mathrm{mod}\ 3) \\ m-(n+1)\equiv 2\ (\mathrm{mod}\ 3) \end{matrix} \quad \Rightarrow \qquad n\equiv m\ (\mathrm{mod}\ 3)$$

and we conclude as for the first candidate. □

Theorem 3.3.9 (case (6) in Fig. 3.1)**.** *The candidate surface branched covers having associated candidate* $S(2,2,2,2)\overset{d:1}{\dashrightarrow}S(2,4,4)$ *are*

$$S \xrightarrow[(2,\ldots,2,1,1,1,1),(4,\ldots,4),(4,\ldots,4)]{4k+4:1} S \qquad S \xrightarrow[(2,\ldots,2,1,1),(4,\ldots,4,2,2),(4,\ldots,4)]{4k+4:1} S$$

$$S \xrightarrow[(2,\ldots,2,1,1),(4,\ldots,4,2),(4,\ldots,4,2)]{4k+2:1} S \qquad S \xrightarrow[(2,\ldots,2),(4,\ldots,4,2,2),(4,\ldots,4,2,2)]{4k+4:1} S$$

$$S \xrightarrow[(2,\ldots,2),(4,\ldots,4,2,2,2),(4,\ldots,4,2)]{4k+6:1} S \qquad S \xrightarrow[(2,\ldots,2),(4,\ldots,4,2,2,2,2),(4,\ldots,4)]{4k+8:1} S$$

for $k\geqslant 1$. The first four are always realizable, the fifth one is never, and the last one is if and only if d is a multiple of 8.

Proof. Again we leave the first assertion to the reader and we proceed with the customary scheme. Since the area of the structure we have chosen on $S(2,4,4)$ is 1, on $S(2,2,2,2)$ we will have a structure generated by the rotations of angle π around points

$$0, \quad \frac{1}{2s}+it, \quad \frac{1}{2s}+i(s+t), \quad is$$

with $s,t\in\mathbb{R}$ and $s>0$. The lattices to which we must apply Lemma 3.3.1 are therefore $\left\langle \tau_{2is}, \tau_{\frac{1}{s}+2it}\right\rangle$ and $\Lambda_{(2,4,4)}=\langle\tau_2,\tau_{2i}\rangle$, so

$$\begin{cases} \lambda\cdot 2is = 2(n+im) \\ \lambda\cdot(\frac{1}{s}+2it)=2(p+iq) \end{cases}$$

for some $n,m,p,q\in\mathbb{Z}$. Whence, after easy computations,

$$s=\sqrt{\frac{n^2+m^2}{2(pm-qn)}}, \qquad t=\frac{sp}{n}-\frac{m}{2sn}, \qquad \lambda=\frac{m-in}{s}.$$

In particular $d=2(pm-qn)$ and the images of the lifts of the cone points of $S(2,2,2,2)$ are

$$\widetilde{f}(0)=\mu \qquad \widetilde{f}\left(\tfrac{1}{2s}+it\right)=p+iq+\mu$$
$$\widetilde{f}(is)=n+im+\mu \qquad \widetilde{f}\left(\tfrac{1}{2s}+i(s+t)\right)=(p+n)+i(q+m)+\mu.$$

The first four candidates are realized respectively with the following choices of n,m,p,q,μ:

n	m	p	q	μ
$k+1$	$k+1$	1	-1	1
k	$k+1$	2	0	0
k	$k+1$	1	-1	0
k	$k+1$	2	0	0

The fifth candidate is always exceptional because we can suppose $\mu=0$ and hence we should have that two of the pairs

$$(p,q), \quad (n,m), \quad (p+n,q+m)$$

consist of even numbers and the third one consists of odd numbers, which is impossible.

For the last candidate we have that p,q,n,m must all be even, so $d=2(pm-qn)$ is a multiple of 8. Conversely if $d=8h$ we can realize the candidate with $n=q=0$, $m=2$ and $p=2h$. $\square$

Theorem 3.3.10 (case (7) in Fig. 3.1)**.** *The candidate surface branched covers having associated candidate* $S(2,2,2,2)\overset{d:1}{\dashrightarrow}S(2,3,6)$ *are*

$$S \xrightarrow[(2,\dots,2,1,1,1,1),(3,\dots,3),(6,\dots,6)]{6k:1} S \qquad S \xrightarrow[(2,\dots,2,1,1,1),(3,\dots,3),(6,\dots,6,3)]{6k+3:1} S$$
$$S \xrightarrow[(2,\dots,2,1,1),(3,\dots,3),(6,\dots,6,3,3)]{6k+6:1} S \qquad S \xrightarrow[(2,\dots,2,1),(3,\dots,3),(6,\dots,6,3,3,3)]{6k+9:1} S$$
$$S \xrightarrow[(2,\dots,2),(3,\dots,3),(6,\dots,6,3,3,3,3)]{6k+12:1} S$$

for $k \geqslant 1$. The first three are always realizable, the fourth one is never, and the last one is if and only if d is a multiple of 12*.*

Proof. Once again we leave the first assertion to the reader and we follow the usual scheme. Since the area of $S(2,3,6)$ is $\sqrt{3}/4$, on $S(2,2,2,2)$ we will have a structure generated by the rotations of angle π around points

$$0, \quad \frac{\sqrt{3}}{8s} + it, \quad \frac{\sqrt{3}}{8s} + i(s+t), \quad is$$

and we apply Lemma 3.3.1 to $\left\langle \tau_{2is}, \tau_{\frac{\sqrt{3}}{8s}+2it} \right\rangle$ and $\Lambda_{(2,3,6)} = \left\langle \tau_{i\sqrt{3}}, \tau_{\frac{3+i\sqrt{3}}{2}} \right\rangle$, so for some $n, m, p, q \in \mathbb{Z}$ we have

$$\begin{cases} \lambda \cdot 2is = ni\sqrt{3} + m\frac{3+i\sqrt{3}}{2} \\ \lambda \cdot \left(\frac{\sqrt{3}}{4s} + 2it \right) = pi\sqrt{3} + q\frac{3+i\sqrt{3}}{2} \end{cases}$$

whence, after some calculations

$$s = \frac{1}{2}\sqrt{\frac{n^2+nm+m^2}{qn-pm}}, \qquad t = \frac{qs}{m} - \frac{m+2n}{8ms},$$
$$\lambda = \frac{\sqrt{3}(m+2n) - 3im}{4s} = \frac{\sqrt{3}}{2s} \cdot ((n+m) - m\omega)$$

so in particular $d = |\lambda|^2 = 3(qn - pm)$. Moreover the following relations will readily allow us to determine the images under $\widetilde{f}$ of the lifts of the cone points of $S(2,2,2,2)$:

$$\lambda \cdot is = \frac{1}{2}((m-n) + (m+2n)\omega),$$
$$\lambda \cdot \left(\frac{\sqrt{3}}{8s} + it \right) = \frac{1}{2}((q-p) + (q+2p)\omega).$$

For the first candidate we choose $\mu = \frac{1}{2}$, $p = q = 2$, $n = k+1$ and $m = 1$. The corresponding $\widetilde{f}$ induces a realization because $d = 6k = 3(qn - pm)$ and the images of the cone points are

$$\begin{array}{ll} \frac{1}{2}(1 + 0\omega), & \frac{1}{2}((m-n+1) + (m+2n)\omega), \\ \frac{1}{2}((q-p+1) + (q+2p)\omega), & \frac{1}{2}((m+q-n-p+1) + (m+q+2n+2p)\omega) \end{array}$$

which are easily recognized to all have the form $\frac{1}{2}(a + b\omega)$ with a, b not both even and $a - b \equiv 1 \pmod 3$, so they equal some $\widetilde{A}_*^{(2)}$.

For the second candidate we choose $\mu = \frac{1}{2}$, $n = 2$, $m = 1$ and

$$q = k,\ p = -1 \quad \text{if } k \equiv 1 \pmod 2, \qquad q = k+1,\ p = 1 \quad \text{if } k \equiv 0 \pmod 2.$$

Then $d = 6k + 3 = 3(qn - pm)$ and the images of the cone points are as before, but now the first three are some $\widetilde{A}_*^{(2)}$, while the last one has the form $a + b\omega$ with $a, b \in \mathbb{Z}$ and $a - b \equiv 2 \pmod 3$, so it is some $\widetilde{C}_*^{(6)}$, so $\widetilde{f}$ induces a realization of the candidate.

For the third candidate we choose $\mu = \frac{1}{2}$, $m = q = 2$ and

$$n = k,\ p = -1 \quad \text{if } k \equiv 0 \pmod 2, \qquad n = k+1,\ p = 1 \quad \text{if } k \equiv 1 \pmod 2.$$

Then $d = 6k + 6 = 3(qn - pm)$ and now the first two images are some $\widetilde{A}_*^{(2)}$ and the last two are some $\widetilde{C}_*^{(6)}$, so $\widetilde{f}$ induces a realization of the candidate.

For the fourth candidate we can once again suppose $\mu = \frac{1}{2}$. Since the images of the last three cone points must be some $\widetilde{C}_*^{(6)}$ we deduce that $m, q, m + q$ should be even and hence $n, p, n + p$ should be odd, which is impossible.

Turning to the last candidate, we can suppose $\mu = \omega$. Then the images of the cone points are

$$\begin{array}{ll} 1 + 0\omega, & \frac{1}{2}((m-n) + (m+2n+2)\omega), \\ \frac{1}{2}((q-p) + (q+2p+2)\omega), & \frac{1}{2}((m+q-n-p) + (m+q+2n+2p+2)\omega) \end{array}$$

and they must all be some $\widetilde{C}_*^{(6)}$, so n, m, p, q should all be even. Therefore $d = 3(qn - pm)$ is a multiple of 12. Conversely, if $d = 12h + 12$ we realize the candidate with the choice $q = 2h$, $n = m = 2$ and $p = -2$. □

Remark 3.3.11. A geometric interpretation of the results exposed in this chapter and in the next one is worth pointing out. Let $\widetilde{\Sigma} \dashrightarrow \Sigma$ be a candidate surface branched cover with associated candidate orbifold cover $\widetilde{X} \dashrightarrow X$, and suppose that $\widetilde{X}$ and X are geometric. Then we have identifications $\widetilde{X} = \mathbb{X}/\widetilde{\Gamma}$ and $X = \mathbb{X}/\Gamma$, where $\mathbb{X}$ is one of the model geometries $\mathbb{S}$, $\mathbb{E}$ or $\mathbb{H}$ (the same for $\widetilde{X}$ and X), and $\widetilde{\Gamma}, \Gamma$ are discrete cocompact groups of isometries of $\mathbb{X}$. A realization of the cover then corresponds to an identification of $\widetilde{\Gamma}$ to a subgroup of Γ. Our results in the spherical, Euclidean, and hyperbolic cases therefore yield a classification of the inclusions, respectively, between finite subgroups of SO(3), between 2-dimensional crystallographic groups, and between discrete, cocompact subgroups of $\mathrm{PSL}(2;\mathbb{R})$ with no more than three singular orbits.

Congruences and density We close this section explaining the reason why our main results are remarkable in view of the prime degree conjecture:

- A prime number of the form $4k + 1$ can always be expressed as $x^2 + y^2$ for $x, y \in \mathbb{N}$ (Fermat);
- A prime number of the form $6k + 1$ (or equivalently $3k + 1$) can always be expressed as $x^2 + xy + y^2$ for $x, y \in \mathbb{N}$ (Gauss);
- The integers that can be expressed as $x^2 + y^2$ or as $x^2 + xy + y^2$ with $x, y \in \mathbb{N}$ have asymptotically zero density in $\mathbb{N}$.

This means that a candidate cover in any of our three statements is "exceptional with probability 1," even though it is realizable when its degree is prime.

We close this section explaining in detail why Theorems 0.0.2 to 0.0.4 are implied by Theorems 3.3.3 to 3.3.6. First of all we have:

$$\begin{aligned}
&\{d \in \mathbb{N}:\ d = x^2 + y^2 \text{ for } x, y \in \mathbb{N},\ x \not\equiv y \pmod 2\}\\
&= \{d \in \mathbb{N}:\ d \equiv 1 \pmod 4,\ d = x^2 + y^2 \text{ for } x, y \in \mathbb{N}\},\\
&\{d \in \mathbb{N}:\ d = x^2 + xy + y^2 \text{ for } x, y \in \mathbb{N} \text{ not both even},\ x \not\equiv y \pmod 3\}\\
&= \{d \in \mathbb{N}:\ d \equiv 1 \pmod 6,\ d = x^2 + xy + y^2 \text{ for } x, y \in \mathbb{N}\},\\
&\{d \in \mathbb{N}:\ d = x^2 + xy + y^2 \text{ for } x, y \in \mathbb{N},\ x \not\equiv y \pmod 3\}\\
&= \{d \in \mathbb{N}:\ d \equiv 1 \pmod 3,\ d = x^2 + xy + y^2 \text{ for } x, y \in \mathbb{N}\}.
\end{aligned}$$

Moreover the statement made in the Introduction that in Theorems 0.0.2 to 0.0.4 the realizable degrees have zero asymptotic density means the following:

$$\begin{aligned}
&\lim_{n\to\infty} \tfrac{1}{n} \cdot \#\{d \in \mathbb{N}:\ d \leqslant n,\ d = x^2 + y^2 \text{ for } x, y \in \mathbb{N}\} = 0,\\
&\lim_{n\to\infty} \tfrac{1}{n} \cdot \#\{d \in \mathbb{N}:\ d \leqslant n,\ d = x^2 + xy + y^2 \text{ for } x, y \in \mathbb{N}\} = 0.
\end{aligned}$$

Chapter 4

Orbifold Covers in $\chi^{\mathrm{orb}} < 0$

In order to carry on as far as possible the investigation of the realizability problem for branched covers also in the case where the associated orbifold cover is hyperbolic, we restrict our analysis to the family of branched covers of S^2 over S^2 with three branching points. More precisely, we stratify this huge family with respect to the number of singular points in the cover sphere. We call $\{C_n\}$, for $n \geqslant 3$ the stratum of all covers like

$$S(\alpha_1, \ldots, \alpha_n) \dashrightarrow S(p, q, r).$$

This chapter contains the analysis of first two strata C_3, C_4. Note that after a long but easy work of enumeration of candidate covers, we provide proofs about realizability and exceptionality using mainly Grothendieck's dessins d'enfant [9, 27], already exploited in [19] and briefly reviewed in chapter 2. However, we conclude this chapter with a section dedicated to an attempt at understanding more about how to exploit the geometric viewpoint also in the hyperbolic case.

4.1 Triangular hyperbolic 2-orbifolds: C_3

The stratum C_3 involves only rigid hyperbolic 2-orbifolds, indeed one knows that a hyperbolic 2-orbifold is rigid if and only if it is *triangular*, namely if it is based on the sphere and it has precisely three cone points. In this section we will show that only very few candidate surface branched covers have associated candidate covers between hyperbolic triangular 2-orbifolds:

Theorem 4.1.1. *The candidate surface branched covers having associated candidate covers between triangular hyperbolic* 2*-orbifolds are precisely:*

$$S \xrightarrow[(5,1),(4,1,1),(2,2,2)]{6:1} S \qquad S \xrightarrow[(5,1,1,1),(4,4),(2,\ldots,2)]{8:1} S \qquad S \xrightarrow[(7,1),(3,3,1,1),(2,\ldots,2)]{8:1} S$$

$$S \xrightarrow[(7,1,1),(3,3,3),(2,\ldots,2,1)]{9:1} S \qquad S \xrightarrow[(8,1,1),(3,3,3,1),(2,\ldots,2)]{10:1} S \qquad S \xrightarrow[(8,2,1,1),(3,\ldots,3),(2,\ldots,2)]{12:1} S$$

$$S \xrightarrow[(9,1,1,1),(3,\ldots,3),(2,\ldots,2)]{12:1} S \qquad S \xrightarrow[(7,7,1,1),(3,\ldots,3,1),(2,\ldots,2)]{16:1} S \qquad S \xrightarrow[(7,7,7,1,1,1),(3,\ldots,3),(2,\ldots,2)]{24:1} S.$$

For the candidate covers of Theorem 4.1.1 the geometric approach is not even necessary, since realizability can be fully analyzed using a completely different technique, namely Grothendieck's *dessins d'enfant.* We will show the following:

Proposition 4.1.2. *Among the candidate covers of Theorem 4.1.1, the second and the eighth are exceptional and all other ones are realizable.*

Let us now establish the results we have stated. The first proof requires the analysis of quite a few cases, some of which we will leave to the reader.

Proof of 4.1.1. Our argument is organized in three steps:

(I) Analysis of the relevant surface candidate covers with degree $d \leqslant 11$;

(II) Restrictions on the base of the associated candidate cover for $d \geqslant 12$;

(III) More restrictions on the cover and conclusion for $d \geqslant 12$.

STEP I. If Π is a partition of an integer d, let us denote by $\ell(\Pi)$ its length (as above), and by $c(\Pi)$ the number of entries in Π which are different from l.c.m.(Π). To have an associated candidate cover between triangular 2-orbifolds (regardless of the geometry), a candidate surface branched cover of degree $d \geqslant 2$ must have the following properties:

- The number of branching points is 3;
- If the partitions of d are Π_1, Π_2, Π_3 then $c(\Pi_1) + c(\Pi_2) + c(\Pi_3) = 3$.

To list all such candidate covers for a given d then one has to:

- List all the partitions Π of d with $c(\Pi) \leqslant 3$;
- Find all possible triples (Π_1, Π_2, Π_3) of partitions with $\ell(\Pi_1)+\ell(\Pi_2)+\ell(\Pi_3) = d + 2$ and $c(\Pi_1) + c(\Pi_2) + c(\Pi_3) = 3$.

We have done this for $2 \leqslant d \leqslant 11$ and then we have singled out the candidate covers having associated hyperbolic 2-orbifold covers, getting the first five items of the statement. To illustrate how this works we will spell out here only the case $d = 8$. The partitions Π of 8 with $c(\Pi) \leqslant 3$ are those described in Table 4.1, with the corresponding values of ℓ and c.

Π	(8)	(6,1,1)	(5,1,1,1)	(4,2,2)	(3,3,1,1)
ℓ	1	3	4	3	4
c	0	2	3	2	2
Π	(7,1)	(5,3)	(4,4)	(4,2,1,1)	(2,2,2,2)
ℓ	2	2	2	4	4
c	1	2	0	3	0
Π	(6,2)	(5,2,1)	(4,3,1)	(3,3,2)	(2,2,2,1,1)
ℓ	2	3	3	3	5
c	1	3	3	3	2

Table 4.1. The partitions Π of 8 with $c(\Pi) \leqslant 3$

The triples of such partitions such that ℓ and c sum up to 10 and 3 respectively are shown in Table 4.2, together with the associated candidate orbifold cover and its geometric type. So we get the second and third item in the statement.

Π_1, Π_2, Π_3			Associated cover	Geometry
(2,2,2,2)	(4,4)	(4,2,1,1)	$S(2,4,4) \dashrightarrow S(2,4,4)$	$\mathbb{E}$
(2,2,2,2)	(4,4)	(5,1,1,1)	$S(5,5,5) \dashrightarrow S(2,4,5)$	$\mathbb{H}$
(2,2,2,2)	(3,3,1,1)	(6,2)	$S(3,3,3) \dashrightarrow S(2,3,6)$	$\mathbb{E}$
(2,2,2,2)	(3,3,1,1)	(7,1)	$S(3,3,7) \dashrightarrow S(2,3,7)$	$\mathbb{H}$

Table 4.2. Triples of partitions of 8 having associated candidate covers between triangular orbifolds

Step II. Let us denote by $\widetilde{X} \overset{d:1}{\dashrightarrow} X$ a candidate orbifold cover with $d \geqslant 12$ and hyperbolic $\widetilde{X} = S(\alpha, \beta, \gamma)$ and $X = S(p, q, r)$. Since

$$0 < -\chi^{\mathrm{orb}}(\widetilde{X}) = 1 - \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) < 1$$

and $\chi^{\mathrm{orb}}(\widetilde{X}) = d \cdot \chi^{\mathrm{orb}}(X)$, we deduce that

$$0 < -\chi^{\mathrm{orb}}(X) = 1 - \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) < \frac{1}{12} \quad \Rightarrow \quad \frac{11}{12} < \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) < 1.$$

Assuming $p \leqslant q \leqslant r$ it is now very easy to check that the last inequality is satisfied only for $p = 2$, $q = 3$, $7 \leqslant r \leqslant 11$ and for $p = 2$, $q = 4$, $r = 5$.
Step III. If $\widetilde{X} \dashrightarrow X$ is a candidate 2-orbifold cover with hyperbolic $\widetilde{X} = S(\alpha, \beta, \gamma)$ and $X = S(p, q, r)$ then the following must happen:

(a) Each of α, β, γ must be a divisor of some element of $\{p, q, r\}$;

(b) $\frac{\chi^{\mathrm{orb}}(\widetilde{X})}{\chi^{\mathrm{orb}}(X)}$ must be an integer d;

(c) There must exist three partitions of d such that associated candidate orbifold cover is $\widetilde{X} \dashrightarrow X$.

Successively imposing these conditions with each of the 5 orbifolds X coming from Step II and restricting to $d \geqslant 12$ we have found the last four items in the statement. Again we only spell out here one example, leaving the other ones to the reader. Let X be $S(2, 3, 8)$. Then the relevant hyperbolic $\widetilde{X}$'s according to (a), excluding X itself, are

$$\begin{array}{ccccc} S(2,4,8) & S(3,3,4) & S(2,8,8) & S(3,3,8) & S(3,8,8) \\ S(3,4,4) & S(4,4,4) & S(4,4,8) & S(4,8,8) & S(8,8,8) \end{array}$$

and $d = \frac{\chi^{\mathrm{orb}}(\widetilde{X})}{\chi^{\mathrm{orb}}(X)}$ is always integer in this case, so point (b) is not an issue. However $d \leqslant 11$ in all cases but the last two (for instance, the case $\widetilde{X} = S(3, 8, 8)$ corresponds to the fifth item in the statement). For $\widetilde{X} = S(4, 8, 8)$ we have $d = 12$ and taking the partitions of 12 as the sixth item in the statement we see that the associated orbifold cover is indeed $S(4, 8, 8) \dashrightarrow S(2, 3, 8)$. For $\widetilde{X} = S(8, 8, 8)$ we have $d = 15$ and it is impossible to find partitions of 15 having the right associated candidate orbifold cover, because the cone point of order 2 in X, being covered by ordinary points of $\widetilde{X}$ only, should require a partition consisting of 2's only, which cannot exist because 15 is odd.

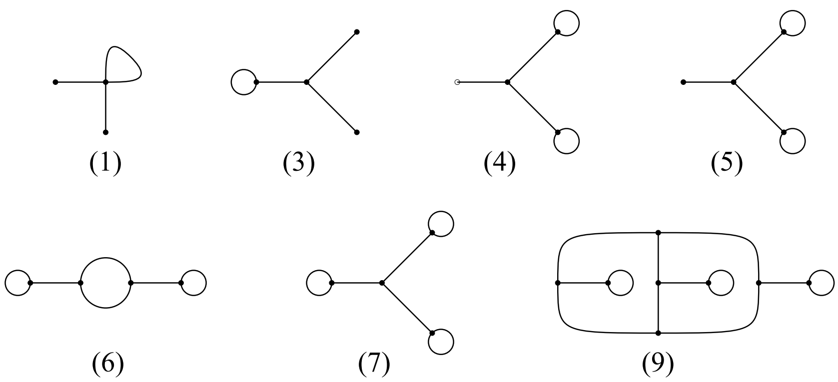

Figure 4.1. Dessins d'enfant for all candidate surfaced branched covers in Theorem 4.1.1 except the second and the eighth

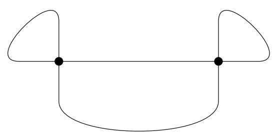

Figure 4.2. Exceptionality of $S \xrightarrow[(5,1,1,1),(4,4),(2,\ldots,2)]{8:1} S$

Carrying out the same analysis one gets the last two items in the statement for $X = S(2,3,7)$, the seventh item for $X = S(2,3,9)$, and nothing new for the other X's. This concludes Step III and the proof. □

As already announced, the next argument is based on a technique different from those used in the rest of this paper, namely Grothendieck *dessins d'enfant*, reviewed in Chapter 2.

Proof of 4.1.2. Dessins d'enfant proving the realizability of all candidate covers claimed to be realizable can be found in Fig. 4.1. The black vertices always correspond to the elements of the second partition in Theorem 4.1.1, and the white vertices to the entries of the third partition, while the regions correspond to the elements of the first partition. However 2-valent white vertices are never shown, except for the single 1-valent one in case (4).

Exceptionality of $S \xrightarrow[(5,1,1,1),(4,4),(2,\ldots,2)]{8:1} S$ is easy: a dessin relative to partitions $(4,4)$ and $(2,2,2,2)$ with at least two complementary regions of length 1 must be as shown in Fig. 4.2, so the third partition is $(4,2,1,1)$, not $(5,1,1,1)$.

For the exceptionality of $S \xrightarrow[(7,7,1,1),(3,\ldots,3,1),(2,\ldots,2)]{16:1} S$ refer to Fig. 4.3. Since it must contain two length-1 regions, a dessin realizing it should be as in (a). The two marked germs of edges cannot be joined together or to the 1-valent vertex, so they go either to the same 3-valent vertex as in (aa) or to different 3-valent vertices as in (ab). Case (aa) is impossible because there is a region with 7 vertices, which will become more than 7 in the complete dessin. In case (ab) we examine where the marked germ of edge could go, getting cases (aba) to (abd), always redrawn in a more convenient way. Cases (aba) and (abb) are impossible because of long regions. In cases (abc) and (abd) we examine where the marked edge could go in order not to create regions of length 5 or longer than 7, and we see that in both cases there is

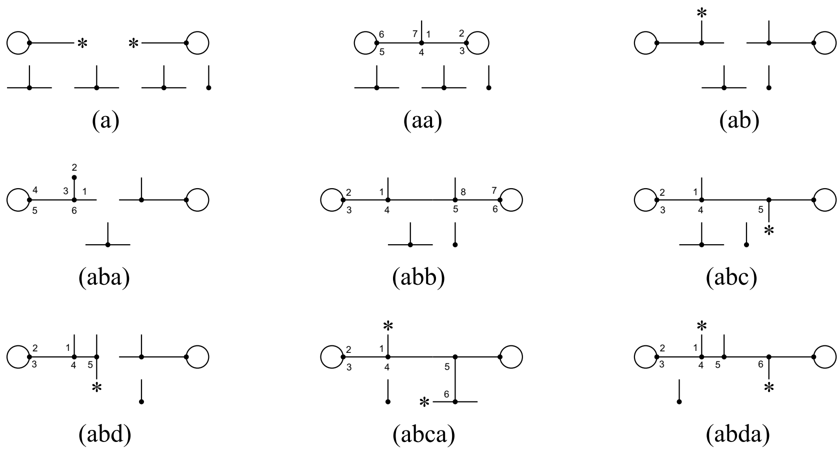

Figure 4.3. Exceptionality of $S \xrightarrow[(7,7,1,1),(3,\ldots,3,1),(2,\ldots,2)]{16:1} S$

only one possibility, namely (abca) and (abda). In both these cases, because of the region of length already 6, the two marked germs of edges should go to one and the same 3-valent vertex, but there are no more available with two free germs of edges, so again we cannot complete the dessin in order to realize (8). Our argument is complete. □

As a conclusion we note that Theorem 4.1.1 and Proposition 4.1.2 are very far from providing a complete analysis of realizability and exceptionality of candidate covers with associated orbifold candidate of hyperbolic type, because most often the orbifolds involved are not triangular. For instance the candidate surface branched cover $S \xrightarrow[(4,4,2),(4,2,2,2),(6,1,1,1,1)]{10:1} S$ considered in the first column of Table 2 in [28] is exceptional, and the associated orbifold candidate $S(6,6,6,6,2,2,2,2) \dashrightarrow S(4,4,6)$ is hyperbolic but not triangular.

4.2 The stratum C_4

In the previous section we have found realizations and exceptions when $\widetilde{X}$ is a hyperbolic triangular orbifold. In this section we increase by one the number of cone points and consider

$$S(\alpha,\beta,\gamma,\delta) \to S(p,q,r).$$

The results of the complete analysis of the stratum C_4 are informally synthetized in Theorem 0.0.8. Even if it represents a little step towards the full comprehension of the hyperbolic case, it produces a lot of examples that could be useful to understand more of the underlying general pattern.

Proposition 4.2.1. *There exist* 141 *candidate surface branched covers having an associated candidate* 2*-orbifold cover* $\widetilde{X} \dashrightarrow X$ *with* $\widetilde{X}$ *being the sphere with four cone points and* X *being a hyperbolic triangular orbifold.*

Proposition 4.2.2. *Among the candidate covers of Theorem 4.2.1 there are* 29 *exceptions; they do not occur in prime degree.*

Proof of 4.2.1. Our argument is organized in two steps:

(I) Analysis of the relevant surface candidate covers with degree $d \leqslant 12$;

(II) Analysis of the relevant surface candidate covers with degree $d \geqslant 13$.

STEP I. If Π is a partition of an integer d, let us denote by $\ell(\Pi)$ its length (as usual), and by $c(\Pi)$ the number of entries in Π which are different from l.c.m.(Π). In order to belong to our family, a candidate surface branched cover of degree $d \geqslant 2$ must have the following properties:

- The number of branching points is 3;
- If the partitions of d are Π_1, Π_2, Π_3 then $c(\Pi_1) + c(\Pi_2) + c(\Pi_3) = 4$.

To list all such candidate covers for a given d then one has to:

- List all the partitions Π of d with $c(\Pi) \leqslant 4$;
- Find all possible triples (Π_1, Π_2, Π_3) of partitions such that $\ell(\Pi_1) + \ell(\Pi_2) + \ell(\Pi_3) = d + 2$ and $c(\Pi_1) + c(\Pi_2) + c(\Pi_3) = 4$.

We have done this for $2 \leqslant d \leqslant 12$ and then we have singled out the candidate covers having associated hyperbolic 2-orbifold covers. For the sake of completeness we have written down all the computations in Appendix A. The 81 candidate covers found by these computations are described in Tables 4.3 and 4.4.

STEP II. Let us denote by $\widetilde{X} \stackrel{d:1}{\dashrightarrow} X$ a candidate orbifold cover with $d \geqslant 13$ and hyperbolic $\widetilde{X} = S(\alpha, \beta, \gamma, \delta)$ and $X = S(p, q, r)$. Since

$$0 < -\chi^{\text{orb}}(\widetilde{X}) = 2 - \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}\right) < 2$$

and $\chi^{\text{orb}}(\widetilde{X}) = d \cdot \chi^{\text{orb}}(X)$, we deduce that

$$0 < -\chi^{\text{orb}}(X) = 1 - \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) < \frac{2}{13} \quad \Rightarrow \quad \frac{11}{13} < \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) < 1.$$

Assuming $p \leqslant q \leqslant r$ it is now very easy to check that the last inequality is satisfied only for (p, q, r) as follows:

A. $(3, 3, 4)$ and $(3, 3, 5)$

B. $(2, 5, 5)$ and $(2, 5, 6)$

C. $(2, 4, r)$ where $5 \leqslant r \leqslant 10$

D. $(2, 3, r)$ where $7 \leqslant r \leqslant 77$

In order to determine the actual candidate covers, especially for group D, the biggest one, we must take care of other restrictions:

(a) Each of $\alpha, \beta, \gamma, \delta$ must be a divisor of some element of $\{p, q, r\}$;

d	Π_1, Π_2, Π_3			Associated cover
5	(2,1,1,1)	(4,1)	(5)	$S(2,2,2,4) \dashrightarrow S(2,4,5)$
	(3,1,1)	(3,1,1)	(5)	$S(3,3,3,3) \dashrightarrow S(3,3,5)$
	(3,1,1)	(4,1)	(4,1)	$S(3,3,4,4) \dashrightarrow S(3,4,4)$
	(2,2,1)	(3,2)	(4,1)	$S(2,2,3,4) \dashrightarrow S(2,4,6)$
6	(2,2,1,1)	(4,1,1)	(6)	$S(2,2,4,4) \dashrightarrow S(2,4,6)$
	(2,2,1,1)	(5,1)	(5,1)	$S(2,2,5,5) \dashrightarrow S(2,5,5)$
	(2,2,1,1)	(4,2)	(5,1)	$S(2,2,2,5) \dashrightarrow S(2,4,5)$
	(3,1,1,1)	(3,3)	(5,1)	$S(3,3,3,5) \dashrightarrow S(3,3,5)$
	(3,1,1,1)	(3,3)	(4,2)	$S(2,3,3,3) \dashrightarrow S(3,3,4)$
	(3,3)	(4,1,1)	(4,1,1)	$S(4,4,4,4) \dashrightarrow S(3,4,4)$
	(2,2,2)	(3,2,1)	(5,1)	$S(2,3,5,6) \dashrightarrow S(2,5,6)$
	(2,2,2)	(3,2,1)	(4,2)	$S(2,2,3,6) \dashrightarrow S(2,4,6)$
7	(2,2,1,1,1)	(3,3,1)	(7)	$S(2,2,2,3) \dashrightarrow S(2,3,7)$
	(2,2,2,1)	(4,1,1,1)	(7)	$S(2,4,4,4) \dashrightarrow S(2,4,7)$
	(3,3,1)	(3,3,1)	(5,1,1)	$S(3,3,5,5) \dashrightarrow S(3,3,5)$
	(3,3,1)	(3,3,1)	(4,2,1)	$S(2,3,3,4) \dashrightarrow S(3,3,4)$
	(2,2,2,1)	(3,3,1)	(5,2)	$S(2,2,3,5) \dashrightarrow S(2,3,10)$
	(2,2,2,1)	(3,3,1)	(4,3)	$S(2,3,3,4) \dashrightarrow S(2,3,12)$
	(2,2,2,1)	(4,2,1)	(6,1)	$S(2,2,4,6) \dashrightarrow S(2,4,6)$
	(2,2,2,1)	(5,1,1)	(6,1)	$S(2,5,5,6) \dashrightarrow S(2,5,6)$
8	(2,2,2,2)	(4,1,1,1,1)	(8)	$S(4,4,4,4) \dashrightarrow S(2,4,8)$
	(2,2,2,1,1)	(3,3,1,1)	(8)	$S(2,2,3,3) \dashrightarrow S(2,3,8)$
	(2,2,2,2)	(3,2,2,1)	(4,4)	$*S(2,3,3,6) \dashrightarrow S(2,4,6)$
	(2,2,2,2)	(5,1,1,1)	(7,1)	$S(5,5,5,7) \dashrightarrow S(2,5,7)$
	(2,2,2,2)	(5,1,1,1)	(6,2)	$*S(3,5,5,5) \dashrightarrow S(2,5,6)$
	(2,2,2,2)	(4,2,1,1)	(7,1)	$S(2,4,4,7) \dashrightarrow S(2,4,7)$
	(2,2,2,2)	(4,2,1,1)	(6,2)	$S(4,4,4,4) \dashrightarrow S(2,4,8)$
	(2,2,2,2)	(3,3,1,1)	(5,3)	$*S(3,3,3,5) \dashrightarrow S(2,3,15)$
	(3,3,1,1)	(3,3,1,1)	(4,4)	$S(3,3,3,3) \dashrightarrow S(3,3,4)$
	(2,2,2,2)	(6,1,1)	(6,1,1)	$S(6,6,6,6) \dashrightarrow S(2,6,6)$
	(2,2,2,2)	(4,2,2)	(6,1,1)	$*S(2,2,6,6) \dashrightarrow S(2,4,6)$
9	(2,2,2,1,1,1)	(3,3,3)	(8,1)	$S(2,2,2,8) \dashrightarrow S(2,3,8)$
	(2,2,2,2,1)	(3,3,1,1,1)	(9)	$S(2,3,3,3) \dashrightarrow S(2,3,9)$
	(3,3,3)	(3,3,3)	(5,1,1,1,1)	$*S(5,5,5,5) \dashrightarrow S(3,3,5)$
	(3,3,3)	(3,3,3)	(4,2,1,1,1)	$*S(2,4,4,4) \dashrightarrow S(3,3,4)$
	(3,3,1,1,1)	(3,3,3)	(4,4,1)	$S(3,3,3,4) \dashrightarrow S(3,3,4)$
	(2,2,2,2,1)	(4,4,1)	(7,1,1)	$S(2,4,7,7) \dashrightarrow S(2,4,7)$
	(2,2,2,2,1)	(4,4,1)	(6,2,1)	$S(2,3,4,6) \dashrightarrow S(2,4,6)$
	(2,2,2,2,1)	(3,3,3)	(5,3,1)	$S(2,3,5,15) \dashrightarrow S(2,3,15)$
	(2,2,2,2,1)	(3,3,3)	(5,2,2)	$*S(2,2,5,5) \dashrightarrow S(2,3,10)$
	(2,2,2,2,1)	(3,3,3)	(4,3,2)	$S(2,3,4,6) \dashrightarrow S(2,3,12)$

Table 4.3. Triples of partitions of $d \leqslant 9$ with hyperbolic associated candidate covers in C_4; asterisks mark exceptional covers, as determined below

d	Π_1, Π_2, Π_3			Associated cover
10	(2,...,2)	(3,3,1,1,1,1)	(10)	$S(3,3,3,3) \dashrightarrow S(2,3,10)$
	(2,2,2,2,1,1)	(4,4,1,1)	(5,5)	$S(2,2,4,4) \dashrightarrow S(2,4,5)$
	(2,2,2,2,1,1)	(3,3,3,1)	(8,2)	$S(2,2,3,4) \dashrightarrow S(2,3,8)$
	(2,2,2,2,1,1)	(3,3,3,1)	(9,1)	$S(2,2,3,9) \dashrightarrow S(3,3,9)$
	(2,...,2)	(5,5)	(6,1,1,1,1)	$S(6,6,6,6) \dashrightarrow S(2,5,6)$
	(2,...,2)	(5,5)	(4,2,2,1,1)	$S(2,2,4,4) \dashrightarrow S(2,4,5)$
	(2,...,2)	(4,4,2)	(7,1,1,1)	$*S(2,7,7,7) \dashrightarrow S(2,4,7)$
	(2,...,2)	(4,4,2)	(6,2,1,1)	$*S(2,3,6,6) \dashrightarrow S(2,4,6)$
	(2,...,2)	(4,4,1,1)	(8,1,1)	$S(4,4,8,8) \dashrightarrow S(2,4,8)$
	(2,...,2)	(4,4,1,1)	(6,3,1)	$S(2,4,4,6) \dashrightarrow S(2,4,6)$
	(2,...,2)	(4,4,1,1)	(6,2,2)	$S(3,3,4,4) \dashrightarrow S(2,4,6)$
	(2,...,2)	(3,3,3,1)	(7,2,1)	$S(2,3,7,14) \dashrightarrow S(2,3,14)$
	(2,...,2)	(3,3,3,1)	(5,4,1)	$S(3,4,5,20) \dashrightarrow S(2,3,20)$
	(2,...,2)	(3,3,3,1)	(5,3,2)	$S(3,6,10,15) \dashrightarrow S(2,3,30)$
	(2,...,2)	(3,3,3,1)	(4,3,3)	$*S(3,3,4,4) \dashrightarrow S(2,3,12)$
	(3,3,3,1)	(3,3,3,1)	(4,4,1,1)	$S(3,3,4,4) \dashrightarrow S(3,3,4)$
11	(2,...,2,1)	(3,3,3,1,1)	(10,1)	$S(2,3,3,10) \dashrightarrow S(2,3,10)$
	(2,...,2,1)	(4,4,2,1)	(5,5,1)	$S(2,2,4,5) \dashrightarrow S(2,4,5)$
12	(2,...,2,1,1)	(3,3,3,3)	(10,1,1)	$S(2,2,10,10) \dashrightarrow S(2,3,10)$
	(2,...,2,1,1)	(3,3,3,3)	(8,2,2)	$S(2,2,4,4) \dashrightarrow S(2,3,8)$
	(2,...,2,1,1)	(4,4,4)	(5,5,1,1)	$S(2,2,5,5) \dashrightarrow S(2,4,5)$
	(2,...,2)	(4,4,1,1,1,1)	(6,6)	$S(4,4,4,4) \dashrightarrow S(2,4,6)$
	(2,...,2)	(3,3,3,1,1,1)	(11,1)	$S(3,3,3,11) \dashrightarrow S(2,3,11)$
	(2,...,2)	(3,3,3,1,1,1)	(10,2)	$S(3,3,3,5) \dashrightarrow S(2,3,10)$
	(2,...,2)	(3,3,3,1,1,1)	(9,3)	$S(3,3,3,3) \dashrightarrow S(2,3,9)$
	(2,...,2)	(3,3,3,1,1,1)	(8,4)	$S(2,3,3,3) \dashrightarrow S(2,3,8)$
	(2,...,2)	(4,4,4)	(8,1,1,1,1)	$*S(8,8,8,8) \dashrightarrow S(2,4,8)$
	(2,...,2)	(4,4,4)	(6,3,1,1,1)	$S(2,6,6,6) \dashrightarrow S(2,4,6)$
	(2,...,2)	(4,4,4)	(6,2,2,1,1)	$S(3,3,6,6) \dashrightarrow S(2,4,6)$
	(2,...,2)	(3,3,3,3)	(7,3,1,1)	$*S(3,7,21,21) \dashrightarrow S(2,3,21)$
	(2,...,2)	(3,3,3,3)	(7,2,2,1)	$*S(2,7,7,14) \dashrightarrow S(2,3,14)$
	(2,...,2)	(3,3,3,3)	(6,4,1,1)	$*S(2,3,12,12) \dashrightarrow S(2,3,12)$
	(2,...,2)	(3,3,3,3)	(5,4,2,1)	$*S(4,5,10,20) \dashrightarrow S(2,3,20)$
	(2,...,2)	(3,3,3,3)	(5,3,3,1)	$*S(3,5,5,15) \dashrightarrow S(2,3,15)$
	(2,...,2)	(3,3,3,3)	(5,3,2,2)	$*S(6,10,15,15) \dashrightarrow S(2,3,30)$
	(2,...,2)	(3,3,3,3)	(4,4,3,1)	$*S(3,3,4,12) \dashrightarrow S(2,3,12)$
	(2,...,2)	(3,3,3,3)	(4,3,3,2)	$*S(3,4,4,6) \dashrightarrow S(2,3,12)$
	(3,3,3,3)	(3,3,3,3)	(4,4,1,1,1,1)	$S(4,4,4,4) \dashrightarrow S(3,3,4)$
	(2,...,2)	(5,5,1,1)	(5,5,1,1)	$S(5,5,5,5) \dashrightarrow S(2,5,5)$
	(2,...,2)	(4,4,2,2)	(5,5,1,1)	$S(2,2,5,5) \dashrightarrow S(2,4,5)$

Table 4.4. Triples of partitions of $10 \leqslant d \leqslant 12$ with hyperbolic associated candidate covers in C_4; asterisks mark exceptional covers, as determined below

(b) $\frac{\chi^{\mathrm{orb}}(\widetilde{X})}{\chi^{\mathrm{orb}}(X)}$ must be an integer d;

(c) There must exist three partitions of d such that associated candidate orbifold cover is $\widetilde{X} \dashrightarrow X$.

We will now carry out our analysis separately for each of the groups A, B, C, D of triples (p, q, r) as described above.

A. First, we want to study

$$\widetilde{X} \dashrightarrow S(3,3,4).$$

Note that $-\chi^{\mathrm{orb}}(X) = \frac{1}{12}$ and, taking into account divisibility between orders of cone points, the cover orbifold with maximum area is $S(4,4,4,4)$, because 4 is the greatest conic order possible for this base. Hence, $d_{\max} := \frac{\max\{-\chi^{\mathrm{orb}}(\widetilde{X})\}}{-\chi^{\mathrm{orb}}(X)}$ implies that the maximum possible degree is 12. And all these cases have already been considered in Step I.

Similar computations for $\widetilde{X} \dashrightarrow S(3,3,5)$ give $d_{\max} = 9$, and lead to the same conclusion.

B. Proceeding as just made for group A, we get the results described in Table 4.5. Then we have no more candidate covers to study.

X	$-\chi^{\mathrm{orb}}(X)$	$\max\{-\chi^{\mathrm{orb}}(\widetilde{X})\}$	$d_{\max}$
$S(2,5,5)$	1/10	6/5	12
$S(2,5,6)$	2/15	4/3	10

Table 4.5. Computation of maximal degrees for group B.

C. Also here we collect useful information about maximal possible degree in Table 4.6. Now we get something to analyse.

X	$-\chi^{\mathrm{orb}}(X)$	$\max\{-\chi^{\mathrm{orb}}(\widetilde{X})\}$	$d_{\max}$
$S(2,4,5)$	1/20	6/5	24
$S(2,4,6)$	1/12	4/3	16
$S(2,4,7)$	3/28	10/7	13
$S(2,4,8)$	1/8	3/2	12
$S(2,4,9)$	5/36	14/9	11
$S(2,4,10)$	3/20	8/5	10

Table 4.6. Computation of maximal degrees for group C.

One can now easily exclude that any interesting cover arises in case $X = S(2,4,7)$. In fact we see that $\widetilde{X} = S(7,7,7,7)$ cannot be a cover orbifold, because the resulting d is not a natural number, and the same happens for the next (with respect to area) admissible orbifold $\widetilde{X} = S(4,7,7,7)$. In this last case, we get that for all other cases d is forced to be strictly less than 13. Then, again, no new covers arises.

For $X = S(2,4,6)$ we first enumerate in Table 4.7 all possible cover orbifolds, in decreasing order with respect to degree. Notice that once you have fixed the base orbifold, the area of the cover orbifold is proportional to the degree of the would-be cover. Successively we exclude those cover orbifolds in the list in which $d \leqslant 12$ (see Table 4.7), because we have already considered these covers in Step I. Among the six remaining covers, only the following two are candidate: $S(6,6,6,6) \stackrel{16:1}{\dashrightarrow} S(2,4,6)$ and $S(4,4,6,6) \stackrel{14:1}{\dashrightarrow} S(2,4,6)$.

$\widetilde{X}$	$-\chi^{\text{orb}}(\widetilde{X})$	d
$S(6,6,6,6)$	$4/3$	16
$S(4,6,6,6)$	$5/4$	15
$S(4,4,6,6)$	$7/6$	14
$S(3,6,6,6)$	$7/6$	14
$S(4,4,4,6)$	$13/12$	13
$S(3,4,6,6)$	$13/12$	13
$S(4,4,4,4)$	1	12
$S(3,3,6,6)$	1	12
$S(3,4,4,6)$	1	12
$S(2,6,6,6)$	1	12
$S(3,3,4,6)$	$11/12$	11
$S(2,4,6,6)$	$11/12$	11
...	...	$\leqslant 12$

Table 4.7. Reduced list of possible cover orbifolds for $X = S(2,4,6)$

Finally, $S(2,4,5)$. We proceed exactly as for $X = S(2,4,6)$: first we order possible cover orbifolds in Table 4.8, then we search among them those that lead to a candidate cover, and then we describe in Table 4.9 all the candidate covers found.

Summing up, in group C we have found 9 candidate hyperbolic orbifold covers, and we will show that only one of them is exceptional (that one labeled with * in Table 4.9).

D. For this last group we must investigate candidate covers having base $X = S(2,3,r)$ with $7 \leqslant r \leqslant 77$. First of all, we make a preliminary study of the function

$$d_{\max}(r) := \frac{-\chi^{\text{orb}}(S(r,r,r,r))}{-\chi^{\text{orb}}(S(2,3,r))} = 12 \cdot \frac{r-2}{r-6},$$

used in the previous paragraph, and we describe our results in Table 4.10. Since d should be an integer and the function $d_{\max}(r)$ is strictly decreasing, for $18 \leqslant r \leqslant 55$, we list in Table 4.10 only those cases in which the maximal degree function takes an integer value. One clearly sees from this table that it is useless to consider $\widetilde{X} = S(2,3,r)$ with $r \geqslant 55$, because these cases, if any, should have been studied in Step I.

$\widetilde{X}$	$-\chi^{\text{orb}}(\widetilde{X})$	d
$S(5,5,5,5)$	$6/5$	24
$S(5,5,5,4)$	$23/20$	23
$S(5,5,4,4)$	$11/10$	22
$S(5,4,4,4)$	$21/20$	21
$S(4,4,4,4)$	1	20
$S(5,5,5,2)$	$9/10$	18
$S(5,5,4,2)$	$17/20$	17
$S(5,4,4,2)$	$4/5$	16
$S(4,4,4,2)$	$3/4$	15
$S(5,5,2,2)$	$3/5$	12
$S(5,4,2,2)$	$11/20$	11
...	...	$\leqslant 12$

Table 4.8. Reduced list of possible cover orbifolds for $X = S(2,4,5)$

Π_1, Π_2, Π_3	Associated cover	d
(2,...,2),(4,4,4,4),(6,6,1,1,1,1)	$S(6,6,6,6) \dashrightarrow S(2,4,6)$	16
(2,...,2),(4,4,4,1,1),(6,6,1,1)	$S(4,4,6,6) \dashrightarrow S(2,4,6)$	14
(2,...,2),(4,...,4),(5,5,5,5,1,1,1,1)	$S(5,5,5,5) \dashrightarrow S(2,4,5)$	24
(2,...,2),(4,...,4,1,1,),(5,5,5,5,1,1)	$S(5,5,4,4) \dashrightarrow S(2,4,5)$	22
(2,...,2),(4,4,4,4,1,1,1,1),(5,...,5)	$S(4,4,4,4) \dashrightarrow S(2,4,5)$	20
(2,...,2),(4,4,4,4,2),(5,5,5,1,1,1)	$*S(5,5,5,2) \dashrightarrow S(2,4,5)$	18
(2,...,2,1),(4,4,4,4,1),(5,5,5,1,1)	$S(5,5,4,2) \dashrightarrow S(2,4,5)$	17
(2,...,2), (4,4,2,1,1),(5,5,5,1)	$S(5,4,4,2) \dashrightarrow S(2,4,5)$	16
(2,...,2,1), (4,4,1,1,1), (5,5,5)	$S(4,4,4,2) \dashrightarrow S(2,4,5)$	15

Table 4.9. Candidate covers in C_4 with base $S(2,4,5)$ and $S(2,4,6)$

r	7	8	9	10	11	12	13	14
$d_{\max}$	60	36	28	24	21,6	20	14,6	18
r	15	16	17	18	22	30	54	55
$d_{\max}$	17,3	16,8	16,36	16	15	14	13	12,97

Table 4.10. Preliminary data about the maximal degree function

We start our analysis from the easiest cases: those with low $d_{\max}(r)$.

For $r = 54$, the only interesting degree in which we have to look for candidate covers is $d = 13$. From the study of the maximal degree function we see that this is a very simple case. On the other hand when we consider $r = 7$, we should look for candidate covers with $12 \leqslant d \leqslant 60$. Hence we need a way to get a better estimation of the maximal possible degree, in order to reduce the range for d.

We exploit congruences: precisely, we group all possible degrees with respect to its congruences classes modulo 2 and 3, and for each group we define a new function that is always lower than $d_{\max}(r)$ (and obviously greater than d). Just an example: if we know that a $d \equiv 1\,(2)$ we infer that a cover orbifold $\widetilde{X}$ should have at least a conic point of order 2; and $-\chi^{\mathrm{orb}}(S(r,r,r,r) \geqslant S(2,r,r,r))$. Hence we know that $d \leqslant d_2(r) := \frac{2-\frac{1}{2}-\frac{3}{r}}{\frac{r-6}{6r}} = 9 \cdot \frac{r-2}{r-6}$. We collect all these auxiliary functions bounding d in Table 4.11.

$d \equiv 0\,(2)$, $d \equiv 0\,(3)$	$d_{\max}(r)$	$12 \cdot \frac{r-2}{r-6}$
$d \equiv 1\,(3)$	$d_3(r)$	$2 \cdot \frac{5r-9}{r-6}$
$d \equiv 1\,(2)$	$d_2(r)$	$9 \cdot \frac{r-2}{r-6}$
$d \equiv 2\,(3)$	$d_{3,3}(r)$	$4 \cdot \frac{2r-3}{r-6}$
$d \equiv 1\,(2)$, $d \equiv 1\,(3)$	$d_{2,3}(r)$	$\frac{7r-12}{r-6}$

Table 4.11. Auxiliary functions bounding degree (in decreasing order)

$r = 54$ As previously remarked, there is one case to be considered when the base orbifold is $X = S(2,3,54)$ namely $d = 13$. Note that $d \equiv 1 \quad (2)$ and $d \equiv 1\,(3)$, and these two conditions imply that there should be at least one singular point of order 2 and one of order 3 in $\widetilde{X}$. Then

$$d \leqslant d_{2,3}(r) = \frac{2 - \frac{1}{2} - \frac{1}{3} - \frac{2}{54}}{-\frac{6-54}{6\cdot 54}} = \frac{7 \cdot 54 - 12}{54 - 6} \approx 7.6$$

and we immediately conclude that there is nothing to analyse.

$31 \leqslant r \leqslant 53$ In all these cases we have $13 < d_{\max}(r) < 14$, then we should only consider $d = 13$ as possible degree. Since $13 \equiv 1\,(2)$, we use $d_2(r)$ in Table 4.11 to reduce the range for d. We get $d \leqslant d_2(31) \approx 10.44$ and then we fall in Step I.

All the other proofs about compatibility are long, but not difficult: so we place them in Appendix B, and here we list in Tables 4.12 and 4.13 the hyperbolic candidate covers arising from case D. Together with those arising from C (9 candidate covers), they are all the hyperbolic candidate covers with $d \geqslant 13$ in the stratum C_4.

In conclusion, in degree $d \leqslant 12$ we have 81 hyperbolic candidate orbifold covers, while in degree $d \geqslant 13$ we have 60 hyperbolic candidate orbifold covers.

□

Summing up, the compatible data corresponding to a hyperbolic orbifold cover of the type $S(\alpha,\beta,\gamma,\delta) \dashrightarrow S(p,q,r)$ can be described as follows:

- There are 81 of them with $d \leqslant 12$ as listed in Tables 4.3 and 4.4;
- There are 60 of them with $d \geqslant 13$, of which 9 arise from group C, and are listed in Table 4.9), and 51 arise from group D and are listed in Tables 4.12 and 4.13).

We will now discuss the realizability of each of the 141 compatible data just mentioned. As a matter of fact we have already the information on realizability in the

Π_1, Π_2, Π_3	Associated cover	d
(2,...,2), (3,...,3), (14,1,1,1,1)	$S(14, 14, 14, 14) \dashrightarrow S(2, 3, 14)$	18
(2,...,2), (3,...,3,1), (13,1,1,1)	$S(3, 13, 13, 13) \dashrightarrow S(2, 3, 13)$	16
(2,...,2),(3,...,3,1,1),(12,1,1)	$S(3, 3, 12, 12) \dashrightarrow S(2, 3, 12)$	14
(2,...,2,1),(3,...,3),(12,1,1,1)	$S(2, 12, 12, 12) \dashrightarrow S(2, 3, 12)$	15
(2,...,2),(3,...,3,1),(12,2,1,1)	$S(3, 6, 12, 12) \dashrightarrow S(2, 3, 12)$	16
(2,...,2),(3,...,3),(12,2,2,1,1)	$S(6, 6, 12, 12) \dashrightarrow S(2, 3, 12)$	18
(2,...,2),(3,...,3),(12,3,1,1,1)	$S(4, 12, 12, 12) \dashrightarrow S(2, 3, 12)$	18
(2,...,2,1),(3,...,3,1),(11,1,1)	$S(2, 3, 11, 11) \dashrightarrow S(2, 3, 11)$	13
(2,...,2,1),(3,...,3,1),(10,2,1)	$S(2, 3, 5, 10) \dashrightarrow S(2, 3, 10)$	13
(2,...,2),(3,...,3,1,1),(10,2,2)	$S(3, 3, 5, 5) \dashrightarrow S(2, 3, 10)$	14
(2,...,2,1),(3,...,3),(10,2,2,1)	$S(2, 5, 5, 10) \dashrightarrow S(2, 3, 10)$	15
(2,...,2),(3,...,3,1),(10,2,2,2)	$^*S(3, 5, 5, 5) \dashrightarrow S(2, 3, 10)$	16
(2,...,2),(3,...,3),(10,5,1,1,1)	$S(2, 5, 5, 5) \dashrightarrow S(2, 3, 10)$	18
(2,...,2),(3,...,3),(10,10,1,1,1,1)	$S(10, 10, 10, 10) \dashrightarrow S(2, 3, 10)$	24
(2,...,2,1),(3,...,3,1),(9,3,1)	$S(2, 3, 3, 9) \dashrightarrow S(2, 3, 9)$	13
(2,...,2),(3,...,3,1),(9,3,3,1)	$S(3, 3, 3, 9) \dashrightarrow S(2, 3, 9)$	16
(2,...,2),(3,...,3,1,1),(9,9,1,1)	$S(3, 3, 9, 9) \dashrightarrow S(2, 3, 9)$	20
(2,...,2,1),(3,...,3),(9,9,1,1,1)	$S(2, 9, 9, 9) \dashrightarrow S(2, 3, 9)$	21
(2,...,2),(3,...,3),(9,9,3,1,1,1)	$^*S(3, 9, 9, 9) \dashrightarrow S(2, 3, 9)$	24

Table 4.12. Triples of partitions for $d \geqslant 13$ arising from case D, with hyperbolic associated candidate covers in C_4 (there are 51 of them); the label * means that the candidate cover will be exceptional

Π_1, Π_2, Π_3	Associated cover	d
(2,...,2,1),(3,...,3,1),(8,4,1)	$S(2,2,3,8) \dashrightarrow S(2,3,8)$	13
(2,...,2),(3,...,3,1,1),(8,4,2)	$S(2,3,3,4) \dashrightarrow S(2,3,8)$	14
(2,...,2,1),(3,...,3),(8,4,2,1)	$S(2,2,4,8) \dashrightarrow S(2,3,8)$	15
(2,...,2),(3,...,3,1),(8,4,2,2)	$*S(2,3,4,4) \dashrightarrow S(2,3,8)$	16
(2,...,2),(3,...,3,1,1,1,1),(8,8)	$S(3,3,3,3) \dashrightarrow S(2,3,8)$	16
(2,...,2,1),(3,...,3,1,1),(8,8,1)	$S(2,3,3,8) \dashrightarrow S(2,3,8)$	17
(2,...,2,1,1),(3,...,3),(8,8,1,1)	$S(2,2,8,8) \dashrightarrow S(2,3,8)$	18
(2,...,2),(3,...,3,1,1,1),(8,8,2)	$S(3,3,3,4) \dashrightarrow S(2,3,8)$	18
(2,...,2,1),(3,...,3,1),(8,8,2,1)	$S(2,3,4,8) \dashrightarrow S(2,3,8)$	19
(2,...,2),(3,...,3,1,1),(8,8,2,2)	$S(3,3,4,4) \dashrightarrow S(2,3,8)$	20
(2,...,2,1),(3,...,3),(8,8,2,2,1)	$*S(2,4,4,8) \dashrightarrow S(2,3,8)$	21
(2,...,2),(3,...,3,1),(8,8,4,1,1)	$*S(2,3,8,8) \dashrightarrow S(2,3,8)$	22
(2,...,2),(3,...,3,1),(8,8,2,2,2)	$*S(3,4,4,4) \dashrightarrow S(2,3,8)$	22
(2,...,2),(3,...,3),(8,8,2,2,2,2)	$S(4,4,4,4) \dashrightarrow S(2,3,8)$	24
(2,...,2),(3,...,3),(8,8,4,2,1,1)	$S(2,4,8,8) \dashrightarrow S(2,3,8)$	24
(2,...,2),(3,...,3,1,1),(8,8,8,1,1)	$S(3,3,8,8) \dashrightarrow S(2,3,8)$	26
(2,...,2,1),(3,...,3),(8,8,8,1,1,1)	$S(2,8,8,8) \dashrightarrow S(2,3,8)$	27
(2,...,2),(3,...,3,1),(8,8,8,2,1,1)	$S(3,4,8,8) \dashrightarrow S(2,3,8)$	28
(2,...,2),(3,...,3),(8,8,8,2,2,1,1)	$S(4,4,8,8) \dashrightarrow S(2,3,8)$	30
(2,...,2),(3,...,3),(8,8,8,8,1,1,1,1)	$S(8,8,8,8) \dashrightarrow S(2,3,8)$	36
(2,...,2,1,1),(3,...,3,1,1),(7,7)	$S(2,2,3,3) \dashrightarrow S(2,3,7)$	14
(2,...,2,1),(3,...,3,1,1,1),(7,7,7)	$S(2,3,3,3) \dashrightarrow S(2,3,7)$	21
(2,...,2,1,1),(3,...,3,1),(7,7,7,1)	$S(2,2,3,7) \dashrightarrow S(2,3,7)$	22
(2,...,2),(3,...,3,1,1,1,1),(7,7,7,7)	$S(3,3,3,3) \dashrightarrow S(2,3,7)$	28
(2,...,2,1),(3,...,3,1,1),(7,7,7,7,1)	$S(2,3,3,7) \dashrightarrow S(2,3,7)$	29
(2,...,2,1,1),(3,...,3),(7,7,7,7,1,1)	$S(2,2,7,7) \dashrightarrow S(2,3,7)$	30
(2,...,2),(3,...,3,1,1,1),(7,7,7,7,7,1)	$S(3,3,3,7) \dashrightarrow S(2,3,7)$	36
(2,...,2,1),(3,...,3,1),	$S(2,3,7,7) \dashrightarrow S(2,3,7)$	37
(2,...,2),(3,...,3,1,1),(7,7,7,7,7,7,1,1)	$*S(3,3,7,7) \dashrightarrow S(2,3,7)$	44
(2,...,2,1),(3,...,3),(7,7,7,7,7,7,1,1,1)	$*S(2,7,7,7) \dashrightarrow S(2,3,7)$	45
(2,...,2),(3,...,3,1),(7,7,7,7,7,7,7,1,1,1)	$*S(3,7,7,7) \dashrightarrow S(2,3,7)$	52
(2,...,2),(3,...,3),(7,7,7,7,7,7,7,7,1,1,1,1)	$S(7,7,7,7) \dashrightarrow S(2,3,7)$	60

Table 4.13. Triples of partitions for $d \geqslant 13$ arising from case D, with hyperbolic associated candidate covers in C_4 (there are 51 of them); the label * means that the candidate cover will be exceptional

lists of data (Tables 4.9, 4.12, and 4.13), marking with an asterisk $*$ the exceptional ones. We prove all realizations using Grothendieck's *dessins d'enfant*, whereas, in order to prove exceptionality we employ some results from [19], together with two new derived techniques that enable us to easily treat the last group of examples.

Proof of 4.1.2. We will draw all the graphs corresponding to exceptions, but not all for realizations; we example one for each degree $d \leqslant 12$, and all realizations occurring for prime $d \geqslant 13$.

Realizations We have chosen to show dessins d'enfant only for the candidate covers in the data listed in the following table. Generally we assign to white vertices valences as in Π_1, and to black ones valences as in Π_3.

Π_1, Π_2, Π_3	Associated cover	d
(3,1,1), (4,1), (4,1)	$S(3,3,4,4) \dashrightarrow S(3,4,4)$	5
(2,2,1,1), (5,1), (5,1)	$S(2,2,5,5) \dashrightarrow S(2,5,5)$	6
(3,3,1), (3,3,1), (5,1,1)	$S(3,3,5,5) \dashrightarrow S(3,3,5)$	7
(2,2,2,2), (5,1,1,1), (7,1)	$S(5,5,5,7) \dashrightarrow S(2,5,7)$	8
(2,2,2,1,1,1), (3,3,3), (8,1)	$S(2,2,2,8) \dashrightarrow S(2,3,8)$	9
(2,2,2,2,1,1), (3,3,3,1), (8,2)	$S(2,2,3,4) \dashrightarrow S(2,3,8)$	10
(2,. . . ,2,1), (4,4,2,1), (5,5,1)	$S(2,2,4,5) \dashrightarrow S(2,4,5)$	11
(2,. . . ,2,1,1), (3,3,3,3), (8,2,2)	$S(2,2,4,4) \dashrightarrow S(2,3,8)$	12

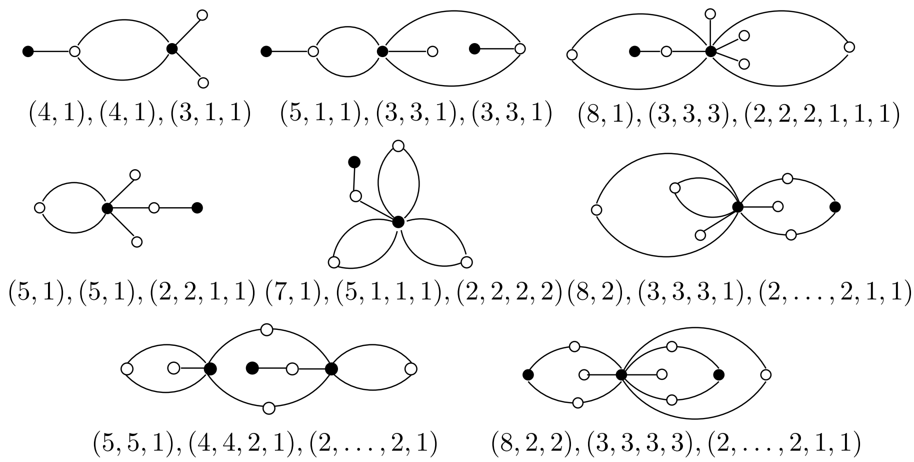

Figure 4.4. A sample of dessins d'enfant realizing candidate covers in $d \leqslant 12$: we assign to black vertices valences in Π_3, and to white ones valences in Π_1, and realize the data listed in the table above

Notice that since there are a lot of candidate covers for $d \geqslant 13$ we decided to show only those in prime degree, which are listed in Table 4.14. In Fig. 4.6 we draw their dessins.

Exceptions We list in Table 4.15 all hyperbolic exceptions.

Instead, we will not show exceptionality for all of them through drawings. We will extensively use two important and efficient tools, both introduced in [19]:

Π_1, Π_2, Π_3	Associated cover	d
(2,...,2,1), (3,...,3,1), (11,1,1)	$S(2,3,11,11) \dashrightarrow S(2,3,11)$	13
(2,...,2,1), (3,...,3,1), (10,2,1)	$S(2,3,5,10) \dashrightarrow S(2,3,10)$	13
(2,...,2,1), (3,...,3,1), (9,3,1)	$S(2,3,3,9) \dashrightarrow S(2,3,9)$	13
(2,...,2,1), (3,...,3,1), (8,4,1)	$S(2,2,3,8) \dashrightarrow S(2,3,8)$	13
(2,...,2,1), (3,...,3,1,1), (8,8,1)	$S(2,3,3,8) \dashrightarrow S(2,3,8)$	17
(2,...,2,1), (3,...,3,1), (8,8,2,1)	$S(2,3,4,8) \dashrightarrow S(2,3,8)$	19
(2,...,2,1),(3,...,3,1,1),(7,7,7,7,1)	$S(2,3,3,7) \dashrightarrow S(2,3,7)$	29
(2,...,2,1),(3,...,3,1),(7,7,7,7,7,1,1)	$S(2,3,7,7) \dashrightarrow S(2,3,7)$	37

Table 4.14. Candidate covers for large, prime degree

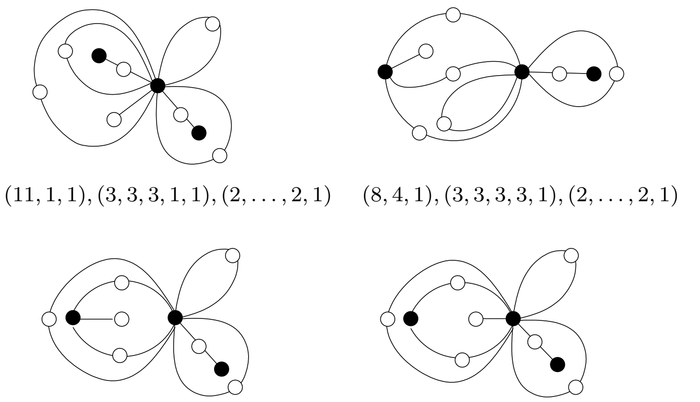

Figure 4.5. Dessins d'enfant realizing the first four candidate surface branched covers with large, prime degree listed in Table 4.14

VED the 'Very Even Data' criterion: let us consider the cover $\widetilde{\Sigma} \xrightarrow[\Pi_1,\Pi_2,\Pi_3]{d:1} \Sigma$; if d is even, and each element of Π_i is also even for $i = 1, 2$ then Π_3 should refine the partition $(d/2, d/2)$.

BD [19] Propositions 4.14 and 4.15, the 'Block Decomposition' method, for the exceptions in $d = 12$.

These two tools apply to many of the candidate covers in Table 4.15. Among the remaining cases, for three of them it is very easy to deduce exceptionality through the classical technique of dessins d'enfant. Precisely:

- $(2,2,2,2),(3,2,2,1),(4,4)$: let us fix 2-valent white vertices as middle points of four edges, and fix two black vertices; then we join them to form one 6-gon, two 4-gons, and one bigon. Since the graph has to be connected, the black vertices should have valence 3 and 5, respectively.

- $(3,3,3),(3,3,3),(5,1,1,1,1)$: here we have six vertices, three white and three black, all 3-valent. Let us start to join them: first form three bigons. While trying to draw the last bigon, we are forced to make the graph disconnected.

Π_1, Π_2, Π_3	Associated cover	d	Exceptionality via
(2,2,2,2), (3,2,2,1), (4,4)	$S(2,3,3,6) \dashrightarrow S(2,4,6)$	8	dessins
(2,2,2,2), (5,1,1,1), (6,2)	$S(3,5,5,5) \dashrightarrow S(2,5,6)$	8	VED
(2,2,2,2), (3,3,1,1), (5,3)	$S(3,3,3,5) \dashrightarrow S(2,3,15)$	8	dessins
(2,2,2,2), (4,2,2), (6,1,1)	$S(2,2,6,6) \dashrightarrow S(2,4,6)$	8	VED
(3,3,3), (3,3,3), (5,1,1,1,1)	$S(5,5,5,5) \dashrightarrow S(3,3,5)$	9	dessins
(3,3,3), (3,3,3), (4,2,1,1,1)	$S(2,4,4,4) \dashrightarrow S(3,3,4)$	9	dessins
(2,2,2,2,1), (3,3,3), (5,2,2)	$S(2,2,5,5) \dashrightarrow S(2,3,10)$	9	dessins
(2,...,2), (4,4,2), (7,1,1,1)	$S(2,7,7,7) \dashrightarrow S(2,4,7)$	10	VED
(2,...,2), (4,4,2), (6,2,1,1)	$S(2,3,6,6) \dashrightarrow S(2,4,6)$	10	VED
(2,...,2), (3,3,3,1), (4,3,3)	$S(3,3,4,4) \dashrightarrow S(2,3,12)$	10	dessins
(2,...,2), (4,4,4), (8,1,1,1,1)	$S(8,8,8,8) \dashrightarrow S(2,4,8)$	12	VED
(2,...,2), (3,3,3,3), (7,3,1,1)	$S(3,7,21,21) \dashrightarrow S(2,3,21)$	12	BD
(2,...,2), (3,3,3,3), (7,2,2,1)	$S(2,7,7,14) \dashrightarrow S(2,3,14)$	12	BD
(2,...,2), (3,3,3,3), (6,4,1,1)	$S(2,3,12,12) \dashrightarrow S(2,3,12)$	12	BD
(2,...,2), (3,3,3,3), (5,4,2,1)	$S(4,5,10,20) \dashrightarrow S(2,3,20)$	12	BD
(2,...,2), (3,3,3,3), (5,3,3,1)	$S(3,5,5,15) \dashrightarrow S(2,3,15)$	12	BD
(2,...,2), (3,3,3,3), (5,3,2,2)	$S(6,10,15,15) \dashrightarrow S(2,3,30)$	12	BD
(2,...,2), (3,3,3,3), (4,4,3,1)	$S(3,3,4,12) \dashrightarrow S(2,3,12)$	12	BD
(2,...,2), (3,3,3,3), (4,3,3,2)	$S(3,4,4,6) \dashrightarrow S(2,3,12)$	12	BD
(2,...,2),(3,...,3,1),(10,2,2,2)	$S(3,5,5,5) \dashrightarrow S(2,3,10)$	16	VED
(2,...,2), (3,...,3,1),(8,4,2,2)	$S(2,3,4,4) \dashrightarrow S(2,3,8)$	16	VED
(2,...,2), (4,4,4,4,2),(5,5,5,1,1,1)	$S(5,5,5,2) \dashrightarrow S(2,4,5)$	18	VED
(2,...,2,1),(3,...,3),(8,8,2,2,1)	$S(2,2,4,5) \dashrightarrow S(2,4,5)$	21	dessins
(2,...,2),(3,...,3,1),(8,8,4,1,1)	$S(2,2,4,5) \dashrightarrow S(2,4,5)$	22	dessins
(2,...,2),(3,...,3,1),(8,8,2,2,2)	$S(2,2,4,5) \dashrightarrow S(2,4,5)$	22	VED
(2,...,2),(3,...,3),(9,9,3,1,1,1)	$S(2,2,4,5) \dashrightarrow S(2,4,5)$	24	dessins
(2,...,2),(3,...,3,1,1),(7,...,7,1,1)	$S(2,2,4,5) \dashrightarrow S(2,4,5)$	44	dessins
(2,...,2,1),(3,...,3),(7,...,7,1,1,1)	$S(2,2,4,5) \dashrightarrow S(2,4,5)$	45	dessins
(2,...,2),(3,...,3,1),(7,...,7,1,1,1,1)	$S(2,2,4,5) \dashrightarrow S(2,4,5)$	52	dessins

Table 4.15. Exceptional covers

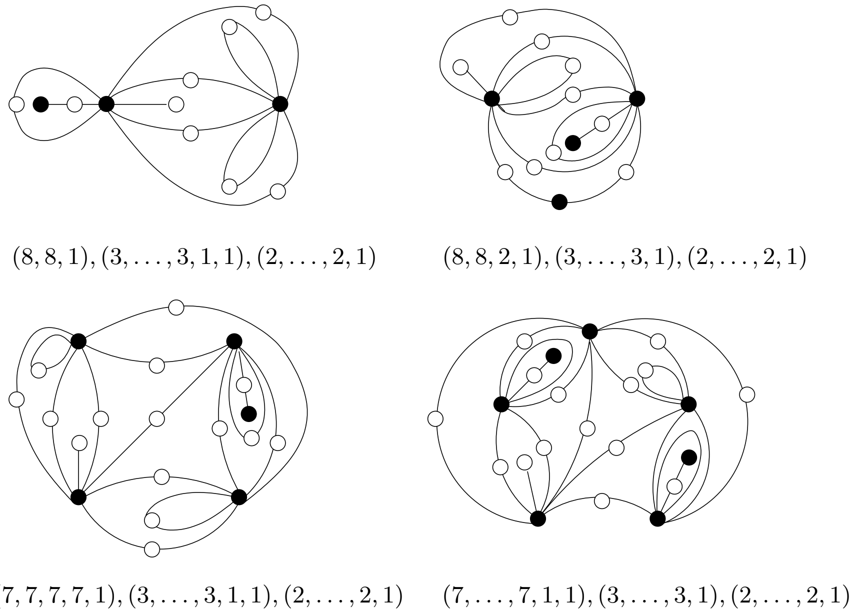

Figure 4.6. Dessins d'enfant realizing the last four candidate surface branched covers with large, prime degree listed in Table 4.14

- $(3,3,3),(3,3,3),(4,2,1,1,1)$: we begin exactly as in the previous case and when drawing the 8-gon we are forced to make the graph disconnected.

Π_1,Π_2,Π_3	Associated cover	d
(2,2,2,2), (3,3,1,1), (5,3)	$S(3,3,3,5) \dashrightarrow S(2,3,15)$	8
(2,2,2,2,1), (3,3,3), (5,2,2)	$S(2,2,5,5) \dashrightarrow S(2,3,10)$	9
(2,...,2), (3,3,3,1), (4,3,3)	$S(3,3,4,4) \dashrightarrow S(2,3,12)$	10
(2,...,2,1),(3,...,3),(8,8,2,2,1)	$S(2,4,4,8) \dashrightarrow S(2,3,8)$	21
(2,...,2),(3,...,3,1),(8,8,4,1,1)	$S(2,3,8,8) \dashrightarrow S(2,3,8)$	22
(2,...,2),(3,...,3),(9,9,3,1,1,1)	$S(3,9,9,9) \dashrightarrow S(2,3,9)$	24
(2,...,2),(3,...,3,1,1),(7,...,7,1,1)	$S(3,3,7,7) \dashrightarrow S(2,3,7)$	44
(2,...,2,1),(3,...,3),(7,...,7,1,1,1)	$S(2,7,7,7) \dashrightarrow S(2,3,7)$	45
(2,...,2),(3,...,3,1),(7,...,7,1,1,1)	$S(3,7,7,7) \dashrightarrow S(2,3,7)$	52

Table 4.16. Hyperbolic covers, whose exceptionality is proved in Appendix C

To conclude the proof of exceptionality of all the data listed in Table 4.15 we are then only left to consider those in Table 4.16. Since proving via dessins d'enfant that a candidate cover is exceptional could be very difficult, especially in large degree, in the next section we expose two more techniques that we employ in the sequel to show exceptionality of the remaining hyperbolic candidate covers in Table 4.16: graph moves and geometric gluings.

We address the reader to Appendix C for the details of each exceptionality proof.

□

4.3 Other techniques: towards a geometric viewpoint

We will now describe two more methods we employ to attack the Hurwitz existence problem. The former is merely a variation on the idea of dessins d'enfant, but it has some practical advantages. The latter should in our opinion open the way to a geometric understanding of the problem in the hyperbolic case too.

Graph moves In order to simplify the discussion of the dessins d'enfant associated to orbifold covers of $S(2,3,r)$, we try to exploit to the greatest extent a sort of rigidity due to the prevalence of 6-gons in all those graphs. In all these cases we assign to the white vertices the valences as prescribed by the partition of the degree over the branching point of order 2 and to the black ones we assign those valences prescribed by the partition over r. The idea is to work locally on these forced situations, performing some graph moves, that just simplify the whole dessin, or reduce it to a known case. Note that performing a local move on a dessin corresponds to changing the associated cover. Moreover, if we can show that a partially constructed dessin for some datum can be trasformed via some moves to a partially constructed dessin for another datum that we know to be exceptional, then the original datum is also exceptional. An example will clarify the exact meaning of this statement.

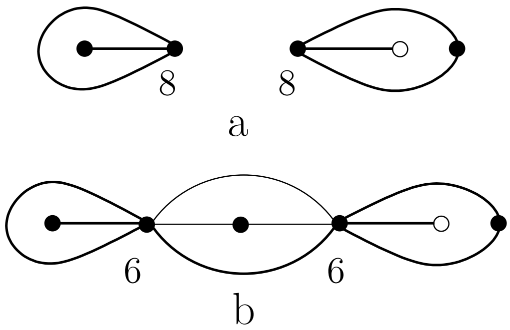

Figure 4.7. From d to $d-2$

Here we provide an example of what we mean: consider the set of partitions $(\Pi_1,\Pi_2,\Pi_3) = ((2,\ldots,2,1),(3,\ldots,3),(8,8,2,2,1))$, associated to the hyperbolic candidate cover $S(2,4,4,8) \overset{21:1}{\dashrightarrow} S(2,3,8)$, listed in Table 4.16. We discuss how a dessin D realizing it has to be; we assume Π_1 be the valences of the white vertices, and Π_3 of the black ones:

- The connectedness required for D prevents the two black vertices of valence 2 from being joined to the same black 8.
- Since there exists one black vertex v of valence 1, and Π_2 contains only 3's, D must be locally, near v, as in Fig. 4.7a-left; similarly, one can show that near the white vertex of valence 1 the graph looks like Fig. 4.7a-right.
- Then, the graph will be as in Fig. 4.7b: we delete the three thin edges.
- Now we conclude, because we have obtained a graph corresponding to a new partition: $(2,\ldots,2,1),(3,\ldots,3),(6,6,2,1)$; and we proved in Theorem 3.3.5 that this is the partition of an exceptional cover in degree 15.

In Appendix C we use this technique to show exceptionality for some candidate covers among those listed in Table 4.15.

Towards a geometric viewpoint via geometric gluings Let D be a fundamental domain of X, namely a hyperbolic $2k$-gon with isometries $\varphi_1, \ldots, \varphi_k$ between pairs of its edges, such that X is D modulo $\{\varphi_1, \ldots, \varphi_k\}$. Notice that only the images in X of the vertices of D can be cone points of X, and one of them v has order p if the angles of D at the preimages of v sum up to $2\pi/p$.

Now take d copies $D_1, \ldots, D_d$ of D arranged in $\mathbb{H}$ so that their union is a (possibly non-convex) $2h$-gon $\widetilde{D}$ and whenever $D_i \cap D_j$ contains more than one point then it consists of an edge of both (that is an isometric gluing along that edge), and upon identifying D_i and D_j with D the superposition of these edges corresponds to one of the isometries φ_m.

Suppose furthermore that other h isometries $\eta_1, \ldots, \eta_h$ between pairs of edges of $\widetilde{D}$ are given, in such a way that upon to identifying with D the D_i's involved in some η_n, this isometry corresponds to some φ_m. Then one has an orbifold $\widetilde{X}$ obtained as the quotient of $\widetilde{D}$ modulo $\{\eta_1, \ldots, \eta_h\}$ and an orbifold cover $f : \widetilde{X} \to X$ of degree d. The orbifold cover f maps the interior of each $D_i \subset \widetilde{D}$ to the interior of D through the identity; moreover, by construction both the identifications between edges in the interior of $\widetilde{D}$, and the isometries $\eta_1, \ldots, \eta_h$ on the boundary of $\widetilde{D}$ project to isometries in $\{\varphi_1, \ldots, \varphi_k\}$. Now we want to recover from f the conic points of $\widetilde{X}$ and their orders. As before, only the preimages through f of the singularities of X can be cone points of $\widetilde{X}$. For any $\tilde{v} \in f^{-1}(v)$, where v is a conic point of order p in X, we count how many D_i's contain $\tilde{v}$ in $\widetilde{X}$: say $q(\tilde{v})$ this number; then the order of $\tilde{v}$ is $p/q(\tilde{v})$ (possibly 1, and in such a case $\tilde{v}$ must be a regular point of $\widetilde{X}$).

On the other hand, consider an orbifold cover $f : \widetilde{X} \to X$ of degree d, where $X = S(p_1, p_2, p_3)$, complemented with the following cover instruction:

$$(p_{11}, \ldots, p_{1m_1}) \dashrightarrow p_1, \qquad \ldots \qquad (p_{31}, \ldots, p_{3m_3}) \dashrightarrow p_3.$$

With these data we are able to recover f as

$$\widetilde{D}/_{\{\eta_1,\ldots,\eta_h\}} \to D/_{\{\varphi_1,\varphi_2\}}.$$

A fundamental domain D of X is the union of two $\Delta(p_1, p_2, p_3)$'s along an edge ℓ, and without loss of generality we suppose that in both the copies of D the edge ℓ joins the cone point of order p_1 to that of order p_2. Then the isometry φ_1 should identify isometrically the two edges containing p_1 different from ℓ, while φ_2 identifies isometrically the remaining two edges. First of all we compute the three partitions of the degree $(d_{ij})_{j=1,\ldots,n_1}$ for $i = 1, 2, 3$, over each conic point of X (remind that $d_{ij} = p_i/p_{ij}$). From the partitions $(d_{ij})^{i=1,2,3}_{j=1,\ldots,n_1}$ we recover how to arrange the d copies of D in $\mathbb{H}$: we start to glue together d_{1j} copies of D around the j-th preimage of the point v_1 of order p_1 in X, and we repeat this procedure for each point in $f^{-1}(v_1)$. At this stage we have d copies of D arranged in n_1 connected components. We continue gluing together this connected components as prescribed by the partition d_{2j}, paying attention to perform only those gluings that decrease the number of connected components. And we do the same for the last partition:

in the end we get one connected component: $\widetilde{D}$. All the other gluings not yet performed correspond to the set of isometries $\{\eta_1, \ldots, \eta_h\}$, and finally we have

$$\widetilde{D}/_{\{\eta_1,\ldots,\eta_h\}} \to D/_{\{\varphi_1,\varphi_2\}}.$$

Notice that if in the instruction we omit p_{ij} when it is equal to 1, then for each of these omitted 1's in $\widetilde{X}$ we should count p_i copies of D around the same (non-singular) point.

Remark 4.3.1. This way of realizing orbifold covers is clearly equivalent to the dessins d'enfant method: actually, it could be seen as its dual. But once you have fixed a geometric structure on the base orbifold X, through the choice of a fundamental domain D and the isometries acting on it, the method just introduced gives us a pull-back of the geometric structure on the base to a geometric structure on $\widetilde{X}$ that is the right choice for realizing the orbifold cover.

We exhibit in Fig. 4.8 and in Fig. 4.9 four examples of realization, respectively:

- $S(2,2,2,4) \xrightarrow[(2,1,1,1),(4,1),(5)]{5:1} S(2,4,5)$;
- $S(4,4,4,4) \xrightarrow[(3,3),(4,1,1),(4,1,1)]{6:1} S(3,4,4)$;
- $S(2,2,2,3) \xrightarrow[(2,2,1,1,1),(3,3,1),(7)]{7:1} S(2,3,7)$;
- $S(5,5,5,5) \xrightarrow[(2,\ldots,2),(4,\ldots,4),(5,\ldots,5,1,1,1,1)]{24:1} S(2,4,5)$.

For each cover we draw $\widetilde{D}$ and D, labeling each vertex in D with its order, and each vertex in $\widetilde{D}$ with the order of its image in X. Hence each angle having as vertex a point labeled with n measures π/n.

This method is effective for proving exceptionality, too. As an easy example we discuss exceptionality of partition

$$(4,4), (3,2,2,1), (2,2,2,2),$$

associated to the hyperbolic candidate cover $S(2,3,3,6) \xrightarrow{8:1} S(2,4,6)$. For these example we refer to Fig. 4.10. We first draw a fundamental domain D of the base orbifold. Then we note that the preimages of the conic point of order 4 in the base are all non-singular; consequently we have two congruent regions P_1 and P_2, as in part I of Fig. 4.10, which have at the center the point projecting to 4 in the base, and at the boundary four points projecting to 6 and four to 2, that alternate.

Now, we want to glue P_1 and P_2 in order to get a sphere, with all points projecting to 2 being non-singular, and with only one singular point of order 6. The first two facts force us to glue these two patches along an edge as in part II of Fig. 4.10; then, without loss of generality, we can choose one conic point of order 6, and we fix it to be the singular one. (See part III of Fig. 4.10). As it is clear from the last group of drawings in Fig. 4.10 the three remaining choices of gluings do not lead to a realization of $S(2,3,3,6) \xrightarrow{8:1} S(2,4,6)$. In fact when the edge labeled a is glued to the edge d we obtain the cover $S(2,2,6,6) \to S(2,4,6)$; in the other two cases (when $a \to b$ or $a \to c$) we have in $\widetilde{X}$ at least a conic point of order 2, and it is forbidden for a cover associated to the partition $((2,2,2,2),(3,2,2,1),(4,4))$.

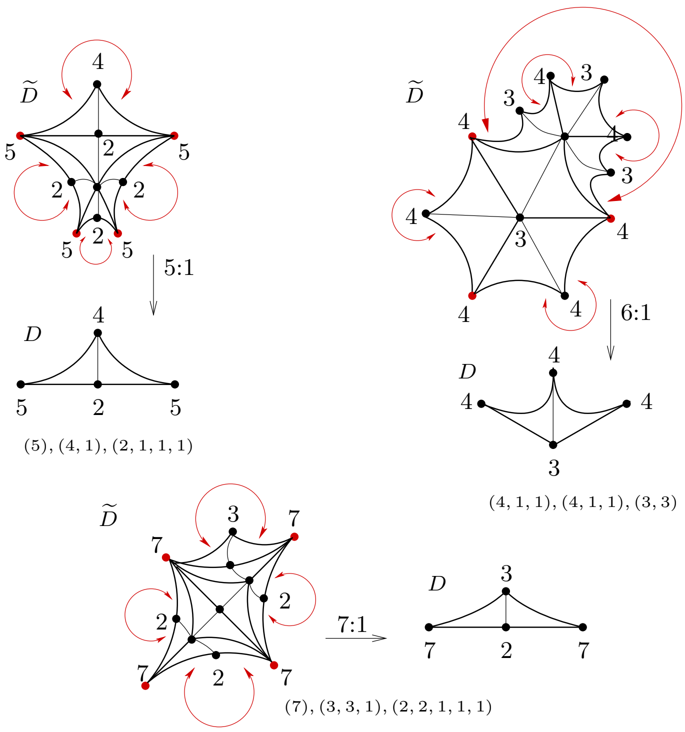

Figure 4.8. Fundamental domains and geometric gluings associated to the partitions ((2, 1, 1, 1), (4, 1), (5)), ((3, 3), (4, 1, 1), (4, 1, 1)), and ((2, 2, 1, 1, 1), (3, 3, 1), (7))

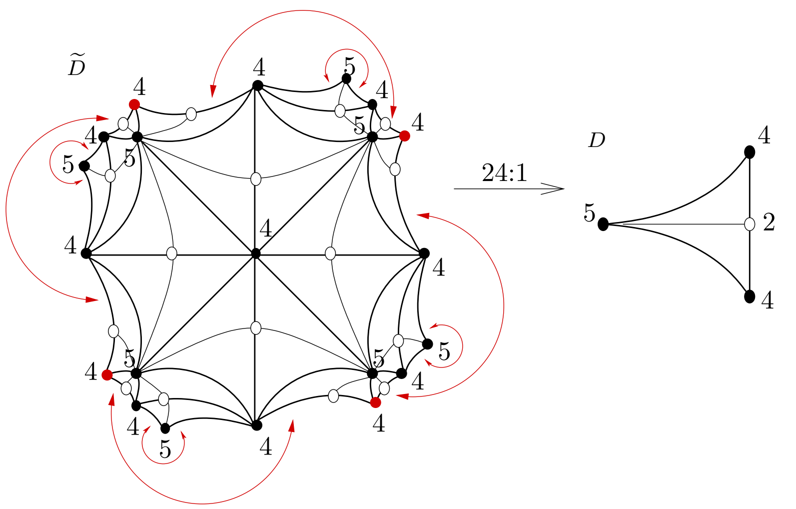

Figure 4.9. Fundamental domains and geometric gluings associated to the partition ((2, . . . , 2), (4, . . . , 4), (5, 5, 5, 5, 1, 1, 1, 1)); all vertices projecting to the cone point of order 2 in the base are white

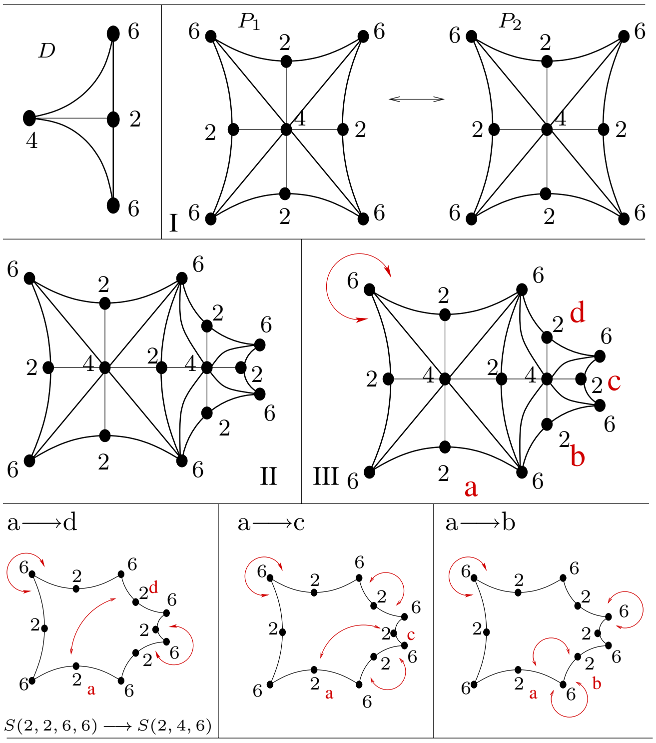

Figure 4.10. Exceptionality of $S(2,3,3,6) \overset{8:1}{\dashrightarrow} S(2,4,6)$

Appendix A

Compatibility I

A.1 Small degree

This appendix contains all detailed computations about compatibility of hyperbolic covers of type $S(\alpha, \beta, \gamma, \delta) \dashrightarrow S(p, q, r)$ with $d \leqslant 12$, divided with respect to degree, that are very similar to those made in Section 4.1. We repeat here the definitions and the scheme used for listing the compatible cases. If Π is a partition of an integer d, let us denote by $\ell(\Pi)$ its length (as usual), and by $c(\Pi)$ the number of entries in Π which are different from l.c.m.(Π). In order to belong to our family, a candidate surface branched cover of degree $d \geqslant 2$ must have the following properties:

- The number of branching points is 3;
- If the partitions of d are Π_1, Π_2, Π_3 then $c(\Pi_1) + c(\Pi_2) + c(\Pi_3) = 4$.

To list all such candidate covers for a given d then one has to:

- List all the partitions Π of d with $c(\Pi) \leqslant 4$;
- Find all possible triples (Π_1, Π_2, Π_3) of partitions such that $\ell(\Pi_1) + \ell(\Pi_2) + \ell(\Pi_3) = d + 2$ and $c(\Pi_1) + c(\Pi_2) + c(\Pi_3) = 4$.

We have done this for $2 \leqslant d \leqslant 12$ and then we have singled out the candidate covers having associated hyperbolic 2-orbifold covers. The output of these computations is the list of the 41 hyperbolic candidate covers in Table 4.3 and the 40 in Table 4.4.

$2 \leqslant d \leqslant 4$ In all these cases, there are no hyperbolic candidate covers.

$d = 5$ The partitions Π of 5 with $c(\Pi) \leqslant 4$ are those described in Table A.1, with the corresponding values of ℓ and c.

Π	(5)	(4,1)	(3,2)	(3,1,1)	(2,2,1)	(2,1,1,1)
ℓ	1	2	2	3	3	4
c	0	1	2	2	1	3

Table A.1. The partitions Π of 5 with $c(\Pi) \leqslant 4$

The triples of such partitions such that ℓ and c sum up to 7 and 4 respectively are shown in Table A.2, together with the associated candidate orbifold cover and its geometric type.

Π_1, Π_2, Π_3			Associated cover	Geometry
(2,1,1,1)	(4,1)	(5)	$S(2,2,2,4) \dashrightarrow S(2,4,5)$	$\mathbb{H}$
(3,1,1)	(3,1,1)	(5)	$S(3,3,3,3) \dashrightarrow S(3,3,5)$	$\mathbb{H}$
(3,1,1)	(4,1)	(4,1)	$S(3,3,4,4) \dashrightarrow S(3,4,4)$	$\mathbb{H}$
(2,2,1)	(3,2)	(4,1)	$S(2,2,3,4) \dashrightarrow S(2,4,6)$	$\mathbb{H}$

Table A.2. Triples of partitions of 5 having associated candidate covers in C_4: there are 4 of them, all hyperbolic

$d = 6$ The partitions Π of 6 with $c(\Pi) \leqslant 4$ are those described in Table A.3, with the corresponding values of ℓ and c.

Π	(6)	(5,1)	(4,2)	(3,3)	(4,1,1)
ℓ	1	2	2	2	3
c	0	1	1	0	2
Π	(3,2,1)	(2,2,2)	(3,1,1,1)	(2,2,1,1)	(2,1,1,1,1)
ℓ	3	3	4	4	5
c	3	0	3	2	4

Table A.3. The partitions Π of 6 with $c(\Pi) \leqslant 4$

Now ℓ and c should sum up to 8 and 4 respectively. From Table A.4 it is easy to deduce that there are 7 associated candidate orbifold covers of the family we are discussing.

Π_1, Π_2, Π_3			Associated cover	Geometry
(2,1,1,1,1)	(3,3)	(6)	$S(2,2,2,2) \dashrightarrow S(2,3,6)$	$\mathbb{E}$
(2,1,1,1)	(4,1,1)	(6)	$S(2,2,4,4) \dashrightarrow S(2,4,6)$	$\mathbb{H}$
(2,2,1,1)	(5,1)	(5,1)	$S(2,2,5,5) \dashrightarrow S(2,5,5)$	$\mathbb{H}$
(2,2,1,1)	(4,2)	(5,1)	$S(2,2,2,5) \dashrightarrow S(2,4,5)$	$\mathbb{H}$
(2,2,1,1)	(4,2)	(4,2)	$S(2,2,2,2) \dashrightarrow S(2,4,4)$	$\mathbb{E}$
(3,1,1,1)	(3,3)	(5,1)	$S(3,3,3,5) \dashrightarrow S(3,3,5)$	$\mathbb{H}$
(3,1,1,1)	(3,3)	(4,2)	$S(2,3,3,3) \dashrightarrow S(3,3,4)$	$\mathbb{H}$
(3,3)	(4,1,1)	(4,1,1)	$S(4,4,4,4) \dashrightarrow S(3,4,4)$	$\mathbb{H}$
(2,2,2)	(3,2,1)	(5,1)	$S(2,3,5,6) \dashrightarrow S(2,5,6)$	$\mathbb{H}$
(2,2,2)	(3,2,1)	(4,2)	$S(2,2,3,6) \dashrightarrow S(2,4,6)$	$\mathbb{H}$

Table A.4. Triples of partitions of 6 having associated candidate covers in C_4: among them there are 8 hyperbolic ones

$d = 7$ The partitions Π of 7 with $c(\Pi) \leqslant 4$ are in Table A.5. As in the previous case, in Table A.6 we list the associate candidate covers.

Π	(7)	(6,1)	(5,2)	(4,3)	(5,1,1)	(4,2,1)	(3,3,1)
ℓ	1	2	2	2	3	3	3
c	0	1	2	2	2	3	1
Π	(3,2,2)	(4,1,1,1)	(3,2,1,1)	(2,2,2,1)	(3,1,1,1,1)	(2,2,1,1,1)	
ℓ	3	4	4	4	5	5	
c	3	3	4	1	4	3	

Table A.5. The partitions Π of 7 with $c(\Pi) \leqslant 4$

Π_1, Π_2, Π_3			Associated cover	Geometry
(2,2,1,1,1)	(3,3,1)	(7)	$S(2,2,2,3) \dashrightarrow S(2,3,7)$	$\mathbb{H}$
(2,2,2,1)	(4,1,1,1)	(7)	$S(2,4,4,4) \dashrightarrow S(2,4,7)$	$\mathbb{H}$
(3,3,1)	(3,3,1)	(5,1,1)	$S(3,3,5,5) \dashrightarrow S(3,3,5)$	$\mathbb{H}$
(3,3,1)	(3,3,1)	(4,2,1)	$S(2,3,3,4) \dashrightarrow S(3,3,4)$	$\mathbb{H}$
(2,2,2,1)	(3,3,1)	(5,2)	$S(2,2,3,5) \dashrightarrow S(2,3,10)$	$\mathbb{H}$
(2,2,2,1)	(3,3,1)	(4,3)	$S(2,3,3,4) \dashrightarrow S(2,3,12)$	$\mathbb{H}$
(2,2,2,1)	(4,2,1)	(6,1)	$S(2,2,4,6) \dashrightarrow S(2,4,6)$	$\mathbb{H}$
(2,2,2,1)	(5,1,1)	(6,1)	$S(2,5,5,6) \dashrightarrow S(2,5,6)$	$\mathbb{H}$

Table A.6. Triples of partitions of 7 having associated candidate covers in C_4: there are 8 of them, all hyperbolic

From now on we will just put the tables corresponding to each case, with the only comment on the notation

$$^*\widetilde{X} \dashrightarrow X$$

which means that the cover is compatible, but not realizable (as proved independently below).

Π	(8)	(7,1)	(6,2)	(5,3)	(4,4)	(6,1,1)
ℓ	1	2	2	2	2	3
c	0	1	1	2	0	2
Π	(5,2,1)	(4,3,1)	(4,2,2)	(5,1,1,1)	(4,2,1,1)	(3,3,1,1)
ℓ	3	3	3	4	4	4
c	3	3	2	3	3	2
Π	(3,2,2,1)	(2,2,2,2)	(4,1,1,1,1)	(2,2,2,1,1)	(2,2,1,1,1,1)	
ℓ	4	4	5	5	6	
c	4	0	4	2	4	

Table A.7. The partitions Π of 8 with $c(\Pi) \leqslant 4$

Π_1, Π_2, Π_3			Associated cover	Geometry
(2,2,1,1,1,1)	(4,4)	(4,4)	$S(2,2,2,2) \dashrightarrow S(2,4,4)$	$\mathbb{E}$
(2,2,2,2)	(4,1,1,1,1)	(8)	$S(4,4,4,4) \dashrightarrow S(2,4,8)$	$\mathbb{H}$
(2,2,2,1,1)	(3,3,1,1)	(8)	$S(2,2,3,3) \dashrightarrow S(2,3,8)$	$\mathbb{H}$
(2,2,2,2)	(3,2,2,1)	(4,4)	$^*S(2,3,3,6) \dashrightarrow S(2,4,6)$	$\mathbb{H}$
(2,2,2,2)	(5,1,1,1)	(7,1)	$S(5,5,5,7) \dashrightarrow S(2,5,7)$	$\mathbb{H}$
(2,2,2,2)	(5,1,1,1)	(6,2)	$^*S(5,5,5,3) \dashrightarrow S(2,5,6)$	$\mathbb{H}$
(2,2,2,2)	(4,2,1,1)	(7,1)	$S(2,4,4,7) \dashrightarrow S(2,4,7)$	$\mathbb{H}$
(2,2,2,2)	(4,2,1,1)	(6,2)	$S(4,4,4,4) \dashrightarrow S(2,4,8)$	$\mathbb{H}$
(2,2,2,2)	(3,3,1,1)	(5,3)	$^*S(3,3,3,5) \dashrightarrow S(2,3,15)$	$\mathbb{H}$
(3,3,1,1)	(3,3,1,1)	(4,4)	$S(3,3,3,3) \dashrightarrow S(3,3,4)$	$\mathbb{H}$
(2,2,2,2)	(6,1,1)	(6,1,1)	$S(6,6,6,6) \dashrightarrow S(2,6,6)$	$\mathbb{H}$
(2,2,2,2)	(4,2,2)	(6,1,1)	$^*S(2,2,6,6) \dashrightarrow S(2,4,6)$	$\mathbb{H}$
(2,2,2,2)	(4,2,2)	(4,2,2)	$S(2,2,2,2) \dashrightarrow S(2,4,4)$	$\mathbb{E}$

Table A.8. Triples of partitions of 8 having associated candidate cover in C_4: there are 11 hyperbolic ones among them

Π	(9)	(8,1)	(7,2)	(6,3)	(5,4)	(7,1,1)
ℓ	1	2	2	2	2	3
c	0	1	2	1	2	2
Π	(6,2,1)	(5,3,1)	(5,2,2)	(4,4,1)	(4,3,2)	(3,3,3)
ℓ	3	3	3	3	3	3
c	2	3	3	1	3	0
Π	(6,1,1,1)	(5,2,1,1)	(4,3,1,1)	(4,2,2,1)	(3,3,2,1)	(3,2,2,2)
ℓ	4	4	4	4	4	4
c	3	4	4	3	4	4
Π	(5,1,1,1,1)	(4,2,1,1,1)	(3,3,1,1,1)	(2,2,2,2,1)	(2,2,2,1,1,1)	
ℓ	5	5	5	5	6	
c	4	4	3	1	3	

Table A.9. The partitions Π of 9 with $c(\Pi) \leqslant 4$

Π_1, Π_2, Π_3			Associated cover	Geometry
(2,2,2,1,1,1)	(3,3,3)	(8,1)	$S(2,2,2,8) \dashrightarrow S(2,3,8)$	$\mathbb{H}$
(2,2,2,1,1,1)	(3,3,3)	(6,3)	$S(2,2,2,2) \dashrightarrow S(2,3,6)$	$\mathbb{E}$
(2,2,2,2,1)	(3,3,1,1,1)	(9)	$S(2,3,3,3) \dashrightarrow S(2,3,9)$	$\mathbb{H}$
(3,3,3)	(3,3,3)	(5,1,1,1,1)	$*S(5,5,5,5) \dashrightarrow S(3,3,5)$	$\mathbb{H}$
(3,3,3)	(3,3,3)	(4,2,1,1,1)	$*S(2,4,4,4) \dashrightarrow S(3,3,4)$	$\mathbb{H}$
(3,3,1,1,1)	(3,3,3)	(4,4,1)	$S(3,3,3,4) \dashrightarrow S(3,3,4)$	$\mathbb{H}$
(2,2,2,2,1)	(4,4,1)	(7,1,1)	$S(2,4,7,7) \dashrightarrow S(2,4,7)$	$\mathbb{H}$
(2,2,2,2,1)	(4,4,1)	(6,2,1)	$S(2,3,4,6) \dashrightarrow S(2,4,6)$	$\mathbb{H}$
(2,2,2,2,1)	(3,3,3)	(5,3,1)	$S(2,3,5,15) \dashrightarrow S(2,3,15)$	$\mathbb{H}$
(2,2,2,2,1)	(3,3,3)	(5,2,2)	$*S(2,2,5,5) \dashrightarrow S(2,3,10)$	$\mathbb{H}$
(2,2,2,2,1)	(3,3,3)	(4,3,2)	$S(2,3,4,6) \dashrightarrow S(2,3,12)$	$\mathbb{H}$

Table A.10. Triples of partitions of 9 having associated candidate cover in C_4: there are 10 hyperbolic ones among them

Π	(10)	(9,1)	(8,2)	(7,3)	(6,4)	(5,5)
ℓ	1	2	2	2	2	2
c	0	1	1	1	2	0
Π	(8,1,1)	(7,2,1)	(6,3,1)	(6,2,2)	(5,4,1)	(5,3,2)
ℓ	3	3	3	3	3	3
c	2	3	2	2	3	3
Π	(7,1,1,1)	(6,2,1,1)	(5,3,1,1)	(5,2,2,1)	(4,4,1,1)	(4,3,2,1)
ℓ	4	4	4	4	4	4
c	3	3	4	4	2	4
Π	(4,4,2)	(4,3,3)	(4,2,2,2)	(3,3,3,1)	(3,3,2,2)	(6,1,1,1,1)
ℓ	3	3	4	3	4	5
c	1	3	3	1	4	4
Π	(4,2,2,1,1)	(2,...,2)	(3,3,1,1,1,1)	(2,2,2,2,1,1)	(2,2,2,1,1,1,1)	
ℓ	5	5	6	6	7	
c	4	0	4	2	4	

Table A.11. The partitions Π of 10 with $c(\Pi) \leqslant 4$

Π_1, Π_2, Π_3			Associated cover	Geometry
(2,...,2)	(3,3,1,1,1,1)	(10)	$S(3,3,3,3) \dashrightarrow S(2,3,10)$	$\mathbb{H}$
(2,2,2,2,1,1)	(4,4,1,1)	(5,5)	$S(2,2,4,4) \dashrightarrow S(2,4,5)$	$\mathbb{H}$
(2,2,2,2,1,1)	(3,3,3,1)	(8,2)	$S(2,2,3,4) \dashrightarrow S(2,3,8)$	$\mathbb{H}$
(2,2,2,2,1,1)	(3,3,3,1)	(9,1)	$S(2,2,3,9) \dashrightarrow S(3,3,9)$	$\mathbb{H}$
(2,2,2,2,1,1)	(4,4,2)	(4,4,2)	$S(2,2,2,2) \dashrightarrow S(2,4,4)$	$\mathbb{E}$
(2,...,2)	(5,5)	(6,1,1,1,1)	$S(6,6,6,6) \dashrightarrow S(2,5,6)$	$\mathbb{H}$
(2,...,2)	(5,5)	(4,2,2,1,1)	$S(2,2,4,4) \dashrightarrow S(2,4,5)$	$\mathbb{H}$
(2,...,2)	(4,4,2)	(7,1,1,1)	$*S(2,7,7,7) \dashrightarrow S(2,4,7)$	$\mathbb{H}$
(2,...,2)	(4,4,2)	(6,2,1,1)	$*S(2,3,6,6) \dashrightarrow S(2,4,6)$	$\mathbb{H}$
(2,...,2)	(4,4,2)	(4,2,2,2)	$S(2,2,2,2) \dashrightarrow S(2,4,4)$	$\mathbb{E}$
(2,...,2)	(4,4,1,1)	(8,1,1)	$S(4,4,8,8) \dashrightarrow S(2,4,8)$	$\mathbb{H}$
(2,...,2)	(4,4,1,1)	(6,3,1)	$S(2,4,4,6) \dashrightarrow S(2,4,6)$	$\mathbb{H}$
(2,...,2)	(4,4,1,1)	(6,2,2)	$S(3,3,4,4) \dashrightarrow S(2,4,6)$	$\mathbb{H}$
(2,...,2)	(3,3,3,1)	(7,2,1)	$S(2,3,7,14) \dashrightarrow S(2,3,14)$	$\mathbb{H}$
(2,...,2)	(3,3,3,1)	(5,4,1)	$S(3,4,5,20) \dashrightarrow S(2,3,20)$	$\mathbb{H}$
(2,...,2)	(3,3,3,1)	(5,3,2)	$S(3,6,10,15) \dashrightarrow S(2,3,30)$	$\mathbb{H}$
(2,...,2)	(3,3,3,1)	(4,3,3)	$*S(3,3,4,4) \dashrightarrow S(2,3,12)$	$\mathbb{H}$
(3,3,3,1)	(3,3,3,1)	(4,4,1,1)	$S(3,3,4,4) \dashrightarrow S(3,3,4)$	$\mathbb{H}$

Table A.12. Triples of partitions of 10 having associated candidate cover in C_4: there are 16 hyperbolic ones among them

Π	(11)	(10,1)	(9,2)	(8,3)	(7,4)	(6,5)
ℓ	1	2	2	2	2	2
c	0	1	2	2	2	2
Π	(9,1,1)	(8,2,1)	(7,3,1)	(7,2,2)	(6,4,1)	(6,3,2)
ℓ	3	3	3	3	3	3
c	2	2	3	3	3	2
Π	(5,5,1)	(5,4,2)	(5,3,3)	(4,4,3)	(8,1,1,1)	(7,2,1,1)
ℓ	3	3	3	3	4	4
c	1	3	3	3	3	4
Π	(6,3,1,1)	(6,2,2,1)	(5,4,1,1)	(5,3,2,1)	(5,2,2,2)	(4,4,2,1)
ℓ	4	4	4	4	4	4
c	3	3	4	4	4	3
Π	(4,3,3,1)	(4,3,2,2)	(3,3,3,2)	(7,1,1,1,1)	(6,2,1,1,1)	(4,4,1,1,1)
ℓ	4	4	4	5	5	5
c	4	4	4	4	4	3
Π	(4,2,2,2,1)	(3,3,3,1,1)	(2,...,2,1)	(2,2,2,2,1,1,1)		
ℓ	5	5	6	7		
c	4	3	1	3		

Table A.13. The partitions Π of 11 with $c(\Pi) \leqslant 4$

Π_1, Π_2, Π_3			Associated cover	Geometry
(2,. . . ,2,1)	(3,3,3,1,1)	(10,1)	$S(2,3,3,10) \dashrightarrow S(2,3,10)$	$\mathbb{H}$
(2,. . . ,2,1)	(4,4,2,1)	(5,5,1)	$S(2,2,4,5) \dashrightarrow S(2,4,5)$	$\mathbb{H}$

Table A.14. Triples of partitions of 11 having associated candidate cover in C_4: both hyperbolic

Π	(12)	(11,1)	(10,2)	(9,3)	(8,4)	(7,5)
ℓ	1	2	2	2	2	2
c	0	1	1	1	1	2
Π	(6,6)	(10,1,1)	(9,2,1)	(8,3,1)	(8,2,2)	(7,4,1)
ℓ	2	3	3	3	3	3
c	0	2	3	3	2	3
Π	(7,3,2)	(6,5,1)	(6,4,2)	(6,3,3)	(5,5,2)	(5,4,3)
ℓ	3	3	3	3	3	3
c	3	3	3	2	3	3
Π	(4,4,4)	(9,1,1,1)	(8,2,1,1)	(7,3,1,1)	(7,2,2,1)	(6,4,1,1)
ℓ	3	4	4	4	4	4
c	0	3	3	4	4	4
Π	(6,3,2,1)	(6,2,2,2)	(5,5,1,1)	(3,3,3,3)	(5,4,2,1)	(5,3,3,1)
ℓ	4	4	4	4	4	4
c	3	3	2	0	4	4
Π	(5,3,2,2)	(4,4,3,1)	(4,4,2,2)	(4,3,3,2)	(8,1,1,1,1)	(6,3,1,1,1)
ℓ	4	4	4	4	5	5
c	4	4	2	4	4	4
Π	(6,2,2,1,1)	(4,4,2,1,1)	(4,2,2,2,2)	(4,4,1,1,1,1)	(3,3,3,1,1,1)	(2,. . . ,2)
ℓ	5	5	5	6	6	6
c	4	3	4	4	3	0
Π	(2,. . . ,2,1,1)	(2,2,2,2,1,1,1,1)				
ℓ	7	8				
c	2	4				

Table A.15. The partitions Π of 12 with $c(\Pi) \leqslant 4$

Π_1, Π_2, Π_3			Associated cover
(2,...,2,1,1)	(3,3,3,3)	(10,1,1)	$S(2,2,10,10) \dashrightarrow S(2,3,10)$
(2,...,2,1,1)	(3,3,3,3)	(8,2,2)	$S(2,2,4,4) \dashrightarrow S(2,3,8)$
(2,...,2,1,1)	(4,4,4)	(5,5,1,1)	$S(2,2,5,5) \dashrightarrow S(2,4,5)$
(2,...,2)	(4,4,1,1,1,1)	(6,6)	$S(4,4,4,4) \dashrightarrow S(2,4,6)$
(2,...,2)	(3,3,3,1,1,1)	(11,1)	$S(3,3,3,11) \dashrightarrow S(2,3,11)$
(2,...,2)	(3,3,3,1,1,1)	(10,2)	$S(3,3,3,5) \dashrightarrow S(2,3,10)$
(2,...,2)	(3,3,3,1,1,1)	(9,3)	$S(3,3,3,3) \dashrightarrow S(2,3,9)$
(2,...,2)	(3,3,3,1,1,1)	(8,4)	$S(2,3,3,3) \dashrightarrow S(2,3,8)$
(2,...,2)	(4,4,4)	(8,1,1,1,1)	$^*S(8,8,8,8) \dashrightarrow S(2,4,8)$
(2,...,2)	(4,4,4)	(6,3,1,1,1)	$S(2,6,6,6) \dashrightarrow S(2,4,6)$
(2,...,2)	(4,4,4)	(6,2,2,1,1)	$S(3,3,6,6) \dashrightarrow S(2,4,6)$
(2,...,2)	(3,3,3,3)	(7,3,1,1)	$^*S(3,7,21,21) \dashrightarrow S(2,3,21)$
(2,...,2)	(3,3,3,3)	(7,2,2,1)	$^*S(2,7,7,14) \dashrightarrow S(2,3,14)$
(2,...,2)	(3,3,3,3)	(6,4,1,1)	$^*S(2,3,12,12) \dashrightarrow S(2,3,12)$
(2,...,2)	(3,3,3,3)	(5,4,2,1)	$^*S(4,5,10,20) \dashrightarrow S(2,3,20)$
(2,...,2)	(3,3,3,3)	(5,3,3,1)	$^*S(3,5,5,15) \dashrightarrow S(2,3,15)$
(2,...,2)	(3,3,3,3)	(5,3,2,2)	$^*S(6,10,15,15) \dashrightarrow S(2,3,30)$
(2,...,2)	(3,3,3,3)	(4,4,3,1)	$^*S(3,3,4,12) \dashrightarrow S(2,3,12)$
(2,...,2)	(3,3,3,3)	(4,3,3,2)	$^*S(3,4,4,6) \dashrightarrow S(2,3,12)$
(3,3,3,3)	(3,3,3,3)	(4,4,1,1,1,1)	$S(4,4,4,4) \dashrightarrow S(3,3,4)$
(2,...,2)	(5,5,1,1)	(5,5,1,1)	$S(5,5,5,5) \dashrightarrow S(2,5,5)$
(2,...,2)	(4,4,2,2)	(5,5,1,1)	$S(2,2,5,5) \dashrightarrow S(2,4,5)$

Table A.16. In these table we omit the triples of partitions of 12 having an associated candidate cover, which is not hyperbolic. There are 22 triples of partitions of 12 having a hyperbolic associated candidate cover in C_4.

Appendix B

Compatibility II

B.1 Large degree

This paragraph is devoted to prove that the data reported in Table 4.12 are the only compatible data for the hyperbolic covers in C_4, with degree $d \geqslant 13$. Recall that within Step II in paragraph D (Chapter 4), we have defined a certain set D of candidate covers of type $S(\alpha,\beta,\gamma,\delta) \dashrightarrow S(p,q,r)$ having base $X = S(2,3,r)$ with $7 \leqslant r \leqslant 77$; there we have already studied all cases where $31 \leqslant r \leqslant 77$. Here we complete the proof of Proposition 4.2.1 by focusing on the cases $7 \leqslant r \leqslant 30$, starting from $r = 30$.

For the reader's convenience we reproduce here in Table B.1 the auxiliary functions bounding the degree, originally described in Table 4.11

$d \equiv 0\,(2),\ d \equiv 0\,(3)$	$d_{\max}(r)$	$12\frac{r-2}{r-6}$
$d \equiv 1\,(3)$	$d_3(r)$	$2\frac{5r-9}{r-6}$
$d \equiv 1\,(2)$	$d_2(r)$	$9\frac{r-2}{r-6}$
$d \equiv 2\,(3)$	$d_{3,3}(r)$	$4\frac{2r-3}{r-6}$
$d \equiv 1\,(2),\ d \equiv 1\,(3)$	$d_{2,3}(r)$	$\frac{7r-12}{r-6}$

Table B.1. Auxiliary functions bounding degree, depending on congruences and divisibility

Remark B.1.1. Notice that, regardless of their applicability as an upper bound for d, the functions defined in Table B.1 satisfy the following inequalities for all r:

$$d_{\max}(r) > d_3(r) > d_2(r) > d_{3,3}(r) > d_{2,3}(r).$$

Just a few words about the notation chosen for these functions of r: when we write $d_{i,j}$, we want to stress that d is such that we must have in the cover a conic point of order i and another of order j.

$r = 30$ Here $d_{\max}(30) = 14$, then we have to consider $d = 13$ and $d = 14$. If $d = 13 \equiv 1\ (2)$, then $d \leqslant d_2(30) = 10.5$, while if $d = 14 \equiv 2\ (3)$, then $d \leqslant d_{3,3}(30) < d_2(30)$. In both cases we do not have anything to consider.

$23 \leqslant r \leqslant 29$ In all these cases we have $14 < d_{\max}(r) < 15$, then we have $d = 13$ and $d = 14$ as possible degrees. As in the previous case, if $d = 13$, then $d \leqslant d_2(23) \approx 11.12$, while if $d = 14$, then $d \leqslant d_{3,3}(23) < d_2(23)$. In both cases we do not have anything to study.

$r = 22$ Here $d_{\max}(22) = 15$, then we get:

$d = 13$ and $d = 15$ lead to $d \leqslant d_2(22) = 11.25$,

$d = 14$ leads to $d \leqslant d_{3,3}(22) < d_2(22) = 11.25$.

As before, nothing to do.

$19 \leqslant r \leqslant 21$ Also here we have to discuss $13 \leqslant d \leqslant 15$.

$d = 13$ and $d = 15$ lead to $d \leqslant d_2(19) \approx 11.77$,

$d = 14$ leads to $d \leqslant d_{3,3}(19) < d_2(19) \approx 11.77$.

Nothing to do.

$r = 18$ Here $d_{\max}(18) = 16$, then we get:

$d = 13$ and $d = 15$ lead to $d \leqslant d_2(18) = 12$,

$d = 14$ leads to $d \leqslant d_{3,3}(18) < d_2(18) = 12$,

$d = 16$ leads to $d \leqslant d_3(18) = 13.5$.

Same conclusion as in the previous cases.

$r = 17$ Here $d_{\max}(17) \approx 16.36 < 17$. As 17 is a prime number, and the orders of cone points over it should divide 17, they can only be in $\{1, 17\}$; then $d = x + y \cdot 17$ with $x,\ y$ non-negative integers, and $d < 17$ implies $y = 0$. Hence we should have x cone points of order 17 in $\widetilde{X}$; since $d \geqslant 13$, we conclude we should have too many cone points in $\widetilde{X}$ (in fact $x = d > 12$). No new cases.

$r = 16$ $d_{\max}(16) = 16, 8$, then we get:

$d = 13$ and $d = 15$ lead to $d \leqslant d_2(16) = 12.6$,

$d = 14$ leads to $d \leqslant d_{3,3}(16) < d_2(16) = 12.6$,

$d = 16$ leads to $d \leqslant d_3(16) = 14.2$.

No interesting items arise.

$r = 15 \quad d_{\max}(15) \approx 17,3 < 18$ implies $d \leqslant 17$; so we have:

$d = 13,\ 15,\ 17$ lead to $d \leqslant d_2(15) = 13,$

$d = 14$ leads to $d \leqslant d_{3,3}(15) < d_2(15) = 13,$

$d = 16$ leads to $d \leqslant d_3(15) \approx 14.6.$

The only item that should be analysed is $d = 13$; for this case we would like to have $\ell(\Pi) = 15$ and $c(\pi) = 4$ (following the notation already used in Chapter 3). Proceeding in the usual study of the length of possible partitions of d, and the number of conic points, from Table B.2 we can easily deduce that the partitions of d associated to the would-be cover with less than five conic points are:

$$(2,\dots,2,1),(3,\dots,3,1),(3,3,3,3,1)$$
$$(2,\dots,2,1),(3,\dots,3,1),(5,5,3)$$
$$(2,\dots,2,1),(3,\dots,3,1),(5,3,3,1,1)$$

Π	(5,5,3)	(5,3,3,1,1)	(3,...,3,1)	(3,...,3,1,1,1,1)	(2,...,2,1)	(2,...,2,1,1,1)
ℓ	3	5	5	7	7	8
c	3	5	1	4	1	3

Table B.2. The only partitions Π of 13 which can be involved in a cover of $S(2,3,15)$

Notice that we cannot have any candidate orbifold covers in C_4.

$r = 14 \quad d_{\max}(14) = 18$ implies $d \leqslant 18$; so we must consider:

$d = 13,\ 15,\ 17$ lead to $d \leqslant d_2(14) = 13.5,$

$d = 14$ leads to $d \leqslant d_{3,3}(14) = 12.5,$

$d = 16$ leads to $d \leqslant d_3(14) = 15.5,$

$d = 18$: no restrictions.

Again, we have to analyse $d = 13$. Here we order the possible triples of partitions of 13 with respect to the total number of cone points:

$$(2,\dots,2,1),(3,\dots,3,1),(7,2,2,2)$$
$$(2,\dots,2,1,1,1),(3,\dots,3,1),(7,2,2,2)$$
$$(2,\dots,2,1),(3,\dots,3,1,1,1,1),(7,2,2,2)$$

None of these is interesting for us.

Also $d = 18$ has to be considered. In Table B.3 we list all possible triples of partitions of 18 having l.c.m. 2, 3, 14 respectively, having less than four conic points; only three of them could lie in C_4. Then we compute the exact degree via the Euler characteristics, we found that only $S(14,14,14,14) \dashrightarrow S(2,3,14)$ is candidate.

Π_1, Π_2, Π_3	$c(\Pi)$	Associated cover	d
$(2,\dots,2),(3,\dots,3),(14,2,2)$	2	$S(7,7) \dashrightarrow S(2,3,14)$	
$(2,\dots,2),(3,\dots,3),(14,2,1,1)$	3	$S(7,14,14) \dashrightarrow S(2,3,14)$	
$(2,\dots,2,1,1),(3,\dots,3),(14,2,2)$	4	$*S(2,2,7,7) \dashrightarrow S(2,3,14)$	15/2
$(2,\dots,2),(3,\dots,3),(7,7,2,2)$	4	$*S(2,2,7,7) \dashrightarrow S(2,3,14)$	15/2
$(2,\dots,2),(3,\dots,3),(14,1,1,1,1)$	4	$S(14,14,14,14) \dashrightarrow S(2,3,14)$	18
$(2,\dots,2),(3,\dots,3,1,1,1),(14,2,2)$	5	$S(3,3,3,7,7) \dashrightarrow S(2,3,14)$	

Table B.3. Partitions and associated cover rising for $r = 14$ and $d = 18$: in those cases labelled with * the associated orbifold cover is even not compatible.

$r = 13$ As in the previous case, $d_{\max}(13) \approx 18.8$ implies $d \leqslant 18$. Moreover, as when we studied the case $r = 17$, we use the fact that r is a prime number: the partition of d over 13 should be made of 13's and 1's. Then for $d \leqslant 18$ we have listed all the possible triples of partitions having the right triple of l.c.m.'s and $c(\Pi) \leqslant 4$:

Π_1, Π_2, Π_3	$c(\Pi)$	d
$(2,\dots,2,1),(3,\dots,3,1),(13)$	2	13
$(2,\dots,2,1,1,1),(3,\dots,3,1),(13)$	4	13
$(2,\dots,2,1),(3,\dots,3,1),(13,1)$	3	14
$(2,\dots,2,1),(3,\dots,3),(13,1,1)$	3	15
$(2,\dots,2),(3,\dots,3,1),(13,1,1,1)$	4	16

There are only two of them that could be associated to an orbifold cover in C_4, but only the last triple $(2,\dots,2),(3,\dots,3,1),(13,1,1,1)$ is associated to a candidate cover: $S(3,13,13,13) \dashrightarrow S(2,3,13)$.

$r = 12$ $d_{\max}(12) = 20$; so:

$d = 13,\ 19$ lead to $d \leqslant d_{2,3}(12) = 12$,

$d = 14,\ 17$ lead to $d \leqslant d_{3,3}(12) = 14$,

$d = 15$ leads to $d \leqslant d_2(12) = 15$,

$d = 16,\ 20$ lead to $d \leqslant d_3(12) = 16$,

$d = 18$: no restrictions.

These conditions let us exclude $d = 13, 19, 17, 20$; hence we have to discuss the cases in which $d \in \{14, 15, 16, 18\}$.

Π_1, Π_2, Π_3	$c(\Pi)$	d	$\ell(\Pi)$
$(2,\ldots,2),(3,\ldots,3,1,1),(12,2)$	3	14	15
$(2,\ldots,2),(3,\ldots,3,1,1),(12,1,1)$	4	14	16
$(2,\ldots,2,1),(3,\ldots,3),(12,1,1,1)$	4	15	17
$(2,\ldots,2,1,1),(3,\ldots,3,1),(12,4)$	4	16	17
$(2,\ldots,2),(3,\ldots,3,1),(12,2,1,1)$	4	16	18
$(2,\ldots,2),(3,\ldots,3,1),(6,6,4)$	4	16	17
$(2,\ldots,2),(3,\ldots,3),(12,6)$	1	18	17
$(2,\ldots,2,1,1),(3,\ldots,3),(12,6)$	3	18	18
$(2,\ldots,2),(3,\ldots,3,1,1,1),(12,6)$	4	18	19
$(2,\ldots,2),(3,\ldots,3),(12,4,2)$	2	18	18
$(2,\ldots,2,1,1),(3,\ldots,3),(12,4,2)$	4	18	19
$(2,\ldots,2),(3,\ldots,3),(12,4,1,1)$	3	18	19
$(2,\ldots,2),(3,\ldots,3),(12,3,3)$	2	18	18
$(2,\ldots,2,1,1),(3,\ldots,3),(12,3,3)$	4	18	19
$(2,\ldots,2),(3,\ldots,3),(12,3,2,1)$	3	18	19
$(2,\ldots,2),(3,\ldots,3),(12,3,1,1,1)$	4	18	20
$(2,\ldots,2),(3,\ldots,3),(12,2,2,2)$	3	18	19
$(2,\ldots,2),(3,\ldots,3),(12,2,2,1,1)$	4	18	20
$(2,\ldots,2),(3,\ldots,3),(6,6,4,2)$	4	18	19

Here, five of them are associated to a candidate cover:

$S(3,3,12,12) \dashrightarrow S(2,3,12);$

$S(2,12,12,12) \dashrightarrow S(2,3,12);$

$S(3,6,12,12) \dashrightarrow S(2,3,12);$

$S(4,12,12,12) \dashrightarrow S(2,3,12);$

$S(6,6,12,12) \dashrightarrow S(2,3,12).$

$r = 11$ $\quad d_{\max}(11) = 21,6$ implies $13 \leqslant d \leqslant 21$. Here we use also the fact that 11 is a prime number: in fact, if we want $c(\pi) \leqslant 4$ and $d \geqslant 13$, the possible partitions over 11 could be only made of one 11 and some 1's, and we list them in the following table.

Π_1, Π_2, Π_3	$c(\Pi)$	d	$\ell(\Pi)$
$(2,\ldots,2,1),(3,\ldots,3,1),(11,1,1)$	4	13	15
$(2,\ldots,2),(3,\ldots,3,1,1),(11,1,1,1)$	5	14	19
$(2,\ldots,2,1),(3,\ldots,3),(11,1,1,1,1)$	5	15	18

Then the only compatible hyperbolic cover of $S(2,3,11)$ in degree greater than 13 is that one associated to the first triple of partitions in the table above:

$$S(2,3,11,11) \dashrightarrow S(2,3,11).$$

$r = 10$ $\quad d_{\max}(10) = 24$ implies $13 \leqslant d \leqslant 24$. Then we analyse the possible degrees:

$d = 13,\ 15,\ 17,\ 19$ lead to $d \leqslant d_2(10) = 18$,

$d = 14,\ 17,\ 20$ leads to $d \leqslant d_{3,3}(10) = 17$,

$d = 16,\ 19,\ 22$ leads to $d \leqslant d_3(10) = 20.5$,

$d = 18,\ 24$: no restrictions.

Hence we can exclude degrees 19, 20, and 22. As previously done, we start our discussion on triples of partitions of the possibile degrees, excluding triples not having the desidered l.c.m.'s, or not having $c(\Pi) = 4$.

Π_1, Π_2, Π_3	d	$\ell(\Pi)$
$(2,\ldots,2,1),(3,\ldots,3,1),(10,2,1)$	13	15
$(2,\ldots,2),(3,\ldots,3,1,1),(10,2,2)$	14	16
$(2,\ldots,2,1,1,1),(3,\ldots,3),(10,5)$	15	16
$(2,\ldots,2,1),(3,\ldots,3),(10,2,2,1)$	15	17
$(2,\ldots,2),(3,\ldots,3,1),(10,2,2,2)$	16	18
$(2,\ldots,2),(3,\ldots,3),(10,5,2,1)$	18	19
$(2,\ldots,2),(3,\ldots,3),(10,5,1,1,1)$	18	20
$(2,\ldots,2,1,1,1),(3,\ldots,3),(10,10,1)$	21	22
$(2,\ldots,2,1),(3,\ldots,3),(10,5,5,1)$	21	22
$(2,\ldots,2),(3,\ldots,3),(10,10,1,1,1,1)$	24	26
$(2,\ldots,2,1,1),(3,\ldots,3),(10,10,2,2)$	24	25
$(2,\ldots,2),(3,\ldots,3),(10,5,5,2,2)$	24	25

And six of them are associated to a candidate cover in C_4:

$S(2,3,5,10) \dashrightarrow S(2,3,10)$;

$S(3,3,5,5) \dashrightarrow S(2,3,10)$;

$S(2,5,5,10) \dashrightarrow S(2,3,10)$;

$S(3,5,5,5) \dashrightarrow S(2,3,10)$;

$S(2,5,5,5) \dashrightarrow S(2,3,10)$;

$S(10,10,10,10) \dashrightarrow S(2,3,10)$.

$r = 9$ $\quad d_{\max}(9) = 28$; then $13 \leqslant d \leqslant 28$. Then we analyse restrictions on the degree:

$d \equiv 1\ (2) \Rightarrow d \leqslant d_2(9) = 21$,

$d \equiv 1\ (2),\ d \equiv 1\ (3) \Rightarrow d \leqslant d_{2,3}(9) = 17$,

$d \equiv 2\ (3) \Rightarrow d \leqslant d_{3,3}(9) = 20$,

$d \equiv 1\ (3) \Rightarrow d \leqslant d_3(9) = 24$,

$d = 18,\ 24$: no restrictions.

Consequently, we deduce that the set of possible degrees we have to consider is

$$\{13, 14, 15, 16, 17, 18, 20, 21, 24\}.$$

Proceeding as in the previous case, we are able to list the admissible triples of partitions of d with $c(\Pi) = 4$:

Π_1, Π_2, Π_3	d	$\ell(\Pi)$
$(2,\ldots,2,1),(3,\ldots,3,1),(9,3,1)$	13	15
$(2,\ldots,2),(3,\ldots,3,1),(9,3,3,1)$	16	18
$(2,\ldots,2,1,1,1,1),(3,\ldots,3),(9,9)$	18	19
$(2,\ldots,2),(3,\ldots,3,1,1),(9,9,1,1)$	20	22
$(2,\ldots,2,1,1,1),(3,\ldots,3),(9,9,3)$	21	22
$(2,\ldots,2,1),(3,\ldots,3),(9,9,1,1,1)$	21	23
$(2,\ldots,2,1,1),(3,\ldots,3),(9,9,3,3)$	24	25
$(2,\ldots,2),(3,\ldots,3),(9,9,3,1,1,1)$	24	26

They give the following candidate covers:

$S(2,3,3,9) \dashrightarrow S(2,3,9);$

$S(3,3,3,9) \dashrightarrow S(2,3,9);$

$S(3,3,9,9) \dashrightarrow S(2,3,9);$

$S(2,9,9,9) \dashrightarrow S(2,3,9);$

$S(3,9,9,9) \dashrightarrow S(2,3,9).$

$r = 8$ $\quad d_{\max}(8) = 36$; then $13 \leqslant d \leqslant 36$. Then we analyse the conditions on d:

$d \equiv 1 \ (2) \Rightarrow d \leqslant d_2(8) = 27,$

$d \equiv 1 \ (3) \Rightarrow d \leqslant d_3(8) = 31,$

$d \equiv 2 \ (3) \Rightarrow d \leqslant d_{3,3}(8) = 26,$

$d \equiv 1 \ (2), \ d \equiv 1 \ (3) \Rightarrow d \leqslant d_{2,3}(8) = 22,$

$d = 18, \ 24, \ 30, \ 36$: no restrictions.

Consequently, the set of possible degrees is

$$\{13, \ldots, 22, 24, 26, 27, 28, 30, 36\}.$$

Proceeding as usual, one gets that the possible triples of partitions with the exact number of cone points are:

$(2,\dots,2,1),(3,\dots,3,1),(8,4,1)$	$(2,\dots,2),(3,\dots,3,1,1),(8,4,2)$
$(2,\dots,2,1),(3,\dots,3),(8,4,2,1)$	$(2,\dots,2),(3,\dots,3,1),(8,4,2,2)$
$(2,\dots,2),(3,\dots,3,1,1,1,1),(8,8)$	$(2,\dots,2,1),(3,\dots,3,1,1),(8,8,1)$
$(2,\dots,2,1,1),(3,\dots,3),(8,8,1,1)$	$(2,\dots,2),(3,\dots,3,1,1,1),(8,8,2)$
$(2,\dots,2,1),(3,\dots,3,1),(8,8,2,1)$	$(2,\dots,2),(3,\dots,3,1,1),(8,8,2,2)$
$(2,\dots,2,1),(3,\dots,3),(8,8,2,2,1)$	$(2,\dots,2),(3,\dots,3,1),(8,8,4,1,1)$
$(2,\dots,2),(3,\dots,3,1),(8,8,2,2,2)$	$^*(2,\dots,2,1,1),(3,\dots,3),(8,8,4,4)$
$(2,\dots,2),(3,\dots,3),(8,8,2,2,2,2)$	$(2,\dots,2),(3,\dots,3),(8,8,4,2,1,1)$
$^*(2,\dots,2),(3,\dots,3),(8,4,4,4,4)$	$(2,\dots,2),(3,\dots,3,1,1),(8,8,8,1,1)$
$(2,\dots,2,1),(3,\dots,3),(8,8,8,1,1,1)$	$(2,\dots,2),(3,\dots,3,1),(8,8,8,2,1,1)$
$^*(2,\dots,2,1,1),(3,\dots,3,1),(8,8,8,4)$	$^*(2,\dots,2),(3,\dots,3,1),(8,8,4,4,4)$
$(2,\dots,2),(3,\dots,3),(8,8,8,2,2,1,1)$	$^*(2,\dots,2),(3,\dots,3),(8,8,4,4,4,2)$
$^*(2,\dots,2,1,1),(3,\dots,3),(8,8,8,4,2)$	$(2,\dots,2),(3,\dots,3),(8,8,8,8,1,1,1,1)$
$^*(2,\dots,2),(3,\dots,3),(8,8,8,4,4,2,2)$	$^*(2,\dots,2),(3,\dots,3,1,1,1),(8,8,8,8,4)$
$^*(2,\dots,2,1,1),(3,\dots,3),(8,8,8,8,2,2)$	

Except of those labelled with a star *, each of them gives a candidate cover (there are 20 of them).

$r = 7$ As noticed in cases $r = 17, 11$, when r is a prime number, its partitions should be made of r's and 1's; we also require $c(\Pi) = 4$. Then we are interested only in degrees $d \equiv a$ (7), with $0 \leqslant a \leqslant 4$. We also see that $d \equiv 4$ implies that $d \equiv 0$ (6). Using both these conditions, and the usual ones (or discussing congruences modulo 2 and 3) we get that the set of all possible degrees is $\{14, 21, 22, 28, 29, 30, 36, 37, 44, 45, 52, 60\}$. As usual, we list the possible triple of partitions of d:

Π_1, Π_2, Π_3	d
$(2,\dots,2,1,1),(3,\dots,3,1,1),(7,7)$	14
$(2,\dots,2,1),(3,\dots,3,1,1,1),(7,7,7)$	21
$(2,\dots,2,1,1),(3,\dots,3,1),(7,7,7,1)$	22
$(2,\dots,2),(3,\dots,3,1,1,1,1),(7,7,7,7)$	28
$(2,\dots,2,1),(3,\dots,3,1,1),(7,7,7,7,1)$	29
$(2,\dots,2,1,1),(3,\dots,3),(7,7,7,7,1,1)$	30
$(2,\dots,2),(3,\dots,3,1,1,1),(7,\dots,7,1)$	36
$(2,\dots,2,1),(3,\dots,3,1),(7,\dots,7,1,1)$	37
$(2,\dots,2),(3,\dots,3,1,1),(7,\dots,7,1,1)$	44
$(2,\dots,2,1),(3,\dots,3),(7,\dots,7,1,1,1)$	45
$(2,\dots,2),(3,\dots,3,1),(7,\dots,7,1,1,1)$	52
$(2,\dots,2),(3,\dots,3),(7,\dots,7,1,1,1,1)$	60

Each of them gives a candidate orbifold cover, listed in Table 4.12.

Appendix C

Exceptionality

In this appendix we provide alternative proofs of exceptionality for some of the candidate covers already shown by other means to be non-realizable. The techniques we will use are:

- GM (Graph Moves): performing moves on partially constructed dessins d'enfant for a candidate branched cover we reduce the proof that it is exceptional to the same statement for a candidate in lower degree;
- GG (Geometric Gluings): let a candidate cover $f : \widetilde{X} \xrightarrow{d:1} X$ be given, where X is a triangular hyperbolic 2-orbifold; we discuss the realizability of f by realizing X as a quotient of a hyperbolic polygon D in $\mathbb{H}^2$ under the action of an isometric pairing of the edges, and by discussing how d copies of D can be glued together to give a fundamental domain of $\widetilde{X}$ compatible with $f : \widetilde{X} \dashrightarrow X$.

The cases treated and the techniques used to treat them are specified in Table C.1. When using GM, we take white and black vertices having valences as in Π_1 and Π_3, respectively. We discuss the two cases labeled with GG in the last paragraph of this appendix.

Π_1, Π_2, Π_3	d	Fig.	Reason
(2,2,2,2), (3,3,1,1),(5,3)	8	C.1a	GM - Thm 3.3.5
(2,2,2,2,1), (3,3,3), (5,2,2)	9	C.1b	GM - VED
(2,...,2),(3,3,3,1),(4,3,3)	10	C.1a	GM - BD
(2,...,2,1),(3,...,3),(8,8,2,2,1)	21	C.2a	GM - BD
(2,...,2),(3,...,3,1),(8,8,4,1,1)	22	C.2b	GM - BD
(2,...,2),(3,...,3),(9,9,3,1,1,1)	24		GG
(2,...,2),(3,...,3,1,1),(7,...,7,1,1)	44		GG
(2,...,2,1),(3,...,3),(7,...,7,1,1,1)	45	3 times C.3c	Thm 3.3.10
(2,...,2),(3,...,3,1),(7,...,7,1,1,1)	52	3 times C.3c, d	Thm 3.3.10

Table C.1. Exceptionality proofs

For each of the data listed in Table C.1 we will now describe in greater detailhow we have establishes realizability or exceptionality.

$(2,2,2,2),(3,3,1,1),(5,3)$ In the first candidate cover, we have two 2-gons; at least one of them is forced to be based at the black vertex of valence 5. Acting on it as described in Fig. C.1a, we get the new partitions: $(2,\dots,2),(3,3,3,1),(6,3,1)$, giving a degree-10 candidate cover of Euclidean type, already proved to be exceptional in Theorem 3.3.5.

$(2,2,2,2,1),(3,3,3),(5,2,2)$ Here, we note that there is a white vertex x of valence 1. We can have 6-gons only, which forces x to be joined to the black vertex of valence 5. Then we act on this vertex as indicated in Fig. C.1b, getting the candidate branch datum $(2,...,2),(3,3,3,1),(6,2,2)$, which is exceptional thanks to the VED criterion.

$(2,\dots,2),(3,3,3,1),(4,3,3)$ Also this partition has a forced local setting: the 2-gon has to be based at the vertex of valence 4, because the graph should be connected. Then we act as indicate in Table C.1 and we get a $\Pi = ((2,\dots,2),(3,3,3,3),(5,3,3,1))$, that is exceptional because of BD.

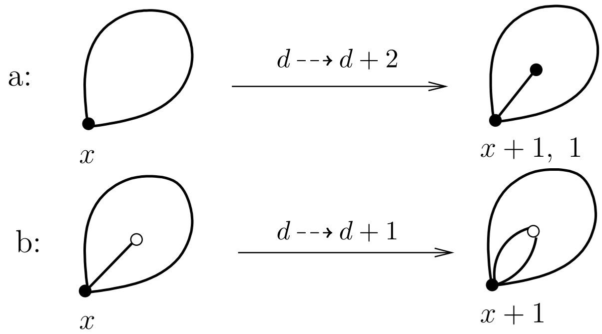

Figure C.1. Graph moves a and b

Other cases via GM In the next figures (Fig. C.2 and C.3) we describe how to do graph moves to prove exceptionality of the remaining covers. More precisely, we first show on the left the forced situation on which we want to act, and then, on the right, the piece of graph after the move. Here we list the original partitions, and the final ones that show exceptionality:

Π	d	Π after GM
(2,...,2,1),(3,...,3),(8,8,2,2,1)	21	(2,...,2),(3,3,3,3), (6,4,1,1)
(2,...,2),(3,...,3,1),(8,8,4,1,1)	22	(2,...,2),(3,3,3,3),(5,4,2,1)
(2,...,2,1),(3,...,3),(7,...,7,1,1,1)	45	(2,...,2,1),(3,...,3),(6,6,6,3,3,3)
(2,...,2),(3,...,3,1),(7,...,7,1,1,1)	52	(2,...,2,1),(3,...,3),(6,6,6,6,3,3,3)

In degrees 21 and 22 we use again Propositions 4.14 and 4.15 of [19]: in fact, after performing the graph moves as described in Fig. C.2a and C.2b, we get two partitions of 12 that were proved to be exceptional in [19].

Also in degrees 45 and 52, the suggested graph moves lead to candidates known to be exceptional: in fact they are reduced to

$$S(2,2,2,2) \overset{18:1}{\dashrightarrow} S(2,3,6) \quad \text{and} \quad \mathrm{S}(2,2,2,2) \overset{33:1}{\dashrightarrow} \mathrm{S}(2,3,6),$$

which are Euclidean and exceptional, as already proved in Chapter 3, Theorem 3.3.10.

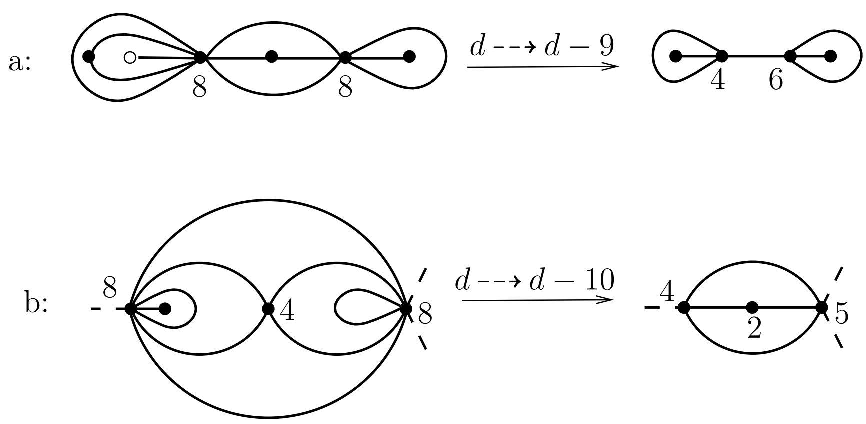

Figure C.2. Other graph moves

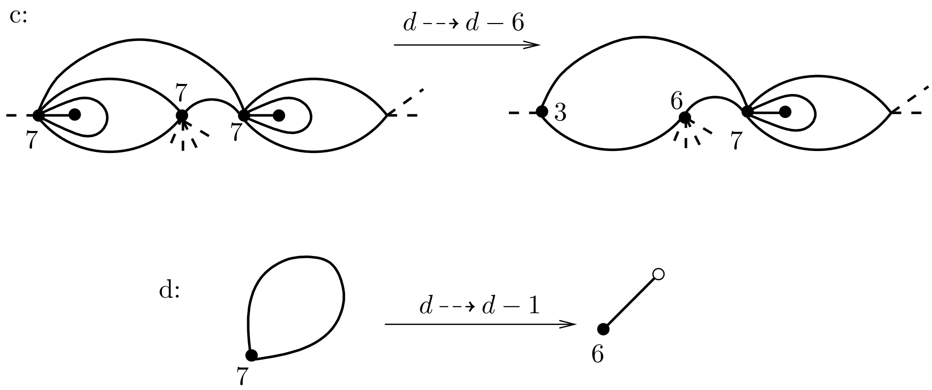

Figure C.3. Other graph moves

Two proofs of exceptionality via GG In Fig. C.4 and Fig. C.5 figures are used to show exceptionality by means of geometric gluings, introduced in the last section of Chapter 4. In order to show exceptionality of a candidate cover $\widetilde{X} \dashrightarrow X$, we will see that once one has the base orbifold as $D/_{\{g_1,g_2\}}$, it is impossible to produce any $\widetilde{D}/_{\{f_1,\ldots,f_h\}}$ covering X and representing the cover orbifold $\widetilde{X}$.

Let us discuss in detail the remaining cases:

I $(9,9,3,1,1,1),(3,\ldots,3),(2,\ldots,2) \quad S(3,9,9,9) \longrightarrow S(2,3,9)$

II $(7,\ldots,7,1,1),(3,\ldots,3,1,1),(2,\ldots,2) \qquad S(3,3,7,7) \longrightarrow S(2,3,7)$

In both of them, first of all we fix X as $D/_{\{g_1,g_2\}}$; then we start constructing $\widetilde{D}$, that will have to be homeomorphic to a disc in the plane, by gluing together the d copies of D and taking into account the partitions of d of the candidate cover. Notice that taking into account the partitions means that we are obliged to group together some copies of D: this operation produces a 'set of tiles' we have to use to build $\widetilde{D}$.

Case I Let $\alpha = \frac{2}{3}\pi$. We want to construct a fundamental domain of $S(3,9,9,9)$ gluing together 24 copies of the fundamental domain of $S(2,3,9)$, that is a triangle T of angles $\pi/3, \pi/3, 2\pi/9$. At the start level, the partitions $(2,\ldots,2), (3,\ldots,3), (9,9,3,1,1,1)$ tell us that there are some prescribed gluings; e.g. we have two 9-gons, a-tile, made of 9 copies of T, each one desingularizing a conic point of order 9 (note that each angle of the two 9-gons is α). In Fig. C.4, first row, we draw the types of tile we have at this level: the b-tile corresponds to 3 in Π_3, and the c-tile corresponds to 1's in Π_3.

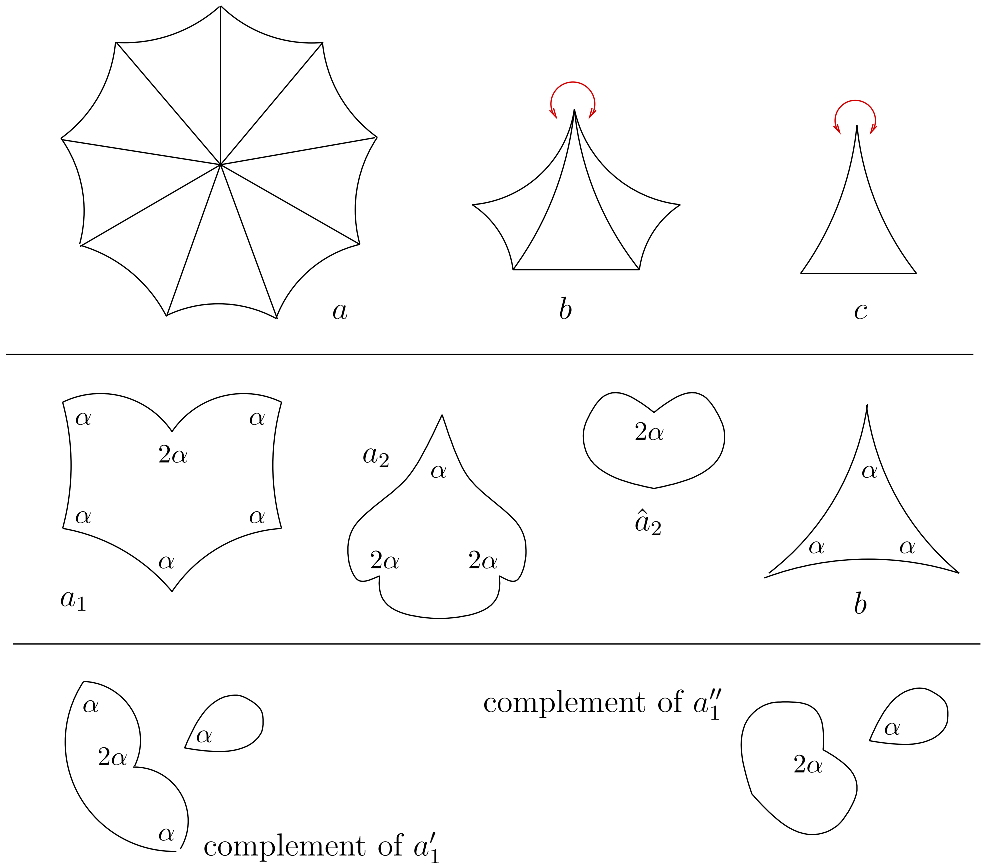

Figure C.4. Set of tiles for $(2,\ldots,2), (3,\ldots,3), (9,9,3,1,1,1)$

Consider now a tile of type c, corresponding to a conic singularity of order 9 in the cover orbifold; we are forced to glue this c-tile on an a-tile, because if we glue a c-tile with a c-tile, or with a b-tile, we close the orbifold and cannot glue anything else. Moreover, note that we have three b-tiles (each one with a cone point of order 9) and two a-tiles. Gluing all of them to the same a-tile corresponds to realizing $S \xrightarrow[(2,\ldots,2),(3,\ldots,3),(9,1,1,1)]{12:1} S$; then the only possibility is to glue one of them with an a-tile, and the remaining two to the other a-tile. Hence we add new tiles to the set with which we have to build a fundamental domain of $S(3,9,9,9)$; see the second row in Fig.C.4; at this level the set is: $\{a_1, a_2, \hat{a}_2, b\}$. Then it is important not to forget those self-gluings not introducing new conic points: there are two of them, namely a_1' and a_1'', see the last row of Fig.C.4.

Since we have no other self-gluing to consider, we have reduced the problem to tile the complement of a b-tile, that is a triangle with each angle 2α, with one a_1, and a_2 or $\hat{a}_2$. Both cases are easily seen to be impossible.

Case II As in the previous case, let $\alpha = \frac{2}{3}\pi$. We want to construct a fundamental domain of $S(3,3,7,7)$ gluing together 44 copies of a fundamental domain of $S(2,3,7)$. As in the previous case we choose as fundamental domain T, a triangle with angles $\pi/3, \pi/3, 2\pi/7$. Even at this level, focusing on the partition of the degree at the preimages of the conic point of order 7, we see that there are some forced gluings: we have six copies of a heptagon whose center projects to the conic point of order 7, with each vertex projecting on the conic singularity in the base of order 3. So we will think of these a-tiles as heptagons, with angles of 2α. Moreover we have two other tiles, that we call b, that are triangles with angles $\pi/3$, $\pi/3$, $2\pi/7$, which carry the conic points of order 7 in $S(3,3,7,7)$, the covering orbifold. In the first row of Fig. C.5 we draw these tiles. Now we pass to consider where the two conic points of order 7 could be: when we glue a b-tile on an a-tile, and we make the resulting identifications (remembering that $3\alpha = 2\pi$), we get a new tile, a_7, with 4 vertices (still projecting to the 3-cone point in the base), and angles $\{2\alpha, \alpha, \alpha, \alpha\}$. Note that gluing another b-tile on this a_7-tile is impossible: we cannot keep on gluing any other tile. Hence we cannot put two b-tiles on the same a-tile.

At the second line of Fig. C.5 we show the tiles we get acting as we perform a self-gluing (not producing a conic point) or a gluing of two adjacent edges, making the common vertex one of the two conic points of order 3 in the cover orbifold, and the same for 7-cone points; in the end, we have tiles of type $\{a, b, a', a'', a_7, a_3\}$. At the third level we introduce one more gluing or self-gluing. After a brief discussion on each case, we can exclude some of them; for instance double self-gluings, and tiles like $a_{3,7}$, namely an a-tile glued with a b-tile and having a 3-conic point. In the end we have the set $\{a, b, a', a_7, a_3, a_{3,3}\}$; we have deleted type a'' because it introduces a region with two edges, and two angles α and 2α, that cannot be filled with our tiles.

We have already shown that we have to glue the two b-tiles on two different a-tiles. Now, taking into account also the two conic points of order 3 in the cover. It is easy to check that a singularity of order 3 should be a vertex corresponding to an angle α; since gluing and self-gluing produce 2α, we reduce only to two a_3's or one $a_{3,3}$.

Hence, we have to discuss realizability for the two sets of tiles:

- $A = \{a_3, a_3, a_7, a_7, a, a\}$;
- $B = \{a_{3,3}, a_7, a_7, a, a, a\}$.

We remark that to realize the cover, now we have to produce an $S(3,3,7,7)$ only by gluing different tiles.

Conclusion in case A We have two a-tiles. Then the problem is to fill the complement of an a-tile (which is a heptagon D' with all angles 2α) with the other tiles in A. Consider the other a-tile and its position: it could have no edges on the boundary of D', or exactly one, or exactly two. Note first that we cannot glue two adjacent edges of the a-tile to two adjacent edges of $\partial D'$, because $2\alpha \neq 2\pi$.

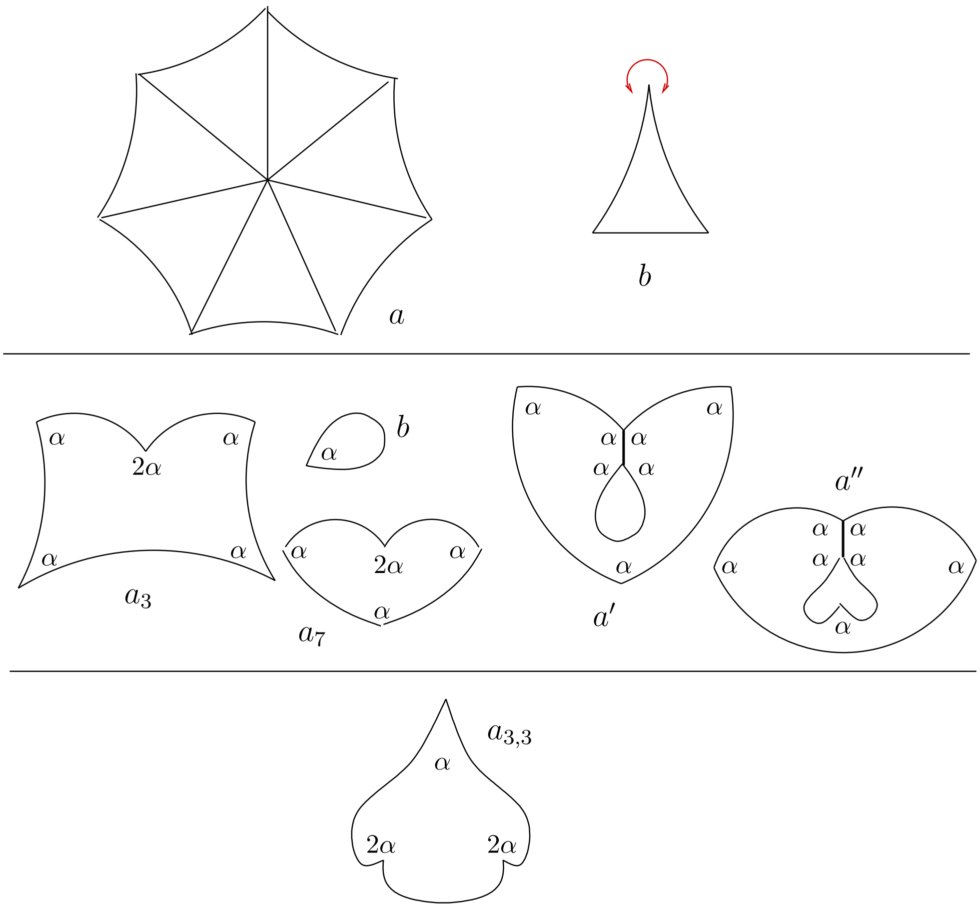

Figure C.5. Set of tiles for $(2,\ldots,2),(3,\ldots,3,1,1),(7,\ldots,7,1,1)$

Moreover, we should exclude the cases of three or more edges on the boundary, because in those cases we are obliged glue the a-tile to two edges at distance one in $\partial D'$, and then to create a region between the a-tile and the boundary, made of $2+n$ edges with 2 angles α, and n angles 2α ($n<4$): it can be easily checked that this kind of region cannot be filled with the tiles in A.

If the a-tile is in the interior, then with the rest of the set A we have to fill a region made of 14 edges and $14\langle 2\alpha\rangle$. We use the brackets $\langle\cdot\rangle$ to say that the angle comes from one angle of a tile and not from the juxtaposition of many. And the set A can offer 18 edges and 4 angles of type $\langle 2\alpha\rangle$. Then, to cover the areas nearby the remaining $10\langle 2\alpha\rangle$, we need to glue together two $\langle\alpha\rangle$ angles. But this requires $10/2$ more edges. This is impossible, because we have 18 edges, and we need $14+5=19$ edges.

If the a-tile has exactly one edge on the boundary, then with the rest of the set A we have to fill a region made of 12 edges, $10\langle 2\alpha\rangle$, and $\langle 2\alpha\rangle$. As above, the set A could offer 18 edges and 4 angles of type $\langle 2\alpha\rangle$. Repeating the discussion as in the previous case, we have no contradiction; but we can estimate better the number of edges required to fill the region. Actually, the existence of many adjacent $\langle 2\alpha\rangle$ on the boundary and the existence of the tile a_3 force us to introduce at least one vertex in the interior. In this way it is easy to see that we need more than 18 edges.

If the a-tile has exactly two edges on the boundary, we have two cases to analyze; the other ones are symmetric. The two edges on the boundary, in common with the tile, could have in between n other edges, with $n = 1$, or 2. We have already discussed at the beginning of this paragraph the case $n = 1$. The case $n = 2$ is slightly more subtle, but also in this case we cannot tile the region in the appropriate way.

Conclusion in case B This case is very easy: if fact here we have an $a_{3,3}$-tile; this tile has two successive $\langle 2\alpha \rangle$ angles. It can only lie in the interior of D', and be glued to an a-tile, forcing it to self-glue. As we carry on gluing we are forced to glue the other a-tile to this object, producing a region R with three edges, one $\langle 2\alpha \rangle$ and two $\langle \alpha \rangle$ angles. If the whole region is in the interior, the problem reduces to fill R with 10 edges, 9 angles $\langle 2\alpha \rangle$ and an angle α with only two a_7-tiles: this is impossible, because two a_7-tiles offer only 8 edges. If R has one edge on the boundary, there remains to fill a region of 8 edges, $5\langle 2\alpha \rangle$ and $\langle 3\alpha \rangle$ with two a_7-tiles, which can only offer 8 edges, $2\langle 2\alpha \rangle$ and $6\langle \alpha \rangle$. Taking into account that two of the $\langle 2\alpha \rangle$ at our disposal to fill R gluing together two pairs of $\langle \alpha \rangle$ need 2/2 more edges, we have the contradiction. In a very similar way we can show that also the case where R has two edges on the boundary D' does not yield a realization of $\widetilde{X}$ as desired.

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