Lorenzo Di Biagio

Pluricanonical systems for 3-folds, 4-folds and n-folds of general type

Dottorato in Matematica, Roma «La Sapienza» (2010).
<http://www.bdim.eu/item?id=tesi_2010_DiBiagioLorenzo_1>

SAPIENZA
UNIVERSITÀ DI ROMA

Pluricanonical systems for 3-folds, 4-folds and n-folds of general type.

Dipartimento di Matematica,
"La Sapienza" - Università di Roma

Dottorato di Ricerca in Matematica – XXI Ciclo

Candidate

Lorenzo Di Biagio
ID number 1083852

Thesis Advisor

Prof. Angelo Felice Lopez

A thesis submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics

July 2010

Ringraziamenti

Al termine di questo lavoro vorrei sentitamente ringraziare il mio relatore, prof. Angelo Lopez, per il grande impegno con cui mi ha seguito in questi anni, per le idee che ha voluto condividere con me, per i consigli e gli incoraggiamenti.
Vorrei ringraziare il dipartimento di matematica dell'università "Roma Tre" per l'ospitalità. Vorrei anche ringraziare Gianluca Pacienza, per avermi fatto conoscere i lavori di Todorov; i miei due "fratelli" Fulvio e Salvatore per le lunghe conversazioni che abbiamo avuto; tutti i colleghi dottorandi (o ormai dottorati) con cui ho avuto scambi di idee e opinioni.
Infine vorrei ringraziare la mia famiglia, per il sostegno, e Aureliana, per il suo supporto morale.

Contents

Introduction

As it is well known, the fundamental problem in algebraic geometry is to classify algebraic varieties, that is: sorting them in isomorphism classes. In its absolute generality this problem goes far beyond the techniques now available, nonetheless it can be considered as a 'guiding problem', that offers impulse to further research in geometry and that allows geometers to measure their achievements.

A first approach to this 'guiding problem' consists, besides restricting the investigation to a particular class of algebraic varieties - in our case complex and projective - of relaxing the notion of isomorphism and focus on the classification of algebraic varieties up to birational equivalence, namely identifying algebraic varieties that have isomorphic Zariski open sets, or, equivalently, that have isomorphic fields of rational functions.

It is then clear the importance of understanding the behavior of certain numerical and geometrical birational invariants that are canonically associated to an algebraic variety. More specifically recall that if X is a smooth complex projective variety of non-negative Kodaira dimension and K_X is its canonical divisor then for every $n \in \mathbb{N}^+$, we can consider both the n-th plurigenus $P_n := h^0(X, nK_X)$ and the associated rational pluricanonical map $\phi_{|nK_X|} : X \dashrightarrow \mathbb{P}(H^0(X, nK_X))$. If we concentrate our attention to varieties of general type (i.e., of maximal Kodaira dimension) of dimension d then, by definition, the plurigenera P_n grow like n^d and ϕ_n is birational (meaning 'birational onto its image') for n sufficiently large. It is then legitimate to wonder if it is possible to find an explicit number n_d, potentially the minimal one, such that n_d does not depend on X (but only on d) and $P_n \neq 0$ or ϕ_n is birational for all $n \geq n_d$.

For curves and surfaces of general type results of this kind are already known since a long time: by simple applications of Riemann-Roch theorem, for curves we have that $P_n \neq 0$ as soon as $n \geq 1$ and $|nK_X|$ is birational as soon as $n \geq 3$; for surfaces Bombieri proved in 1973 (see [5]) that $P_n \neq 0$ for $n \geq 2$ and $|nK_X|$ is birational for $n \geq 5$.

For varieties of higher dimension recent advances have been made independently by Hacon-McKernan (see [20]) and Takayama (see [36]) using ideas of Tsuji. They proved that actually, and for every d, this n_d exists, even if their methods do not directly allow us to compute it. In the case of threefolds J.A. Chen and M. Chen proved in [8] that $P_n > 0$ for every $n \geq 27$ and that $|nK_X|$ is birational for all

$n \geq 73$.

On the other hand it is reasonable to expect effective statements, and even better ones in the case of threefolds, if one requires in addition that some invariant of X is big. This is the content of an article of G.T. Todorov (see [37]) who proved that if the volume of X (see def. 1.3) is sufficiently large then $P_2 \neq 0$ and $|5K_X|$ is birational. Notice that the hypothesis about the largeness of the volume is not too restrictive since varieties of general type of dimension d and volume bounded by a constant M are birationally bounded (see [20, cor. 1.2]); or, in other words, varieties of general type of dimension d and bounded volume are birational to subvarieties of $\mathbb{P}^{2d+1}$ with bounded degree, hence they can be birationally arranged in a finite number of families.

In this work we develop a strategy to effectively study non-vanishing (and size) of pluricanonical systems and birationality of pluricanonical maps for varieties of general type of any dimension and large volume, also with respect to the genus of the curves lying on the variety.

As a matter of fact we succeed in improving Todorov's results for threefolds (also studying higher plurigenera and higher pluricanonical maps) and in finding effective results even for fourfolds and, partially, for fivefolds.

We also manage to give characterizations for threefolds of general type with birational fourth pluricanonical map. We just need to assume that the volume is sufficiently large: as far as we know this approach has never been considered before.

We will now give some details about the most significant results that we obtain.

In the case of threefolds of general type we prove the following theorems:

Theorem 0.1. *(see theorem 2.2). Let X be a smooth projective threefold of general type such that vol$(X) > \alpha^3$. If $\alpha \geq 879$ then $h^0(2K_X) \geq 1$ and if $\alpha \geq 432(n+1) - 3$ then $h^0((n+1)K_X) \geq n$, for all $n \geq 2$.*

Theorem 0.2. *(see theorem 3.3). Let X be a smooth projective threefold of general type such that vol$(X) > \alpha^3$. If $\alpha > 1917\sqrt[3]{2}$ then $|lK_X|$ gives a birational map for every $l \geq 5$.*

In both cases we have much more precise estimates on α, depending on l and on the genus of the curves lying on X. See theorem 2.2 and theorem 3.3, respectively.

We find analogous results also for fourfolds of general type. Using general facts (see theorem 4.1 and theorem 4.6) and a lower bound on the volume of threefolds of general type given by J.Chen and M.Chen (see [8]) we prove:

Theorem 0.3. *(see corollary 4.4). Let X be a smooth projective fourfold of general type such that vol$(X) > \alpha^4$. If $\alpha \geq 1709$ then $h^0(X, (1+m)K_X) \geq n$ for all $n \geq 1$ and all $m \geq 191n$.*

Theorem 0.4. *(see corollary 4.8). Let X be a smooth projective fourfold of general type such that vol$(X) > \alpha^4$. If $\alpha \geq 2816$ then $|lK_X|$ gives a birational map for every $l \geq 817$.*

As before, we have more precise estimates on α, depending on the genus of the curves lying on X. See corollary 4.4 and corollary 4.8, respectively.

In the case of varieties of general type of dimension d, when l is sufficiently large, we also find functions $\alpha_1(d,l), \alpha_2(d,l)$ such that if $\mathrm{vol}(X) > \alpha_1(d,l)^d$ then either $h^0(lK_X) \neq 0$ or X is birational to a fibre space over a curve such that the general fibre has small volume (see theorem 4.12) and if $\mathrm{vol}(X) > \alpha_2(d,l)^d$ then either $|lK_X|$ is birational or X is birational to a fibre space over a curve such that the general fibre has small volume (see theorem 4.13); both these functions depend on the lower bounds of the volume of varieties of dimension equal or smaller than $d-2$, thus allowing us to find explicit results also in the case of fivefolds.

Unlike the fourfold case, the results about threefolds of general type are optimal in the sense that there exist threefolds of arbitrarily large volume with $P_1 = 0$ and $|4K_X|$ not birational. Therefore another interesting question that arises naturally when dealing with threefolds of general type is to study when $|4K_X|$ is birational. It is clear that $|4K_X|$ cannot be birational if X is birationally equivalent to a fibration over a curve B such that the general fibre is a minimal surface S with $K_S^2 = 1$ and with geometric genus $= 2$, since in this case $|4K_S|$ is not birational. In general the converse does not hold (see remark 3.11), but it turns out that it actually holds when the volume of X is sufficiently large. We prove the following:

Theorem 0.5. *(see corollary 3.10). Let X be a smooth projective threefold of general type such that vol(X) $> \alpha^3$. If $\alpha > 6141\sqrt[3]{2}$ then $|4K_X|$ does not give a birational map if, and only if, X is birational to a fibre space X'', with $f : X'' \to B$, where B is a curve, such that the general fiber X''_b is a smooth minimal surface of general type with volume* 1 *and geometric genus $p_g = 2$.*

Again, we have better estimates on α depending on the genus of the curves on X: see corollary 3.10.

The birationality of $|4K_X|$ has already been analyzed also by Lee, Dong, M.Chen, Zhang. Actually both Dong in [13] and Chen-Zhang in [10], requiring that the geometric genus (rather than the volume) of X is sufficiently large ($h^0(K_X) \geq 7$ for Dong, $h^0(K_X) \geq 5$ for Chen-Zhang), give characterizations for the birationality of the fourth pluricanonical map. Note, however, that the largeness of the geometric genus is not implied by the largeness of the volume (see remark 2.4).

The proofs of the above-mentioned results rest on the algebro-geometric techniques of the minimal model program. More precisely they rest on the study of singularities of pairs. The basic idea (already successfully applied by Angehrn-Siu, Hacon-McKernan, Takayama, Todorov and many others) is to produce log canonical centers of certain divisors, cut their dimension and then pull back sections from the log canonical centers to the ambient variety. When for threefolds of general type this cutting-down process is not fruitful then, following Todorov, we apply some results by McKernan about family of tigers to come down to study fibre spaces.

This thesis is organized as follows:

in Chapter 1 we fix the notation, recall the standard definitions and introduce new ones. We also present some of the relevant techniques in the generality we will need: when exact references are not available we will give also proofs;

in Chapter 2 we examine plurigenera for threefolds of general type and large volume, studying their non-vanishing and their size;

in Chapter 3 we deal with the birationality of the nth canonical maps ($n \geq 5$) for threefolds of general type and large volume. As direct consequences we characterize also such threefolds with birational 4th, 3rd or 2nd canonical map;

in Chapter 4 we study plurigenera and pluricanonical maps for varieties of general type of large volume and any dimension. Giving up optimality, we give explicit estimates for fourfolds and, in the sense already explained, also for fivefolds. As direct corollaries we also recover some well-known results about surfaces of general type;

in Chapter 5 we present a first attempt to treat adjoint pluricanonical systems for varieties of general type, but proving some results only for surfaces.

Chapter 1

Preliminaries

1.1 Notation

We will work over the field of complex numbers, $\mathbb{C}$. As in [31], [32] a *scheme* is a separated algebraic scheme of finite type over $\mathbb{C}$. A *variety* is a reduced, irreducible scheme. A *curve* is a variety of dimension 1. A *surface* is a variety of dimension 2. A *d-fold* is a variety of dimension d. We will usually deal with closed points of schemes, unless otherwise specified.

In our terminology a *countable set* is a set that has a bijection over a subset of $\mathbb{N}$, the set of natural numbers. Thus it can be finite or infinite. By $\mathbb{N}^+$ we will denote the set $\mathbb{N} \setminus \{0\}$; by $\mathbb{Q}^+$ the set $\{q \in \mathbb{Q} | q > 0\}$.

Unless otherwise specified a *divisor* or a $\mathbb{Q}$-*divisor* is meant to be Weil. A divisor is called $\mathbb{Q}$-*Cartier* if an integral multiple is a Cartier divisor. Of course when we work on smooth varieties Weil and Cartier divisors coincide.

Let $q \in \mathbb{Q}$: we write $[q] = \llcorner q \lrcorner$, $\ulcorner q \urcorner$, $\{q\}$ for the round-down (or integral part), round-up and fractional part of q, respectively. Recall that $[q]$ is the greatest integer $\leq q$, $\ulcorner q \urcorner$ is the least integer $\geq q$ and $\{q\} = q - [q]$.

If X is a variety and D a Weil-$\mathbb{Q}$-divisor on X, when writing $D = \sum_i q_i D_i$ we will assume that the D_i's are distinct prime divisors. Given the case, we also define the round-down of D, $[D] = \llcorner D \lrcorner$, and the round-up of D, $\ulcorner D \urcorner$, as $[D] := \sum_i [q_i] D_i$, $\ulcorner D \urcorner := \sum_i \ulcorner q_i \urcorner D_i$.

A projective morphism $f : X \to Y$ is called an (algebraic) *fibre space* (according to Mori) if X, Y are smooth projective varieties, f is surjective and $f_*(\mathcal{O}_X) = \mathcal{O}_Y$. Notice that, under this definition, $f_*(\mathcal{O}_X) = \mathcal{O}_Y$ is the same as requiring f to have connected fibres.

1.2 Topological issues

In this section we will recall some basic definitions and state some easy results of topological flavour that will be used in the proof of the main theorems.

First of all we just recollect how the operation of closure behaves in relation with the operation of union and intersection and what this implies if we consider the Zariski topology:

Lemma 1.1. *Let X be a topological space and A_i subsets of X for all $i \in I$. Then $\overline{\cup_{i\in I} A_i} \supseteq \cup_{i\in I}\overline{A_i}$ and $\overline{\cap_{i\in I} A_i} \subseteq \cap_{i\in I}\overline{A_i}$. If $|I| < \infty$ then $\overline{\cup_{i\in I} A_i} = \cup_{i\in I}\overline{A_i}$.*

Proof. $\overline{\cup_{i\in I} A_i} \supseteq \cup_{i\in I} A_i \supseteq A_i$ for all $i \in I$, therefore $\overline{\cup_{i\in I} A_i} \supseteq \overline{A_i}$ for all $i \in I$. Hence $\overline{\cup_{i\in I} A_i} \supseteq \cup_{i\in I}\overline{A_i}$.

Since $A_i \subseteq \overline{A_i}$ for all $i \in I$, $\cap_{i\in I} A_i \subseteq \cap_{i\in I}\overline{A_i}$. An arbitrary intersection of closed sets is closed, hence $\overline{\cap_{i\in I} A_i} \subseteq \cap_{i\in I}\overline{A_i}$.

If $|I| < +\infty$ then $\cup_{i\in I}\overline{A_i}$ is closed and contains $\cup_{i\in I} A_i$. Therefore $\overline{\cup_{i\in I} A_i} \subseteq \cup_{i\in I}\overline{A_i}$ and we are done. $\square$

Lemma 1.2. *Let X be a scheme and $A \subseteq X$. If $\overline{A} = \cup_{i=1}^{k} Z_i$, irredundant decomposition into irreducible closed sets (cf. [21, I, 1.5]), then, for all $1 \leq i \leq k$, $\overline{A \cap Z_i} = Z_i$.*

Proof. First of all, let us recall that if an irreducible closed set Y is contained in a finite union of closed sets W_j, $0 \leq j \leq s < +\infty$, then, by the irreducibility hypothesis, there exists $\overline{j}$ such that $Y \subseteq W_{\overline{j}}$.

Fix $j \in \{1, \dots, k\}$. $Z_j \subseteq \overline{A} = \overline{A \cap (\cup_{i=1}^{k} Z_i)} = \overline{\cup_{i=1}^{k}(A \cap Z_i)} = \cup_{i=1}^{k}\overline{A \cap Z_i}$ by lemma 1.1. Therefore there exists $\overline{i}$ such that $Z_j \subseteq \overline{A \cap Z_{\overline{i}}}$. Since $Z_j \subseteq Z_{\overline{i}}$, by the irredundancy hypothesis $\overline{i}$ must be equal to j, hence $Z_j \subseteq \overline{A \cap Z_j}$.

Also, since Z_j is closed, $Z_j \supseteq \overline{A \cap Z_j}$ and so we can conclude. $\square$

The following lemma asserts that every dense set (in the Zariski topology) is separable, in the sense of general topology (cf. [28, p. 49]):

Lemma 1.3. *Let X be a scheme of dimension n. Let $A \subseteq X$ be a Zariski-dense set. Then there exists a Zariski-dense countable set $B \subseteq A$.*

Proof. Without loss of generality we can suppose X to be irreducible: actually if $X = \cup_{i=1}^{k} Z_i$, irredundant decomposition into irreducible closed sets, then, by lemma 1.2, $A \cap Z_i$ is Zariski-dense in Z_i for all i.

Now assume X irreducible. If $n = 0$ the statement is clear, otherwise we will prove that if $A \subseteq X$ is a Zariski-dense set then, for all $1 \leq d \leq n$, there exists a countable set $B_d \subseteq A$ such that $\overline{B_d}$ is an irreducible variety of dimension at least d. $d = n$ is our thesis.

Let $d = 1$. Since A is Zariski-dense, A has at least infinitely countable many elements. Choose $B \subseteq A$ with $|B| = |\mathbb{N}|$. $\overline{B}$ has dimension at least 1. $\overline{B} = \cup_{i=1}^{k} C_i$, irredundant decomposition into irreducible closed sets. There exists j such that C_j

has dimension at least 1. Let $B_1 := B \cap C_j$. By lemma 1.2, $\overline{B \cap C_j} = C_j$ and so the case $d = 1$ is proved.

For $1 \leq d \leq n-1$ suppose we have constructed B_d^A for every $A \subseteq X$ Zariski-dense. Let us construct B_{d+1}^A. Let $B_d^1 \subseteq A$ be a countable set such that $Y_1 = \overline{B_d^1}$ is an irreducible closed set of dimension at least d. If the dimension is at least $d+1$ we are done. Otherwise let us consider $A_1 := A \setminus Y_1$. A_1 is Zariski-dense, in fact $X = \overline{A} = \overline{(A \cap Y_1^c) \cup (A \cap Y_1)} = \overline{A \cap Y_1^c} \cup \overline{A \cap Y_1}$ by lemma 1.1 and, since X is irreducible and $\overline{A \cap Y_1} \subseteq Y_1 \subsetneq X$, we have that $\overline{A_1} = \overline{A \setminus Y_1} = \overline{A \cap Y_1^c} = X$. There exists $B_d^2 \subseteq A_1$ a countable set such that $Y_2 := \overline{B_d^2}$ is irreducible and of dimension at least d. Note that Y_2 is different from Y_1, since Y_2 contains points not in Y_1. If $\dim Y_2 \geq d+1$ then we are done, otherwise we go on: $A_k := A \setminus (\cup_{i=1}^k Y_i)$ is a Zariski-dense set; let $B_d^{k+1} \subseteq A_k$ a countable set such that $Y_{k+1} := \overline{B_d^{k+1}}$ is an irreducible closed set of dimension at least d. Note that Y_{k+1} is different from $Y_1, \ldots, Y_k$. If $\dim Y_{k+1} \geq d+1$ we are done, otherwise at the end we have countably many countable sets B_d^i such that $Y_i = \overline{B_d^i}$ are irreducible, distinct, closed sets ($i \in \mathbb{N}$). Let $B' = \cup_{i \in \mathbb{N}} B_d^i$. B' is a countable set and $\overline{B'} \supseteq \cup_{i \in \mathbb{N}} \overline{B_d^i} = \cup_{i \in \mathbb{N}} Y_i$, therefore $\overline{B'}$ contains an irreducible closed set of dimension at least $d+1$. As before, a suitable choice (cf. lemma 1.2) of $B_{d+1} \subseteq B'$ will do.

$\square$

Definition 1.4. Let X be a variety. Let $P \subseteq X$. P is called *very general* if it is the complement of a countable union of proper closed subvarieties of X. P is called *countably dense* if it is not contained in the union of countably many proper closed subvarieties of X.

As we will see in the following lemma, countable density is a property stronger than Zariski-density but not as much constraining as being very general. If (very) general sets will usually be the starting point of our analysis it is also true that manipulating these sets leads us to face countably dense sets rather than other (very) general sets. For example if we randomly decompose a very general set into a finite (or countable) union of disjoint sets then we loose information about being very general but we rest assured that at least one of the new sets is countably dense.

Lemma 1.5. *Let X be a variety of dimension $d \geq 1$ and let $A, B, C \subseteq X$.*

1. *If A is countably dense then A is Zariski-dense.*
2. *If A is very general then A is countably dense (and hence Zariski-dense).*
3. *If A is countably dense and B is very general, then $A \cap B$ is countably dense.*
4. *If $A \setminus B \subseteq C$, with A very general, then either B is countably dense or C contains a very general subset of X.*

Proof. 1. Every closed set is a finite union of closed irreducible subsets (cf. [21, I, 1.5]). So, by the very definition of countably dense, $\overline{A} = X$.

2. By hypothesis $A = X \setminus \cup_{i\in\mathbb{N}} V_i$, $V_i \subsetneq X$ proper closed subsets of X. If A is not countably dense, then $A \subseteq \cup_{i\in\mathbb{N}} Z_i$, where Z_i are proper closed subsets of X. Therefore $X = \cup_{i\in\mathbb{N}} V_i \cup \cup_{i\in\mathbb{N}} Z_i$, a countable union of proper closed subsets. But this is not possible, since X is irreducible.

3. By hypothesis $B = X \setminus \cup_{i\in\mathbb{N}} V_i$, $V_i \subsetneq X$ proper closed subsets of X. If $A \cap B$ is not countably dense, then $A \cap B \subseteq \cup_{i\in\mathbb{N}} Z_i$, where Z_i are proper closed subsets of X. Then $A \cap (X \setminus \cup_{i\in\mathbb{N}} V_i) \subseteq \cup_{i\in\mathbb{N}} Z_i$, that is $(A \cap (X \setminus \cup_{i\in\mathbb{N}} V_i)) \cup \cup_{i\in\mathbb{N}} V_i \subseteq \cup_{i\in\mathbb{N}} Z_i \cup \cup_{i\in\mathbb{N}} V_i$, therefore $A \subseteq \cup_{i\in\mathbb{N}} Z_i \cup \cup_{i\in\mathbb{N}} V_i$, contradiction.

4. By hypothesis $A = X \setminus \cup_{i\in\mathbb{N}} V_i$, $V_i \subsetneq X$ proper closed subsets of X. If B is not countably dense then, by definition, $B \subseteq \cup_{i\in\mathbb{N}} Z_i$, where Z_i are proper closed subsets of X. Hence $X \setminus (\cup_{i\in\mathbb{N}} V_i \cup \cup_{i\in\mathbb{N}} Z_i) \subseteq A \setminus B$ and we are done.

□

If we have a family of points and divisors through them, then the countable density of the set of points is the right property that allows us to extract a finite number of divisors that are "unrelated", in a certain sense:

Lemma 1.6. *Let X be a variety of dimension ≥ 1 and let A be a countably dense subset of X. Suppose that for all $x \in A$ there exists a divisor D_x such that $x \in$ Supp(D_x). Then there exist $x_1, x_2 \in A$ such that $x_1 \notin$ Supp(D_{x_2}) and $x_2 \notin$ Supp(D_{x_1}).*

More generally, under the same hypotheses, for every $n \in \mathbb{N}$ there exist $x_1, \ldots, x_n \in A$ such that $x_i \notin$ Supp(D_{x_j}) for every $i \neq j$, $1 \leq i, j \leq n$.

Proof. By lemma 1.5 and lemma 1.3, there exists a countable, Zariski-dense set $B \subset A$. For all $b \in B$ consider D_b. $V := A \setminus \cup_{b\in B}\mathrm{Supp}(D_b)$ is non-empty (otherwise $A \subseteq \cup_{b\in B}\mathrm{Supp}(D_b)$, contradiction). Let $x_1 \in V$. Since B is Zariski-dense, D_{x_1} cannot pass through b for every $b \in B$. Let x_2 such that $x_2 \notin \mathrm{Supp}(D_{x_1})$ and we are done.

For the general case choose $B_1 := B$ as before. We define $B_2, \ldots, B_{n-1}$ inductively: suppose we have already defined $B_2, \ldots, B_i$; since $V_i := A \setminus \cup_{k=1}^{i} \cup_{b\in B_k} \mathrm{Supp}(D_b)$ is still countably dense, and hence Zariski-dense, we can choose a countable Zariski-dense set $B_{i+1} \subset V_i$. Now we define $x_1, \ldots, x_n$ inductively. Choose a point $x_n \in V_{n-1}$. Suppose we have already defined $x_n, x_{n-1}, \ldots, x_{i+1}$. Since B_i is Zariski-dense, there exists a point x_i such that it does not belong to $\cup_{k=i+1}^{n}\mathrm{Supp}(D_{x_k})$. $x_1, \ldots x_n$, defined in this way, respect the requirements on the associated divisors, and we are done. □

Note that Zariski-density is not enough to obtain the same conclusion: for example consider, on a curve, a countable infinity of points $\{x_1, \ldots, x_n, \ldots\}$ and, for every x_n, the divisor $D_n = x_1 + \ldots + x_n$.

When we will study pluricanonical systems on a projective variety X it will be clear that we can have better explicit results if we know that we do not need to deal with curves of small volume (i.e., of small genus). That is why we define *g-countably dense* varieties as varieties in which we cannot avoid curves of genus $< g$, not even "very generally". More precisely we give the following

Definition 1.7. Let X be a projective variety. Let $g \in \mathbb{N}^+$. Let

$$\Omega_g := \bigcup_{\substack{C \text{ curve } \subseteq X, \\ g(C) < g}} C$$

(where $g(C)$ is the geometric genus of the (possibly singular) curve C). Then we will say that X is g-countably dense if Ω_g is countably dense, that is: Ω_g is not contained in the union of countably many proper closed subvarieties of X.

Remark 1.8. Clearly, if X is not g-countably dense then X is not g'-countably dense for every $g' \leq g$. Moreover if X is not g-countably dense then, by definition, there exists a very general subset Λ of X such that for every $x \in \Lambda$ and every curve C through x then $g(C) \geq g$.

Remark 1.9. Being g-countably dense is a birational property.

If X is a projective variety and Λ a very general subset of X, we can define

$$g_{X,\Lambda} := \min_{\left\{\substack{x \in \Lambda, C \text{ curve} \subseteq X, \\ x \in C}\right\}} g(C)$$

and

$$g_X := \max_{\left\{\substack{\Lambda \subseteq X, \\ \Lambda \text{ very general}}\right\}} \{g_{X,\Lambda}\}.$$

Note that the max is well defined: just consider a very ample divisor A on X and observe that for every point x there is a curve passing through x and of geometric genus $\leq p_a(A^{n-1})$, where p_a is the arithmetic genus.

Remark 1.10. Every projective variety X is not g_X-countably dense but it is (g_X+1)-countably dense. In fact by the definition of g_X there exists Λ very general such that $g_{X,\Lambda} = g_X$, hence $\Omega_{g_X} \subseteq X \setminus \Lambda$, which is not countably dense. Conversely, if X were not (g_X+1)-countably dense then by remark 1.8 there would exist a very general subset Λ for which $g_{X,\Lambda} \geq g_X + 1$: contradiction.

To prove that the maps we will consider are birational we will essentially prove that they separate two very general points. Note that, by the following lemma, this is actually enough.

Lemma 1.11. *Let X and Y be projective varieties, with $\dim(X) \geq 1$ and let $\phi : X \dashrightarrow Y$ be a dominant rational map. If there exists a very general set $U \subseteq X$ such that ϕ separates every couple of points in U then ϕ is birational.*

Proof. By resolution of the locus of indeterminacy and resolution of singularities we can suppose that X is smooth and that ϕ is a projective morphism. Let d be the dimension of X. By definition, there exist countable many proper closed subvarieties of X, $\{V_i\}_{i\in\mathbb{N}}$, such that $U = X \setminus \cup_{i\in\mathbb{N}} V_i$. By generic smoothness (see [21, cor. III.10.7]) there exists a nonempty open set $W \subseteq Y$ such that $\overline{\phi} := \phi|_{\phi^{-1}(W)}$ is a smooth morphism. Hence, in particular, $\overline{\phi}$ is flat and all the fibres are equidimensional of a certain dimension k (see [21, thm. III.10.2]). Since $\phi^{-1}(W)$ is open and nonempty (because ϕ is dominant), then $\phi^{-1}(W) \cap U \neq \emptyset$. Let $p \in \phi^{-1}(W) \cap U$. By hypothesis we know that for any other point $u \in \phi^{-1}(W) \cap U$, $u \neq p$, we have that $\overline{\phi}(p) \neq \overline{\phi}(u)$, hence p is an irreducible component of $\overline{\phi}^{-1}(\overline{\phi}(p))$, thus $k = 0$ and so $\dim(Y) = d$. Moreover $\exists s \in \mathbb{N}^+$ such that $|\overline{\phi}^{-1}(w)| = s$ for any $w \in W$. Let $U' := Y \setminus \cup\phi(V_i)$. Since $\dim(Y) = d$ then U' is a very general subset of Y, hence $U' \cap W \neq \emptyset$. If $q \in U' \cap W$ then $\overline{\phi}^{-1}(q) \subset \phi^{-1}(W) \cap U$. Since $\overline{\phi}$ separates points in $\phi^{-1}(W) \cap U$, then $s = 1$, i.e., $\overline{\phi}$ is an isomorphism and so ϕ is birational. □

1.3 Volume and big divisors

Definition 1.12. Let X be a projective variety of dimension d and let D be a Cartier integral divisor. Then the *volume of D*, $\mathrm{vol}(D)$, is just

$$\limsup_{m\to+\infty} \frac{h^0(X, mD) \cdot d!}{m^d}.$$

It turns out that this limsup is actually a limit. Moreover by the homogeneous property of the volume we can extend this definition to $\mathbb{Q}$-Cartier divisors. It occurs that the volume of a divisor does depend only on its numerical class. If X is nonsingular and K_X is its canonical bundle then $\mathrm{vol}(X) := \mathrm{vol}(K_X)$. Since the volume of a divisor is a birational invariant then if X is singular take any desingularization X' of X and set $\mathrm{vol}(X) := \mathrm{vol}(X')$. If $\mathrm{vol}(D) > 0$ then D is called *big*. If K_X is big then X is called a variety of *general type*. For all these matters see [31, 2.2.C].

Example 1.13. Let X be a projective variety of dimension d and let D be a big and nef Cartier integral divisor. Then, by asymptotic Riemann-Roch (see [31, cor. 1.4.41]), we have that $\mathrm{vol}(D) = D^d$.

Definition 1.14. Let X be a projective variety, let $V \subseteq X$ be a subvariety of dimension $d \geq 1$ and let D be a Cartier integral divisor. The *restricted volume of D along V* is

$$\mathrm{vol}_{X|V}(D) := \limsup_{m\to+\infty} \frac{h^0(X|V, mD) \cdot d!}{m^d},$$

where

$$H^0(X|V, mD) := Im\left(H^0(X, mD) \to H^0(V, mD|_V)\right)$$

and $h^0(X|V, mD)$ is its dimension.

Notice that if $V = X$ then $\mathrm{vol}_{X|V}(D) = \mathrm{vol}_X(D) = \mathrm{vol}(D)$. If $V \not\subseteq \mathbb{B}_+(D)$ then the limsup is a limit and it actually depends only on the numerical class of D.

Moreover, as in the case of the volume, the definition can be extended to $\mathbb{Q}$-Cartier divisors. For all these matters see [17, sect. 2].

Example 1.15. Let X be a projective variety and let A be an ample integral Cartier divisor. By Serre's theorem (see [31, thm. 1.2.6]) the restriction maps are eventually surjective. Moreover $A|_V$ is ample on V, hence $\mathrm{vol}_{X|V}(A) = \mathrm{vol}_V(A|_V) = (A|_V)^d = A^d \cdot V$.

Example 1.16. Clearly $\mathrm{vol}_{X|V}(D) \leq \mathrm{vol}_V(D|_V)$, but it may very well happen that $\mathrm{vol}_{X|V}(D) < \mathrm{vol}_V(D|_V)$. For example, let X be a smooth projective surface and let D be a Cartier integral divisor on X. Consider $\mu_1 : X' \to X$, the blow-up of X at a point, and its exceptional divisor E_1. Finally, consider $\mu_2 : X'' \to X'$ the blow-up of X' at a point living on E_1. Call E_2 the exceptional divisor of μ_2 and, by a slight abuse of notation, denote by E_1 the proper transform of E_1 under μ_2. Let $\mu := \mu_1 \circ \mu_2$. Let $B := \mu^*(D) + 2E_1 + E_2$. $\mathcal{O}_{E_2}(B|_{E_2}) = \mathcal{O}_{E_2}(1)$, hence $B|_{E_2}$ is big on E_2. On the contrary, for every $m \in \mathbb{N}^+$, consider

$$0 \to \mathcal{O}_{X''}(mB - E_2) \to \mathcal{O}_{X''}(mB) \to \mathcal{O}_{E_2}(mB|_{E_2}) \to 0.$$

By Fujita's lemma (see [27, lemma 1-3-2(3)]):

$$H^0(X'', mB - E_2) = H^0(X, mD) = H^0(X'', mB),$$

hence for every $m \in \mathbb{N}^+$ the restriction map $H^0(X'', mB) \to H^0(E_2, mB|_{E_2})$ is the null map. Therefore $\mathrm{vol}_{X''|E_2}(B) = 0$.

Thus the volume of an integral divisor measures the number of its sections, but only asymptotically, while the restricted volume measures the number of sections of the restriction $mD|_V$ that can be lifted to X, but, again, only asymptotically. Even so, one can hope (in certain cases) to obtain information also about actual multiples of the divisor.

The technique to pass from the limit to effective numbers has been developed by Angehrn and Siu for their famous theorem: the key point is to find a specific subvariety and then prove that the restriction map (for the given divisor) is surjective. Both to produce the subvariety and to study the surjectivity of the restriction map, one needs to use particular techniques, such as the Tie Breaking (see [29, prop. 8.7.1] and [7, thm. 3.7]) or Nadel's vanishing theorem (see [32, thm. 9.4.8]), that require the divisor to be ample (or big and nef). When the divisor is not ample but only big then we can use local analogues: in fact a big divisor is ample outside a closed subset. The following definitions and lemma will make this clearer:

Definition 1.17. Let X be a variety, let D be a $\mathbb{Q}$-Cartier divisor and let $p \in \mathbb{N}^+$ be such that pD is integral. The *stable base locus* of D is the algebraic set $\mathbb{B}(D) = \bigcap_{m \geq 1} Bs(|mpD|)$, where $Bs(|mpD|)$ is the set-theoretic base locus of the linear system $|mpD|$ with the convention that $Bs(|mpD|) = X$ if $|mpD|$ is empty (cf. [16, §1] or [31, def. 2.1.20, rmk. 2.1.24]). By [31, prop. 2.1.21] actually there exists an integer m_0 such that $\mathbb{B}(D) = Bs(|km_0D|)$ for all $k \geq 0$.

Unfortunately these loci do not depend only on the numerical class of D. Nakamaye then suggested to slightly perturb D: the *augmented base locus* of D is defined as $\mathbb{B}_+(D) = \mathbb{B}(D - \epsilon A)$ for any ample A and sufficiently small $\epsilon \in \mathbb{Q}^+$. This definition does not depend on A or on ϵ (provided it is sufficiently small). Moreover D is big if and only if $\mathbb{B}_+(D)$ is a proper closed subset of X (see [16, §1 and ex. 1.7]). The augmented base locus and restricted volumes are closely related notions: in fact, generalizing Nakamaye's theorem from big and nef $\mathbb{Q}$-Cartier divisors to arbitrary $\mathbb{Q}$-Cartier divisors, Ein et al. proved in [17, thm. C] that

$$\mathbb{B}_+(D) = \bigcup_{\left\{\substack{V:V \text{ subv. of } X, \\ \dim(V) \geq 1, \\ \mathrm{vol}_{X|V}(D)=0}\right\}} V$$

Lemma 1.18. *Let X be a projective variety and D a big $\mathbb{Q}$-Cartier divisor on X. Then there exists $\epsilon > 0$ such that for every ample $\mathbb{Q}$-Cartier divisor A, $||A|| < \epsilon$, and for every $x \notin \mathbb{B}_+(D)$ there is an effective $\mathbb{Q}$-Cartier divisor E such that $x \notin$ Supp(E) and $D \sim_{\mathbb{Q}} A + E$.*

Proof. By [16, §1], [32, 10.3.2] and [31, 2.1.21], there exists $m \in \mathbb{N}$ such that mD, mA are integral divisors and $\mathbb{B}_+(D) = \mathbb{B}(D-A) = Bs(|mD - mA|)$. Since $x \notin \mathbb{B}_+(D)$ then there exists an effective divisor $F \in |mD - mA|$ such that $x \notin \mathrm{Supp}(F)$. Set $E := F/m$. $D \sim_{\mathbb{Q}} A + E$ and we are done. ☐

Remark 1.19. We could have chosen E to skip n points not in $\mathbb{B}_+(D)$.

Remark 1.20. If D is a big $\mathbb{Q}$-Cartier divisor then we have

$$\mathbb{B}_+(D) = \bigcap_{\left\{\substack{E:E \ \mathbb{Q}\text{-Cartier divisor}, \\ E \text{ effective}, \\ D-E \text{ ample}}\right\}} \mathrm{Supp}(E),$$

where $\supseteq$ comes from lemma 1.18 and $\subseteq$ from its converse: if there exists an effective $\mathbb{Q}$-Cartier divisor E such that $x \notin \mathrm{Supp}(E)$ and $D \sim_{\mathbb{Q}} A + E$, for a certain ample $\mathbb{Q}$-Cartier divisor A, then $x \notin \mathbb{B}_+(D)$. In fact $\mathbb{B}_+(D) = \mathbb{B}(D - \epsilon A)$ (for a sufficiently small $\epsilon \in \mathbb{Q}^+$) $= Bs(|mD - m\epsilon A|)$ (for a sufficiently large and divisible $m \in \mathbb{N}$); since for a sufficiently large and divisible m we have that $m(1-\epsilon)A$ is very ample and that $mD - m\epsilon A \sim_{\mathbb{Z}} m(1-\epsilon)A + mE$ then $Bs(|mD - m\epsilon A|) \subseteq \mathrm{Supp}(E)$ and thus $x \notin \mathbb{B}_+(D)$.

Notice that, because of the properties recalled above, some authors refer to $\mathbb{B}_+(D)$ as the *non ample locus of D*.

Since, as explained before, our analysis will heavily use the fact that varieties and subvarieties have sufficiently large volumes, it is fundamental, when dealing with varieties of general type, to accurately avoid subvarieties not of general type. This is possible by the following well-known fact (see, for example, [20, proof of 1.1]):

Lemma 1.21. *Let X be a projective variety of general type. Let W be the union of all subvarieties of X not of general type. Then W is contained in a countable union of proper subvarieties of X.*

Proof. Possibly considering a resolution of singularities, we can suppose that X is smooth.

For any polynomial $h = h(t) \in \mathbb{Q}[t]$ consider the Hilbert scheme $\mathrm{Hilb}^h(X)$ that parametrizes closed subschemes of X with Hilbert polynomial h. Consider also its universal family $\mathcal{Y}^h \subseteq \mathrm{Hilb}^h(X) \times X$ and the two natural projections $\pi : \mathcal{Y}^h \to X$ and $f : \mathcal{Y}^h \to \mathrm{Hilb}^h(X)$ (see [34, par. 2]):

$$\begin{tikzcd} \mathcal{Y}^h \arrow[r, "\pi"] \arrow[d, "f"] & X \\ \mathrm{Hilb}^h(X) & \end{tikzcd}$$

If a fibre of f (and hence, by upper semicontinuity, a general fibre) is not of general type then we will show that π is not dominant. This is enough to conclude, since $\mathbb{Q}[t]$ is a countable set.

By contradiction, suppose that a general fibre of f is not of general type but π is dominant. Potentially taking general hyperplane sections of $\mathrm{Hilb}^h(X)$ and restricting $\mathcal{Y}^h$ to their preimages through f, we can suppose that $\dim(\mathcal{Y}^h) = \dim(X)$.

By Stein's factorization (see [21, cor. III.11.5]), π can be factored as follows:

$$\begin{tikzcd} \mathcal{Y}^h \arrow[r, "\pi"] \arrow[d, "\gamma"] & X \\ \mathcal{I}^h \arrow[ur, "\psi"'] & \end{tikzcd}$$

where γ has connected fibres and ψ is a finite morphism. Clearly, in our case, ψ is dominant and, since $\dim(\mathcal{Y}^h) = \dim(X)$, γ is also birational.

Since X is a variety of general type then by [31, prop. 1.2.13] $\psi^* K_X$ is big on $\mathcal{I}^h$. Passing to a resolution $\phi : \widetilde{\mathcal{I}^h} \to \mathcal{I}^h$ we have that $\phi^*\psi^*(K_X)$ is big on $\widetilde{\mathcal{I}^h}$. Since $\psi \circ \phi$ is dominant and X, $\widetilde{\mathcal{I}^h}$ are smooth then we can apply [12, par. 1.41, (1.11)]: $K_{\widetilde{\mathcal{I}^h}} \sim \phi^*\psi^*(K_X) + \mathrm{Ram}(\psi \circ \phi)$, where $\mathrm{Ram}(\psi \circ \phi)$ is an effective divisor. Hence $K_{\widetilde{\mathcal{I}^h}}$ is big. Thus since $\mathcal{Y}^h$ and $\widetilde{\mathcal{I}^h}$ are birational then also $\mathcal{Y}^h$ is a variety of general type, i.e., $K_{\mathcal{Y}^h}$ is big. Therefore if F is a general fibre of f we have that $K_F = K_{\mathcal{Y}^h}|_F$ is big. Contradiction. □

Remark 1.22. If X is a projective variety of general type then, by lemma 1.21, there exists a very general subset Λ of X such that every subvariety through any point of Λ is of general type. Hence such an X is not 2-countably dense (see definition 1.7).

1.4 Multiplier ideals and singularities of pairs

First of all we recall some standard definitions:

Definition 1.23. (cf. [32, 9.1.10, 9.3.55] and [30, 0.4]). A *pair* (X, Δ) consists of a normal variety X and a $\mathbb{Q}$-divisor Δ such that $K_X + \Delta$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor.

The pair (X, Δ) is said to be *effective* if Δ is effective.

A projective birational morphism $\mu : X' \to X$ is said to be a *log resolution of the pair* (X, Δ) if X' is smooth, $\mathrm{Exc}(\mu)$ is a divisor and $\mu^{-1}(\mathrm{Supp}(\Delta)) \cup \mathrm{Exc}(\mu)$ is a divisor with simple normal crossing support.

Let L be an integral Cartier divisor on X, let $V \subseteq H^0(X, L)$ and let $|V|$ be the corresponding linear series. A projective birational morphism $\mu : X' \to X$ is said to be a *log resolution for* $(X, |V|)$ if X' is smooth, $\mathrm{Exc}(\mu)$ is a divisor and $\mu^*|V| = |W| + F$, where $W \subseteq H^0(X', \mathcal{O}_{X'}(\mu^*L - F))$ defines a base point free linear series and $F + \mathrm{Exc}(\mu)$ is a divisor with simple normal crossing supports.

Definition 1.24. (cf. [32, §9.2.A]). Let X be a smooth variety and let D be a $\mathbb{Q}$-divisor on X. The *multiplier ideal sheaf* $\mathcal{J}(D) = \mathcal{J}(X, D)$ is defined in the following way: fix any log resolution $\mu : X' \to X$ of (X, D); then $\mathcal{J}(D) := \mu_*\mathcal{O}_{X'}\left(K_{X'/X} - [\mu^*D]\right)$. The definition does not depend on the chosen log resolution and if D is effective then $\mathcal{J}(D)$ is actually an ideal sheaf.

The multiplier ideal associated to an effective $\mathbb{Q}$-divisor D should be seen as a way to measure, in a subtle way, the singularities of D (or, better, of the pair (X, D)). For example consider $X = \mathbb{A}^2$, $C_1 = \{y^2 = x^3 - x^2\}$ a rational curve with a node in $(0, 0)$ and $C_2 = \{y^2 = x^3\}$ a rational curve with a cusp in $(0, 0)$. To compute $\mathcal{J}(C_1)$ we can just use, as a log resolution of (X, C_1), a single blow up at $(0, 0)$ (we need it since C_1 is singular, hence it has not simple normal crossing support): $\mathcal{J}(X, C_1) = \mathcal{O}_X(-C_1)$. Moreover $\mathcal{J}(cC_1) = \mathcal{O}_X$ for every $0 \leq c < 1$. To compute $\mathcal{J}(C_2)$ we need three successive blow-ups to make C_2 with simple normal crossing support. As before, $\mathcal{J}(C_2) = \mathcal{O}_X(-C_2)$, but this time $\mathcal{J}(cC_2) = \mathcal{O}_X$ for every $0 \leq c < 5/6$, while $\mathcal{J}(cC_2) \subsetneq \mathcal{O}_X$ as soon as $c \geq 5/6$. Therefore in a certain sense multiplier ideals have recognized C_2 to have worse singularities than C_1.

It is important also to stress out that the use of the language of multiplier ideals allows us to apply Kodaira-type non-vanishing theorems (such, for example, Kawamata-Viehweg - see [32, thm. 9.1.18]) without the need to pass to a new space every time in order to resolve singularities and obtain simple normal crossing divisors.

Definition 1.25. Let (X, Δ) be a pair and $\mu : X' \to X$ be a log resolution of the pair. We can canonically write $K_{X'} - \mu^*(K_X + \Delta) \equiv \sum a(E)E$, where the sum is taken over all prime divisors E. These $a(E) = a(E, X, \Delta) \in \mathbb{Q}$ are called discrepancies of (X, Δ).

Given $x \in X$, (X, Δ) is said to be *klt at x* or *kawamata log terminal at x* (respectively: *lc at x* or *log canonical at x*) if for every E such that $x \in \mu(E)$ we have that $a(E) > -1$ (resp.: $a(E) \geq -1$). (X, Δ) is *klt* or *kawamata log terminal* (respectively: *lc* or *log canonical*) if it is klt (resp.: lc) at x for every $x \in X$. These definitions do not depend on the log resolution μ.

We say that a subvariety $V \subset X$ is a *lc centre* or *log canonical centre* for the pair (X, Δ) if it is the image, through a certain μ, of a divisor E of discrepancy ≤ -1.

The valuation corresponding to this divisor is called a *log canonical place*.

A log canonical centre V for the pair (X, Δ) is *pure* if it is log canonical at the generic point of V.

A log canonical centre V for the pair (X, Δ) is *exceptional* if it is pure and there is a unique log canonical place lying over the generic point of V.

We will denote by $LLC(X, \Delta, x)$ the set of all lc centres for (X, Δ) that pass through the point x.

If (X, D) is effective and X is smooth, then we can use equivalent definitions for klt and lc, based on multiplier ideals:

Lemma 1.26. *Let X be a smooth variety and let (X, D) be an effective pair. (X, D) is klt (resp. lc) if and only if $\mathcal{J}(D) = \mathcal{O}_X$ (resp. $\mathcal{J}((1-\epsilon)D) = \mathcal{O}_X \ \forall\ 0 < \epsilon < 1$).*

Proof. Let $\mu : X' \to X$ be a log resolution of (X, D). Let $K_{X'} - \mu^*(K_X) = \sum_E a(E)E$ and let $K_{X'} - \mu^*(K_X + D) = \sum_E (a(E) + b(E))E$. Hence $K_{X'/X} - [\mu^*((1-\epsilon)D)] = \sum_E (a(E) + \ulcorner(1-\epsilon)b(E)\urcorner)E$. Notice that $a(E)$ is always a non-negative integer, $a(E) > 0$ implies that E is exceptional and that $b(E)$ is always non-positive.

(X, D) is klt, by definition, if $\forall E\ a(E) + b(E) > -1 \Leftrightarrow \ulcorner a(E) + b(E)\urcorner \geq 0 \Leftrightarrow a(E) + \ulcorner b(E)\urcorner \geq 0 \Leftrightarrow \mathcal{J}(D) = \mathcal{O}_X$, in fact if $a(E) = 0$ then $\ulcorner b(E)\urcorner \geq 0$ and this implies $\ulcorner b(E)\urcorner = 0$; if $a(E) > 0$ then E is exceptional, hence just apply Fujita's lemma (see [27, lemma 1-3-2(3)]).

(X, D) is lc, by definition, if $\forall E\ a(E) + b(E) \geq -1$. Since $a(E)$ is non-negative $a(E)+b(E) = -1 \Rightarrow b(E) < 0 \Leftrightarrow (1-\epsilon)b(E) > b(E)$ for all $0 < \epsilon < 1 \Leftrightarrow a(E) + (1-\epsilon)b(E) > -1$ for all $0 < \epsilon < 1$. Hence $a(E) + b(E) \geq -1 \Leftrightarrow a(E) + (1-\epsilon)b(E) > -1$ for all $0 < \epsilon < 1 \Leftrightarrow (X, (1-\epsilon)D)$ is klt for all $0 < \epsilon < 1 \Leftrightarrow \mathcal{J}((1-\epsilon)D) = \mathcal{O}_X$ for all $0 < \epsilon < 1$. □

Remark 1.27. Also local analogues hold: (X, D) is klt (resp. lc) at a point $x \in D$ if and only if $\mathcal{J}(D)_x = \mathcal{O}_{X,x}$ (resp. $\mathcal{J}((1-\epsilon)D)_x = \mathcal{O}_{X,x}$ for every $0 < \epsilon < 1$).

These considerations justify the following:

Definition 1.28. Let (X, D) be an effective pair, with X smooth and let $x \in X$. The *log canonical threshold at x*, $lct(D, x) = lct(X, D, x)$, is just $\inf\{c > 0 | \mathcal{J}(X, cD)_x \subsetneq \mathcal{O}_{X,x}\}$.

We will denote by $Nklt(X, D)$ the *non-klt locus for* (X, D), that is $\operatorname{Supp}(\mathcal{O}_X/\mathcal{J}(X, D)) \subset X$ with the reduced structure.

For practical purposes it is convenient to give a way to compute log canonical thresholds using discrepancies (see [32, 9.3.16]):

Lemma 1.29. *Let (X, D) be an effective pair, with X smooth and let $x \in X$. Fix any log resolution $\mu : X' \to X$ of the pair (X, D). Let $K_{X'/X} = \sum_E a(E)E$ and let $-\mu^*(D) = \sum_E b(E)E$. Then*

$$lct(X, D, x) = min_{\left\{\substack{E: x \in \mu(E), \\ b(E) \neq 0}\right\}} \left\{ -\frac{a(E) + 1}{b(E)} \right\}.$$

Proof. By remark 1.27, $lct(D,x) = \inf\{c > 0 | \exists E \text{ s.t. } x \in \mu(E) \text{ and } a(E) + cb(E) \leq -1\}$. Since $a(E) \geq 0$, then $a(E)+cb(E) \leq -1 \Rightarrow b(E) \neq 0$, hence $lct(D,x) = \inf\{c > 0 | c \geq -\frac{a(E)+1}{b(E)}$ for a certain E s.t. $x \in \mu(E)$ and $b(E) \neq 0\} = \min_{\left\{\substack{E: x \in \mu(E), \\ b(E) \neq 0}\right\}} \left\{-\frac{a(E)+1}{b(E)}\right\}$. ◻

Remark 1.30. As a byproduct of lemma 1.29 we have that the log canonical threshold is a positive rational number.

Log canonical centres will be our main tool to produce subvarieties from which it is possible to pull back forms. Log canonical centres, in our case, are quite well behaved from this point of view: they can be made exceptional (using the Tie Breaking method) and their dimension can be cut down (see [20, thm. 4.1], [36, §5], [32, 10.4.10]).

It turns out to be quite convenient to use, sometimes, the language of multiplier ideals when dealing with log canonical centres. That is why, first of all, we collect here some useful results about multiplier ideals. The next few lemmas are used to prove proposition 1.35 that essentially states that the multiplier ideal of a divisor D is trivial outside its support: though very immediate, it will be very useful several times.

Lemma 1.31. *Let X be a topological space and $\mathcal{F}$ a sheaf of abelian groups on X. Let $U \subseteq X$ and $W \subseteq U$ be open sets. Then $\mathcal{F}|_U(W) = \mathcal{F}(W)$.*

Proof. By definition (cf. [21, II.1]), $\mathcal{F}|_U$ is the sheaf associated to the presheaf $V \mapsto \lim_{O \supseteq V} \mathcal{F}(O)$, where V is any open set of U and the direct limit is taken over all open sets O of X containing V. Since V is an open set of U, and hence of X, $\lim_{O \supseteq V} \mathcal{F}(O) = \mathcal{F}(V)$, therefore $V \mapsto \lim_{O \supseteq V} \mathcal{F}(O) = \mathcal{F}(V)$ is a sheaf and thus $\mathcal{F}|_U(W) = \mathcal{F}(W)$. ◻

Lemma 1.32. *Let X, X' be topological spaces, $\mu : X' \to X$ a continuous function, $\mathcal{F}$ a sheaf of abelian groups on X' and $U \subseteq X$ an open set. Let $\mu|_U : \mu^{-1}(U) \to U$ be the restriction of μ to $\mu^{-1}(U)$. Then $(\mu_* \mathcal{F})|_U$ and $(\mu|_U)_*(\mathcal{F}|_{\mu^{-1}(U)})$ are naturally isomorphic.*

Proof. Let $W \subseteq U$ be an open set. Note that $\mu^{-1}(W) \subseteq \mu^{-1}(U)$. By definition (cf. [21, II.1]) and by 1.31 $(\mu_* \mathcal{F})|_U(W) = (\mu_* \mathcal{F})(W) = \mathcal{F}(\mu^{-1}(W))$.

On the other hand, $(\mu|_U)_*(\mathcal{F}|_{\mu^{-1}(U)})(W) = \mathcal{F}|_{\mu^{-1}(U)}(\mu^{-1}(W)) = \mathcal{F}(\mu^{-1}(W))$. ◻

Lemma 1.33. *Let X, X' be varieties, D be a Cartier divisor on X and $\mu : X' \to X$ be a morphism such that it does not map X' into the support of D. Then $Supp(\mu^* D) \subseteq \mu^{-1}(Supp(D))$.*

Proof. Let D be represented by $\{(U_i, f_i)\}$, where $\{U_i\}$ is an open cover of X and f_i is a rational function on U_i. Recall that, by definition, the support of D is the set of points $x \in X$ at which a local equation of D is not a unit in $\mathcal{O}_{X,x}$ (cf. [31, 1.1.1]).

Since X' is reduced and irreducible μ does not map X' into the support of D, then, by [31, p. 10], $\mu^*(D)$ is represented by $\{(\mu^{-1}(U_i), f_i \circ \mu|_{\mu^{-1}(U_i)})\}$.

Let $z \in \mathrm{Supp}(\mu^* D)$, i.e. there exists j such that $z \in \mu^{-1}(U_j)$ and $(f_j \circ \mu|_{\mu^{-1}(U_i)}))_z$ is not a unit in $\mathcal{O}_{X',z}$.

Since $\mu_z^{\#} : \mathcal{O}_{X,\mu(z)} \to \mathcal{O}_{X',z}$ is a local homomorphism of local rings (cf. [21, p. 72]), then $(f_j)_{\mu(z)}$ is not a unit in $\mathcal{O}_{X,\mu(z)}$, hence $\mu(z) \in \mathrm{Supp}(D)$ and the thesis follows.

□

Before stating the proposition, since it does not require X to be smooth, we recall the following generalization of the definition of multiplier ideals to singular varieties:

Definition 1.34. (multiplier ideal on singular varieties, cf. [32, §9.3.G]) Let (X, Δ) be a pair and D a $\mathbb{Q}$-Cartier $\mathbb{Q}$ divisor on X. Let $a(E)$'s be the discrepancies of the pair $(X, \Delta + D)$. Then we define $\mathcal{J}((X, \Delta); D)$ as $\mu_* \mathcal{O}_{X'}(\sum \ulcorner a(E) \urcorner E)$. Clearly if X is non-singular $\mathcal{J}((X, \Delta); D) = \mathcal{J}(X, \Delta + D)$.

Proposition 1.35. *Let (X, Δ) be a pair and let D, E be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X.*

Let us consider the multiplier ideals $\mathcal{J}((X, \Delta); D)$ and $\mathcal{J}((X, \Delta); D + E)$. Set $U := X \setminus Supp(E)$. We have that $\mathcal{J}((X, \Delta); D)|_U \cong \mathcal{J}((X, \Delta); D + E)|_U$ and, in particular, for every $z \in U$, $\mathcal{J}((X, \Delta); D)_z = \mathcal{J}((X, \Delta); D + E)_z$.

Proof. Let $\mu : X' \to X$ be a log resolution of (X, Δ), D, E and $D + E$. Let $K'_X \equiv \mu^*(K_X + \Delta) + \sum_F a(F)F$, $\mu^*(D) = \sum_F b(F)F$ and $\mu^*(E) = \sum_F c(F)F$, the sums running over all prime divisor of X'. By definition of multiplier ideal we must show that $(\mu_* \mathcal{O}_{X'}(\sum_F \ulcorner a(F) - b(F) \urcorner F))|_U \cong (\mu_* \mathcal{O}_{X'}(\ulcorner a(F) - b(F) - c(F) \urcorner F))|_U$. By lemma 1.32 we need only to show that

$$\mathcal{O}_{X'}\left(\sum_F \ulcorner a(F) - b(F) \urcorner F\right)|_{\mu^{-1}U} \cong \mathcal{O}_{X'}\left(\sum_F \ulcorner a(F) - b(F) - c(F) \urcorner F\right)|_{\mu^{-1}U}.$$

Note that if $c(F) \neq 0$ then $F \subseteq \mathrm{Supp}(\mu^* E)$ and thus, by lemma 1.33, $F \subseteq \mu^{-1}(\mathrm{Supp}(E))$. Therefore $F \cap \mu^{-1}U = \emptyset$ and we are done. □

One of the main operation that can be performed on multiplier ideals is the restriction to appropriate divisors. Since this is a little bit subtle, after reintroducing the definition and the main property, we investigate how it relates to the classical tensor product.

Definition 1.36. (cf. [32, §9.5.A]) Let X be a nonsingular variety and let D be an effective $\mathbb{Q}$-divisor on X and $H \subset X$ a smooth irreducible divisor that does not appear in the support of D. Then we define $\mathcal{J}(X, D)_H := \mathrm{Im}(\mathcal{J}(X, D) \hookrightarrow \mathcal{O}_X \overset{\pi}{\twoheadrightarrow} \mathcal{O}_H)$.

Theorem 1.37 (Restriction theorem - cf. [32], 9.5.1)**.** *Using the notation introduced before, there is an inclusion $\mathcal{J}(H, D|_H) \subseteq \mathcal{J}(X, D)_H$.*

Lemma 1.38. *Let X be a variety, let $Y \subseteq X$ be a closed subscheme of X, defined in X by the ideal sheaf $\mathcal{J}_Y$, and let D be an integral divisor on X, with D effective. Let $\pi_D : \mathcal{O}_X \to \mathcal{O}_D$ be the natural map. If $D|_Y$ is effective on Y then we have a short exact sequence:*

$$0 \to \mathcal{J}_Y(-D) \to \mathcal{J}_Y \to \pi_D(\mathcal{J}_Y) \to 0.$$

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{J}_Y(-D) \arrow[r] \arrow[d] & \mathcal{J}_Y \arrow[r] \arrow[d, "i"] & \ker \overline{\pi}_Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X(-D) \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r, "\pi_D"] \arrow[d] & \mathcal{O}_D \arrow[r] \arrow[d, "\overline{\pi}_Y"] & 0 \\
0 \arrow[r] & \mathcal{O}_X(-D) \otimes \mathcal{O}_Y \arrow[r] \arrow[d] & \mathcal{O}_Y \arrow[r] \arrow[d] & \mathcal{O}_D \otimes \mathcal{O}_Y \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The second row and all the columns are exact. Since $D|_Y$ is effective by hypothesis, then also the third row is exact. Therefore by the snake lemma also the first row is exact. Note that, by the commutativity of the diagram, $\pi_D(i(\mathcal{J}_Y)) \subseteq \ker \overline{\pi}_Y$. Moreover, considering the stalks for every $x \in X$, we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{J}_{Y,x} \arrow[r, two heads] \arrow[d, hook, "i_x"] & (\ker \overline{\pi}_Y)_x \arrow[d, hook] \\
\mathcal{O}_{X,x} \arrow[r, two heads, "\pi_{D,x}"] & \mathcal{O}_{D,x}
\end{tikzcd}$$

and it is clear that, for every $x \in X$, $(\ker \overline{\pi}_Y)_x \subseteq \pi_{D,x}(i_x(\mathcal{J}_{Y,x}))$. Therefore $\ker \overline{\pi}_Y = \pi_D(i(\mathcal{J}_Y)) = \pi_D(\mathcal{J}_Y)$.

□

Remark 1.39. If $\mathcal{J}_Y = \mathcal{J}(B) = \mathcal{J}(X, B)$ (with $Y = Z(\mathcal{J}(B))$) is a multiplier ideal of an effective $\mathbb{Q}$-divisor B and D is an effective, integral divisor on X such that $D|_Y$ is effective, then by lemma 1.38 we have the following exact sequence (cf. definition 1.36):

$$0 \to \mathcal{J}(B) \otimes \mathcal{O}_X(-D) \to \mathcal{J}(B) \to \mathcal{J}(B)_D \to 0. \tag{1.1}$$

In a similar manner, we have a natural exact sequence $\mathcal{J}(B) \otimes \mathcal{O}_X(-D) \xrightarrow{\psi} \mathcal{J}(B) \to \mathcal{J}(B) \otimes \mathcal{O}_D \to 0$. Let $\mu : X' \to X$ be a log resolution of (X, B). Since $0 \to \mu^*(\mathcal{O}_X(-D)) \otimes \mathcal{O}_{X'}(K_{X'/X} - [\mu^* B]) \to \mu^*(\mathcal{O}_X) \otimes \mathcal{O}_{X'}(K_{X'/X} - [\mu^* B])$ is exact, then, by the projection formula and the definition of multiplier ideal, we can conclude that ψ is injective. Hence we also have the following short exact sequence:

$$0 \to \mathcal{J}(B) \otimes \mathcal{O}_X(-D) \xrightarrow{\psi} \mathcal{J}(B) \otimes \mathcal{O}_X \to \mathcal{J}(B) \otimes \mathcal{O}_D \to 0. \tag{1.2}$$

Therefore in this case, being $D|_Y$ effective, $\mathcal{J}(B)|_D \cong \mathcal{J}(B)_D$.

The following lemma about log canonical centres of codimension 1 will be needed later:

Lemma 1.40. *Let (X, Δ) be a pair, $\Delta = \sum_{i=1}^{s} d_i \Delta_i$ with Δ_i prime divisors and $d_i \in \mathbb{Q}$. If W is a lc centre for (X, Δ) of codimension 1 then there exists $\bar{i} \in \{1, \ldots, s\}$ such that $W = \Delta_{\bar{i}}$ and $d_{\bar{i}} \geq 1$. If moreover W is pure then $d_{\bar{i}} = 1$.*

Proof. By definition of lc centre, there exists $\mu : X' \to X$ a log resolution of (X, Δ) and a prime divisor $E \subset X'$ such that $\mu(E) = W$ and of discrepancy $a(E, X, \Delta) \leq -1$. Since the codimension of W is 1 then E cannot be exceptional for μ, hence (cf. [32, 9.3G, footnote 14] or [30, 2.25-2.26]) E is a strict transform of one of the $\Delta_i's$, i.e., $\exists \bar{i}$ such that $W = \mu(E) = \Delta_{\bar{i}}$ and $a(E, X, \Delta) = -d_{\bar{i}} \Rightarrow d_{\bar{i}} \geq 1$.

If moreover W is pure, i.e., it is lc at the generic point of W, then, since $\mu(E) = W$ actually contains the generic point of W, $-d_{\bar{i}} = a(E, X, \Delta) \geq -1 \Rightarrow d_{\bar{i}} = 1$. □

1.5 Some techniques

In this section we list some of the techniques that will be involved later. Most of them are already well-known but since they are needed in more particular settings we include also proofs.

First of all we state the classical Tie Breaking theorem, but in its local version, using big divisors to perturb the log canonical centre, instead of ample ones. Check also [29, prop. 8.7.1] and [7, thm. 3.7].

Lemma 1.41 (local Tie Breaking with a big divisor)**.** *(cf. [37, lemma 2.6]). Let X be a complex smooth projective variety and let Δ be an effective $\mathbb{Q}$-divisor and assume that (X, Δ) is lc but not klt at some point $x \in X$. Then:*

a. If $W_1, W_2 \in LLC(X, \Delta, x)$ and W is an irreducible component of $W_1 \cap W_2$ containing x, then $W \in LLC(X, \Delta, x)$.

b. By the item before, $LLC(X, \Delta, x)$ has a unique minimal irreducible element, say V.

c. If L is a big divisor and $x \notin \mathbb{B}_+(L)$ then there exist a positive rational number a and an effective $\mathbb{Q}$-divisor M such that $M \sim_{\mathbb{Q}} L$ and such that for all $0 < \epsilon \ll 1$, $(X, (1-\epsilon)\Delta + \epsilon a M)$ is lc at x and $LLC(X, (1-\epsilon)\Delta + \epsilon a M, x) = \{V\}$.

d. If Δ is big, $x \notin \mathbb{B}_+(\Delta)$ and $\Delta \sim_{\mathbb{Q}} \lambda D$ with $\lambda < c$, $\lambda \in \mathbb{Q}^+$ and D a $\mathbb{Q}$-divisor, then there exists an effective $\mathbb{Q}$-divisor Δ' such that (X, Δ') is lc, not klt at x, $LLC(X, \Delta', x) = \{V\}$ and $\Delta' \sim_{\mathbb{Q}} \lambda' D$ with $\lambda' < c$, $\lambda' \in \mathbb{Q}^+$.

e. In every case, we can also assume that there is a unique place lying above V.

Proof. We will prove only items c.,d. and e., referring to [37, lemma 2.6] and [20, lemma 2.5] for the item a. Check also [29, proposition 8.7.1].

First of all, simply notice that if (X, Δ) is lc but not klt at x then every element in $LLC(X, \Delta, x)$ is a log canonical centre image of a divisor of discrepancy -1.

c. Since L is big and $x \notin \mathbb{B}_+(L)$ then, by lemma 1.18, we can find an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor G such that $L \sim_{\mathbb{Q}} A+G$ and $x \notin \mathrm{Supp}(G)$.

   Let $\mathcal{I}_V$ be the ideal of V. Let $p \gg 0$ such that pA, pG, $p\Delta$ are integral divisors and $\mathcal{I}_V(pA+pG)$ is globally generated outside $\mathrm{Supp}(G)$. Let $|\mathcal{I}_V(pA+pG)| \subseteq |pA+pG|$ be the linear series of divisors corresponding to sections in $H^0(\mathcal{I}_V(pA+pG))$.

   Let $\mu : X' \to X$ be a log resolution of $(X, p\Delta + |\mathcal{I}_V(pA+pG)|)$. Let $K_{X'} - \mu^*(K_X) = \sum a(E)E$ ($a(E) \geq 0$) and $K_{X'} - \mu^*(K_X + \Delta) = \sum (a(E)+b(E))E$. By hypothesis if $x \in \mu(E)$ then $a(E)+b(E) \geq -1$ and there exists a divisor E_0 such that $\mu(E_0) = V$ and $a(E_0)+b(E_0) = -1$.

   By definition of log resolution, $\mu^*|\mathcal{I}_V(pA+pG)| = |S| + B$, where $|S|$ is base point free and $B+\mathrm{Exc}(\mu)+$(strict transforms of components of Δ) is a divisor with simple normal crossing supports.

   If we pick a general effective divisor $M' \in |\mathcal{I}_V(pA+pG)|$ then we can suppose that μ is a log resolution for $(X, \Delta + M')$ and, since $\mathcal{I}_V(pA+pG)$ is locally free outside V and V is minimal, we can also suppose that M' is nonsingular outside $V \cup \mathrm{Supp}(G)$ and μ^*M' does not contain divisors E with $x \in \mu(E)$ and $a(E)+b(E) = -1$, unless $\mu(E) = \mu(E_0) = V$: in fact if μ^*M' contains such an E then, by the generality of the choice of M' in $|\mathcal{I}_V(pA+pG)|$, $\mathrm{Supp}(E) \subseteq \mathrm{Supp}(B)$ and hence $\mu(E) \subseteq \mu(B) \subseteq V \cup \mathrm{Supp}(G)$; by the minimality of V and the fact that $x \in \mu(E)$ but $x \notin \mathrm{Supp}(G)$ then we have $\mu(E) = V$.

   Let $M := M'/p \sim_{\mathbb{Q}} L$. Let $-\mu^*M = \sum c(E)E$. Thus $c(E) \leq 0$, $c(E_0) < 0$ since $\mu(E_0) = V \subseteq \mathrm{Supp}(M)$ and $c(E) = 0$ if $x \in \mu(E)$, $a(E)+b(E) = -1$ and $\mu(E) \neq \mu(E_0)$.

   Let ϵ be a sufficiently small rational number and

$$a := \min_{\substack{\{E \,:\, \mu(E)=V, \\ a(E)+b(E)=-1\}}} \left\{ \frac{b(E)}{c(E)} \right\}.$$

   $a > 0$, since $b(E) < 0$ (in fact $a(E)+b(E) = -1$ and $a(E) \geq 0$) and, as before, $\mu(E) = V$ implies $c(E) < 0$. Let $\tilde{E}$ be a divisor that attains the minimum for a. We have that $K_{X'} - \mu^*(K_X + (1-\epsilon)\Delta + \epsilon a M) = \sum (a(E) + (1-\epsilon)b(E) + \epsilon a c(E))E$.

   First of all, let us check that V is still a lc-centre for $(X, (1-\epsilon)\Delta + \epsilon a M)$: $a(\tilde{E}) + (1-\epsilon)b(\tilde{E}) + \epsilon b(\tilde{E}) = a(\tilde{E}) + b(\tilde{E}) = -1$ and $\mu(\tilde{E}) = V$ by definition.

Then let us check that $(X, (1-\epsilon)\Delta + \epsilon a M)$ is lc in a neighbourhood of x: for every E such that $x \in \mu(E)$, since Δ is effective and hence $b(E) \leq 0$, we have that $a(E) + (1-\epsilon)b(E) + a\epsilon c(E) \geq a(E) + b(E) + a\epsilon c(E) =: d(E)$. If $a(E)+b(E) > -1$ then $d(E) > -1$, since ϵ is sufficiently small. If $a(E)+b(E) = -1$ and $\mu(E) \neq \mu(E_0)$ then $c(E) = 0$ and thus $d(E) \geq -1$. If $a(E)+b(E) = -1$ and $\mu(E) = \mu(E_0) = V$ then, by definition, $a \leq \frac{b(E)}{c(E)}$ that implies that $ac(E) \geq b(E) \Rightarrow -\epsilon b(E) + \epsilon ac(E) \geq 0$ that is equivalent to saying that $a(E) + (1-\epsilon)b(E) + \epsilon ac(E) \geq -1$.

Eventually let us check that V is the unique element of $LLC(X, (1-\epsilon)\Delta + \epsilon a M, x)$: if $a(E) + b(E) > -1$ then, since ϵ is sufficiently small, $a(E) + (1-\epsilon)b(E) + a\epsilon c(E) \geq a(E) + b(E) + a\epsilon c(E) > -1$; if $a(E) + b(E) = -1$, $x \in \mu(E)$ and $\mu(E) \neq \mu(E_0) = V$ then, as we have already seen, $c(E) = 0$ and $a(E) + b(E) = -1, a(E) \geq 0 \Rightarrow b(E) < 0$. Thus, since $\epsilon > 0$, we have that $a(E) + (1-\epsilon)b(E) > -1$.

d. Let us apply c. with $L = \Delta$ and let $a \in \mathbb{Q}^+$ and $M \sim_{\mathbb{Q}} \Delta$ be as in c. Let $\Delta' := (1-\epsilon)\Delta + \epsilon a M$ and $\lambda' := \lambda - \epsilon\lambda + \epsilon a \lambda$. Hence $\Delta' \sim_{\mathbb{Q}} \lambda' D$. Since $\lambda < c$, choosing ϵ sufficiently small we can manage to have $\lambda' < c$, and we are done.

e. Let us start from the outcome of item c., that is, setting $\Delta' := (1-\epsilon)\Delta + \epsilon a M$, we are in the following situation: $LLC(X, \Delta', x) = \{V\}$, and (X, Δ') is lc at x.

Since L is big then we can write $L = A + \overline{E}$ with $A, \overline{E}$ $\mathbb{Q}$-divisors, A ample, $\overline{E}$ effective.

Let $\mu : X' \to X$ be a log resolution of $(X, \Delta' + \overline{E})$. Fixing the notation, we will also write $K_{X'} - \mu^*(K_X) = \sum a(E)E$ and $-\mu^*(\Delta') = \sum f(E)E$.

If V is the image of a strict transform of a component of Δ' then it is clear that there exists a unique place lying above V. Therefore we can suppose that $\operatorname{codim}(V) > 1$ and thus that V is the image of at least one exceptional divisor.

Following the proof of [29, prop. 8.7.1], we will prove the thesis in two steps: first of all we will produce a divisor H $\mathbb{Q}$-linearly equivalent to L, that passes through V and such that μ^*H contains an ample divisor E_{amp}; this divisor H, as in c., will be used for another tie breaking. Then, using E_{amp} we will slightly modify H so that we can be sure there is only one exceptional divisor above V.

Recall that for every Weil divisor W on X', $\mu_*(W)$ is well defined as a Weil divisor on X (cf. [19, §1.4]). Moreover, since μ is projective, recall that if $W \sim W'$ then $\mu_*(W) \sim \mu_*(W')$ (cf. [19, theorem 1.4]) and that, since μ is also birational, $\mu_*\mu^*Y = Y$, for every divisor Y on X.

Since by [12, par. 1.40] $\operatorname{Exc}(\mu)$ is a divisor whose irreducible components are μ-exceptional then by [12, par. 1.42] there exists an effective and μ-exceptional $\mathbb{Q}$-divisor F such that $\mu^*A - F$ is ample. Choose an effective $\mathbb{Q}$-divisor $E_{\text{amp}} \sim_{\mathbb{Q}} \mu^*A - F$ such that it is irreducible, with nonsingular support and such that it has simple normal crossings with $\mathrm{Exc}(\mu)$ and with all the strict transforms of the components of $\Delta' + \overline{E}$. Let E be a prime divisor such that $\mu(E) = V$; then E is exceptional and, in particular, the map $\mu|_E : E \to V$ has positive dimensional fibres, hence notice that $\mu(E_{\mathrm{amp}}) \supset V$. Let us now consider the effective $\mathbb{Q}$-divisor $H := \mu_*(E_{\mathrm{amp}} + F + \mu^*\overline{E}) \sim_{\mathbb{Q}} L$. Since F is exceptional then $H = \mu_*(E_{\mathrm{amp}}) + \overline{E}$: hence, by the choice of μ and E_{amp}, we have that μ is a log resolution also of $\Delta' + H$. Set $-\mu^*(H) = \sum g(E)E$. Let $\epsilon' \in \mathbb{Q}^+$, $\epsilon' \ll 1$ and

$$a' := \min_{\substack{\{E\,:\,\mu(E)=V,\\ a(E)+f(E)=-1\}}} \left\{ \frac{f(E)}{g(E)} \right\}.$$

Therefore $K_{X'} - \mu^*(K_X + (1-\epsilon')\Delta' + \epsilon' a' H) = \sum(a(E) + (1-\epsilon')f(E) + \epsilon' a' g(E))E$ and as in the proof of item c., it can be easily seen that $(X, (1-\epsilon')\Delta' + \epsilon' a' H)$ is lc at x and $LLC(X, (1-\epsilon')\Delta' + \epsilon' a' H, x) = \{V\}$ (in this case, however, $a(E) + f(E) = -1$ and $x \in \mu(E)$ directly imply $\mu(E) = V$). And so the first step is done.

As for the second step, let E_0 be an exceptional divisor such that $\mu(E_0) = V$ and $a(E_0) + (1-\epsilon')f(E_0) + \epsilon' a' g(E_0) = -1$. Since E_{amp} is ample, then for a small $\xi \in \mathbb{Q}^+$ we have that $E_{\mathrm{amp}} - \xi E_0$ is ample too and $\mathbb{Q}$-linearly equivalent to an effective, irreducible divisor A_ξ. We can choose A_ξ to have nonsingular support and simple normal crossings with $\mathrm{Exc}(\mu)$ and with all the strict transforms of the components of $\Delta' + H$. Let us consider $H' := \mu_*(\mu^* H - E_{\mathrm{amp}} + \xi E_0 + A_\xi) \sim_{\mathbb{Q}} \mu_*\mu^* H = H$. Notice that μ is a log resolution also for $\Delta' + H'$. Since $H' = \mu_*(\mu^* H - E_{\mathrm{amp}} + \xi E_0 + A_\xi) = H - \mu_*(E_{\mathrm{amp}}) + \mu_*(A_\xi)$ then $\mu^*(H') = \mu^* H - E_{\mathrm{amp}} - exc_1 + A_\xi + exc_2 \sim_{\mathbb{Q}} \mu^* H$, where exc_1 and exc_2 are effective sums of exceptional divisors. Since $\xi E_0 \sim_{\mathbb{Q}} E_{\mathrm{amp}} - A_\xi$ then $\xi E_0 + exc_1 \sim_{\mathbb{Q}} exc_2$, that is: $\xi E_0 + exc_1 = exc_2$. Therefore $\mu^*(H') = \mu^* H + A_\xi - E_{\mathrm{amp}} + \xi E_0$. Set $-\mu^*(H') = \sum g'(E)$. Note that $g'(E) = g(E)$ unless $E = E_{\mathrm{amp}}, A_\xi, E_0$; notice, however, that E_0 is the only exceptional divisor for which g and g' are different: namely $g'(E_0) = g(E_0) - \xi$. Let $\epsilon'' \in \mathbb{Q}^+$, $\epsilon'' \ll 1$ and

$$a'' := \min_{\substack{\{E\,:\,\mu(E)=V,\\ a(E)+f(E)=-1\}}} \left\{ \frac{f(E)}{g'(E)} \right\}.$$

Then $K_{X'} - \mu^*(K_X + (1-\epsilon'')\Delta' + \epsilon'' a'' H') = \sum(a(E) + (1-\epsilon'')f(E) + \epsilon'' a'' g'(E))E$. As before $(X, (1-\epsilon'')\Delta' + \epsilon'' a'' H')$ is lc at x, $LLC(X, (1-\epsilon'')\Delta' + \epsilon'' a'' H', x) = \{V\}$, but this time E_0 is the only exceptional divisor with discrepancy -1 and such that $\mu(E_0) = V$: in fact, since for any E such that $\mu(E) = V$ and $a(E) + f(E) = -1$ we have that $g'(E) < g(E) \Rightarrow f(E)/g'(E) < f(E)/g(E)$ then E_0 is the only exceptional divisor above V that attains the minimum in the definition of a''.

□

The following lemma just deals about intersections of codimension 1 log canonical centres:

Lemma 1.42 ([37], 3.3)**.** *Let X be a smooth projective variety and let (X, Δ_1), (X, Δ_2) be effective pairs such that at some point $x \in X$ there exists W_i a pure log canonical centre of codimension 1 at x for (X, Δ_i), with Δ_i with smooth support at x ($i = 1, 2$). Then there exists $Z \subseteq W_1 \cap W_2$ a minimal pure log canonical centre at x for the pair (X, Δ), where $\Delta = k(\Delta_1 + \Delta_2)$ for some rational number $0 < k \leq 1$.*

Proof. Let $\Delta_1 = \sum_j a_{1j} D_{1j}$, where $a_{1j} \neq 0$, D_{1j} prime divisors. Since $\mathrm{Supp}(\Delta_1)$ is non singular at x we can suppose that D_{11} is the unique divisor in Δ_1 passing through x. Analogously let $\Delta_2 = \sum_j a_{2j} D_{2j}$, where $a_{2j} \neq 0$, D_{2j} prime divisors, where D_{21} is the unique divisor in Δ_2 passing through x.

By lemma 1.40 $W_1 = D_{11}$, $W_2 = D_{21}$ and $a_{11} = a_{21} = 1$.

Let $\mu : X' \to X$ be a log resolution of $\Delta_1 + \Delta_2$. Since it can be easily seen that a log resolution of the support of a divisor is also a log resolution of the divisor itself then, by Hironaka's theorem (cf. [31, thm. 4.1.3]), we can construct μ as a sequence of blowing-ups along smooth centers contained in the singular locus of $\mathrm{Supp}(\Delta_1 + \Delta_2)$ and X, that is contained, under our hypotheses, in $\mathrm{Sing}(\mathrm{Supp}(\Delta_1)) \cup \mathrm{Sing}(\mathrm{Supp}(\Delta_2)) \cup (\mathrm{Supp}(\Delta_1) \cap \mathrm{Supp}(\Delta_2))$. Therefore, since $\mathrm{Supp}(\Delta_i)$ is nonsingular at x for $i = 1, 2$ and $x \in D_{ij} \Leftrightarrow j = 1$, then every exceptional divisor Ξ of μ such that $x \in \mu(\Xi)$ goes, under μ, in a subvariety of $D_{11} \cap D_{21}$. Now set $K_{X'} - \mu^* K_X = \sum_E a(E) E$ (so that $a(E) \neq 0 \Leftrightarrow E$ is exceptional), $\mu^* \Delta_1 = \sum_E b_1(E) E$, $\mu^* \Delta_2 = \sum_E b_2(E) E$. Consider $\widetilde{D_{11}}$, the strict transform of D_{11}: since it is not an exceptional divisor, then $a(\widetilde{D_{11}}) = 0$ and $b_1(\widetilde{D_{11}}) = 1$, therefore if we set $c := lct(\Delta_1 + \Delta_2, x)$, we have

$$c = \min_{\substack{\{E \,:\, x \in \mu(E), \\ b_1(E) + b_2(E) > 0\}}} \left\{ \frac{a(E) + 1}{b_1(E) + b_2(E)} \right\} \leq \frac{a(\widetilde{D_{11}}) + 1}{b_1(\widetilde{D_{11}}) + b_2(\widetilde{D_{11}})} \leq 1.$$

Clearly $c > 0$.

$K_{X'} - \mu^*(K_X + c(\Delta_1 + \Delta_2)) = \sum_E (a(E) - c b_1(E) - c b_2(E)) E$. Let $\overline{E}$ be such that $a(\overline{E}) - c b_1(\overline{E}) - c b_2(\overline{E}) = -1$, $x \in \mu(\overline{E})$ and $b_1(\overline{E}) + b_2(\overline{E}) > 0$.

If $\overline{E}$ is exceptional then $\mu(\overline{E}) \subseteq D_{11} \cap D_{21} = W_1 \cap W_2$.

If $\overline{E}$ is not exceptional, since $b_1(\overline{E}) + b_2(\overline{E}) > 0$ then $\overline{E}$ is a strict transform, $E = \widetilde{D_{ij}}$ for some i, j. Since $x \in \mu(\overline{E})$ then $\overline{E} = \widetilde{D_{11}}$ or $\overline{E} = \widetilde{D_{21}}$. If $\widetilde{D_{11}} = \widetilde{D_{21}}$ then $\mu(\overline{E}) = W_1 = W_2$ and we are done. If $\widetilde{D_{11}} \neq \widetilde{D_{21}}$ then, supposing $\overline{E} = \widetilde{D_{11}}$, we have that $b_2(\widetilde{D_{11}}) = 0, a(\widetilde{D_{11}}) = 0$ and hence $c = 1$. This implies that $\widetilde{D_{21}}$ has discrepancy -1 for $(X, c(\Delta_1 + \Delta_2))$ and so W_1, W_2 are both lc centers of $(X, c(\Delta_1 + \Delta_2))$. We can now conclude applying lemma 1.41, b. □

The next lemma, due to Hacon-McKernan (cf. [20, lemma 2.6]), essentially explains how to pull back sections from log canonical centres when we already know that these centres have dimension 0. The main ingredient is Nadel's Vanishing theorem, that, under particular conditions, assures the surjectivity of the restriction map. For the convenience of the reader we start by enunciating Nadel's theorem:

Theorem 1.43 (Nadel vanishing theorem)**.** *(see [32, thm. 9.4.8]). Let X be a smooth projective variety, let D be any $\mathbb{Q}$-divisor on X and let L be any integral divisor such that $L - D$ is big and nef. Then*

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(X, D)) = 0 \qquad \textit{for every } i > 0.$$

Lemma 1.44. *Let X be a smooth projective variety and D a big and integral divisor on X. Let $x, y \notin \mathbb{B}_+(D)$. Assume that there exists an effective $\mathbb{Q}$-divisor $\Delta_x \sim_{\mathbb{Q}} \lambda_x D$ with $\lambda_x \in \mathbb{Q}^+$ and such that $LLC(X, \Delta_x, x) = \{\{x\}\}$. Then for every $m \in \mathbb{N}^+$ such that $m > [\lambda_x]$,*

$$h^0(\mathcal{O}_X(K_X + mD)) > 0.$$

If moreover there exists another effective $\mathbb{Q}$-divisor $\Delta_y \sim_{\mathbb{Q}} \lambda_y D$ with $\lambda_y \in \mathbb{Q}^+$, such that $LLC(X, \Delta_y, y) = \{\{y\}\}$ and such that $x \notin Supp(\Delta_y)$ and $y \notin Supp(\Delta_x)$, then for every $m \in \mathbb{N}^+$ such that $m > [\lambda_x + \lambda_y]$,

$$h^0(\mathcal{O}_X(K_X + mD)) \geq 2.$$

More generally, let $x_1, \ldots, x_n \notin \mathbb{B}_+(D)$. If for every $1 \leq i \leq n$ there exists an effective $\mathbb{Q}$-divisor $\Delta_i \sim_{\mathbb{Q}} \lambda_i D$ with $\lambda_i \in \mathbb{Q}^+$, such that $LLC(X, \Delta_i, x_i) = \{x_i\}$ and such that $x_i \notin \cup_{j \neq i} Supp(\Delta_j)$ then for every $m \in \mathbb{N}^+$ such that $m > [\sum_{i=1}^n \lambda_i]$,

$$h^0(\mathcal{O}_X(K_X + mD)) \geq n.$$

Proof. Since $x \notin \mathbb{B}_+(D)$, by lemma 1.18 there exist an ample $\mathbb{Q}$-divisor A_x of sufficiently small norm and an effective $\mathbb{Q}$-divisor E_x such that $D \sim_{\mathbb{Q}} A_x + E_x$ and $x \notin \mathrm{Supp}(E_x)$. Let us consider the multiplier ideal associated to Δ_x, $\mathcal{J}(\Delta_x)$. Let us notice that, by the hypothesis that $\{x\}$ is an isolated lc-centre at x, there exists an open neighbourhood U_x of x such that $\mathcal{J}(\Delta_x)_x \subsetneq \mathcal{O}_{X,x}$ but $\mathcal{J}(\Delta_x)_z = \mathcal{O}_{X,z}$ for all $z \in U_x - \{x\}$ (cf. [32, def. 9.3.9]).

Let B_x be the $\mathbb{Q}$-divisor $\Delta_x + (m - \lambda_x)E_x$. Since $x \notin \mathrm{Supp}(E_x)$, using proposition 1.35 we can conclude that $\mathcal{J}(B_x)_x \subsetneq \mathcal{O}_{X,x}$ and $\mathcal{J}(B_x)_z = \mathcal{O}_{X,z}$ for every $z \in U'_x := U_x \cap (X - \mathrm{Supp}(E_x))$, that is: the set of zeroes $Z(\mathcal{J}(B_x))$ has x as an isolated point.

Let us consider the following exact sequence:

$$0 \to \mathcal{J}(B_x) \to \mathcal{O}_X \to \mathcal{O}_{Z(\mathcal{J}(B_x))} \to 0$$

Tensoring it by $\mathcal{O}_X(K_X + mD)$ we obtain:

$$0 \to \mathcal{J}(B_x) \otimes \mathcal{O}_X(K_X + mD) \to \mathcal{O}_X(K_X + mD) \to \mathcal{O}_{Z(\mathcal{J}(B_x))} \otimes \mathcal{O}_X(K_X + mD) \to 0$$

Since x is an isolated point in $Z(\mathcal{J}(B_x))$ we have that $h^0(\mathcal{O}_{Z(\mathcal{J}(B_x))} \otimes \mathcal{O}_X(K_X + mD)) > 0$.

Let us notice that since m is an integer greater than $[\lambda_x]$ then $m > \lambda_x$, hence $mD - B_x \sim_{\mathbb{Q}} (m - \lambda_x)A_x$ is big and nef. Therefore we can apply Nadel's theorem to conclude that $H^1(\mathcal{O}_X(K_X + mD) \otimes \mathcal{J}(B_x)) = 0$ and thus the first part of the lemma is proved.

Since $x, y \notin \mathbb{B}_+(D)$ then, by remark 1.19, there exist an ample $\mathbb{Q}$-divisor A of sufficiently small norm and an effective $\mathbb{Q}$-divisor E such that $D \sim_{\mathbb{Q}} A + E$ and $x, y \notin \mathrm{Supp}(E)$. Let B be the $\mathbb{Q}$-divisor $\Delta_x + (m - \lambda_x - \lambda_y)E + \Delta_y$. Since $x, y \notin \mathrm{Supp}(E)$, $x \notin \mathrm{Supp}(\Delta_y)$, $y \notin \mathrm{Supp}(\Delta_x)$, using proposition 1.35, as before we can conclude that $Z(\mathcal{J}(B))$ has x, y as two isolated points.

Let us consider the following exact sequence:

$$0 \to \mathcal{J}(B) \otimes \mathcal{O}_X(K_X + mD) \to \mathcal{O}_X(K_X + mD) \to \mathcal{O}_{Z(\mathcal{J}(B))} \otimes \mathcal{O}_X(K_X + mD) \to 0$$

Since x, y are two isolated points in $Z(\mathcal{J}(B))$ we have that $h^0(\mathcal{O}_{Z(\mathcal{J}(B))} \otimes \mathcal{O}_X(K_X + mD)) \geq 2$ and since $mD - B \sim_{\mathbb{Q}} (m - \lambda_x - \lambda_y)A$ we have that $mD - B$ is big and nef (by hypothesis $\lambda_x + \lambda_y < m$). Therefore we can conclude as before, simply applying Nadel's theorem.

The general case in analogous. □

When dealing with more that one point, the previous lemma will be applied together with lemma 1.6.

As we have already said, we will use log canonical centres to pull back sections of multiples of the canonical divisor. Clearly the first step is to produce log canonical centres. For this purpose we will use a standard tool, by means of divisors that are sufficiently singular at given points and whose existence is guaranteed by assuming some hypotheses on their volumes:

Lemma 1.45. *(see [31, prop. 1.1.31], [32, lemma 10.4.12], [37, lemma 2.2]). Let X be a projective variety of dimension d and let D be a $\mathbb{Q}$-Cartier divisor. Fix a positive real number α with*

$$vol(D) > \alpha^d.$$

Then for any sufficiently large and divisible $k \in \mathbb{N}$ there exists for any smooth point $x \in X$ a divisor $A_x \in |kD|$ such that $mult_x(A_x) > k\alpha$.

Proof. Assume k sufficiently divisible so that kD is integral. Since x is a smooth point we can fix a system of local parameters $\{x_1, \ldots, x_d\}$ at x so that any section s of $H^0(kD)$ can be written as $\sum_{e_1,\ldots,e_d} a_{e_1,\ldots,e_d} x_1^{e_1} \ldots x_d^{e_d}$ $(e_1, \ldots e_d \geq 0)$.

It is clear that s vanishes at x with order $> c$ if all the partial derivatives of s at x of degree $0, \ldots, c$ vanish. These conditions are linear in $a_{e_1,\ldots,e_d}$ and their number is

$$\binom{d+c}{d} = \frac{c^d}{d!} + O(c^{d-1}).$$

By the definition of volume, there exists $\beta > \alpha^d$ (for example $\beta = (\mathrm{vol}(D) + \alpha^d)/2)$) such that for k sufficiently large (and independent of x)

$$h^0(kD) > \frac{\beta k^d}{d!} > \frac{\alpha^d k^d}{d!}.$$

Therefore, if $c = \alpha k$ (k large) we have that

$$h^0(kD) > \binom{d+c}{d}.$$

Hence there exists a section in $H^0(kD)$ - and thus a divisor $A_x \in |kD|$ - such that $\mathrm{ord}_x(s) = \mathrm{mult}_x(A_x) > c = \alpha k$. □

Remark 1.46. Lemma 1.45 can be applied also to two distinct smooth points x, y. In this case clearly the number of conditions for the vanishing of partial derivatives is doubled, hence we can say that if $\mathrm{vol}(D) > 2\alpha^d$ then there exists $A_{x,y} \in |kD|$ such that $\mathrm{mult}_x(A_{x,y}) > k\alpha$ and $\mathrm{mult}_y(A_{x,y}) > k\alpha$.

Divisors with high multiplicities at a given point naturally give rise to log canonical centres:

Proposition 1.47. *(see [32, prop. 9.3.2]). Let X be a smooth projective variety of dimension d, let D be an effective $\mathbb{Q}$-divisor and let $x \in X$ be a point. If $mult_x(D) \geq d$ then $\mathcal{J}(X, D)_x \subsetneq \mathcal{O}_{X,x}$.*

If $\mathcal{J}(X, D)$ is not trivial at x then $x \in Nklt(X, D)$, i.e., there exists a log canonical centre (not necessarily pure) of (X, D) that passes through x.

Pulling back sections from log canonical centres is not easy to do, unless the lc centres are points. Unfortunately when the volume is low, cutting down the dimension of lc centres does not allow us to have information about small multiple of the canonical divisor. That is why Todorov in [37], using ideas of McKernan (see [33]) has developed another strategy in the case of threefold, that is to produce a morphism from the threefold to a curve and use this to have sections. The next lemma shows how to obtain such a fibration from lc centres, while the next proposition shows how to create sections in this way:

Lemma 1.48 (McKernan-Todorov)**.** *([33, lemma 3.2] and [37, lemma 3.2]). Let X be a smooth projective variety and suppose that for every point $x \in P$, where P is a countably dense subset of X, we may find an effective $\mathbb{Q}$-divisor Δ_x and a subvariety V_x such that V_x is a pure log canonical centre for (X, Δ_x) at x and $\Delta_x \sim_{\mathbb{Q}} \Delta/w_x$ for some big $\mathbb{Q}$-divisor Δ on X and a rational positive number w_x. Then there exists a diagram*

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[d, "f"] & X \\ B & \end{tikzcd}$$

such that f is a dominant morphism of normal projective varieties with connected fibres and for a general fibre X'_b of f there exists $x \in \pi(X'_b)$ such that $\pi(X'_b)$ is a pure log canonical centre for (X, Δ_x) at x, with $\Delta_x \sim_{\mathbb{Q}} \Delta/w$ for some $w \in \mathbb{Q}^+$. We also have that π is a generically finite and dominant morphism of normal varieties.

Proposition 1.49. *Let X be a smooth projective threefold of general type. Suppose that there exist a smooth projective curve B and a dominant morphism with connected fibres $f : X \to B$ such that the general fibre X_b is a minimal, smooth surface of general type. Moreover suppose there exist $\lambda \in \mathbb{Q}^+$ and, for a general $b \in B$, an effective $\mathbb{Q}$-divisor D_b on X such that $D_b \sim_{\mathbb{Q}} \lambda K_X$ and such that X_b is a lc centre for (X, D_b). Suppose also that, for general b, there exists $\beta \in \mathbb{Q}^+$ such that vol$(X_b) \leq \beta^2$. Then, given $b_1, \ldots, b_k$ general points on B, the restriction map gives a surjection*

$$H^0(\mathcal{O}_X((n+1)K_X) \to H^0(\mathcal{O}_{X_{b_1}}((n+1)K_{X_{b_1}})) \oplus \ldots \oplus H^0(\mathcal{O}_{X_{b_k}}((n+1)K_{X_{b_k}}))$$

as long as $\lambda k(4(n+1)[\beta^2] - 1) < 1$ and $n > \lambda k$.

Proof. By Kawamata's theorem A (cf. [25], taking $S = \{pt\}$) for every $1 \leq i \leq k$ and every positive integer m the restriction maps

$$H^0(\mathcal{O}_X(m(K_X + X_{b_i}))) \to H^0(\mathcal{O}_{X_{b_i}}(mK_{X_{b_i}}))$$

are surjective. Since for every i we have an injection

$$H^0(\mathcal{O}_X(m(K_X + X_{b_i}))) \hookrightarrow H^0(\mathcal{O}_X(m(K_X + X_{b_1} + \ldots + X_{b_k}))),$$

then the restriction maps

$$H^0(\mathcal{O}_X(m(K_X + X_{b_1} + \ldots + X_{b_k}))) \to H^0(\mathcal{O}_{X_{b_i}}(mK_{X_{b_i}}))$$

are surjective. Since X_{b_i} are minimal surfaces of general type then, by [5] for m large enough (namely $m \geq 4$) $|mK_{X_{b_i}}|$ is base point free, hence a general $G \in |m(K_X + X_{b_1} + \ldots + X_{b_k})|$ is such that for every i, $G|_{X_{b_i}}$ is a general divisor in the base-point-free linear system $|mK_{X_{b_i}}|$.

Since K_X is big then $K_X = A + E$ where A is an ample $\mathbb{Q}$-divisor and E an effective $\mathbb{Q}$-divisor.

Let now b' be a general point on B, m be a sufficiently large integer, G a general divisor in $|m(K_X + X_{b_1} + \ldots + X_{b_k})|$, ϵ a rational number, $0 < \epsilon \ll 1$. Let

$$h :=: h_{n,k} := \frac{k(n+1) - \epsilon k}{\lambda k + 1},$$

$$j :=: j_{n,k} := -h_{n,k},$$

$$i :=: i_{n,k} := -1 + \frac{h_{n,k}}{k},$$

and consider the $\mathbb{Q}$-divisor

$$F :=: F_{n,k} := hD_{b'} + \frac{i}{m}G + jX_{b'} + \epsilon E.$$

For ϵ sufficiently small $h > 0$. Since $X_{b'}$ is an exceptional log canonical centre of $D_{b'}$ then $D_{b'} = X_{b'} +$ other surfaces. Therefore if $i \geq 0$ then F is an effective divisor: in order to have $i \geq 0$ it is enough to ask that $n > \lambda k$.

Moreover, by the choices of h, i, j, we have that

$$nK_X - (X_{b_1} + \ldots + X_{b_k}) - F \equiv \epsilon A.$$

Start with the following short exact sequence:

$$0 \to \mathcal{O}_X(-(X_{b_1} + \ldots + X_{b_k})) \to \mathcal{O}_X \to \mathcal{O}_{X_{b_1}} \oplus \ldots \oplus \mathcal{O}_{X_{b_k}} \to 0$$

After tensoring it by $\mathcal{O}_X((n+1)K_X)$ and using remark 1.39 with $\mathcal{J}_Y = \mathcal{J}(F)$ (X_{b_i} are general fibres, so their restriction to $Y = Z(\mathcal{J}(F))$ is effective) we have the following exact sequence:

$$0 \to \mathcal{O}_X((n+1)K_X - (X_{b_1} + \ldots + X_{b_k})) \otimes \mathcal{J}(F) \to \mathcal{O}_X((n+1)K_X) \otimes \mathcal{J}(F) \to$$

$$\to \mathcal{O}_{X_{b_1}}((n+1)K_{X_{b_1}}) \otimes \mathcal{J}(F)_{X_{b_1}} \oplus \ldots \oplus \mathcal{O}_{X_{b_k}}((n+1)K_{X_{b_k}}) \otimes \mathcal{J}(F)_{X_{b_k}} \to 0$$

By Nadel's vanishing theorem (theorem 1.43),

$$H^1(\mathcal{O}_X((n+1)K_X - (X_{b_1} + \ldots + X_{b_k})) \otimes \mathcal{J}(F)) = 0.$$

Moreover, since F is effective, $\mathcal{J}(F) \subseteq \mathcal{O}_X$, hence

$$\mathcal{O}_X((n+1)K_X) \otimes \mathcal{J}(F) \subseteq \mathcal{O}_X((n+1)K_X).$$

Therefore to prove the theorem it is now sufficient only to prove that, under the hypotheses, $\mathcal{J}(F)_{X_{b_i}}$ is trivial for every i.

To ease the notation, let $b = b_i$. By theorem 1.37, since $X_b \nsubseteq \mathrm{Supp}(F)$, we have that $\mathcal{J}(F)_{X_b} \supseteq \mathcal{J}(F|_{X_b})$, therefore we have to prove only that $\mathcal{J}(F|_{X_b})$ is trivial. Set

$$\Delta := D_{b'}|_{X_b} \text{ and } \Gamma := E|_{X_b}.$$

Δ and Γ are effective divisors, with $\Delta \sim_{\mathbb{Q}} \lambda K_{X_b}$. $F|_{X_b} = h\Delta + \frac{i}{m}G|_{X_b} + \epsilon\Gamma$. Since m is large enough and $G|_{X_b}$ is a general divisor in the base-point-free linear system $|mK_{X_b}|$, then, by Kollar-Bertini (cf. [32, 9.2.29]),

$$\mathcal{J}\left(h\Delta + \frac{i}{m}G|_{X_b} + \epsilon\Gamma\right) = \mathcal{J}(h\Delta + \epsilon\Gamma).$$

But, by [32, prop. 9.2.32.i],

$$\mathcal{J}(h\Delta+\epsilon\Gamma) \supseteq \mathcal{J}\left(h\Delta + \frac{\epsilon k}{\lambda k+1}\Delta + \epsilon\Gamma\right) = \mathcal{J}\left(\frac{k(n+1)}{\lambda k+1}\Delta + \epsilon\Gamma\right) = \mathcal{J}\left(\frac{k(n+1)}{\lambda k+1}\Delta\right),$$

where the last equality is due to [32, ex. 9.2.30]. Set

$$h' :=: h'_{n,k} = \frac{k(n+1)}{\lambda k+1}.$$

Now for every $x \in \Delta$, pick a curve $C \subset X_b$ passing through x that is a component of a divisor in $|4K_{X_b}|$ but it is not a component of Δ (cf. [37, proof of claim 1]). Then $\mathrm{mult}_x(h'\Delta) = h'\mathrm{mult}_x(\Delta) \leq h'\Delta.C$. Since $\Delta \sim_{\mathbb{Q}} \lambda K_{X_b}$ is nef (X_b is minimal and of general type) then $h'\Delta.C \leq 4h'\Delta.K_{X_b} = 4h'\lambda K^2_{X_b}$. If $\mathrm{mult}_x(h'\Delta) < 1$ for all $x \in \Delta$ then $\mathcal{J}(h'\Delta)$ is trivial, as wanted (cf. [32, prop. 9.5.13]). Therefore we need only to impose $4\lambda h' K^2_{X_b} < 1$. By hypothesis, and since $\mathrm{vol}(X_b)$ is an integer, it is enough to ask that $\lambda k(4(n+1)[\beta^2] - 1) < 1$. □

Chapter 2

Plurigenera for 3-folds of general type

In this chapter we will be dealing with plurigenera for threefolds of general type. Before stating and proving the main theorem, for the convenience of the reader we quote Hacon-McKernan's theorem about the lifting of log canonical centres:

Theorem 2.1 (Hacon-McKernan)**.** *(see [20, theorem 4.1]). Let (X, Δ) be an effective pair, with X $\mathbb{Q}$-factorial. Let V be an exceptional log canonical centre of (X, Δ). Let $f : W \to V$ be a resolution of singularities V and suppose that W is a variety of general type. Let Θ be an effective $\mathbb{Q}$-divisor on W. Suppose that there are positive rational numbers λ and μ such that $\Delta \sim_{\mathbb{Q}} \lambda K_X$ and $\Theta \sim_{\mathbb{Q}} \mu K_W$. Let*

$$\nu := (\lambda + 1)(\mu + 1) - 1.$$

There is a very general subset U of V with the following property:

Suppose that $W' \subset W$ is a pure log canonical centre of (W, Θ) whose image $V' \subset V$ intersects U.

Then for every positive rational number δ, we may find an effective $\mathbb{Q}$-divisor Δ' on X such that V' is a pure log canonical centre of (X, Δ'), where $\Delta' \sim_{\mathbb{Q}} (\nu + \delta)K_X$. Now suppose that we may write $K_X \sim_{\mathbb{Q}} A + E$, where A is an ample $\mathbb{Q}$-divisor, E is an effective $\mathbb{Q}$-divisor and V is not contained in Supp(E). Then we may also assume that V' is an exceptional log canonical centre of (X, Δ').

Theorem 2.2. *Let X be a smooth projective threefold of general type such that vol$(X) > \alpha^3$. If $\alpha \geq 879$ then $h^0(2K_X) \geq 1$ and if $\alpha \geq 432(n + 1) - 3$ then $h^0((n + 1)K_X) \geq n$, for all $n \geq 2$. More generally, if X is not g-countably dense and if g, n, α are as in Table 2.1 or, in the other cases, $\alpha \geq 48(n + 1) - 3$, then $h^0((n+1)K_X) \geq n$, for all $n \geq 1$. Moreover, under the same bounds on α and g given by the case $n = 1$, we have that $h^0(lK_X) \geq 1$ for all $l \geq 2$.*

Remark 2.3. This improves [37, theorem 1.1].

Table 2.1.

g	n	α	g	n	α
2	1	≥ 879	10	$1,\ldots,304$	$\geq 60(n+1)-3$
	≥ 2	$\geq 432(n+1)-3$		$305,\ldots,381$	≥ 18354
3	≥ 1	$\geq 132(n+1)-3$	11	$1,\ldots,8$	$\geq 60(n+1)-3$
4	$1,\ldots,6$	$\geq 96(n+1)-3$		$9,10$	≥ 550
	7	≥ 714	12	$1,\ldots,4$	$\geq 60(n+1)-3$
	≥ 8	$\geq 84(n+1)-3$		5	≥ 306
5	1	≥ 165	13	$1,2$	$\geq 60(n+1)-3$
	2	≥ 242		3	≥ 223
	≥ 3	$\geq 72(n+1)-3$	14	$1,2$	$\geq 60(n+1)-3$
6	$1,\ldots,43$	$\geq 72(n+1)-3$	15	1	≥ 117
	$44,\ldots,52$	≥ 3234		2	≥ 156
	≥ 53	$\geq 60(n+1)-3$	$16,17,18$	1	≥ 117
7	1	≥ 141	19	1	≥ 111
	2	≥ 184	20	1	≥ 105
	≥ 3	$\geq 60(n+1)-3$	21	1	≥ 101
$8,9$	≥ 1	$\geq 60(n+1)-3$	22	1	≥ 97

Proof. We will follow [37] very closely. Since we need to obtain explicit numbers from an asymptotic measure (the volume) the idea is to use the hypothesis about the volume to produce singular divisors and, in this way, log canonical centres. Then we would like to pull back sections from the log canonical centres, using Nadel's vanishing theorem. Unfortunately we do not have information about sections of systems of divisors on lc centres, unless lc centres are points: thus we need a technique by Hacon-McKernan to cut down the dimension of the lc centres (cf. thm. 2.1). But when lc centres have codimension 1 and small volume this cutting-down process does not lead to have a bicanonical section: therefore Todorov's idea is to apply, in this case, a theorem of McKernan about family of tigers and so produce a fibration of X onto a curve and then, from this fibration, produce bicanonical sections (cf. proposition 1.49).

By remark 1.22 X is at least not 2-countably dense, hence $g \geq 2$. Furthermore, since X is not g-countably dense then by remark 1.8 there exists a very general subset Λ such that every curve passing through any point of Λ has geometric genus $\geq g$. Let X_0 be the intersection between Λ and the complement of the union of all subvarieties of X not of general type and $\mathbb{B}_+(K_X)$. X_0 is a very general subset of X, hence countably dense.

Since $\mathrm{vol}(K_X) > \alpha^3$, by lemma 1.45, for every $x \in X$ and every $k \gg 0$ there exists a divisor $A_x \in |kK_X|$ with $\mathrm{mult}_x(A_x) > k\alpha$. Let $\Delta'_x := A_x \frac{\lambda'_x}{k}$, with $\lambda'_x < \frac{3}{\alpha}$, $\lambda'_x \in \mathbb{Q}^+$, but close enough to $\frac{3}{\alpha}$ so that $\mathrm{mult}_x(\Delta'_x) > 3$. Note that $\Delta'_x \sim \lambda'_x K_X$. Let

$s_x := lct(X, \Delta'_x, x)$. Since for a small ϵ we have that $\text{mult}_x((1-\epsilon)\Delta'_x)$ is still > 3 then, by proposition 1.47, $s_x < 1$. Moreover, by remark 1.30, $s_x \in \mathbb{Q}^+$. Therefore, without loss of generality, we can suppose that (X, Δ'_x) is lc, not klt in x.

By lemma 1.41, d., for every $x \in X_0$ there exists an effective $\mathbb{Q}$-divisor $D_x \sim \lambda_x K_X$, with $\lambda_x < \frac{3}{\alpha}$, $\lambda_x \in \mathbb{Q}^+$, such that (X, D_x) is lc, not klt in x and $LLC(X, D_x, x) = \{V_x\}$, where V_x is the unique minimal irreducible element of $LLC(X, \Delta'_x, x)$. Moreover we can also assume that V_x is an exceptional lc centre.

Fix $\beta \in \mathbb{Q}^+$. Set

$$Y_0 := \{x \in X_0 \text{ s.t. } \dim(V_x) = 0\},$$

$$Y_1 := \{x \in X_0 \text{ s.t. } \dim(V_x) = 1\},$$

$$Y_{2,a} := \{x \in X_0 \text{ s.t. } \dim(V_x) = 2 \text{ and } \text{vol}(K_{V_x}) > \beta^2\},$$

$$Y_{2,b} := \{x \in X_0 \text{ s.t. } \dim(V_x) = 2 \text{ and } \text{vol}(K_{V_x}) \leq \beta^2\}.$$

Since X_0 is countably dense then at least one between Y_0, Y_1, $Y_{2,a}$ and $Y_{2,b}$ is countably dense. We will therefore analyze these cases.

Before continuing with the proof we remark that since given a variety of general type (i.e., vol > 0), all the subvarieties that pass through a very general point are of general type (i.e., vol > 0), one is tempted to argue that if the volume is sufficiently large then all the subvarieties that pass through a general point have large volume. Unfortunately this is not the case: for example let C_g be a smooth curve of genus g. Then $\text{vol}(C_2 \times C_g) = \text{vol}(C_2)\text{vol}(C_g) = 4(2g-2)$. Hence we can construct examples of varieties of volume as large as one likes but such that for every point there is a curve of vol $= 2$ (namely C_2).

Back in track: first of all, let us assume that Y_0 is countably dense. For every $x \in Y_0$ we have that $V_x = \{x\}$, in fact $\dim(V_x) = 0$ and V_x is irreducible. $x \in \text{Supp}(D_x)$. Therefore we can apply lemma 1.6 and lemma 1.44 (with $m = n$) to conclude that for every $n \geq 1$, as soon as

$$\lambda_x < \frac{3}{\alpha} \leq 1 \Leftrightarrow \alpha \geq 3, \tag{2.1}$$

$h^0(K_X + nK_X) = h^0((n+1)K_X) \geq n$.

Let us now consider the case Y_1 countably dense. We wish to apply theorem 2.1, to cut down the dimension of the lc centers. For every $x \in Y_1$ consider V_x and a resolution $f_x : W_x \to V_x$. As we have already seen, V_x is an exceptional lc centre of (X, D_x). Since $x \in X_0$, V_x, and hence W_x, are of general type and V_x is not contained in the augmented base locus of K_X. Moreover $g(W_x) \geq g \geq 2$. Let U_x be the very general subset of V_x defined as in [20, thm. 4.1]. Set $U'_x := U_x \cap X_0$. U'_x is still a very general and non-empty subset of V_x. We also have that $\text{vol}(W_x) \geq 2g-2$. Let $\epsilon \ll 1$, $\epsilon \in \mathbb{Q}^+$. Then $\text{vol}(W_x) > 2g-2-\epsilon$. Set $t_1 := 1/(2g-2-\epsilon)$: $\text{vol}(t_1 K_{W_x}) > 1$. For every $y \in U'_x$ let us consider $y' \in f_x^{-1}(y) \subset W_x$. Since y' is a smooth point, by lemma 1.45 and proposition 1.47, there exists $\Theta_{y'} \sim t_1 K_{W_x}$ such that $(W_x, \Theta_{y'})$ is not klt in y'. As before, since $lct(W_x, \Theta_{y'}, y') < 1$, we can suppose that $(W_x, \Theta_{y'})$ is lc, not klt in y' and $\Theta_{y'} \sim \mu_{y'} K_{W_x}$ with $\mu_{y'} \in \mathbb{Q}^+$ and $\mu_{y'} \leq 1/(2g-2-\epsilon)$. Since W_x is a curve and lc centres are irreducible, $LLC(W_x, \Theta_{y'}, y') = \{y'\}$. We can now apply theorem 2.1 since $f_x(y') = y \in U'_x$ and $\{y'\}$ is a pure lc centre: for every $\delta \in \mathbb{Q}^+$, there exists a divisor D'_y such that $\{y\}$ is an exceptional lc centre for (X, D'_y) and $D'_y \sim ((\lambda_x + 1)(\mu_{y'} + 1) - 1 + \delta)K_X$. Let us notice that since $\{y\}$ is an exceptional lc centre then $LLC(X, D'_y, y) = \{\{y\}\}$.

At the end we are in the following situation: for every point $z \in \cup_{x \in Y_1} U'_x$ there exists a $\mathbb{Q}$-divisor D'_z such that $LLC(X, D'_z, z) = \{\{z\}\}$ and such that $D'_z \sim ((\lambda_z + 1)(\mu_z + 1) - 1 + \delta)K_X$, with $\lambda_z < \frac{3}{\alpha}$ and $\mu_z \leq 1/(2g-2-\epsilon)$. Let us prove that $\cup_{x \in Y_1} U'_x$ is still a countably dense subset of X: if $\cup_{x \in Y_1} U'_x \subseteq \cup_{i \in \mathbb{N}} Z_i$, where Z_i are closed proper subsets of X, then, for every $x \in Y_1$, $U'_x \subseteq \cup_{i \in \mathbb{N}} Z_i$, hence $U'_x \subseteq (\cup_{i \in \mathbb{N}} Z_i) \cap V_x = \cup_{i \in \mathbb{N}} (Z_i \cap V_x)$. But U'_x is very general in V_x, hence countably dense in V_x. Therefore for every $x \in Y_1$ there exists $i \in \mathbb{N}$ such that $Z_i \supseteq V_x \ni x$, i.e., $Y_1 \subseteq \cup_{i \in \mathbb{N}} Z_i$, but this is a contradiction.

We can now apply lemma 1.6 and lemma 1.44 (with $m = n$) to conclude that for every $n \geq 1$ if

$$\left(\frac{3}{\alpha} + 1\right)(1 + 1/(2g-2-\epsilon)) - 1 + \delta \leq 1 \Leftrightarrow \alpha \geq \frac{6g-3-3\epsilon}{(2g-2-\epsilon)(1-\delta) - 1} \tag{2.2}$$

(we are considering ϵ, δ very small) then $h^0((n+1)K_X) \geq n$.

Let us now suppose that $Y_{2,a}$ is countably dense. Again, we want to apply theorem 2.1. As before, for every $x \in Y_{2,a}$ we have V_x, a resolution $f_x : W_x \to V_x$ and U_x the very general subset of V_x defined as in [20, thm. 4.1]. As before, consider $U'_x := U_x \cap X_0$. U'_x is still a very general and non-empty subset of V_x. For every $y \in U'_x$ consider $y' \in f_x^{-1}(y)$. Since $\mathrm{vol}(V_x) > \beta^2$ then $\mathrm{vol}(W_x) = \mathrm{vol}(V_x) > \beta^2$. Set $t_1 = 2/\beta$. Then $\mathrm{vol}(t_1 K_{W_x}) > 2^2$. Hence there exists $\Theta_{y'} \sim t_1 K_{W_x}$ such that $(W_x, \Theta_{y'})$ is not klt in y'. Since $lct(W_x, \Theta_{y'}, y') < 1$, we can suppose that $(W_x, \Theta_{y'})$ is lc, not klt in y' and $\Theta_{y'} \sim \mu_{y'} K_{W_x}$ with $\mu_{y'} \in \mathbb{Q}^+$ and $\mu_{y'} < 2/\beta$. Therefore there exists a pure lc centre $W'_{y'} \in LLC(W_x, \Theta_{y'}, y')$. Set $V'_y := f_x(W'_{y'}) \ni y$. By thm. 2.1, for every $\delta \in \mathbb{Q}^+$ there exists a $\mathbb{Q}$-divisor D'_y such that V'_y is an exceptional lc centre for (X, D'_y) and such that

$$D'_y \sim ((\lambda_x + 1)(\mu_{y'} + 1) - 1 + \delta)K_X.$$

Recall that $\lambda_x < \frac{3}{\alpha}$ and $\mu_{y'} < \frac{2}{\beta}$. Consider

$$J_0 := \left\{ y \in \cup_{x \in Y_{2,a}} U'_x \ s.t.\ \dim(V'_y) = 0 \right\}$$

and

$$J_1 := \left\{ y \in \cup_{x \in Y_{2,a}} U'_x \ s.t.\ \dim(V'_y) = 1 \right\}.$$

Note that if V'_y is a point, i.e. $V'_y = \{y\}$, then $LLC(X, D'_y, y) = \{\{y\}\}$, while if V'_y is a curve then it is of general type because it passes through $y \in X_0$. Since $\cup_{x \in Y_{2,a}} U'_x$ is countably dense in X, then either J_0 or J_1 is countably dense.

If J_0 is countably dense then we can apply lemma 1.6 and lemma 1.44 (with $m = n$) to conclude that, assuming ϵ, δ very small and

$$\beta > \frac{2}{1-\delta}, \tag{2.3}$$

for every $n \geq 1$ if

$$\left(\frac{3}{\alpha} + 1\right)(1 + 2/\beta) - 1 + \delta \leq 1 \Leftrightarrow \alpha \geq \frac{3\beta + 6}{\beta(1-\delta) - 2} \tag{2.4}$$

then $h^0((n+1)K_X) \geq n$. If J_1 is countably dense then we can argue exactly in the same way as we did before for Y_1 countably dense: simply re-read the proof substituting Y_1 with J_1 and λ_x with $(\lambda_x + 1)(\mu_{y'} + 1) - 1 + \delta$. We can conclude that, assuming ϵ, δ very small and

$$\beta > \frac{2}{(2-\delta)\left(\frac{2g-2-\epsilon}{2g-1-\epsilon}\right) - 1 - \delta}, \tag{2.5}$$

for every $n \geq 1$ if

$$\left(\left(\left(\frac{3}{\alpha} + 1\right)\left(\frac{2}{\beta} + 1\right) - 1 + \delta\right) + 1\right)\left(1 + \frac{1}{2g-2-\epsilon}\right) - 1 + \delta \leq 1 \Leftrightarrow \tag{2.6}$$

$$\Leftrightarrow \alpha \geq \frac{3\beta + 6}{\beta\left((2-\delta)\left(\frac{2g-2-\epsilon}{2g-1-\epsilon}\right) - 1 - \delta\right) - 2} \tag{2.7}$$

then $h^0((n+1)K_X) \geq n$.

Let us now suppose that $Y_{2,b}$ is countably dense. Recall that for every $x \in Y_{2,b}$ we have a divisor $D_x \sim \lambda_x K_X$ such that $LLC(X, D_x, x) = \{V_x\}$, V_x is an exceptional log canonical centre and $\dim(V_x) = 2$. Since if we decompose a countably dense set as a countable union of subsets then at least one of the subsets is countably dense, we can suppose that $\lambda_x = \lambda$ for a fixed $\lambda \in \mathbb{Q}^+$. Recall that $\lambda < \frac{3}{\alpha}$. By lemma 1.48, we are in the following situation:

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[d, "f"] & X \\ B & \end{tikzcd}$$

where X', B are normal projective varieties, f is a dominant morphism with connected fibres, π is a dominant and generically finite morphism and the image under π of a general fibre of f is V_x. Arguing exactly as in [37] we can suppose that there exists a proper closed subset $X_1 \subset X$ such that for all $x \notin X_1$, D_x is smooth at x. Either π is birational or the inverse image of a general $x \in X \setminus X_1$ under π is contained in at least two different fibres of f: in fact if π is not birational, since π is a dominant generically finite morphism, then there exists an open set $O' \subseteq X'$ and $m \in \mathbb{N}$, $m > 1$, such that for every $x \in O'$, $\#\pi^{-1}(x) = m$; let $F_x \subset X'$ be a fibre of f such that $\pi(F_x) = V_x$; F_x and V_x are birational through $\pi|_{F_x}$ by [33], hence, since V_x is smooth at x, $\pi|_{F_x}^{-1}(x)$ is connected, that is $\#(\pi^{-1}(x) \cap F_x) = 1$. Using the above fact and by the construction of B then if π is not birational there are at least two log canonical centres through x. In this case we can apply lemma 1.42 and lemma 1.41, d., e., to conclude that there exists a countably dense set $Y := Y_{2,b} \cap (X \setminus X_1)$ such that for all $y \in Y$ there exists a divisor $S_y \sim k(2\lambda K_X)$ ($0 < k \leq 1$, $\lambda < \frac{3}{\alpha}$) such that $LLC(S_y, y) = \{C_y\}$, where C_y is an irreducible variety of dimension at most 1. Therefore, as in the case of Y_0 and Y_1, we can apply lemma 1.6 and lemma 1.44 (with $m = n$) to conclude that for every $n \geq 1$, if $2\lambda k < \frac{6}{\alpha} \leq 1$, that is

$$\alpha \geq 6 \tag{2.8}$$

and

$$\left(\frac{6}{\alpha} + 1\right)(1 + 1/(2g - 2 - \epsilon)) - 1 + \delta \leq 1 \Leftrightarrow \alpha \geq \frac{12g - 6 - 6\epsilon}{(2g - 2 - \epsilon)(1 - \delta) - 1} \tag{2.9}$$

(we are considering ϵ, δ very small) then $h^0((n + 1)K_X) \geq n$.

We can now suppose that π is birational. Again arguing as in [37], we can suppose $X' = X$ and that the general fibre of f over a point $b \in B$, X_b, is minimal and smooth (and of general type). Since f is projective, B is smooth and the general fibre of f is connected, then by Zariski's connectedness theorem (see [11, 2.3.7]) f has connected fibres. Hence, by Stein factorization, f is a fibre space. Moreover for every $b \in B$ there exists a divisor $D_b \sim \lambda K_X$ for which we have $\mathcal{J}(D_b) \subset \mathcal{O}_X(-X_b)$ (since the fibre is an exceptional lc centre for (X, D_b)). Hence we are exactly in the situation of proposition 1.49: setting $k = 1$, we know that if $\lambda(4(n + 1)[\beta^2] - 1) < 1$ and $n > \lambda$ then there is a surjection

$$H^0(X, \mathcal{O}_X((n + 1)K_X)) \to H^0(X_b, \mathcal{O}_{X_b}((n + 1)K_{X_b}))$$

and thus the theorem is proved, because, by [2, VII.5.4],

$$h^0(X_b, \mathcal{O}_{X_b}((n + 1)K_{X_b})) \geq n.$$

Since $\lambda < \frac{3}{\alpha}$ then the numerical conditions are satisfied as long as

$$\alpha \geq 12(n + 1)[\beta^2] - 3 \tag{2.10}$$

$$\alpha \geq \frac{3}{n} \tag{2.11}$$

It is now time to put everything together, that is to find the best possible value for β such that we have the lowest inferior bound for α.

Set $b := \frac{2g-3}{2g-1}$. Note that (2.5) implies (2.3) and that $\beta > \frac{2}{b} = \frac{4g-2}{2g-3}$ implies (2.5). Moreover (2.9) $\Rightarrow$ (2.2) $\Rightarrow$ (2.1) $\Rightarrow$ (2.11), (2.9) $\Rightarrow$ (2.8) and (2.7) $\Rightarrow$ (2.4). Besides

(2.7) $\Rightarrow$ (2.9) if $\beta < \frac{12g-10}{2g-3}$. Therefore we are left to consider only the conditions (2.7) and (2.10). Set

$$\beta' := \frac{1+\sqrt{1+\frac{b(b+1)}{4(n+1)}}}{b}$$

and, finally, choose $\beta := \sqrt{[\beta'^2]+1-\epsilon'}$ with $0 < \epsilon' \ll 1$ and such that $\beta \in \mathbb{Q}$. In this way $\beta > \beta' > \frac{2}{b} = \frac{4g-2}{2g-3}$, $[\beta^2] = [\beta'^2]$ and, actually, $[\beta^2] = 4$ for every $n \geq 1$ as soon as $g \geq 19$.

Since (2.7) does not depend on n, for n sufficiently large (2.10) $\Rightarrow$ (2.7). Moreover, with that choice of β and with g sufficiently large (namely $g \geq 23$), we have that (2.10) $\Rightarrow$ (2.7) for every n. Therefore in general (2.10) $\Rightarrow$ (2.7), except for a finite number of couples (g, n) listed below:

1. $g = 2, n = 1$;
2. $g = 4, n = 7$;
3. $g = 5, n = 2$;
4. $g = 6, 44 \leq n \leq 52$;
5. $g = 7, n = 2$;
6. $g = 10, 305 \leq n \leq 381$;
7. $g = 11, n = 9, 10$;
8. $g = 12, n = 5$;
9. $g = 13, n = 3$;
10. $g = 15, n = 2$;
11. $19 \leq g \leq 22, n = 1$.

The theorem now follows by simple computations.

For the last statement just notice that if we go back over the above proof but using lemma 1.44 with $n = 1$ and $m = 2$ (instead of $n = 1$ and $m = 1$) then we can conclude that $h^0(3K_X) > 0$ when $g = 2, \alpha \geq 141$, or $g = 3, \alpha \geq 69$, or $g = 4, \alpha \geq 47$, or $g \geq 5, \alpha \geq 33$. Therefore, for $n = 1$, if g, α are as in the hypotheses of the theorem then not only $h^0(2K_X) > 0$ but also $h^0(3K_X) > 0$ so, in these cases, we can say that $h^0(lK_X) \geq 1$ for every $l \geq 2$.

$\square$

Remark 2.4. There are examples of smooth threefolds X with arbitrarily large volume but $h^0(K_X) = 0$: in fact just choose a smooth surface of general type S with $H^0(K_S) = 0$, for example a numerical Godeaux surface (see [2, VII, 10.1]) and a smooth curve C of genus g. Then set $X := S \times C$: $\mathrm{vol}(X) = 3\mathrm{vol}(C)\mathrm{vol}(S) = 3(2g - 2)\mathrm{vol}(S) \xrightarrow[g \to +\infty]{} +\infty$, but by Kunneth's formula $H^0(K_X) \cong H^0(K_S) \otimes H^0(K_C) = 0$.

Remark 2.5. In [23] there are many examples of threefolds of small volume that do not verify some of the theses of theorem 2.2. They are all constructed as weighted complete intersections $X_{d_1,\dots d_c}$ canonically embedded in a weighted projective space $\mathbb{P}(a_1, \dots, a_{c+3})$ and with at worst canonical singularities. For instance $X_{46} \subset \mathbb{P}(4, 5, 6, 7, 23)$, a general element in the family of all degree 46 hypersurfaces in $\mathbb{P}(4, 5, 6, 7, 23)$, has volume $= 1/420$ and p_g, P_2, P_3 all equal zero (see [23, 15.3] and [23, Table 15.1 No. 23]). $X_{10,12,18} \subseteq \mathbb{P}(3, 4, 5, 5, 6, 7, 9)$ has volume $2/105$ and $P_n < n - 1$ for all n between 2 and 14 (see [23, Table 18.16 No. 35]). $X_{10,12} \subseteq \mathbb{P}(2, 2, 3, 4, 5, 5)$ has volume $= 1/10$ and $P_2 = 2$, but $P_3 = 1$ (see [23, Table 15.4 No. 43]). Notice that all the plurigenera have been computed using the formula in [23, par. 18].

Chapter 3

Pluricanonical maps for 3-folds of general type

3.1 Pluricanonical maps of order ≥ 5

In order to have effective estimates on which pluricanonical system determines a birational map, by generic smoothness we should only understand when pluricanonical systems separate very general points (see lemma 1.11). Since we now need to keep track of two points and not only one, in this case to have the best results we cannot argue exactly in the same way as before (that is, applying [20, thm. 4.1] (see thm. 2.1)).

Therefore, following [37], we will use a slightly different technique by Takayama to inductively lower the dimension of lc centers on a birational modification of the original variety. Before stating and proving the main theorem we will therefore quote Takayama's results:

Theorem 3.1 (Takayama)**.** *([36, prop. 5.3]). Let X be a smooth projective variety of general type and of dimension d, let $0 < \epsilon < 1$ and let $\mu : X' \to X$ be a birational morphism from X' smooth and projective such that $\mu^*(K_X) \sim_{\mathbb{Q}} A_\epsilon + E_\epsilon$, where A_ϵ is an ample $\mathbb{Q}$-divisor, E_ϵ is an effective $\mathbb{Q}$-divisor and they verify the properties of [36, theorem 3.1]. Let Q be the union of all subvarieties of X of general type. Let us take two distinct points $x_1, x_2 \in X' \setminus (\mu^{-1}(Q) \cup Supp(E_\epsilon))$. Then the following induction statement $(*_j)$ holds for $1 \leq j \leq d$:*

There exists a positive constant

$$a_j < s_j + t_j / \sqrt[d]{vol(X)}$$

and a non-empty subset $I_j \subseteq \{1, 2\}$ with the following properties. There exists an effective $\mathbb{Q}$-divisor D_j on X' such that $D_j \sim_{\mathbb{Q}} a_j A_\epsilon$ and such that

(i) (X', D_j) is lc at x_i for $i \in I_j$,

(ii) (X', D_j) *is not klt at* x_i *for* $i \in I_j$*,*

(iii) (X', D_j) *is not lc at* x_i *for* $i \in \{1, 2\} \setminus I_j$*,*

(iv) *codim*$(Nklt(X', D_j)) \geq d$ *at* x_i *for* $i \in I_j$*.*

The constants s_j, t_j are defined inductively in [36, Notation 5.2(3)]. We do not recollect them here, since we will need a slightly different inductive definition that can be found in the proof of theorem 3.3 and, more explicitly and generally, in the proof of theorem 4.6.

Lemma 3.2 (Takayama)**.** *([36, lemma 5.4])* $(*_1)$ *holds.*

Theorem 3.3. *Let X be a smooth, not g-countably dense, projective threefold of general type and such that $vol(X) > \alpha^3$. Let $l \in \mathbb{N}$, $l \geq 5$. Let*

$$f(l, g) := 3\sqrt[3]{2}\left(4l\left[\frac{32g^2}{((g(l-1)-(l+1))^2}\right]-1\right).$$

If l, g, α are as in Table 3.1 or, in the other cases,

$$\alpha > \frac{3\sqrt[3]{2}g(1+2\sqrt{2})}{g(l-1)-(l+1)-4\sqrt{2}g}$$

then $|lK_X|$ gives a birational map.

Table 3.1.

l	g	α	l	g	α
5	$\neq 9$	$> f(l,g)$	9	≤ 4	$> f(l,g)$
	9	$118\sqrt[3]{2}$	10	≤ 3	$> f(l,g)$
6	$\neq 8$	$> f(l,g)$	11	2	$> f(l,g) = 261\sqrt[3]{2}$
	8	$73\sqrt[3]{2}$	12	2	$> f(l,g) = 141\sqrt[3]{2}$
7	≤ 23	$> f(l,g)$	13	2	$> f(l,g) = 153\sqrt[3]{2}$
	$24, \ldots, 39$	$81\sqrt[3]{2}$	14	2	$> f(l,g) = 165\sqrt[3]{2}$
8	≤ 6	$> f(l,g)$			
	7	$93\sqrt[3]{2}$			

Corollary 3.4. *If $\alpha > 1917\sqrt[3]{2}$ (or, in case $g \geq 10$, $\alpha > 117\sqrt[3]{2}$) then $|lK_X|$ gives a birational map for every $l \geq 5$. More generally, if $g \neq 9$, $\alpha > 3\sqrt[3]{2}\left(20\left[\frac{8g^2}{(2g-3)^2}\right]-1\right)$ or $g = 9$, $\alpha > 118\sqrt[3]{2}$ then $|lK_X|$ gives a birational map for every $l \geq 5$.*

Remark 3.5. This improves [37, theorem 1.2].

Remark 3.6. By [8] we know that if $l \geq 73$ then $|lK_X|$ is always birational, independently of the volume of X.

Proof. We will follow [37]. By [36, theorem 3.1], for every $0 < \epsilon < 1$ there exists a smooth projective variety X', a birational morphism $\mu : X' \to X$ and an approximate Zariski decomposition $\mu^*(K_X) \sim_{\mathbb{Q}} A + E$ where $A = A_\epsilon$ is an ample $\mathbb{Q}$-divisor and $E = E_\epsilon$ is an effective $\mathbb{Q}$-divisor that satisfy condition (1),(2),(3) of Takayama's theorem (cf. [36, theorem 3.1]).

First of all let us notice that we can argue on X' instead of X: in fact, for every $l \in \mathbb{N}^+$, μ induces an isomorphism $\mu^* : H^0(lK_X) \to H^0(lK_{X'})$ (cf. [21, II.8.19 and ex. 8.8]), hence if $U' := X' \setminus \cup_{i\in\mathbb{N}} V_i$ is a very general subset of X' (with V_i proper subvarieties of X') and $|lK_{X'}|$ separates every couple of points in U', then $|lK_X|$ separates every couple of points in $U = X \setminus (\cup_{i\in\mathbb{N}} \mu(V_i))$, a very general subset of X: actually, if $x, y \in U$ then, since μ is surjective, $\exists x', y' \in U'$ such that $x = \mu(x'), y = \mu(y')$; then, given $s \in H^0(lK_{X'})$ separating x' and y', there must exist a section $\sigma \in H^0(lK_X)$ with $s = \mu^*(\sigma)$ and therefore σ separates x and y.

By remark 1.22 X' is at least not 2-countably dense, hence $g \geq 2$. Furthermore, since X' is not g-countably dense then by remark 1.8 there exists a very general subset $\Lambda \subseteq X'$ such that every curve passing through any point of Λ has geometric genus $\geq g$. Now we would like to simply apply theorem 3.1, but in order to have better numerical conditions, as in the proof of theorem 2.2 we will distinguish two different cases depending on the volume of lc centres.

By the proof of lemma 3.2 (see [36, lemma 5.4]), there exists a very general subset U of X' such that for every two distinct points $x, y \in U$ we can construct, depending on x, y, an effective $\mathbb{Q}$-divisor $D_1 \sim_{\mathbb{Q}} a_1 A$, with $a_1 < \frac{3\sqrt[3]{2}}{\alpha(1-\epsilon)}, a_1 \in \mathbb{Q}^+$, such that $x, y \in Z(\mathcal{J}(D_1))$,$(X', D_1)$ is lc not klt at one of the points, say $p(x,y) \in \{x, y\}$, and either $\mathrm{codim} Z(\mathcal{J}(D_1)) > 1$ at $p(x,y)$ or there is one irreducible component of $Z(\mathcal{J}(D_1))$, say $V_{p(x,y)}$, that passes through $p(x,y)$ and such that $\mathrm{codim} V_{p(x,y)} = 1$. We can suppose $U \subseteq \Lambda$.

Fix $\beta \in \mathbb{Q}^+$.

Let $U' := \{p(x,y) | \mathrm{codim} Z(\mathcal{J}(D_1)) = 1 \text{ at } p(x,y) \text{ and } \mathrm{vol}(V_{p(x,y)}) \leq \beta^2\}$. Since $U = U' \cup (U \setminus U')$, then by lemma 1.5, 4., we are in one of these two cases:

1. $U \setminus U'$ contains a very general subset U'' of X;
2. U' is countably dense.

In the first case we know that $\forall x, y \in U''$ either $\mathrm{codim} Z(\mathcal{J}(D_1)) > 1$ at $p(x,y)$ or $\mathrm{vol}(V_{p(x,y)}) > \beta^2$. Applying the inductive steps of theorem 3.1 (see [36, prop. 5.3]), we can conclude that given two very general points $x, y \in X'$ there exists (depending on x, y) an effective $\mathbb{Q}$-divisor D on X' and $a \in \mathbb{Q}^+$ with $D \sim_{\mathbb{Q}} aA$ such that $x, y \in Z(\mathcal{J}(X', D))$ with $\dim Z(\mathcal{J}(X', D)) = 0$ around x or y, that is x or y is an isolated point of $Z(\mathcal{J}(X', D))$, and

$$a < \left(1 + \frac{1}{(1-\epsilon)(g-1)}\right)\left(1 + \frac{2\sqrt{2}}{(1-\epsilon)\beta}\right)\left(2 + \frac{3\sqrt[3]{2}}{(1-\epsilon)\alpha}\right) - 2 + \epsilon f,$$

where $f = \left(1 + \frac{1}{(1-\epsilon)(g-1)}\right)\left(2 + \frac{2\sqrt{2}}{(1-\epsilon)\beta}\right) > 0$.

By [12, 1.41], $K_{X'} \sim_{\mathbb{Z}} \mu^*(K_X) + \mathrm{Exc}(\mu) \sim_{\mathbb{Q}} A + E + \mathrm{Exc}(\mu)$, where $\mathrm{Exc}(\mu)$ is the exceptional locus and it is an effective divisor by [12, 1.40]. Therefore, replacing D with $D + (l-1)(E + \mathrm{Exc}(\mu))$ (with $l \in \mathbb{N}^+$), as in the proof of lemma 1.44 we can conclude that by Nadel's vanishing theorem and proposition 1.35, $|lK_{X'}|$ separates two very general points in X' as soon as $l \geq [a] + 2$.

Hence, in the first case, considering $l \geq 5$, we now need only to estimate α (depending on g, β, ϵ) in order to have $[a] \leq l - 2$, that is $a < l - 1$. To that purpose, choosing ϵ sufficiently small and

$$\beta > \frac{4\sqrt{2}g}{g(l-1) - (l+1)}, \tag{3.1}$$

it is enough to ask that

$$\left(1 + \frac{1}{g-1}\right)\left(1 + \frac{2\sqrt{2}}{\beta}\right)\left(2 + \frac{3\sqrt[3]{2}}{\alpha}\right) < l + 1$$

$$\Leftrightarrow \alpha > \frac{3\sqrt[3]{2}g(\beta + 2\sqrt{2})}{\beta(g(l-1) - (l+1)) - 4\sqrt{2}g}. \tag{3.2}$$

If, otherwise, the second case occur then $\beta \geq 1$ since the volume of a surface of general type is at least 1. Moreover by lemma 1.48, we are in the following situation:

$$\begin{tikzcd} X'' \arrow[r, "\pi"] \arrow[d, "f"] & X' \\ B & \end{tikzcd}$$

where X'', B are normal projective varieties, f is a dominant morphism with connected fibres, π is a dominant and generically finite morphism and the image under π of a general fibre of f is V_x where V_x is a surface through x, a general point. Moreover there exists a divisor D_x such that V_x is a pure log canonical centre of (X', D_x). In addition, setting $\overline{a} := \frac{3\sqrt[3]{2}}{\alpha(1-\epsilon)} \in \mathbb{Q}^+$ we have that $D_x \sim_{\mathbb{Q}} \overline{a}A$. Moreover for every p in a countably dense subset of U', V_p is the image through π of a fiber of f.

Again as in [37], we can suppose that there exists a proper closed subset $X'_1 \subset X'$ such that for all $x \notin X'_1$, D_x is smooth at x.

As in [37] and the proof of theorem 2.2, we will distinguish two other different subcases, depending on the birationality of π: in fact, as we have already proved in theorem 2.2, either π is birational or for a general $x \in X' \setminus X'_1$ there are at least two log canonical centres through x.

In the latter case we claim that, using these two log canonical centres, we can still apply the inductive steps of theorem 3.1, starting from codimension 2. That is: we claim that given x, y general points of X' we can find an effective $\mathbb{Q}$-divisor $D_{x,y}$ on X' and $\overline{a}'' \in \mathbb{Q}$ such that $D_{x,y} \sim_{\mathbb{Q}} \overline{a}''A$ and such that $D_{x,y}$ satisfies the induction statement $(*_2)$ of theorem 3.1.

Now we will prove the claim. Just consider x, y general points of X' and consider also the divisors D_x and D_y. By [32, prop. 9.2.32(i)], since D_x is not klt at x and D_y is not klt at y, then $D_x + D_y$ is not klt at x and y. Since D_x and D_y are nonsingular outside X'_1 then we can write

$$D_x = D^1_x + \theta_x D^2_x + G_x,$$

where $\theta_x \in \mathbb{Q}^+$, D^1_x is the only prime divisor in D_x passing through x and D^2_x is the only prime divisor in $D_x - D^1_x$ passing through y (or $D^2_x = 0$ if $y \in D^1_x$) and $G_x = D_x - D^1_x - \theta_x D^2_x$ is effective. Analogously

$$D_y = D^1_y + \theta_y D^2_y + G_y$$

with D^1_y the only prime divisor in D_y passing through y, D^2_y the only prime divisor in $D_y - D^1_y$ passing through x (or $D^2_y = 0$ if $x \in D^1_y$) and $G_y = D_y - D^1_y - \theta_y D^2_y$ is effective. After rescaling with a positive rational number $q \leq 1$ we can suppose that $qD_x + qD_y$ is lc but not klt at one of the points, say x, and it is not klt at the other point y. Therefore, depending on x, y there exists $\overline{a}' \in \mathbb{Q}^+$, $\overline{a}' \leq \frac{6\sqrt[3]{2}}{\alpha(1-\epsilon)} < \epsilon + \frac{6\sqrt[3]{2}}{\alpha(1-\epsilon)}$, such that $qD_x + qD_y \sim_{\mathbb{Q}} \overline{a}' A$.

If there exists, for the pair $(X', qD_x + qD_y)$, a lc centre through x of codimension ≥ 2 then by 1.41, b., just consider the unique minimal element W of $LLC(X', qD_x + qD_y, x)$ and proceed as follows:

1. if $qD_x + qD_y$ is not lc at y then we can apply 1.41, d., with ϵ' sufficiently small so that $(1-\epsilon')(qD_x + qD_y)$ is still not lc at y. After the tie-breaking, potentially taking a smaller ϵ', the new divisor is $\sim_{\mathbb{Q}} \overline{a}'' A$ with $\overline{a}''$ still $< \epsilon + \frac{6\sqrt[3]{2}}{\alpha(1-\epsilon)}$. This new divisor verifies the induction statement $(*_2)$ of theorem 3.1 with $a_2 = \overline{a}'', s_2 = \epsilon, t_2 = \frac{6\sqrt[3]{2}}{\alpha(1-\epsilon)}\mathrm{vol}(X)$;

2. if $qD_x + qD_y$ is lc, not klt both at x and y but for the pair $(X', qD_x + qD_y)$ there exists a lc centre Z through y such that $x \notin Z$, then take a sufficiently small positive rational number ϵ' and an ample divisor $H \sim_{\mathbb{Q}} \epsilon' A$. Let B be a general divisor given by a general section in $H^0(\mathcal{O}_{X'}(mH) \otimes \mathcal{I}_Z)$ for m sufficiently large and divisible. By Kollar-Bertini theorem (see [32, ex. 9.2.29]) $\mathcal{J}(X', qD_x + qD_y + \frac{1}{m}B) = \mathcal{J}(X', qD_x + qD_y)$ outside Z, hence $qD_x + qD_y + \frac{1}{m}B$ is still lc, not klt at x but it is not lc at y. Hence we can apply 1.;

3. if $qD_x + qD_y$ is lc, not klt both at x and y, if every lc centre for $(X', qD_x + qD_y)$ through y passes through x and $y \in W$, then W is the minimal lc centre at y as well. Hence, as in 1., we can apply the tie-breaking and produce a new divisor that is lc, not klt at x, y and whose Nklt locus has codimension ≥ 2 both at x and y. Therefore, as in 1., the induction statement $(*_2)$ of theorem 3.1 is satisfied;

4. if $qD_x + qD_y$ is lc, not klt both at x and y, if every lc centre for $(X', qD_x + qD_y)$ through y passes through x and $y \notin W$, then as in 2., using $\mathcal{I}_W$ we can suppose

that

$$D' := qD_x + qD_y + \frac{1}{m}B$$

is still lc, not klt at y but not lc at x. If the minimal lc centre of (X', D') at y has codimension ≥ 2 then, after switching x and y, we can apply 1. Otherwise there is a unique lc centre of codimension 1 at y for the pair (X', D'). Call it Y_1. Since y is a general point then there exists an effective $\mathbb{Q}$-divisor $F_y \sim_{\mathbb{Q}} \overline{a}A$ that passes through y, that is smooth at y, such that (X', F_y) is lc, not klt at y and such that $LLC(X', F_y, y) = \{Y_2\}$ where Y_2 is an irreducible subvariety of codimension 1. Since, by hypothesis, there are at least two different V's passing through y then we can suppose that $Y_2 \neq Y_1$. Consider $D' + kF_y$ where

$$k = \max\{c : D' + cF_y \text{ is lc at } y\}.$$

Clearly $k \geq 0$ and thus $D' + kF_y$ is an effective divisor that is still not lc at x but it is lc, not klt at y. Let $\mu : \tilde{X} \to X'$ be a log resolution of $D' + F_y$. Thus we can write $K_{\tilde{X}/X'} = \sum a(E)E$, $K_{\tilde{X}/X'} - \mu^*D' = \sum(a(E) + b(E))E$, $K_{\tilde{X}/X'} - \mu^*F_y = \sum(a(E) + c(E))E$, so that

$$K_{\tilde{X}/X'} - \mu^*(D' + kF_y) = \sum(a(E) + b(E) + kc(E))E.$$

By hypothesis there exists a non-exceptional E_0 (i.e. $a(E_0) = 0$) such that $b(E_0) = -1$ and $\mu(E_0) = Y_1$. Since F_y is smooth at y, then the only irreducible component of F_y passing through y is $Y_2 \neq Y_1$, hence $c(E_0) = 0$ and thus $\mu(E_0) = Y_1$ is a lc centre at y also for $D' + kF_y$. By the definition of k there exists a prime divisor $\overline{E}$ such that $y \in \mu(\overline{E})$ and $a(\overline{E}) + b(\overline{E}) + (k+1)c(\overline{E}) < -1$; but we must also have that $a(\overline{E}) + b(\overline{E}) + kc(\overline{E}) \geq -1$. Hence, in particular, $c(\overline{E}) \neq 0$ and thus $\overline{E} \neq E_0$ and this implies that $\mu(\overline{E}) \neq \mu(E_0) = Y_1$. Thus now, considering $Y_1 \cap \mu(\overline{E})$ and using 1.41,a., we know that there exists a lc centre of $D' + kF_y$ at y of codimension ≥ 2. Moreover since $D' + kF_y \geq D'$ then $D' + kF_y$ is still not lc at x. Hence, after switching x and y, we can apply 1. and produce a new divisor $\sim_{\mathbb{Q}} \overline{a}''A$ with $\overline{a}'' < \epsilon + \frac{9\sqrt[3]{2}}{\alpha(1-\epsilon)}$ and satisfying $(*_2)$ of theorem 3.1.

If, on the contrary, every lc centre for $(X', qD_x + qD_y)$ through x has codimension 1 then, by 1.41, a., there exists only one lc centre for $(X', qD_x + qD_y)$ that passes through x. Since it has codimension 1 then this lc centre must be the support of a prime divisor in $qD_x + qD_y$. Since

$$qD_x + qD_y = qD_x^1 + q\theta_x D_x^2 + qD_y^1 + q\theta_y D_y^2 + qG_x + qG_y$$

then, identifying a prime divisor with its support, this lc centre can be:

i. D_x^1;

ii. D_y^1;

iii. D_y^2.

We analyze these different cases.

i. a. if $D_x^1 = D_y^1$ then just see ii.;

b. if $D_x^1 \neq D_y^1$ and $D_x^1 = D_y^2$ then D_y^1 does not pass through x, or otherwise $D_y^2 = 0$. Since $D_x^1 = D_y^2$ does not pass through y then the components through y of $\mathrm{Nklt}(X', qD_x + qD_y)$ are contained in $D_x^2 \cup D_y^1$. Since neither D_x^2 nor D_y^1 passes through x then there exists an irreducible component of $\mathrm{Nklt}(X', qD_x + qD_y)$ that passes through y but not through x: hence, as in 2., we can possibly add a small divisor to make sure that $qD_x + qD_y$ is not lc at y, but still lc, not klt at x. At this point, as in the second part of 4., we can add another divisor through x in order to have a lc centre at x of codimension ≥ 2 and then apply 1. to produce a new divisor $\sim_{\mathbb{Q}} \overline{a}''A$ with $\overline{a}'' < \epsilon + \frac{9\sqrt[3]{2}}{\alpha(1-\epsilon)}$ and satisfying $(*_2)$ of theorem 3.1;

c. if $D_x^1 \neq D_y^1$ and $D_x^1 \neq D_y^2$ then $q = 1$ and thus D_y^1 is a lc centre through y not only for the divisor D_y but also for the divisor $qD_x + qD_y = D_x + D_y$. Therefore D_y^1 cannot pass through x. We can conclude as in ib.;

ii. since $x \in D_y^1$ then $D_y^2 = 0$ and hence D_y is lc, not klt also at x. Therefore instead of $qD_x + qD_y$ we can just consider D_y and its lc centre D_y^1. Now, as in 4., we can add another divisor to obtain a lc centre at x of codimension ≥ 2 and thus falling in one of the cases 1, 2, 3, 4. Eventually we produce a new divisor $\sim_{\mathbb{Q}} \overline{a}''A$ with $\overline{a}'' < \epsilon + \frac{9\sqrt[3]{2}}{\alpha(1-\epsilon)}$ and satisfying $(*_2)$ of theorem 3.1;

iii. $D_y^2 \neq D_x^1$, otherwise see ib. We also know that $x \notin D_y^1$, otherwise $D_y^2 = 0$. If $y \notin D_x^1$ then there exists an irreducible component of $\mathrm{Nklt}(X', qD_x + qD_y)$ at y that does not pass through x. We can conclude as in ib. If $y \in D_x^1$ then $D_x^2 = 0$. Since the lc centre at x is unique then the discrepancy $a(D_x^1, X', qD_x + qD_y)$ is > -1. Hence the irreducible components of $\mathrm{Nklt}(X', qD_x + qD_y)$ through y are contained in D_y^1. Therefore we can conclude as in ib.

Summing up: if π is not birational we can conclude that for every x, y general points in X' there exists an effective $\mathbb{Q}$-divisor $D_{x,y}$ on X' and a positive rational number (depending on x, y) $\overline{a}'' < \epsilon + \frac{9\sqrt[3]{2}}{\alpha(1-\epsilon)}$ such that $D_{x,y} \sim_{\mathbb{Q}} \overline{a}''A$ and such that $D_{x,y}$ satisfies the induction statement $(*_2)$ of theorem 3.1.

We can now apply the inductive steps of Takayama (see [36, prop. 5.3; in particular lemmas 5.5, 5.8]) and conclude that for every x, y general points in X' there exists an effective $\mathbb{Q}$-divisor $D'_{x,y}$ on X' and a positive rational number $\overline{a}'''$ such that $D'_{x,y} \sim_{\mathbb{Q}} \overline{a}'''A$, $x, y \in Z(\mathcal{J}(X', D'_{x,y}))$ with $\dim Z(\mathcal{J}(X', D'_{x,y})) = 0$ around x or y and

$$\overline{a}''' < \left(2 + \frac{2}{(1-\epsilon)(g-1)}\right)\left(1 + \frac{\frac{9}{2}\sqrt[3]{2}}{(1-\epsilon)\alpha}\right) - 2 + 2\epsilon h,$$

where $h = \left(1 + \frac{1}{(1-\epsilon)(g-1)}\right) > 0$.

As before, we conclude that $|lK'_X|$ separates x and y as soon as $\overline{a}''' < l-1$. To that purpose, choosing ϵ sufficiently small, it is enough to ask that

$$\left(2+\frac{2}{g-1}\right)\left(1+\frac{\frac{9}{2}\sqrt[3]{2}}{\alpha}\right) < l+1 \Leftrightarrow \alpha > \frac{9\sqrt[3]{2}g}{g(l-1)-(l+1)}. \tag{3.3}$$

We can now assume that π is birational. Moreover since $K_{X''} \sim \pi^*(K'_X)+\mathrm{Exc}(\pi)$, then we can suppose that the general fiber X''_b is a pure log canonical centre of $D''_b \sim_{\mathbb{Q}} \overline{a}K_{X''}$. Arguing as in [37] we can suppose X'' is smooth and that the general fiber X''_b of f is smooth and minimal (and of general type). As in the proof of 2.2, f is a fibre space. In addition, since the fibers of f are all numerically equivalent, we also know that $\mathrm{vol}(X''_b) \leq \beta^2$.

As before to prove that $|lK_{X'}|$ separates two very general points it is enough to show that $|lK_{X''}|$ separates two very general points on X''.

Choose x, y general points on the same fiber. Since for all $l \geq 5$, $|lK_{X''_b}|$ gives a birational map on X''_b by a result of Bombieri (cf. [5]), in order to separate x and y we can simply apply proposition 1.49 with $k=1$, $n=l-1$, obtaining the following conditions:

$$\overline{a}(4l[\beta^2]-1) < 1, \tag{3.4}$$

$$\overline{a} < l-1, \tag{3.5}$$

that are implied by

$$\alpha > 3\sqrt[3]{2}(4l[\beta^2]-1), \tag{3.6}$$

$$\alpha > \frac{3\sqrt[3]{2}}{l-1}. \tag{3.7}$$

(Recall that $\beta \geq 1$ and hence $[\beta^2] \geq 1$).

If x, y are on different fibers then, since by a result of Bombieri $H^0(2K_{X''_b}) \neq 0$, we can apply proposition 1.49 with $k=2$ and $n=1$, obtaining the following conditions:

$$2\overline{a}(8[\beta^2]-1) < 1, \tag{3.8}$$

$$\overline{a} < \frac{1}{2}, \tag{3.9}$$

that are implied by

$$\alpha > 6\sqrt[3]{2}(8[\beta^2]-1), \tag{3.10}$$

$$\alpha > 6\sqrt[3]{2}. \tag{3.11}$$

Under these assumptions $H^0(2K_{X''})$ separates x and y. Then if l is even also $H^0(lK_{X''})$ separates x and y. If l is odd then if moreover

$$H^0(3K_{X''}) \neq 0 \tag{3.12}$$

we can conclude in the same way.

To deal with condition (3.12) we could simply use theorem 2.2, but since we do not need $h^0(3K_{X''}) \geq 2$ (because for our purposes it is enough to ask that $h^0(3K_{X''}) \geq 1$) then instead of applying theorem 2.2 in its full extent we can simply use the results about $h^0(3K_X)$ stated at the end of the proof of theorem 2.2.

It is now time to put everything together, that is to find the best possible value for β such that we have the lowest inferior bound for α.

If $\beta < 1$ we need only to consider (3.1) and (3.2).

If $\beta \geq 1$ then, since $l \geq 5$ and $g \geq 2$, (3.6) $\Rightarrow$ (3.3) $\Rightarrow$ (3.7) and (3.6) $\Rightarrow$ (3.10) $\Rightarrow$ (3.11). Moreover, by (3.1), (3.6) $\Rightarrow$ (3.12). Thus, if $\beta \geq 1$, we are left to consider only these conditions: (3.1), (3.2), (3.6).

Set

$$\beta' := \frac{4g\sqrt{2}}{g(l-1)-(l+1)}.$$

Since $l \geq 5$, $\beta' > 0$. Finally, if $\beta' \geq 1$ choose $\beta := \sqrt{[\beta'^2]+1-\epsilon'}$, if $\beta' < 1$ choose $\beta := 1-\epsilon'$, with $0 < \epsilon' \ll 1$ and such that $\beta \in \mathbb{Q}$. (3.1) is obviously verified. Besides, in this way $[\beta^2] = [\beta'^2]$.

Now some simple computations allow us to conclude: for (l, g) not as in Table 3.1 we have that $\beta' < 1$ and hence that $|lK_X|$ gives a birational map for

$$\alpha > \frac{3\sqrt[3]{2}g(1+2\sqrt{2})}{g(l-1)-(l+1)-4\sqrt{2}g}.$$

For $l = 7, g = 24, \ldots, 39$ and $l = 8$, $g = 7$ it turns out that it is better to take a larger value for β', namely $\beta' = 1$. Hence for all (l, g) as in Table 3.1 we have that (3.6) $\Rightarrow$ (3.2) except for $l = 5, g = 9$ and $l = 6, g = 8$.

$\square$

Remark 3.7. Notice that, in general, the birationality of $|nK_X|$ does not imply the birationality of $|nK_X|$ for every $n \geq m$. For example consider the threefold $X_{46} \subset \mathbb{P}(4,5,6,7,23)$ defined in remark 2.5: in this case $|nK_X|$ is birational if and only if $n = 23$ or $n \geq 27$ (see [20, question 1.6]).

Remark 3.8. In theorem 3.3, whenever $l \geq 21$ then we could simply apply [9, theorem 0.1] to prove that $\mathrm{vol}(X) \gg 1$ implies that $|lK_X|$ yields a birational map: in fact, if $\mathrm{vol}(X)$ is sufficiently large then by theorem 2.2 we know that $h^0(3K_X) \geq 2$. Anyway, the estimates on the volumes obtained directly applying Takayama's techniques are much better.

3.2 Pluricanonical maps of order ≤ 4

Remark 3.9. As already Todorov pointed out in [37], we cannot expect to have results about $|4K_X|$ analogous to those listed in theorem 3.3. In fact just choose a smooth surface of general type S such that $|4K_S|$ does not give a birational map, for example a smooth minimal surface S with $K_S^2 = 1$ and $h^0(K_S) = 2$ (cf. [2, VII, 7.1]), and a smooth curve C of genus g. Then set $X := S \times C$.

$\mathrm{vol}(X) = 3(2g-2)\mathrm{vol}(S) \xrightarrow[g\to+\infty]{} +\infty$, but since the map $\phi_{|4K_X|}$ given by $|4K_X|$ is, by Kunneth's formula, essentially constructed with the two maps $\phi_{|4K_S|}$ and $\phi_{|4K_C|}$ followed by a Segre's embedding, then $\phi_{|4K_X|}$ is never birational.

In the wake of remark 3.9, and when the volume is sufficiently large, we can characterize threefolds for which $|4K_X|$ does not give a birational map. In fact if a threefold X satisfies the conditions on α as listed in the proof of theorem 3.3 (imposing, this time, $l = 4$) but, at the same time, $\phi_{|4K_X|}$ is not birational, then X must necessarily be birational to a threefold fibered by surfaces for which the fourth pluricanonical map is not birational. Such surfaces X''_b have volume 1 and geometric genus $p_g = 2$ by [2, prop. VII.7.1 and VII.7.3]. Therefore we can state the following:

Corollary 3.10. *Let X be a smooth projective threefold of general type such that $vol(X) > \alpha^3$. If $\alpha > 6141\sqrt[3]{2}$ then $|4K_X|$ does not give a birational map if, and only if, X is birational to a fibre space X'', with $f : X'' \to B$, where B is a curve, such that the general fiber X''_b is a smooth minimal surface of general type with volume 1 and geometric genus $p_g = 2$. More generally, if X is not g-countably dense and if g, α are as in Table 3.2 or, in the other cases, $\alpha > 3\sqrt[3]{2}\left(16\left[\frac{32g^2}{(3g-5)^2}\right] - 1\right)$, then $|4K_X|$ does not give a birational map if, and only if, X is birational to a fibre space X'' as above.*

Table 3.2.

g	α	g	α
11	$> 237\sqrt[3]{2}$	39	$> 168\sqrt[3]{2}$
$30,\ldots,37$	$> 189\sqrt[3]{2}$	40	$> 156\sqrt[3]{2}$
38	$> 182\sqrt[3]{2}$	41	$> 146\sqrt[3]{2}$

Proof. The "if" part is trivial (and not depending on g, α). For the "only if", simply consider again all the conditions on α as in the proof of theorem 3.3, but with $l = 4$ instead of $l \geq 5$; moreover, instead of the usual value for β', it is better to take a larger value in some cases: for $g = 11$ $\beta' := \sqrt{5}$, for $g = 30, \ldots, 37$, $\beta' := 2$. Note also that the condition (3.12) is not needed. Now, for $g = 2, \ldots 37$ and $g \geq 42$ we have that (3.6) $\Rightarrow$ (3.2), while for $g = 38, 39, 40, 41$ (3.2) $\Rightarrow$ (3.6). □

Remark 3.11. In [13] and [10] there is an example of a smooth canonical threefold X with volume $= 2$ and such that $|4K_X|$ does not give a birational map. For this X the thesis of corollary 3.10 does not apply: in fact for a generic irreducible curve C_0 in any family of curves on X we have $K_X \cdot C_0 \geq 2$ (see [10, Ex. 6.3]), but if X were birationally fibred by surfaces of volume 1 and $p_g = 2$, we would have, on a general fibre, a family of curves for which $K_X \cdot C_0 \leq 1$.

Analogously, dealing this time with the 3rd pluricanonical map, considering the characterization of surfaces with a birational 3rd pluricanonical map (cf. [2, prop.

VII.7.1, VII.7.2 and VII.7.3]) and requiring X not to be 3-countably dense, we can state also the following:

Corollary 3.12. *Let X be a smooth, not 3-countably dense, projective threefold of general type such that vol$(X) > \alpha^3$. If $\alpha > 5178\sqrt[3]{2}$ then $|3K_X|$ does not give a birational map if, and only if, X is birational to a fibre space X'', with $f : X'' \to B$, where B is a curve, such that the general fibre X''_b is a smooth minimal surface of general type and either it has volume 1 and geometric genus $p_g = 2$ or it has volume 2 and $p_g = 3$. More generally, if X is not g-countably dense, with $g \geq 3$, and if g, α are as in Table 3.3 or, in the other cases, $\alpha > 6\sqrt[3]{2}\left(12\left[\frac{8g^2}{(g-2)^2}\right] - 1\right)$ then $|3K_X|$ does not give a birational map if, and only if, X is birational to a fibre space X'' as above.*

Table 3.3.

g	α	g	α
11	$> 858\sqrt[3]{2}$	$35, \ldots, 37$	$> 642\sqrt[3]{2}$
19	$> 714\sqrt[3]{2}$	38	$> 640\sqrt[3]{2}$

Proof. Consider again the proof of theorem 3.3, but with $l = 3$: this time, however, if x, y are on different fibers then we need to apply prop. 1.49 with $k = 2, n = 2$ obtaining a new condition (3.10), namely $\alpha > 6\sqrt[3]{2}(12[\beta^2] - 1)$, and a new condition (3.11), namely $\alpha > 3\sqrt[3]{2}$. As before, (3.12) is no longer needed. Moreover instead of the usual value for β', it is better to take a larger value in some cases: for $g = 11$ $\beta' := \sqrt{12}$, for $g = 19$ $\beta' := \sqrt{10}$, for $g = 35, \ldots, 37$, $\beta' := 3$. Therefore this time we have that (3.10)$\Rightarrow$ (3.6) and we are left to consider only conditions (3.1), (3.2) and (3.10). For $g \neq 38$ we have that (3.10) $\Rightarrow$ (3.2), while for $g = 38$ (3.2) $\Rightarrow$ (3.10). □

Remark 3.13. There are examples of threefolds X of general type with large volume and $|3K_X|$ birational even if X is covered by curves of genus 2: just consider the product $C_2 \times C_g \times C_g$ (where C_a is a smooth curve of genus a) and let g go to infinity.

Remark 3.14. By corollaries 3.10 and 3.12 we have that if X is a threefold of general type, not 3-countably dense and of sufficiently large volume then the birationality of $|3K_X|$ implies the birationality of $|4K_X|$.

We can say something also for the second pluricanonical map, even if in this case we need to suppose that X is not 4-countably dense. Note that the classification of surfaces for which the second pluricanonical map is not birational has not been completed yet. You can refer to [3, §2] for a survey on this subject and to [6, theorem 0.7, remark 0.8] for a partial classification (however notice that by our assumption about countably density the *standard case* and the symmetric product case cannot occur).

Corollary 3.15. *Let X be a smooth, not 4-countably dense, projective threefold of general type and such that vol$(X) > \alpha^3$. If $\alpha > 24570\sqrt[3]{2}$ then $|2K_X|$ does not give a birational map if, and only if, X is birational to a fibre space X'', with $f : X'' \to B$, where B is a curve, such that such that the general fiber X''_b is a smooth minimal surface of general type and $|2K_{X''_b}|$ does not give a birational map. More generally, if X is not g-countably dense, with $g \geq 4$, and if g, α are as in Table 3.4 or, in the other cases, $\alpha > 6\sqrt[3]{2}\left(8\left[\frac{32g^2}{(g-3)^2}\right] - 1\right)$, then $|2K_X|$ does not give a birational map if, and only if, X is birational to a fibre space X'' as above.*

Table 3.4.

g	α	g	α
8	$> 3930\sqrt[3]{2}$	$43, 44$	$> 1770\sqrt[3]{2}$
12	$> 2730\sqrt[3]{2}$	$53, 54$	$> 1722\sqrt[3]{2}$
14	$> 2490\sqrt[3]{2}$	$69, \ldots, 72$	$> 1674\sqrt[3]{2}$
22	$> 2058\sqrt[3]{2}$	73	$> 1630\sqrt[3]{2}$
24	$> 2010\sqrt[3]{2}$	$101, \ldots, 110$	$> 1626\sqrt[3]{2}$
26	$> 1962\sqrt[3]{2}$	$197, \ldots, 241$	$> 1578\sqrt[3]{2}$
29	$> 1914\sqrt[3]{2}$	242	$> 1560\sqrt[3]{2}$
32	$> 1866\sqrt[3]{2}$	243	$> 1532\sqrt[3]{2}$
37	$> 1818\sqrt[3]{2}$		

Proof. Consider the proof of theorem 3.3, but with $l = 2$ instead of $l \leq 5$. (3.12) is not needed. Moreover instead of the usual value for β', it is better to take a larger value in some cases: $g = 8$ $\beta' := \sqrt{82}$, $g = 12$ $\beta' := \sqrt{57}$, $g = 14$ $\beta' := \sqrt{52}$, $g = 22$ $\beta' := \sqrt{43}$, $g = 24$ $\beta' := \sqrt{42}$, $g = 26$ $\beta' := \sqrt{41}$, $g = 29$ $\beta' := \sqrt{40}$, $g = 32$ $\beta' := \sqrt{39}$, $g = 37$ $\beta' := \sqrt{38}$, $g = 43, 44$ $\beta' := \sqrt{37}$, $g = 53, 54$ $\beta' := 6$, $g = 69, \ldots, 72$ $\beta' := \sqrt{35}$, $g = 101, \ldots, 110$ $\beta' := \sqrt{34}$, $g = 197, \ldots, 241$ $\beta' := \sqrt{33}$. This time we have that (3.10)$\Rightarrow$ (3.6) and we are left to consider only conditions (3.1), (3.2) and (3.10). For $g \neq 73, 242, 243$ we have that (3.10) $\Rightarrow$ (3.2), while for $g = 73, 242, 243$ (3.2) $\Rightarrow$ (3.10). □

Remark 3.16. The birationality of the fourth pluricanonical map for threefolds of general type has been studied by, among the others, Lee, Dong, M.Chen, Zhang. Actually it is still an open problem when $\phi_{|4K_X|}$ not birational implies that X is birational to an X'' as in corollary 3.10 (cf. [10, 6.4]). Both Dong in [13] and Chen-Zhang in [10] give characterizations for the birationality of the fourth pluricanonical map, but instead of using the volume of X, they suppose that the canonical bundle has a sufficient number of sections ($h^0(K_X) \geq 7$ in Dong's paper, $h^0(K_X) \geq 5$ in Chen-Zhang's). As we have already seen (cf. remark 2.4) this is not implied by the largeness of the volume.

As for the birationality of the third pluricanonical map, explicit characterizations not depending on the volume are not known (cf. [10, Open problems 6.4]).

Chapter 4

Higher dimensional results

We know that there exists a positive lower bound on the volume of any variety of general type of a given dimension (see, for example, [36, theorem 1.2]). If only we knew these lower bounds explicitly then the ideas we exploited for threefolds to find estimates for the non-vanishing of pluricanonical systems or the birationality of pluricanonical maps could be generalized to varieties of any dimension. Unfortunately this is not the case. Anyway, we did explicit calculations in the case of fourfolds, since in [8] J. Chen and M. Chen computed a lower bound for the volume for threefolds of general type. However notice that since we do not have the technique of the fibration at our disposal, these estimates are probably far from being optimal.

4.1 Plurigenera

Theorem 4.1. *Let X be a smooth projective variety of general type and of dimension d, such that vol$(X) > \alpha^d$. Let Π be a very general subset of X and, for $i = 1, \dots, d-1$, let $v_i \in \mathbb{Q}^+$ such that vol$(Z) > v_i$ for every $Z \subset X$ subvariety of dimension i passing through a point $x \in \Pi$ and let $\mu_i := \frac{i}{\sqrt[i]{v_i}}$. Set*

$$M := \left[\left(\frac{d}{\alpha} + 1\right) \cdot (\mu_{d-1} + 1) \cdot (\mu_{d-2} + 1) \cdot \ldots \cdot (\mu_1 + 1)\right].$$

Then for all $n \geq 1$, for all $m \geq nM$, $h^0((m+1)K_X) \geq n$.

Remark 4.2. By [20, corollary 1.3] or [36, theorem 1.2], we know that there exists η_i such that for every variety Z of dimension i and of general type, then vol$(Z) \geq \eta_i$. Therefore for every $i = 1, \dots, d-1$, the v_i's exist and are greater than 0.

Proof. As in [37] and as before, to prove the theorem we will essentially produce lc centres and then, using theorem 2.1, cut their dimensions until they are points; then we can apply Nadel's vanishing theorem to pull back sections from the points to the variety.

Let X_0 be the intersection between Π and $X \setminus \mathbb{B}_+(K_X)$. X_0 is a very general subset of X, hence countably dense. Note that for every $x \in X_0$, every subvariety through x is of general type, since its volume is strictly positive by hypothesis.

Since $\mathrm{vol}(K_X) > \alpha^d$, by lemma 1.45, for every $x \in X$ and every $k \gg 0$ there exists a divisor $A_x \in |kK_X|$ with $\mathrm{mult}_x(A_x) > k\alpha$. Let $\Delta'_x := A_x \frac{\lambda'_x}{k}$, with $\lambda'_x < \frac{d}{\alpha}$, $\lambda'_x \in \mathbb{Q}^+$, but close enough to $\frac{d}{\alpha}$ so that $\mathrm{mult}_x(\Delta'_x) > d$. Note that $\Delta'_x \sim \lambda'_x K_X$. Let $s_x := lct(X, \Delta'_x, x)$. By lemma 1.47, $s_x < 1$. Moreover, by remark 1.30, $s_x \in \mathbb{Q}^+$. Therefore, without loss of generality, we can suppose that (X, Δ'_x) is lc, not klt in x.

By lemma 1.41, d., for every $x \in X_0$ there exists an effective $\mathbb{Q}$-divisor $D_x \sim \lambda_x K_X$, with $\lambda_x < \frac{d}{\alpha}, \lambda_x \in \mathbb{Q}^+$, such that (X, D_x) is lc, not klt in x and $LLC(X, D_x, x) = \{V_x\}$, where V_x is the unique minimal element of $LLC(X, \Delta'_x, x)$. Moreover we can also assume that V_x is an exceptional lc centre.

For every $0 \leq i \leq d-1$, set

$$Y_i := \{x \in X_0 \text{ s.t. } \dim(V_x) = i\}.$$

Since X_0 is countably dense then at least one between the Y_i's is countably dense. Moreover we can assume that Y_{d-1} is countably dense - in fact, numerically, this is the “worst” possible scenario, as it will be clear further on in the proof.

Now we apply theorem 2.1: for every $x \in Y_{d-1}$ consider V_x and a resolution $f_x : W_x \to V_x$. As we have already seen, V_x is an exceptional lc centre of (X, D_x). Since $x \in X_0$, V_x, and hence W_x, are of general type and V_x is not contained in the augmented base locus of K_X. Moreover $\mathrm{vol}(W_x) > v_{d-1}$ by hypothesis, since the volume is a birational invariant. Let U_x be the very general subset of V_x defined as in [20, thm. 4.1]. Set $U'_x := U_x \cap X_0$. U'_x is still a very general and non-empty subset of V_x. Moreover $\mathrm{vol}(\mu_{d-1}K_{W_x}) > (d-1)^{d-1}$. For every $y \in U'_x$ let us consider $y' \in f_x^{-1}(y) \subset W_x$. Since y' is a smooth point, by lemma 1.45 and remark 1.47, there exists $\Theta_{y'} \sim \mu_{d-1}K_{W_x}$ such that $(W_x, \Theta_{y'})$ is not klt in y'. As before, since $lct(W_x, \Theta_{y'}, y') < 1$, we can suppose that $(W_x, \Theta_{y'})$ is lc, not klt in y' and $\Theta_{y'} \sim \mu_{y'}K_{W_x}$ with $\mu_{y'} \in \mathbb{Q}^+$ and $\mu_{y'} < \mu_{d-1}$. Therefore there exists a pure lc centre $W'_{y'} \in LLC(W_x, \Theta_{y'}, y')$. Set $V'_y := f_x(W'_{y'}) \ni y$. By theorem 2.1, for every $\delta \in \mathbb{Q}^+$ there exists a $\mathbb{Q}$-divisor D'_y such that V'_y is an exceptional lc centre for (X, D'_y) and such that $D'_y \sim ((\lambda_x+1)(\mu_{y'}+1) - 1 + \delta)K_X$. At the end we are in the following situation: $\cup_{x \in Y_{d-1}} U'_x$ is countably dense in X and for every $z \in \cup_{x \in Y_{d-1}} U'_x$ there exists a $\mathbb{Q}$-divisor D'_z such that $LLC(X, D'_z, z) = \{V'_z\}$ with V'_z exceptional lc centre, $\dim(V'_z) < d-1$ and $D'_z \sim_{\mathbb{Q}} ((\lambda_z+1)(\mu_z+1) - 1 + \delta)K_X$ with $\lambda_z < \frac{d}{\alpha}$ and $\mu_z < \mu_{d-1}$.

We can now apply theorem 2.1 again and again and conclude that there exists a countably dense set $\Gamma \subseteq X$ such that for every $x \in \Gamma$ there exists a $\mathbb{Q}$-divisor B_x such that $LLC(X, B_x, x) = \{\{x\}\}$ and $B_x \sim_{\mathbb{Q}} \gamma K_X$ with

$$\gamma < \left(\frac{d}{\alpha} + 1\right) \cdot (\mu_{d-1}+1) \cdot (\mu_{d-2}+1) \cdot \ldots \cdot (\mu_1+1) - 1 + \delta q,$$

where q is a positive rational number.

Taking δ sufficiently small, we can conclude that $\gamma < M$, therefore, by lemma 1.6 and lemma 1.44, for all $n \geq 1$, for all $m \geq nM$, $h^0((m+1)K_X) \geq n$. $\square$

Remark 4.3. In the above proof it is clear that

$$M \geq [(\mu_{d-1}+1)\cdot(\mu_{d-2}+1)\cdot\ldots\cdot(\mu_1+1)]$$

and that "=" holds as soon as

$$\left(\frac{d}{\alpha}+1\right)\cdot(\mu_{d-1}+1)\cdot\ldots\cdot(\mu_1+1)-[(\mu_{d-1}+1)\cdot\ldots\cdot(\mu_1+1)]<1$$

i.e.

$$\frac{d}{\alpha}<\frac{1-\{(\mu_{d-1}+1)\cdot\ldots\cdot(\mu_1+1)\}}{(\mu_{d-1}+1)\cdot\ldots\cdot(\mu_1+1)}$$

i.e.

$$\alpha>\frac{d(\mu_{d-1}+1)\cdot\ldots\cdot(\mu_1+1)}{1-\{(\mu_{d-1}+1)\cdot\ldots\cdot(\mu_1+1)\}}.$$

Corollary 4.4. *Let X be a smooth, not g-countably dense, projective variety of general type of dimension d and such that $vol(X)>\alpha^d$. If $d=3$, if*

$$\alpha>\frac{9^{\frac{2g-1}{2g-2}}}{1-\left\{3^{\frac{2g-1}{2g-2}}\right\}}$$

then we have that $h^0((1+m)K_X)\geq n$ for all $n\geq 1$ and all $m\geq\left[3^{\frac{2g-1}{2g-2}}\right]n$. If $d=4$,

$$\alpha>\frac{12(3\sqrt[3]{2660}+1)^{\frac{2g-1}{2g-2}}}{1-\left\{3(3\sqrt[3]{2660}+1)^{\frac{2g-1}{2g-2}}\right\}}$$

then $h^0(X,(1+m)K_X)\geq n$ for all $n\geq 1$ and all $m\geq\left[3(3\sqrt[3]{2660}+1)^{\frac{2g-1}{2g-2}}\right]n$. In general: if $d=3$, if $\alpha>27$ then $h^0(X,(1+m)K_X)\geq n$ for all $n\geq 1$ and all $m\geq 4n$; if $d=4$, if $\alpha\geq 1709$ then $h^0(X,(1+m)K_X)\geq n$ for all $n\geq 1$ and all $m\geq 191n$.

Proof. For every X and for every $0<\epsilon\ll 1$ we can take $v_1=2g-2-\epsilon$ (by remark 1.8), $v_2=1-\epsilon$ (the minimal model of a surface is nonsingular, hence the volume is an integer) and, by [8], $v_3=\frac{1}{2660}-\epsilon$. Therefore $\mu_1=\frac{1}{2g-2}+o(1)$, $\mu_2=2+o(1)$ and $\mu_3=3\sqrt[3]{2660}+o(1)$ (with $o(1)>0$, $\lim_{\epsilon\to 0}o(1)=0$).

If X is a threefold we have that $(\mu_2+1)\cdot(\mu_1+1)=3^{\frac{2g-1}{2g-2}}+o(1)$ therefore, by 4.1 and 4.3, if

$$\alpha>\frac{9^{\frac{2g-1}{2g-2}}}{1-\left\{3^{\frac{2g-1}{2g-2}}\right\}}$$

then $h^0((1+m)K_X)\geq n$ for every $n\geq 1$ and every $m\geq\left[3^{\frac{2g-1}{2g-2}}\right]n$. In general, taking $g=2$ by remark 1.22, we can conclude that if $\alpha>27$ then $h^0(X,(1+m)K_X)\geq n$ for all $n\geq 1$ and all $m\geq 4n$.

If X is a fourfold we have that $(\mu_3+1)\cdot(\mu_2+1)\cdot(\mu_1+1) = 3(3\sqrt[3]{2660}+1)\frac{2g-1}{2g-2}+o(1)$, therefore we can conclude that if

$$\alpha > \frac{12(3\sqrt[3]{2660}+1)\frac{2g-1}{2g-2}}{1-\left\{3(3\sqrt[3]{2660}+1)\frac{2g-1}{2g-2}\right\}}$$

then $h^0\left(X,(1+m)K_X\right) \geq n$ for all $n \geq 1$ and all $m \geq \left\lceil 3(3\sqrt[3]{2660}+1)\frac{2g-1}{2g-2}\right\rceil n$. In general, taking $g=2$, we can conclude that if $\alpha \geq 1709$ then $h^0(X,(1+m)K_X) \geq n$ for all $n \geq 1$ and all $m \geq 191n$.

□

Remark 4.5. It is not known which is the least integer n for which $P_n \neq 0$ for any fourfold of sufficiently large volume, but by remark 2.5 and taking products we know that this n must be greater or equal to 4.

4.2 Pluricanonical maps

For the birationality of pluricanonical systems, using - as for the 3-fold case - Takayama's result instead of Hacon-McKernan's, under the same notation and hypotheses of 4.1, we can state that

Theorem 4.6. *Let X be a smooth projective variety of general type and of dimension d, such that vol$(X) > \alpha^d$. Let Π be a very general subset of X and, for $i = 1,\ldots,d-1$, let $v_i \in \mathbb{Q}^+$ such that vol$(Z) > v_i$ for every $Z \subset X$ subvariety of dimension i passing through a point $x \in \Pi$ and let $\mu_i := \frac{i}{\sqrt[i]{v_i}}$. Setting, for every $i = 1,\ldots,d-1$, $r_i := \sqrt[i]{2}\mu_i$,*

$$\overline{s} := 2\prod_{i=1}^{d-1}(1+r_i) - 2,$$

$$\overline{t} := \sqrt[d]{2}d\prod_{i=1}^{d-1}(1+r_i),$$

we have that if $l \geq \left\lceil \overline{s} + \frac{\overline{t}}{\alpha}\right\rceil + 2$ then the linear system $|lK_X|$ gives a birational map.

Proof. As in the proof of theorem 3.3, we can reduce ourselves to the following situation: for every $0 < \epsilon < 1$ there exists a smooth projective variety X' and a birational morphism $\pi : X' \to X$ and a decomposition $\mu^*(K_X) \sim_{\mathbb{Q}} A + E$ where $A = A_\epsilon$ is an ample $\mathbb{Q}$-divisor and $E = E_\epsilon$ is an effective $\mathbb{Q}$-divisor. As in thm. 3.3 we will argue on X'. By theorem 3.1, we know that given two very general points $x, y \in X'$ there exists an effective $\mathbb{Q}$-divisor D on X' and a positive constant a with $D \sim_{\mathbb{Q}} aA$ such that $x, y \in Z(\mathcal{J}(X', D))$ with $\dim Z(\mathcal{J}(X', D)) = 0$ around x or y, that is x or y is an isolated point of $Z(\mathcal{J}(X', D))$. Besides, by the same theorem, we also know that $a < s + t/\sqrt[d]{\text{vol}(X)} \leq s + t/\alpha$ where s, t are non-negative constants defined as follows. Let s_i, s'_i, t_i $(i = 1, \dots, d)$ be non-negative constants determined inductively as (cf. [36, notation 5.2]): $s_1 = 0$, $t_1 = \sqrt[d]{2}d/(1-\epsilon)$, $s'_i = s_i + \epsilon$,

$$s_{i+1} = \left(1 + \sqrt[d-i]{2}\frac{\mu_{d-i}}{1-\epsilon}\right) s'_i + 2 \sqrt[d-i]{2}\frac{\mu_{d-i}}{1-\epsilon},$$

$$t_{i+1} = \left(1 + \sqrt[d-i]{2}\frac{\mu_{d-i}}{1-\epsilon}\right) t_i.$$

Finally, set $s := s_d, t := t_d$.

As in the proof of thm. 3.3, we can say that, given $l \in \mathbb{N}$, $|lK_{X'}|$ separates two very general points in X' as soon as $l \geq [a] + 2$.

It can be easily seen that $s = \overline{s} + o(1)$ and $t = \overline{t} + o(1)$, with $o(1) > 0$ and such that $\lim_{\epsilon \to 0} o(1) = 0$. Note that $\overline{s}$ and $\overline{t}$ do not depend on ϵ.

Since $a < s + t/\alpha$ then $a < \overline{s} + \overline{t}/\alpha + o(1)$, therefore, taking ϵ sufficiently small, $[a] \leq [\overline{s} + \overline{t}/\alpha]$ and thus we can conclude.

□

Remark 4.7. In the above proof it is clear that

$$\left[\overline{s} + \frac{\overline{t}}{\alpha}\right] \geq [\overline{s}]$$

and that "=" holds as soon as

$$\frac{\overline{t}}{\alpha} < 1 - \{\overline{s}\}$$

(where $\{\cdot\}$ is the fractional part), that is

$$\alpha > \frac{\overline{t}}{1 - \{\overline{s}\}}.$$

We can now do explicit calculations in the case of fourfolds, using the same notation and estimates as in 4.4.

Corollary 4.8. *Let X be a smooth, not g-countably dense, projective fourfold of general type such that vol$(X) > \alpha^4$. If*

$$\alpha > \frac{4\sqrt[4]{2}\left(\frac{g}{g-1}\right)(1+2\sqrt{2})(1+3\sqrt[3]{5320})}{1 - \left\{2\left(\frac{g}{g-1}\right)(1+2\sqrt{2})(1+3\sqrt[3]{5320})\right\}}$$

we have that the linear system $|lK_X|$ gives a birational map for every

$$l \geq \left[2\left(\frac{g}{g-1}\right)(1+2\sqrt{2})(1+3\sqrt[3]{5320})\right].$$

In general, if $\alpha \geq 2816$ then $|lK_X|$ gives a birational map for every $l \geq 817$.

Proof. For every X and every $0 < \epsilon \ll 1$, as in 4.4 we can take $r_1 = \frac{1}{g-1} + o(1)$, $r_2 = 2\sqrt[2]{2} + o(1)$, $r_3 = 3\sqrt[3]{5320} + o(1)$. Therefore

$$\overline{s} = 2\left(\frac{g}{g-1}\right)(1+2\sqrt{2})(1+3\sqrt[3]{5320}) - 2 + o(1)$$

and

$$\overline{t} = 4\sqrt[4]{2}\left(\frac{g}{g-1}\right)(1+2\sqrt{2})(1+3\sqrt[3]{5320}) + o(1).$$

Hence, by 4.6 and its remark, if

$$\alpha > \frac{4\sqrt[4]{2}\left(\frac{g}{g-1}\right)(1+2\sqrt{2})(1+3\sqrt[3]{5320})}{1-\left\{2\left(\frac{g}{g-1}\right)(1+2\sqrt{2})(1+3\sqrt[3]{5320})\right\}}$$

then $|lK_X|$ gives a birational map for every

$$l \geq \left[2\left(\frac{g}{g-1}\right)(1+2\sqrt{2})(1+3\sqrt[3]{5320})\right].$$

In general, taking $g = 2$, we can conclude that if $\alpha \geq 2816$ then $|lK_X|$ gives a birational map for every $l \geq 817$. □

Remark 4.9. It is not known which is the least integer n for which $|nK_X|$ gives a birational map for any fourfold X of sufficiently large volume, but by remark 3.7 and taking products we know that this n must be greater or equal to 27.

In the case of a surface of general type S, many things about its pluricanonical maps are already known. For example $|5K_S|$ gives always a birational map, while $|4K_S|$ gives a birational map if $\text{vol}(S) \geq 2$ (cf. [2, thm. 5.1]). Anyway we can apply theorem 4.6 to obtain, in another way, similar (but weaker) results:

Corollary 4.10. *Let S be a smooth, not g-countably dense, projective surface of general type. If vol(S) $> \frac{2g^2}{(2g-3)^2}$ then $|5K_S|$ gives a birational map. In general, if vol(S) ≥ 9 then $|5K_S|$ gives a birational map.*

Proof. Simply apply theorem 4.6 with $\overline{s} = \frac{2}{g-1}$, $\overline{t} = 2\sqrt{2} + \frac{2\sqrt{2}}{g-1}$: we want $\overline{s} + \frac{\overline{t}}{\alpha} < 4$ and this implies $\alpha > \frac{\sqrt{2}g}{2g-3}$. □

Corollary 4.11. *Let S be a smooth, not g-countably dense, projective surface of general type. If vol(S) $> \frac{8g^2}{(3g-5)^2}$ then $|4K_S|$ gives a birational map. In general, if vol(S) ≥ 33 then $|4K_S|$ gives a birational map.*

Proof. As before, simply apply theorem 4.6. This time we want $\overline{s} + \frac{\overline{t}}{\alpha} < 3$ and this implies $\alpha > \frac{2\sqrt{2}g}{3g-5}$. □

Note that even if this corollary does not add anything new to the geography of surfaces of general type (in fact if $\text{vol}(S) = 1$ we have $g \leq 11$ and so the inequality is not verified), however it gives another direct way to prove that $|4K_S|$ is birational if $\text{vol}(S) = 2$ and $g \geq 6$ or $\text{vol}(S) = 3, 4$ and $g \geq 4$ or $\text{vol}(S) = 5, \ldots, 32$ and $g \geq 3$.

4.3 Fibrations

In the threefold case, for studying both the non-vanishing of plurigenera and the birationality of pluricanonical maps, we avoided log canonical centres of codimension 1 and small volume by passing to a fibration (see lemma 1.48) and then producing sections of pluricanonical bundles (see proposition 1.49). It is important to notice that this reducing to the fibration is not exclusive to the three-dimensional case, even if, for varieties of dimension greater or equal to 4, we were not able to have both the ambient variety smooth and the general fibre minimal, thus not permitting us to apply something similar to proposition 1.49.

Anyway this approach by means of lemma 1.48 allows us to say something about varieties of any dimension but without the need to take into accounts log canonical centres of codimension 1 and small volume.

Hence we can state the following two theorems. Their proofs are not essentially different from the ones given in this and previous chapters, but for the convenience of the reader we rewrite them in their generality.

Notice that now explicit numbers can be easily obtained also in the case of fivefolds.

Theorem 4.12. *Let X be a smooth projective variety of general type of dimension d and such that vol$(X) > \alpha^d$. Let Π be a very general subset of X and, for $i = 1, \dots, d-2$, let $v_i \in \mathbb{Q}^+$ such that vol$(Z) > v_i$ for every $Z \subset X$ subvariety of dimension i passing through a point $x \in \Pi$. Let $\mu_i := \frac{i}{\sqrt[i]{v_i}}$ and $R := \prod_{i=1}^{d-2}(\mu_i + 1)$. Let l be a positive integer, $l > R$. Let*

$$\beta_1 := \frac{(d-1)R}{l-R},$$

$$\beta_2 := \frac{(d-1)(l+R)}{l-R}.$$

For all $\overline{\beta} > \beta_1$, setting $\tilde{\beta} := \min\{\overline{\beta}, \beta_2\}$, if

$$\alpha > \frac{d(1+(d-1)/\tilde{\beta})R}{l-(1+(d-1)/\tilde{\beta})R}$$

then either $h^0(lK_X) \geq 1$ (and for all $n \in \mathbb{N}^+$, $h^0(mK_X) \geq n$ for all $m \geq n(l-1)+1$) or X is birational to a fibre space X', with $f : X' \to B$, where B is a curve, such that the volume of the general fibre is $\leq \overline{\beta}^{d-1}$.

Proof. Let X_0 be the intersection between Π and $X \setminus \mathbb{B}_+(K_X)$. X_0 is a very general subset of X, hence countably dense. Note that for every $x \in X_0$, every subvariety through x is of general type, since its volume is strictly positive by hypothesis.

Since $\mathrm{vol}(K_X) > \alpha^d$, by lemma 1.45, for every $x \in X$ and every $k \gg 0$ there exists a divisor $A_x \in |kK_X|$ with $\mathrm{mult}_x(A_x) > k\alpha$. Let $\Delta'_x := A_x \frac{\lambda'_x}{k}$, with $\lambda'_x < \frac{d}{\alpha}$, $\lambda'_x \in \mathbb{Q}^+$, but close enough to $\frac{d}{\alpha}$ so that $\mathrm{mult}_x(\Delta'_x) > d$. Note that $\Delta'_x \sim \lambda'_x K_X$. Let $s_x := lct(X, \Delta'_x, x)$. By proposition 1.47, $s_x < 1$. Moreover, by remark 1.30,

$s_x \in \mathbb{Q}^+$. Therefore, without loss of generality, we can suppose that (X, Δ'_x) is lc, not klt in x.

By lemma 1.41, d., for every $x \in X_0$ there exists an effective $\mathbb{Q}$-divisor $D_x \sim \lambda_x K_X$, with $\lambda_x < \frac{d}{\alpha}$, $\lambda_x \in \mathbb{Q}^+$, such that (X, D_x) is lc, not klt in x and $LLC(X, D_x, x) = \{V_x\}$, where V_x is the unique minimal element of $LLC(X, \Delta'_x, x)$. Moreover we can also assume that V_x is an exceptional lc centre.

For every $0 \leq i \leq d-2$, set

$$Y_i := \{x \in X_0 \text{ s.t. } \dim(V_x) = i\}.$$

Then fix $\beta \in \mathbb{Q}^+$ and set

$$Y_{d-1,a} := \{x \in X_0 \text{ s.t. } \dim(V_x) = d-1 \text{ and } \mathrm{vol}(V_x) > \beta^{d-1}\},$$

$$Y_{d-1,b} := \{x \in X_0 \text{ s.t. } \dim(V_x) = d-1 \text{ and } \mathrm{vol}(V_x) \leq \beta^{d-1}\}.$$

Since X_0 is countably dense then at least one between the Y_i's, $Y_{d-1,a}$, $Y_{d-1,b}$ is countably dense. If one of the Y_i, $0 \leq i \leq d-2$, or $Y_{d-1,a}$ is countably dense then we can assume that $Y_{d-1,a}$ is countably dense (in fact, numerically, this is the "worst" possible scenario) and we apply theorem 2.1: for every $x \in Y_{d-1,a}$ consider V_x and a resolution $f_x : W_x \to V_x$. As we have already seen, V_x is an exceptional lc centre of (X, D_x). Since $x \in X_0$, V_x, and hence W_x, are of general type and V_x is not contained in the augmented base locus of K_X. Moreover $\mathrm{vol}(W_x) > \beta^{d-1}$ by hypothesis, since the volume is a birational invariant. Let U_x be the very general subset of V_x defined as in [20, thm. 4.1]. Set $U'_x := U_x \cap X_0$. U'_x is still a very general and non-empty subset of V_x. Moreover $\mathrm{vol}(\frac{d-1}{\beta} K_{W_x}) > (d-1)^{d-1}$. For every $y \in U'_x$ let us consider $y' \in f_x^{-1}(y) \subset W_x$. Since y' is a smooth point, by lemma 1.45 and proposition 1.47, there exists $\Theta_{y'} \sim \frac{d-1}{\beta} K_{W_x}$ such that $(W_x, \Theta_{y'})$ is not klt in y'. As before, since $lct(W_x, \Theta_{y'}, y') < 1$, we can suppose that $(W_x, \Theta_{y'})$ is lc, not klt in y' and $\Theta_{y'} \sim \mu_{y'} K_{W_x}$ with $\mu_{y'} \in \mathbb{Q}^+$ and $\mu_{y'} < \frac{d-1}{\beta}$. Therefore there exists a pure lc centre $W'_{y'} \in LLC(W_x, \Theta_{y'}, y')$. Set $V'_y := f_x(W'_{y'}) \ni y$. By theorem 2.1, for every $\delta \in \mathbb{Q}^+$ there exists a $\mathbb{Q}$-divisor D'_y such that V'_y is an exceptional lc centre for (X, D'_y) and such that

$$D'_y \sim ((\lambda_x + 1)(\mu_{y'} + 1) - 1 + \delta) K_X.$$

At the end we are in the following situation: $\cup_{x \in Y_{d-1}} U'_x$ is countably dense in X and for every $z \in \cup_{x \in Y_{d-1}} U'_x$ there exists a $\mathbb{Q}$-divisor D'_z such that $LLC(X, D'_z, z) = \{V'_z\}$ with V'_z exceptional lc centre, $\dim(V'_z) < d-1$ and $D'_z \sim_{\mathbb{Q}} ((\lambda_z+1)(\mu_z+1)-1+\delta)K_X$ with $\lambda_z < \frac{d}{\alpha}$ and $\mu_z < \frac{d-1}{\beta}$.

We can now apply theorem 2.1 again and again and conclude that there exists a countably dense set $\Gamma \subseteq X$ such that for every $x \in \Gamma$ there exists a $\mathbb{Q}$-divisor B_x such that $LLC(X, B_x, x) = \{\{x\}\}$ and $B_x \sim_{\mathbb{Q}} \gamma K_X$ with

$$\gamma < \left(\frac{d}{\alpha} + 1\right) \cdot \left(\frac{d-1}{\beta} + 1\right) \cdot R - 1 + o(1),$$

where $o(1) > 0$ and $\lim_{\delta\to 0} o(1) = 0$.

By lemma 1.6 and lemma 1.44, $h^0(lK_X) \geq 1$ as soon as $l > \gamma + 1$ (and we also have that for all $n \in \mathbb{N}^+$, $h^0(mK_X) \geq n$ for all $m \geq n(l-1)+1$). To that purpose, imposing

$$l > \left(\frac{d-1}{\beta} + 1\right) R, \tag{4.1}$$

it is enough to ask that

$$\alpha > \frac{d\left(\frac{d-1}{\beta} + 1\right) R}{l - \left(\frac{d-1}{\beta} + 1\right) R}. \tag{4.2}$$

Let us now suppose that $Y_{d-1,b}$ is countably dense. Recall that for every $x \in Y_{d-1,b}$ we have a divisor $D_x \sim \lambda_x K_X$ such that $LLC(X, D_x, x) = \{V_x\}$, V_x is an exceptional log canonical centre and $\dim(V_x) = d-1$. Since if we decompose a countably dense set as a countable union of subsets then at least one of the subsets is countably dense, we can suppose that $\lambda_x = \lambda$ for a fixed $\lambda \in \mathbb{Q}^+$. Recall that $\lambda < \frac{d}{\alpha}$. By lemma 1.48, we are in the following situation:

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[d, "f"] & X \\ B & \end{tikzcd}$$

where X', B are normal projective varieties, f is a dominant morphism with connected fibres, π is a dominant and generically finite morphism and the image under π of a general fibre of f is V_x. Arguing exactly as in [37] we can suppose that there exists a proper closed subset $X_1 \subset X$ such that for all $x \notin X_1$, D_x is smooth at x. Either π is birational or the inverse image of a general $x \in X \setminus X_1$ under π is contained in at least two different fibres of f. In this case we can apply lemma 1.42 and lemma 1.41, d.,e., to conclude that there exists a countably dense set $Y := Y_{d-1,b} \cap (X \setminus X_1)$ such that for all $y \in Y$ there exists a divisor $S_y \sim k(2\lambda K_X)$ ($0 < k \leq 1$, $\lambda < \frac{d}{\alpha}$) such that $LLC(S_y, y) = \{C_y\}$, where C_y is an irreducible variety of dimension at most $d-2$. Taking δ any sufficiently small positive rational number, we can now apply theorem 2.1 again and again and conclude that there exists a countably dense set $\Gamma' \subseteq X$ such that for every $x \in \Gamma'$ there exists a $\mathbb{Q}$-divisor B'_x such that $LLC(X, B'_x, x) = \{\{x\}\}$ and $B'_x \sim_{\mathbb{Q}} \gamma' K_X$ with

$$\gamma' < \left(\frac{2d}{\alpha} + 1\right) \cdot R - 1 + o(1),$$

where $o(1) > 0$ and $\lim_{\delta\to 0} o(1) = 0$. Hence, as before, $h^0(lK_X) \geq 1$ as soon as $l > \gamma' + 1$ (and we also have that for all $n \in \mathbb{N}^+$, $h^0(mK_X) \geq n$ for all $m \geq n(l-1)+1$). To that purpose, imposing

$$l > R, \tag{4.3}$$

it is enough to ask that

$$\alpha > \frac{2dR}{l - R}. \tag{4.4}$$

Summing up: given $l \in \mathbb{N}^+$, if (4.1), (4.2), (4.3), (4.4) are satisfied then we can conclude that either $h^0(lK_X) \geq 1$ (and for all $n \in \mathbb{N}^+$, $h^0(mK_X) \geq n$ for all $m \geq n(l-1)+1$) or X is birational to the fibre space $X' \to B$, where B is a curve, such that the volume of the general fibre is $\leq \beta^{d-1}$.

Note that (4.1) $\Rightarrow$ (4.3). Moreover (4.1) is equivalent to requiring $\beta > \frac{(d-1)R}{l-R}$ and, if (4.1) holds,((4.2) $\Rightarrow$ (4.4)) $\Leftrightarrow \beta \leq \frac{(d-1)(l+R)}{l-R}$. The thesis follows.

$\square$

Theorem 4.13. *Let X be a smooth projective variety of general type of dimension d and such that vol$(X) > \alpha^d$. Let Π be a very general subset of X and, for $i = 1, \dots, d-2$, let $v_i \in \mathbb{Q}^+$ such that vol$(Z) > v_i$ for every $Z \subset X$ subvariety of dimension i passing through a point $x \in \Pi$. Let $\mu_i := \frac{i}{\sqrt[i]{v_i}}$, $r_i := \sqrt[i]{2}\mu_i$ and $P := \prod_{i=1}^{d-2}(1+r_i)$. Let l be a positive integer, $l > 2P-1$. Let*

$$\beta_1 := \frac{2\sqrt[d-1]{2}(d-1)P}{l+1-2P},$$

$$\beta_2 := \frac{\sqrt[d-1]{2}(d-1)(l+1+4P)}{2(l+1-2P)}.$$

For all $\overline{\beta} > \beta_1$, setting $\tilde{\beta} := \min\{\overline{\beta}, \beta_2\}$, if

$$\alpha > \frac{d\sqrt[d]{2}(1+\sqrt[d-1]{2}(d-1)/\tilde{\beta})P}{l+1-2(1+\sqrt[d-1]{2}(d-1)/\tilde{\beta})P}$$

then either $|lK_X|$ gives a birational map or X is birational to a fibre space X'', with $f : X'' \to B$, where B is a curve, such that the volume of the general fibre is $\leq \overline{\beta}^{d-1}$.

Proof. By [36, theorem 3.1], for every $0 < \epsilon < 1$ there exists a smooth projective variety X', a birational morphism $\mu : X' \to X$ and an approximate Zariski decomposition $\mu^*(K_X) \sim_{\mathbb{Q}} A + E$ where $A = A_\epsilon$ is an ample $\mathbb{Q}$-divisor and $E = E_\epsilon$ is an effective $\mathbb{Q}$-divisor that satisfy condition (1),(2),(3) of Takayama's theorem (cf. [36, theorem 3.1]).

As in the proof of theorem 3.3, we can argue on X' instead of X and regard Π as a subset of X'.

By lemma 3.2 (see also its proof in [36, lemma 5.4]), there exists a very general subset U of X' such that for every two distinct points $x, y \in U$ we can construct, depending on x, y, an effective divisor $D_1 \sim_{\mathbb{Q}} a_1 A$, with $a_1 < \frac{d\sqrt[d]{2}}{\alpha(1-\epsilon)}, a_1 \in \mathbb{Q}^+$, such that $x, y \in Z(\mathcal{J}(D_1))$,$(X', D_1)$ is lc not klt at one of the points, say $p(x,y) \in \{x, y\}$, and either codim$Z(\mathcal{J}(D_1)) > 1$ at $p(x,y)$ or there is one irreducible component of $Z(\mathcal{J}(D_1))$, say $V_{p(x,y)}$, that passes through $p(x,y)$ and such that codim$V_{p(x,y)} = 1$. We can suppose $U \subseteq \Pi$.

Fix $\beta \in \mathbb{Q}^+$.

Let $U' := \{p(x,y) | \text{codim} Z(\mathcal{J}(D_1)) = 1 \text{ at } p(x,y) \text{ and } \text{vol}(V_{p(x,y)}) \leq \beta^{d-1}\}$. Since $U = U' \cup (U \setminus U')$, then by lemma 1.5, 4., we are in one of these two cases:

1. $U \setminus U'$ contains a very general subset U'' of X;

2. U' is countably dense.

In the first case we know that $\forall x, y \in U''$ either $\operatorname{codim} Z(\mathcal{J}(D_1)) > 1$ at $p(x,y)$ or $\operatorname{vol}(V_{p(x,y)}) > \beta^{d-1}$. Applying the inductive steps of theorem 3.1 (and as in theorem 4.6) we can conclude that given two very general points $x, y \in X'$ there exist (depending on x, y) an effective $\mathbb{Q}$-divisor D on X' and $a \in \mathbb{Q}^+$ with $D \sim_{\mathbb{Q}} aA$ such that $x, y \in Z(\mathcal{J}(X', D))$ with $\dim Z(\mathcal{J}(X', D)) = 0$ around x or y, that is x or y is an isolated point of $Z(\mathcal{J}(X', D))$, and $a < \overline{s} + \frac{\overline{t}}{\alpha} + o(1)$ with

$$\overline{s} = 2\left(1 + \frac{(d-1)\sqrt[d-1]{2}}{\beta}\right) \cdot P - 2,$$

$$\overline{t} = d\sqrt[d]{2}\left(1 + \frac{(d-1)\sqrt[d-1]{2}}{\beta}\right) \cdot P,$$

and $o(1) > 0, \lim_{\epsilon \to 0} o(1) = 0$.

By [12, 1.41], $K_{X'} \sim_{\mathbb{Z}} \mu^*(K_X) + \operatorname{Exc}(\mu) \sim_{\mathbb{Q}} A + E + \operatorname{Exc}(\mu)$, where $\operatorname{Exc}(\mu)$ is the exceptional locus and it is an effective divisor by [12, 1.40]. Therefore, replacing D with $D + (l-1)(E + \operatorname{Exc}(\mu))$, as in the proof of lemma 1.44 we can conclude that by Nadel's vanishing theorem and proposition 1.35, $|lK_{X'}|$ separates two very general points in X' as soon as $l \geq [\overline{s} + \frac{\overline{t}}{\alpha}] + 2 \Leftrightarrow l > \overline{s} + \frac{\overline{t}}{\alpha} + 1$. To that purpose, imposing

$$l > 2\left(1 + \frac{(d-1)\sqrt[d-1]{2}}{\beta}\right) P - 1, \tag{4.5}$$

it is enough to ask that

$$\alpha > \frac{d\sqrt[d]{2}\left(1 + \frac{(d-1)\sqrt[d-1]{2}}{\beta}\right) P}{l + 1 - 2\left(1 + \frac{(d-1)\sqrt[d-1]{2}}{\beta}\right) P}. \tag{4.6}$$

In the second case, by lemma 1.48, we are in the following situation:

$$\begin{tikzcd} X'' \arrow[r, "\pi"] \arrow[d, "f"] & X' \\ B & \end{tikzcd}$$

where X'', B are normal projective varieties, f is a dominant morphism with connected fibres, π is a dominant and generically finite morphism and the image under π of a general fibre of f is V_x where V_x is a hypersurface through x, a general point. Moreover there exists a divisor D_x such that V_x is a pure log canonical centre of (X', D_x). In addition, setting $\overline{a} := \frac{d\sqrt[d]{2}}{\alpha(1-\epsilon)} \in \mathbb{Q}^+$ we have that $D_x \sim_{\mathbb{Q}} \overline{a}A$. Moreover for every $p(x,y)$ in a countably dense subset of U', $V_{p(x,y)}$ is the image through π of a fiber of f.

Again as in [37], we can suppose that there exists a proper closed subset $X_1' \subset X'$ such that for all $x \notin X_1'$, D_x is smooth at x.

As in [37] and the proof of theorem 2.2, we know that either π is birational or for a general $x \in X' \setminus X_1'$ there are at least two log canonical centres through x.

In the latter case consider x, y two general points of X' and consider also the divisors D_x and D_y. By [32, prop. 9.2.32(i)], since D_x is not klt at x and D_y is not klt at y, then $D_x + D_y$ is not klt at x and y. After rescaling with a constant ≤ 1 we can suppose that $D_x + D_y$ is lc but not klt at one of the points, say x, and it is not klt at the other point y. Therefore, depending on x, y there exists $\overline{a}' \in \mathbb{Q}^+$, $\overline{a}' < \frac{2d\sqrt[d]{2}}{\alpha} + o(1)$, such that $D_x + D_y \sim_{\mathbb{Q}} \overline{a}' A$. At this point, arguing exactly as in the proof of theorem 3.3, we can say that if π is not birational then for every x, y general points in X' there exists an effective $\mathbb{Q}$-divisor $D_{x,y}$ on X' and a positive rational number (depending on x, y) $\overline{a}'' < \frac{3d\sqrt[d]{2}}{\alpha} + o(1)$ such that $D_{x,y} \sim_{\mathbb{Q}} \overline{a}'' A$ and such that $D_{x,y}$ satisfies the induction statement $(*_2)$ of theorem 3.1 (with $a_2 = \overline{a}''$, $s_2 = o(1)$, $t_2 = \frac{3d\sqrt[d]{2}}{\alpha}\mathrm{vol}(X)$).

Now applying the inductive steps of Takayama (see [36, prop. 5.3, lemma 5.5, lemma 5.8]), we can conclude that for every general point $x, y \in X'$ there exists a divisor $D' \sim a'A$ such that $\mathrm{Nklt}(X', D')$ has dimension 0 in x or y and D' is not klt at the other point and $a' < \overline{s}' + \frac{\overline{t}'}{\alpha} + o(1)$ with

$$\overline{s}' = 2P - 2,$$

$$\overline{t}' = 3d\sqrt[d]{2}P.$$

As before, $|lK_{X'}|$ separates two very general points in X' as soon as $l \geq [\overline{s}' + \frac{\overline{t}'}{\alpha}] + 2 \Leftrightarrow l > \overline{s}' + \frac{\overline{t}'}{\alpha} + 1$. To that purpose, imposing

$$l > 2P - 1, \tag{4.7}$$

it is enough to ask that

$$\alpha > \frac{3d\sqrt[d]{2}P}{l + 1 - 2P}. \tag{4.8}$$

Summing up: given $l \in \mathbb{N}^+$, if (4.5), (4.6), (4.7), (4.8) are satisfied then we can conclude that either $|lK_{X'}|$ is birational or X', and hence X, is birational to the fibre space $f : X'' \to B$, with the general fibre having volume $\leq \beta^{d-1}$.

Note that (4.5) $\Rightarrow$ (4.7). Moreover (4.5) is equivalent to requiring $\beta > \frac{2(d-1)\sqrt[d-1]{2}P}{l+1-2P}$ and, if (4.5) holds, ((4.6) $\Rightarrow$ (4.8)) $\Leftrightarrow \beta \leq \frac{(d-1)\sqrt[d-1]{2}(l+1+4P)}{2(l+1-2P)}$. The thesis follows.

□

Chapter 5

Adjoint pluricanonical systems

Until now we have only dealt with pluricanonical systems on varieties of general type, but of course we could investigate also other kinds of big divisors. However, if L is a line bundle on a variety X of dimension d, the simple assumptions "L big" and "$\mathrm{vol}(L) \gg 0$" are not enough to allow us to conclude that there exists an $m \in \mathbb{N}^+$ such that $H^0(cL) \neq 0$ for a certain $0 < c \leq m$:

Example 5.1. Let X be a nonsingular curve of genus g and let $a \in \mathbb{N}^+$. Let $\nu_n : \mathrm{Pic}(X)^a \to \mathrm{Pic}(X)^{na}$ be the map such that $\nu_n(D) = nD$ for every $D \in \mathrm{Pic}(X)^a$. Since ν_n is dominant, then if D is general of degree a then nD is general of degree na. Moreover consider the natural map: $X^{(na)} \to \mathrm{Pic}(X)^{na}$, where $X^{(na)}$ is the symmetric product of X. Since $\dim(X^{(na)}) = na$ while $\dim(\mathrm{Pic}(X)^{na}) = g$, then the generic line bundle of degree na is not effective as long as $na < g$. Hence if L is a general line bundle on X of degree (i.e.: volume) $a > 0$ then $H^0(nL) = 0$ as long as $na < g$. Therefore, for every $a > 0$, allowing g to go to infinity it is clear that such an m cannot exist.

Taking products $X \times \ldots \times X$ the same conclusion is still valid for varieties of arbitrary dimension.

By virtue of the above example, we will consider only a particular class of big divisors on a variety X, namely big *adjoint divisors*.

5.1 Adjoint plurigenera

The non-vanishing of the space of sections of adjoint divisors have been extensively studied during the last years: we can mention Beltrametti-Sommese's conjecture (see [4, conj. 7.2.7]), Kawamata's conjecture (see [26, conj. 2.1]), Höring's theorem on adjoint divisors (see [22, thm. 1.2]) that is a generalized version of Beltrametti-Sommese's conjecture.

For a threefold X, Kawamata proved the following proposition (see also [22, thm. 1.7] for a similar statement, more general in some sense):

Proposition 5.2 (Kawamata)**.** *(see [26, prop. 4.1]). Let X be a projective threefold* *with at worst $\mathbb{Q}$-factorial canonical singularities. Let D be an integral Cartier divisor. Assume that K_X is nef and that $D - K_X$ is big and nef. Then $H^0(X, D) \neq 0$.*

Remark 5.3. Notice that the fact that D must be integral does not allow us to use Kawamata's theorem to prove non-vanishing results about pluricanonical systems on smooth threefolds of general type by simply reducing ourselves to their minimal model.

For surfaces Kawamata's conjecture is true; in fact Kawamata proved a generalized version of the following theorem:

Theorem 5.4 (Kawamata)**.** *(see [26, conj. 2.1 and thm. 3.1]) Let X be a smooth surface. Let D be a nef integral Cartier divisor such that $D - K_X$ is big and nef. Then $H^0(X, D) \neq 0$.*

Instead of considering $K_X + L$, where L is a big and nef line bundle, it is natural to try to relax some of the hypotheses on L, that is considering L big but maybe not nef, and, at the same time, bring in other hypotheses about the volume of X and L, in order to obtain explicit results about non-vanishing of adjoint linear systems.

Unfortunately the techniques that we used did not lead to significant effective results for threefolds of general type; anyway if we would like to apply something similar to proposition 1.49, the first step is to study adjoint systems on surfaces of general type.

We could prove the following theorem. Before stating it, we put before an easy generalization of lemma 1.45 (cf. [35, exercise 3.5], but applied only to a single point).

Lemma 5.5. *Let X be a projective variety and let $V \subseteq X$ be a subvariety of X of dimension d and let D be a $\mathbb{Q}$-Cartier divisor. Fix a positive real number α with*

$$vol_{X|V}(D) > \alpha^d.$$

Then for any sufficiently large and divisible $k \in \mathbb{N}$ there exists for any smooth point $x \in V$ a divisor $A_x \in |kD|$ such that $mult_x(A_x|_V) > k\alpha$ and such that $V_x \nsubseteq Supp(A_x)$.

Proof. Assume k sufficiently divisible so that kD is integral. The proof is very much similar to lemma 1.45. We need only to point out that instead of considering $H^0(kD)$ we are now considering sections in $H^0(kD|_V)$ that come from $H^0(kD)$ by the natural restriction map $r : H^0(kD) \to H^0(kD|_V)$. Clearly if $s = r(t)$ is not zero then t gives a divisor whose support does not contain V. □

Remark 5.6. *Mutatis mutandis* remark 1.46 holds also in this case.

Theorem 5.7. *Let X be a smooth, projective surface of general type and let L be a big line bundle on X such that $vol(L) > \alpha^2$. If $\alpha > 4$ then $h^0(2K_X + L) \geq 1$ and if $\alpha > 3n + (-1)^{n+1}n$ then $h^0\left(K_X + \left(\left[\frac{n}{2}\right] + 1\right)(K_X + L)\right) \geq n$ for all $n \geq 1$. More* *generally, if X is not g-countably dense, if $\alpha > \frac{4g-4}{2g-3}$ then $h^0(2K_X + L) \geq 1$ and if $\alpha > \frac{2n}{1-\left\{\frac{n}{2g-2}\right\}}$ then*

$$h^0\left(K_X + \left(\left[\frac{n}{2g-2}\right] + 1\right)(K_X + L)\right) \geq n$$

for all $n \geq 1$.

Proof. By remark 1.8 there exists a very general subset $\Lambda \subseteq X$ such that for every $x \in \Lambda$ and every curve C through x then $g(C) \geq g$. Set $X_0 := \Lambda \cap (X \setminus (\mathbb{B}_+(L) \cup \mathbb{B}_+(K_X)))$.

As in the proof of theorem 2.2, for every $x \in X_0$ we have an effective $\mathbb{Q}$-divisor $D_x \sim_{\mathbb{Q}} \lambda_x L$, with $\lambda_x < \frac{2}{\alpha}, \lambda_x \in \mathbb{Q}^+$, such that (X, D_x) is lc, not klt in x and $LLC(X, D_x, x) = \{V_x\}$, where V_x is an exceptional lc centre at x. As in the proof of theorem 2.2, set

$$Y_0 := \{x \in X_0 \text{ s.t. } \dim(V_x) = 0\},$$

$$Y_1 := \{x \in X_0 \text{ s.t. } \dim(V_x) = 1\}.$$

Either Y_0 or Y_1 is countably dense.

If Y_0 is countably dense then for every $x \in Y_0$ take an effective $\mathbb{Q}$-divisor $F_x \sim_{\mathbb{Q}} \lambda_x K_X$ such that $x \notin \mathrm{Supp}(F_x)$ (this is possible by lemma 1.18). Consider $D'_x := D_x + F_x \sim_{\mathbb{Q}} \lambda_x(K_X + L)$. Since for every $x \in Y_0$, $x \notin \mathrm{Supp}(F_x)$ then by prop. 1.35 we have that $V_x = \{x\}$ is still an exceptional lc centre for (X, D'_x) and thus we can apply lemma 1.6 and lemma 1.44 to conclude that for all $n \geq 1$, for all $m > \left[\frac{2n}{\alpha}\right]$ we have that

$$h^0(K_X + m(K_X + L)) \geq n.$$

If Y_1 is countably dense then for every $x \in Y_1$ we need to cut down the dimension of V_x. But this time we cannot simply apply the specific theorem by Hacon-McKernan (as we did in theorem 4.1) or the specific theorem by Takayama (as we did in theorem 4.6), since we are not dealing with pluricanonical systems.

First of all, since $x \notin \mathbb{B}_+(L)$, we can pick an effective $\mathbb{Q}$-divisor E_x such that $L - E_x$ is ample and $x \notin \mathrm{Supp}(E_x)$. Then we can write L as $L = (1-\lambda_x)A_x + D_x + (1-\lambda_x)E_x$, with A_x ample. This is needed to invoke [35, theorem 4.4]: $D_x + (1-\lambda_x)E_x$ is an effective divisor and, since $x \notin \mathrm{Supp}(E_x)$, $(X, D_x + (1-\lambda_x)E_x)$ is still lc at x and $LLC(X, D_x + (1-\lambda_x)E_x, x) = \{V_x\}$, therefore, since $g(V_x) \geq g$,

$$\mathrm{vol}_{X|V_x}(K_X + L) \geq \mathrm{vol}(V_x) \geq 2g - 2.$$

At this point we would like to apply [35, lemma A.4]. Let $Z(\mathcal{J}(X, D_x)) = V_x \cup Z$, where Z is closed and, since V_x is an exceptional lc centre at x, $x \notin Z$. For all $x \in Y_1$, let us consider $U \cap X_0 \cap (V_x \setminus Z)$, where U is the dense subset of V_x defined in the above-mentioned lemma (notice that $V_x \setminus Z$ is open and non-empty in V_x and that U can be chosen open, thus $U \cap X_0 \cap (V_x \setminus Z) \neq \emptyset$ and is very general in V_x). For all $x \in Y_1$, for all $y \in U \cap X_0 \cap (V_x \setminus Z)$, since y in particular is a smooth point on V_x, by lemma 5.5 we have that for every $t > \frac{1}{2g-2}$ there exists an effective

$\mathbb{Q}$-divisor $B \sim_{\mathbb{Q}} t(K_X + L)$ such that $\mathrm{mult}_y(B|_{V_x}) > 1$ and such that $V_x \not\subseteq \mathrm{Supp}(B)$. In particular take

$$t := \frac{1}{2g-2} + \frac{1}{\alpha} - \frac{\lambda_x}{2}.$$

For k sufficiently large, $\mathcal{I}_{V_x}(kB)$ is globally generated outside $\mathbb{B}_+(B) \subseteq \mathbb{B}_+(K_X) \cup \mathbb{B}_+(L)$, thus the possible choices of $B' \sim kB$ such that $B'|_{V_x} = kB|_{V_x}$ and $V_x \not\subseteq \mathrm{Supp}(B')$ form a free linear series off $V_x \cup \mathbb{B}_+(K_X) \cup \mathbb{B}_+(L)$. By Kollar-Bertini's theorem, if B' is general then $\mathcal{J}(X, D_x + \frac{1}{k}B') = \mathcal{J}(X, D_x)$ off $V_x \cup \mathbb{B}_+(K_X) \cup \mathbb{B}_+(L)$. Let $B'' := \frac{1}{k}B'$. Since $\mathrm{mult}_y(B''|_{V_x}) > 1$ still holds then, by [35, lemma A.4] and for every $\epsilon \in \mathbb{Q}^+$, $\epsilon \ll 1$, we have that $Z(\mathcal{J}((1-\epsilon)D_x + B''))$ contains y as an isolated point (in fact (X, D_x) is lc at the generic point of V_x and $V_x \not\subseteq \mathrm{Supp}(B'')$). As before, for every x, for every y, take an effective $\mathbb{Q}$-divisor F'_y such that $F'_y \sim_{\mathbb{Q}} (1-\epsilon)\lambda_x K_X$ and $y \notin \mathrm{Supp}(F'_y)$. Then y is still an isolated point of $Z(\mathcal{J}((1-\epsilon)D_x + B'' + F'_y))$ where

$$(1-\epsilon)D_x + B'' + F'_y \sim_{\mathbb{Q}} \mu_{x,y}(K_X + L),$$

with $\mu_{x,y} \in \mathbb{Q}^+$, $\mu_{x,y} < \frac{1}{2g-2} + \frac{2}{\alpha}$.

Notice that since this point y can be chosen in a very general subset of V_x, then by the same argument as in the proof of theorem 2.2 we know that, varying $x \in Y_1$, the set of all possible points y is countably dense in X.

Therefore we can now apply lemma 1.6 and lemma 1.44: for all $n \geq 1$, for all $m > \left[\frac{n}{2g-2} + \frac{2n}{\alpha}\right]$, $h^0(K_X + m(K_X + L)) \geq n$.

In particular, for all $n \geq 1$, if

$$\alpha > \frac{2n}{1 - \left\{\frac{n}{2g-2}\right\}},$$

we have that $\left[\frac{n}{2g-2}\right] + 1 > \frac{n}{2g-2} + \frac{2n}{\alpha}$. This is equivalent to $\left[\frac{n}{2g-2}\right] + 1 > \left[\frac{n}{2g-2} + \frac{2n}{\alpha}\right]$ and thus

$$h^0\left(K_X + \left(\left[\frac{n}{2g-2}\right] + 1\right)(K_X + L)\right) \geq n.$$

□

Remark 5.8. For threefolds of general type, by the same arguments, it can be proved a theorem analogous to 5.7: if L is a big line bundle of sufficiently large volume then $H^0(6K_X + 5L) \neq 0$.

5.2 Adjoint pluricanonical maps

When do adjoint pluricanonical systems separate general points? A celebrated conjecture by Fujita says that if X is any smooth projective variety of dimension d and A is an ample integral divisor then (i) $K_X + (l-1)A$ is base point free and (ii) $K_X + lA$ is very ample for any $l \geq d+2$ (see, for example, [32, conj. 10.4.1]).

In the case of surfaces Fujita conjecture has been proved: it actually follows from more precise results due to Reider. It is interesting to note that in [14] Ein gave a proof for Reider's theorem (more precisely for the base point free part) using multiplier ideals and non-vanishing theorems. (i) has been proved also in dimension three and four (see [15], [18], [24]).

Unfortunately this kind of techniques do not seem very adept at separating tangent vectors, and for this reason not very much is known about very ampleness of adjoint divisors. However, if we just focus on points separation, then probably the most notable progress up to now was made by Angehrn and Siu, who proved that for any smooth projective variety X of dimension d and any A ample (or big and nef) divisor on X, if $l \geq \frac{1}{2}(n^2 + 2nr - n + 2)$ then $|K_X + lA|$ separates any set of r distinct points of X (see [1, cor. 0.4]).

As in the previous section, we can try to relax hypotheses on A adding hypotheses on X in order to obtain explicit results about the birationality of adjoint linear systems. We could prove the following:

Theorem 5.9. *Let X be a smooth, projective surface of general type and let L be a big line bundle on X such that $vol(L) > \alpha^2$. If $\alpha > 2\sqrt{2}$ then $|K_X + l(K_X + L)|$ gives a birational map for every $l \geq 2$. If X is not g-countably dense, with $g \geq 3$, if $\alpha > 2\sqrt{2}\frac{g-1}{g-2}$ then $|K_X + l(K_X + L)|$ gives a birational map for every $l \geq 1$.*

Proof. Let Λ and X_0 be the same sets as in the proof of theorem 5.7. For every $x_1, x_2 \in X_0$, by remark 1.46 and as in the proof of theorem 2.2, we have an effective $\mathbb{Q}$-divisor $D_{1,2} \sim_{\mathbb{Q}} \lambda_{1,2}L$, with $\lambda_{1,2} < \frac{2\sqrt{2}}{\alpha}$, $\lambda_{1,2} \in \mathbb{Q}^+$, such that $x_1, x_2 \in Nklt(X, D_{1,2})$ and $(X, D_{1,2})$ is lc at (at least) one of the two points.

If either x_1 or x_2 is an isolated component of $Nklt(X, D_{1,2})$ then we are done (see proof of lemma 1.44). Otherwise, as usual, we need to cut its dimension. Applying [36, lemma 5.5] and [35, simplifying ass. 4.5,4.6] we can suppose that $D_{1,2}$ is lc both at x and y and that there exists a unique irreducible component V of $Nklt(X, D_{1,2})$ through x and that $y \in V$; we can also assume that $\dim(V) = 1$ and that x, y are smooth points in V.

As in the proof of theorem 5.7, we have that $\mathrm{vol}_{X|V}(K_X + L) \geq 2g - 2$ and thus by remark 5.6 for every $t > \frac{2}{2g-2} = \frac{1}{g-1}$ there exists an effective $\mathbb{Q}$-divisor $B \sim_{\mathbb{Q}} t(K_X + L)$ such that $\mathrm{mult}_{x_1}(B|_V) > 1$, $\mathrm{mult}_{x_2}(B|_V) > 1$ and $V \nsubseteq \mathrm{Supp}(B)$. Take

$$t := \frac{1}{g-1} + \frac{\sqrt{2}}{\alpha} - \frac{\lambda_{1,2}}{2}.$$

As in the proof of thm. 5.7 we can also suppose that $\mathcal{J}(X, D_{1,2} + B) = \mathcal{J}(X, D_{1,2})$ off $V \cup \mathbb{B}_+(K_X) \cup \mathbb{B}_+(L)$. Hence by [35, lemma A.4] for every $\epsilon \in \mathbb{Q}^+$, $\epsilon \ll 1$ we have that $Z(\mathcal{J}((1-\epsilon)D_{1,2} + B))$ contains x, y as isolated points.

To ease calculations, take an effective $\mathbb{Q}$-divisor $F_{1,2} \sim_{\mathbb{Q}} (1-\epsilon)\lambda_{1,2}K_X$ such that $x_1, x_2 \notin \mathrm{Supp}(F_{1,2})$ (this is possible by remark 1.19). x_1, x_2 are still isolated points of $Z(\mathcal{J}((1-\epsilon)D_{1,2} + B + F))$ and

$$(1-\epsilon)D_{1,2} + B + F \sim_{\mathbb{Q}} \mu_{1,2}(K_X + L),$$

with $\mu_{1,2} \in \mathbb{Q}^+$, $\mu_{1,2} < \frac{1}{g-1} + \frac{2\sqrt{2}}{\alpha}$.

Therefore, as in the proof of lemma 1.44, for every $l > \left[\frac{1}{g-1} + \frac{2\sqrt{2}}{\alpha}\right]$ we have that $|K_X + l(K_X + L)|$ gives a birational map.

In particular if
$$\alpha > \frac{2\sqrt{2}}{1 - \left\{\frac{1}{g-1}\right\}},$$
we have that
$$\left[\frac{1}{g-1}\right] + 1 > \left[\frac{1}{g-1} + \frac{2\sqrt{2}}{\alpha}\right]$$
and we can conclude: if $g = 2$ then $\left[\frac{1}{g-1}\right] = 1$, while if $g \geq 3$ then $\left[\frac{1}{g-1}\right] = 0$. $\square$

Remark 5.10. For threefolds of general type, by the same arguments, it can be proved a theorem analogous to 5.9: if L is a big line bundle of sufficiently large volume then $|8K_X + 7L|$ gives a birational map.

Bibliography

[1] Urban Angehrn and Yum Tong Siu. Effective freeness and point separation for adjoint bundles. *Invent. Math.*, 122(2):291–308, 1995.

[2] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 2004.

[3] Ingrid C. Bauer, Fabrizio Catanese, and Roberto Pignatelli. Complex surfaces of general type: some recent progress. In *Global aspects of complex geometry*, pages 1–58. Springer, Berlin, 2006.

[4] Mauro C. Beltrametti and Andrew J. Sommese. *The adjunction theory of complex projective varieties*, volume 16 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 1995.

[5] Enrico Bombieri. Canonical models of surfaces of general type. *Inst. Hautes Études Sci. Publ. Math.*, (42):171–219, 1973.

[6] Giuseppe Borrelli. The classification of surfaces of general type with nonbirational bicanonical map. *J. Algebraic Geom.*, 16(4):625–669, 2007.

[7] Amaël Broustet. Non-annulation effective et positivité locale des fibrés en droites amples adjoints. *Math. Ann.*, 343(4):727–755, 2009.

[8] Jungkai A. Chen and Meng Chen. Explicit birational geometry of threefolds of general type, II. *arXiv:0810.5044*, 2008.

[9] M. Chen. On the $\mathbb{Q}$-divisor method and its application. *J. Pure Appl. Algebra*, 191(1-2):143–156, 2004.

[10] Meng Chen and De-Qi Zhang. Characterization of the 4-canonical birationality of algebraic threefolds. *Math. Z.*, 258(3):565–585, 2008.

[11] V. I. Danilov. Algebraic varieties and schemes. In *Algebraic geometry, I*, volume 23 of *Encyclopaedia Math. Sci.*, pages 167–297. Springer, Berlin, 1994.

[12] Olivier Debarre. *Higher-dimensional algebraic geometry.* Universitext. Springer-Verlag, New York, 2001.

[13] Jia-qi Dong. The 4-canonical maps of 3-folds of general type. *Northeast. Math. J.*, 22(1):119–126, 2006.

[14] Lawrence Ein. Adjoint linear systems. In *Current topics in complex algebraic geometry (Berkeley, CA, 1992/93)*, volume 28 of *Math. Sci. Res. Inst. Publ.*, pages 87–95. Cambridge Univ. Press, Cambridge, 1995.

[15] Lawrence Ein and Robert Lazarsfeld. Global generation of pluricanonical and adjoint linear series on smooth projective threefolds. *J. Amer. Math. Soc.*, 6(4):875–903, 1993.

[16] Lawrence Ein, Robert Lazarsfeld, Mircea Mustaţă, Michael Nakamaye, and Mihnea Popa. Asymptotic invariants of base loci. *Ann. Inst. Fourier (Grenoble)*, 56(6):1701–1734, 2006.

[17] Lawrence Ein, Robert Lazarsfeld, Mircea Mustaţă, Michael Nakamaye, and Mihnea Popa. Restricted volumes and base loci of linear series. *Amer. J. Math.*, 131(3):607–651, 2009.

[18] Takao Fujita. Remarks on Ein-Lazarsfeld criterion of spannedness of adjoint bundles of polarized threefolds. *arXiv:alg-geom/9311013v1*, 1993.

[19] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].* Springer-Verlag, Berlin, second edition, 1998.

[20] Christopher D. Hacon and James McKernan. Boundedness of pluricanonical maps of varieties of general type. *Invent. Math.*, 166(1):1–25, 2006.

[21] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[22] Andreas Hoering. On a conjecture of Beltrametti and Sommese. *arXiv:0912.1295v1*, 2009.

[23] A. R. Iano-Fletcher. Working with weighted complete intersections. In *Explicit birational geometry of 3-folds*, volume 281 of *London Math. Soc. Lecture Note Ser.*, pages 101–173. Cambridge Univ. Press, Cambridge, 2000.

[24] Yujiro Kawamata. On Fujita's freeness conjecture for 3-folds and 4-folds. *Math. Ann.*, 308(3):491–505, 1997.

[25] Yujiro Kawamata. On the extension problem of pluricanonical forms. In *Algebraic geometry: Hirzebruch 70 (Warsaw, 1998)*, volume 241 of *Contemp. Math.*, pages 193–207. Amer. Math. Soc., Providence, RI, 1999.

[26] Yujiro Kawamata. On effective non-vanishing and base-point-freeness. *Asian J. Math.*, 4(1):173–181, 2000. Kodaira's issue.

[27] Yujiro Kawamata, Katsumi Matsuda, and Kenji Matsuki. Introduction to the minimal model problem. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 283–360. North-Holland, Amsterdam, 1987.

[28] John L. Kelley. *General topology.* Springer-Verlag, New York, 1975. Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.], Graduate Texts in Mathematics, No. 27.

[29] János Kollár. Kodaira's canonical bundle formula and adjunction. In *Flips for 3-folds and 4-folds*, volume 35 of *Oxford Lecture Ser. Math. Appl.*, pages 134–162. Oxford Univ. Press, Oxford, 2007.

[30] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[31] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].* Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[32] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].* Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.

[33] James McKernan. Boundedness of log terminal Fano pairs of bounded index. *arXiv:math/0205214v1*, 2002.

[34] Yoichi Miyaoka and Thomas Peternell. *Geometry of higher-dimensional algebraic varieties*, volume 26 of *DMV Seminar.* Birkhäuser Verlag, Basel, 1997.

[35] Gianluca Pacienza. *Pluricanonical systems on projective varieties of general type.* Lecture notes on the work of Hacon-McKernan, Takayama, Tsuji, prepared for the Summer School "Geometry of complex projective varieties and the MMP". Grenoble, 2007.

[36] Shigeharu Takayama. Pluricanonical systems on algebraic varieties of general type. *Invent. Math.*, 165(3):551–587, 2006.

[37] Gueorgui Tomov Todorov. Pluricanonical maps for threefolds of general type. *Ann. Inst. Fourier (Grenoble)*, 57(4):1315–1330, 2007.