Filippo Cerocchi

Applicazioni dinamiche e spettrali della teoria di Gromov-Hausdorff

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THÈSE

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Filippo CEROCCHI

Thèse co-dirigée par **S. GALLOT** et **A. SAMBUSETTI**

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M. S. GALLOT
Professeur, Université de Grenoble I, Directeur de Thèse
M. M. HERZLICH
Professeur, Université de Montpellier II, Rapporteur
M. C. PETRONIO
Professeur, Università di Pisa, Rapporteur
M. R. PIERGALLINI
Professeur, Università di Camerino, Examinateur
M. A. SAMBUSETTI
Professeur, Università di Roma « Sapienza », Directeur de Thèse

Université Joseph Fourier / Université Pierre Mendès France / Université Stendhal / Université de Savoie / Grenoble INP

Al cuore fragile di mio padre,
alla fiducia incrollabile di mia madre.

Contents

Présentation

Cette Thèse est divisée en deux parties. Dans la première partie on s'intéresse à une nouvelle version de la méthode du barycentre, introduite initialement par G. Besson, G. Courtois et S. Gallot, et à de nouvelles applications de cette méthode aux problèmes de rigidité dynamique et de comparaison entre spectres du Laplacien de deux variétés riemanniennes. Dans la deuxième partie on aborde un thème classique de la Géométrie riemannienne concernant les actions de groupes discrets : le Lemme de Margulis, dont nous donnons une version sans courbure, reposant uniquement sur les propriétés algébriques du groupe qui agit.

La première partie de cette thèse est composée de trois chapitres :

Dans le Chapitre 1 nous présentons la méthode du barycentre de G. Besson, G. Courtois et S. Gallot. Cette méthode a été introduite par ses auteurs avec l'objectif de démontrer la conjecture de l'Entropie Minimale (voir [**BCG1**]). La méthode du barycentre est une technique qui permet, à partir de la donnée d'une application (éventuellement non continue, éventuellement définie seulement sur l'orbite d'un groupe discret), de construire une application régulière dont le déterminant jacobien est majoré de manière optimale. Un exemple d'application de cette méthode (développé par G. Besson, G. Courtois et S. Gallot) est le suivant :
Soit (Y, g) une variété riemannienne compacte, connexe, orientée, de dimension n, qui admet une application continue f à valeurs dans un espace localement symétrique (X, g_0) de courbure négative et de même dimension, alors la méthode du barycentre produit une famille d'applications $\tilde{F}_c : \tilde{Y} \to \tilde{X}$ entre les revêtements universels, de classe C^∞, qui dépendent d'un paramètre c (qu'il faut choisir plus grand que l'entropie volumique de (Y, g)), qui sont équivariantes par rapport à l'homomorphisme entre les groupes fondamentaux de Y et de X induit par l'application f de départ, et dont le jacobien est majoré optimalement selon l'inégalité :

$$|\operatorname{Jac}(\tilde{F}_c)| \leq (1+\varepsilon(c)) \left(\frac{\operatorname{Ent}_{\mathrm{vol}}(Y,g)}{\operatorname{Ent}_{\mathrm{vol}}(X,g_0)} \right)^n$$

où $\operatorname{Ent}_{\mathrm{vol}}(Y,g)$ (resp. $\operatorname{Ent}_{\mathrm{vol}}(X,g_0)$) désigne l'entropie volumique de (Y,g) (resp. de (X,g_0)) et où $\varepsilon(c) \to 0$ quand c tend vers l'entropie volumique de (Y,g). L'équivariance permet aux applications $\tilde{F}_c$ d'induire des applications F_c entre les quotients Y et X, qui sont toutes homotopes à l'application f de départ et qui sont presque-contractantes.

Dans le cas général (voir [**BCG2**], [**BCG5**] et [**BCG6**]), on définit de même une famille d'applications $\tilde{F}_c : \tilde{Y} \to \tilde{X}$, chaque application $\tilde{F}_c$ se construisant comme suit :

- on construit une application $\tilde{y} \mapsto \nu^c_{\tilde{y}}$ de $\tilde{Y}$ dans l'espace des mesures de probabilité sur $\tilde{Y}$, choisie de telle manière que la dérivée logarithmique de $\nu^c_{\tilde{y}}$ par rapport à $\tilde{y}$ soit donnée par c et que $\nu^c_{\gamma\tilde{y}} = \gamma_* \nu^c_{\tilde{y}}$ pour tout $\gamma \in \pi_1(Y)$ (ici $\pi_1(Y)$ est vu comme l'ensemble des transformations de revêtement correspondant au revêtement universel $\tilde{Y} \to Y$),

- on transporte chacune de ces mesures $\nu_{\tilde{y}}^c$ en une mesure[1] $\mu_{\tilde{y}}^c$ définie sur $\tilde{X}$ (ou sur son bord idéal $\partial\tilde{X}$),
- on définit $\tilde{F}_c(\tilde{y})$ comme le barycentre de la mesure $\mu_{\tilde{y}}^c$.

Il y a plusieurs choix possibles pour la définition du barycentre sur l'espace $(\tilde{X}, \tilde{g}_0)$, chacun de ces choix correspondant à des applications différentes de la méthode : G. Besson, G. Courtois et S. Gallot ([**BCG1**] et [**BCG2**]) définissaient le barycentre d'une mesure μ supportée par le bord idéal $\partial\tilde{X}$ comme l'unique point où la fonction $\tilde{x} \mapsto \int_{\partial\tilde{X}} B(\tilde{x}, \theta)\, d\mu(\theta)$ atteint son minimum (où B est la fonction de Busemann de $\tilde{X}$) et, dans un travail plus récent ([**BCG5**] et [**BCG6**]), il définissent le barycentre d'une mesure μ supportée par $\tilde{X}$ (lorsque la courbure sectionnelle de $\tilde{X}$ est inférieure ou égale à $-K^2$) comme l'unique point où la fonction $\tilde{x} \mapsto \int_{\tilde{X}} \cosh\left(K\, d_{\tilde{X}}(\tilde{x}, \tilde{z})\right) d\mu(\tilde{z})$ atteint son minimum (pour une mesure μ telle que cette fonction soit finie).
Toujours lorsque la courbure sectionnelle de $\tilde{X}$ est inférieure ou égale à $-K^2$, A. Sambusetti ([**Samb2**]) définissait le barycentre d'une mesure μ supportée par $\tilde{X}$ comme l'unique point où la fonction $\tilde{x} \mapsto \int_{\tilde{X}} d_{\tilde{X}}(\tilde{x}, \tilde{z})^2\, d\mu(\tilde{z})$ atteint son minimum (pour une mesure μ telle que cette fonction soit finie).
Ces auteurs définissaient le barycentre dans le cas où la courbure sectionnelle de $(\tilde{X}, \tilde{g}_0)$ est strictement négative ([**BCG1**], [**BCG2**], [**BCG5**], [**BCG6**] et [**Samb2**]) ou dans le cas où $(\tilde{X}, \tilde{g}_0)$ est un produit d'espaces symétriques, dont chacun est de courbure strictement négative ([**BCG4**]).

L. Sabatini ([**Saba1**], [**Saba2**]) a ensuite remarqué que la notion de barycentre introduite par A. Sambusetti se généralise au cas de courbure négative ou nulle (redonnant, dans le cas plat, la notion classique de barycentre de l'espace euclidien); il a démontré ensuite que cette notion de barycentre se généralise au cas de courbure bornée de signe quelconque, lorsque le support de chacune des mesures $\mu_{\tilde{y}}^c$ est de diamètre suffisamment petit (pour plus de précisions voir ([**Saba1**] et le Théorème 1.4.1 de la présente thèse). Dans le cas où la donnée initiale est une $\varepsilon-$approximation de Hausdorff $f : (Y, g) \to (X, g_0)$ (et quand ε est suffisamment petit), il invente une nouvelle manière (plus directe) de définir l'application $\tilde{y} \mapsto \mu_{\tilde{y}}^c$ de $\tilde{Y}$ dans l'espace des mesures de probabilité sur $\tilde{X}$ de sorte que $\mu_{\tilde{y}}^c$ soit de densité contrôlée par rapport à la mesure riemannienne $dv_{\tilde{g}_0}$. Ceci lui permet de comparer optimalement les volumes des deux variétés (Y, g) et (X, g_0) sans faire d'hypothèses sur la courbure de (Y, g) .

Dans cette thèse nous utilisons une variante de la méthode de L. Sabatini pour atteindre deux objectifs qui semblaient a priori hors de portée de sa méthode :

- établir un résultat du type "conjugaison des flots $\Longrightarrow$ isométrie"
(ce résultat exige une version de la méthode de L. Sabatini qui soit valable pour les $(1, C)-$quasi isométries, lorsque C n'est pas petit),
- comparer les spectres des laplaciens de (Y, g) et de (X, g_0) sans faire d'hypothèse sur la courbure de (Y, g) , ce qui exige une étude précise de ce que devient l'application F_c (construite ci-dessus) dans le cas de presque-égalité entre volumes de (Y, g) et de (X, g_0) , ainsi que des estimées des normes L^∞ des fonctions propres du Laplacien de (X, g_0) , que l'on transplante sur (Y, g) en les composant par F_c .

[1]Dans le cas où la donnée initiale est une application $f : Y \to X$, on pose $\nu_{\tilde{y}}^c = e^{-c\, d_{\tilde{Y}}(\tilde{y}, \tilde{z})} dv_{\tilde{g}}(\tilde{z})$ et on définit $\mu_{\tilde{y}}^c$ comme le transporté de cette mesure par l'application $\tilde{f} : \tilde{Y} \to \tilde{X}$ relevée de l'application f . Dans le cas où la donnée initiale est une représentation $\rho : \Gamma = \pi_1(Y) \to \pi_1(X)$ (voir [**BCG2**] et [**BCG6**]), on peut faire de même en utilisant comme mesure $\nu_{\tilde{y}}^c$ une combinaison linéaire de mesures de Dirac $\delta_{\gamma.\tilde{y}_0}$ supportée par une orbite $\Gamma.\tilde{y}_0$, qu'on transporte (par la représentation) en une combinaison linéaire $\mu_{\tilde{y}}^c$ de mesures de Dirac $\delta_{\rho(\gamma).\tilde{x}_0}$ supportée par une orbite $\rho(\Gamma).\tilde{x}_0 \subset \tilde{X}$; plus précisément, on peut par exemple définir

$$\nu_{\tilde{y}}^c = \sum_{\gamma \in \Gamma} e^{-c\, d_{\tilde{Y}}(\tilde{y}, \gamma.\tilde{y}_0)} \delta_{\gamma.\tilde{y}_0} \quad , \quad \mu_{\tilde{y}}^c = \sum_{\gamma \in \Gamma} e^{-c\, d_{\tilde{Y}}(\tilde{y}, \gamma.\tilde{y}_0)} \delta_{\rho(\gamma).\tilde{x}_0} \quad .$$

On définit *l'énergie au point* $y \in Y$ *d'une application* $k : (Y, g) \to (X, g_0)$, de classe C^1, comme le nombre

$$e_y(k) = \sum_{i=1}^{n} \| \, d_y k(e_i) \, \|_{g_0}^2 \quad ,$$

où $\{e_i\}_{i=1}^n$ est n'importe quelle base $g-$orthonormée de $(T_y Y, g_y)$. Notre version personnelle de la méthode de L. Sabatini s'énonce alors comme suit :

THEOREM 0.1. (Version simplifiée, pour une version complète voir le Théorème 1.2.1) *Soient* (X, g_0) *et* (Y, g) *deux variétés riemanniennes connexes, complètes, de dimension* n, *telles que* X *soit simplement connexe; notons* ρ *et* d *les distances riemanniennes sur* X *et sur* Y *associées respectivement aux métriques* g_0 *et* g. *On suppose*[2] *que* $\mathrm{inj}(X, g_0) > 0$ *et que* $\mathrm{inj}(Y, g) > 0$. *On note respectivement* σ *et* σ_0 *les courbures sectionnelles de* (Y, g) *et de* (X, g_0). *Supposons que* $|\sigma_0| \leq \kappa_0^2$ *et que* $|\sigma|$ *est bornée.*
Soient Γ_Y *(resp* Γ_X*) un sous-groupe discret du groupe des isométries de* (Y, g) *(resp. de* (X, g_0)*), agissant de façon libre et proprement discontinue sur* Y *(resp. sur* X*). Supposons qu'il existe un isomorphisme* $\lambda : \Gamma_Y \to \Gamma_X$ *et deux applications* $f : X \to Y$ *et* $h : Y \to X$, *équivariantes par rapport à* λ^{-1} *et* λ *respectivement, qui vérifient, pour tout* $x \in X$ *et pour tout* $y \in Y$:

$$\rho(h(y), x) \leq \alpha \cdot d(y, f(x)) + \varepsilon \ , \quad d(y, f(x)) \leq \alpha \cdot \rho(h(y), x) + \varepsilon \ ,$$
$$\rho(x, (h \circ f)(x)) \leq \varepsilon, \quad d(y, (f \circ h)(y)) \leq \varepsilon \ .$$

pour des constantes $\alpha \geq 1$ *et* ε *données.*
Pour toute donnée de constantes strictement positives ε, c, R *et* α *telles que*

$$\alpha \geq 1 \quad , \quad R < \frac{1}{7\,\alpha} \cdot \min \left\{ \mathrm{inj}(X, g_0), \frac{\pi}{2\,\kappa_0} \right\} \quad , \quad \varepsilon < \frac{R}{5} \quad ,$$

il existe une application λ*-équivariante* $H_c^R : Y \to X$, *de classe* C^1, *telle que*

$$\forall y \in Y \qquad \rho(H_c^R(y), h(y)) \leq A(n, \alpha, c, \varepsilon, \kappa_0, R) \cdot \left(\frac{\varepsilon}{\kappa_0} \right)^{\frac{1}{2}} \quad ,$$

(où la constante universelle $A(n, \alpha, c, \varepsilon, \kappa_0, R)$ *est une fonction explicite*[3] *de* $(n, \alpha, c, \varepsilon, \kappa_0, R)$, *qui est bornée supérieurement lorsqu'il existe trois constantes positives* B_1, B_2 *et* B_3 *telles que* $c\,\varepsilon \leq B_1$, $c\,\varepsilon^{\frac{1}{2}} \geq B_2\,\kappa_0^{\frac{1}{2}}$ *et* $\alpha \leq B_3$*) et dont l'énergie vérifie, en tout point* $y \in Y$,

$$e_y(H_c^R) \leq n \cdot (1 + \eta(n, \alpha, c, \varepsilon, \kappa_0, R)) \quad ,$$

où $\eta(n, \alpha, c, \varepsilon, \kappa_0, R)$ *est une fonction explicite*[4] *de* $(n, \alpha, c, \varepsilon, \kappa_0, R)$, *qui tend vers zéro lorsqu'il existe deux constantes positives* B_1', *et* B_2' *telles que* $c\,R \geq B_1'$ *et* $c\,\varepsilon^{\frac{9}{10}} \leq B_2'\,\kappa_0^{\frac{1}{10}}$ *et lorsque* $\kappa_0\,\varepsilon$, $(\alpha - 1)$ *et* $\kappa_0\,R$ *tendent simultanément vers zéro.*
Si de plus l'application h *de départ est continue, alors* H_c^R *est homotope à* h *par une homotopie* λ*-équivariante.*

Il est important, dans les applications (afin que celles-ci soient optimales), que le majorant de l'énergie ponctuelle soit proche de n comme c'est le cas dans le théorème précédent, car la valeur n est précisément l'énergie d'une isométrie.
La démonstration de ce théorème est donnée dans le chapitre 1, sections 1.2 et 1.3.

Dans la section 3.1, nous montrons que ce théorème redonne le résultat suivant de L. Sabatini ([**Saba1**], Théorème 4.6.1). C'est ce corollaire que nous utiliserons dans la suite du chapitre 3 pour comparer les spectres de deux variétés riemanniennes :

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[2] L'hypothèse $\mathrm{inj}(X, g_0) > 0$ (resp. $\mathrm{inj}(Y, g) > 0$) est automatiquement vérifiée lorsque (X, g_0) (resp. (Y, g)) possède un groupe cocompact d'isométries, par exemple lorsque (X, g_0) (resp. (Y, g)) est le revêtement universel riemannien d'une variété riemannienne compacte.

[3] Pour une valeur explicite de $A(n, \alpha, c, \varepsilon, \kappa_0, R)$, voir l'énoncé du Théorème 1.2.1. Remarquer que l'introduction de la constante κ_0 complique un peu les expressions, mais qu'elle est utile pour fixer l'échelle et rendre les constantes et estimées invariantes par les homothéties.

[4] Pour une valeur explicite de $\eta(n, \alpha, c, \varepsilon, \kappa_0, R)$, voir l'énoncé du Théorème 1.2.1. Ici aussi l'introduction de la constante κ_0 est nécessaire pour fixer l'échelle et rendre les constantes et estimées invariantes par les homothéties.

THEOREM 0.2. *Soit (X, g_0) une variété riemannienne compacte, connexe, de dimension n, dont la courbure sectionnelle vérifie $|\sigma_0| \le \kappa_0^2$. Soit (Y, g) une variété riemannienne compacte connexe de dimension n. Supposons que $d_{GH}((Y,g), (X,g_0)) < \varepsilon$ où*

$$\kappa_0\,\varepsilon < C(n)^{-4}\,\mathrm{Min}\left([\kappa_0\ \mathrm{inj}(X,g_0)]^4\,,\,1\right) \quad \text{où} \quad C(n) = (n+1)^8\,2^{8n}\ .$$

Il existe alors une application $H_\varepsilon : (Y^n, g) \to (X^n, g_0)$, de classe C^1, qui possède les propriétés suivantes:

(i) *H_ε est une* $\left(10 \cdot C(n)\ \dfrac{(\kappa_0\,\varepsilon)^{\frac{3}{4}}}{\kappa_0}\right)$*-approximation de Gromov-Hausdorff;*

(ii) *en tout point $y \in Y$ l'énergie ponctuelle de cette application vérifie :*

$$e_y(H_\varepsilon) \le n\,\left(1 + 20\,(\kappa_0\,\varepsilon)^{\frac{1}{4}}\right)\ ;$$

(iii) *en tout point $y \in Y$ le déterminant Jacobien de cette application vérifie :*

$$\mathrm{Jac}(H_\varepsilon)(y) \le \left(1 + 20\,(\kappa_0\,\varepsilon)^{\frac{1}{4}}\right)^{\frac{n}{2}}\ .$$

Si de plus il existe une ε-approximation de Gromov-Hausdorff continue $h : (Y,g) \to (X,g_0)$, alors H_ε est homotope à h.

Pour obtenir le Théorème 0.2 à partir du Théorème 0.1, il faut utiliser la théorie des τ-revêtements (voir G. Reviron [**Rev**], où il est expliqué comment on peut relever une approximation de Gromov-Hausdorff entre deux variétés riemanniennes compactes en une “bonne” quasi-isométrie entre leurs revêtements universels riemanniens). Nous introduirons rapidement des éléments de cette théorie dans la section 1.4.2.

Le Chapitre 2 est consacré au problème de la rigidité par conjugaison du flot géodésique. Nous nous intéresserons au problème suivant, posé à l'origine par E. Hopf : quelles sont les variétés riemanniennes compactes qui sont déterminées (à isométries près) par leur flot géodésique ? Pour être plus précis, en faisant la convention qu'un difféomorphisme C^0 est un homéomorphisme, introduisons la:

DEFINITION. Une variété riemannienne compacte (X, g_0) est dite C^k-rigide par conjugaison si et seulement si toute variété riemannienne dont le flot géodésique est conjugué à celui de (X, g_0) (par un difféomorphisme C^k) est isométrique à (X, g_0).

Nous pouvons maintenant reformuler le problème posé par E. Hopf de la manière suivante:

QUESTION. Quelles sont les variétés riemanniennes compactes qui sont C^1-rigides par conjugaison ? Quelles sont celles qui sont C^0-rigides par conjugaison ?

Très tôt, des contre-exemples à cette conjecture ont été trouvés : en effet A. Weinsten a remarqué que les surfaces de Zoll sont C^∞-conjuguées à la sphère canonique bien qu'elles ne lui soient pas isométriques (voir [**Besse**], §4.F).

Il s'est avéré plus difficile de trouver des réponses positives à la Question , nous tentons de donner ici un panorama de ces réponses positives :

- dans [**Besse**], Appendix D, Marcel Berger a donné le premier exemple de variété C^0-rigide par conjugaison : le projectif réel $\mathbb{R}P^n$ ($n \ge 2$) muni de sa métrique canonique.
- C. Croke et J. P. Otal ([**Cr1**], [**Ot**]) ont démontré indépendamment la C^1-rigidité par conjugaison des surfaces de courbure négative. Leur résultat a été ensuite étendu à la C^0-rigidité (par conjugaison) par C. Croke, A. Fathi et J. Feldman ([**CFF**]).
- En 1994 C. Croke et B. Kleiner ont montré la C^1-rigidité par conjugaison de toute variété riemannienne qui admet un champ de vecteurs parallèle (voir [**Cr-Kl**]); ce qui implique la C^1-rigidité de toute variété plate.

- En 1995, G. Besson, G. Courtois et S. Gallot ont démontré la C^1-rigidité par conjugaison des variétés localement symétriques de courbure strictement négative et de dimension $n \geq 3$, comme conséquence de la résolution de la conjecture de l'Entropie Minimale. Leur résultat a été ensuite repris par U. Hamenstädt qui a prouvé la C^0-rigidité (par conjugaison) des variétés localement symétriques de courbure strictement négative et de dimension $n \geq 3$ dans l'ensemble des variétés de courbure sectionnelle négative, comme corollaire de l'invariance du Volume par conjugaison C^0.
- Plus récemment P. Eberlein ([**Eb**]) et d'autres mathématiciens ont commencé à s'intéresser au cas des nilvariétés[5]. En particulier C. Gordon et Y. Mao ([**Go-Ma**]) ont prouvé la C^0-rigidité par conjugaison (dans l'ensemble des nilvariétés compactes) de certaines familles de nilvariétés compactes d'ordre de nilpotence égal à 2.

Les preuves de ces résultats sont très différentes les unes des autres; en particulier, le résultat de rigidité par conjugaison de Besson Courtois et Gallot est basé sur la preuve de la conjecture de l'Entropie Minimale et il n'est pas évident de voir si leur argument peut être généralisé en une stratégie permettant de résoudre d'autres problèmes de rigidité dynamique. L'existence d'une telle stratégie serait très intéressante au vu des récents développements de la méthode du barycentre.
Je travaille actuellement dans cette direction, et montre déjà dans cette thèse qu'une généralisation de cette méthode (en modifiant la notion de barycentre utilisée) donne une preuve de la C^1-rigidité par conjugaison des variétés plates (Chapitre 2, sections 2.3, 2.4):

THEOREM 2.3.1. *Soit (X, g_0) une variété riemannienne compacte, plate de dimension $n \geq 3$. Alors (X, g_0) est C^1-rigide par conjugaison des flots.*

La méthode est constructive : en effet, à l'aide de la méthode du barycentre (modifié), on construit l'isométrie comme limite d'une suite d'applications $H_c^R : Y \to X$ (construites comme dans le Théorème 0.1) en partant d'une quasi-isométrie $\tilde{f} : (\tilde{Y}, \tilde{g}) \to (\tilde{X}, \tilde{g}_0)$. L'existence de cette quasi-isométrie et ses propriétés seront établies dans la proposition suivante :

PROPOSITION 2.2.2. *Soient (Y, g) et (X, g_0) deux variétés connexes, compactes de même dimension $n \geq 3$, dont les flots géodésiques sont C^0-conjugués. On note λ l'isomorphisme entre les groupes fondamentaux de Y et de X induit par la conjugaison des flots géodésiques*[6].

(i) *Il existe une constante $C > 0$ et une $(1, C)$-quasi isométrie $\tilde{f} : (\tilde{Y}, \tilde{g}) \to (\tilde{X}, \tilde{g}_0)$.*

(ii) *Si de plus (X, g_0) est un $K(\pi, 1)$, il existe une constante $C > 0$ et une $(1, C)$-quasi isométrie $\tilde{f} : (\tilde{Y}, \tilde{g}) \to (\tilde{X}, \tilde{g}_0)$, continue et λ-équivariante (dont le $(1, C)$-quasi inverse sera noté $\tilde{h} : (\tilde{X}, \tilde{g}_0) \to (\tilde{Y}, \tilde{g})$), telles que les applications f et h (induites par passage au quotient) soient des équivalences d'homotopie entre les variétés de base Y et X et telles que les morphismes induits f_* et h_* entre les groupes fondamentaux de Y et de X coïncident respectivement avec les isomorphismes λ et λ^{-1}.*

Ce dernier résultat donne par ailleurs une preuve quasi-immédiate de la rigidité C^0 (par conjugaison) des variétés plates dans l'ensemble des variétés de courbure de Ricci positive ou nulle (nous donnons ici deux versions de cette preuve).

[5]Considérons les groupes de Lie nilpotents, munis chacun d'une métrique riemannienne invariante à gauche, une *nilvariété* est un quotient (riemannien) d'un tel groupe par l'action d'un réseau.

[6]Si $\varphi : UY \to UX$ est la C^0-conjugaison, et si $\pi_X : UX \to X$ et $\pi_Y : UY \to Y$ sont les projections canoniques, alors $(\pi_Y)_*$ est un isomorphisme entre $\pi_1(UY, v)$ et $\pi_1(Y, \pi_Y(v))$ et on définit alors λ comme égal à $(\pi_X)_* \circ \varphi_* \circ ((\pi_Y)_*)^{-1}$.

Le Chapitre 3 traite de la comparaison entre spectres de deux variétés. Pour comprendre les résultats de ce chapitre, rappelons que, dans le cas d'une variété riemannienne compacte, connexe, sans bord, quelconque (Y, g) , on appelle "*Spectre de cette variété*" la suite $\{\lambda_i(Y, g)\}_{i\in\mathbb{N}}$ des valeurs propres du laplacien de (Y, g) , classées dans l'ordre croissant et répétées autant de fois que l'exige leur multiplicité. Ce spectre étant discret, on a donc

$$0 = \lambda_0(Y, g) < \lambda_1(Y, g) \leq \lambda_2(Y, g) \leq \ldots \leq \lambda_i(Y, g) \leq \ldots$$

e $\lambda_i(Y, g) \to +\infty$ quand $i \to +\infty$.

Énoncé de manière naïve, le problème qui nous intéresse dans ce Chapitre 3 est le suivant:

QUESTION NAÏVE : *Que peut-on dire du spectre d'une variété qui se trouve à distance de Gromov-Hausdorff bornée d'une autre variété compacte fixée ?*

Il existe de nombreuses estimées (asymptotiquement optimales à une constante universelle multiplicative près) de chacune des valeurs propres du Laplacien d'une variété riemannienne, estimées qui sont universelles sur l'ensemble des variétés riemanniennes de courbure de Ricci minorée et de diamètre majoré (voir par exemple [**Cheng**], [**Gro6**], [**Li-Yau**], [**Ga1**], [**Ga2**], [**Ga3**]). Les résultats établissant la convergence des spectres quand on prend des limites de Gromov-Hausdorff sont plus rares ([**Dod-Pat**], [**Ch-Co**]); le résultat le plus achevé sur ce sujet est celui de J. Cheeger et T. Colding ([**Ch-Co**]), qui résout notre question naïve initiale en faisant deux hypothèses supplémentaires :

- la distance de Gromov-Hausdorff est supposée tendre vers zéro (et non plus bornée), i. e. on considère (quand $k \to +\infty$) les spectres d'une suite de variétés riemanniennes (Y_k, g_k) quand cette suite de variétés converge (au sens de la distance de Gromov-Hausdorff) vers une variété riemannienne (X, g_0) de même dimension, et on compare cette suite de spectres avec le spectre de la variété-limite,
- on suppose qu'il existe un minorant uniforme (i. e. indépendant de n) de la courbure de Ricci des (Y_k, g_k) ;

La conclusion est alors que chacune des valeurs propres $\lambda_i(Y_k, g_k)$ du spectre de (Y_k, g_k) converge (quand $k \to +\infty$) vers la valeur propre correspondante de (X, g_0) .

Le résultat que nous démontrerons dans cette thèse a des objectifs différents et plus proches de notre question naïve initiale : en effet, si (X, g_0) désigne une variété compacte fixée et si (Y, g) désigne une variété quelconque qui se trouve à distance de Gromov-Hausdorff bornée de (X, g_0) :

- la distance de Gromov-Hausdorff entre (Y, g) et (X, g_0) n'est plus supposée tendre vers zéro, mais seulement bornée par une constante ε donnée, la seule restriction étant que cette constante ε doit être inférieure à une constante universelle ε_0 (calculable) pour que le résultat soit non trivial.
- aucune hypothèse n'est faite sur la courbure de (Y, g) ,
- le résultat n'est pas un résultat de convergence, mais un résultat d'approximation, où l'on donne une majoration de l'erreur d'approximation.

Nos hypothèses étant beaucoup plus faibles, elles ne nous permettent pas de redémontrer le résultat de J. Cheeger et T. Colding ([**Ch-Co**]); en particulier nous démontrons qu'il est impossible, sous nos hypothèses, d'obtenir une minoration du spectre de (Y, g) (voir l'Exemple 3.4.3 et la Proposition 3.4.4). Une autre démonstration de la faiblesse de nos hypothèses est qu'elles n'imposent quasiment aucune restriction topologique à la variété Y : on peut par exemple, pour toute donnée de la variété (X, g_0) , trouver une suite (Y_k, g_k) de variétés riemanniennes qui convergent (au sens de la distance de Gromov-Hausdorff) vers (X, g_0) , qui vérifient nos hypothèses, et telles que les Y_k soient deux à deux non homotopes (voir l'Exemple 3.4.1 et la Proposition 3.4.2) alors que, si une telle suite de variétés riemanniennes vérifiait les hypothèses du Théorème de J. Cheeger et T. Colding, cela impliquerait que tous les Y_k , sauf un nombre fini, sont difféomorphes à X . Plus précisément, voici l'énoncé de notre résultat de comparaison des spectres:

THEOREM 3.1.2. *Soit* (X^n, g_0) *une variété riemannienne compacte, connexe satisfaisant les conditions:*

$$\mathrm{Diam}(X, g_0) \leq D \ , \quad \mathrm{inj}(X, g_0) \geq i_0 \ , \quad |\sigma_0| \leq \kappa^2,$$

où D, i_0, κ *sont des constantes positives arbitraires.*
Soit (Y^n, g) *une variété riemannienne compacte, connexe telle qu'il existe une* ε*-approximation de Gromov-Hausdorff continue* $f : (Y, g) \to (X, g_0)$ *de degré absolu différent de zéro, où*

$$\varepsilon < \varepsilon_1(n, i_0, \kappa) = \frac{1}{\kappa} \cdot \min \left\{ \left[\frac{\min\{ 1 \ ; \ \kappa \ \mathrm{inj}(X, g_0)\}}{2^{8n} \, (n+1)^8} \right]^4 \ ; \ \left(\frac{\left(\frac{10}{9}\right)^{\frac{2}{n}} - 1}{20(n+1)} \right)^4 \right\} \quad ;$$

on suppose que le volume de (Y, g) *vérifie*

$$[1 - 10\, n\, (n+1)\, (\kappa\varepsilon)^{\frac{1}{4}}] \cdot \mathrm{Vol}_g(Y) < \mathrm{Vol}_{g_0}(X)$$

alors, pour tout $i \in \mathbb{N}$, *on a*

$$(1) \quad \lambda_i(Y, g) \leq \left(1 + C_1(n)(\kappa\varepsilon)^{\frac{1}{16}} \right) \cdot \left(1 + C_2(n, \, \kappa \, D, \, D^2 \cdot \lambda_i(X, g_0))(\kappa\varepsilon)^{\frac{1}{8}} \right) \cdot \lambda_i(X, g_0) \quad ,$$

où

$$C_1(n) = 14(n-1)\sqrt{n+1} \quad ,$$

$$C_2(n, \, \alpha, \Lambda) = 4(n+1) \, \left[(2n+1) \, e^n \, [1 + B(\alpha) \sqrt{\Lambda + (n-1)\alpha^2}]^n + 2 \right] \quad ,$$

où $B(\alpha)$ *est la constante isopérimétrique définie dans la Proposition 3.2.4. On remarquera que le membre de droite de l'inégalité (1) tend vers* $\lambda_i(X, g_0)$ *quand* $\varepsilon \to 0_+$.

Le point faible de ce résultat est évidemment l'hypothèse "Volume de (Y, g) presque inférieur au Volume de (X, g_0)". Remarquons cependant que cette hypothèse est également présente dans le résultat de J. Cheeger et T. Colding cité ci-dessus, puisque leurs hypothèses assurent que la suite des volumes $(\mathrm{Vol}(Y_k, g_k))_{k\in\mathbb{N}}$ converge vers le volume $\mathrm{Vol}(X, g_0)$ de la variété-limite.

La seconde partie de cette thèse (constituée du seul chapitre 4) est consacrée à l'étude d'un problème classique : le Lemme de Margulis.
Pour comprendre ce problème, rappelons que la "*systole*" (ou "*systole globale*") $\mathrm{sys}(X, g)$ d'une variété riemannienne quelconque (X, g) est définie comme l'infimum de la longueur de toutes les courbes fermées non homotopes à zéro sur X ; introduisons la notion de "*systole ponctuelle*" $\mathrm{syst}(x)$ en un point $x \in X$, définie comme l'infimum de la longueur des lacets non homotopes à zéro de point-base x : la systole (globale) est donc l'infimum (quand x parcourt X) de la systole ponctuelle $\mathrm{syst}(x)$. En revanche, la "*diastole*" $\mathrm{dias}(X, g)$ de (X, g) sera définie comme le supremum (quand x parcourt X) de la systole ponctuelle $\mathrm{syst}(x)$. Le lemme de Margulis classique minore la diastole de (X, g) en fonction de sa dimension et de bornes supérieure et inférieure de sa courbure sectionnelle; une version ultérieure (due à M. Gromov) minore la systole de (X, g) en fonction de sa dimension, de bornes supérieure et inférieure de sa courbure sectionnelle et d'une borne supérieure de son diamètre.
Après avoir présenté rapidement le Lemme de Margulis classique, nous expliquerons la généralisation proposée par G. Besson, G. Courtois et S. Gallot dans leur prépublication datée de 2003 : *Un lemme de Margulis sans courbure et ses applications* ([**BCG3**]).
Nous présenterons ensuite la preuve d'un nouveau Lemme de Margulis, sans hypothèse de courbure, qui s'applique à toutes les variétés connexes, de dimension $n \geq 3$, dont l'entropie volumique est bornée supérieurement et dont le groupe fondamental est isomorphe à un produit libre et est supposé sans torsion d'ordre 2 :

THEOREM 4.1.1. *Soit* H *un nombre réel positif quelconque et soit* (X, g) *une variété riemannienne connexe quelconque telle que* $\mathrm{Ent}(X, g) \leq H$, *dont le groupe fondamental est un produit libre sans élément de torsion d'ordre* 2 . *Alors*

$$\mathrm{dias}(X, g) \geq \frac{\log(3)}{6H} \quad .$$

Nous donnerons ensuite la borne inférieure universelle suivante de la systole :

THEOREM 4.1.2. *Soient* H, D *deux nombres réels positifs quelconques et soit* (X, g) *une variété riemannienne compacte connexe quelconque telle que* $\mathrm{Ent}(X, g) \leq H$ *et* $\mathrm{Diam}(X, g) \leq D$*, dont le groupe fondamental est un produit libre sans torsion. Alors*

$$\mathrm{sys}(X, g) \geq \frac{1}{H} \cdot \log\left(1 + \frac{4}{e^{2DH} - 1}\right) \quad .$$

À la différence des résultats de G. Margulis et de M. Gromov cités ci-dessus, il n'est fait ici (comme c'est aussi le cas dans [**BCG3**]) aucune hypothèse sur la courbure de (X, g). Le résultat de G. Besson, G. Courtois et S. Gallot [**BCG3**] peut être réinterprété comme un résultat de transport de propriétés, i. e. si une variété (X_0, g_0) vérifie les hypothèses du théorème de Margulis (ou de Gromov) alors, sur toute variété X dont le groupe fondamental est isomorphe à un sous-groupe non cyclique du groupe fondamental de X_0, toute métrique g sur X (d'entropie volumique majorée) hérite de propriétés "à la Margulis", i. e. la diastole de (X, g) est minorée et la systole de (X, g) est minorée si son diamètre est majoré.
À la différence des résultats de [**BCG3**], les propriétés "à la Margulis" démontrées ici dans les deux théorèmes ci-dessus découlent directement des propriétés algébriques du groupe fondamental et non de l'existence d'une autre action (par isométries) de ce groupe sur une autre variété de courbure sectionnelle $\sigma \leq -1$, dont le quotient est de rayon d'injectivité minoré.

En suivant la même démarche que G. Besson, G. Courtois et S. Gallot dans [**BCG3**], citons deux applications du Théorème 4.1.2:

PRECOMPACTNESS THEOREM. *Soient* H, D, V, l *quatre nombres réels positifs quelconques. Notons* $\mathfrak{M}_n^{dec}(D, V, H; l)$ *l'ensemble des variétés riemanniennes compactes, connexes* (X, g) *(modulo isométries) de dimension* n*, dont le diamètre, le volume et l'entropie volumique sont plus petits de* D*,* V *et* H *respectivement, dont le groupe fondamental est isomorphe à un produit libre sans torsion, et telles que la longueur du plus court lacet géodésique du revêtement universel* $(\tilde{X}, \tilde{g})$ *de* (X, g) *soit plus grande que* l.
Cet ensemble est précompact quand il est muni de la topologie associée à la distance de Gromov-Hausdorff.
De plus $\mathfrak{M}_n^{dec}(D, V, H; l)$ *contient un nombre fini de:*
(i) *types d'homotopie, pour tout* $n \in \mathbb{N}$,
(ii) *types topologiques (à homéomorphismes près) lorsque* $n = 4$,
(iii) *types différentiables (à difféomorphismes près) lorsque* $n \geq 5$.

VOLUME ESTIMATE. *Soient* H, D *deux nombres réels positifs quelconques; pour toute variété riemannienne compacte connexe* (X, g)*, 1-essentielle, dont le groupe fondamental est un produit libre sans torsion, et dont l'entropie volumique et le diamètre sont majorés respectivement par* H *et* D*, on a :*

$$\mathrm{Vol}(X, g) \geq \frac{C_n}{H^n} \cdot \log\left(1 + \frac{4}{e^{2DH} - 1}\right)^n$$

où $C_n > 0$ *est une constante universelle qui ne dépend que de la dimension* n *(voir* [**Gro5**]*, Theorem 0.1.A pour une valeur explicite -non optimale- de la constante* C_n *).*

Presentazione

Questa Tesi di Dottorato è divisa in due parti. Nella prima parte ci si interessa ad una nuova versione del metodo del baricentro, una tecnica originariamente introdotta da G. Besson, G. Courtois e S. Gallot, e ad alcune nuove applicazioni di questo metodo ai problemi di rigidità dinamica e al paragone tra gli spettri del Laplaciano di due varietà Riemanniane. Nella seconda parte invece affronteremo un tema classico della Geometria Riemanniana: il Lemma di Margulis, del quale forniamo una versione senza curvatura che si basa unicamente sulle proprietà algebriche del gruppo che agisce.

La prima parte è divisa in tre capitoli. Descriverò brevemente il loro contenuto.

Nel Capitolo 1 presento il metodo del baricentro di G. Besson, G. Courtois e S. Gallot. Questo metodo fu originariamente introdotto dai suoi autori per risolvere la congettura dell'Entropia Minimale (vedere [**BCG1**]). Il metodo del baricentro è una tecnica che permette, a partire dal dato iniziale di una mappa (possibilmente non continua, possibilmente definita unicamente sull'orbita di un gruppo discreto) di costruire una applicazione regolare il cui determinante Jacobiano è maggiorato in modo ottimale.
Un esempio di applicazione di questo metodo (sviluppato da G. Besson, G. Courtois et S. Gallot) è il seguente:
Sia (Y, g) una varietà Riemanniana, compatta, connessa, orientata di dimensione n che ammette una applicazione continua f a valori in uno spazio localmente simmetrico (X, g_0) di curvatura strettamente negativa della stessa dimensione di Y, allora il metodo produce una famiglia di applicazioni $\tilde{F}_c : (\tilde{Y}, \tilde{g}) \to (\tilde{X}, \tilde{g}_0)$ di classe C^∞, dipendenti da un parametro c (che bisogna scegliere strettamente maggiore dell'entropia volumica di (Y, g)), che sono equivarianti rispetto all'omomorfismo tra i gruppi fondamentali indotto dalla mappa f di partenza, e il cui determinante Jacobiano è stimato ottimalmente secondo la disuguaglianza:

$$|\operatorname{Jac}(\tilde{F}_c)| \leq (1 + \varepsilon(c)) \cdot \left(\frac{\operatorname{Ent}_{\mathrm{vol}}(Y, g)}{\operatorname{Ent}_{\mathrm{vol}}(X, g_0)} \right)^n$$

dove $\operatorname{Ent}_{\mathrm{vol}}(Y, g)$ (risp. $\operatorname{Ent}_{\mathrm{vol}}(X, g_0)$) denota l'entropia volumica di (Y, g) (risp. (X, g_0)) e dove $\varepsilon \to 0$ quando $c \to \operatorname{Ent}_{\mathrm{vol}}(Y, g)$. Per equivarianza le mappe $\tilde{F}_c$ inducono delle applicazioni F_c tra i quozienti Y ed X, tutte omotope all'applicazione f di partenza e che sono quasi contraenti.

Nel caso generale (vedere [**BCG1**], [**BCG2**] e [**BCG6**]) si può ugualmente definire una famiglia di applicazioni $\tilde{F}_c : \tilde{Y} \to \tilde{X}$, ciascuna costruita come segue:

- si costruisce una applicazione $\tilde{y} \to \nu^c_{\tilde{y}}$ da $\tilde{Y}$ nello spazio delle misure di probabilità su $\tilde{Y}$, scelta in modo tale che la derivata logaritmica di $\nu^c_{\tilde{y}}$ rispetto a $\tilde{y}$ sia data da c e tale che $\nu^c_{\gamma \tilde{y}} = \gamma_* \nu^c_{\tilde{y}}$ per ogni $\gamma \in \pi_1(Y)$ (qui $\pi_1(Y)$ è visto come l'insieme delle trasformazioni di rivestimento corrispondente al rivestimento universale),

- si trasporta ciascuna di queste misure $\nu_{\tilde{y}}^c$ in una misura[7] $\mu_{\tilde{y}}^c$ definita su $\tilde{X}$ (o sul suo bordo ideale $\partial\tilde{X}$),
- si definisce $\tilde{F}_c(\tilde{y})$ come il baricentro della misura $\mu_{\tilde{y}}^c$.

Ci sono molte scelte possibili per la definizione del baricentro sullo spazio $(\tilde{X}, \tilde{g}_0)$, ciascuna di queste scelte corrisponde a delle applicazioni differenti del metodo: G. Besson, G. Courtois e S. Gallot ([**BCG1**], [**BCG2**]) hanno definito il baricentro di una misura μ a supporto sul bordo ideale $\partial\tilde{X}$ come l'unico punto dove la funzione $\tilde{x} \to \int_{\partial\tilde{X}} B(\tilde{x}, \theta)\, d\mu(\theta)$ raggiunge il suo minimo (dove B è la funzione di Busemann di $\tilde{X}$) e, in un lavoro più recente ([**BCG5**], [**BCG6**]), definiscono il baricentro di una misura μ a supporto su $\tilde{X}$ (quando la curvatura sezionale di $\tilde{X}$ è inferiore o uguale a $-K^2$)come l'unico punto di minimo della funzione $\tilde{x} \to \int_{\tilde{X}} \cosh(d_{\tilde{X}}(\tilde{x}, \tilde{z}))\, d\mu(\tilde{z})$ (ovviamente per una misura μ per la quale questa funzione sia finita).
Sempre nel caso in cui la curvatura sezionale $\tilde{X}$ è inferiore o uguale a $-K^2$, A. Sambusetti ([**Samb2**]) ha definito il baricentro di una misura μ a supporto su $\tilde{X}$ come l'unico punto di minimo della funzione $\tilde{x} \to \int_{\tilde{X}} d_{\tilde{X}}(\tilde{x}, \tilde{z})^2\, d\mu(\tilde{z})$ (per una misura μ per la quale questa funzione sia finita).
Questi autori hanno definito il baricentro nel caso in cui la curvatura sezionale di $(\tilde{X}, \tilde{g}_0)$ sia strettamente negativa ([**BCG1**], [**BCG2**], [**BCG5**], [**BCG6**] e[**Samb2**]) o nel caso in cui la varietà $(\tilde{X}, \tilde{g}_0)$ sia un prodotto di spazi simmetrici, ciascuno di curvatura strettamente negativa ([**BCG4**]).

L. Sabatini ([**Saba1**], [**Saba2**]) ha successivamente osservato che la nozione di baricentro introdotta da A. Sambusetti può essere generalizzata al caso di curvatura negativa o nulla (nozione che, nel caso piatto, restituisce la nozione classica di baricentro nello spazio euclideo); successivamente ha mostrato che questa nozione si generalizza al caso di curvatura limitata senza ipotesi di segno, qualora il supporto di ciascuna delle misure $\mu_{\tilde{y}}^c$ sia di diametro sufficientemente piccolo (vedere [**Saba1**] e il Teorema 1.2.1 di questa tesi). Nel caso dove il dato iniziale sia una Gromov-Hausdorff ε-approssimazione $f : (Y, g) \to (X, g_0)$ (e quando ε è sufficientemente piccolo), egli fornisce una nuova maniera (più diretta) di definire l'applicazione $\tilde{y} \to \mu_{\tilde{y}}^c$ di $\tilde{Y}$ nello spazio delle misure di probabilità su $\tilde{X}$ in modo tale che $\mu_{\tilde{y}}^c$ sia di densità controllata rispetto alla misura Riemanniana $dv_{\tilde{g}_0}$. Questo gli permette di confrontare in modo ottimale i volumi delle due varietà (Y, g) e (X, g_0) senza fare ipotesi sulla curvatura della varietà (Y, g) .

In questa tesi utilizzeremo una variante del metodo di L. Sabatini per ottenere due obiettivi che sembrano a priori fuori dalla portata del suo metodo:

- stabilire un risultato del tipo "coniugio del flusso $\Rightarrow$ isometria" (questo risultato richiede una versione del metodo di Sabatini che sia valida per le $(1, C)$-quasi isometrie, quando C non è piccolo),
- paragonare gli spettri dei Laplaciani di (Y, g) ed (X, g_0) senza fare ipotesi sulla curvatura di (Y, g) , cosa che esige uno studio preciso della natura delle applicazioni F_c , costruite come sopra, nel caso di quasi-uguaglianza tra i volumi di (Y, g) ed (X, g_0) , così come delle stime delle norme L^∞ delle autofunzioni del Laplaciano di (X, g_0) , che trasportiamo su (Y, g) , per composizione con le mappe F_c .

[7]Nel caso in cui il dato iniziale sia una applicazione $f : Y \to X$, si pone $\nu_{\tilde{y}}^c = e^{-c\, d_{\tilde{Y}}(\tilde{y}, \tilde{z})}\, dv_{\tilde{g}}(\tilde{z})$ e si definisce $\mu_{\tilde{y}}^c$ come il trasportato di questa misura attraverso l'applicazione $\tilde{f} : \tilde{Y} \to \tilde{X}$, sollevamento della mappa f . Nel caso in cui il dato iniziale sia una rappresentazione $\rho : \Gamma = \pi_1(Y) \to \pi_1(X)$ si procede in modo analogo utilizzando come misure $\nu_{\tilde{y}}^c$ una combinazione lineare di misure di Dirac $\delta_{\gamma\, \tilde{y}_0}$ a supporto su un'orbita $\Gamma\, \tilde{y}_0$, che vengono trasportate (tramite la rappresentazione ρ) su combinazioni lineari $\mu_{\tilde{y}}^c$ di misure di Dirac $\delta_{\rho(\gamma)\, \tilde{x}_0}$, combinazioni supportate su un'orbita $\rho(\Gamma)\, \tilde{x}_0 \subset \tilde{X}$; più precisamente si può semplicemente definire

$$\nu_{\tilde{y}}^c = \sum_{\gamma \in \Gamma} e^{-c\, d_{\tilde{Y}}(\tilde{y}, \gamma\, \tilde{y}_0)} \quad , \quad \mu_{\tilde{y}}^c = \sum_{\gamma \in \Gamma} e^{-c\, d_{\tilde{X}}(\tilde{x}, \rho(\gamma)\, \tilde{x}_0)}$$

Si definisce *l'energia al punto* $y \in Y$ *di una applicazione* $k : (Y,g) \to (X,g_0)$, di classe C^1, come il numero $e_y(k) = \sum_{i=1}^n \| d_yk(e_i) \|_{g_0}^2$, dove $\{e_i\}$ è una qualsiasi base g-ortonormale di T_yY. La nostra personale versione del metodo di L. Sabatini può essere enunciata come segue:

THEOREM 0.1. (Versione semplificata, per una versione completa vedere il Teorema 1.2.1) *Siano* (X,g_0) *e* (Y,g) *due varietà Riemanniane connesse, complete, di dimensione* n*, tali che* X *sia semplicemente connessa; denotiamo* ρ *e* d *le distanze Riemanniane su* X *e su* Y *associate rispettivamente alle metriche* g_0 *e* g*. Supponiamo* [8] *che* $\mathrm{inj}(X,g_0) > 0$ *e che* $\mathrm{inj}(Y,g) > 0$*. Denotiamo rispettivamente* σ *e* σ_0 *le curvature sezionali di* (Y,g) *e di* (X,g_0)*. Supponiamo che* $|\sigma_0| \le \kappa_0^2$ *e che* $|\sigma|$ *sia limitata.*
Siano Γ_Y *(risp.* Γ_X *) un sottogruppo discreto del gruppo di isometrie di* (Y,g) *(risp. di* (X,g_0) *), che agiscono in modo libero e propriamente discontinuo su* Y *(risp. sur* X *).*
Supponiamo che esista un isomorfismo $\lambda : \Gamma_Y \to \Gamma_X$ *e due applicazioni* $f : X \to Y$ *e* $h : Y \to X$*, equivarianti rispetto a* λ^{-1} *e* λ *rispettivamente, che verificano, per ogni* $x \in X$ *e per ogni* $y \in Y$*:*

$$\rho(h(y),x) \le \alpha \cdot d(y,f(x)) + \varepsilon \ , \quad d(y,f(x)) \le \alpha \cdot \rho(h(y),x) + \varepsilon \ ,$$

$$\rho(x,(h\circ f)(x)) \le \varepsilon, \quad d(y,(f\circ h)(y)) \le \varepsilon \ .$$

per delle constanti $\alpha \ge 1$ *e* ε *date.*
Per ogni assegnazione di costanti strettamente positive ε*,* c*,* R *e* α *tali che*

$$\alpha \ge 1 \quad , \quad R < \frac{1}{7\,\alpha} \cdot \min\left\{\mathrm{inj}(X,g_0), \frac{\pi}{2\,\kappa_0}\right\} \quad , \quad \varepsilon < \frac{R}{5} \ ,$$

esiste una applicazione λ*-equivariante* $H_c^R : Y \to X$*, di classe* C^1*, tale che*

$$\forall y \in Y \qquad \rho(H_c^R(y), h(y)) \le A(n,\alpha,c,\varepsilon,\kappa_0,R) \cdot \left(\frac{\varepsilon}{\kappa_0}\right)^{\frac{1}{2}} \ ,$$

(dove la costante universale $A(n,\alpha,c,\varepsilon,\kappa_0,R)$ *è una funzione esplicita*[9] *di* $(n,\alpha,c,\varepsilon,\kappa_0,R)$*, che è limitata superiormente quando esistono tre costanti* B_1*,* B_2*,* B_3 *tali che* $c\,\varepsilon \le B_1$*,* $B_2\,\kappa_0^{\frac{1}{2}} \le c\,\varepsilon^{\frac{1}{2}}$ *e* $\alpha \le B_3$*) e la cui energia verifica, in ogni punto* $y \in Y$*,*

$$e_y(H_c^R) \le n \cdot (1 + \eta(n,\alpha,c,\varepsilon,\kappa_0,R)) \ ,$$

dove $\eta(n,\alpha,c,\varepsilon,\kappa_0,R)$ *è una funzione esplicita*[10] *di* $(n,\alpha,c,\varepsilon,\kappa_0,R)$*, che tende a zero quando esistono due costanti positive* B'_1 *e* B'_2 *tali che* $c\,R \ge B'_1$ *e* $c\,\varepsilon^{\frac{9}{10}} \le B'_2\kappa_0^{\frac{1}{10}}$ *e* $\kappa_0\,\varepsilon$*,* $\alpha - 1$ *e* $\kappa_0\,R$ *tendono simultaneamente a zero.*
Inoltre se l'applicazione h *di partenza è continua, allora* H_c^R *è omotopa a* h *attraverso una omotopia* λ*-equivariante.*

È importante, nelle applicazioni (affinché queste siano ottimali), che il maggiorante dell'energia puntuale sia prossima ad n come nel caso del teorema precedente, perché tale valore è precisamente l'energia di una isometria.
La dimostrazione di questo teorema è data nel Capitolo 1, sezioni 1.2, 1.3.

Nella sezione 1.4 mostriamo che questo teorema restituisce il risultato di L. Sabatini ([**Saba1**], Teorema 4.6.1) . È questo corollairo che utilizzeremo nel Capitolo 3 per paragonare gli spettri di due varietà Riemanniane:

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[8] L'ipotesi $\mathrm{inj}(X,g_0) > 0$ (risp. $\mathrm{inj}(Y,g) > 0$) è automaticamente verificata quando (X,g_0) (risp. (Y,g)) possiede un gruppo cocompatto di isometrie, per esempio quando (X,g_0) (risp. (Y,g)) è il rivestimento universale Riemanniano di una varietà Riemanniana compatta.

[9] Per un valore esplicito di $A(n,\alpha,c,\varepsilon,\kappa_0,R)$, vedere l'enunciato del Teorema 1.2.1. Si osservi che l'introduzione della costante κ_0 complica un po' le espressioni, ma essa è utile per fissare la scala e rendere le costanti e le stime invarianti per omotetie.

[10] Per un valore esplicito di $\eta(n,\alpha,c,\varepsilon,\kappa_0,R)$, vedere l'enunciato del Teorema 1.2.1. Anche in questo caso fissare il parametro κ_0 ha la funzione di fissare la scala e rendere costanti e stime invarianti per omotetie.

THEOREM 0.2. *Sia* (X, g_0) *una varietà Riemanniana compatta, connessa di dimensione n la cui curvatura sezionale verifichi* $|\sigma_0| \leq \kappa_0^2$. *Sia* (Y, g) *una varietà Riemanniana compatta n-dimensionale. Assumiamo che* $d_{GH}((Y, g), (X, g_0)) < \varepsilon$ *dove*

$$\kappa_0\, \varepsilon < C(n)^{-4} \min\{[\kappa_0 \operatorname{inj}(X, g_0)]^4\,,\, 1\} \qquad \textit{dove} \qquad C(n) = (n+1)^8\, 2^{8n}$$

Esiste allora una mappa $H_\varepsilon : (Y^n, g) \to (X^n, g_0)$, *di classe* C^1, *che possiede le seguenti proprietà:*

(i) H_ε *è una* $\left(10\, C(n)\, \frac{(\kappa_0\, \varepsilon)^{\frac{3}{4}}}{\kappa_0}\right)$*-approssimazione di Gromov-Hausdorff;*

(ii) *in ogni punto* $y \in Y$ *l'energia puntuale verifica:*

$$e_y(H_\varepsilon) \leq n \left(1 + 20\, (\kappa_0\, \varepsilon)^{\frac{1}{4}}\right) \;;$$

(iii) *in ogni punto* $y \in Y$ *il determinante Jacobiano di questa applicazione verifica:*

$$\operatorname{Jac}(H_\varepsilon)(y) \leq \left(1 + 20\, (\kappa_0\, \varepsilon)^{\frac{1}{4}}\right)^{\frac{n}{2}} \;.$$

Se inoltre esiste una ε*-approssimazione di Gromov-Hausdorff continua* $h : (Y, g) \to (X, g_0)$, *allora* H_ε *è omotopa ad* h.

Per ottenere il Teorema 0.1 a partire dal Teorema 0.2, è necessario richiede l'utilizzo dei τ-rivestimenti (vedere [**Rev**], dove è spiegato come sia possibile sollevare in una approssimazione di Gromov-Hausdorff tra due varietà ad una "buona" quasi-isometria tra il rivestimento universale di X ed un opportuno rivestimento della varietà Y). Introdurremo rapidamente degli elementi di questa teoria nella sottosezione 1.4.2.

Il Capitolo 2 è dedicato al problema della rigidità per coniugio del flusso geodetico. Ci interesseremo cioè alla seguente questione, originariamente posta da E. Hopf: quali sono le varietà Riemanniane compatte che sono unicamente determinate (a meno di isometrie) dal loro flusso geodetico? Per essere più precisi, utilizzando la convenzione che un diffeomorfismo di classe C^0 è un omeomorfismo, intorduciamo la:

DEFINITION. Un varietà Riemanniana compatta (X, g_0) è detta C^k-rigida per coniugio se e soltanto se ogni varietà Riemanniana compatta (Y, g) il cui flusso geodetico è coniugato a quello di (X, g_0) (attraverso un diffeomorfismo di classe C^k) è isometrica a (X, g_0).

Possiamo ora riformulare il problema posto da E. Hopf nel modo seguente:

QUESTION. *Quali sono le varietà Riemanniane compatte che sono* C^1 *rigide per coniugio? Quali sono quelle che sono* C^0*-rigide per coniugio?*

Molto presto furono trovati dei controesempi a questa congettura: A. Weinstein osservò che le superfici di Zoll sono C^∞-coniugate alla sfera canonica benché non siano isometriche ad essa (vedere [**Besse**], §4.F).

Si è rivelato più difficile invece trovare risposte affermative. Cerchiamo di fornire qui di seguito un panorama di queste risposte positive:

- Marcel Berger in [**Besse**], Appendix D, ha fornito il primo esempio di varietà C^0-rigida: $\mathbb{R}P^n$, dotata della sua metrica standard è C^0-rigida per $n \geq 2$.
- C. Croke e J. P. Otal ([**Cr1**],[**Ot**]) hanno dimostrato indipendentemente la C^1-rigidità delle superfici di curvatura negativa ; successivamente questo risultato è stato esteso alla C^0-rigidità da C. Croke, A. Fathi, J. Feldman in [**CFF**].
- Nel 1994 C. Croke e B. Kleiner hanno dimostrato la C^1-rigidità delle varietà che ammettono un campo di vettori parallelo ([**Cr-Kl**]); questo in particolare dimostra la C^1-rigidità di ogni varietà piatta.

- Nel 1995 G. Besson, G. Courtois e S. Gallot hanno fornito una prova della C^1-rigidità delle varietà localmente simmetriche di curvatura strettamente negativa e di dimensione $n \geq 3$, come conseguenza della risoluzione della congettura dell'Entropia Minimale. Il loro risultato è stato recentemente ripreso da U. Hamenstädt che ha dimostrato la C^0-rigidità (per coniugio) delle varietà localmente simmetriche di curvatura strettamente negativa all'interno della classe delle varietà Riemanniane di curvatura strettamente negativa, come corollario dell'invarianza del volume attraverso il C^0-coniugio.
- Più recentemente P. Eberlein ([**Eb**]) ed altri matematici hanno cominciato ad investigare il caso delle nilvarietà [11].
  In particolare C. Gordon e Y. Mao hanno dimostrato in [**Go-Ma**] la C^0-rigidità (all'interno della classe delle nilvarietà compatte) di determinate famiglie di nilvarietà compatte con ordine di nilpotenza uguale a 2.

Le dimostrazioni di questi vari risultati sono di spirito molto differente l'una dall'altra; in particolare il risultato di rigidità dinamica di Besson Courtois e Gallot si basa sulla dimostrazione della congettura dell'Entropia Minimale e non è facile capire se il loro argomento possa fornire una strategia generale per risolvere i problemi di rigidità dinamica. L'esistenza di un tale strategia sarebbe particolarmente interessante anche alla luce degli ultimi sviluppi del metodo del baricentro.
Lavoro in questa direzione, e fornisco una dimostrazione basata sul metodo del baricentro della C^1-rigidità per coniugio del flusso delle varietà piatte (Capitolo 2, sezione 2.3) :

THEOREM 2.3.1. *Sia (X, g_0) una varietà Riemanniana, compatta, piatta di dimension $n \geq 3$. Allora (X, g_0) è C^1-rigida per coniugio del flusso.*

Il metodo è costruttivo: grazie al metodo del baricentro (modificato), si costruisce l'isometria come limite di una successione di applicazioni $H_c^R : Y \to X$ (costruite come nel Teorema 0.1) e partendo da una quasi isometria $\bar{f} : (\tilde{Y}, \tilde{g}) \to (\tilde{X}, \tilde{g}_0)$. L'esistenza di questa quasi isometria e le sue proprietà sono stabilite nella seguente proposizione:

La dimostrazione si basa essenzialmente sul risultato seguente:

PROPOSITION 2.2.2. *Siano (Y, g) ed (X, g_0) due varietà Riemanniane compatte, connesse di dimensione $n \geq 3$, con flussi geodetici C^0-coniugati. Denotiamo con λ l'isomorfismo tra i gruppi fondamentali di Y e di X indotto dal coniugio dei flussi geodetici*[12].

(i) *Esiste una costante $C > 0$ ed una $(1, C)$-quasi isometria $\tilde{f} : (\tilde{Y}, \tilde{g}) \to (\tilde{X}, \tilde{g}_0)$.*

(ii) *Inoltre, se (X, g_0) è uno spazio $K(\pi, 1)$ esiste una costante $C > 0$ ed una $(1, C)$-quasi isometria continua e λ-equivariante, $\tilde{f} : \tilde{Y} \to \tilde{X}$, $\tilde{h} : \tilde{Y} \to \tilde{X}$ (la cui $(1, C)$-quasi inversa è denotata $\tilde{h} : (\tilde{X}, \tilde{g}_0) \to (\tilde{Y}, \tilde{g})$). Le mappe $\tilde{f}$ e $\tilde{h}$ inducono due equivalenze omotopiche f, h tra le varietà di base Y , X , tali che i morfismi indotti f_* e h_* tra i gruppo fondamentali di Y ed X coincidono rispettivamente con gli isomorfismi λ , λ^{-1} .*

Una applicazione di questo risultato è una dimostrazione quasi immediata della C^0-rigidità per coniugio del flusso delle varietà piatte all'interno della classe delle varietà di curvatura di Ricci positiva (forniremo due versioni di questa dimostrazione).

Nel Capitolo 3 viene invece dimostrato un teorema di paragone per lo spettro del Laplaciano. Per comprendere i risultati di questo capitolo, ricordiamo che, nel caso di una varietà Riemanniana compatta, connessa, senza bordo, (Y, g) , si chiama "*Spettro di questa varietà*" la successione $\{\lambda_i(Y, g)\}_{i \in \mathbb{N}}$ degli autovalori del Laplaciano di (Y, g) , in ordine

[11] Una nilvarietà è il quoziente di un gruppo di Lie nilpotente semplicemente connesso, dotato di una metrica invariante a sinistra, per l'azione di un suo sottogruppo discreto.

[12] Se $\varphi : UY \to UX$ è il C^0-coniugio e se $\pi_X : UX \to X$ e $\pi_Y : UY \to Y$ sono le proiezioni canoniche allora $(\pi_Y)_*$ è un isomorfismo tra $\pi_1(UY, v)$ e $\pi_1(Y, \pi(v))$ e si definisce allora λ come $(\pi_X)_* \circ \varphi_* \circ ((\pi_Y)_+)^{-1}$.

crescente e ripetuti tante volte quante è la loro molteplicità. Tale spettro è discreto e dunque si ha:

$$0 = \lambda_0(Y,g) < \lambda_1(Y,g) \leq \lambda_2(Y,g) \leq \cdots \leq \lambda_i(Y,g) \leq \cdots$$

e $\lambda_i(Y,g) \to +\infty$ quando $i \to \infty$.

DOMANDA INGENUA: *Cosa possiamo dire dello spettro di una varietà che si trova a distanza di Gromov-Hausdorff limitata da un'altra varietà compatta fissata?*

Esistono numerose stime (asintoticamente ottimali a meno di una costante moltiplicativa) per ciascuno degli autovalori del Laplaciano di una varietà Riemanniana, stime che sono universali sull'insieme delle varietà Riemanniane con curvatura di Ricci limitata inferiormente e diametro limitato superiormente (vedere per esempio [**Cheng**], [**Gro6**], [**Li-Yau**], [**Ga1**], [**Ga2**], [**Ga3**]). I risultati che stabiliscono la convergenza dello spettro quando si prendono limiti di Gromov-Hausdorff sono invece più rari (ad esempio [**Dod-Pat**], [**Ch-Co**] e più recentemente [**Aub2**]); il risultato più completo a questo proposito è quello di J. Cheeger e T. Colding ([**Ch-Co**]), che risolve la domanda ingenua dalla quale eravamo partiti facendo due ipotesi supplementari:

- si suppone che la distanza di Gromov-Hausdorff stia tendendo a zero (dunque non più semplicemente limitata), ovvero si considerano per $k \to \infty$ gli spettri di una successione di varietà Riemanniane (Y_k, g_k) quando questa successione converge (rispetto alla distanza di Gromov-Hausdorff) ad una varietà Riemanniana (X, g_0) della stessa dimensione, e si paragona questa successione di spettri con lo spettro della varietà limite.
- si assume che esista un limite inferiore uniforme (cioè non dipendente dall'indice k) per la curvatura delle varietà (Y_k, g_k).

La conclusione allora è che ciascun autovalore $\lambda_i(Y_k, g_k)$ dello spettro di (Y_k, g_k) converge (per $k \to \infty$) verso l'autovalore corrispondente di (X, g_0).
Il risultato che dimostreremo in questa tesi a degli obiettivi differenti, più vicini a quella che era la nostra domanda iniziale: infatti, se (X, g_0) è una varietà compatta fissata e se (Y, g) è una qualunque varietà che si trova a distanza limitata da (X, g_0)

- la distanza di Gromov-Hausdorff tra (Y, g) e (X, g_0) non è più supposta tendere a zero, ma si richiede solamente che sia limitata da una costante ε assegnata, inferiore di una costante universale ε_1 (calcolabile) per la quale il risultato sia non banale.
- nessuna ipotesi viene fatta sulla curvatura di (Y, g)
- il risultato non è un risultato di convergenza ma di approssimazione, dove viene fornita una maggiorazione dell'errore di approssimazione.

Poiché le nostre ipotesi sono molto più deboli, esse non ci permettono di ridimostrare il risultato di J. Cheeger e T. Colding ([**Ch-Co**]); in particolare dimostreremo che è impossibile, sotto le nostre ipotesi, fornire una stima inferiore dello spettro di (Y, g) (vedere Example 3.4.3 e Proposition 3.4.4). Un'ulteriore dimostrazione della debolezza delle nostre ipotesi è che non impongono pressoché alcuna restrizione topologica alla varietà Y: per ogni varietà (X, g_0) fissata è possibile trovare una successione (Y_k, g_k) di varietà Riemanniane che convergono (nel senso della distanza di Gromov-Hausdorff) a (X, g_0), che verificano le nostre ipotesi e tali che le Y_k sono a due a due non omotope (vedere Example 3.4.1 e Proposition 3.4.2), mentre, se una successione di varietà Riemanniane verifica le ipotesi del Teorema di J. Cheeger e T. Colding, questo implica il fatto che tutte le Y_k ad eccezione di un numero finito debbano essere diffeomorfe ad X).
Più precisamente questo è l'enunciato del nostro risultato di paragone dello spettro:

THEOREM 3.1.2. *Sia (X^n, g_0) una varietà compatta, connessa che verifica le ipotesi:*

$$\mathrm{Diam}(X, g_0) \leq D, \quad \mathrm{inj}(X, g_0) \geq i_0, \quad |\sigma_0| \leq \kappa^2,$$

dove D, i_0, κ sono costanti positive arbitrarie.
Sia (Y^n, g) una varietà Riemanniana compatta, connessa che ammette una ε-approssimazione

di Gromov-Hausdorff, $f : (Y, g) \to (X, g_0)$ *, di grado assoluto non nullo, dove*

$$\varepsilon < \varepsilon_1(n, i_0, \kappa) = \frac{1}{\kappa} \min \left\{ \left[\frac{\min\{ 1 \,;\, \kappa \operatorname{inj}(X, g_0)\}}{2^{8n}\,(n+1)^8} \right]^4 ; \left(\frac{\left(\frac{10}{9}\right)^{\frac{2}{n}} - 1}{20} \right)^4 \right\}$$

Se assumiamo la seguente disuguaglianza per il volume di (Y^n, g)

$$[1 - 10\,n\,(\kappa\varepsilon)^{\frac{1}{4}}] \cdot \operatorname{Vol}_g(Y) < \operatorname{Vol}_{g_0}(X) \cdot \operatorname{Adeg}(f)$$

allora, per ogni $i \in \mathbb{N}$ *, abbiamo*

$$\lambda_i(Y, g) \le \left(1 + C_1(n)(\kappa\varepsilon)^{\frac{1}{16}}\right) \cdot \left(1 + C_2(n,\, \kappa\, D,\, D^2 \cdot \lambda_i(X, g_0))(\kappa\varepsilon)^{\frac{1}{8}}\right) \cdot \lambda_i(X, g_0)$$

dove

$$C_1(n) = 14(n-1)\sqrt[4]{n}$$

$$C_2(n,\, \alpha, \Lambda) = 4\sqrt{n}\, \left[(2n+1)\, e^n\, [1 + B(\alpha)\sqrt{\Lambda + (n-1)\alpha^2}]^n + 2\right]$$

Qui $B(\alpha)$ *è la costante isoperimetrica della Proposizione 3.2.4. Si osservi che il lato destro dell'ultima disuguaglianza converge verso* $\lambda_i(X, g_0)$ *quando* $\varepsilon \to 0_+$ *.*

Il punto debole di questo risultato è evidentemente la condizione imposta sul volume della varietà (Y, g): "Volume di (Y, g) quasi inferiore al Volume di (X, g_0)". Osserviamo tuttavia che questa ipotesi è implicitamente presente nel risultato di J. Cheeger e T. Colding succitato, poiché le loro ipotesi assicurano la convergenza della successione dei volumi $\{\operatorname{Vol}_{g_k}(Y_k)\}$ verso il volume $\operatorname{Vol}_{g_0}(X)$ della varietà limite.

La seconda parte della Tesi (costituita unicamente dal Capitolo 4), è invece dedicata ad un argomento classico: il Lemma di Margulis.

Per comprendere questo problema, ricordiamo che la "*sistole*" (o "*sistole globale*") $\operatorname{sys}(X, g)$ di una varietà Riemanniana qualunque (X, g) è definita come l'estremo inferiore delle lunghezze di tutte le curve chiuse non omotope a zero su X; introduciamo la nozione di "*sistole puntuale*" $\operatorname{sys}(x)$ in un punto $x \in X$, definita come l'estremo inferiore di tutti i cappi non omotopi a zero di punto base x: la sistole (globale) è allora l'estremo inferiore (al variare di x in X) della sistole puntuale $\operatorname{sys}(x)$. Definiremo invece la "*diastole*" $\operatorname{dias}(X, g)$ di (X, g) come l'estremo superiore (al variare di x in X) della sistole puntuale $\operatorname{sys}(x)$. Il Lemma di Margulis classico fornisce un limite inferiore alla diastole di (X, g) in funzione della sua dimensione e dei limiti, superiore e inferiore, della sua curvatura sezionale; una versione ulteriore (dovuta a M. Gromov) fornisce una stima inferiore per la sistole di (X, g) in funzione della dimensione, dei limiti, superiore e inferiore, alla curvatura sezionale e di un limite superiore sul diametro.

Dopo aver rapidamente presentato il Lemma di Margulis classico, spiegheremo la generalizzazione proposta da G. Besson, G. Courtois e S. Gallot nella loro prepubblicazione del 2003, *Un lemme de Margulis sans courbure et ses applications*, ([**BCG4**]) .

Presenterò la dimostrazione di un nuovo Lemma di Margulis senza ipotesi di curvatura che si applica a tutte le varietà Riemanniane connesse di dimensione $n \ge 3$ la cui entropia volumica è limitata superiormente e il cui gruppo fondamentale è decomponibile (cioè isomorfo ad un prodotto libero) e privo di elementi di torsione di ordine 2:

THEOREM 4.1.1. *Sia* $H > 0$ *e sia* X *varietà Riemanniana connessa di dimensione* n *, tale che* $\operatorname{Ent}(X) \le H$ *e il cui gruppo fondamentale sia decomponibile e privo di* 2-*torsione. Allora:*

$$\operatorname{dias}(X) \ge \frac{\log(3)}{6H}.$$

Inoltre, esibiremo un limite inferiore universale per la sistole (omotopica):

THEOREM 4.1.2. *Siano* $H, D > 0$ *. Sia* X *una varietà Riemanniana compatta tale che* $\operatorname{Ent}(X) \le H$ *,* $\operatorname{Diam}(X) \le D$ *, il cui gruppo fondamentale è decomponibile e privo di torsione. Allora:*

$$\operatorname{sys} \pi_1(X) \ge \frac{1}{H} \cdot \log\left(1 + \frac{4}{e^{2DH} - 1}\right).$$

A differenza dei risultati di G. Margulis e di M. Gromov precedentemente citati, non si fa alcuna ipotesi sulla curvatura di (X, g) (come nel caso di [**BCG4**]). Il risultato di G. Besson, G. Courtois e S. Gallot può essere reinterpretato come un risultato di trasporto di proprietà, ovvero se una varietà (X_0, g_0) verifica le ipotesi del teorema di Margulis (o di Gromov) allora per tutte le varietà X il cui gruppo fondamentale è isomorfo ad un sottogruppo non ciclico del gruppo fondamentale di X_0 e per ogni metrica g su X (di entropia volumica limitata) eredita la proprietà *à la Margulis*, cioè la diastole di (X, g) è limitata inferiormente e se il suo diametro è limitato superiormente lo è anche la sistole. Differentemente dal risultato dimostrato in [**BCG3**], le proprietà “alla Margulis” che sono stabilite nei due teoremi precedenti, seguono direttamente dalle proprietà algebriche del gruppo fondamentale e non dall'esistenza di un'altra azione (per isometrie) di questo gruppo su un altro spazio di curvatura sezionale $\sigma \leq -1$, il cui quoziente abbia raggio di iniettività limitato inferiormente.

Seguendo un procedimento analogo a quello di G. Besson, G. Courtois e S. Gallot in [**BCG4**] citiamo due applicazioni del Teorema 4.1.2:

PRECOMPACTNESS THEOREM. *Sia $\mathfrak{M}_n^{dec}(D, V, H; l)$ la famiglia delle varietà Riemanniane compatte di dimensione n i cui gruppi fondamentali sono decomponibili e privi di torsione e tali che diametro, volume e entropia volumica siano rispettivamente più piccoli di D, V, H, e tali che la lunghezza del più corto laccio geodetico nel rivestimento universale Riemanniano sia più grande di l. Tale famiglia è precompatta with rispetto alla topologia di Gromov-Hausdorff.*
Inoltre $\mathfrak{M}_n^{dec}(D, V, H; l)$ è finita a meno di:
(i) *omotopia, per ogni $n \in \mathbb{N}$;*
(ii) *omeomorfismo, per $n = 4$;*
(iii) *diffeomorfismo, per $n \geq 5$.*

VOLUME ESTIMATE. *Per ogni varietà Riemanniana compatta, connessa, 1-essenziale di dimensione n, X, di gruppo fondamentale decomponibile e senza torsione e la cui entropia volumica e diametro sono limitati superiormente da $H, D > 0$ rispettivamente, abbiamo la seguente stima:*

$$\mathrm{Vol}(X) \geq \frac{C_n}{H^n} \cdot \log\left(1 + \frac{4}{e^{2DH} - 1}\right)^n$$

dove $C_n > 0$ è una costante universale dipendente esclusivamente dalla dimensione n (un valore esplicito per C_n -seppur non ottimale- può essere trovato in [**Gro5**]*, Theorem 0.1.A).*

Introduction

This Ph.D. Thesis is divided into two parts. In the first part we are interested by a new version of the barycenter method, a technique originally introduced by G. Besson, G. Courtois and S. Gallot, and by its applications to dynamical rigidity problems and to the comparison between the Laplace spectrum of two Riemannian manifolds. In the second part we shall deal with a classical subject in Riemannian Geometry: the Margulis Lemma. We shall provide a version of Margulis Lemma without curvature assumptions, based only on the algebraic properties of the group acting on the manifold.

The first part is divided into three chapters. We shall quickly give a description of their contents.

In Chapter 1 we present the Barycenter Method of G. Besson G. Courtois and S. Gallot. This method has been introduced by its authors in order to solve the Minimal Entropy conjecture (see [**BCG1**]). The method is a technique that allows, starting from the initial data of a map (possibly non continuous, possibly defined only on the orbit of a discrete group), to construct a regular map whose Jacobian determinant is sharply bounded from above.
To give an example of application of this method (developed by G. Besson, G. Courtois and S. Gallot) let us consider a compact, connected, oriented Riemannian manifold of dimension n , (Y, g) , and suppose that it admits a continuous map which takes its values in a n-dimensional locally symmetric space of negative curvature (X, g_0) . Then the method produces a family of smooth maps $\tilde{F}_c : (\tilde{Y}, \tilde{g}) \to (\tilde{X}, \tilde{g}_0)$, depending on a parameter c (greater than the volume entropy of (Y, g)), which are equivariant with respect to the homomorphism between the fundamental groups induced by the map f , and whose Jacobian is sharply bounded by:

$$| \operatorname{Jac}(\tilde{F}_c)| \leq (1 + \varepsilon(c)) \cdot \left(\frac{\operatorname{Ent}_{\mathrm{vol}}(Y, g)}{\operatorname{Ent}_{\mathrm{vol}}(X, g_0)} \right)^n$$

where $\operatorname{Ent}_{\mathrm{vol}}(Y, g)$ (resp. $\operatorname{Ent}_{\mathrm{vol}}(X, g_0)$) denotes the volume entropy of (Y, g) (resp. (X, g_0)) and where $\varepsilon \to 0$ when $c \to \operatorname{Ent}_{\mathrm{vol}}(Y, g)$. By the equivariance of the maps $\tilde{F}_c$, these maps induce maps F_c between the quotients Y , X , which are all homotopic to the original map f , and which almost contracts volumes.

In the general case (see [**BCG2**], [**BCG5**] and [**BCG6**]) we can define a family of maps $\tilde{F}_c : \tilde{Y} \to \tilde{X}$, each of them constructed as follows:

- we construct a map $\tilde{y} \to \nu^c_{\tilde{y}}$ from $\tilde{Y}$ into the space of probability measures on $\tilde{Y}$, chosen in such a way that the logarithmic derivative of $\nu^c_{\tilde{y}}$ with respect to $\tilde{y}$, is given by c , and such that $\nu^c_{\gamma \tilde{y}} = \gamma_* \nu^c_{\tilde{y}}$, for any $\gamma \in \pi_1(Y)$ (here $\pi_1(Y)$ is considered as the group of deck transformations of the universal covering),
- we send each of these measures $\nu^c_{\tilde{y}}$ onto a measure[13] $\mu^c_{\tilde{y}}$ defined on $\tilde{X}$ (or on the ideal boundary $\partial \tilde{X}$),

[13]In the case where the initial data is a map $f : Y \to X$ we define $\nu^c_{\tilde{y}} = e^{-c\, d_{\tilde{Y}}(\tilde{y}, \tilde{z})}\, dv_{\tilde{g}}(\tilde{z})$ and we define $\mu^c_{\tilde{y}}$ as the transportation of this measure by the map $\tilde{f} : \tilde{Y} \to \tilde{X}$, where $\tilde{f}$ is the lift of the map f . When the initial data is a representation $\rho : \Gamma = \pi_1(Y) \to \pi_1(X)$ (see [**BCG2**] and [**BCG6**]) we proceed in an analogous way, using measures $\nu^c_{\tilde{y}}$ which are linear combinations of Dirac measures $\delta_{\gamma\, \tilde{y}_0}$, supported by the orbit of $\tilde{y}_0$ under the action of Γ , that we transport (via the representation ρ) onto linear combinations of Dirac measures $\delta_{\rho(\gamma)\, \tilde{x}_0}$, whose support is $\rho(\Gamma).\tilde{x}_0 \subset \tilde{X}$; more precisely

- we define $\tilde{F}_c$ as the barycenter of the measure $\mu^c_{\tilde{y}}$.

There are many possible choices for the definition of the barycenter on the space $(\tilde{X}, \tilde{g}_0)$, each of them corresponding to a different application of this method: G. Besson, G. Courtois and S. Gallot ([**BCG1**] and [**BCG2**]) defined the barycenter of a measure μ with support on the ideal boundary $\partial\tilde{X}$ as the unique minimum point of the function $\tilde{x} \to \int_{\partial\tilde{X}} B(\tilde{x}, \theta)\, d\mu(\theta)$ (here B stands for the Busemann function of $\tilde{X}$) and, in a more recent work ([**BCG5**] and [**BCG6**]), they define the barycenter of a measure μ supported on $\tilde{X}$ (when the sectional curvature of $\tilde{X}$ is less or equal to $-K^2$) as the unique point in $\tilde{X}$ where the function $\tilde{x} \to \int_{\tilde{X}} \cosh(d_{\tilde{X}}(\tilde{x}, \tilde{z}))\, d\mu(\tilde{z})$ attains its minimum.
Still in the case where the sectional curvature is bounded from above by $-K^2$, A. Sambusetti ([**Samb2**]) has defined the barycenter of a measure supported on $\tilde{X}$ as the unique minimum point of the function $\tilde{x} \to \int_{\tilde{X}} d_{\tilde{X}}(\tilde{x}, \tilde{z})^2\, d\mu(\tilde{z})$.
These authors have defined the barycenter in the case where the sectional curvature of $(\tilde{X}, \tilde{g}_0)$ is negative ([**BCG1**], [**BCG2**], [**BCG5**], [**BCG6**], [**Samb2**]) or in the case where the manifold $(\tilde{X}, \tilde{g}_0)$ is a product of symmetric spaces, each one of negative curvature ([**BCG4**]).

L. Sabatini ([**Saba1**],[**Saba2**]) has successively remarked that the notion of barycenter introduced by A. Sambusetti may be generalized to the case of non-positive curvature (applying this notion of barycenter in the flat case, one gets the standard notion of barycenter in the euclidean space); afterwards he proved that this notion of barycenter can be extended to the case where the sectional curvature is only bounded (without any assumption on the sign of this curvature), provided that the support of the measures $\mu^c_{\tilde{y}}$ has diameter sufficiently small (see [**Saba1**] and Theorem 1.2.1 of this Thesis). In the case where the initial data is a Gromov-Hausdorff ε-approximation $f : (Y, g) \to (X, g_0)$ (and when ε is sufficiently small), he provides a new (and more direct) manner of defining the map $\tilde{y} \to \mu^c_{\tilde{y}}$ from $\tilde{Y}$ into the space of probability measures on $\tilde{X}$, so that the image measures $\mu^c_{\tilde{y}}$ have controlled densities with respect to the Riemannian measure $dv_{\tilde{g}_0}$. This allows him to compare the volumes of the manifolds (Y, g) and (X, g_0) without making any curvature-assumption on (Y, g).

In the present thesis we shall modify the method of L. Sabatini in order to attain two aims:

- establish a result of the type "conjugacy of geodesic flows $\Rightarrow$ isometry" (this result requires a version of Sabatini's method which is valid for $(1, C)$-quasi isometries, when C is not a small number).
- compare the Laplace Spectra of two Riemannian manifolds (Y, g) and (X, g_0) when their Gromov-Hausdorff distance is bounded, without assuming any condition on the curvature of the manifold (Y, g); this last result requires the maps F_c (constructed above) to be "almost isometric" (in a rather weak sense) in the case where the volumes of (Y, g) and (X, g_0) are almost equal.

We define the *pointwise energy at the point* $y \in Y$ *of a* C^1 *map* $k : (Y, g) \to (X, g_0)$, as the number $e_y(k) = \sum_{i=1}^n \| d_y k(e_i) \|^2_{g_0}$, where $\{e_i\}$ is any g-orthonormal basis of $T_y Y$. Our version of L. Sabatini's method may be stated as follows:

Theorem 0.1. (Simplified version, for a complete statement see the Theorem 1.2.1) *Let (X, g_0) and (Y, g) be two complete, connected Riemannian manifolds of dimension n such that X is simply connected; let us denote by ρ and d the Riemannian distances on X and Y associated to the metrics g_0 and g respectively. Let us assume* [14] *that* $\mathrm{inj}(X, g_0) > 0$ *and that* $\mathrm{inj}(Y, g) > 0$. *We denote by σ and σ_0 the sectional curvatures*

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we can define:
$$\nu^c_{\tilde{y}} = \sum_{\gamma \in \Gamma} e^{-c\, d_{\tilde{Y}}(\tilde{y}, \gamma\, \tilde{y}_0)} \quad , \quad \mu^c_{\tilde{y}} = \sum_{\gamma \in \Gamma} e^{-c\, d_{\tilde{X}}(\tilde{x}, \rho(\gamma)\, \tilde{x}_0)} \quad .$$

[14] The assumption $\mathrm{inj}(X, g_0) > 0$ (resp. $\mathrm{inj}(Y, g) > 0$) is automatically verified when (X, g_0) (resp. (Y, g)) possesses a cocompact group of isometries, for example when (X, g_0) (resp. (Y, g)) is the Riemannian universal covering of a compact Riemannian manifold.

of (Y,g) and (X,g_0) respectively. Let us suppose that $|\sigma_0| \le \kappa_0^2$ and that $|\sigma|$ is bounded. Let Γ_Y (resp. Γ_X) be a discrete subgroup of the isometry group of (Y,g) (resp. of (X,g_0)), which acts freely and properly discontinuously on Y (resp. on X). Assume that there exists an isomorphism $\lambda : \Gamma_Y \to \Gamma_X$ and two maps $f : X \to Y$ and $h : Y \to X$, equivariant with respect to λ^{-1} and λ respectively, which verifies, for any $x \in X$ and for any $y \in Y$:

$$\rho(h(y),x) \le \alpha \cdot d(y,f(x)) + \varepsilon \ , \quad d(y,f(x)) \le \alpha \cdot \rho(h(y),x) + \varepsilon \ ,$$

$$\rho(x,(h\circ f)(x)) \le \varepsilon, \quad d(y,(f\circ h)(y)) \le \varepsilon \ .$$

for some given constants $\alpha \ge 1$ and ε.
For any data of positive constants ε, c, R and α such that

$$\alpha \ge 1 \quad , \quad R < \frac{1}{7\,\alpha} \cdot \min\left\{ \mathrm{inj}(X,g_0), \frac{\pi}{2\,\kappa_0} \right\} \quad , \quad \varepsilon < \frac{R}{5} \ ,$$

there exists a λ-equivariant C^1 map $H_c^R : Y \to X$ such that

$$\forall y \in Y \qquad \rho(H_c^R(y),h(y)) \le A(n,\alpha,c,\varepsilon,\kappa_0,R) \cdot \left(\frac{\varepsilon}{\kappa_0}\right)^{\frac{1}{2}} \ ,$$

(where the universal constant $A(n,\alpha,c,\varepsilon,\kappa_0,R)$ is an explicit function[15] of $(n,\alpha,c,\varepsilon,\kappa_0,R)$, which is bounded from above when there exist three positive constants B_1, B_2, B_3 such that $B_1 \ge c\,\varepsilon$, $B_2\,\kappa_0^{\frac{1}{2}} \le c\,\varepsilon^{\frac{1}{2}}$ and $\alpha \le B_3$) and whose pointwise energy verifies, for any point $y \in Y$,

$$e_y(H_c^R) \le n \cdot (1 + \eta(n,\alpha,c,\varepsilon,\kappa_0,R)) \ ,$$

where $\eta(n,\alpha,c,\varepsilon,\kappa_0,R)$ is an explicit function[16] of $(n,\alpha,c,\varepsilon,\kappa_0,R)$, which goes to zero when there exist two positive constants B_1', B_2' such that $c\,R \ge B_1'$ and $c\,\varepsilon^{\frac{9}{10}} \le B_2'\,\kappa_0^{\frac{1}{10}}$ and when $\kappa_0\,\varepsilon$, $(\alpha-1)$ and $\kappa_0\,R$ simultaneously go to zero.
Moreover if the initial data h is a continuous map, then H_c^R is homotopic to h by a λ-equivariant homotopy.

It is important in many applications (in order to maintain the sharpness of the estimates) that the upper bound of the pointwise energy would be almost equal to n (as in the previous theorem), since the value n is precisely the energy of an isometry.
The proof of this theorem is given in sections 1.2, 1.3.

In the section 1.4 we show how Theorem 1.2.1 can be used to obtain L. Sabatini's result ([**Saba1**], Theorem 4.6.1). The corollary that we shall need in the Chapter 3 to compare the spectra of two Riemannian manifolds is the following:

Theorem 0.2. *Let (X,g_0) be a n-dimensional compact, connected Riemannian manifold with bounded sectional curvature $|\sigma_0| \le \kappa_0^2$. Let (Y,g) be any n-dimensional compact Riemannian manifold. Assume that $d_{GH}((Y,g),\,(X,g_0)) < \varepsilon$ where*

$$\kappa_0\,\varepsilon < C(n)^{-4} \cdot \min\{[\kappa_0\ \mathrm{inj}(X,g_0)]^4\,;\,1\} \quad \text{where} \quad C(n) = (n+1)^8\,2^{8n} \ .$$

Then there exists a C^1 map $H_\varepsilon : (Y^n,g) \to (X^n,g_0)$ which satisfies the following properties:

(i) *H_ε is a Gromov-Hausdorff $\left(10.C(n)\,\frac{(\kappa_0\,\varepsilon)^{\frac{3}{4}}}{\kappa_0}\right)$-approximation;*

[15]For an explicit value of $A(n,\alpha,c,\varepsilon,\kappa_0,R)$, see the statement of Theorem 1.2.1. Remark that the presence of the parameter κ_0 makes the estimates a little more complicated, but is needed in order to make the estimates invariant by changes of scale.

[16]For an explicit value of $\eta(n,\alpha,c,\varepsilon,\kappa_0,R)$, see the statement of Theorem 1.2.1. As before, we fix the parameter κ_0 in order to make the estimates invariant by changes of scale.

(ii) *For any* $y \in Y$ *we have the following pointwise energy estimate:*

$$e_y(H_\varepsilon) \leq n\,\left(1 + 20\,(\kappa_0\,\varepsilon)^{\frac{1}{4}}\right)\;;$$

(iii) *For any* $y \in Y$ *we have the following bound for the Jacobian determinant of* H_r *:*

$$\mathrm{Jac}(H_\varepsilon)(y) \leq \left(1 + 20\,(\kappa_0\,\varepsilon)^{\frac{1}{4}}\right)^{\frac{n}{2}}\;.$$

Moreover if there exists a continuous Gromov-Hausdorff ε*-approximation* $h\,:\,(Y,g) \to (X,g_0)$ *, then* H_ε *is homotopic to* h *.*

To obtain Theorem 0.2 from Theorem 0.1 we must deal with the theory of the τ-coverings (see G. Reviron, [**Rev**], where it is explained how it is possible to lift a Gromov-Hausdorff approximation between two manifolds to a "good" quasi-isometry between a suitable covering of Y and the universal covering of X). We shall briefly introduce some elements of this theory in section 1.4.2.

Chapter 2 is devoted to the Conjugacy Rigidity problem. Namely, we are concerned with the following question, originally asked by E. Hopf: which are the compact Riemannian manifolds that are uniquely determined by their geodesic flow? To be more precise we give the following

DEFINITION. A compact Riemannian manifold (X,g_0) is said to be C^k-conjugacy rigid if and only if any compact Riemannian manifold (Y,g) whose geodesic flow is conjugate to the one of (X,g_0) (via a C^k-diffeomorphism) is isometric to (X,g_0)

We can now restate E. Hopf's conjecture as follows:

QUESTION. *Which compact Riemannian manifolds are* C^1 *conjugacy rigid? Which ones are* C^0 *conjugacy rigid?*

Counterexamples to the rigidity conjecture were soon discovered: A. Weinstein pointed out that Zoll surfaces are C^∞-conjugate to the standard sphere though they are not isometric to it (see [**Besse**], §4.F).
It turned out to be more difficult to find positive answers to the previous Question. We try to give here a brief panorama of positive answers to the conjugacy rigidity problem:

- Marcel Berger in [**Besse**], Appendix D, provided the first example of C^0-conjugacy rigid manifold: $\mathbb{R}P^n$, endowed with the canonical metric, is C^0-conjugacy rigid for $n \geq 2$.
- C. Croke J. P. Otal (see [**Cr1**], [**Ot**]), independently proved the C^1-conjugacy rigidity for surfaces of nonpositive curvature ; successively this result has been improved to the C^0-rigidity by C.Croke, A. Fathi and J. Feldman in [**CFF**].
- In 1994 Croke and Kleiner proved the C^1-conjugacy rigidity for $n-$dimensional manifolds ($n \geq 2$) which admit a parallel vector field (see [**Cr-Kl**]); in particular this implies the C^1-rigidity of flat manifolds.
- In 1995 G. Besson, G. Courtois and S. Gallot proved the C^1-conjugacy rigidity of locally symmetric $n-$dimensional manifolds of negative curvature ($n \geq 3$), as a consequence of the solution of the Minimal Entropy conjecture (see [**BCG1**]). Their result has been recently used by U. Hamenstädt who proved the C^0-rigidity of locally symmetric manifolds of negative curvature within the class of compact Riemannian manifolds of negative curvature as a consequence of the invariance of volumes under C^0-conjugacy of geodesic flows (see [**Ham**]).
- More recently P. Eberlein ([**Eb**]) and other mathematicians started to investigate the case of nilmanifolds [17].

[17] A nilmanifold is the quotient of a simply connected nilpotent Lie group, endowed with a left invariant metric, by the action of a lattice.

C. Gordon and Y. Mao proved in [**Go-Ma**] the C^0-rigidity (within the class of all compact nilmanifolds) of some families of compact 2-step nilmanifolds.

The proofs of these results use quite different approaches; in particular the conjugacy rigidity result of G. Besson, G. Courtois and S. Gallot relies on the solution of the Minimal Entropy conjecture, and it is not easy to understand whether or not their argument provides a general approach for solving dynamical rigidity problems. The existence of such a strategy would be of great interest in view of the more recent developements of the barycenter method (see [**BCG4**], [**Saba1**]).
We work in this direction and we show that a generalization of this technique (up to modifications of the notion of barycenter) gives a proof of the C^1-rigidity of flat manifolds (Chapter 2, section 2.3)

THEOREM 2.3.1. *Let (X, g_0) be a compact flat Riemannian manifold of dimension $n \geq 3$. Then (X, g_0) is C^1-conjugacy rigid.*

The method is constructive: in fact, thanks to the (modified) barycenter method we can construct the isometry as the limit of a sequence of maps $H_c^R : Y \to X$ (constructed as in Theorem 0.1) obtained by deformation of a quasi-isometry $\tilde{f} : (\tilde{Y}, \tilde{g}) \to (\tilde{X}, \tilde{g}_0)$. The existence of this quasi-isometry and its properties are stated in the following proposition:

PROPOSITION 2.2.2. *Let (Y, g) and (X, g_0) be two compact connected Riemannian manifolds of dimension $n \geq 3$, with C^0-conjugate geodesic flows. Let us denote by λ the isomorphism between the fundamental groups of Y and X induced by the conjugacy of the geodesic flows*[18].

(i) *There exists a constant $C > 0$ and a $(1, C)$-quasi isometry $\tilde{f} : (\tilde{Y}, \tilde{g}) \to (\tilde{X}, \tilde{g}_0)$.*

(ii) *Moreover, if (X, g_0) is a $K(\pi, 1)$-space we can find a continuous, λ-equivariant $(1, C)$-quasi isometry $\tilde{f} : \tilde{Y} \to \tilde{X}$ (whose $(1, C)$-quasi inverse is denoted by $\tilde{h} : \tilde{Y} \to \tilde{X}$). This induces two homotopy equivalences f, h between the basis-manifolds Y and X, such that the induced morphisms f_* and h_* between the fundamental groups of Y and X coincide with the isomorphisms λ and λ^{-1}.*

The latter result gives a quasi immediate proof of the C^0-conjugacy rigidity of flat manifolds among the manifolds of non-negative Ricci curvature (we shall give two versions of this proof).

In Chapter 3 we are concerned with comparing the spectra of two Riemannian manifolds. To understand the results of this chapter, we recall that in the case of a compact, connected Riemannian manifold without boundary (Y, g), we call "*spectrum of* (Y, g)" the sequence $\{\lambda_i(Y, g)\}_{i\in\mathbb{N}}$ of the eigenvalues of the Laplacian of (Y, g), in increasing order and counted with multiplicities. Since this spectrum is discrete we have:

$$0 = \lambda_0(Y, g) < \lambda_1(Y, g) \leq \lambda_2(Y, g) \leq \cdots \leq \lambda_i(Y, g) \leq \cdots$$

and $\lambda_i(Y, g) \to +\infty$ when $i \to +\infty$.
Naively we can state the problem as follows:

NAÏVE QUESTION: *what can be said about the spectrum of a Riemannian manifold if we only know that its Gromov-Hausdorff distance to another fixed compact Riemannian manifold is bounded ?*

There exist a large number of estimates (which are asymptotically sharp up to a multiplicative constant) for each of the eigenvalues of the Laplacian of a compact Riemannian manifold. These estimates are universal in the set of compact Riemannian manifolds with

[18] If $\varphi : UY \to UX$ is a C^0-conjugacy of geodesic flows and if $\pi_X : UX \to X$ and $\pi_Y : UY \to Y$ are the canonical projections, then $(\pi_Y)_*$ is an isomorphism between $\pi_1(UY, v)$ and $\pi_1(Y, \pi_Y(v))$, and we define λ as the composition $(\pi_X)_* \circ \varphi_* \circ ((\pi_Y)_*)^{-1}$

Ricci curvature uniformly bounded from below and diameter uniformly bounded from above (see for example [**Cheng**], [**Gro6**], [**Li-Yau**], [**Ga1**], [**Ga2**], [**Ga3**]). On the other hand there are few results that establish the convergence of the spectrum when we take Gromov-Hausdorff limits (for example [**Dod-Pat**], [**Ch-Co**] and more recently [**Aub2**]); between these convergence results, the most celebrated one is the result of J. Cheeger and T. Colding ([**Ch-Co**]), which solves our naive question under the following additional assumptions:

- instead of assuming the Gromov-Hausdorff distance to be bounded, they assume that it goes to zero, i. e. they consider (when $k \to \infty$) a sequence of Riemannian manifold $(Y_k, g_k)_{k\in\mathbb{N}}$ which converges to a Riemannian manifold of the same dimension (X, g_0) and they compare the limit of the spectra of the (Y_k, g_k)'s with the spectrum of the limit manifold,
- they assume the existence of a uniform lower bound of the Ricci curvature of all the (Y_k, g_k)'s;

they conclude that each eigenvalue $\lambda_i(Y_k, g_k)$ of the spectrum of (Y_k, g_k) converges (when $k \to \infty$) to the corresponding eigenvalue of (X, g_0) .

The result that we shall prove in this Thesis has a different aim, more similar to our initial naive question: if (X, g_0) is a fixed compact Riemannian manifold and if (Y, g) is any Riemannian manifold (of the same dimension as (X, g_0)) lying at bounded Gromov-Hausdorff distance from (X, g_0) :

- the Gromov-Hausdorff distance between (Y, g) and (X, g_0) is not assumed to go to zero, but is just assumed to be bounded by a given constant ε , which must be chosen smaller than some universal constant (that we compute in section 3),
- no curvature assumptions are made on the manifold (Y, g) ,
- it is not a convergence result but an approximation result, because we give an explicit bound of the error $(\lambda_i(Y, g) - \lambda_i(X, g_0))$ (for other, more complete, approximation results, see [**Aub2**]).

Since our assumptions are much weaker than the ones of J. Cheeger and T. Colding's result, our conclusions are also weaker and we cannot recover their result ([**Ch-Co**]); in particular we prove that it is impossible, under our assumptions, to bound from below the spectrum of (Y, g) (see Example 3.4.3 and Proposition 3.4.4). Another illustration of the weakness of our assumptions is that they do not impose any topological restriction on the manifold Y : in fact, for any given manifold (X, g_0) we can find a sequence of Riemannian manifolds (Y_k, g_k) , converging to (X, g_0) (with respect to the Gromov-Hausdorff distance), which verify our assumptions, and such that the Y_k's are pairwise non homotopic (see Example 3.4.1 and Proposition 3.4.2), whereas the assumptions made by J. Cheeger and T. Colding imply that, except for a finite number, all the Y_k's are diffeomorphic to X .
More precisely, here is the statement of our comparison result:

THEOREM 3.1.2. *Let (X^n, g_0) be a compact, connected, Riemannian manifold satisfying the assumptions:*

$$\mathrm{Diam}(X, g_0) \leq D, \quad \mathrm{inj}(X, g_0) \geq i_0, \quad |\sigma_0| \leq \kappa^2,$$

where D, i_0, κ are arbitrary positive constants.
Let (Y^n, g) be any compact, connected Riemannian manifold such that there exists a continuous Gromov-Hausdorff ε-approximation $f : (Y, g) \to (X, g_0)$ of non zero absolute degree, where

$$\varepsilon < \varepsilon_1(n, i_0, \kappa) = \frac{1}{\kappa} \min \left\{ \left[\frac{\min\{ 1 \,;\, \kappa\ \mathrm{inj}(X, g_0)\}}{2^{8n}\,(n+1)^8} \right]^4 ; \left(\frac{\left(\frac{10}{9}\right)^{\frac{2}{n}} - 1}{20} \right)^4 \right\}$$

If we assume that

$$[1 - 10\, n\, (\kappa\varepsilon)^{\frac{1}{4}}] \cdot \mathrm{Vol}_g(Y) < \mathrm{Vol}_{g_0}(X) \cdot \mathrm{Adeg}(f)$$

then, for every $i \in \mathbb{N}$, we have

$$\lambda_i(Y, g) \leq \Big(1 + C_1(n)(\kappa\varepsilon)^{\frac{1}{16}}\Big) \cdot \Big(1 + C_2(n,\, \kappa D,\, D^2 \cdot \lambda_i(X, g_0))(\kappa\varepsilon)^{\frac{1}{8}}\Big) \cdot \lambda_i(X, g_0)$$

where

$$C_1(n) = 14(n-1)\sqrt[4]{n}$$

$$C_2(n,\,\alpha,\Lambda) = 4\sqrt{n}\,\left[(2n+1)\,e^n\,[1+B(\alpha)\sqrt{\Lambda+(n-1)\alpha^2}]^n+2\right]$$

where $B(\alpha)$ is the isoperimetric constant defined in the Proposition 3.2.4 and where the right hand side of the latter inequality goes to $\lambda_i(X,g_0)$ when $\varepsilon\to 0_+$.

The weak point of this result is clearly the assumption which (in degree one) writes "Volume of (Y,g) almost smaller than Volume of (X,g_0)". We remark however that this assumption is implicit in the result of J. Cheeger and T. Colding, since their assumptions imply that the sequence of volumes $\{\mathrm{Vol}_{g_k}(Y_k)\}_{k\in\mathbb{N}}$ converges to the volume $\mathrm{Vol}_{g_0}(X)$ of the limit manifold.

In order to prove the Theorem 3.1.2, we first prove that the maps F_c (constructed as above) are "almost isometric" (in a rather weak sense which will be specified in the section 3.3.1) in the case where the volumes of (Y,g) and (X,g_0) are almost equal. This allows to compare the L^2 norms of f and df on (X,g_0) to the L^2 norms of $f\circ F_c$ and $d(f\circ F_c)$ on (Y,g) for any function f of the vector space $\mathcal{A}_X(\lambda)$ spanned by the eigenfunctions of the Laplacian of (X,g_0) corresponding to eigenvalues $\lambda_k\le\lambda$ (sections 3.3.2 and 3.3.3). This comparison is possible only if we know how to bound (from above) the L^∞ norms of f and df in terms of their L^2 norms[19]. This is done using iterations of Sobolev quantitative inequalities (see section 3.2).

The second part of this Thesis (section 4) is devoted to a new approach of a very classical subject: the Margulis Lemma.
In order to illustrate the problem we recall that the "*systole*" (or "*global systole*") $\mathrm{sys}(X,g)$ of a Riemannian manifold (X,g) is defined as the infimum of the lengths of the homotopically non trivial loops in X ; we introduce the notion of "*pointwise systole*" $\mathrm{sys}(x)$ at a point $x\in X$, which is defined as the infimum of the lengths of the homotopically non trivial loops based at x: thus we observe that the (global) systole is the infimum of the pointwise systole $\mathrm{sys}(x)$, when x runs in X . On the other hand, the "*diastole*" $\mathrm{dias}(X,g)$ of (X,g) is defined as the supremum (when x runs in X) of the pointwise systole, $\mathrm{sys}(x)$.

Let us also recall that the "*Entropy*" (or "*Volume Entropy*") of a compact Riemannian manifold (X,g) (denoted by $\mathrm{Ent}(X,g)$) is defined as:

$$\mathrm{Ent}(X,g) = \liminf_{R\to\infty}\frac{1}{R}\log(\mathrm{Vol}(B(\tilde{x},R)))$$

where $B(\tilde{x},R)$ denotes the geodesic ball of radius R and centered at $\tilde{x}$ in the Riemannian universal covering $(\tilde{X},\tilde{g})$ (the limit exists and this definition does not depend on the choice of the point $\tilde{x}$).

The classical Margulis Lemma (see for instance [**Bu-Za**] chapter 37) gives a lower bound for the diastole of (X,g) in terms of its dimension and of lower and upper bounds of the sectional curvature; a more recent version, due to M. Gromov (see [**Bu-Ka**], Proposition 2.5.3), gives a lower bound for the global systole of (X,g) in terms of its dimension, of lower and upper bounds of the sectional curvature and of an upper bound of the diameter.
After a quick introduction to the classical Margulis Lemma, we shall explain the generalization introduced by G. Besson, G. Courtois and S. Gallot in a preprint of 2003, *Un lemme de Margulis sans courbure et ses applications*, ([**BCG3**]).
We shall give a proof of a new Margulis Lemma which gives a lower bound for the diastole of (X,g) without assuming any curvature condition, and which writes:

[19]Notice that it is important that, for any function $f\in\mathcal{A}_X(\lambda)$, the bounds of $\frac{\|f\|_{L^\infty}}{\|f\|_{L^2}}$ and of $\frac{\|df\|_{L^\infty}}{\|df\|_{L^2}}$ that we obtain depend on λ , but not on the dimension of $\mathcal{A}_X(\lambda)$.

THEOREM 4.1.1. *Let $H > 0$ and let (X, g) be any connected Riemannian n-manifold such that* $\mathrm{Ent}(X, g) \leq H$ *, whose fundamental group is decomposable and without 2-torsion. Then:*

$$\mathrm{dias}(X, g) \geq \frac{\log(3)}{6H}.$$

From this we deduce our new version of Gromov's Margulis Lemma, i. e. the following lower bound for the global systole:

THEOREM 4.1.2. *Let $H, D > 0$. Let (X, g) be any compact Riemannian manifold such that* $\mathrm{Ent}(X, g) \leq H$, $\mathrm{Diam}(X, g) \leq D$ *, whose fundamental group is decomposable and torsion-free. Then we have:*

$$\mathrm{sys}(X, g) \geq \frac{1}{H} \cdot \log\left(1 + \frac{4}{e^{2DH} - 1}\right).$$

Contrary to the results of G. Margulis and M. Gromov, we do not assume any curvature condition on (X, g) : in fact, following [**BCG3**], we replace these classical assumptions on curvature by a much weaker assumption on the Entropy, whose effect is to fix the scale. As this lack of curvature assumptions is already present in [**BCG3**], we have to explain why our result is new with respect to [**BCG3**] : the result of G. Besson, G. Courtois and S. Gallot may be viewed as a kind of heredity property, i. e. (roughly speaking) if a manifold (X_0, g_0) verifies the assumptions of the classical Margulis Lemma, for any manifold X whose fundamental group is isomorphic to a noncyclic subgroup of the fundamental group of X_0, any metric g on X (with bounded Entropy) inherits the following "Margulis property": the diastole of (X, g) is bounded from below (resp. the systole of (X, g) is bounded from below if the diameter is bounded).
Contrary to the results of [**BCG3**], the "Margulis property" proved by theorems 4.1.1 and 4.1.2 follows directly from the algebraic properties of the fundamental group, and not from the existence of an action (by isometries) of this group on another manifold of sectional curvature $\sigma \leq -1$, whose quotient has injectivity radius bounded from below.

Following the ideas of G. Besson, G. Courtois and S. Gallot in [**BCG3**] we state two applications of Theorem 4.1.2:

PRECOMPACTNESS THEOREM. *Let $\mathfrak{M}_n^{dec}(D, V, H; l)$ denote the family of compact, Riemannian n-manifolds whose fundamental groups are decomposable and torsion-free, whose diameter, volume and volume-entropy are respectively smaller than D, V, H and such that the length of the shortest geodesic loop in the universal covering is greater than l. This family is precompact with respect to the Gromov-Hausdorff topology.*
Moreover $\mathfrak{M}_n^{dec}(D, V, H; l)$ is finite up to:
(i) *homotopy, for all $n \in \mathbb{N}$;*
(ii) *homeomorphism, for $n = 4$;*
(iii) *diffeomorphism, for $n \geq 5$.*

VOLUME ESTIMATE. *For any connected and compact, 1-essential Riemannian n-manifold, X with decomposable, torsion-free fundamental group and whose volume-entropy and diameter are bounded above by $H, D > 0$ respectively, we have the following estimate:*

$$\mathrm{Vol}(X) \geq \frac{C_n}{H^n} \cdot \log\left(1 + \frac{4}{e^{2DH} - 1}\right)^n$$

where $C_n > 0$ is a universal constant depending only on the dimension n (an explicit – although not sharp – upper bound to C_n can be found in [**Gro5**]*, Theorem 0.1.A).*

Part 1

The Barycenter Method. Applications to Rigidity Problems and Spectral Geometry

CHAPTER 1

The Barycenter Method

Aperçu du chapitre 1 : *Dans ce chapitre nous allons présenter la méthode du barycentre en courbure de signe variable (sections 1.2, 1.3). Pour démontrer le théorème principal (Théorème 1.2.1), l'idée est de construire, à partir d'une quasi-isométrie* h *donnée, une application* $H_{r,\,c}^R$ *(homotope à* h *, dépendant de paramètres*[1] $R,\ c,\ r$ *et dont l'énergie est bornée de manière optimale, voir la preuve du Théorème 1.2.1) en associant, à chaque point* y *de la variété de départ* (Y, g) *, une mesure* μ_y^c *sur la variété* (X, g_0) *, mesure dont le support est de diamètre contrôlé, de manière à ce qu'elle admette une notion de barycentre*[2]*. On définit alors* $H_{r,\,c}^R(y)$ *comme le barycentre*[3] *de cette mesure.*
Dans l'espace euclidien $\mathbb{E}$*, le barycentre* b_μ *d'une mesure* μ *est défini comme le point où la fonction de Leibniz*[4] $x \to \mathcal{B}_\mu(x)$ *atteint son minimum et il est classique d'estimer la distance de* x *à* b_μ *(le barycentre) en termes de l'écart* $\mathcal{B}_\mu(x) - \mathcal{B}_\mu(b_\mu)$*.*
Dans la section 1.2.3, des propriétés similaires sont étudiés dans le cadre général des variétés Riemanniennes: à toute mesure finie μ *(dont le support est de diamètre contrôlé) nous associons une fonction de Leibniz* $\mathcal{B}_\mu$ *de façon analogue au cas euclidien et nous allons définir le barycentre comme le point où la fonction* $\mathcal{B}_\mu$ *atteint son minimum*[5] *(voir la Proposition 1.2.13); même dans le cas Riemannien nous allons contrôler la distance entre les barycentres* $H_{r,\,c}^R(y)$ *et* $H_{r,\,c}^R(y')$ *des mesures* $\mu_y^c,\ \mu_{y'}^c$ *en termes de l'écart entre les dénsités associés à ces mesures, et de la distance entre* y *et* y' *.*
L'idée est simple mais, a priori, donne uniquement une éstimée assez grossier pour la constante de Lipschitz de l'application barycentre $H_{r,\,c}^R$ *(il est simple s'apperçoir que, dans le cas général, il est impossible de fournir un borne supérieur proche de* 1 *pour cette constante). En revanche, l' éstimée pour l'energie ponctuelle (resp. pour le determinant Jacobien) de* $H_{r,\,c}^R$ *donnée dans le Théorème 1.2.1 est optimale, dans le sense que le borne supérieur approche*[6] *la valeur optimale* n *(resp.* 1 *) quand l'application initiale* h *est une* ε*-approximation de Gromov-Hausdorff.*
Une des difficultés dans la preuve du Théorème 1.2.1 réside dans le fait que il est nécessaire se mettre en coordonnées exponentielles centrées en $H_{r,\,c}^R(y)$ *(le barycentre de la mesure*

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[1] Le paramètre c est le coefficient de décroissance exponentielle de la densité de chacune de ces mesures μ_y^c en fonction de la distance au point y , le paramètre R donne la valeur de cette distance à partir de laquelle on applique un "cut off" à la densité de la mesure μ_y^c , de telle sorte que R majore le diamètre maximal des supports des mesures μ_y^c , le paramètre r est imposé par la présence du cut-locus de (Y, g) : il mesure l'erreur introduite en remplaçant la distance au point y par une fonction régulière (voir la section 1.3.1.1).

[2] Ceci permet également d'éviter les problèmes causés par la présence du cut-locus de (X, g_0) , problèmes qu'on ne rencontre pas dans le cadre classique de courbure négative ou nulle.

[3] La notion de barycentre utilisée ici sera précisée dans la Proposition 1.2.13.

[4] La fonction de Leibniz $\mathcal{B}_\mu$ est définie par la formule: $\mathcal{B}_\mu(x) = \int_{\mathbb{E}} d(x,\ z)^2\, d\mu(z)$.

[5] Dans le cas de courbure sectionelle négative ce procédé marche sans aucune hypothèse sur la taille du support de la mesure μ. En revanche, sans hypothèse sur le signe de la courbure sectionnelle, le raisonnement marche uniquement si le support est de diamètre contrôlé, car ceci permet d'eviter les problèmes causés par la presence du cut-locus dans la variété de reference (X, g_0) .

[6] Il est en effet important, dans les applications (afin que celles-ci soient optimales), que le majorant de l'énergie ponctuelle (resp. du déterminant jacobien) obtenu soit proche de n (resp. de 1) pour des valeurs bien choisies des paramètres, comme c'est le cas dans les théorèmes 1.2.1 et 1.4.1, car la valeur n (resp. 1) est précisément l'énergie (resp. le déterminant jacobien) d'une isométrie.

μ_y^c). Nous sommes donc forcés à substituer dans les formules $h(y)$ par $H_{r,c}^R$ et borner l'erreur commis en faisant cette substitution, ce qui rend nécessaire une éstimée plus fine de la distance entre $h(y)$ et $H_{r,c}^R(y)$: ceci est achévé[7] *dans la Proposition 1.3.4. L'une des conséquences de la Proposition 1.3.4 est que, quand l'application initiale d'une ε-approximation de Gromov-Hausdorff, $H_{r,c}^R$ est aussi une ε'-approximation de Gromov-Hausdorff (où $\varepsilon' = C \cdot \varepsilon^{\frac{3}{4}}$).*
Corollaire du Théorème 1.2.1 et de la théorie des τ-revêtements developpée par G. Reviron ([**Rev**]*) est le Théorème 1.4.1 (prouvé initialment par L. Sabatini* [**Saba1**]*) qui donne un borne supérieur explicit et simple pour l'energie ponctuelle (resp. le determinant Jacobien) de $H_{r,c}^R$ qui est proche de n (resp. proche de 1) quand l'application initiale est une ε-approximation de Gromov-Hausdorff.*

Prospetto del capitolo 1: *In questo capitolo presenteremo il metodo del baricentro in curvatura di segno variabile (sezioni 1.2, 1.3). Per dimostrare il teorema principale (Teorema 1.2.1) l'idea è quella di costruire, a partire da una quasi isometria h assegnata, una applicazione $H_{r,c}^R$ (omotopa ad h, dipendente dai parametri*[8] *R, c, r e la cui energia è limitata in modo ottimale, vedere la dimostrazione del Teorema 1.2.1) associando ad ogni punto y della varietà di partenza (Y, g), una misura μ_y^c sulla varietà (X, g_0), misura il cui supporto è di diametro controllato, per far sì che sia possibile definirne il baricentro*[9]*. Si definisce allora $H_{r,c}^R(y)$ come il baricentro*[10] *di questa misura.*
Nello spazio euclideo $\mathbb{E}$, il baricentro b_μ di una misura μ è definito come il punto dove la funzione di Leibniz[11] *$x \to \mathcal{B}_\mu(x)$ raggiunge il suo minimo, ed è un argomento classico quello che consente di stimare la distanza tra x e b_μ (il baricentro) in termini della differenza $\mathcal{B}_\mu(x) - \mathcal{B}_\mu(b_\mu)$.*
Nella sezione 1.2.3 studiamo delle proprietà simili nel caso generale delle varietà Riemanniane: ad ogni misura finita μ (il cui supporto è di diametro controllato) associamo una funzione di Leibniz $\mathcal{B}_\mu$ in modo analogo al caso euclideo ed andiamo a definire il baricentro come il punto di minimo della funzione $\mathcal{B}_\mu$ [12] *(vedere la Proposition 1.2.13); anche nel caso Riemanniano abbiamo un controllo sulla distanza tra i baricentri $H_{r,c}^R(y)$ e $H_{r,c}^R(y')$ delle misure μ_y^c, $\mu_{y'}^c$ in termini della differenza tra le densità di queste misure e della distanza tra y e y' .*
L'idea è semplice ma, a priori, fornisce una stima abbastanza grezza per la costante di Lipschitz dell'applicazione baricentro $H_{r,c}^R$ (è facile accorgersi che, nel caso generale, è impossibile fornire un limite superiore prossimo ad 1 per questa costante). Tuttavia, la

[7] Là aussi, l'idée est simple, mais sa réalisation délicate : il s'agit de faire apparaître le point $h(y)$ lui-même comme le barycentre d'une mesure ν_y^c (centrée en $h(y)$) proche de la mesure μ_y^c, puis de contrôler (comme dans le cas euclidien) la distance entre les barycentres $H_{r,c}^R(y)$ et $h(y)$ des mesures μ_y^c et ν_y^c à l'aide de l'écart entre ces deux mesures.

[8] Il parametro c è il coefficiente di decrescita esponenziale della densità di ciascuna delle misure μ_y^c in funzione della distanza dal punto y, il parametro R fornisce il valore della distanza a partire dalla quale si applica un "cut off" alla densità della misura μ_y^c in modo tale che R sia un maggiorante per il diametro massimale dei supporti delle misure μ_y^c, il parametro r invece è imposto dalla presenza del cut-locus di (Y, g): esso misura l'errore introdotto quando si rimpiazza la distanza al punto y con una sua regolarizzazione (vedere la sezione 1.3.1.1).

[9] Questo permette di evitare i problemi causati dalla presenza del cut-locus di (X, g_0), problema che non si incontra nel caso classico di curvatura negativa o nulla.

[10] La nozione di baricentro qui utilizzata verrà precisata nella Proposizione 1.2.13.

[11] La funzione di Leibniz è definita dalla formula: $\mathcal{B}_\mu(x) = \int_{\mathbb{E}} d(x, z)^2 \, d\mu(z)$.

[12] Nel caso di curvatura negativa o nulla questo procedimento funziona senza porre alcuna ipotesi sulla taglia del supporto della misura μ . D'altra parte, senza ipotesi sul segno della curvatura sezionale, il ragionamento funziona unicamente se il supporto è di diametro controllato, poiché ciò permette di evitare i problemi causati dalla presenza del cut-locus sulla varietà di referenza (X, g_0) .

stima per l'energia puntuale (resp. per il determinante Jacobiano) di $H^R_{r,c}$ *data nel Teorema 1.2.1 è ottimale, nel senso che il limite superiore è vicino*[13] *al valore ottimale* n *(risp.* 1 *) quando l'applicazione* h *è una* ε*-approssimazione di Gromov-Hausdorff.*
Una delle difficoltà nella dimostrazione del Teorema 1.2.1 sta nel fatto che per fare i conti è necessario mettersi in coordinate esponenziali centrate in $H^R_{r,c}(y)$ *(il baricentro della misura* μ^c_y *). Siamo quindi costretti a sostituire nelle formule* $h(y)$ *con* $H^R_{r,c}(y)$ *e a stimare l'errore commesso effettuando questa sostituzione, il che rende necessaria una stima più fine della distanza tra* $h(y)$ *e* $H^R_{r,c}(y)$ *: questo è fatto*[14] *nella Proposizione 1.3.4. Una delle conseguenze della Proposizione 1.3.4 è che, quando l'applicazione iniziale è una* ε*-approssimazione di Gromov-Hausdorff,* $H^R_{r,c}$ *è anch'essa una* ε'*-approssimazione di Gromov-Hausdorff (dove* $\varepsilon' = C \cdot \varepsilon^{\frac{3}{4}}$ *).*
Corollario del Teorema 1.2.1 e della teoria dei τ*-rivestimenti sviluppata da G. Reviron (*[**Rev**]*) è il Teorema 1.4.1 (dimostrato originariamente da L. Sabatini* [**Saba1**]*) che fornisce un limite superiore esplicito e semplice per l'energia puntuale (risp. il determinante Jacobiano) di* $H^R_{r,c}$ *che è prossimo ad* n *(risp. ad* 1 *) quando l'applicazione iniziale è una* ε*-approssimazione di Gromov-Hausdorff.*

Sketch of the chapter 1: *In this chapter we shall introduce the barycenter method in curvature of variable sign (sections 1.2, 1.3). To prove the main theorem (Theorem 1.2.1) the idea is the following: starting from a given quasi isometry we construct a map* $H^R_{r,c}$ *(homotopic to* h *, depending on the parameters*[15] R, c, r *and whose energy is sharply bounded from above, see the proof of the Theorem 1.2.1) by associating to any* y *in* (Y, g) *a measure* μ^c_y *on* X *such that one can define a notion of barycenter for this measure; we then define* $H^R_{r,c}(y)$ *as the barycenter*[16] *of the measure* μ^c_y *.*
In the euclidean space $\mathbb{E}$ *, the barycenter* b_μ *of some measure* μ *can be defined as the point where the Leibniz function*[17] $x \mapsto \mathcal{B}_\mu(x)$ *attains its minimum and it is then classical to estimate the distance from* x *to the barycenter* b_μ *in terms of the gap* $\mathcal{B}_\mu(x) - \mathcal{B}_\mu(b_\mu)$ *.*
In section 1.2.3, the similar properties are studied in the case of Riemannian manifolds: to any finite measure μ *(with support of controlled diameter), we associate a Leibnitz function* $\mathcal{B}_\mu$ *in a similar way and define the barycenter as the point where* $\mathcal{B}_\mu$ *attains its minimum*[18] *(see Proposition 1.2.13); as in the euclidean case, we can then estimate the distance from* x *to the barycenter* b_μ *in terms of the gap* $\mathcal{B}_\mu(x) - \mathcal{B}_\mu(b_\mu)$ *(see the stability/concavity inequality given by Lemma 1.2.12). The idea is then to use this estimate to control the distance between the barycenters* $H^R_{r,c}(y)$ *and* $H^R_{r,c}(y')$ *of the measures* μ^c_y*,* $\mu^c_{y'}$ *in terms of the difference between the densities of these two measures, and then in terms*

[13] Ai fini delle applicazioni, affinché esse siano ottimali, è importante che il maggiorante dell'energia puntuale (risp. del determinante Jacobiano) ottenuto sia vicino ad n (risp. di 1) per dei valori opportuni dei parametri, come nel caso dei teoremi 1.2.1 e 1.4.1, poiché il valore n (risp. 1) è precisamente l'energia (risp. il determinante Jacobiano) di una isometria.

[14] Anche in questo caso, l'idea è semplice ma la sua realizzazione è delicata: si tratta di far apparire il punto $h(y)$ anch'esso come baricentro di una opportuna misura ν^c_y (centrata in $h(y)$) vicina alla misura μ^c_y , quindi di controllare (come nel caso euclideo) la distanza tra i baricentri $H^R_{r,c}(y)$ e $h(y)$ delle misure μ^c_y e ν^c_y attraverso lo scarto tra le due misure.

[15] The parameter c is a coefficient which measures the exponential rate of decreasing of the density of any measure μ^c_y in function of the distance from the point y , the parameter R gives the value of the distance at which me must apply a cut-off to the density of the measure μ^c_y in such a way that R is an upper bound for the diameter of the support of the measures μ^c_y . The parameter r is imposed by the existence of the cut-locus of the manifold (Y, g) : it measures the error done when we replace the distance at y by a regularization of this distance-function (see section 1.3.1.1).

[16] We shall explain the notion of barycenter that we shall use in Proposition 1.2.13.

[17] The Leibniz function is the function $\mathcal{B}_\mu$ defined by $\mathcal{B}_\mu(x) = \int_{\mathbb{E}} d(x, z)^2 \, d\mu(z)$.

[18] In nonpositive curvature this works without any assumption on the diameter of the support of the measure μ . On the contrary, in curvature of any sign, this only works for measures whose support has controlled diameter, in order to overcome the problems arising the existence of a cut-locus on the reference manifold (X, g_0) .

of the distance between y *and* y' *.*
This idea is simple, but a priori it gives just a rough estimate for the Lipschitz constant of the map $H^R_{r,c}$ *(it is easy to see that, in the general case, it is not possible to bound from above this constant by a number close to* 1 *). On the contrary, the upper bounds for the pointwise energy (resp. for the Jacobian determinant) of* $H^R_{r,c}$ *provided by the main theorem 1.2.1 are sharp, in the sense that these upper bounds are close*[19] *to* n *(resp. to* 1 *) when the initial map* h *is a Gromov-Hausdorff* $\varepsilon-$*approximation.*
One of the difficulties, in the proof of Theorem 1.2.1, is that it is necessary to compute everything with respect to a unique system of exponential coordinates centered at $H^R_{r,c}(y)$ *(the barycenter of* μ^c_y *). This compels to change* $h(y)$ *for* $H^R_{r,c}(y)$ *and to bound the error induced by this change, we thus have to get a small upper bound for the distance between* $h(y)$ *and* $H^R_{r,c}(y)$ *: this is obtained*[20] *in the Proposition 1.3.4. Another consequence of the Proposition 1.3.4 is to prove that, when the initial map* h *is a Gromov-Hausdorff* ε*-approximation, then* $H^R_{r,c}$ *is a Gromov-Hausdorff* ε'*-approximation (where* $\varepsilon' = C \cdot \varepsilon^{\frac{3}{4}}$ *).*
Once proved Theorem 1.2.1 we obtain, as a corollary of the Theorem 1.2.1 and of the theory of τ*-coverings developed by G. Reviron (*[**Rev**]*), the Theorem 1.4.1 (initially due to L. Sabatini* [**Saba1**]*) which provides a simple explicit upper bound for the pointwise energy (resp. for the Jacobian determinant) of* $H^R_{r,c}$ *which is close to* n *(resp. close to* 1 *) when the initial map* h *is a Gromov-Hausdorff* ε*-approximation.*

1.1. Strategic foreword

The Barycenter Method is a technique introduced during the nineties by G. Besson, G. Courtois and S. Gallot ([**BCG1**],[**BCG2**]), in order to give an answer to the Minimal Entropy conjecture (conjectured by M. Gromov and implicitely by A. Katok, see [**Gro5**], [**Katok**]) and the Minimal Volume conjecture (conjectured by M. Gromov , [**Gro4**])

We are not interested into a detailed description of the proofs of these conjectures, but we find useful to give a synthetic overview on the strategy, in view of the detailed treatment of the last generalization of the Barycenter Method that we shall present in sections §§1.2, 1.3, 1.4 (see also [**Saba1**]).

We try to summarize the original technique.

First they construct an isometric action ρ of $\pi_1(X)$ on the space $L^2(\partial\tilde{X}, d\vartheta)$ the space of L^2-functions on $(\partial\tilde{X}, d\vartheta)$ (the ideal boundary of the symmetric space $(\tilde{X}, \tilde{g}_0)$ endowed with $d\vartheta$, the standard probability measure on $S^{n-1} \simeq \partial\tilde{X}$).

Then they exhibit a family $\{\Phi_c\}$ of ρ-equivariant, Lipschitz (with an explicit control on the Lipschitz constant) embeddings of the Riemannian manifold $(\tilde{X}, \tilde{g})$ into the space $L^2(\partial\tilde{X}, d\vartheta)$. The parameter c is chosen to be strictly greater than the volume entropy of the metric g.

Next they show that there is a well defined notion of the *barycenter* of a positive measure of finite mass, without atoms on $\partial\tilde{X}$. By *barycenter* they intend a law which assigns to any positive measure of finite mass, without atoms μ on $\partial\tilde{X}$ a unique point $x = \mathrm{bar}(\mu)$. In this case the assignement is the following: we associate to any positive measure of finite

[19] In view of the applications to be sharp, it is important that the upper bound for the pointwise energy (resp. for the Jacobian determinant) that we obtain is close to n (resp. close to 1) for suitable values of the parameters (as it is the case in Theorems 1.2.1 and 1.4.1) because the value n (resp. 1) is precisely the energy (resp. the Jacobian determinant) of an isometry.

[20] Also in this case the idea is simple. The idea is to show that the point $h(y)$ is the barycenter of a suitable measure ν^c_y (centered at $h(y)$) close to the measure μ^c_y, and to control (as in the euclidean case) the distance between the barycenters of $H^R_{r,c}$ and $h(y)$ of the measures μ^c_y and ν^c_y by means of the difference between the two measures.

mass, without atoms μ the unique point in $(\tilde{X}, \tilde{g}_0)$ such that the following equalities hold:

$$\int_{\partial \tilde{X}} dB_{(\tilde{x},\vartheta)}(e_i)\, d\mu(\vartheta) = 0 \tag{1.2}$$

where $B : \tilde{X} \times \partial\tilde{X} \to \mathbb{R}$ denotes the Busemann function of the locally symmetric metric $\tilde{g}_0$ and where $\{e_i\}$ is a basis of $T_{\tilde{x}}\tilde{X}$. Such a point exists and is unique. In particular the *barycenter* is equivariant with respect to the actions of the isometry group of $(\tilde{X}, \tilde{g}_0)$ on the space of measures by push-forward and on the Riemannian manifold $(\tilde{X}, \tilde{g}_0)$ by isometries.

The next step is to observe that if μ is a positive measure of finite mass, without atoms and $\nu = \alpha \cdot \mu$ for $\alpha > 0$, then the barycenter of ν coincide with the barycenter of μ. Moreover, if μ has a density which is in $L^2(\partial\tilde{X}, d\vartheta)$, it makes sense to study the barycenter in restriction to the intersection S^{∞}_{+} of the positive cone with the unit sphere in $L^2(\partial\tilde{X}, d\vartheta)$. Let $\pi : S^{\infty}_{+} \to \tilde{X}$ be the map $\pi(\varphi) = \mathrm{bar}(\varphi(\vartheta)^2\, d\vartheta)$. They show that π is a $\pi_1(X)$-equivariant, C^1-submersion and that the differential of π can be explicitely computed with respect to two orthonormal bases, one of $\mathcal{H}_{\varphi}$, the horizontal subspace of the submersion π at $\varphi \in \pi^{-1}(\tilde{x})$, and the other in $T_{\tilde{x}}\tilde{X}$.

Taking the barycenter of the image measures of the immersions Φ_c they obtain a family of equivariant C^1-maps $H_c : \tilde{X} \to \tilde{X}$, $H_c = \pi \circ \Phi_c$. The map H_c is called the *barycenter map*. Moreover, in §5 of [**BCG1**] they show that the differentials of the maps H_c can be explicitely controlled, and the control is sharp as the parameter c tends to the volume entropy of $(\tilde{X}, \tilde{g})$.

As a consequence in [**BCG1**] the authors go further, obtaining other inequalities and rigidity results.

We stress the fact that the *whole* argument strongly relies on the assumption $\sigma_0 \leq 0$.

It is evident that there is no hope of finding analogous rigidity results for manifolds with curvature of variable sign, however, keeping in mind the previous *road map*, we shall see how to treat this case.

1.2. Barycenter Method in curvature of variable sign

1.2.1. Introduction. We shall present a new version of Besson Courtois and Gallot's Barycenter Method. During the years the original method has been slightly modified in order to answer different questions (see for example ([**BCG4**], [**Samb2**]).
In 2009 L. Sabatini gave a first version of the Barycenter Method in curvature of variable sign ([**Saba1**]). Unfortunatly, since he was looking for a different kind of result (namely he proved some volume estimates without curvature assumptions), the whole description of the machinery became rather technical. The purpose of this section (and of the next one) is to present a simpler description of the barycenter method in curvature of variable sign. We state the main theorem:

THEOREM 1.2.1. *Let (X, g_0) and (Y, g) be two connected, complete Riemannian manifolds of dimension n. Let us denote σ and σ_0 respectively the sectional curvature of (Y, g) and (X, g_0). Assume that (X, g_0) is simply connected and that $|\sigma_0| \leq \kappa_0^2$ and $|\sigma| \leq \kappa^2$. As X, Y could be non-compact we assume* $\mathrm{inj}(X, g_0) > 0, \mathrm{inj}(Y, g) > 0$. *Let us denote by ρ, d the Riemannian distances associated to g_0, g respectively.*
Let Γ_X and Γ_Y be two discrete subgroups of isometries acting freely and properly discontinuously on (X, g_0) and (Y, g) respectively. Suppose that there exist $\lambda : \Gamma_Y \to \Gamma_X$ isomorphism and two maps $f : X \to Y$, $h : Y \to X$ which are equivariant with respect to λ, λ^{-1} respectively and which satisfy for every $x \in X$ and $y \in Y$:

$$\rho(h(y), x) \leq \alpha \cdot d(y, f(x)) + \varepsilon, \quad d(y, f(x)) \leq \alpha \cdot \rho(h(y), x) + \varepsilon, \tag{1.3}$$

$$\rho(x,(h\circ f)(x))\le\varepsilon,\quad d(y,(f\circ h)(y))\le\varepsilon. \tag{1.4}$$

for some $\alpha\ge1$ *and*

$$0\le\varepsilon\le\frac{2\ \mathrm{rad}(X,g_0)}{5\alpha+1} \tag{1.5}$$

where

$$\mathrm{rad}(X,g_0)=\frac{1}{5}\cdot\min\left\{\mathrm{inj}(X,g_0),\frac{\pi}{2\,\kappa_0}\right\}\ .$$

Let $R_1=\alpha\cdot(R+\varepsilon+2r)$ *and consider the function*

$$\delta_{n,\alpha,\varepsilon,\kappa_0}(R,r,c)=\min\Bigg\{2R_1\,c\ ;$$

$$(n+1)^2\,2^{\frac{n-1}{9}}\ \alpha^{n+1}\,e^{2c\,(\varepsilon+r)}\cdot\left[c\left(\varepsilon+\frac{3\,r}{2}\right)+\alpha^{2n+3}\,(\alpha-1)+\kappa_0^2\,R_1^2+3^{n+4}\,\frac{\varepsilon}{R}\right]^{\frac12}\Bigg\}$$

We associate to $(n,\,\alpha,\,\varepsilon,\,\kappa_0)$ *the subset* $\mathcal{D}_{n,\alpha,\varepsilon,\kappa_0}\subset\mathbb{R}^3_+$ *of those elements* $(R,\,r,\,c)\in\mathbb{R}^3_+$ *satisfying the conditions*

$$\alpha\,(R+2\varepsilon)<\mathrm{rad}(X,g_0)\ . \tag{1.6}$$

$$r\le\min\left\{\frac{1}{\kappa}\ ;\ \mathrm{inj}(Y,g)\ ;\ \frac12\left[\frac{\mathrm{rad}(X,g_0)}{\alpha}-(R+2\varepsilon)\right]\ ;\ \frac{\varepsilon}{4}\right\}\ . \tag{1.7}$$

$$\varepsilon+2r<\frac{R}{2}\ . \tag{1.8}$$

Assume that $\mathcal{D}_{n,\alpha,\varepsilon,\kappa_0}\neq\varnothing$. *Then for any* $(R,\,r,\,c)\in\mathcal{D}_{n,\alpha,\varepsilon,\kappa_0}$ *there exists a* λ-*equivariant,* C^1-*map* $H^R_{r,\,c}$ *such that* $\rho(H^R_{r,c}(y),h(y))\le\frac{\delta_{n,\alpha,\varepsilon,\kappa_0}(R,r,c)}{c}$ *and whose point-wise energy* ($e_y(H^R_{r,c})=\sum_{i=1}^n\,\|\ d_yH^R_{r,c}(e_i)\ \|^2$ *for any* $\{e_i\}_{i=1}^n$ *orthonormal basis of* T_yY) *satisfies:*

$$e_y(H^R_{r,\,c})\le n\cdot\alpha^{4n+2}\cdot e^{\frac{\delta}{\alpha}+c(4\varepsilon+6r)}\cdot(1+(\kappa\,r)^2)^2\cdot\left[1+\frac92\left(\alpha\,(R+\varepsilon+2r)+\frac{\delta}{c}\right)^2\kappa_0^2\right]^2\cdot$$

$$\cdot\left[1+\frac{\kappa_0^2}{2}\left(\alpha\,(R+\varepsilon+2r)+\frac{\delta}{c}\right)^2\right]^{2(n-1)}\cdot\left[1+(n+1)\cdot3^{n+1}\,\frac{2\varepsilon+3r}{R+\varepsilon+2r}\right]^2\cdot$$

$$\cdot\left[1+\frac{\delta}{n\,\alpha}\left(1+\frac{n+1}{c\,(R+\varepsilon)}\right)\right] \tag{1.9}$$

where $\delta=\delta_{n,\alpha,\varepsilon,\kappa_0}(R,r,c)$. *Thus, for* r *sufficiently small we obtain the following bound:*

$$e_y(H^R_{r,\,c})\le n\cdot\alpha^{4n+2}\cdot e^{\frac{\delta}{\alpha}+5\,c\varepsilon}\cdot\left[1+\frac{\kappa_0^2}{2}\left(\alpha\,(R+\varepsilon)+\frac{\delta}{c}\right)^2\right]^{2(n-1)}\cdot$$

$$\cdot\left[1+\frac92\left(\alpha\,(R+\varepsilon)+\frac{\delta}{c}\right)^2\kappa_0^2\right]^2\cdot\left[1+2\,(n+1)\,3^{n+1}\cdot\frac{\varepsilon}{R+\varepsilon}\right]^2\cdot\left[1+\frac{\delta}{n\,\alpha}\left(1+\frac{n+1}{c\,(R+\varepsilon)}\right)\right]$$

(1.10)

Moreover, if h *is continuous* $H^R_{r,\,c}$ *is* λ-*equivariantly homotopic to* h.

Remark 1.2.2 (The set $\mathcal{D}_{n,\alpha,\varepsilon,\kappa_0}$). Conditions (1.6), (1.7), (1.8), show that when $\alpha\to1$ and $\varepsilon\to0$ there exists a non empty open set $\mathcal{D}_{n,\alpha,\varepsilon,\kappa_0}\subset\mathbb{R}^3_+$ of values $(R,r,c)\in\mathbb{R}^3_+$ satisfying all the conditions: when ε it is sufficiently small and $\alpha=\left(1+\frac{1}{M\,(2\,(n+2))}\right)$, with $M\in\mathbb{N}$ sufficiently large, the triple $(\varepsilon^{1/4}\,\kappa_0^{-1},\,[\kappa_0\,\varepsilon]^{-1/2},\,\varepsilon^2\,\kappa_0^{-1})$ is in $\mathcal{D}_{n,\alpha,C,\kappa_0}$.

REMARK 1.2.3 (Optimality). We underline the fact that the bound for the pointwise energy of the map $H_{r,\,c}^{R}$ is *sharp* in the following sense: when the intial data tend to be sharp (that is $\alpha \to 1$, $\varepsilon \to 0$) we can choose the parameters $(R,\, c,\, r) \in \mathcal{D}_{n,\alpha,\varepsilon,\kappa_0}$ to obtain the convergence of the right hand term of (1.9) to n: as we shall show in section §1.4 the choices made for the parameters in the subsection §1.4.3 give the estimate announced in the Theorem 1.4.1, which tends to be sharp as $\varepsilon \to 0$ (in §1.4 α appears as an explicit function of the Gromov-Hausdorff distance ε, converging to 1 when ε goes to 0).

REMARK 1.2.4 (The geometric assumptions on (Y,g)). We want to stress the fact that the geometric assumptions on (Y,g), $\operatorname{inj}(Y,g) > 0$, $|\sigma| \le \kappa^2$ are automatically satisfied when (Y,g) is a Riemannian covering of some compact Riemannian manifold (*i.e.* when the action of Γ_Y is cocompact).

REMARK 1.2.5 (The parameter r). The dependence of r on the geometry of (Y,g) is necessary, but, when the sectional curvature of (Y,g) is bounded at infinity (and in particular when (Y,g) is a Riemannian covering of some compact manifold), there exists some κ such that $|\sigma| \le \kappa^2$ and then the parameter r can be chosen arbitrarily small, thus this parameter has a arbitrarily small (though non zero) influence on the condition (1.8).

REMARK 1.2.6 (Best values and Lipschitz maps). Assume that Γ_X and Γ_Y act cocompactly on X, Y respectively and assume that $\mathcal{D}_{n,\alpha,\varepsilon,\kappa_0}$ is not empty. Consider a sequence $\{(R_k, r_k, c_k)\}_{k\in\mathbb{N}} \subset \mathcal{D}_{n,\alpha,\varepsilon,\kappa_0}$ such that

$$e_y(H_{r_k,c_k}^{R_k}) \longrightarrow \inf_{(R,r,c)\in\mathcal{D}_{n,\alpha,\varepsilon,\kappa_0}} \{e_y(H_{r,\,c}^{R})\}$$

Since the actions are equivariant the maps $H_{r_k,c_k}^{R_k}$'s induce maps $\bar{H}_{r_k,c_k}^{R_k} : Y/\Gamma_Y \to X/\Gamma_X$. An Ascoli-Arzelà argument shows that, up to extraction of a subsequence, the maps $\bar{H}_{r_k,c_k}^{R_k}$ converge uniformly to a continuous map $\bar{H}$. It follows that the maps $H_{r_k,c_k}^{R_k}$ converge uniformly to a continuous map H (lift of $\bar{H}$). Moreover the condition satisfied by the pointwise energy of the maps $H_{r_k,c_k}^{R_k}$ implies that the $H_{r_k,c_k}^{R_k}$ are Lipschitz with bounded Lipschitz constant, and thus the map H is Lipschitz, where the Lipschitz constant is given by the square root of the infimum of the values of $e_y(H_{r,\,c}^{R})$ for $(R,r,c) \in \mathcal{D}_{n,\alpha,\varepsilon,\kappa_0}$.

The lack of strong assumptions on the sectional curvature of (X, g_0) (*i.e.* constancy, constancy of sign) induces two different problems:

(A) As the curvature is not constant we do not have explicit formulas for the volume form and for the Hessian of the distance function; we have to use Rauch's Theorem and Bishop-Gunther's Theorem to obtain estimates of these invariants.

(B) The Hessian of the distance function is even not defined when the distance function is greater than $\operatorname{inj}(X, g_0)$. Thus *a priori* the barycenter of a measure μ is not defined when $\operatorname{supp}(\mu) \not\subset B_{g_0}(x_0, 2\cdot \operatorname{inj}(X,g_0))$ for some $x_0 \in X$.

We shall produce a map $y \mapsto \mu_y^c$ from Y into a space of *localized measures of finite mass* on X, i. e. a space of measures of finite mass whose supports have sufficiently small diameter in order that each of these measures admits a "barycenter" (the precise definition of the "barycenter" will be given in the Proposition 1.2.13). The barycenter map will then be $y \mapsto$ barycenter of μ_y^c.

1.2.2. Two results from comparison geometry. Throughout this section we shall use the following comparison results: Rauch's Theorem and Bishop-Gunther's Theorem. We give statements which are adapted to our purposes (proofs can be found in [**Ch-Eb**]).

THEOREM 1.2.7 (Rauch). *Let* (X, g_0) *be a Riemannian manifold of dimension* n. *Let us denote by* ρ *the Riemannian distance associated to the metric* g_0 *and by* σ_0 *the sectional curvature. For any* $(x, z) \in X \times X$ *such that* $\rho(x, z) \leq \text{inj}(X, g_0)$:

(i) *if* $\sigma_0 \leq \kappa_0^2$ *and* $\rho(x, z) < \frac{\pi}{\kappa_0}$:

$$Dd(\rho)|_{(x,z)} \geq \frac{\kappa_0}{\text{tg}(\kappa_0\, \rho(x, z))} \cdot (g_0 - d\rho \otimes d\rho) \; ;$$

(ii) *if* $\sigma_0 \geq -\kappa_0^2$:

$$Dd(\rho)|_{(x,z)} \leq \frac{\kappa_0}{\text{th}(\kappa_0\, \rho(x, z))} \cdot (g_0 - d\rho \otimes d\rho) \; ;$$

where the derivatives are computed with respect to the first variable x.

THEOREM 1.2.8 (Bishop-Gunther). *Let* (X, g_0) *be a Riemannian manifold and let* $x \in X$. *The exponential map* $\exp_x$ *is a diffeomorphism between* $B(0_x, \text{inj}(X, g_0)) \subset T_xX$ *and* $B(x, \text{inj}(X, g_0))$. *We define*

$$\varphi : (0, \text{inj}(X, g_0)) \times S^{n-1} \to B(x, \text{inj}(X, g_0)) \setminus \{x\}, \; \varphi(t, v) = \exp_x(t\,v)$$

Let us write $\varphi^* dv_{g_0} = \vartheta(t, v)\, dt\, dv$ *(where* dv *denotes the standard measure on* S^{n-1}*) and* σ_0 *for the sectional curvature. Then we have:*

(i) *if* $\sigma_0 \leq \kappa_0^2$ *and* $t \leq \frac{\pi}{\kappa_0}$ *then* $\vartheta(t, v) \geq \left(\frac{\sin(\kappa_0 t)}{\kappa_0}\right)^{n-1}$;

(ii) *if* $\sigma_0 \geq -\kappa_0^2$ *then* $\vartheta(t, v) \leq \left(\frac{\sinh(\kappa_0 t)}{\kappa_0}\right)^{n-1}$.

1.2.3. Localized measures of finite mass. In this subsection we shall give a sufficient condition for a finite measure on (X, g_0) to admit a barycenter. First of all let us give the definition of barycentric radius of the manifold (X, g_0).

DEFINITION 1.2.9. Let (X, g_0) be a complete Riemannian manifold of dimension n and sectional curvature $|\sigma_0| \leq \kappa_0^2$. The *barycentric radius* of (X, g_0), $\text{rad}(X, g_0)$, is defined as: $\text{rad}(X, g_0) = \frac{1}{5} \cdot \min\{\text{inj}(X, g_0), \frac{\pi}{2\kappa_0}\}$.

Let $x_0 \in X$ and let μ be a finite measure whose support is contained into $\overline{B(x_0, R)}$ where $R < \text{rad}(X, g_0)$. Let us define the following "Leibniz function" from X to $\mathbb{R}^+$:

$$\mathcal{B}_\mu(x) = \int_X \rho(x, z)^2\, d\mu(z)$$

LEMMA 1.2.10. *The function* $\mathcal{B}_\mu$ *is continuous and* C^2 *in* $B(x_0, \text{inj}(X, g_0) - R)$. *Moreover, for any* $x \in B(x_0, \text{inj}(X, g_0) - R)$ *we have:*

$$\nabla \mathcal{B}_\mu(x) = \int_X \nabla(\rho_z^2)(x)\, d\mu(z)$$

and, for every $u, v \in T_xX$,

$$Dd\mathcal{B}_\mu(u, v) = \int_X Dd(\rho_z^2)|_x(u, v)\, d\mu(z) \; .$$

Proof. The function $x \to \rho(x, z)$ is continuous so for any $x_1 \in X$ and $z \in \text{supp}(\mu)$ if $x \in B(x_1, R)$ we have:

$$\rho(x, z)^2 \leq [\rho(x, x_0) + R]^2 \leq [\rho(x_1, x_0) + 2R]^2$$

Hence $\rho_x^2 = \rho(x, \cdot)^2$ is μ-integrable, and $\mathcal{B}_\mu$ is continuous. For what concerns the regularity of $\mathcal{B}_\mu$ observe that $\rho(x, z)^2$ is C^∞ on $B(x_0, i_0 - R) \times \overline{B(x_0, R)}$ (where $i_0 = \text{inj}(X, g_0)$). Moreover, the second derivatives of this function with respect to x are bounded by a constant (independent from x) when $(x, z) \in \overline{B(x_0, i_0 - R')} \times \overline{B(x_0, R)}$, for any $R' > R$; this follows from a standard compactness argument. By the Lebesgue's dominated derivation theorem for integrals, it follows that $\mathcal{B}_\mu$ is C^2 on $B(x_0, i_0 - R)$, and its derivatives are computed as shown in the statement. □

LEMMA 1.2.11. *The function* $\mathcal{B}_\mu$ *attains its minimum; every minimum point is in* $B(x_0, 2R)$ *and verifies:*

$$\int_X \nabla(\rho^2)|_{(x,z)}\, d\mu(z) = 0_x \tag{1.11}$$

Proof. A classical compactness argument proves that the restriction of $\mathcal{B}_\mu$ to $\overline{B(x_0, 2R)}$ attains its minimum at some point $b \in \overline{B(x_0, 2R)}$. Let $x \in X \setminus \overline{B(x_0, 2R)}$. Then for any $z \in \operatorname{supp}(\mu)$ we have:

$$\rho(x, z) \geq \rho(x, x_0) - \rho(z, x_0) > 2R - R = R$$

it follows that $\mathcal{B}_\mu(x) > R^2 \cdot \mu(X) \geq \mathcal{B}_\mu(x_0) \geq \mathcal{B}_\mu(b)$, and thus b is a global absolute minimum for $\mathcal{B}_\mu$. In the Lemma 1.2.10 we proved that $\mathcal{B}_\mu$ is C^2 in $B(x_0, \operatorname{inj}(X, g_0) - R)$ for every $x \in B(x_0, 4R) \subset B(x_0, \operatorname{inj}(X, g_0) - R)$. We thus have $\nabla \mathcal{B}_\mu(b) = 0$. □

Let us give a proof of the unicity of the minimum point of $\mathcal{B}_\mu$:

LEMMA 1.2.12. *Let* (X, g_0) *be a complete Riemannian manifold of dimension* n *and sectional curvature* $|\sigma_0| \leq \kappa_0^2$. *Let* μ *be a finite measure on* (X, g_0) *whose support is contained into* $B(x_0, R)$ *for* $R < \operatorname{rad}(X, g_0)$, $x_0 \in X$. *For any couple of points* $x, b \in B(x_0, 2\operatorname{rad}(X, g_0))$ *if* b *satisfies equation (1.11) then:*

$$\mathcal{B}_\mu(x) - \mathcal{B}_\mu(b) \geq \frac{\kappa_0 R'}{\operatorname{tg}(\kappa_0 R')} \cdot \rho(x, b)^2 \cdot \mu(X) \tag{1.12}$$

where $R' = R + \rho(x_0, b) + \rho(x, x_0)$.

Proof. First we observe that since $\rho(x, b) \leq 4 \cdot \operatorname{rad}(X, g_0) < \operatorname{inj}(X, g_0)$ there exists a unique minimizing geodesic from x to b. Let c be the geodesic defined by the conditions $c(0) = b$, $c(t_0) = x$ (with $t_0 = \rho(x, b)$); for all $t \in [0, t_0]$ and $z \in \operatorname{supp}(\mu)$:

$$\rho(c(t), z)^2 - \rho(c(0), z)^2 = t\, d\rho^2|_{(c(0),z)}(\dot{c}(0)) + \int_0^t (t - s)\, Dd\rho^2|_{(c(s),z)}(\dot{c}(s), \dot{c}(s)), ds$$

and, applying the version of the Rauch's Theorem (Theorem 1.2.7) mentioned above we obtain

$$\rho(c(t), z) \leq \rho(x_0, z) + \min[\rho(x_0, b) + \rho(c(t), b);\ \rho(x_0, x) + \rho(c(t), x)] \leq$$
$$\leq R + \frac{1}{2} \cdot [\rho(x_0, b) + \rho(b, x) + \rho(x, x_0)] \leq$$
$$\leq R + \rho(x_0, b) + \rho(x, x_0) = R' < 5R < \operatorname{inj}(X, g_0)$$

Now since $\rho(c(t), z) < \operatorname{inj}(X, g_0)$ we have that the function $t \to \rho(c(t), z)^2$ is regular. We develop the function at the first order in $t = 0$ and we find:

$$\rho(c(t), z)^2 - \rho(b, z)^2 \geq t \cdot d(\rho^2)|_{(b,z)}(\dot{c}(0)) + \int_0^t (t - s)\, \frac{2\kappa_0\, \rho(c(s), z)}{\operatorname{tg}(\kappa_0\, \rho(c(s), z))}\, ds$$

On the other hand the function $t \to \frac{2\kappa_0\, t}{\operatorname{tg}(\kappa_0 t)}$ is decreasing on $[0, \frac{\pi}{2\kappa_0}]$ and $\rho(c(t), z) \leq R' < \frac{\pi}{2\kappa_0}$ and thus: $\frac{2\kappa_0\, \rho(c(t),z)}{\operatorname{tg}(\kappa_0\, \rho(c(t),z))} \geq \frac{2\kappa_0\, R'}{\operatorname{tg}(\kappa_0\, R')}$. We plug this estimate into the previous inequality and we obtain:

$$\rho(c(t), z)^2 - \rho(b, z)^2 \geq t \cdot d(\rho^2)|_{(b,z)}(\dot{c}(0)) + \frac{\kappa_0\, R'}{\operatorname{tg}(\kappa_0\, R')} \cdot t^2$$

thus integrating with respect to μ:

$$\mathcal{B}_\mu(c(t)) - \mathcal{B}_\mu(b) \geq t \cdot \int_X d(\rho^2)|_{(b,z)}(\dot{c}(0))\, d\mu(z) + \frac{\kappa_0\, R'}{\operatorname{tg}(\kappa_0 R')} \cdot t^2 \cdot \mu(X)$$

Since by assumption b satisfies equation (1.11) we deduce that:

$$\mathcal{B}_\mu(c(t)) - \mathcal{B}_\mu(b) \geq \frac{\kappa_0\, R'}{\operatorname{tg}(\kappa_0\, R')} \cdot t^2 \cdot \mu(X)$$

and we conclude taking $t = t_0 = \rho(b, x)$. □

PROPOSITION 1.2.13. *Let μ be a finite measure on (X, g_0) whose support is contained in a geodesic ball $B(x_0, R)$ of radius R (for some $R < \mathrm{rad}(X, g_0)$). The measure μ admits a barycenter i.e. there exists a unique minimum point x for $\mathcal{B}_\mu$, denoted by* $\mathrm{bar}(\mu)$. *Moreover,* $\mathrm{bar}(\mu) \in B(x_0, 2R)$.

Proof. By the Lemma 1.2.10, $\mathcal{B}_\mu$ attains its minimum and every point where the minimum is attained astisfies the critical equation 1.2.11 and lies in $B(x_0, 2R)$; by the Lemma 1.2.12 $x,\, x' \in B(x_0, 2R)$ and $0 = \mathcal{B}_\mu(x') - \mathcal{B}_\mu(x) \geq \frac{\kappa_0\, R'}{\mathrm{tg}(\kappa_0\, R')}\, \mu(X)\, \rho(x, x')^2$ which implies that $x = x'$ and thus that the minimum is attained at a unique point. □

DEFINITION 1.2.14. We define the set of localized measures $\mathcal{L}(X, g_0)$ as the set of all finite measures without atoms on X whose support is contained in some geodesic ball of radius R (for some $R < \mathrm{rad}(X, g_0)$).

1.3. Proof of the Theorem 1.2.1

1.3.1. The barycenter map: definition and first properties. Let us recall that $\lambda : \Gamma_Y \to \Gamma_X$ is an isomorphism such that the map h (resp. f) is equivariant with respect to λ (resp. λ^{-1}). We construct a family of maps $\{H^R_{r,\,c}\}$ satisfying the following properties:

- **Equivariance:** $\forall \gamma \in \Gamma_Y,\, \forall y \in Y,\, H^R_{r,\,c}(\gamma\, y) = \lambda(\gamma)\, H^R_{r,\,c}(y)$.
- **Regularity:** $H^R_{r,\,c} \in C^1(Y, X)$.
- **Homotopy:** whenever h is continuous, $H^R_{r,\,c}$ is homotopic to h by an homotopy which is λ-equivariant.

We follow the *road-map* of section 1.1. The idea is to embed (Y, g) into $\mathcal{L}(X, g_0)$ in the most isometric way as possible and to take the barycenter of the push-forward measures.

1.3.1.1. *Regularized cut-off distance function and the barycenter map.* Let d denote the Riemannian distance on (Y, g), let r be arbitrarily small, and let d_r denote a function $d_r : Y \times Y \to \mathbb{R}^+$, C^∞ with respect to the first variable, satisfying:

(1) $d - r \leq d_r \leq d + r$;
(2) $\forall \gamma \in \mathrm{Is}(Y, g)$, and $\forall y,\, y' \in Y$, $d_r(\gamma\, y, \gamma\, y') = d_r(y, y')$;
(3) $\| \nabla d_r \|_g \leq 1 + c_1 \cdot (\kappa\, r)^2 < 1 + (\kappa\, r)^2$, for any $r < \min\{\mathrm{inj}(Y, g),\, \frac{1}{\kappa}\}$, where we put $c_1 = \frac{\cosh(1)}{2}$;

the construction of such an approximation of the distance function d is classical; it was originally introduced by K. Grove and K. Shiohama in [**Gr-Sh**]. For the sake of completeness we shall explain how such a regularization is constructed and we shall prove the aforementioned properties.

Let us consider a C^∞ function $\psi : [0, +\infty) \to [0, 1]$ such that $\psi \equiv 1$ in a neighbourhood of 0, $\psi(x) = 0$ for $x \geq 1$ and $\psi' \leq 0$. Take $r < \mathrm{inj}(Y, g)$ and define the function:

$$\psi_r(t) = \frac{\psi\left(\frac{t}{r}\right)}{\omega_{n-1} \int_0^r \psi\left(\frac{s}{r}\right)\, s^{n-1} dt}$$

where ω_{n-1} denotes the volume of the standard $(n-1)$-dimensional unit sphere $\omega_{n-1} = \mathrm{Vol}_{\mathrm{can}}(S^{n-1})$. Consider now a point $y \in Y$ and $B(0_y, r) \subset T_y Y$. If $f : Y \to \mathbb{R}$ is a Lipschitz function we define:

$$f_r(y) = \int_{B(0_y, r)} f(\exp_y(w))\, \psi_r(\| w \|)\, dw$$

By the Proposition 2.1 in [**Gr-Sh**] we know that f_r is a C^∞ function and that its differential is given by the formula:

$$(df_r)|_y = \int_0^r \int_{S_y^{n-1}} (df)_{\exp_y(tv)}(J_u(t))\, \psi_r(t)\, t^{n-1}\, dv\, dt \tag{1.13}$$

where J_u is the Jacobi field along the geodesic $c_v : t \to \exp_y(t\, v)$ satisfying the initial conditions $J_u(0) = u$, $\nabla_v J_u = 0$.

We take $f(y) = d(y,z)$ and we define

$$d_r(y,z) = \int_0^r \int_{S_y^{n-1}} d(\exp_y(t\,v), z)\, \psi_r(t)\, t^{n-1}\, dv\, dt\ .$$

Proof of property (1). By the definition of $d_r(y,z)$ we have that:

$$d_r(y,z) - d(y,z) = \int_{B(0_y,r)} \big(d(\exp_y(v), z) - d(y,z)\big)\cdot \psi_r(\| v \|)\, dv$$

By the triangle inequality $|d(\exp_y(v), z) - d(y,z)| < d(y, \exp_y(v)) \leq \| v \| < r$. □

Proof of property (2). Let γ be an isometry of (Y,g) and $d\gamma$ its differential. We remark that $d\gamma$ is an isometry from $B(0_y, r)$ to $B(0_{\gamma\,y}, r)$ (whose Jacobian determinant is equal to ± 1), hence, making the change of variables $w = d\gamma\, v$:

$$\int_{B(0_{\gamma\,y},r)} d(\exp_{\gamma\,y}(w), \gamma\, z) dw = \int_{B(0_y,r)} d(\exp_y(d\gamma(v)), \gamma\, z)\, |\operatorname{Jac}(d\gamma)|\, dv =$$

$$= \int_{B(0_y,r)} d(\exp_y(v), z)\, dv$$

which proves the property (2). □

Proof of property (3). Let us consider the differential of the function $d_r(y,z)$ (taken with respect to the variable y), which is given by the formula:

$$\nabla d_r(y,z) = \int_0^r \int_{S_y^{n-1}} g(\nabla d(\exp_y(t\,v), z), J_u(t))\, \psi_r(t)\, t^{n-1}\, dv\, dt$$

where J_u is the Jacobi field along $c_v : t \to \exp_y(t\,v)$ determined by the initial conditions $J_u(0) = u$, $\nabla_v J_u = 0$; by the properties of the distance function ∇d is a unitary vector field and $|g(\nabla d(\exp_y(t\,v), z), J_u(t))| \leq \| J_u(t) \|$. Since $r < \frac{1}{\kappa} < \frac{\pi}{2\kappa}$ and $\sigma < \kappa^2$ we know that there are no focal points for J_u in $(0,r)$ and, on the other hand we know that $\sigma \geq -\kappa^2$. Hence the assumptions of Rauch II ([**Ch-Eb**], Theorem 1.29) are satisfied and we can conclude that $\| J_u(t) \| \leq \| J_u^\kappa(t) \|_\kappa$ where J_u^κ is the corresponding Jacobi field in the model space of constant curvature $-\kappa^2$; we conclude by remarking that $\| J_u^\kappa(t) \|_\kappa = \cosh(\kappa\, t) \leq (1 + (\kappa\, t)^2) \leq (1 + (\kappa\, r)^2)$ because of the assumptions made on r . □

Given an approximation of the distance function, d_r , we can define a *cut-off distance* d_r^{cut} by composition with a cut-off function, *i.e.* a C^∞-function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

- $\forall t \geq 0$, $0 \leq \varphi'(t) \leq 1$;
- $\varphi(t) = t$ for $t \in [0, R - r]$;
- $\varphi(t) = R$ for $t \in [R + r, +\infty)$;

hence the cut-off distance $d_r^{cut} = \varphi \circ d_r$ satisfies the following properties:

(i) d_r^{cut} is a C^∞ function.
(ii) For any $\gamma \in \Gamma_Y$ and for any $y,\, y' \in Y$: $d_r^{cut}(\gamma\, y, \gamma\, y') = d_r^{cut}(y, y')$.
(iii) $\| \nabla d_r^{cut} \|_g \leq 1 + (\kappa\, r)^2$ for $d_r(y, y') \leq R + r$ ($= 0$ otherwise).

The next Lemma provides a simple but crucial estimate:

Lemma 1.3.1. *Let $f_+(x) = \max\{0,\, f(x)\}$. We have the following inequality:*

$$e^{-c\,(\varepsilon + r)} \left(e^{-c\,\alpha \rho(h(y),\,z)} - e^{-c\,(R - \varepsilon - r)}\right)_+ \leq e^{-c\, d_r^{cut}(y,\, f(z))} - e^{-c\, R} \leq$$

$$\leq e^{c\,(\varepsilon + 2r)} \left(e^{-\frac{c}{\alpha}\rho(h(y),\,z)} - e^{-c\,(R + \varepsilon + 2r)}\right)_+$$

Proof. By the definition of φ we have:

$$\min\{t-r\ ;\ R\} \leq \varphi(t) \leq \min\{t\ ;\ R\}$$

hence we obtain:

$$\min\{d_r(y,y')-r\ ;\ R\} \leq d_r^{cut}(y,y') \leq \min\{d_r(y,y')\ ;\ R\} \tag{1.14}$$

and finally:

$$\min\{\alpha^{-1}\cdot\rho(h(y),z)-\varepsilon-2r;\ R\} \leq d_r^{cut}(y,f(z)) \leq \min\{\alpha\cdot\rho(h(y),z)+\varepsilon+r\ ;\ R\}. \quad \square$$

Definition 1.3.2. Let $d\mu_y(z) = \left(e^{-c\,d_r^{cut}(y,\,f(z))} - e^{-c\,R}\right) dv_0(z)$. By the Lemma 1.3.1 the support of μ_y is contained into the geodesic ball $B(h(y),R_1)$ where $R_1 = \alpha\cdot(R+\varepsilon+2r)$. By assumption $R_1 < \mathrm{rad}(X,g_0)$. The *barycenter map* is the map given by:

$$H^R_{r,\,c}(y) = \mathrm{bar}^q(\mu_y)$$

Remark 1.3.3. Let $d\nu_y(z) = \left(e^{-c\,\alpha\rho(h(y),\,z)} - e^{-c\,(R-\varepsilon-r)}\right)_+ d((\exp_{h(y)})_*\mathcal{L}_{h(y)})$, where $\mathcal{L}_{h(y)}$ is the Lebesgue measure on $T_{h(y)}X$. Then we have

$$\mathrm{supp}(\nu_y) \subset B(h(y),\alpha^{-1}\,(R-\varepsilon-r))\,.$$

Hence, since $\alpha \geq 1$, $\mathrm{supp}(\nu_y) \subset B(h(y),R)$. We shall show that $h(y) = \mathrm{bar}(\nu_y)$. In fact, by the definition of ν_y we have

$$\frac{1}{2}\int_X \nabla(\rho_z^2)(h(y))\,d\nu_y(z) =$$

$$= \frac{1}{2}\int_{T_{h(y)}Y} \nabla(\rho^2_{\exp_{h(y)}(v)})(h(y))\left(e^{-c\,\alpha\rho(h(y),\exp_{h(y)}(v))} - e^{-c\,(R-\varepsilon-r)}\right)_+ d\mathcal{L}_{h(y)}(v) =$$

$$= \int_{T_{h(y)}} (-v)\,\left(e^{-c\,\alpha\|v\|} - e^{-c\,(R-\varepsilon-r)}\right)_+ d\mathcal{L}_{h(y)}(v) = 0_{h(y)}$$

which proves the equality.

1.3.1.2. *Distance of the barycenter map from the initial data.* The aim of this section is to give an estimate of the C^0-distance between the maps $H^R_{r,c}$ and h. We shall prove the following

Proposition 1.3.4 (Distance between $h(y)$ and $H^R_{r,c}(y)$). *Assume that*

$$\alpha\,(R+\varepsilon+2r) < \mathrm{rad}(X,g_0),$$

then we have the following estimate for the distance between $H^R_{r,\,c}$ and h:

$$\rho(h(y),H^R_{r,\,c}(y)) \leq \frac{1}{c}\cdot(n+1)^2\,2^{\frac{n-1}{9}}\,\alpha^{n+1}\,e^{c\,(2\varepsilon+3r)}\cdot$$

$$\cdot\left[c\left(\varepsilon+\frac{3}{2}r\right)+\alpha^{2n+3}\,(\alpha-1)+\kappa_0^2\,R_1^2+\frac{(R+\varepsilon+2r)^{n+2}}{(R-\varepsilon-r)^{n+3}}\,(2\varepsilon+3r)\right]^{\frac{1}{2}}$$

where $R_1 = \alpha\,(R+\varepsilon+2r)$. Moreover if $\varepsilon < \frac{R}{2}$ and $r < \frac{\varepsilon}{4}$:

$$\rho(h(y),H^R_{r,c}(y)) \leq \frac{1}{c}\,(n+1)^2\,2^{\frac{n-1}{9}}\,\alpha^{n+1}\,e^{2c\,(\varepsilon+3r)}\cdot$$

$$\cdot\left[c\left(\varepsilon+\frac{3\,r}{2}\right)+\alpha^{2n+3}\,(\alpha-1)+\kappa_0^2\,R_1^2+3^{n+4}\,\frac{\varepsilon}{R}\right]^{\frac{1}{2}}$$

and if moreover r is sufficiently small:

$$\rho(h(y),H^R_{r,c}(y)) \leq \frac{1}{c}\,(n+1)^2\,2^{\frac{n-1}{9}}\,\alpha^{n+1}\,e^{2c\,\varepsilon}\cdot\left[c\,\varepsilon+\alpha^{2n+3}\,(\alpha-1)+\kappa_0^2\,R_1^2+3^{n+4}\,\frac{\varepsilon}{R+\varepsilon}\right]^{\frac{1}{2}}$$

In order to prove the Proposition 1.3.4 we need to prove the following

LEMMA 1.3.5. *Under the assumptions of the Theorem 1.2.1 we have the following estimate for the distance between* $H^R_{r,\,c}$ *and* h *:*

$$\rho(h(y), H^R_{r,\,c}(y)) \le \frac{\delta'_{n,\varepsilon,\alpha,\kappa_0,}(R,r,c)}{c} = \frac{1}{c}\cdot\sqrt{\frac{\mathrm{tg}(3R_1\,\kappa_0)}{3R_1\,\kappa_0}\cdot\frac{n\,(n+1)}{2\,\alpha^2}}\cdot$$

$$\cdot\left(e^{c\,(2\varepsilon+3r)}\,\alpha^{2n+4}\left(\frac{\sinh(\kappa_0\,R_1)\,R}{\sin(\kappa_0\,R)\,R_1}\right)^{n-1}\left(\frac{\int_0^{c\,(R+\varepsilon+2r)}\,t^{n+2}\,e^{-t}\,dt}{\int_0^{c(R-\varepsilon-r)}\,t^{n+2}\,e^{-t}\,dt}\right)-1\right)^{\frac{1}{2}}$$

where $R_1 = \alpha\cdot(R+\varepsilon+2r)$ *.*

Proof. Let us remark that we have $H^R_{r,\,c}(y)\in B(h(y),2R_1)$. In particular we have $\rho(H^R_{r,\,c}(y),h(y)) < 2R_1$; however we would like to show that the distance between h and $H^R_{r,\,c}$ is small with respect to R_1 . Since $H^R_{r,\,c}(y)$ and $h(y)$ are respectively the barycenters of the measures μ_y , ν_y , we have by equation (1.12):

$$\mathcal{B}_{\mu_y}(h(y)) \ge \mathcal{B}_{\mu_y}(H^R_{r,\,c}(y)) + \frac{3R_1\,\kappa_0}{\mathrm{tg}(3R_1\,\kappa_0)}\cdot\rho(h(y),H^R_{r,\,c}(y))^2\cdot\mu_y(X) \tag{1.15}$$

$$\mathcal{B}_{\nu_y}(H^R_{r,\,c}(y)) \ge \mathcal{B}_{\nu_y}(h(y)) + \frac{3R_1\,\kappa_0}{\mathrm{tg}(3R_1\,\kappa_0)}\cdot\rho(h(y),H^R_{r,\,c}(y))^2\cdot\nu_y(X) \tag{1.16}$$

Using Bishop-Gunther's Theorem we can write:

$$\exp^*_{h(y)}\,dv_0 = \vartheta(t,v)\,dt\,dv \ge \left(\frac{\sin(\kappa_0\,t)}{\kappa_0\,t}\right)^{n-1} t^{n-1}\,dt\,dv \ge \left(\frac{\sin(\kappa_0\,t)}{\kappa_0\,t}\right)^{n-1} d\mathcal{L}_{h(y)}$$

From the previous inequality and using the Lemma 1.3.1 we see that

$$\left(\frac{\sin(\kappa_0\,R)}{\kappa_0\,R}\right)^{n-1} e^{-c\,(\varepsilon+r)}\mathcal{B}_{\nu_y}(H^R_{r,\,c}(y)) \le \mathcal{B}_{\mu_y}(H^R_{r,\,c}(y))$$

hence, if we sum to (1.15) inequality (1.16) times $\left[-\left(\frac{\sin(\kappa_0\,R)}{\kappa_0\,R}\right)^{n-1} e^{-c\,(\varepsilon+r)}\right]$, we get:

$$\frac{3R_1\,\kappa_0}{\mathrm{tg}(3R_1\,\kappa_0)}\,\rho(H^R_{r,\,c}(y),h(y))^2\left[\mu_y(X)+e^{-c\,(\varepsilon+r)}\left(\frac{\sin(\kappa_0\,R)}{\kappa_0\,R}\right)^{n-1}\nu_y(X)\right]\le$$

$$\le \mathcal{B}_{\mu_y}(h(y)) - \left(\frac{\sin(\kappa_0\,R)}{\kappa_0\,R}\right)^{n-1} e^{-c\,(\varepsilon+r)}\mathcal{B}_{\nu_y}(h(y)) \tag{1.17}$$

We want to estimate the unknown terms in (1.17). We start with $\mathcal{B}_{\mu_y}(h(y))$:

$$\mathcal{B}_{\mu_y}(h(y)) = \int_X \rho(h(y),z)^2\left(e^{-c\,d^{cut}_r(y,\,f(z))}-e^{-cR}\right)dv_0(z)\le$$

$$\le e^{c\,(\varepsilon+2r)}\int_X \rho(h(y),z)^2\left(e^{-\frac{c}{\alpha}\,\rho(h(y),\,z)}-e^{-c\,(R+\varepsilon+2r)}\right)_+dv_0(z)=$$

$$= e^{c\,(\varepsilon+2r)}\int_0^{+\infty} t^2\left(e^{-\frac{c}{\alpha}\,t}-e^{-\frac{c}{\alpha}\,R_1}\right)_+\cdot\int_{S^{n-1}}\vartheta(t,v)\,dv\,dt\le$$

$$\le\left(\frac{\sinh(\kappa_0\,R_1)}{\kappa_0\,R_1}\right)^{n-1}\frac{e^{c\,(\varepsilon+2r)}\,\omega_{n-1}\,\alpha^{n+2}}{(n+2)\,c^{n+2}}\int_0^{\frac{c}{\alpha}\,R_1} e^{-t}\,t^{n+2}\,dt$$

where the last inequality is proved using the fact that, for $a>0$, we have:

$$\int_0^{+\infty} t^n\left(e^{-a\,t}-e^{-a\,R}\right)_+dt = \frac{1}{(n+1)\,a^{n+1}}\int_0^{aR} t^{n+1}e^{-t}\,dt \tag{1.18}$$

Now we give an estimate of $\mu_y(X)$:

$$\mu_y(X) = \int_X\left(e^{-c\,d^{cut}_r(y,\,f(z))}-e^{-cR}\right)dv_0(z)\ge$$

$$\ge e^{-c\,(\varepsilon+r)}\left(\frac{\sin(\kappa_0\,R)}{\kappa_0\,R}\right)^{n-1}\omega_{n-1}\int_0^{+\infty} t^{n-1}\left(e^{-c\,\alpha t}-e^{-c\,(R-\varepsilon-r)}\right)_+dt\ge$$

$$\geq \left(\frac{\sin(\kappa_0\, R)}{\kappa_0\, R}\right)^{n-1} \cdot \frac{\omega_{n-1}\, e^{-c\,(\varepsilon+r)}}{n\cdot c^n\, \alpha^n} \cdot \int_0^{c\,(R-\varepsilon-r)} t^n\, e^{-t}\, dt$$

Computations in polar coordinates, together with equation (1.18):

$$\mathcal{B}_{\nu_y}(h(y)) = \frac{\omega_{n-1}}{(n+2)\,(c\,\alpha)^{n+2}} \cdot \int_0^{c\,(R-\varepsilon-r)} t^{n+2}\, e^{-t}\, dt$$

$$\nu_y(X) = \frac{\omega_{n-1}}{n\, c^n\, \alpha^n} \cdot \int_0^{c\,(R-\varepsilon-r)} t^n\, e^{-t}\, dt$$

Plugging these estimates into equation (1.17) and using the fact that the function $x \to \left(\frac{\int_0^x t^{n+2}\, e^{-t}\, dt}{\int_0^x t^n\, e^{-t}\, dt}\right)$ is an increasing function bounded above by $(n+2)\,(n+1)$ we obtain:

$$\rho(h(y), H^R_{r,\,c}(y))^2 \leq \frac{\mathrm{tg}(3R_1\,\kappa_0)}{3R_1\,\kappa_0} \cdot \frac{n\,(n+1)}{2\,\alpha^2} \cdot$$

$$\cdot \frac{1}{c^2} \cdot \left[e^{c\,(2\varepsilon+3r)}\, \alpha^{2n+4} \left(\frac{\sinh(\kappa_0\, R_1)\, R}{\sin(\kappa_0\, R)\, R_1}\right)^{n-1} \left(\frac{\int_0^{c\,(R+\varepsilon+2r)} t^{n+2}\, e^{-t}\, dt}{\int_0^{c\,(R-\varepsilon-r)} t^{n+2}\, e^{-t}\, dt}\right) - 1\right]$$

which is the desired inequality. □

Proof of the Proposition 1.3.4. Let us write

$$C_1 = \frac{\mathrm{tg}(3R_1\,\kappa_0)}{6R_1\,\kappa_0}, \quad C_2 = \frac{n(n+1)}{c^2\,\alpha^2}, \quad C_3 = e^{c\,(2\varepsilon+3r)},$$

$$C_4 = \alpha^{2n+4}, \quad C_5 = \left(\frac{\sinh(\kappa_0\, R_1)\, R}{\sin(\kappa_0\, R)\, R_1}\right)^{n-1}, \quad C_6 = \left(\frac{\int_0^{c\,(R+\varepsilon+2r)} t^{n+2}\, e^{-t}\, dt}{\int_0^{c\,(R-\varepsilon-r)} t^{n+2}\, e^{-t}\, dt}\right).$$

In the Lemma 1.3.5 we proved that:

$$\rho(h(y), H^R_{r,c}(y)) \leq \sqrt{C_1} \cdot \sqrt{C_2} \cdot (C_3\, C_4\, C_5\, C_6 - 1)^{\frac{1}{2}} \tag{1.19}$$

where $C_3,\, C_4,\, C_5,\, C_6 \geq 1$. First of all let us remark that:

$$\frac{C_3\, C_4\, C_5\, C_6 - 1}{C_3\, C_4\, C_5} \leq (C_3 - 1) + (C_4 - 1) + (C_5 - 1) + (C_6 - 1) \tag{1.20}$$

as is shown by the following computation:

$$[(C_3 - 1) + (C_4 - 1) + (C_5 - 1) + (C_6 - 1)]\, C_3\, C_4\, C_5 \geq$$

$$\geq (C_6 - 1)\, C_3\, C_4\, C_5 + (C_5 - 1)\, C_3\, C_4 + (C_4 - 1)\, C_3 + (C_3 - 1) =$$

$$= C_3\, C_4\, C_5\, C_6 - C_3\, C_4\, C_5 + C_3\, C_4\, C_5 - C_3\, C_4 + C_3\, C_4 - C_3 + C_3 - 1 = C_3\, C_4\, C_5\, C_6 - 1$$

We shall provide more readable estimates. By elementary computations we see that $\sqrt{C_2} \leq \frac{n+1}{c\,\alpha}$ and $\sqrt{C_1} \leq \sqrt{\frac{\mathrm{tg}(\frac{3\pi}{10})}{\frac{6\pi}{10}}} \leq \sqrt{0,855} < 0,95$. Now let us prove the following inequality:

$$C_5 = \left(\frac{\sinh(\kappa_0\, R_1)}{\kappa_0\, R_1} \cdot \frac{\kappa_0\, R}{\sin(\kappa_0\, R)}\right)^{n-1} \leq 1 + (n-1)\, 2^{\frac{n-1}{6}}\, \kappa_0^2\, R_1^2 \tag{1.21}$$

In order to prove (1.21) let us observe that:

$$\frac{\sinh(\kappa_0\, R_1)}{\kappa_0\, R_1} \leq \cosh(\kappa_0\, R_1) \leq 1 + \frac{\cosh(\kappa_0\, R_1)}{2}\, \kappa_0^2\, R_1^2 \leq 1 + 0,525\, \kappa_0^2\, R_1^2$$

and

$$\frac{\sin(\kappa_0\, R)}{\kappa_0\, R} \geq \cos(\kappa_0\, R) \geq 1 - \frac{\kappa_0^2\, R^2}{2}\,.$$

Since $R < R_1$ and $\kappa_0\, R_1 < \frac{\pi}{10}$ we obtain:

$$\frac{\sinh(\kappa_0\, R_1)}{\kappa_0\, R_1} \cdot \frac{\kappa_0\, R}{\sin(\kappa_0\, R)} \leq 1 + \left(\frac{205}{200 - \pi^2}\right) \kappa_0^2\, R_1^2$$

hence

$$\left(\frac{\sinh(\kappa_0\, R_1)}{\kappa_0\, R_1} \cdot \frac{\kappa_0\, R}{\sin(\kappa_0\, R)}\right)^{n-1} \leq \left(1 + \left(\frac{205}{200 - \pi^2}\right) \kappa_0^2\, R_1^2\right)^{n-1} \leq$$

$$\leq 1 + (n-1)\left(1 + \left(\frac{205}{200 - \pi^2}\right) \kappa_0^2\, R_1^2\right)^{n-2} \cdot \left(\frac{205\, \kappa_0^2\, R_1^2}{200 - \pi^2}\right) \leq$$

$$\leq 1 + (n-1)\left(1 + \left(\frac{205}{200-\pi^2}\right)\frac{\pi^2}{100}\right)^{n-1} \kappa_0^2\, R_1^2 \leq 1 + (n-1)\, 2^{\frac{n-1}{6}}\, \kappa_0^2\, R_1^2$$

where the last inequality follows from the fact that: $\left(\frac{205}{200-\pi^2}\right)\frac{\pi^2}{100} + 1 \leq \sqrt[6]{2}$. Finally we get the following inequality for C_5:

$$C_5 \leq \left(\frac{\sinh\left(\frac{\pi}{10}\right)}{\sin\left(\frac{\pi}{10}\right)}\right)^{n-1} \leq (1,03)^{n-1} \leq 2^{\frac{n-1}{20}} \tag{1.22}$$

The term C_3 can easily be estimated by:

$$C_3 \leq 1 + e^{c(2\varepsilon+3r)} \cdot c\,(2\varepsilon + 3r) \tag{1.23}$$

Finally let us estimate C_6:

$$C_6 \leq 1 + (n+3)\,\frac{(R+\varepsilon+2r)^{n+2}}{(R-\varepsilon-r)^{n+3}}\,(2\varepsilon+3r) \tag{1.24}$$

Let us prove inequality (1.24). First of all let us remark that

$$\int_0^x t^{n+2}\, e^{-t}\, dt \geq e^{-x} \int_0^x t^{n+2}\, dt = \frac{x^{n+3}\, e^{-x}}{n+3}, \quad \forall x \in \mathbb{R}_+.$$

We deduce that $\frac{x^{n+2}\, e^{-x}}{\int_0^x t^{n+2}\, e^{-t}\, dt} \leq \frac{n+3}{x}$, for any $x \geq 0$. On the other hand we have $\forall x,\, h \in \mathbb{R}_+$:

$$\int_x^{x+h} t^{n+2}\, e^{-t}\, dt \leq e^{-x}\,(x+h)^{n+2}\, h$$

hence, using the last two inequalities, and taking $x = (R-\varepsilon-r)$ and $h = 2\varepsilon+3r$ we obtain:

$$C_6 - 1 = \frac{\int_x^{x+h} t^{n+2}\, e^{-t}\, dt}{\int_0^x t^{n+2}\, e^{-t}\, dt} \leq (n+3)\frac{(x+h)^{n+2}}{x^{n+3}}\, h$$

which is exactly (1.24). Moreover, when $\varepsilon < \frac{R}{2}$ and r is such that $\varepsilon + 2r < \frac{R}{2}$ we obtain

$$C_6 \leq 1 + (n+3)\, 3^{n+3}\left(\frac{2\varepsilon+3r}{R+\varepsilon+2r}\right) \tag{1.25}$$

Now take inequality (1.19). Using inequality 1.20, the estimates find for C_1 , C_2 and (1.21), (1.22), (1.23), (1.24) we find:

$$\rho(h(y), H^R_{r,c}(y)) \leq \frac{1}{c} \cdot (n+1)^2 \cdot 2^{\frac{n-1}{9}}\, \alpha^{n+1} e^{2c\,(\varepsilon+\frac{3}{2}r)}.$$

$$\cdot \left[c\,(\varepsilon + \frac{3}{2}r) + \alpha^{2n+3}\,(\alpha-1) + \kappa_0^2\, R_1^2 + \frac{(R+\varepsilon+2r)^{n+2}}{(R-\varepsilon-r)^{n+3}}\,(2\varepsilon+3r)\right]^{\frac{1}{2}} \tag{1.26}$$

which is the desired inequality. The second inequality follows from (1.26) using (1.25). □

1.3.2. Equivariance. We verify the equivariance of the map $H^R_{r,\,c}$ with respect to the isomorphism $\lambda : \Gamma_Y \to \Gamma_X$.

PROPOSITION 1.3.6 (Equivariance of $H^R_{r,\,c}$). *The map $H^R_{c,\,r}$ is λ-equivariant.*

Proof. Is sufficient to show that $\mathcal{B}_{\mu_{\gamma\,y}}(\lambda(\gamma)\,x) = \mathcal{B}_{\mu_y}(x)$:

$$\mathcal{B}_{\mu_{\gamma\,y}}(\lambda(\gamma)\,x) = \int_X \rho(\lambda(\gamma)\,x, z)^2\, d\mu_{\gamma\,y}(z) =$$

$$= \int_X \rho(\lambda(\gamma)\,x, z)^2 \left(e^{-c\, d_r^{cut}(\gamma\, y,\, f(z))} - e^{-cR}\right) dv_0(z) =$$

$$= \int_X \rho(x, \lambda(\gamma)^{-1}\, z)^2 \left(e^{-c\, d_r^{cut}(y,\, f(\lambda(\gamma)^{-1}\, z))} - e^{-c\,R}\right) dv_0\left(\lambda(\gamma)^{-1} \cdot z\right) = \mathcal{B}_{\mu_y}(x)$$

where in the last equality we used the invariance of the Riemannian measure under isometries and the invariance of d_r^{cut}. □

1.3.3. Regularity. In this subsection we shall prove that $H^R_{r,\,c} \in C^1(Y, X)$. Let us fix a point $y_0 \in Y$ and let $x_0 = H^R_{r,\,c}(y_0)$. We denote $U_\sigma = B(x_0, \sigma)$, $V_\sigma = B(y_0, \sigma)$, where σ is assumed to be strictly smaller than the injectivity radius of (X, g_0) . Let B_0 denote the geodesic ball $B(x_0, \frac{10}{3} \cdot \mathrm{rad}(X, g_0))$

LEMMA 1.3.7. *Assume that* $\varepsilon < \frac{\mathrm{rad}(X,g_0)}{3\,(\alpha+1)}$ *and* $R_1 < \mathrm{rad}(X, g_0)$. *Let*

$$\sigma < \frac{1}{\alpha} \cdot \left(\frac{10}{3} \cdot \mathrm{rad}(X, g_0) - 3R_1 - (1+\alpha)\,\varepsilon \right) .$$

For any $y \in V_\sigma$ *we have* $\mathrm{supp}(\mu_y) \subset B_0$.

Proof. For any $y \in V_\sigma$ and for any $z \in \mathrm{supp}_{\mu_y}$, recalling that $\rho(h(y_0), H^R_{r,c}(y_0)) \leq 2R_1$ we have:

$$\rho(x_0, z) \leq \rho(x_0, h(y_0)) + \rho(h(y_0), h(y)) + \rho(h(y), z) \leq$$
$$\leq 2R_1 + R_1 + \alpha\, d(y_0, y) + (1+\alpha)\,\varepsilon$$

Since $d(y_0, y) < \sigma$ we obtain the desired estimate. □

Let σ be sufficiently small that there exist coordinates $\{x_i\}_{i=1,..,n}$, $\{y_i\}_{i=1,..,n}$ on U_σ, V_σ respectively. Let

$$\Phi = (\Phi_1, .., \Phi_n), \quad \Phi_i(x, y) = \int_X d(\rho^2)|_{(x,z)} \left(\frac{\partial}{\partial x_i} \right) d\mu_y(z)$$

LEMMA 1.3.8. *Let* $R_1 + \left(\frac{1+\alpha}{2}\right)\varepsilon < \mathrm{rad}(X, g_0)$ *and assume that*

$$\sigma < \frac{1}{1+\alpha} \cdot (2 \cdot \mathrm{rad}(X, g_0) - 2R_1) - \varepsilon .$$

For any $(x, y) \in U_\sigma \times V_\sigma$, $\Phi(x, y) = 0$ *if and only if* $x = H^R_{r,\,c}(y)$.

Proof. $\Phi(x, y) = 0$ if and only if x verifies $\int_X d(\rho^2)|_{(x,z)}\, d\mu_y(z) = 0_x$. Since $\rho(x, h(y)) \leq \rho(x_0, x) + \rho(x_0, h(y_0)) + \rho(h(y_0), h(y))$ we get

$$\rho(x, h(y)) \leq \sigma + 2R_1 + \alpha\,\sigma + (1+\alpha)\varepsilon < 2 \cdot \mathrm{rad}(X, g_0)$$

By the Lemma 1.2.12 $\Phi(x, y) = 0$ has a unique solution in $B(h(y), 2\,\mathrm{rad}(X, g_0))$, because if $\Phi(x, y) = \Phi(x', y)$ and $x \neq x'$, then the Lemma 1.2.12 implies that $\mathcal{B}_{\mu_y}(x') - \mathcal{B}_{\mu_y}(x) > 0$ and that $\mathcal{B}_{\mu_y}(x) - \mathcal{B}_{\mu_y}(x') > 0$, thus if x_1, $H^R_{r,\,c}(y)$ are two solutions of $\Phi(x, y) = 0$ one must have $x_1 = H^R_{r,\,c}(y)$. □

We shall show that there exists the partial derivatives of Φ_i with respect to x_k and with respect to y_j :

LEMMA 1.3.9. $\frac{\partial \Phi_i}{\partial y_j}$ *exists and is continuous in* $U_\sigma \times V_\sigma$. *Moreover,*

$$\frac{\partial \Phi_i}{\partial y_j} = -c \int_X d(\rho^2)|_{(x,\,z)} \left(\frac{\partial}{\partial x_i} \right) \cdot g \left(\nabla d^{cut}_r(y, f(z)), \frac{\partial}{\partial y_j} \right) d\mu_y(z)$$

Proof. Let us first remark that, since $d(d^{cut}_r)|_{(x,z)} = 0$ when $d_r(y, f(z)) \geq R + r$ (which is the case when $y \in V_\sigma$ and $z \notin B_0$ by the same proof given in the Lemma 1.3.7), the previous integral can be actually computed over B_0 . Let

$$\varphi_i(x, y, z) = d(\rho^2)|_{(x,\,z)} \left(\frac{\partial}{\partial x_i} \right) \cdot \left(e^{-c\, d^{cut}_r(y,\, f(z))} - e^{-cR} \right)$$

we can think of $\varphi_i(x, y, \cdot)$ as a function over B_0 . Since $\rho(x, z) < \mathrm{inj}(X, g_0)$ for any $(x, z) \in U_\sigma \times B_0$, ρ^2 is a C^∞ function; moreover, since d^{cut}_r is C^∞ (by construction), for any $z \in \mathrm{supp}(\mu_y) \subset B_0$, $\varphi_i(\cdot, \cdot, z)$ is C^∞ on $U_\sigma \times V_\sigma$ thus the derivative $\partial\varphi_i/\partial y_j$ exists and satisfies:

$$\frac{\partial \varphi_i}{\partial y_j}(x, y, z) = -c\, d(\rho^2)|_{(x,\,z)} \left(\frac{\partial}{\partial x_i} \right) g \left(\nabla d^{cut}_r(y, f(z)), \frac{\partial}{\partial y_j} \right) e^{-c\, d^{cut}_r(y,\, f(z))}$$

The absolute value of the previous expression is bounded above by

$$2c\,\rho(x, z)\ \| \partial/\partial x_i \|_{g_0} \| \partial/\partial y_j \|_g\ (1 + (\kappa\, r)^2)$$

We choose σ sufficiently small. There are two constants $A(\sigma)$, $B(\sigma)$ such that $\| \partial/\partial x_i \|_{g_0} \leq A(\sigma)$ and $\| \partial/\partial y_j \|_g \leq B(\sigma)$. On the other hand $\rho(x,z)$, for $z \in B_0$ is bounded above by two times the radius of B_0. Hence, over B_0,

$$\left| \frac{\partial \varphi_i}{\partial y_j} \right| \leq 3c \cdot \frac{20}{3} \operatorname{rad}(X, g_0) \cdot A(\sigma) \cdot B(\sigma)$$

where the right hand side of this inequality is independent of (x,y) and integrable on B_0, thus we can use Lebesgue's theorem to differentiate under the integral sign and we obtain the existence and the continuity of $\frac{\partial \Phi_i}{\partial y_j}$ and the formula $\frac{\partial \Phi_i}{\partial y_j} = \int_{B_0} \frac{\partial \varphi_i}{\partial y_j}(x,y,z)\, dv_0(z)$.
□

LEMMA 1.3.10. *$\frac{\partial \Phi_i}{\partial x_k}$ exists and is continuous. Moreover, for $x = H^R_{r,c}(y)$ we have:*

$$\frac{\partial \Phi_i}{\partial x_k}(x,y) = \int_X Dd(\rho^2)|_{(x,\,z)} \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k} \right) d\mu_y(z)$$

Proof. Let φ_i denote the function defined in the proof of the Lemma 1.3.9. We know that φ_i is C^∞ with respect to $(x,y) \in U_\sigma \times V_\sigma$. Moreover, we have

$$\frac{\partial}{\partial x_k} \left[d(\rho^2) \left(\frac{\partial}{\partial x_i} \right) \right] = Dd(\rho^2) \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k} \right) + d(\rho^2) \left(D_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_i} \right)$$

and thus

$$\frac{\partial \varphi_i}{\partial x_k} = \left[Dd(\rho^2) \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k} \right) + d(\rho^2) \left(D_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_i} \right) \right] \cdot \left(e^{-c\, d_r^{cut}(y,\, f(z))} - e^{-cR} \right)$$

Since $\forall x \in \overline{U_\sigma}$ and $\forall z \in B\left(x_0, \frac{10}{3} \operatorname{rad}(X, g_0)\right)$ we have

$$\rho(x,z) \leq \rho(x,x_0) + \rho(x_0,z) \leq \sigma + \frac{10}{3} \operatorname{rad}(X,g_0) < \operatorname{inj}(X,g_0)$$

this implies that the $(x,z) \to \rho(x,z)^2$ is C^∞ on the (compact) closure of $U_\sigma \times B\left(x_0, \frac{10}{3}\right)$ thus $\| Dd\rho^2|_{(x,z)} \|$ is bounded above by some constant independent of (x,z). On the other hand, the existence of a C^∞-coordinate system on $U_{2\sigma}$ implies that $\| D_{\frac{\partial}{\partial x_k}} \frac{\partial}{dex_i} \|$ is bounded above by a constant independent of x on $\overline{U_\sigma}$. Hence $|\frac{\partial \varphi_i}{\partial x_k}|$ is bounded (on $U_\sigma \times V_\sigma \times B_0$) by some constant independent of (x,y,z).
As in the proof of the Lemma 1.3.9 we can use Lebesgue's theorem to differentiate under the integral sign. Thus we obtain the existence of the derivatives with respect to x and their continuity. To obtain the formula just observe that if $x = H^R_{r,c}(y) = \operatorname{bar}(\mu_y)$, equation (1.11) implies:

$$\int_{B_0} d(\rho^2)|_{(x,\,z)} \left(D_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_i} \right) d\mu_y(z) = 0. \quad \square$$

PROPOSITION 1.3.11 (Regularity). *The map $H^R_{r,c}$ is $C^1(Y,X)$.*

Proof. Consider the function Φ. By the Lemmas 1.3.9, 1.3.10 we know that the function Φ is $C^1(Y \times X)$. Let y_0 be any point of Y and assume $x_0 = H^R_{r,c}(y_0)$. We want to show that the matrix $\left(\frac{\partial \Phi_i}{\partial x_j}(x_0,y_0) \right)_{i,j}$ is invertible. Consider the corresponding quadratic form:

$$\sum_{i,j} \left(\frac{\partial \Phi_i}{\partial x_j}(x_0,y_0) \right) u_i\, u_j = \int_X Dd(\rho^2)|_{(x_0,z)}(u,u)\, d\mu_{y_0}(z)$$

where $u = \sum_1^n u_i \cdot \frac{\partial}{\partial x_i}$. Observe that since $\rho(x_0,z) < \frac{10}{3} \operatorname{rad}(X,g_0)$ by Rauch's Theorem (Theorem 1.2.7):

$$Dd(\rho^2)|_{(x,\,z)}(u,u) \geq \frac{\frac{20}{3} \cdot \operatorname{rad}(X,g_0) \cdot \kappa_0}{\operatorname{tg}\left(\frac{10}{3} \cdot \operatorname{rad}(X,g_0) \cdot \kappa_0 \right)} \cdot g_0$$

Thus the quadratic form si positive definite, which implies that the matrix $\left(\frac{\partial \Phi_i}{\partial y_j} \right)_{i,j}$ is invertible. By the implicit function theorem we know that there exists an open neighbourhood $V_{\sigma'}$ of y_0 and a C^1 map $h_r : V_{\sigma'} \to X$, uniquely defined by the equation

$\Phi(h_r(y), y) = 0$. It follows that $H^R_{r,\,c}|_{V_{\sigma'}} = h_r$; hence $H^R_{r,\,c} \in C^1(Y, X)$. □

1.3.4. The pointwise energy estimate. This subsection is devoted to the proof of the estimate (1.9) of the Theorem 1.2.1. We shall denote by R_1 the quantity $\alpha\,(R+\varepsilon+2r)$.

N.B. All derivatives of ρ^2 and d_r^{cut} are computed with respect to the first variable.

1.3.4.1. 2^{nd} *derivatives of* Φ . Let us fix $y_0 \in Y$ and let $x_0 = h(y_0)$; we consider $\Phi = (\Phi_1, .., \Phi_n)$,

$$\Phi_i(x, y) = \int_X d(\rho^2)|_{(x,\,z)} \left(\frac{\partial}{\partial x_i}\right) d\mu_y(z)$$

where $\{x_i\}_{i=1,..,n}$ is a coordinate system around x_0 . Let $\{y_j\}_{j=1,..,n}$ be a coordinate system around y_0 . For any $y \in Y$ we have $\Phi(H^R_{r,\,c}(y), y) = 0$. Let us derive the previous identity with respect to $\frac{\partial}{\partial y_j}$:

$$0 = \frac{\partial}{\partial y_j}(\Phi_i(H^R_{r,\,c}(y), y)) =$$

$$= \frac{\partial \Phi_i}{\partial y_j}(H^R_{r,\,c}(y), y) + \sum_{k=1}^n \frac{\partial \Phi_i}{\partial x_k}(H^R_{r,\,c}(y), y) \cdot \left(d_y H^R_{r,\,c}\left(\frac{\partial}{\partial y_j}\right)\right)_k$$

where u_k is the k-th component of the vector $u \in T_{H^R_{r,c}(y)}X$ with respect to the frame $\left\{\frac{\partial}{\partial x_k}\right\}_{1\le k\le n}$. Using the explicit expressions for the partial derivatives of Φ_i derived from the previous subsection (we can apply the Lemmas 1.3.9, 1.3.10 because $(H^R_{r,c}(y), h(y))$ lies in a small neighbourhood of $(H^R_{r,c}(y_0), y_0)$) and writing $\partial_{x_i} = \frac{\partial}{\partial x_i}$, $\partial_{y_j} = \frac{\partial}{\partial y_j}$ we obtain:

$$0 = \sum_{k=1}^n \int_X Dd(\rho^2)|_{(H^R_{r,\,c}(y),\,z)}(\partial_{x_i}, \partial_{x_k}) \cdot \left(d_y H^R_{r,\,c}\left(\partial_{y_j}\right)\right)_k d\mu_y(z) -$$

$$-c \int_X d(\rho^2)|_{(H^R_{r,\,c}(y),\,z)}(\partial_{x_i}) \cdot d(d_r^{cut})|_{(y,\,f(z))}(\partial_{y_j})\, e^{-c\, d_r^{cut}(y,\,f(z))}\, dv_0(z)$$

We multiply both sides for v_i , u_j and we sum over i, j :

$$\frac{1}{2}\int_X Dd(\rho^2)|_{(H^R_{r,\,c}(y),\,z)}(v, d_y H^R_{r,\,c}(u))\, d\mu_y(z) =$$

$$= \frac{c}{2}\int_X d(\rho^2)|_{(H^R_{r,\,c}(y),z)}(v) \cdot d(d_r^{cut})|_{(y,\,f(z))}(u)\, e^{-c\, d_r^{cut}(y,\,f(z))}\, dv_0(z) \tag{1.27}$$

where $u = \sum_1^n u_j\, \partial_{y_j}$, $v = \sum_1^n v_i\, \partial_{x_i}$. We remark that the second integral in equation (1.27) is computed on $B(h(y), R_1)$ since $\rho(h(y), z) \geq R_1$ implies $d_r(y, f(z)) \geq R+r$ and thus $d_r^{cut} \equiv R$ which implies that $\nabla d_r^{cut}(y, f(z)) \equiv 0$ when z outside $B(h(y), R_1)$.

1.3.4.2. *Inequality between bilinear forms.* We shall work on equation (1.27).
Let $u \in T_yY$ and let $v = \frac{d_y H^R_{r,\,c}(u)}{\|d_y H^R_{r,\,c}(u)\|}$. We recall that $\rho(H^R_{r,\,c}(y), h(y)) \leq \frac{\delta}{c}$ (see the Proposition 1.3.4), hence the triangle inequality gives $\rho(H^R_{r,\,c}(y), z) \leq R_1 + \frac{\delta}{c}$.
Since $R_1 + \frac{\delta}{c} < \min\{\mathrm{inj}(X, g_0), \frac{\pi}{2\kappa_0}\}$ it follows that we can apply Rauch's Theorem (Theorem 1.2.7):

$$\frac{1}{2}Dd(\rho^2)|_{(H^R_{r,\,c}(y),\,z)}(v, d_y H^R_{r,\,c}(u)) \geq \frac{\left(R_1 + \frac{\delta}{c}\right)\kappa_0}{\mathrm{tg}\left(\left(R_1 + \frac{\delta}{c}\right)\kappa_0\right)} \cdot \|\, d_y H^R_{r,\,c}(u) \,\| \tag{1.28}$$

Using the estimate (1.28) and applying the Cauchy-Schwarz inequality to the second integral in equation (1.27) we find:

$$\frac{\left(R_1 + \frac{\delta}{c}\right)\kappa_0}{\mathrm{tg}\left(\left(R_1 + \frac{\delta}{c}\right)\kappa_0\right)} \cdot \|\, d_y H^R_{r,\,c}(u) \,\| \cdot \mu_y(X) \leq$$

$$\leq \left(\int_{B(h(y),R_1)} g_0(\nabla\rho(H^R_{r,c}(y),z),v)^2\, \rho(H^R_{r,c}(y),z)\, e^{-c\, d^{cut}_r(y,\,f(z))}\, dv_0(z) \right)^{\frac{1}{2}} \cdot$$

$$\cdot \left(\int_{B(h(y),R_1)} g(\nabla d^{cut}_r(y,f(z)),u)^2\, \rho(H^R_{r,c}(y),z)\, e^{-c\, d^{cut}_r(y,\,f(z))}\, dv_0(z) \right)^{\frac{1}{2}} \cdot c \tag{1.29}$$

1.3.4.3. *An intermediate estimate.* We want to estimate the two integrals in the right hand side of (1.29). Let us remark that if $z \in B(h(y), R_1)$ we have, using inequality (1.14):

$$d^{cut}_r(y,f(z)) \geq \min\{d(y,f(z)) - 2r,\ R\} \geq \min\left\{ \frac{(\rho(h(y),z) - \varepsilon)}{\alpha} - 2r,\ R \right\} \geq$$

$$\geq \min\left\{ \frac{\rho(h(y),z)}{\alpha} - \varepsilon - 2r,\ R \right\} = \frac{\rho(h(y),z)}{\alpha} - \varepsilon - 2r \geq \frac{\rho(H^R_{r,c}(y),z)}{\alpha} - \frac{\delta}{c\,\alpha} - \varepsilon - 2r$$

where the last inequality comes from the Proposition 1.3.4. Thus we can write:

$$\int_{B(h(y),R_1)} g_0(\nabla\rho(H^R_{r,c}(y),z),v)^2\, \rho(H^R_{r,c}(y),z) e^{-c\, d^{cut}_r(y,\,f(z))}\, dv_0(z) \leq$$

$$\leq \exp\left(\frac{\delta}{\alpha} + c\,(\varepsilon + 2r) \right) \cdot$$

$$\cdot \int_{B(H^R_{r,c}(y),R_1+\frac{\delta}{c})} g_0(\nabla\rho(H^R_{r,c}(y),z),v)^2\, \rho(H^R_{r,c}(y),z)\, e^{-\frac{c}{\alpha}\,\rho(H^R_{r,c}(y),\,z)}\, dv_0(z)$$

Denoting by S^{n-1} the unit sphere of $T_{H^R_{r,c}(y)}X$, we use the exponential map centered at $H^R_{r,c}(y)$ to define the parametrization

$$\phi : \left(0, R_1 + \frac{\delta}{c}\right) \times S^{n-1} \to X, \quad \phi(t,w) = \exp_{H^R_{r,c}(y)}(t\,w)$$

of $B(H^R_{r,c}(y), R_1 + \frac{\delta}{c}) \setminus \{H^R_{r,c}(y)\}$. By Bishop-Gunther's Theorem (Theorem 1.2.8) we have:

$$\int_{B(h(y),R_1)} g_0(\nabla\rho(H^R_{r,c}(y),z),v)^2\, \rho(H^R_{r,c}(y),z)\, e^{-c\, d^{cut}_r(y,\,f(z))}\, dv_0(z) \leq$$

$$\leq e^{\frac{\delta}{\alpha}+c\,(\varepsilon+2r)} \left(\frac{\sinh\left((\frac{\delta}{c} + R_1)\,\kappa_0\right)}{(\frac{\delta}{c} + R_1)\,\kappa_0} \right)^{n-1} \cdot \int_0^{R_1+\frac{\delta}{c}} \left(\int_{S^{n-1}} g_0(w,v)^2\, dw \right) t^n\, e^{-\frac{c}{\alpha}\,t}\, dt$$

where we have used the fact that, for every $w \in S^{n-1}$ and any $t \in \left(0, R_1 + \frac{\delta}{c}\right)$, $w = \nabla\rho(H^R_{r,c}(y), \phi(t,v))$. Thus, since $\int_{S^{n-1}} g_0(u,v)^2\, du = \frac{\omega_{n-1}}{n} \cdot g_0(v,v) = \frac{\omega_{n-1}}{n}$, we get

$$\int_{B(h(y),R_1)} g_0(\nabla\rho(H^R_{r,c}(y),z),v)^2\, \rho(H^R_{r,c}(y),z)\, e^{-c\, d^{cut}_r(y,\,f(z))}\, dv_0(z) \leq$$

$$\leq e^{\frac{\delta}{\alpha}+c(\varepsilon+2r)}\, \frac{\alpha^{n+1}\,\omega_{n-1}}{n\,c^{n+1}} \cdot \left(\frac{\sinh\left((\frac{\delta}{c} + R_1)\,\kappa_0\right)}{(\frac{\delta}{c} + R_1)\,\kappa_0} \right)^{n-1} \int_0^{\frac{cR_1+\delta}{\alpha}} t^n\, e^{-t}\, dt \tag{1.30}$$

Plugging (1.30) into (1.29), and using the estimate for $\mu_y(X)$ given in the proof of the Proposition 1.3.4 we find:

$$\|\, d_y H^R_{r,c}(u)\, \|^2 \leq \left(\frac{\mathrm{tg}\left((R_1 + \frac{\delta}{c})\,\kappa_0\right)}{(R_1 + \frac{\delta}{c})\,\kappa_0} \right)^2 \cdot \frac{n}{\omega_{n-1}}\, c^{n-1} \cdot c^2\, \alpha^{3n+1}\, e^{\frac{\delta}{\alpha}+c\,(3\varepsilon+4r)} \cdot$$

$$\cdot \left(\frac{\sinh\left((\frac{\delta}{c} + R_1)\,\kappa_0\right)}{(\frac{\delta}{c} + R_1)\,\kappa_0} \right)^{n-1} \cdot \left(\frac{\kappa_0\, R}{\sin(\kappa_0\, R)} \right)^{2(n-1)} \cdot \left(\frac{\int_0^{c(R+\varepsilon+2r)+\frac{\delta}{\alpha}} t^n\, e^{-t}\, dt}{\left(\int_0^{c(R-\varepsilon-r)} t^n\, e^{-t}\, dt \right)^2} \right) \cdot$$

$$\cdot \int_{B(h(y),R_1)} g(\nabla d^{cut}_r(y,f(z)),u)^2\, \rho(H^R_{r,c}(y),z)\, e^{-c\, d^{cut}_r(y,\,f(z))}\, dv_0(z)$$

We can estimate the last integral as follows:

$$\int_{B(h(y),R_1)} g(\nabla d_r^{cut}(y,f(z)),u)^2\,\rho(H_{r,\,c}^R(y),z)\,e^{-c\,d_r^{cut}(y,\,f(z))}\,dv_0(z)\le$$

$$\le e^{c\,(\varepsilon+2r)}\cdot\int_{B(h(y),R_1)} g(\nabla d_r^{cut}(y,f(z)),u)^2\,\rho(H_{r,\,c}^R(y),\,z)\,e^{-\frac{c}{\alpha}\,\rho(h(y),z)}\,dv_0(z)$$

Now let $\{e_i\}_{i=1,..,n}$ be a g-orthonormal base for T_yY. Using property (iv) of the cut-off distance d_r^{cut} and the Bishop-Gunther's Theorem (Theorem 1.2.8) we can write:

$$\sum_{i=1}^{n}\int_{B(h(y),R_1)} g(\nabla d_r^{cut}(y,f(z)),e_i)^2\,\rho(H_{r,\,c}^R(y),z)\,e^{-\frac{c}{\alpha}\,\rho(h(y),\,z)}\,dv_0(z)\le$$

$$\le (1+(\kappa\,r)^2)^2\int_{B(h(y),R_1)}\rho(H_{r,\,c}^R(y),z)\,e^{-\frac{c}{\alpha}\rho(h(y),\,z)}\,dv_0(z)\le$$

$$\le (1+(\kappa\,r)^2)^2\,\omega_{n-1}\left(\frac{\sinh(R_1\,\kappa_0)}{R_1\,\kappa_0}\right)^{n-1}\int_0^{R_1}\left(\frac{\delta}{c}t^{n-1}\,e^{-\frac{c}{\alpha}t}+t^n\,e^{-\frac{c}{\alpha}t}\right)dt$$

where in the last inequality we used the fact that $\rho(H_{r,\,c}^R(y),h(y))\le\frac{\delta}{c}$ and we have computed the integral using the parametrization $(t,w)\to\exp_{h(y)}(t\,w)$. Thus we get the following estimate for the pointwise energy:

$$e_y(H_{r,\,c}^R)\le n\cdot\alpha^{4n+2}\cdot e^{\frac{\delta}{\alpha}+c(4\varepsilon+6r)}\cdot(1+(\kappa\,r)^2)^2\left(\frac{\operatorname{tg}\left((R_1+\frac{\delta}{c})\,\kappa_0\right)}{(R_1+\frac{\delta}{c})\,\kappa_0}\right)^2\cdot$$

$$\cdot\left(\frac{\sinh(R_1\,\kappa_0)\cdot\sinh\left((\frac{\delta}{c}+R_1)\kappa_0\right)}{\sin(\kappa_0\,R)^2}\right)^{n-1}\cdot\left(\frac{R^2}{R_1\cdot(\frac{\delta}{c}+R_1)}\right)^{(n-1)}\cdot$$

$$\cdot\left(\frac{\int_0^{c(R+\varepsilon+2r)+\frac{\delta}{\alpha}}t^n\,e^{-t}\,dt}{\int_0^{c(R-\varepsilon-r)}t^n\,e^{-t}\,dt}\right)\cdot\left[\frac{\int_0^{\frac{c}{\alpha}\,R_1}\left(\frac{\delta}{\alpha}\,t^{n-1}\,e^{-t}+t^n\,e^{-t}\right)dt}{\int_0^{c(R-\varepsilon-r)}t^n\,e^{-t}\,dt}\right] \tag{1.31}$$

1.3.4.4. *Proof of the estimates (1.9) and (1.10).* We shall now show how to manipulate the inequality (1.31) in order to prove (1.9) and (1.10).

LEMMA 1.3.12.

$$\frac{\sinh(\kappa_0\,R_1)}{\sin(\kappa_0\,R_1)}\le\left(1+\frac{\kappa_0^2\,R_1^2}{2}\right)^{n-1}\left(\frac{R_1}{R}\right)^{n-1} \tag{1.32}$$

Proof. In order to prove (1.32) let us remark that:

$$\frac{\sinh(\kappa_0\,R_1)}{\sin(\kappa_0\,R_1)}=\frac{\sinh(\kappa_0\,R_1)}{\sin(\kappa_0\,R_1)}\cdot\frac{\sin(\kappa_0\,R_1)}{\sin(\kappa_0\,R)}\le\frac{\sinh(\kappa_0\,R_1)}{\sin(\kappa_0\,R_1)}\cdot\frac{R_1}{R}$$

thus, since $\kappa_0\,R_1\le\frac{\pi}{10}$ it is sufficient to prove that $\forall x\in(0,\frac{\pi}{10})$

$$\frac{\sinh(x)}{\sin(x)}\le 1+\frac{x^2}{2} \tag{1.33}$$

Using the Taylor's formula with remainder we get:

$$\sinh(x)=x+\int_0^x\frac{(x-t)^2}{2}\cdot\cosh(t)\cdot dt\le x+\cosh(x)\cdot\frac{x^3}{6}$$

and, analogously, we have $\cos(t)\le 1$ and thus

$$\sin(x)=x-\int_0^x\frac{(x-t)^2}{2}\cdot\cos(t)\cdot dt\ge x-\frac{x^3}{6}\,.$$

Hence we obtain:

$$\frac{\sinh(x)}{\sin(x)}\le\frac{1+\cosh(x)\cdot\frac{x^2}{6}}{1-\frac{x^2}{6}}\le\frac{1+\cosh(\frac{\pi}{10})\cdot\frac{x^2}{6}}{1-\frac{x^2}{6}}\le\frac{1+0,525\cdot\frac{x^2}{3}}{1-0,5\cdot\frac{x^2}{3}}\le$$

$$\le 1+\left(\frac{205}{200-\pi^2}\right)\cdot\frac{x^2}{3}\le 1+\frac{x^2}{2}\quad\square$$

Lemma 1.3.13.

$$\left(\frac{\sinh\left(\kappa_0\ \left(R_1+\frac{\delta}{c}\right)\right)}{\sin(\kappa_0\, R)}\right)^{n-1} \le \left[1+\frac{\kappa_0^2\ \left(R_1+\frac{\delta}{c}\right)^2}{2}\right]^{n-1}\cdot\left(\frac{R_1+\frac{\delta}{c}}{R}\right)^{n-1} \tag{1.34}$$

Proof. The proof of this inequality is analogous to the one of inequality (1.32). We just need to notice that here we have $R_1+\frac{\delta}{c}\le 3\,R_1\le\frac{3\pi}{10}$. □

$$\left(\frac{\operatorname{tg}\left(\left(R_1+\frac{\delta}{c}\right)\kappa_0\right)}{\left(R_1+\frac{\delta}{c}\right)\ \kappa_0}\right)^2 \le \left(1+\frac{9}{2}\left(R_1+\frac{\delta}{c}\right)^2\kappa_0^2\right)^2 \tag{1.35}$$

As in the previouses cases we use the Taylor's formula with remainder:

$$\operatorname{tg}(x)=x+\int_0^x\frac{(x-t)^2}{2}\cdot\frac{d^3}{dt^3}\left[\operatorname{tg}(t)\right]\,dt\le x+\frac{x^2}{2}\cdot\int_0^x\frac{d^3}{dt^3}\left[\operatorname{tg}(t)\right]\,dt\le$$

$$\le x+x^2\cdot\left[\operatorname{tg}(t)\left(1+\operatorname{tg}^2(t)\right)\right]_0^x\le x+\left(\frac{\operatorname{tg}(x)}{x}\cdot\left(1+\operatorname{tg}^2(x)\right)\right)\cdot x^3$$

if $x\le\frac{3\pi}{10}$ we have

$$\frac{\operatorname{tg}(x)}{x}\le 1+\left(\frac{\operatorname{tg}\left(\frac{3\pi}{10}\right)}{\frac{3\pi}{10}}\right)\cdot\left[1+\operatorname{tg}^2\left(\frac{3\pi}{10}\right)\right]\cdot x^2\le 1+\frac{9}{2}\cdot x^2$$

Since $R_1+\frac{\delta}{c}\le 3\,R_1\le\frac{3\pi}{10}$, this concludes the proof of inequality (1.35). □

Lemma 1.3.14.

$$\left(\frac{\sinh(R_1\,\kappa_0)\cdot\sinh\left(\left(R_1+\frac{\delta}{c}\right)\ \kappa_0\right)}{\sin(\kappa_0\,R)^2}\right)^{n-1}\cdot\left(\frac{R^2}{R_1\ \left(R_1+\frac{\delta}{c}\right)}\right)^{n-1}\le$$

$$\le\left[1+\frac{1}{2}\kappa_0^2\left(R_1+\frac{\delta}{c}\right)^2\right]^{2(n-1)} \tag{1.36}$$

Proof. Using (1.32) and (1.34) we obtain:

$$\left(\frac{\sinh(R_1\,\kappa_0)\cdot\sinh\left(\left(R_1+\frac{\delta}{c}\right)\ \kappa_0\right)}{\sin(\kappa_0\,R)^2}\right)^{n-1}\cdot\left(\frac{R^2}{R_1\ \left(R_1+\frac{\delta}{c}\right)}\right)^{n-1}\le$$

$$\le\left(1+\frac{\kappa_0^2\,R_1^2}{2}\right)^{n-1}\cdot\left(\frac{R^2}{R_1\ \left(\frac{\delta}{c}+R_1\right)}\right)^{n-1}\cdot\left(1+\frac{\kappa_0^2\ \left(R_1+\frac{\delta}{c}\right)^2}{2}\right)^{n-1}\cdot$$

$$\cdot\left(\frac{R_1+\frac{\delta}{c}}{R}\right)^{n-1}\cdot\left(\frac{R}{R_1}\right)^{n-1}\cdot\left(\frac{R}{R_1+\frac{\delta}{c}}\right)^{n-1}\le$$

$$\le\left(1+\frac{\kappa_0^2\,R_1^2}{2}\right)^{n-1}\cdot\left(1+\frac{\kappa_0^2\ \left(R_1+\frac{\delta}{c}\right)^2}{2}\right)^{n-1}\le\left(1+\frac{1}{2}\,\kappa_0^2\left(R_1+\frac{\delta}{c}\right)^2\right)^{2\,(n-1)}.\quad\square$$

Lemma 1.3.15.

$$\frac{\int_0^{c(R+\varepsilon+2r)+\frac{\delta}{\alpha}}t^n\,e^{-t}\,dt}{\int_0^{c(R-\varepsilon-r)}t^n\,e^{-t}\,dt}\le 1+(n+1)\,3^{n+1}\,\frac{2\varepsilon+3r}{R+\varepsilon+2r} \tag{1.37}$$

Proof. Since $\int_0^x t^n\, e^{-t}\, dt \geq e^{-x} \int_0^x t^n\, e^{-t}\, dt = \frac{x^{n+1}\, e^{-x}}{n+1}$ we get the inequality:

$$\frac{x^n\, e^{-x}}{\int_0^x t^n\, e^{-t}\, dt} \leq \frac{n+1}{x}\,. \tag{1.38}$$

Thus we have:

$$\frac{\int_0^{x+h} t^n\, e^{-t}\, dt}{\int_0^x t^n\, e^{-t}\, dt} \leq 1 + \frac{\int_x^{x+h} t^n\, e^{-t}\, dt}{\int_0^x t^n\, e^{-t}\, dt} \leq 1 + \frac{(x+h)^n e^{-x}\, h}{\int_0^x t^n\, e^{-t}\, dt} \leq 1 + (n+1)\frac{(x+h)^n}{x^{n+1}} \cdot h$$

We conclude taking $x = R - \varepsilon - r$ and $h = 2\varepsilon + 3r$ and using the inequality $\varepsilon + 2r < \frac{R}{2}$. $\square$

LEMMA 1.3.16.

$$\frac{\int_0^{\frac{c}{\alpha}R_1} \left(\frac{\delta}{\alpha}\, t^{n-1}\, e^{-t} + t^n\, e^{-t}\right)\, dt}{\int_0^{R-\varepsilon-r} t^n\, e^{-t}\, dt} \leq$$

$$\leq \left[1 + \frac{\delta}{n\,\alpha}\left(1 + \frac{n+1}{c\,(R+\varepsilon)}\right)\right] \cdot \left[1 + (n+1)\, 3^{n+1} \left(\frac{2\varepsilon + 3r}{R + \varepsilon + 2r}\right)\right] \tag{1.39}$$

Proof. We recall that $\frac{c}{\alpha}R_1 = c(R + \varepsilon + 2r)$ hence,

$$\frac{\int_0^{\frac{c}{\alpha}R_1} \left(\frac{\delta}{\alpha}\, t^{n-1}\, e^{-t} + t^n\, e^{-t}\right)\, dt}{\int_0^{R-\varepsilon-r} t^n\, e^{-t}\, dt} \leq \frac{\delta}{\alpha} \cdot \frac{\int_0^{c(R+\varepsilon+2r)} t^{n-1}\, e^{-t}\, dt}{\int_0^{c(R-\varepsilon-r)} t^n\, e^{-t}\, dt} + \frac{\int_0^{c(R+\varepsilon+2r)} t^n\, e^{-t}\, dt}{\int_0^{c(R-\varepsilon-r)} t^n\, e^{-t}\, dt}$$

Now, integrating by parts, we have for any $x \in (0 + \infty)$:

$$\int_0^x t^{n-1}\, e^{-t}\, dt = \frac{1}{n}\left(1 + \frac{x^n\, e^{-x}}{\int_0^x t^n\, e^{-t}\, dt}\right) \leq \frac{x^n\, e^{-x}}{n} + \frac{1}{n}\int_0^x t^n\, e^{-t}\, dt$$

Hence, using the inequality (1.38) we get:

$$\frac{\int_0^x t^{n-1}\, e^{-t}\, dt}{\int_0^x t^n\, e^{-t}\, dt} \leq \frac{1}{n}\left(1 + \frac{x^n\, e^{-x}}{\int_0^x t^n\, e^{-t}\, dt}\right) \leq \frac{1}{n}\left(1 + \frac{n+1}{x}\right)$$

Now (1.39) follows from (1.37). $\square$

We apply the inequalities (1.32), (1.34), (1.35), (1.36) (1.37), (1.39) and we obtain the inequality (1.9). The inequality (1.10) follows from (1.9) by taking r sufficiently small. $\square$

1.3.5. Homotopy. Let us now prove the following:

PROPOSITION 1.3.17 (Homotopy between $H^R_{r,c}$ and h). *If h is continuous, then there exists a λ-equivariant homotopy between h and $H^R_{r,c}$.*

Proof. Let us consider the following probability measures:

$$\bar{\mu}_y = \frac{1}{\mu_y(X)} \cdot \mu_y, \quad d\mu_y(z) = \left(e^{-c\, d_r^{cut}(y, f(z))} - e^{-c\, R}\right) dv_0(z)$$

$$\bar{\nu}_y = \frac{1}{\nu_y(X)} \cdot \nu_y, \quad d\nu_y(z) = \left(e^{-c\,\alpha\rho(h(y),\, z)} - e^{-c\,(R-\varepsilon-r)}\right)_+ d((\exp_{h(y)})_* \mathcal{L}_{h(y)}),$$

where $\mathcal{L}_{h(y)}$ denotes the Lebesgue measure on $T_{h(y)}X$. Since they are just rescalings of the μ_y's and the ν_y's we still have: $H^R_{r,c}(y) = \operatorname{argmin} \mathcal{B}_{\bar{\mu}_y}$ and $h(y) = \operatorname{argmin} \mathcal{B}_{\bar{\nu}_y}$. Let us define the probability measures

$$\nu^t_y = (1-t)\,\bar{\mu}_y + t\,\bar{\nu}_y.$$

Since $\operatorname{supp}(\bar{\mu}_y) = \operatorname{supp}(\mu_y)$ and $\operatorname{supp}(\bar{\nu}_y) = \operatorname{supp}(\nu_y)$, we see that the support of ν^t_y is contained into $B(h(y), R_1)$. In particular ν^t_y admits a barycenter. Hence we are allowed to define the following map,

$$\Psi_t(y) = \operatorname{argmin} \mathcal{B}_{\nu^t_y}$$

We shall show that Ψ_t is a λ-equivariant homotopy between h and $H^R_{r,c}$.
To prove the equivariance of Ψ_t we need to show that $\mathcal{B}_{\nu^t_{\gamma\, y}}(\lambda(\gamma)\, x) = \mathcal{B}_{\nu^t_y}(x)$. Now

$$\mathcal{B}_{\nu^t_{\gamma\, y}}(\lambda(\gamma)\, x) = \int_X \rho(\lambda(\gamma)\, x, z)^2 d\nu^t_{\gamma\, y} =$$

$$= (1-t)\int_X \rho(\lambda(\gamma)\,x,z)^2\, d\bar{\mu}_{\gamma\, y}(z) + t\int_X \rho(\lambda(\gamma)\,x,z)^2\, d\bar{\nu}_{\gamma\, y}(z) =$$
$$= \frac{(1-t)}{\mu_y(X)}\int_X \rho(x,\lambda(\gamma)^{-1}z)^2 \left(e^{-c\, d_r^{cut}(y,\, f(\lambda(\gamma)^{-1}z))} - e^{-c\, R}\right) dv_0(\lambda(\gamma)^{-1}z) +$$
$$+ \frac{t}{\nu_y(X)}\int_X \rho(x,\lambda(\gamma)^{-1}z)^2\, d\nu_y(\lambda(\gamma)^{-1}z) = \mathcal{B}_{\nu_y^t}(x)$$

which proves the λ-equivariance.

Now observe that $\Psi_0(y) = \operatorname{argmin} \mathcal{B}_{\bar{\mu}_y} = \operatorname{argmin} \mathcal{B}_{\mu_y} = H^R_{r,\,c}(y)$. On the other hand $\Psi_1(y) = \operatorname{argmin} \mathcal{B}_{h(y)} = \operatorname{argmin} \rho(x,h(y))^2 = h(y)$.

Finally let us show that $\Psi_t(y)$ is continuous in (t,y). First of all let us observe that if $(t,y)\to(t_0,y_0)$ then for any measurable set $A\subset X$ we have $\nu_y^t(A)\to\nu_{y_0}^{t_0}(A)$. It follows that the associated functions $\mathcal{B}_{\nu_y^t}$ converge to $\mathcal{B}_{\nu_{y_0}^{t_0}}$ in the sense C^0; in fact, $\nu_y^t - \nu_{y_0}^{t_0}$ are signed measures such that $|\nu_y^t - \nu_{y_0}^{t_0}|\to 0$ as $(t,y)\to(t_0,y_0)$. We write

$$|\mathcal{B}_{\nu_y^t}(x) - \mathcal{B}_{\nu_{y_0}^{t_0}}(x)| \le \left|\int_X \rho(x,z)^2\, d\nu_y^t(z) - \int_X \rho(x,z)^2\, d\nu_{y_0}^{t_0}\right| \le$$
$$\le \int_X \rho(x,z)^2\, d|\nu_y^t - \nu_{y_0}^{t_0}|(z)$$

and since $\nu_y^t\to\nu_{y_0}^{t_0}$, and $\rho(x,\cdot)$ is bounded on

$$B(h(y), R_1 + \rho(h(y), H^R_{r,\,c}(y))) \supset \operatorname{supp}(|\nu_y^t - \nu_{y_0}^{t_0}|)\ ,$$

we get the C^0 convergence of $\mathcal{B}_{\nu_y^t}$ to $\mathcal{B}_{\nu_{y_0}^{t_0}}$. Hence it follows that $\operatorname{argmin}\mathcal{B}_{\nu_y^t}\to \operatorname{argmin}\mathcal{B}_{\nu_{y_0}^{t_0}}$, which proves the continuity of $\Psi_t(y)$ in (t_0,y_0). Since we did not make any assumption on the point (t_0,y_0), it follows that $\Psi_t(y)$ is continuous in (t,y). □

This Proposition concludes the proof of the Theorem 1.2.1. □

1.4. Regularized Gromov-Hausdorff ε-approximations via the Barycenter Method in variable curvature

1.4.1. Introduction. The aim of this section is to show how the Barycenter Method in variable curvature (see [**Saba1**] and the Theorem 1.2.1) can be used in order to obtain regularized Gromov-Hausdorff ε-approximations between compact manifolds, satisfying jacobian and pointwise energy bounds which are sharp. We shall give examples and counterexamples. The results in this section are due to L. Sabatini (see [**Saba1**], Ch.4), however, for completeness of the present work, for easier reference and since in the Ph.D. Thesis of L. Sabatini they are stated in a different form (in particular the homogeneity of the result is hidden), we found useful to restate and prove them in this section.

THEOREM 1.4.1. *Let (X,g_0) be a n-dimensional compact Riemannian manifold with bounded sectional curvature $|\sigma_0|\le\kappa_0^2$. Let (Y,g) be any n-dimensional compact Riemannian manifold. Assume $d_{GH}((Y,g),(X,g_0))<\varepsilon$ where*

$$\kappa_0\,\varepsilon < \left[\frac{\min\{\,1\ ;\ \kappa_0\ \operatorname{inj}(X,g_0)\,\}}{2^{8n}\,(n+1)^8}\right]^4 \tag{1.40}$$

There exists a C^1 map $H_\varepsilon : (Y^n,g)\to(X^n,g_0)$ satisfying the following properties:

(i) *H_ε is a Gromov-Hausdorff $(\frac{10}{\kappa_0}\,(n+1)^4\,2^{4n}\,(\kappa_0\,\varepsilon)^{\frac34})$-approximation;*

(ii) *For any $y\in Y$ we have the following pointwise energy estimate:*

$$e_y(H_\varepsilon) \le n\,\left(1 + 20\,(\kappa_0\,\varepsilon)^{\frac14}\right)\ ; \tag{1.41}$$

(iii) *For any $y\in Y$ we have the following bound for the Jacobian of H_r :*

$$\operatorname{Jac}(H_\varepsilon)(y) \le \left(1 + 20\,(\kappa_0\,\varepsilon)^{\frac14}\right)^{\frac{n}{2}}\ . \tag{1.42}$$

Moreover if $h:(Y,g)\to(X,g_0)$ is a continuous Gromov-Hausdorff ε-approximation H_ε is homotopic to h.

1.4.2. τ-coverings and the Reviron Lemma.

DEFINITION 1.4.2. Let (Y,g) be a Riemannian manifold and let $\tau > 0$. The τ-covering of (Y,g) is the normal covering corresponding to the normal subgroup $\pi_1^\tau(Y,y_0) \lhd \pi_1(Y,y_0)$ generated by the τ-lassos that is, $\pi_1^\tau(Y,y_0)$ is the subgroup generated by those elements γ that admit a representation $c^{-1}\,\beta\, c$ where β is a closed loop contained in a geodesic ball of radius τ, whereas c is a continuous path connecting y_0 with $\beta(0)=\beta(1)$.

In [**Rev**] G. Reviron proved an interesting result reconstructing the large scale metric structure of τ-coverings (for sufficiently small values of τ) of a compact (connected) length space (Y,d_Y) which is at small Gromov-Hausdorff distance from a compact (connected) length space (X,d_X), admitting a universal covering, whose systole is bounded below by some constant.

The following result follows directly from [**Rev**], Theorem 16:

LEMMA 1.4.3 (Reviron). *Let (X,g_0) be a closed (connected) n-dimensional Riemannian manifold and let $(\tilde X,\tilde g_0)$ be the Riemannian universal covering. Let $i_0 = \mathrm{inj}(X,g_0)$ and let ε, $\tau > 0$ be such that*

$$5\varepsilon < \tau < \frac{i_0}{2} - \frac{3\,\varepsilon}{2}\,. \tag{1.43}$$

Let (Y,g) be a n-dimensional compact (connected) Riemannian manifold, and let $\bar p : (\bar Y,\bar g) \to (Y,g)$ be a τ-covering for (Y,g).
If $d_{GH}((Y,g),(X,g_0)) < \varepsilon$ there exists a triple $(\lambda,\, \tilde f,\, \tilde h)$ such that:

(i) *λ is an isomorphism between the group of deck transformations of the τ-covering $\bar p : \bar Y \to Y$ and the fundamental group of X (thought as the group of the automorphisms of the universal covering).*

(ii) *There exists two maps $\tilde f : \tilde X \to \bar Y$, $\tilde h : \bar Y \to \tilde X$ which are λ^{-1} and λ equivariant respectively; they are the lifts of the Gromov-Hausdorff ε-approximations $f : (X,g_0) \to (Y,g)$, $h : (Y,g) \to (X,g_0)$, and they satisfy:*

$$\tilde\rho(\tilde h(\bar y),\tilde x) \le \left(1 - \frac{3\varepsilon}{2\,\tau}\right)^{-1} \bar d(\bar y, \tilde f(\tilde x)) + 2\varepsilon\ , \tag{1.44}$$

$$\bar d(\bar y, \tilde f(\tilde x)) \le \left(1 - \frac{3\varepsilon}{2\,\tau}\right)^{-1} \tilde\rho(\tilde h(\bar y),\tilde x) + 2\varepsilon\ ; \tag{1.43'}$$

$$\tilde\rho((\tilde h\circ\tilde f)(\tilde x),\tilde x) \le \varepsilon\ , \quad \bar d((\tilde f\circ\tilde h)(\bar y),\bar y) \le \varepsilon\ . \tag{1.45}$$

1.4.3. Proof of the Theorem 1.4.1. Consider the covering $(\bar Y,\bar g)$ of the manifold (Y,g) provided by the Lemma 1.4.3, corresponding to $\tau = \frac{\mathrm{inj}(X,g_0)}{3}$. The idea is to apply the Theorem 1.2.1 to the two manifolds $(\tilde X,\tilde g_0)$ and $(\tilde Y,\tilde g)$, the maps $\tilde h$, $\tilde f$ and the isomorphism λ produced by the Lemma 1.4.3 of G. Reviron, where we have identified the groups Γ_X, Γ_Y of the Theorem 1.2.1 respectively with the fundamental group of X and the group of deck transformations of the covering $\bar Y \to Y$.

Let us make the following choices for the parameters:

$$\kappa_0 R = \frac{(\kappa_0\,\varepsilon)^{\frac14}}{2^{2n}\,(n+1)^2}\,; \quad \frac{c}{\kappa_0} = \frac{(\kappa_0\,\varepsilon)^{-\frac12}}{(n+1)^4\,2^{4n}}\,;$$

$$\kappa_0 r < \min\left\{\frac{\kappa_0\,\varepsilon}{100},\ \kappa_0\ \mathrm{inj}(Y,g),\ \frac{\kappa_0}{2^{32n}\,(n+1)^{32}\kappa}\right\}.$$

We show that for these choices of the parameters R, c, r the triple (R,r,c) is in the set $\mathcal{D}_{n,\left(\frac{2i_0}{2i_0-9\varepsilon}\right),2\varepsilon,\kappa_0}$. By the definition of ε we have that:

$$\varepsilon < \frac{5}{2^{32n}\,(n+1)^{32}} \cdot \mathrm{rad}(\tilde{X},\tilde{g}_0) \tag{1.46}$$

and

$$\frac{\varepsilon}{\mathrm{inj}(X,g_0)} < \frac{\min\{1\,;\,\kappa_0^3\,\mathrm{inj}(X,g_0)^3\}}{2^{32n}\,(n+1)^{32}} \le 2^{-32n}\,(n+1)^{-32} \tag{1.47}$$

We may then apply the Lemma 1.4.3, choosing $\tau = \frac{i_0}{3}$ (we recall that $i_0 = \mathrm{inj}(X,g_0)$); it implies that the two maps $\tilde{f} : \tilde{X} \to \bar{Y}$, $\tilde{h} : \bar{Y} \to \tilde{X}$ satisfy the assumptions of the Theorem 1.2.1, i. e.

$$\rho(h(y),x) \le \alpha \cdot d(y,f(x)) + \varepsilon', \quad d(y,f(x)) \le \alpha \cdot \rho(h(y),x) + \varepsilon',$$
$$\rho(x,(h\circ f)(x)) \le \varepsilon'\ , \quad d(y,(f\circ h)(y)) \le \varepsilon'\ .$$

where

$$\alpha = \left(1 - \frac{3\varepsilon}{2\,\tau}\right)^{-1} = \left(1 - \frac{9\varepsilon}{2i_0}\right)^{-1} \quad \text{and} \quad \varepsilon' = 2\,\varepsilon$$

This gives the following estimate:

$$\alpha \le \frac{1}{1 - \frac{9}{2}\cdot 2^{-32n}\,(n+1)^{-32}} \tag{1.48}$$

We apply (1.48) and we obtain:

$$\frac{1}{5\alpha+1} > \frac{1 - \frac{9}{2}\cdot 2^{-32n}\,(n+1)^{-32}}{6} > \frac{5}{2^{32n}\,(n+1)^{32}}\quad ,$$

and thus (1.46) implies that the assumption $\varepsilon' \le \dfrac{2\,\mathrm{rad}(X,g_0)}{5\alpha+1}$ of the Theorem 1.2.1 is fulfilled.

From the definitions of $\kappa_0\,R$ and $\kappa_0\,\varepsilon$ we deduce that

$$R < \frac{\min\{\mathrm{inj}(X,g_0)\,;\,\frac{1}{\kappa_0}\}}{2^{10n}\,(n+1)^{10}} < \frac{5\,\mathrm{rad}(\tilde{X},\tilde{g}_0)}{2^{10n}\,(n+1)^{10}}$$

The last inequality and inequality (1.46) imply that

$$R + \frac{5}{2}\,\varepsilon' = R + 5\varepsilon < \left(\frac{5}{2^{10n}\,(n+1)^{10}} + \frac{25}{2^{32n}\,(n+1)^{32}}\right)\cdot \mathrm{rad}(\tilde{X},\tilde{g}_0) <$$
$$< \left(1 - \frac{9}{2\cdot 2^{32n}\,(n+1)^{32}}\right)\cdot \mathrm{rad}(\tilde{X},\tilde{g}_0) \le \frac{\mathrm{rad}(\tilde{X},\tilde{g}_0)}{\alpha}$$

and this proves that the assumption $\alpha\,(R+2\varepsilon') = \alpha\,(R+4\varepsilon) < \mathrm{rad}(X,g_0)$ of the Theorem 1.2.1 is fulfilled, and simultaneously that

$$\frac{1}{2}\left[\frac{\mathrm{rad}(\tilde{X},\tilde{g}_0)}{\alpha} - R - 2\varepsilon'\right] = \frac{1}{2}\left[\frac{\mathrm{rad}(\tilde{X},\tilde{g}_0)}{\alpha} - R - 4\varepsilon\right] > \frac{\varepsilon}{2} \ge 50\,r\quad ;$$

hence, to show that the assumptions of the Theorem 1.2.1 concerning the parameter r (*i.e.* inequality (1.7)) are fulfilled it is sufficient to show that $r \le \min\{\mathrm{inj}(\bar{Y},\bar{g})\ ;\ \frac{1}{\kappa}\}$, which follows directly from the definition of r. The last assumption of the Theorem 1.2.1, i. e. $\varepsilon' + 2r = 2\,\varepsilon + 2r < \frac{R}{2}$ is also satisfied : this follows directly from the definitions of ε, R and r. Finally we remark that:

$$4(\varepsilon + r) < \frac{404}{100}\cdot\varepsilon < \frac{404}{100\,\kappa_0}\cdot 2^{-32n}\,(n+1)^{-32}.$$

The previous discussion shows that we can apply the Theorem 1.2.1; this theorem says that there exists a C^1-map, $\bar{H}^R_{r,c} : \bar{Y} \to \tilde{X}$ satisfying the following properties:

(i) $\rho_0(\bar{H}^R_{r,c}(\bar{y}),\tilde{h}(\bar{y})) < \frac{\delta}{c}$, where the expression for $\delta = \delta_{n,\left(\frac{2i_0}{2i_0-9\varepsilon}\right),2\varepsilon,\kappa_0}(R,r,c)$ is given in the statement of Theorem 1.2.1 ;

(ii) The pointwise energy of the map $\bar{H}^R_{r,c}$ satisfies the following inequality:

$$e_{\bar{y}}(\bar{H}^R_{r,c}) \le n \cdot \alpha^{4n+2} \cdot e^{\frac{\delta}{\alpha}+10\,c\varepsilon} \cdot \left[1+\frac{\kappa_0^2}{2}\left(\alpha\,(R+2\varepsilon)+\frac{\delta}{c}\right)^2\right]^{2(n-1)} \cdot$$

$$\cdot \left[1+\frac{9}{2}\left(\alpha\,(R+2\varepsilon)+\frac{\delta}{c}\right)^2\kappa_0^2\right]^2 \cdot \left[1+2\,(n+1)\,3^{n+1}\cdot\frac{2\varepsilon}{R+2\varepsilon}\right]^2 \cdot$$

$$\cdot \left[1+\frac{\delta}{n\,\alpha}\left(1+\frac{n+1}{c\,(R+2\varepsilon)}\right)\right]$$

where $\alpha = \left(\frac{2i_0}{2i_0-9\varepsilon}\right) = \left(1-\frac{9\varepsilon}{2i_0}\right)^{-1}$.

(iii) When $\tilde{h}$ is continuous, $\bar{H}^R_{r,c}$ is λ-equivariantly homotopic to $\tilde{h}$.

In order to simplify the notation and to stress the dependence on the parameters R, r, c from ε we shall use the notation $\bar{H}_\varepsilon$ instead of $\bar{H}^R_{r,c}$. We shall prove the estimate (1.41) (and, as a consequence, by the arithmetic-geometric inequality, the estimate for the jacobian determinant (1.42)) and we shall show that the induced quotient map $H_\varepsilon : Y \to X$ is a Gromov-Hausdorff $\left(\frac{10}{\kappa_0}(n+1)^4\,2^{4n}\,(\kappa_0\,\varepsilon)^{\frac{3}{4}}\right)$-approximation.

In the sequel we shall need the following inequalities:

(a) $e^x \le (1+x\,e^x)\,,\ \forall x \in \mathbb{R}^+$;

(b) $(1+x)^n \le (1-x)^{-n} \le (1-n\,x)^{-1} \le (1+2\,n\,x)\,,\ \forall x \in [0, \frac{1}{2n}]$;

(c) $\prod_{i=1}^n (1+x_i) \le \left(1+\frac{1}{n}\sum_{i=1}^n x_i\right)^n \le 1+2\sum_{i=1}^n x_i$, when $x_i \ge 0$ and $\sum x_i \le \frac{1}{2}$;

Lemma 1.4.4. *Under the assumptions of the Theorem 1.4.1 and making the aforementioned choices for the parameters R, r, c we find:*

$$\delta < 3\cdot(\kappa_0\,\varepsilon)^{\frac{1}{4}} \tag{1.49}$$

$$\frac{\delta}{c} < \frac{3}{\kappa_0}(n+1)^4\,2^{4n}\cdot(\kappa_0\,\varepsilon)^{\frac{3}{4}} \tag{1.50}$$

In particular H_ε is a Gromov-Hausdorff $\left(\frac{10}{\kappa_0}(n+1)^4\,2^{4n}\cdot(\kappa_0\,\varepsilon)^{\frac{3}{4}}\right)$-approximation.

Proof. The definition of $\delta = \delta_{n,\left(\frac{2i_0}{2i_0-9\varepsilon}\right),2\varepsilon,\kappa_0}(R,r,c)$ in the Theorem 1.2.1 gives us:

$$\delta < (n+1)^2\,2^{\frac{n-1}{9}}\left(1-\frac{9\varepsilon}{2i_0}\right)^{-(n+1)} e^{5c\,\varepsilon}\cdot$$

$$\cdot\left[3c\,\varepsilon+\left(1-\frac{9\varepsilon}{2i_0}\right)^{-(2n+3)}\frac{9\varepsilon}{2i_0-9\varepsilon}+\kappa_0^2\,R_1^2+3^{n+4}\,\frac{2\varepsilon}{R}\right]^{\frac{1}{2}}$$

where $R_1 = \left(1-\frac{9\varepsilon}{2i_0}\right)^{-1}(R+2\varepsilon+2r)$. Since $2^{\frac{1}{9}}$ is an upper bound for $\left(1-\frac{9\varepsilon}{2i_0}\right)^{-1}$ and since $2^{\frac{8}{9}}$ is an upper bound for $e^{5c\varepsilon}$ and $\frac{9\varepsilon}{2i_0-9\varepsilon} < \frac{9\varepsilon}{i_0}$, we obtain:

$$\delta < (n+1)^2\,2^{n+1}\cdot\left[3c\varepsilon+2^{2n+3}\cdot\frac{9\varepsilon}{i_0}+(\kappa_0\,R_1)^2+3^{n+4}\,\frac{2\varepsilon}{R}\right]^{\frac{1}{2}}. \tag{1.51}$$

We remark that:

$$3c\varepsilon = 3\cdot\left[(n+1)^2\cdot 2^{2n}\right]^{-2}\cdot(\kappa_0\,\varepsilon)^{\frac{1}{2}} \tag{1.52}$$

$$\frac{9\varepsilon}{i_0} < \frac{9}{(n+1)^{16}\,2^{16n}}\cdot(\kappa_0\,\varepsilon)^{\frac{1}{2}} \tag{1.53}$$

$$(\kappa_0\,R_1) < \frac{3}{2}\cdot\kappa_0 R = \frac{3}{2}\cdot\left[(n+1)^2\,2^{2n}\right]^{-1}\cdot(\kappa_0\,\varepsilon)^{\frac{1}{4}} \tag{1.54}$$

$$\frac{2\varepsilon}{R} < 2\cdot\frac{(n+1)^2\,2^{2n}}{(n+1)^8\,2^{8n}}\cdot(\kappa_0\,\varepsilon)^{\frac{1}{2}} \tag{1.55}$$

Thus using (1.52), (1.53), (1.54), (1.55) in (1.51) we obtain:

$$\delta < (n+1)^2\,2^{n+1}\cdot\left[\frac{3\,(\kappa_0\,\varepsilon)^{\frac{1}{2}}}{(n+1)^4\,2^{4n}}+\frac{2^{2n+3}\,9\,(\kappa_0\,\varepsilon)^{\frac{1}{2}}}{(n+1)^{16}\,2^{16n}}+\frac{4\,(\kappa_0\,\varepsilon)^{\frac{1}{2}}}{(n+1)^4\,2^{4n}}+\frac{2\cdot 3^{n+4}\cdot(\kappa_0\,\varepsilon)^{\frac{1}{2}}}{(n+1)^6\,2^{6n}}\right]^{\frac{1}{2}}$$

$$< (n+1)^2\,2^{n+1}\left(\frac{19\,(\kappa_0\,\varepsilon)^{\frac{1}{2}}}{(n+1)^4\,2^{4n}}\right)^{\frac{1}{2}} < 3\cdot(\kappa_0\,\varepsilon)^{\frac{1}{4}}$$

which proves (1.49), (1.50) follows from this estimate and from the choice made for the parameter c. □

Let us now prove the pointwise energy estimate:

LEMMA 1.4.5. *Under the assumptions of the Theorem 1.4.1 and making the aforementioned choices for the parameters R, r, c we find the following pointwise energy estimate:*

$$e_{\bar{y}}(\bar{H}_\varepsilon) \le n\cdot(1+20\cdot(\kappa_0\,\varepsilon)^{\frac{1}{4}}) \tag{1.56}$$

Proof. We recall that, by the Theorem 1.2.1 the pointwise energy verifies the following estimate:

$$e_{\bar{y}}(\bar{H}_\varepsilon) \le n\cdot\left(1-\frac{9\varepsilon}{2i_0}\right)^{-(4n+2)}\cdot e^{\delta+10\,c\varepsilon}\cdot\left[1+\frac{\kappa_0^2}{2}\left(\left(1-\frac{9\varepsilon}{2i_0}\right)^{-1}(R+3\varepsilon)+\frac{\delta}{c}\right)^2\right]^{2(n-1)}\cdot$$

$$\cdot\left[1+\frac{9}{2}\left(\left(1-\frac{9\varepsilon}{2i_0}\right)^{-1}(R+3\varepsilon)+\frac{\delta}{c}\right)^2\kappa_0^2\right]^2\cdot\left[1+2\,(n+1)\,3^{n+1}\cdot\frac{3\varepsilon}{R+3\varepsilon}\right]^2\cdot$$

$$\cdot\left[1+\frac{\delta}{n\left(1-\frac{9\varepsilon}{2i_0}\right)^{-1}}\left(1+\frac{n+1}{c\,(R+2\varepsilon)}\right)\right] \tag{1.57}$$

Let us remark that, by the assumptions made on the parameters we can apply equality (b) to the terms $\left(1-\frac{9\varepsilon}{2i_0}\right)^{-(4n+2)}$ and $\left[1+\frac{\kappa_0^2}{2}\left(\frac{2i_0}{2i_0-9\varepsilon}\cdot(R+3\varepsilon)+\frac{\delta}{c}\right)\right]^{2(n-1)}$,

$$\left(1-\frac{9\varepsilon}{2i_0}\right)^{-(4n+2)} < \left(1+9\,(4n+2)\,\frac{\varepsilon}{i_0}\right) \tag{1.58}$$

$$\left[1+\frac{\kappa_0^2}{2}\left(\frac{2i_0}{2i_0-9\varepsilon}\cdot(R+3\varepsilon)+\frac{\delta}{c}\right)^2\right]^{2(n-1)} < 1+2\,(n-1)\kappa_0^2\left(2R+\frac{\delta}{c}\right)^2 \tag{1.59}$$

Using inequality (a) for the term $e^{\delta+10c\,\varepsilon}$ we obtain:

$$e^{\delta+10c\,\varepsilon} < (1+2\,(\delta+10c\,\varepsilon)) \tag{1.60}$$

Using these inequalities and the fact that $c\,R \ge (n+1)^2\,2^{2n}$ (by the choice of the parameters) we get the following estimate for $e_{\bar{y}}(\bar{H}_\varepsilon)$:

$$e_{\bar{y}}(\bar{H}_\varepsilon) \le n\cdot\left(1+9\,(4n+2)\,\frac{\varepsilon}{i_0}\right)\cdot(1+2(\delta+10c\,\varepsilon))\cdot\left(1+18\kappa_0^2\left(2R+\frac{\delta}{c}\right)^2\right)\cdot$$

$$\cdot\left(1+18\cdot 3^{n+1}\,(n+1)\,\frac{\varepsilon}{R}\right)\cdot\left(1+2(n-1)\,\kappa_0^2\left(2R+\frac{\delta}{c}\right)^2\right)\cdot\left(1+\frac{\delta}{n}\left(1+\frac{1}{n}\right)\right)$$

the assumptions made on ε, R, r, c allow us to use inequality (c) and thus:

$$e_{\bar{y}}(\bar{H}_\varepsilon) \le n\cdot\left(1+2\cdot\left(9\,(4n+2)\frac{\varepsilon}{i_0}+2(\delta+10c\,\varepsilon)+18\kappa_0^2\left(2R+\frac{\delta}{c}\right)^2+\right.\right.$$

$$\left.\left.+18\,(n+1)\,3^{n+1}\,\frac{\varepsilon}{R}+2\,(n-1)\,\kappa_0^2\left(2R+\frac{\delta}{c}\right)^2+\frac{\delta}{n}\cdot\left(1+\frac{1}{n}\right)\right)\right)$$

We observe that:

$$9\,(4n+2)\cdot\frac{\varepsilon}{i_0} < \frac{9\,(4n+2)}{(n+1)^{24}\,2^{24n}}\cdot(\kappa_0\,\varepsilon)^{\frac{1}{4}}$$

$$2(\delta+10c\,\varepsilon) < \frac{13}{2}\cdot(\kappa_0\,\varepsilon)^{\frac{1}{4}}$$

$$\kappa_0^2\left(2R+\frac{\delta}{c}\right)^2 < \left[\frac{2\,(\kappa_0\,\varepsilon)^{\frac{1}{4}}}{(n+1)^2\,2^{2n}} + \frac{3\,(n+1)^4\,2^{4n}\,(\kappa_0\,\varepsilon)^{\frac{1}{4}}}{(n+1)^{16}\,2^{16n}}\right]^2 < \frac{5\,(\kappa_0\,\varepsilon)^{\frac{1}{4}}}{(n+1)^{12}\,2^{12n}}$$

$$18\,(n+1)\,3^{n+1}\,\frac{\varepsilon}{R} < 18\,(n+1)\,3^{n+1}\,(n+1)^2\,2^{2n}\,\frac{(\kappa_0\,\varepsilon)^{\frac{1}{4}}}{(n+1)^{16}\,2^{16n}} < \frac{(\kappa_0\,\varepsilon_0)^{\frac{1}{4}}}{(n+1)^{11}\,2^{11n}}$$

$$\frac{\delta}{n}\cdot\left(1+\frac{1}{n}\right) < 3\,(\kappa_0\,\varepsilon)^{\frac{1}{4}}\ .$$

Thus we get:

$$e_{\bar{y}}(\bar{H}_\varepsilon) \le n\cdot\left(1+20\,(\kappa_0\,\varepsilon)^{\frac{1}{4}}\right)$$

which proves the estimate (1.56) of the energy. □

1.4.4. End of the proof of the Theorem 1.4.1. We proved that the map $\bar{H}_\varepsilon$ is a λ-equivariant C^1-map satisfying the pointwise energy bound (1.56) and such that the induced quotient map $H_\varepsilon\,:\,(Y,g)\to(X,g_0)$ is a Gromov-Hausdorff $\left(\frac{10}{\kappa_0}\,(n+1)^4\,2^{4n}\right)$-approximation (Lemma 1.4.4). Clearly, the pointwise energy estimate (1.56) holds also for the induced quotient map H_ε . Moreover, when h is continuous $\tilde{h}$ is continuous and when $\tilde{h}$ is continuous, by the Theorem 1.2.1, the map $\bar{H}_\varepsilon$ is λ-equivariantly homotopic to $\tilde{h}$, hence H_ε is homotopic to h and this concludes the proof of the Theorem 1.4.1. □

CHAPTER 2

The Conjugacy Rigidity Problem

Aperçu du chapitre 2 : *Dans ce chapitre nous nous intéressons à un problème classique de la Géométrie Riemannienne et des Systèmes Dynamiques : quand peut-on dire que deux variétés riemanniennes dont les flots géodésiques sont conjugués sont isométriques ?*
Nous commencerons (dans la section 2.2) par démontrer un résultat général (Proposition 2.2.2) : deux variétés riemanniennes dont les flots géodésiques sont C^0-*conjugués ont leurs revêtements universels riemanniens quasi-isométriques*[1]*. Nous utiliserons ensuite ce résultat et la nouvelle version de l'application barycentre (définie au chapitre 1) pour obtenir une nouvelle démonstration du fait que toute variété riemannienne dont le flot géodésique est* C^0-*conjugué à celui d'une variété plate est isométrique à cette variété plate (Theorem 2.3.1), nous appellerons cette propriété la "rigidité dynamique" des variétés plates. La méthode est constructive :*

- *Pour démontrer la Proposition 2.2.2, l'idée est d'utiliser l'isomorphisme entre les groupes fondamentaux des deux variétés (notées* Y *et* X*) pour construire une bijection entre l'orbite de l'action de* $\pi_1(Y)$ *sur le revêtement universel riemannien* $\tilde{Y}$ *et l'orbite de l'action de* $\pi_1(X)$ *sur le revêtement universel riemannien* $\tilde{X}$ *; on démontre que, lorsque les flots géodésiques sont* C^1-*conjugués, cette bijection est une* $C'-$*approximation de Gromov-Hausdorff entre ces orbites (où* C' *est une constante finie), qui est équivariante par rapport aux deux actions (voir le Lemme 2.2.8)*[2]*; comme ces deux orbites sont respectivement à distances de Gromov-Hausdorff de* $\tilde{Y}$ *et de* $\tilde{X}$ *inférieures aux diamètres de* Y *et de* X*, on a construit, à partir de la bijection entre les orbites, une* $C-$ *approximation de Gromov-Hausdorff equivariante* $\tilde{f}$ *entre les revêtements universels riemanniens* $\tilde{Y}$ *et* $\tilde{X}$ *(où* C *est une constante finie). Si de plus une des deux variétés est un espace* $K(\pi,1)$ [3]*, nous pouvons choisir* $\tilde{f}$ *de sorte qu'elle soit continue et induise (en passant au quotient) une équivalence d'homotopie entre* Y *et* X *(voir le Lemma 2.2.9 et la preuve du point (ii) de la Proposition 2.2.2).*
  Une conséquence presque immédiate de la Proposition 2.2.2 est une preuve (dont l'on donne ici deux versions différentes) du fait que toute variété de courbure de Ricci non-négative, dont le flot géodésique est C^0*-conjugué à celui d'une variété plate, est isométrique à celle-ci.*
- *Pour démontrer le Theorem 2.3.1, on applique une variante de la méthode du barycentre décrite au chapitre 1 en utilisant comme donnée initiale la* C*-approximation de Gromov-Hausdorff construite dans la Proposition 2.2.2 (ii)*[4]*. Nous exhibons ainsi une suite d'applications Lipschitziennes presque isométriques*

[1] Dans ce chapitre, cela signifie que la distance de Gromov-Hausdorff entre ces deux revêtements est bornée. Voir la Définition 2.2.1 pour plus de précisions.

[2] La comparaison entre les distances sur les deux orbites repose fortement sur le fait que la conjugaison des flots préserve le temps de parcours le long de toute géodésique et sur la cocompacité des groupes fondamentaux.

[3] Un espace topologique est un $K(\pi,1)$ si tous ses groupes d'homotopie (excepté le premier) sont triviaux; lorsque le groupe π est spécifié, on suppose de plus que son groupe fondamental est isomorphe à π.

[4] Il faut remarquer ici que l'application de la méthode c'est pas totalement triviale: bien que la constante multiplicative de notre quasi isométrie soit égale à 1 (ce qui est la valeur optimale pour le paramètre α dans le Théorème 1.2.1), nous ne sommes pas capables d'avoir un contrôle suffisamment bon de la constante additive C. C'est alors la géométrie du revêtement universel riemannien $\tilde{X}$ de l'espace de référence X (et en particulier le fait que son rayon d'injectivité soit infini, puisqu'il est

explicites (sections 2.3.2 et 2.3.3 et Proposition 2.3.8) dont la limite est l'isométrie recherchée [5].

Prospetto del capitolo 2: *In questo capitolo ci interessiamo ad un problema classico di Geometria Riemanniana e Sistemi Dinamici: quando due varietà Riemanniane i cui flussi geodetici sono coniugati sono isometriche?*
Cominceremo (nella sezione 2.2) dimostrando un risultato di carattere generale (Proposizione 2.2.2): due varietà Riemanniane i cui flussi geodetici sono C^0-coniugati hanno i rivestimenti universali Riemanniani quasi isometrici[6]. *Utilizzeremo successivamente questo risultato e la nuova versione dell'applicazione baricentro (definita nel capitolo 1) per ottenere una nuova dimostrazione del fatto che ogni varietà Riemanniana il cui flusso geodetico è C^1-coniugato a quello di una varietà piatta è isometrica a questa varietà piatta (Teorema 2.3.1), chiameremo questa proprietà "rigidità dinamica" delle varietà piatte. Il metodo è costruttivo:*

- *Per dimostrare la Proposizione 2.2.2, l'idea è quella di sfruttare l'isomorfismo tra i gruppi fondamentali delle due varietà (denotate Y ed X) per costruire una biiezione tra l'orbita dell'azione di $\pi_1(Y)$ sul rivestimento universale Riemanniano $\tilde{Y}$ e l'orbita dell'azione di $\pi_1(X)$ sul rivestimento universale Riemanniano $\tilde{X}$; si dimostra che, quando i flussi geodetici sono C^0-coniugati, tale biiezione è una C'-approssimazione di Gromov-Hausdorff tra queste orbite (dove C' è una costante fissata), che è equivariante rispetto alle due azioni (vedere il Lemma 2.2.8)*[7]*; poiché le due orbite sono rispettivamente a distanza di Gromov-Hausdorff limitata da $\tilde{Y}$ e $\tilde{X}$ inferiori ai diametri di Y ed X, abbiamo costruito, a partire dalle biiezioni tra le orbite, una C-approssimazione di Gromov-Hausdorff equivariante $\tilde{f}$ tra i rivestimenti universali Riemanniani $\tilde{Y}$ e $\tilde{X}$ (dove C è una costante finita). Se inoltre una delle due varietà è uno spazio $K(\pi,1)$*[8] *possiamo scegliere $\tilde{f}$ in modo tale che essa sia continua ed induca (passando al quoziente) una equivalenza omotopica tra Y e X (vedere il Lemma 2.2.9 e la dimostrazione del punto (ii) della Proposizione 2.2.2).*
  Una conseguenza quasi immediata della Proposizione 2.2.2 è una dimostrazione (della quale forniamo due differenti versioni) del fatto che ogni varietà di curvatura di Ricci non negativa il cui flusso geodetico è C^0-coniugato a quello di una varietà piatta, è isometrica a tale varietà.
- *Per dimostrare il Teorema 2.3.1 si applica una variante del metodo del baricentro descritto nel capitolo 1 utilizzando come dato iniziale la C-approssimazione di Gromov-Hausdorff costruita nella Proposizione 2.2.2 (ii)*[9]. *Esibiremo dunque una successione di applicazioni Lipschitziane, quasi isometriche esplicite (sezioni 2.3.2 e 2.3.3 e Proposizione 2.3.8) il cui limite è l'isometria cercata.*

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euclidien) qui nous permet d'appliquer la méthode.

[5]Remarquons que ceci implique que les deux résultats de rigidité dynamique des variétés plates d'une part et des espaces localement symétriques de courbure strictement négative d'autre part se démontrent par une même méthode.

[6]In questo capitolo ciò significa che la distanza di Gromov-Hausdorff tra i due rivestimenti è limitata. Vedere la Definizione 2.2.1 per una definizione più precisa.

[7]Il paragone tra le distanze su queste orbite poggia fortemente sul fatto che il coniugio del flusso preserva il tempo di percorrenza lungo ogni geodetica e sulla cocompattezza dei gruppi fondamentali.

[8]Uno spazio topologico è un $K(\pi,1)$ se tutti i suoi gruppi di omotopia (ad eccezione del primo) sono nulli; quando π è un gruppo specifico, si suppone in più che il suo gruppo fondamentale è isomorfo a π.

[9]Bisogna osservare che l'applicazione del metodo non è del tutto banale: difatti, benché la costante moltiplicativa della nostra quasi isometria sia uguale a 1 (che è il valore ottimale per il parametro α nel Teorema 1.2.1), non siamo capaci di ottenere un controllo sufficientemente preciso sulla costante additiva C. È dunque la geometria del rivestimento universale Riemanniano $\tilde{X}$ dello spazio di riferimento X a permetterci di applicare il metodo.

Sketch of the chapter 2: *In this chapter we are concerned with a classical subject in Riemannian Geometry and Dynamical Systems: in which cases two Riemannian manifolds whose geodesic flows are conjugate are isometric?*
We shall start (in section 2.2) with the proof of a general result (Proposition 2.2.2): two Riemannian manifolds whose geodesic flows are C^0*-conjugate have quasi isometric*[10]*. Afterwards we shall use this result and the new version of the barycenter map (defined in chapter* 1 *) to obtain a new proof of the fact that any Riemannian manifold whose geodesic flow is* C^1*-conjugate to the one of a flat manifold is isometric to this flat manifold (Theorem 2.3.1), we shall call this property "dynamic rigidity" of flat manifolds. The method is constructive:*

- *To prove Proposition 2.2.2 the idea is to use the isomorphism between the fundamental groups of the two manifolds (denoted* Y *and* X *) to construct a bijection between the orbit of the action of* $\pi_1(Y)$ *on the Riemannian universal covering* $\tilde{Y}$ *and the orbit of the action* $\pi_1(X)$ *on* $\tilde{X}$ *; we show that, when the geodesic flows are* C^0*-conjugate this bijection is a Gromov-Hausdorff* C'*-approximation between these orbits (here* C' *is a fixed constant), equivariant with respect to the two actions (see the Lemma 2.2.8)*[11]*; since the two orbits are at Gromov-Hausdorff distance (from* $\tilde{Y}$ *and* $\tilde{X}$ *respectively) bounded above by the maximum between the diameters of* Y *and* X *, we have constructed an equivariant Gromov-Hausdorff* C*-approximation (where* C *is a fixed constant). Moreover, when one of the two manifold is a* $K(\pi,1)$*-space*[12] *we can choose* $\tilde{f}$ *such that it is continuous and such that it induces (passing to the quotients) a homotopy equivalence between* Y *and* X *(see the Lemma 2.2.9 and the proof of point (ii) of the Proposition 2.2.2).*
  A quasi immediate consequence of the Proposition 2.2.2 is a proof (of which we provide two different versions) of the fact that any Riemannian manifold of non negative Ricci curvature whose geodesic flow is C^0*-conjugate to the one of a flat manifold, is isometric to this manifold.*
- *To prove Theorem 2.3.1 we apply a slightly different version of the barycenter method described in chapter* 1 *using as initial data the Gromov-Hausdorff* C*-approximation constructed in Proposition 2.2.2 (ii)*[13]*. We shall exhibit a sequence of quasi isometric Lipschitz maps (sections 2.3.2 and 2.3.3 and Proposition 2.3.8), the limit of which is an isometry.*

2.1. Introduction

In this chapter we shall be concerned with the following question: which compact Riemannian manifolds are uniquely determined by their geodesic flow[14]? In order to state the problem in a more precise way let us introduce some definitions:

Definition 2.1.1. Let (X, g_0) and (Y, g) be two Riemannian manifolds. We will say that they have C^r-conjugate geodesic flows if there exists a C^r-diffeomorphism $\varphi : U_gY \to U_{g_0}X$ between their unit tangent bundles, which commutes with the geodesic flows: $\varphi \circ g^t = g_0^t \circ \varphi$ where we denoted by g^t (resp. g_0^t) the geodesic flow of (Y, g) (resp. (X, g_0)) at the time t.

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[10]In this chapter this means that the Gromov-Hausdorff distance between the two coverings is bounded. See Definition 2.2.1 for the precise notion of quasi isometry.

[11]The comparison between the distances on the two orbits strongly relies on the fact that the conjugacy preserves the travel time along the geodesic paths , and on the cocompactness of the actions.

[12]A topological space is a $K(\pi, 1)$ if all its homotopy groups (except for the first) are trivial; when the π is a specific group we assume morever that the fundamental group is isomorphic to π.

[13]We remark here that the application of the barycenter method is not completely trivial: in fact, although the multiplicative constant of our quasi isometry is equal to 1 (the optimal value for the parameter α in the Theorem 1.2.1), we have no effective control on the additive constant C. It is the geometry of the Riemannian universal covering of the reference manifold that allows us to apply the method.

[14]Let us recall that, the geodesic flow g^t of (Y, g) is defined for any $v \in U_gY$, by $g^t(v) = \dot{c}_v(t)$, where c_v is the geodesic with initial data $\dot{c}_v(0) = v$.

Definition 2.1.2. Let (X, g_0) be a complete Riemannian manifold. We say that (X, g_0) is C^r-conjugacy rigid if any Riemannian manifold (Y, g) whose geodesic flow is C^r-conjugate to the one of (X, g_0) is isometric to (X, g_0). We say that (X, g_0) is C^r-rigid within some class $\mathcal{M}$ of Riemannian manifold if any Riemannian manifold $(Y, g) \in \mathcal{M}$ whose geodesic flow is C^r-conjugate to the one of (X, g_0) is isometric to (X, g_0).

Early counterexamples to conjugacy rigidity were discovered by Weinstein, who pointed out that Zoll surfaces and the standard sphere have C^∞-conjugate geodesic flows though they are not isometric manifolds (see [**Besse**], §4.F). On the other hand, as a consequence of the Blashke's Conjecture for spheres, proved by Marcel Berger (see [**Besse**], Appendix D), a first example of C^0-conjugacy rigid manifold was found: $\mathbb{R}P^n$ endowed with the canonical metric is C^0-conjugacy rigid for $n \geq 2$. A more general result was given by C. Croke and independently by J.P. Otal (see [**Cr1**], [**Ot**]), who proved the C^1-conjugacy rigidity for surfaces of nonpositive curvature, this was later improved to the C^0 case by C.Croke, A. Fathi and J. Feldman in [**CFF**]. In fact, when both the surfaces X and Y have negative curvature the conjugacy-rigidity problem turned out to be linked with the question whether the isospectrality of X and Y (with respect to the marked length spectrum) implies the existence of an isometry between X and Y (see [**Cr1**],[**CFF**] and [**Ot**]). In 1994 C. Croke and B. Kleiner proved the C^1-conjugacy rigidity for manifolds which admit a parallel vector field (see [**Cr-Kl**]), and one year later G. Besson, G. Courtois and S. Gallot gave a proof of the C^1-conjugacy rigidity of locally symmetric manifolds of negative curvature of dimension $n \geq 3$, as a consequence of the solution of the minimal entropy conjecture (see [**BCG1**]). The result of G. Besson, G. Courtois and S. Gallot has been later improved: in [**Ham**] U. Hamenstädt proved the C^0-rigidity of locally symmetric manifolds of negative curvature within the class of Riemannian manifolds of negative curvature. In fact, she has shown the C^0-conjugacy invariance of volumes when one manifold has negative curvature and the other one is a locally symmetric space of negative curvature. In 1996 another paper by C. Croke, P. Eberlein and B. Kleiner, [**CEK**], proved the C^0-conjugacy rigidity of locally symmetric manifolds of non-positive curvature of dimension $n \geq 3$ and rank $d \geq 2$, within the class of manifolds of non-positive curvature.

During the nineties a fundamental paper of P. Eberlein ([**Eb**]) brought the attention of some mathematicians to the investigation of 2-step nilmanifolds (*i.e.* compact quotients of a 2-step nilpotent Lie Group -endowed with a left invariant metric- obtained by the action of a lattice). A result by C. Gordon and W. Mao (see [**Go-Ma**]) states the C^0-conjugacy rigidity of a particular class of compact 2-step nilmanifolds (namely non-singular of Heisenberg type) within the class of compact nilmanifolds.

The proofs of these results are very different theorethically; in particular the conjugacy rigidity result of Besson, Courtois and Gallot relies on the study if the equality case in the proof of the minimal entropy conjecture, and it is not evident whether or not their argument gives a general strategy for solving the conjugacy rigidity problems. The existence of such a strategy would be of great interest in view of the more recent generalizations of the Barycenter Method (see [**BCG4**], [**Saba1**]).

This Chapter is organized as follows: in section 2.2 we present a general result of independent interest which shows how a C^0-conjugacy of geodesic flows imply the existence of a $(1, C)$-quasi isometry between the Riemannian universal coverings. This result is stronger when the initial data (X, g_0) is a $K(\pi, 1)$-space. As a consequence we shall give two short proofs of the C^0-conjugacy rigidity of flat manifolds within the class of manifolds of non-negative Ricci curvature. In section 2.3 we combine our general result with the Barycenter Method, to give a new proof of the C^1-conjugacy rigidity of flat manifolds.

2.2. Geodesic conjugacies and quasi isometries

The purpose of this section is to present a general remark that we shall use in the section 2.3 to provide a new proof of the C^1-conjugacy rigidity of flat manifolds.

This general result has an independent interest: starting from a C^0-conjugacy of geodesic flows, it gives rather precise informations on the large scale geometry of the universal covering of the unknown manifold (Y, g).

Let us recall the following definitions:

DEFINITION 2.2.1. A map $f : (Y, d) \to (X, \rho)$ between two non compact metric spaces is said to be (α, C)-quasi Lipschitz if it satisfies

$$\rho(f(y),\, f(y')) \leq \alpha \cdot d(y,\, y') + C, \quad \forall y, y' \in Y. \tag{2.61}$$

The map f is a (α, C)-quasi-isometry if there exists $h : (X, d_X) \to (Y, d_Y)$ such that both f and h are (α, C)-quasi Lipschitz and

$$\forall x \in X,\ \rho((f \circ h)(x),\, x) \leq C, \quad \forall y \in Y,\ d((h \circ f)(y),\, y) \leq C. \tag{2.62}$$

and we shall say that h is a (α, C)-quasi inverse for f. We remark that quasi-Lipschitz maps and quasi-isometries are in general non-continuous.

As $\pi : UM \to M$ induces an isomorphism $\pi_1(UM, v) \to \pi_1(M, \pi(v))$ when the dimension of M is greater or equal to 3 we may state the following result:

PROPOSITION 2.2.2. *Let (Y, g) and (X, g_0) be two compact connected Riemannian manifolds of dimension $n \geq 3$, with C^0-conjugate geodesic flows. Let us denote by λ the isomorphism between the fundamental groups of Y and X induced by the conjugacy of the geodesic flows.*

(i) *There exists a constant $C > 0$ and a $(1, C)$-quasi isometry $\tilde{f} : (\tilde{Y}, \tilde{g}) \to (\tilde{X}, \tilde{g}_0)$ (whose $(1, C)$-quasi inverse is denoted by $\tilde{h} : (\tilde{X}, \tilde{g}_0) \to (\tilde{Y}, \tilde{g})$).*

(ii) *Moreover, if (X, g_0) is a $K(\pi, 1)$-space we can find two continuous, λ-equivariant $(1, C)$-quasi isometries $\tilde{f} : \tilde{Y} \to \tilde{X}$, $\tilde{h} : \tilde{X} \to \tilde{Y}$ which are quasi inverses. They induce two homotopy equivalences f, h between the base manifolds Y, X, such that $h_*^{-1} = f_* = (\pi_X)_* \circ \varphi_* \circ ((\pi_Y)_*)^{-1}$ where $\pi_X : UX \to X$ and $\pi_Y : UY \to Y$ are the projections and $\varphi : UY \to UX$ the C^0-conjugacy.*

Let us give the definition of a non-principal ultrafilter and of an asymptotic cone of a metric space in order to state an immediate consequence of the Proposition 2.2.2:

DEFINITION 2.2.3. A non-principal ultrafilter on $\mathbb{N}$ is a finitely additive probability measure ω such that every subset $S \subset \mathbb{N}$ is ω-measurable and satisfies $\omega(S) \in \{0, 1\}$ and $\omega(S) = 0$ if $\#S < \infty$.

The reason why we introduce non-principal ultrafilters is that, roughly speaking, they pick out a convergent subsequence from any given bounded sequence of real numbers. Here we give a precise statement of this property:

Property. Let ω be a non-principal ultrafilter on $\mathbb{N}$. For every bounded sequence of real numbers $\{a_n\}_{n \in \mathbb{N}}$ there exists a unique point $l \in \mathbb{R}$ such that $\omega(\{n : |a_n - l| < \varepsilon\}) = 1$ for every $\varepsilon > 0$. We shall write $l = \lim_\omega a_n$.

DEFINITION 2.2.4. Let ω be a non-principal ultrafilter on $\mathbb{N}$. Let (X_n, d_n) a sequencence of metric spaces with basepoints p_n and let X_∞ denote the set of sequences $\{x_n\}$ where $x_n \in X_n$ and $d_n(x_n, p_n)$ is bounded independently of n. Consider the equivalence relation $[\{x_n\} \sim \{y_n\}$ iff $\lim_\omega d_n(x_n, y_n) = 0]$. Let X_ω denote the set of equivalence classes and endow X_ω with the metric $d_\omega(\{x_n\}, \{y_n\}) = \lim_\omega d_n(x_n, y_n)$. One writes $(X_\omega, d_\omega) = \lim_\omega (X_n, d_n)$ (if the choice of the basepoints is not important, otherwise we can stress the dependence from the sequence of the basepoints).
When we take $X_n = X$, $d_n = \frac{d}{n}$ and we fix the basepoint $p \in X$, (X_ω, d_ω) is called the *asymptotic cone* of X and it is also denoted by $\mathrm{Cone}_\omega(X, p)$.

The following corollary follows immediately from the Proposition 2.2.2:

COROLLARY 2.2.5 (C^0-conjugacy $\Rightarrow$ asymptotic isometry). *Let (Y,g) and (X,g_0) be two compact, connected Riemannian manifolds of dimension $n\geq 3$ whose geodesic flows are C^0-conjugate; let $(\tilde{Y},\tilde{g})$ and $(\tilde{X},\tilde{g}_0)$ be their Riemannian universal covering and let ω be any non-principal ultrafilter. For any choice of a base point $\tilde{y}_0\in\tilde{Y}$ the asymptotic cone $\mathrm{Cone}_\omega(\tilde{Y},\tilde{y}_0)$ is isometric to $\mathrm{Cone}_\omega(\tilde{X},\tilde{f}(\tilde{y}_0))$ - and the isometry is induced by the map $\tilde{f}$ provided by the Proposition 2.2.2.*

Before going into the proof of the Proposition 2.2.2 we point out two facts:

LEMMA 2.2.6. *Let $\varphi:U_gY\to U_{g_0}X$ be a C^k-conjugacy of geodesic flows $k\geq 0$. Then φ lifts to a C^k-conjugacy $\tilde{\varphi}:U_{\tilde{g}}\tilde{Y}\to U_{\tilde{g}_0}\tilde{X}$.*

Proof. Let $p_Y:\tilde{Y}\to Y$ and $p_X:\tilde{X}\to X$ be the universal coverings, then $dp_Y:\widetilde{U_gY}=U_{\tilde{g}}\tilde{Y}\to U_gY$ and $dp_X:\widetilde{U_{g_0}X}=U_{\tilde{g}_0}\tilde{X}\to U_{g_0}X$ are the universal coverings of U_gY and $U_{g_0}X$ and the homeomorphism $\varphi:U_gY\to U_{g_0}X$ lifts to an homeomorphism $\tilde{\varphi}$ between the universal coverings $U_{\tilde{g}}\tilde{Y}$ and $U_{\tilde{g}_0}\tilde{X}$ such that $dp_X\circ\tilde{\varphi}=\varphi\circ dp_Y$. Moreover, as the lift of a geodesic is a geodesic we have $dp_Y\circ\tilde{g}^t=g^t\circ dp_Y$ and $dp_X\circ\tilde{g}_0^t=g_0^t\circ dp_X$ and thus, for any $v\in U\tilde{Y}$:

$$dp_X\circ\tilde{\varphi}\circ\tilde{g}^t(v)=\varphi\circ dp_Y\circ\tilde{g}^t(v)=\varphi\circ g^t\circ dp_Y(v)=g_0^t\circ\varphi\circ dp_Y(v)$$

where we used the fact that φ is a conjugacy of geodesic flows. On the other hand we have:

$$dp_X\circ\tilde{g}_0^t\circ\tilde{\varphi}(v)=g_0^t\circ dp_X\circ\tilde{\varphi}(v)=g_0^t\circ\varphi\circ dp_Y(v)$$

Hence both $t\to\tilde{\varphi}\circ\tilde{g}^t(v)$ and $t\to\tilde{g}_0^t\circ\tilde{\varphi}(v)$ are liftings (via the covering map dp_X) of the same path $t\to g_0^t\circ\varphi\circ dp_Y(v)$. Moreover, these two liftings satisfy the same initial condition, since $\tilde{\varphi}\circ\tilde{g}^0(v)=\tilde{\varphi}(v)=\tilde{g}_0^0\circ\tilde{\varphi}(v)$, hence

$$\tilde{\varphi}\circ\tilde{g}^t=\tilde{g}_0^t\circ\tilde{\varphi},\quad\forall t\in\mathbb{R}\quad\square \tag{2.63}$$

LEMMA 2.2.7. *Let (Y,g) and (X,g_0) be two compact connected Riemannian manifolds of dimension $n\geq 3$, and assume that (X,g_0) is a $K(\pi,1)$-space. Let $\pi_X:UX\to X$ and $\pi_Y:UY\to Y$ be the unit tangent bundles and assume that UY and UX are homeomorphic by means of an homeomorphism φ. Then there is a homotopy equivalence $f:Y\to X$, which induces the isomorphism $(\pi_X)_*\circ\varphi_*\circ(\pi_Y)_*^{-1}$.*

Proof. It is known that any orientable Riemannian manifold of dimension $n\geq 3$ whose unit tangent bundle is homeomorphic to the one of a $K(\pi,1)$ manifold, is itself a $K(\pi,1)$-space (see for instance [**BCG1**], Proposition D.1). Remark that the orientability assumption made in [**BCG1**], Proposition D.1, is inessential in the proof.
Now we observe that the homeomorphism φ induces an isomorphism between the fundamental groups of the unit tangent bundles $\varphi_*:\pi_1(U_gY)\to\pi_1(U_{g_0}X)$. Consider the isomorphism $(p_X)_*\circ\varphi_*\circ((p_Y)_*)^{-1}:\pi_1(Y)\to\pi_1(X)$. By the Proposition 1.B.9 in [**Hat**], there must exist a continuous map f which induces this isomorphism between fundamental groups. Hence by Whitehead's Theorem (see [**Hat**], Theorem 4.5) f is an homotopy equivalence between Y and X. $\square$

LEMMA 2.2.8. *Let (Y,g) and (X,g_0) be compact, connected Riemannian manifolds of dimension $n\geq 3$. Assume that $\varphi:UY\to UX$ is a C^0-conjugacy of geodesic flows. Let $\tilde{\varphi}:U\tilde{Y}\to U\tilde{X}$ be the lift of φ to the universal coverings and let $[\tilde{w}_0,\tilde{x}_0]=\tilde{\varphi}[\tilde{v}_0,\tilde{y}_0]$ (where we shall denote a point from $T\tilde{Y}$ sometimes by $[\tilde{v},\tilde{y}]$ and sometimes simply by $\tilde{v}$). Let us denote by $\lambda=(\pi_X)_*\circ\varphi_*\circ((\pi_Y)_*)^{-1}$ the isomorphism between the fundamental groups. The map induced between the orbits*

$$\tilde{f}_\lambda:\pi_1(Y)\cdot\tilde{y}_0\to\pi_1(X)\cdot\tilde{x}_0,\quad \textit{given by}\quad \tilde{f}_\lambda(\gamma\cdot\tilde{y}_0)=\lambda(\gamma)\cdot\tilde{x}_0$$

is a λ-equivariant, $(1,C)$-quasi isometry between orbits (that we consider as metric subspaces of $(\tilde{Y},\tilde{g})$ and $(\tilde{X},\tilde{g}_0)$ respectively), where C is the constant:

$$C=2\cdot\max\{\mathrm{Diam}(\pi_{\tilde{X}}(\tilde{\varphi}((U_{\tilde{y}_0}\tilde{Y}))),\ \mathrm{Diam}(\pi_{\tilde{Y}}(\tilde{\varphi}^{-1}(U_{\tilde{x}_0}\tilde{X})))\}.$$

Proof. To simplify the notation observe that $((\pi_Y)_*)^{-1}\gamma = d\gamma$, *i.e.* the inverse of the isomorphism induced by the natural projection $\pi_Y : UY \to Y$ send the deck transformation γ into its differential $d\gamma$. We remark that $\tilde{f}_\lambda(\gamma\cdot\tilde{y}_0) = \pi_{\tilde{X}}(\tilde{\varphi}(d\gamma\cdot[\tilde{v}_0,\tilde{y}_0]))$, where

$$d\gamma\cdot[\tilde{v},\tilde{y}] = [d\gamma\cdot\tilde{v},\gamma\cdot\tilde{y}]\ .$$

This follows directly from the definition of the isomorphism λ. The map $\tilde{f}_\lambda$ is trivially equivariant with respect to the isomorphism λ and is a bijection and the inverse map is given by:

$$\tilde{f}_\lambda^{-1}(\lambda(\gamma)\cdot\tilde{x}_0) = \gamma\cdot\tilde{y}_0\ .$$

In analogy with $\tilde{f}_\lambda$ we remark that we have

$$\tilde{f}_\lambda^{-1}(\lambda(\gamma)\cdot\tilde{x}_0) = \pi_{\tilde{Y}}(\tilde{\varphi}^{-1}(((\pi_X)_*)^{-1}\lambda(\gamma)\cdot[\tilde{w}_0,\tilde{x}_0]))\ .$$

The roles of Y, X being interchangeable it is sufficient to show that $\tilde{f}_\lambda$ is $(1,C)$-quasi Lipschitz for a suitable $C>0$. Let $\tilde{c}:[0,T]\to\tilde{Y}$, a minimizing geodesic of unit speed between $\gamma\cdot\tilde{y}_0$ and $\gamma'\cdot\tilde{y}_0$ (hence $T = d(\gamma\,\tilde{y}_0,\gamma'\,\tilde{y}_0)$) and consider the path $\tilde{\alpha} = \pi_{\tilde{X}}\circ\tilde{\varphi}[\dot{\tilde{c}}]$. Since $\tilde{\varphi}$ is a C^0-conjugation of geodesic flows we have that

$$\tilde{\alpha}(t) = \pi_{\tilde{X}}(\tilde{\varphi}(\dot{\tilde{c}}(t))) = \pi_{\tilde{X}}\circ\tilde{\varphi}\circ\tilde{g}^t[\dot{\tilde{c}}(0)] = \pi_{\tilde{X}}\circ\tilde{g}_0^t\circ\tilde{\varphi}[\dot{\tilde{c}}(0)]$$

thus $\tilde{\alpha}$ is the geodesic of $(\tilde{X},\tilde{g}_0)$ with initial speed $\tilde{\varphi}[\dot{\tilde{c}}(0)]$ and length $\ell(\tilde{\alpha}) = T$. Observe that $\tilde{\alpha}(0) = \pi_{\tilde{X}}(\tilde{\varphi}[\dot{\tilde{c}}(0)])$ and $\tilde{\alpha}(T) = \pi_{\tilde{X}}(\tilde{\varphi}[\dot{\tilde{c}}(T)])$. As $\dot{\tilde{c}}(0)\in U_{\gamma\tilde{y}_0}\tilde{Y}$ and $\dot{\tilde{c}}(T)\in U_{\gamma'\tilde{y}_0}\tilde{Y}$ we have

$$\rho(\tilde{f}_\lambda(\gamma'\cdot\tilde{y}_0),\tilde{f}_\lambda(\gamma\cdot\tilde{y}_0))\le$$
$$\le\rho(\tilde{f}_\lambda(\gamma\cdot\tilde{y}_0),\tilde{\alpha}(0)) + \rho(\tilde{\alpha}(0),\tilde{\alpha}(T)) + \rho(\tilde{\alpha}(T),\tilde{f}_\lambda(\gamma'\cdot\tilde{y}_0))$$

By the λ-equivariance of $\tilde{f}_\lambda$ and by the definition of the isomorphism λ we have

$$\rho(\tilde{f}_\lambda(\gamma\cdot\tilde{y}_0),\,\tilde{\alpha}(0)) = \rho(\pi_{\tilde{X}}\circ\tilde{\varphi}[d\gamma\cdot\tilde{v}_0],\,\pi_{\tilde{X}}\circ\tilde{\varphi}(\dot{\tilde{c}}(0)))$$

and, as $d\gamma\cdot\tilde{v}_0$ and $\dot{\tilde{c}}(0)$ both belong to $U_{\gamma\tilde{y}_0}\tilde{Y}$, this is bounded above by the $\mathrm{Diam}(\pi_{\tilde{X}}\circ\tilde{\varphi}[d\gamma\cdot U_{\tilde{y}_0}\tilde{Y}]) = \mathrm{Diam}(\pi_{\tilde{X}}\circ\tilde{\varphi}[U_{\tilde{y}_0}\tilde{Y}])$, because $(\pi_{\tilde{X}}\circ\tilde{\varphi})[d\gamma\cdot U_{\tilde{y}_0}\tilde{Y}] = [(\pi_X\circ\varphi)_*]\cdot(\pi_{\tilde{X}}\circ\tilde{\varphi})(U_{\tilde{y}_0}\tilde{Y}) = \lambda(\gamma)\cdot(\pi_{\tilde{X}}\circ\tilde{\varphi})(U_{\tilde{y}_0}\tilde{Y})$. Analogously we can prove that

$$\rho(\tilde{f}_\lambda(\gamma'\cdot\tilde{y}_0),\tilde{\alpha}(T))\le\mathrm{Diam}(\pi_{\tilde{X}}(\tilde{\varphi}(U_{\tilde{y}_0}Y)))$$

hence we obtain the following inequality

$$\rho(\tilde{f}_\lambda(\gamma'\cdot\tilde{y}_0),\,\tilde{f}_\lambda(\gamma\cdot\tilde{y}_0))\le d(\gamma\cdot\tilde{y}_0,\,\gamma'\cdot\tilde{y}_0) + 2\cdot\mathrm{Diam}(\pi_{\tilde{X}}(\tilde{\varphi}(U_{\tilde{y}_0}\tilde{Y})))\ .$$

In analogy with the previous inequality we obtain:

$$d(\tilde{f}_\lambda^{-1}(\lambda(\gamma)\cdot\tilde{x}_0),\,\tilde{f}_\lambda^{-1}(\lambda(\gamma')\cdot\tilde{x}_0))\le\rho(\lambda(\gamma)\cdot\tilde{x}_0,\lambda(\gamma')\cdot\tilde{x}_0) + 2\cdot\mathrm{Diam}(\pi_{\tilde{Y}}(\tilde{\varphi}^{-1}(U_{\tilde{x}_0}\tilde{X})))\ .$$

Setting $C = 2\cdot\max\{\mathrm{Diam}(\pi_{\tilde{X}}(\tilde{\varphi}((U_{\tilde{y}_0}\tilde{Y}))),\,\mathrm{Diam}(\pi_{\tilde{Y}}(\tilde{\varphi}^{-1}(U_{\tilde{x}_0}\tilde{X})))\}$, this proves that $\tilde{f}_\lambda$, $\tilde{f}_\lambda^{-1}$ are two $(1,C)$-quasi isometries between the orbits $\pi_1(Y)\cdot\tilde{y}_0$ and $\pi_1(X)\cdot\tilde{x}_0$, which are equivariant with respect to λ, λ^{-1} respectively. □

LEMMA 2.2.9. *Let (X,g_0), (Y,g) be compact Riemannian manifolds and $\lambda:\pi_1(Y)\to\pi_1(X)$ an isomorphism. Assume that there exists a λ-equivariant map between the orbits*

$$\tilde{f}_\lambda:\pi_1(Y)\cdot\tilde{y}_0\to\pi_1(X)\cdot\tilde{x}_0$$

which is (α,C)-quasi Lipschitz (when considering the orbits as metric subspaces of $(\tilde{Y},\tilde{g})$ and $(\tilde{X},\tilde{g}_0)$). Then any continuous map $f:Y\to X$ such that the induced morphism $f_*:\pi_1(Y)\to\pi_1(X)$ is equal to λ lifts to a $(\alpha,C+D)$-quasi Lipschitz continuous map $\tilde{f}:\tilde{Y}\to\tilde{X}$, for some $D>0$ (depending on f).

Proof. Let K_0 be the compact closure of a Dirichlet domain centered at $\tilde{y}_0$ for the action of $\pi_1(Y)$ on $\tilde{Y}$. Let us denote

$$D_0 = \sup_{\tilde{y}\in K_0} d(\tilde{y}_0,\,\tilde{y}),\quad D_0' = \sup_{\tilde{y}\in K_0}\rho(\tilde{x}_0,\,\tilde{f}(\tilde{y})).$$

For any $\tilde{y},\tilde{y}'\in\tilde{Y}$, let $\gamma,\gamma'\in\pi_1(Y)$ be such that $\gamma^{-1}\tilde{y},\,(\gamma')^{-1}\tilde{y}'\in K_0$. As $\tilde{f}$ is λ-equivariant we have:

$$\rho(\tilde{f}(\tilde{y}),\,\tilde{f}(\tilde{y}'))\le$$

$$\leq \rho(\tilde{f}(\tilde{y}),\, \tilde{f}_\lambda(\gamma\,\tilde{y}_0)) + \rho(\tilde{f}_\lambda(\gamma\,\tilde{y}_0),\, \tilde{f}_\lambda(\gamma'\,\tilde{y}_0)) + \rho(\tilde{f}_\lambda(\gamma'\,\tilde{y}_0),\, \tilde{f}(\tilde{y}')) \leq$$

$$\leq \alpha \cdot d(\gamma\,\tilde{y}_0,\, \gamma'\,\tilde{y}_0) + 2 \cdot D_0' + C \leq \alpha \cdot d(\tilde{y},\, \tilde{y}') + 2 \cdot (D_0' + \alpha\, D_0) + C$$

Hence $\tilde{f}$ is $(\alpha, C + D)$-quasi Lipschitz for $D = 2 \cdot (D_0' + \alpha\, D_0)$. □

We can now prove the Proposition 2.2.2:

Proof of the Proposition 2.2.2, (i). By the Lemma 2.2.8 the existence of a C^0-conjugacy of geodesic flows implies the existence of a λ-equivariant, $(1, C)$-quasi isometry between the orbits of two points $\tilde{y}_0 \in \tilde{Y}$, $\tilde{x}_0 \in \tilde{X}$ under the isometric actions of $\pi_1(Y)$, $\pi_1(X)$ on $\tilde{Y}$, $\tilde{X}$ respectively. Since both fundamental groups act cocompactly $(\tilde{X}, \tilde{g}_0)$ is $(1, C')$-quasi isometric to $\pi_1(X) \cdot \tilde{x}_0$ and $(\tilde{Y}, \tilde{g})$ is $(1, C'')$-quasi isometric to $\pi_1(\tilde{Y}) \cdot \tilde{y}_0$. Hence $(\tilde{X}, \tilde{g}_0)$ and $(\tilde{Y}, \tilde{g})$ are $(1, 3(C + C' + C''))$-quasi isometric. □

Proof of the Proposition 2.2.2, (ii). Take the map $\tilde{f}$, which is a lift of the homotopy equivalence f given by the Lemma 2.2.7 for the isomorphism $\lambda = (\pi_X)_* \circ \varphi_* \circ (\pi_Y)_*^{-1}$. The map $\tilde{f}_\lambda : \pi_1(Y) \cdot \tilde{y}_0 \to \pi_1(X) \cdot \tilde{x}_0$ defined in the Lemma 2.2.8 is a $(1, C')$-quasi isometry. It follows by the Lemma 2.2.9 that there exists $D > 0$ such that $\tilde{f}$ is $(1, C' + D)$-quasi Lipschitz. Changing Y with X, the same argument shows that $\tilde{h} : \tilde{X} \to \tilde{Y}$, lift of the homotopy equivalence $h : X \to Y$ given by the isomorphism λ^{-1}, is $(1, C' + D)$-quasi Lipschitz.
We need to show that there exists a constant C'' such that

$$\rho(\tilde{f} \circ \tilde{h}(\tilde{x}), \tilde{x}) \leq C'', \quad d(\tilde{h} \circ \tilde{f}(\tilde{y}), \tilde{y}) \leq C''$$

Let $D' = d(\tilde{h} \circ \tilde{f}(\tilde{y}_0), \tilde{y}_0)$, for a fixed $\tilde{y}_0$ and let D_0 be the diameter of the Dirichlet domain K_0 centered at $\tilde{y}_0$. By definition $(h \circ f)_* = \mathrm{Id}_{\pi_1(Y)}$, so $\tilde{h} \circ \tilde{f}(\gamma \cdot \tilde{y}) = \gamma \cdot \tilde{h} \circ \tilde{f}(\tilde{y})$, hence $\forall \gamma \in \pi_1(Y)$

$$d(\tilde{h} \circ \tilde{f}(\gamma \cdot \tilde{y}_0), \gamma \cdot \tilde{y}_0) = d(\tilde{h} \circ \tilde{f}(\tilde{y}_0), \tilde{y}_0) = D'$$

Let γ be such that $\gamma^{-1} \cdot \tilde{y}$ belongs to K_0. Then:

$$d(\tilde{h} \circ \tilde{f}(\tilde{y}), \tilde{y}) \leq d(\tilde{h} \circ \tilde{f}(\tilde{y}), \tilde{h} \circ \tilde{f}(\gamma \cdot \tilde{y}_0)) + d(\tilde{y}, \gamma \cdot \tilde{y}_0) + d(\tilde{h} \circ \tilde{f}(\tilde{y}_0), \tilde{y}_0) \leq$$

$$\leq D' + D_0 + D_0'$$

where D_0' is the diameter of $\tilde{h} \circ \tilde{f}(K_0)$. By the same argument we get an analogous bound for $\rho(\tilde{f} \circ \tilde{h}(\tilde{x}), \tilde{x})$. Let C'' be the maximum between these two numbers. Then the maps $\tilde{f}$, $\tilde{h}$ are $(1, C)$-quasi isometries for $C = \max\{C'',\, C' + D\}$. □

As an example of the flexibility of the Proposition 2.2.2 we shall present two different proofs of the C^0-conjugacy rigidity of flat manifolds within the class of manifolds of non-negative Ricci curvature:

THEOREM 2.2.10. *Let (X, g_0) be a compact flat manifold of dimension $n \geq 3$. Let (Y, g) be a compact Riemannian manifold whose Ricci curvature satisfies* $\mathrm{Ricci}_g \geq 0$. *If (Y, g) and (X, g_0) have*
, C^0-conjugate geodesic flows they are isometric.

"Böchner-type" Proof. Let $\lambda = (p_X)_* \circ \varphi_* \circ (p_X)_*^{-1}$. By Proposition 2.2.2, (ii) it follows that there exists a continuous λ-equivariant, $(1, C)$-quasi isometry $\tilde{f} : (\tilde{Y}, \tilde{g}) \to (\tilde{X}, \tilde{g}_0)$ which induces an homotopy equivalence $f : Y \to X$. Since there exists a flat torus $(\bar{X}, \bar{g}_0)$ which covers (X, g_0) we consider the corresponding covering $(\bar{Y}, \bar{g})$ of (Y, g) and the lift of $\varphi : U_g Y \to U_{g_0} X$, $\bar{\varphi} : U_{\bar{g}} \bar{Y} \to U_{\bar{g}_0} \bar{X}$. We observe that in particular the first Betti number of $(\bar{Y}, \bar{g})$ is equal to n. By assumption $\mathrm{Ricci}_{\bar{g}} \geq 0$ and we can apply a classical result of Böchner (see [**Böch**], [**Böch-Ya**]):

THEOREM (for instance, [**GHL**], Theorem 4.37). Let (Y, g) be a compact n-dimensional Riemannian manifold. If the Ricci curvature is non-negative and the first Betti number is equal to n, then (Y, g) is an n-dimensional flat torus.

It follows that (Y,g) is a flat manifold. Since $(\tilde{Y},\tilde{g})$ and $(\tilde{X},\tilde{g}_0)$ are two simply connected flat manifolds they must be isometric both to $\mathbb{E}^n$. In particular (Y,g) and (X,g_0) are quotients of $\mathbb{E}^n$ by the actions of two isomorphic $(1,C)$-quasi isometric lattices. An easy metric argument shows that the two maximal rank free abelian subgroups of the two isomorphic, (1,C)-quasi isometric lattices are realized on $\mathbb{E}^n$ one from the other by a rotation. Moreover the Second Bieberbach Theorem (see for instance Theorem 2.1.2 in [**Dek**]) says that the isomorphism λ is actually a conjugation by an affine transformation, $(A,a)\in \mathrm{Aff}(\mathbb{R}^n)$. Combining the two informations we find that the linear transformation A is actually a rotation, hence $(A,a)\in \mathrm{Isom}(\mathbb{E}^n)$. We conclude that (Y,g) is isometric to (X,g_0) . □

"Asymptotic" Proof. By Proposition 2.2.2, (ii) we have a λ-equivariant, continuous, $(1,C)$ -quasi isometry which induces an homotopy equivalence between the quotient spaces. Since $(\tilde{Y},\tilde{g})$ is $(1,C)$-quasi isometric to $(\tilde{X},\tilde{g}_0)$ which is a Euclidean space it follows that the asymptotic cone of $(\tilde{Y},\tilde{g})$ is isometric to $(\tilde{X},\tilde{g}_0)$, hence is a Euclidean space. Since $\mathrm{Ricci}_g \geq 0$ we can use the following result:

THEOREM ([**Col**], Theorem 0.3). Let M^n be a Riemannian manifold of non-negative Ricci curvature and let M_∞ denote an asymptotic cone for M. If M_∞ is isometric to $\mathbb{E}^n$, then M^n is isometric to $\mathbb{E}^n$.

Since $(\tilde{Y},\tilde{g})$ and $(\tilde{X},\tilde{g}_0)$ are two simply connected flat manifolds it follows that they are both isometric to $\mathbb{E}^n$. On the other hand a $(1,C)$-quasi isometry between two isomorphic lattices into $\mathbb{E}^n$ must be an isometry and it follows that (Y,g) and (X,g_0) are isometric. □

REMARK 2.2.11. The C^0-rigidity of flat manifolds within the class of compact manifolds with non-negative Ricci curvature follows from Böchner's Theorem and the results in [**CEK**].

2.3. C^1-conjugacy rigidity of flat manifolds

In this section we shall give a proof of the C^1-conjugacy rigidity of flat manifolds of dimension $n\geq 3$ based on the Barycenter Method. We state the theorem:

THEOREM 2.3.1. *Let (X,g_0) be a compact flat Riemannian manifold of dimension $n\geq 3$. Then (X,g_0) is C^1-conjugacy rigid.*

The idea is to use the map which is guaranteed by the Proposition 2.2.2 as starting map for the Barycenter Method adapted to the flat case.

First of all let us remark that a flat Riemannian manifold (X,g_0) is a $K(\pi,1)$-space. In fact its Riemannian universal covering is diffeomorphic to $\mathbb{R}^n$ where $n=\dim(X)$, so it is contractible, hence (X,g_0) is an aspherical manifold. It follows that its k^{th} homotopy group is trivial for $k\geq 2$, hence it is a $K(\pi,1)$-space.

As a corollary of the Proposition 2.2.2 we have:

COROLLARY 2.3.2. *Let (Y,g) and (X,g_0) be two compact Riemannian manifolds of dimension $n\geq 3$ with C^0-conjugate geodesic flows and assume that (X,g_0) is flat. Then there exist two λ-equivariant, continuous $(1,C)$-quasi-isometries $\tilde{f}:(\tilde{Y},\tilde{g})\to(\tilde{X},\tilde{g}_0)$ and $\tilde{h}:(\tilde{X},\tilde{g}_0)\to(Y,\tilde{g})$, which induce two homotopy equivalences $f:(Y,g)\to(X,g_0)$ and $h:(X,g_0)\to(Y,g)$ (with $f_*=(h_*)^{-1}=\lambda$).*

We will use these maps to construct a family of maps $\{\tilde{F}_{c,r}\}_{c\in(0,+\infty),r>0}$ between $\tilde{Y}$ and $\tilde{X}$ with the following properties:

- **Equivariance:** for any $\gamma\in\pi_1(Y)$, $\tilde{F}_{c,r}(\gamma\cdot\tilde{y})=f_*(\gamma)\cdot\tilde{F}_{c,r}(\tilde{y})$.
- **Regularity:** we shall verify that $\tilde{F}_{c,r}$ are $C^1(\tilde{Y},\tilde{X})$, $\forall c\in(0,+\infty)$, $0<r<\min\{\mathrm{inj}(Y,g),\ \frac{1}{\kappa}\}$.

- **Homotopy equivalence:** the induced maps $F_{c,r} : Y \to X$ are all homotopic to f (in particular they are homotopy equivalences).

Moreover, we shall show that the induced maps $F_{c,r}$ converge to an isometry, $F : (Y,g) \to (X,g_0)$, as the parameters c and r go to 0.

2.3.1. The barycenter map. Let $c \in (0,+\infty)$, $0 < r < \min\left\{\mathrm{inj}(Y,g)\,,\,\frac{1}{\kappa}\right\}$ and let ρ and d denote the Riemannian distances of $(\tilde{X},\tilde{g}_0)$ and $(\tilde{Y},\tilde{g})$ respectively and let d_r be a regularization of d constructed as in §1.2.1.1. For any $\tilde{y} \in \tilde{Y}$ let us define the following function:

$$\mathcal{B}^c_{\tilde{y}}(\tilde{x}) = \int_{\tilde{X}} \rho(\tilde{x},\tilde{z})^2 \cdot e^{-c\,d_r(\tilde{y},\,\tilde{h}(\tilde{z}))} dv_0(\tilde{z})$$

Lemma 2.3.3. *For any $\tilde{y} \in \tilde{Y}$ the function $\mathcal{B}^c_{\tilde{y}}$ is well defined. Moreover, it is a strictly convex, C^∞ function, such that $\mathcal{B}^c_{\tilde{y}}(\tilde{x}) \to +\infty$ as $\rho(\tilde{x},0) \to \infty$.*

Proof. Since we have chosen $c > \mathrm{Ent}_{\mathrm{vol}}(X,g_0) = 0$, using properties (2.61), (2.62) of quasi isometries, it is easy to see that:

$$\mathcal{B}^c_{\tilde{y}}(\tilde{x}) = \int_{\tilde{X}} \rho(\tilde{x},\tilde{z})^2 \cdot e^{-c\,d_r(\tilde{y},\,\tilde{h}(\tilde{z}))}\, dv_0(\tilde{z}) \leq$$

$$\leq e^{c\,(d(\tilde{y},\,\tilde{h}(\tilde{x}))+3C+r)} \int_{\tilde{X}} \rho(\tilde{x},\tilde{z})^2 \cdot e^{-c\,\rho(\tilde{x},\,\tilde{z})}\, dv_0(\tilde{z}) < \infty$$

where the last integral is finite by the definition of volume entropy. For what concerns the regularity, since $(\tilde{X},\tilde{g}_0)$ is the Euclidean space of dimension n, we observe that $\rho(\tilde{x},\tilde{z})^2 = \tilde{g}_0(\tilde{x}-\tilde{z},\tilde{x}-\tilde{z})$. In particular it follows that $\mathcal{B}^c_{\tilde{y}}$ is a polynomial of degree 2. Hence $\mathcal{B}^c_{\tilde{y}}$ is C^∞ and that $\mathcal{B}^c_{\tilde{y}} \to \infty$ as $\rho(\tilde{x},0) \to \infty$. Convexity follows once you notice that:

$$Dd\mathcal{B}^c_{\tilde{y}}(\tilde{x}) = \left(2\int_{\tilde{X}} e^{-c\,d_r(\tilde{y},\tilde{h}(\tilde{z}))}\, dv_0(\tilde{z})\right) \cdot \tilde{g}_0 \quad \square$$

As a consequence of the Lemma 2.3.3 the function $\mathcal{B}^c_{\tilde{y}}$ admits a global minimum on $\tilde{X}$, and this minimum is attained at a unique point. So we define the map $\tilde{F}_{c,r} : \tilde{Y} \to \tilde{X}$ as $\tilde{F}_{c,r}(\tilde{y}) = \mathrm{argmin}(\mathcal{B}^c_{\tilde{y}})$. We observe that since $\mathcal{B}^c_{\tilde{y}}$ is C^∞ and $\tilde{F}_{c,r}(\tilde{y})$ is the only point in $\tilde{X}$ where the minimum of $\mathcal{B}^c_{\tilde{y}}$ is attained, we have:

$$0 = \nabla\mathcal{B}^c_{\tilde{y}}(\tilde{F}_{c,r}(\tilde{y})) = 2\cdot\int_{\tilde{X}} (\tilde{F}_{c,r}(\tilde{y}) - \tilde{z}) \cdot e^{-c\,d_r(\tilde{y},\,\tilde{h}(\tilde{z}))}\, dv_0(\tilde{z}) \tag{2.64}$$

Lemma 2.3.4 (Equivariance). *The maps $\tilde{F}_{c,r}$ are equivariant with respect to the isomorphism $\lambda = f_* : \pi_1(Y) \to \pi_1(X)$.*

Proof. First we observe that from the characterization of $\tilde{F}_{c,r}$ given by equation (2.64) we obtain an explicit expression for the baycenter map:

$$\tilde{F}_{c,r}(\tilde{y}) = \frac{\int_{\tilde{X}} \tilde{z}\cdot e^{-c\,d_r(\tilde{y},\,\tilde{h}(\tilde{z}))}\, dv_0(\tilde{z})}{\int_{\tilde{X}} e^{-c\,d_r(\tilde{y},\,\tilde{h}(\tilde{z}))}\, dv_0(\tilde{z})}$$

The function $\tilde{y} \to \int_{\tilde{X}} e^{-c\,d_r(\tilde{y},\,\tilde{h}(\tilde{z}))}\, dv_0(\tilde{z})$ is invariant by the action of $\pi_1(Y)$, by the invariance of v_0 with respect to the action of the isometries of $(\tilde{X},\tilde{g}_0)$. So we only need to prove the equivariance of $\tilde{y} \to \int_{\tilde{X}} \tilde{z}\cdot e^{-c\,d_r(\tilde{y},\,\tilde{h}(\tilde{z}))}\, dv_0(\tilde{z})$:

$$\int_{\tilde{X}} \tilde{z}\cdot e^{-c\,d_r(\gamma\cdot\tilde{y},\,\tilde{h}(\tilde{z}))}\, dv_0(\tilde{z}) = \int_{\tilde{X}} \tilde{z}\cdot e^{-c\,d_r(\tilde{y},\,\gamma^{-1}\tilde{h}(\tilde{z}))}\, dv_0(\tilde{z}) =$$

$$= \int_{\tilde{X}} \tilde{z}\cdot e^{-c\,d_r(\tilde{y},\,\tilde{h}(\lambda(\gamma)^{-1}\tilde{z}))}\, dv_0(\lambda(\gamma)^{-1}\tilde{z}) = \int_{\tilde{X}} (\lambda(\gamma)\cdot\tilde{z})\cdot e^{-c\,d_r(\tilde{y},\,\tilde{h}(\tilde{z}))}\, dv_0(\tilde{z})$$

for all $\gamma \in \pi_1(Y)$. So $\tilde{F}_{c,r}(\gamma\cdot\tilde{y}) = \lambda(\gamma)\cdot\tilde{F}_{c,r}(\tilde{y})$, for all $\gamma \in \pi_1(Y)$. $\square$

Lemma 2.3.5 (Regularity). *The maps $\tilde{F}_{c,r}$ are $C^1(\tilde{Y},\tilde{X})$.*

Proof. It is sufficient to verify that both $\tilde{y} \to \int_{\tilde{X}} \tilde{z}\, e^{-c\, d_r(\tilde{y}, \tilde{h}(\tilde{z}))}\, dv_0(\tilde{z})$ and $\tilde{y} \to \int_{\tilde{X}} e^{-c\, d_r(\tilde{y}, \tilde{h}(\tilde{z}))} dv_0(\tilde{z})$ are regular functions. To this purpose we remark that the functions

$$\frac{|e^{-c\, d_r(\tilde{y},\, \tilde{h}(\tilde{z}))} - e^{-c\, d_r(\tilde{y}',\, \tilde{h}(\tilde{z}))}|}{d(\tilde{y},\, \tilde{y}')}$$

are dominated by $2c{\cdot}e^{-c\, d_r(\tilde{y},\, \tilde{h}(\tilde{z}))}$ (here we use the regularity of the function $e^{-c\, d_r(\tilde{y}, \tilde{h}(\tilde{z}))}$ and the property (3) of the regularized distance -see subsection 1.2.1.1-).
Since the function $e^{-c\, d_r(\tilde{y}, \tilde{z})}$ is C^1 (actually C^∞) in $\tilde{y}$ we apply Lebesgue's dominated convergence Theorem to obtain the C^1-regularity of $\tilde{y} \to \int_{\tilde{X}} \tilde{z}\, e^{-c\, d_r(\tilde{y}, \tilde{h}(\tilde{z}))}\, dv_0(\tilde{z})$ and $\tilde{y} \to \int_{\tilde{X}} e^{-c\, d_r(\tilde{y}, \tilde{h}(\tilde{z}))} dv_0(\tilde{z})$ and thus of $\tilde{F}_{c,r}$. $\square$

LEMMA 2.3.6 (Homotopy). *The maps* $\tilde{F}_{c,r}$ *are all homotopic to* $\tilde{f}$ *, by an homotopy which is equivariant with respect to the isomorphism* $\lambda = f_* : \pi_1(Y) \to \pi_1(X)$ *.*

Proof. Take the homotopy $H(\tilde{y}, t) = (1-t)\, \tilde{F}_{c,r}(\tilde{y}) + t\, \tilde{f}(\tilde{y})$. $\square$

2.3.2. The jacobian and pointwise energy estimates. In this subsection we shall prove optimal estimates for the jacobian and the pointwise energy of the maps $\tilde{F}_{c,r}$. The first step is to estimate the distance between the map $\tilde{F}_{c,r}$ and the map $\tilde{f}$.

PROPOSITION 2.3.7. *The following estimate holds:*

$$\rho(\tilde{f}(\tilde{y}), \tilde{F}_{c,r}(\tilde{y})) \le \sqrt{n}\, (2C + r)\, e^{4c\left(C + \frac{r}{2}\right)} = A(n, c, C, r)$$

Proof. By the definition of $\tilde{F}_{c,r}$ we have that:

$$\tilde{F}_{c,r}(\tilde{y}) \cdot \int_{\tilde{X}} e^{-c\, d_r(\tilde{y}, \tilde{h}(\tilde{z}))}\, dv_0(\tilde{z}) = \int_{\tilde{X}} \tilde{z}\, e^{-c\, d_r(\tilde{y}, \tilde{h}(\tilde{z}))}\, dv_0(\tilde{z}) \tag{2.65}$$

hence, since $(\tilde{X}, \tilde{g}_0) = (\mathbb{R}^n, \langle \cdot\,, \cdot \rangle)$, for any unitary vector $v \in \mathbb{R}^n$,

$$\left(\int_{\mathbb{R}^n} e^{-c\, d_r(\tilde{y}, \tilde{h}(\tilde{z}))}\, dv_0(\tilde{z})\right) \cdot \langle \tilde{F}_{c,r}(\tilde{y}) - \tilde{f}(\tilde{y}), v \rangle = \int_{\mathbb{R}^n} \langle \tilde{z} - \tilde{f}(\tilde{y}), v \rangle\, e^{-c\, d_r(\tilde{y}, \tilde{h}(\tilde{z}))}\, dv_0(\tilde{z}) =$$

$$= \int_{\mathbb{R}^n} \left\langle \frac{\tilde{z} - \tilde{f}(\tilde{y})}{\| \tilde{z} - \tilde{f}(\tilde{y}) \|}\, , v \right\rangle \cdot \| \tilde{z} - \tilde{f}(\tilde{y}) \|\ e^{-c\, \| \tilde{f}(\tilde{y}) - \tilde{z} \|}\, dv_0(\tilde{z}) +$$

$$+ \int_{\mathbb{R}^n} \left\langle \frac{\tilde{z} - \tilde{f}(\tilde{y})}{\| \tilde{z} - \tilde{f}(\tilde{y}) \|}\, , v \right\rangle \cdot \| \tilde{z} - \tilde{f}(\tilde{y}) \| \cdot e^{-c \| \tilde{f}(\tilde{y}) - \tilde{z} \|} \cdot \left(e^{c(\| \tilde{f}(\tilde{y}) - \tilde{z} \| - d_r(\tilde{y}, \tilde{h}(\tilde{z})))} - 1 \right)\, dv_0(\tilde{z})$$

Writing the first integral of the right hand side of the previous equality in polar coordinates centered at $\tilde{f}(\tilde{y})$ we find

$$\int_0^{+\infty} \left(\int_{S^{n-1}_{\tilde{f}(\tilde{y})}} \langle w, v \rangle\, dw \right)\, t\, e^{-c\, t}\, t^{n-1}\, dt = 0$$

Plugging this equality in the previous one and calculating the equality for $v = \frac{\| \tilde{F}_{c,r}(\tilde{y}) - \tilde{f}(\tilde{y}) \|}{\| \tilde{F}_{c,r}(\tilde{y}) - \tilde{f}(\tilde{y}) \|}$ we obtain:

$$\int_{\mathbb{R}^n} e^{-c\, d_r(\tilde{y}, \tilde{h}(\tilde{z}))}\, dv_0(\tilde{z}) {\cdot} \parallel \tilde{F}_{c,r}(\tilde{y}) - \tilde{f}(\tilde{y}) \parallel \le$$

$$\le \left(e^{c(2C+r)} - 1 \right) \int_{\mathbb{R}^n} \left| \left\langle \frac{\tilde{z} - \tilde{f}(\tilde{y})}{\| \tilde{z} - \tilde{f}(\tilde{y}) \|}\, , v \right\rangle \right| \cdot \| \tilde{z} - \tilde{f}(\tilde{y}) \| \cdot e^{-c\, \| \tilde{z} - \tilde{f}(\tilde{y}) \|} \tag{2.66}$$

where we used the fact that $\tilde{h}$, $\tilde{f}$ are quasi-inverses $(1, C)$-quasi isometries and that $|d_r - d| \le r$. The same estimates gives

$$\int_{\mathbb{R}^n} e^{-c\, d_r(\tilde{y}, \tilde{h}(\tilde{z}))}\, dv_0(\tilde{z}) \ge e^{-c\, (2C+r)} \int_{\mathbb{R}^n} e^{-c\, \| \tilde{f}(\tilde{y}) - \tilde{z} \|}\, dv_0(\tilde{z}) \ge$$

$$\ge e^{-c(2C+r)}\, \omega_{n-1} \int_0^{+\infty} e^{-ct}\, t^{n-1}\, dt$$

we plug the new estimate into (2.66) and computing the right hand term in polar coordinates centered at $\tilde f(\tilde y)$ we get:

$$\| \tilde F_{c,r}(\tilde y) - \tilde f(\tilde y) \| \leq e^{c(2C+r)} \cdot \left(e^{c(2C+r)} - 1 \right) \cdot \frac{\int_0^{+\infty} \left(\int_{S^{n-1}_{\tilde f(\tilde y)}} |\langle w, v\rangle| \, dw \right) e^{-c\,t}\, t^n \, dt}{\omega_{n-1} \int_0^{+\infty} e^{-c\,t}\, t^{n-1}\, dt}$$

Since $\frac{1}{\omega_{n-1}} \int_{S^{n-1}} |\langle v, w\rangle| \, dw \leq \left(\frac{1}{\omega_{n-1}} \int_{S^{n-1}} |\langle v, w\rangle|^2 \, dw \right) \leq \frac{1}{\sqrt{n}}$ we get:

$$\| \tilde F_{c,r}(\tilde y) - \tilde f(\tilde y) \| \leq \sqrt{n}\, e^{c(2C+r)} \left(\frac{e^{c(2C+r)} - 1}{c} \right) \leq \sqrt{n}\, 2\, e^{2c(2C+r)} \cdot \left(C + \frac{r}{2} \right) \; \square$$

We are now able to prove the following:

PROPOSITION 2.3.8. *Let us define the pointwise energy of the map* $\tilde F_{c,r}$ *as* $e_{\tilde y}(\tilde F_{c,r}) = \sum_1^n \tilde g_0(d\tilde F_{c,r}(e_i), d\tilde F_{c,r}(e_i))$, *where* $\{e_i\}_1^n$ *is a* $\tilde g$*-orthonormal basis of* $T_{\tilde y}\tilde Y$. *We have the following estimates for the pointwise energy and the jacobian of* $\tilde F_{c,r}$ *at the point* $\tilde y \in \tilde Y$ *:*

$$e_{\tilde y}(\tilde F_{c,r}) \leq e^{4c\,(2C+r)}\, (1 + (\kappa\, r)^2)^2 \cdot n \cdot e^{2c\, A(n,\, c,\, C,\, r)} \tag{2.67}$$

$$|\, \mathrm{Jac}(\tilde F_{c,r})(\tilde y)| \leq e^{2nc\,(2C+r)}\, (1 + (\kappa\, r)^2)^n \cdot e^{n\, c\, A(n,\, c,\, C,\, r)} \tag{2.68}$$

Proof. We differentiate equation (2.3.7) we obtain:

$$\left(\int_{\mathbb{R}^n} e^{-c\, d_r(\tilde y, \tilde h(\tilde z))}\, dv_0(\tilde z) \right) d_{\tilde y} \tilde F_{c,r}(u) =$$

$$= -c \int_{\mathbb{R}^n} (\tilde z - \tilde F_{c,r}(\tilde y)) \cdot \tilde g(\nabla d_r(\tilde y, \tilde h(\tilde y)), u)\, e^{-c\, d_r(\tilde y, \tilde h(\tilde z))}\, dv_0(\tilde z)$$

Let $v = \frac{d_{\tilde y} \tilde F_{c,r}(u)}{\| d_{\tilde y} \tilde F_{c,r}(u) \|}$ and consider the scalar product of the previous equation with v:

$$\left(\int_{\mathbb{R}^n} e^{-c\, d_r(\tilde y, \tilde h(\tilde z))}\, dv_0(\tilde z) \right) \| d_{\tilde y} \tilde F_{c,r}(u) \| =$$

$$= -c \int_{\mathbb{R}^n} \langle \tilde z - \tilde F_{c,r}(\tilde y), v\rangle\, \tilde g(\nabla d_r(\tilde y, \tilde h(\tilde z)), u)\, e^{-c\, d_r(\tilde y, \tilde h(\tilde z))}\, dv_0(\tilde z)$$

and using Cauchy-Schwartz together with the fact that $|\, \| \tilde f(\tilde y) - \tilde x \| - d(\tilde y, \tilde h(\tilde x))| \leq 2C$ and $|\, d_r - d| \leq r$ we obtain

$$\left(\int_{\mathbb{R}^n} e^{-c\, d_r(\tilde y, \tilde h(\tilde z))} \right)^2 \| d_{\tilde y} \tilde F_{c,r}(u) \|^2 \leq$$

$$\leq c^2 \left(\int_{\mathbb{R}^n} \left\langle \frac{\tilde z - \tilde F_{c,r}(\tilde y)}{\| \tilde z - \tilde F_{c,r}(\tilde y) \|}, v \right\rangle^2 \| \tilde z - \tilde F_{c,r}(\tilde y) \| \; e^{-c\, d_r(\tilde y, \tilde h(\tilde z))}\, dv_0(\tilde z) \right) \cdot$$

$$\cdot \left(\int_{\mathbb{R}^n} \tilde g(\nabla d_r(\tilde y, \tilde h(\tilde z)), u)^2 \; \| \tilde z - \tilde F_{c,r}(\tilde y) \| \; e^{-c\, d_r(\tilde y, \tilde h(\tilde z))}\, dv_0(\tilde z) \right) \leq$$

$$\leq c^2\, e^{2c(2C+r)} \left(\int_{\mathbb{R}^n} \left\langle \frac{\tilde z - \tilde F_{c,r}(\tilde y)}{\| \tilde z - \tilde F_{c,r}(\tilde y) \|}, v \right\rangle^2 \| \tilde z - \tilde F_{c,r}(\tilde y) \| \; e^{-c\, \|\tilde z - \tilde f(\tilde y)\|}\, dv_0(\tilde z) \right) \cdot$$

$$\cdot \left(\int_{\mathbb{R}^n} \tilde g(\nabla d_r(\tilde y, \tilde h(\tilde z)), u)^2 \; \| \tilde z - \tilde F_{c,r}(\tilde y) \| \; e^{-c\, \|\tilde z - \tilde f(\tilde y)\|}\, dv_0(\tilde z) \right) \leq$$

We sum over an orthonormal basis $\{e_i\}_{1 \leq i \leq n}$ of $T_{\tilde y}\tilde Y$ and we obtain:

$$e^{-2c(2C+r)} \left(\int_{\mathbb{R}^n} e^{-c\|\tilde f(\tilde y) - \tilde z\|} dv_0(\tilde z) \right)^2 \cdot \sum_{i=1}^n \| d_{\tilde y} \tilde F_{c,r}(e_i) \|^2 \leq c^2\, e^{2c\,(2C+r)}\, e^{2c\, \|\tilde F_{c,r}(\tilde y) - \tilde f(\tilde y)\|} \cdot$$

$$\cdot \left(\int_{\mathbb{R}^n} \left\langle \frac{\tilde z - \tilde F_{c,r}(\tilde y)}{\| \tilde z - \tilde F_{c,r}(\tilde y) \|}, v \right\rangle^2 \| \tilde z - \tilde F_{c,r}(\tilde y) \| \; e^{-c\, \|\tilde z - \tilde F_{c,r}(\tilde y)\|}\, dv_0(\tilde z) \right) \cdot$$

$$\cdot \left(\int_{\mathbb{R}^n} \| \nabla d_r(\tilde y, \tilde h(\tilde z)) \|^2 \, \| \tilde z - \tilde F_{c,r}(\tilde y) \| \; e^{-c\|\tilde z - \tilde F_{c,r}(\tilde y)\|}\, dv_0(\tilde z) \right) \leq$$

$$\leq c^2\, e^{2c\,(2C+r)}\, e^{2c\, A(n,\, c,\, C,\, r)}\, (1 + (\kappa\, r)^2)^2 \omega_{n-1} \cdot$$

$$\cdot \left(\int_0^{+\infty} \left(\int_{S^{n-1}_{\tilde{F}_{c,r}(\tilde{y})}} \langle w, v \rangle^2 \, dw \right) e^{-ct} \, t^n \, dt \right) \cdot \left(\int_0^{+\infty} e^{-ct} \, t^n \, dt \right)$$

thus we get:

$$\sum_{i=1}^n \| \, d_{\tilde{y}} \tilde{F}_{c,r}(e_i) \, \|^2 \le \frac{c^2}{n} \, e^{4c\,(2C+r)+2c\,A(n,c,C,r)} \cdot (1+(\kappa\, r)^2)^2 \left(\frac{\int_0^{+\infty} e^{-ct} \, t^n \, dt}{\int_0^{+\infty} e^{-ct} \, t^{n-1} \, dt} \right)^2$$

since $\left(\frac{\int_0^{+\infty} e^{-ct} \, t^n \, dt}{\int_0^{+\infty} e^{-ct} \, t^{n-1} \, dt} \right) = \frac{n}{c}$ which proves (2.67). The Jacobian estimate (2.68) follows directly from (2.68). $\square$

2.3.3. End of the proof of the Theorem 2.3.1. In the previous subsections we have constructed a family of C^1 maps from $\tilde{Y}$ to $\tilde{X}$, depending on the parameters c, r, equivariant with respect to the isomorphism $\lambda = f_* : \pi_1(Y) \to \pi_1(X)$ and satisfying the bounds for the pointwise energy and the jacobian proved in the Proposition 2.3.8. Moreover, the induced maps $F_{c,r} : Y \to X$ are all homotopic to the starting homotopy equivalence f (and obviously satisfy the same bounds for the pointwise energy and the jacobian). We shall show that we can extract a subsequence $\{F_{c_n,r_n}\}_{n\in\mathbb{N}}$ converging to an isometry $F : Y \to X$.

The explicit formula for $A(n,c,C,r)$ shows that $4c\,(2C+r)+2c\,A(n,c,C,r) \to 0$ when $c,\, r \to 0$. Hence, if we denote $\varepsilon(c,r) = (1+(\kappa\, r)^2)^n \, e^{2nc\,(2C+r)+nc\,A(n,c,C,r)} - 1$, the estimates in the Proposition 2.3.8 can be written as follows:

$$e_y(F_c) \le n \cdot (1+\varepsilon(c,r))^{\frac{2}{n}}, \quad |\operatorname{Jac}(F_c)| \le (1+\varepsilon(c,r)) \quad \text{with } \varepsilon(c,r) \to 0 \text{ for } c,r \to 0\,.$$

We put $r = c$. Take a sequence $\{c_n\}_{n\in\mathbb{N}}$ converging to 0. We shall denote by $\tilde{F}_{c_n}$ the maps $\tilde{F}_{c_n,c_n}$. The family $\{F_{c_n}\}_{n\in\mathbb{N}}$ is a uniformly bounded, equicontinuous family of $C^1(Y,X)$ functions; we extract a sequence $\{F_{c_n}\}_{n\in\mathbb{N}}$ converging to a map $F \in C^0(Y,X)$. We remark that F is also homotopic to the map f, hence it is a homotopy equivalence. In particular $\operatorname{Adeg}(F) = \operatorname{Adeg}(F_{c_n}) = 1$. By the C^1-conjugacy rigidity of volumes (see the Proposition 1.2 in [**Cr-Kl**]) we know that $\operatorname{Vol}(Y,g) = \operatorname{Vol}(X,g_0)$. Hence we can use the coarea formula (Theorem 13.4.2, [**Bu-Za**]) and the Jacobian estimates for the maps F_{c_n} obtaining:

$$\operatorname{Vol}(X,g_0) = \operatorname{Vol}(Y,g) \ge \lim_{n\to\infty} \int_Y |\operatorname{Jac}(F_{c_n})(y)| \, dv_g(y) \ge$$

$$\ge \lim_{n\to\infty} \int_X \#(F^{-1}_{c(n)}(x)) \, dv_{g_0}(x) \ge \lim_{n\to\infty} \operatorname{Adeg}(F_{c_n}) \operatorname{Vol}(X,g_0) = \operatorname{Vol}(X,g_0)$$

where in the last inequality we used the fact that $\operatorname{Adeg}(F_{c_n}) = 1$. It follows that $|\operatorname{Jac}(F_{c_n})| \to 1$ *a.e.* which implies that $e_y(F_{c_n}) \to n$, *a.e.*. Since $e_y(F_{c_n}) \to n$ and $\operatorname{Jac}(F_c) \to 1$ *a.e.* we deduce the convergence *a.e.* of $\| \, d_y F_{c_n} \, \|_{g,\,g_0} = \sup_{v\in T_yY} \frac{\|d_yF_{c_n}\cdot v\|_{g_0}}{\|v\|_g}$: $\| \, d_y F_{c_n} \, \|_{g,\,g_0} \to 1$, *a.e.*. On the other hand the upper bound for the pointwise energy says that $\| \, d_y F_{c_n} \, \|_{g,\,g_0}$ is uniformly bounded above by $2n$. It follows that we have a sequence of maps $\{F_{c_n}\}$ uniformly converging to a map F, such that the sequence $\{\| \, d_y F_{c_n} \, \|_{g,\,g_0}\}_{y\in Y,\, n\in\mathbb{N}}$ is bounded and coverges to 1 *a.e.*; hence (see [**BCG1**], Lemme 7.8) F must be a Lipschitz function and the Lipschitz constant is ≤ 1, *i.e.* is a contraction from (Y,g) to (X,g_0).
Moreover, F is an homotopy equivalence and $\operatorname{Adeg}(F) = 1$.
Since we have equality of volumes, we can deduce that it is an isometry, as in the Proposition C.1 in [**BCG1**] (as modified for non orientable manifolds in [**Samb1**]). $\square$

Remark 2.3.9. Observe that the C^1-regularity is only needed to provide the equality of volumes. In fact, by the same argument we can prove that a flat manifold (X,g_0) is C^0-conjugacy rigid within the class of Riemannian manifolds with volume less or equal to $\operatorname{Vol}(X,g_0)$. The reason is that the opposite (large) inequality $\operatorname{Vol}(Y,g) \ge \operatorname{Vol}(X,g_0)$ is provided by the Jacobian estimate in the subsection 2.3.2, hence, within this class, the volume is preserved and we can conclude as in the previous case.

CHAPTER 3

A Spectra Comparison Theorem and its applications

Aperçu du chapitre 3: *Dans ce chapitre nous présentons une application de la méthode du barycentre à la Géométrie Spectrale. Énoncé de manière naïve, notre but est de comparer le spectre du Laplacien d'une variété Riemannienne* **quelconque** *à celui d'une variété de référence fixée lorsque ces deux variétés sont* $\varepsilon-$*proches (au sens de la distance Gromov-Hausdorff et pour une valeur fixée de la constante* $\varepsilon > 0$*).*
De façon plus précise: étant donnée une variété compacte connexe de référence (X^n, g_0)*, de géométrie bornée*[1]*, considérons n'importe quelle variété riemannienne* (Y, g) *de même dimension qui admet une* $\varepsilon-$*approximation de Gromov-Hausdorff*[2]*, continue et de degré absolu différent de zéro, sur* (X^n, g_0)*. Nous supposerons également que le volume de* (Y, g) *est presque inférieur à celui de* (X, g_0)*.*
Dans le Théorème principal 3.1.2 nous montrons que chaque valeur propre $\lambda_i(Y, g)$ *du Laplacien de* (Y, g) *est alors presque inférieure à la valeur propre correspondante de* (X^n, g_0)*, plus précisément on a* $\lambda_i(Y, g) \leq (1 + \eta(\varepsilon))\, \lambda_i(X, g_0)$*, où la fonction* $\eta(\varepsilon)$ *est explicite (donnée dans l'énoncé du Théorème 3.1.2) et tend vers zéro quand* ε *tend vers zéro.*

Après avoir, dans la section 3.1, fait un bref rappel historique sur le problème de l'estimation des valeurs propres du spectre du Laplacien dans le cadre des variétés Riemanniennes et avoir présenté et discuté l'énoncé du Théorème principal 3.1.2, nous allons (dans la section 3.2) faire des rappels sur les inégalités de Sobolev quantitatives (précisant comment les constantes qui y interviennent dépendent des bornes de la géométrie globale, voir section 3.2.1) et, à l'aide de la méthode d'itération de Moser, nous allons montrer comment en déduire des bornes supérieures du rapport entre norme L^∞ *et norme* L^2 *de* f *et de* df *pour toute fonction* f *de l'espace vectoriel* $\mathcal{A}_X(\lambda)$ *engendré par les fonctions propres du Laplacien de* (X, g_0) *associées aux valeurs propres* $\lambda_i(X, g_0) \leq \lambda$ *(voir section* [3] *3.2.2).*
Dans la section 3.3 nous allons démontrer un résultat de comparaison "technique" (la Proposition 3.1.4) qui nous permettra, à l'aide de la méthode du barycentre, de démontrer le Théorème 3.1.2. Ce résultat compare le spectre d'une variété Riemannienne quelconque (Y, g) *à celui d'une variété* (X, g_0) *de "géométrie faiblement bornée*[4]*" lorsque ces deux*

---

[1] Ici "géométrie bornée" signifie que le diamètre et la valeur absolue de la courbure sectionnelle de (X^n, g_0) sont majorés et que son rayon d'injectivité est minoré. La condition de "géométrie bornée" n'impose aucune restriction à la géométrie de (X^n, g_0), puisque toute variété compacte admet de telles bornes; cette "condition" indique seulement que toutes les constantes universelles qui interviendront dans le résultat seront calculées en fonction de ces bornes de la géométrie de (X^n, g_0).

[2] Bien que ε ne soit pas supposé petit, il faut cependant le choisir inférieur à une valeur critique, calculable en fonction des bornes de la géométrie de la variété de réference (X, g_0), pour que les résultats que nous obtenons ne soient pas vides.

[3] Les résultats de cette section constituent une référence détaillée pour les résultats démontrés par S. Gallot dans deux notes aux Comptes Rendus de l'Académie des Sciences ([**Ga1**], [**Ga2**].
Notons une contrainte importante : pour toute fonction $f \in \mathcal{A}_X(\lambda)$, les bornes que nous obtiendrons pour les rapports $\dfrac{\|f\|_{L^\infty}}{\|f\|_{L^2}}$ et $\dfrac{\|df\|_{L^\infty}}{\|df\|_{L^2}}$ (qui dépendront évidemment de λ) ne doivent pas dépendre de la dimension de $\mathcal{A}_X(\lambda)$.

[4] En effet, dans la Proposition 3.1.4, l'hypothèse de "géométrie bornée" faite sur la variété (X, g_0) peut être affaiblie en une hypothèse de "géométrie faiblement bornée", cette dernière hypothèse signifiant que la courbure de Ricci de (X, g_0) est minorée et son diamètre majoré.

variétés sont $\varepsilon-$proches au sens de la distance Gromov-Hausdorff (pour une valeur fixée de la constante $\varepsilon > 0$) et lorsqu'il existe une application Lipschitzienne $F : (Y, g) \to (X, g_0)$ (de degré absolu non trivial), dont l'énergie ponctuelle est presque inférieure à n (la valeur n étant critique, puisqu'il s'agit de la valeur de l'énergie d'une isométrie).
La preuve de la Proposition 3.1.4 se fait en trois étapes : dans la section 3.3.1, nous montrons que la variété Y de départ est divisée en deux parties : une première partie où le déterminant Jacobien de F est presque égal à 1 (ce qui implique que l'application F est presque isométrique sur cette partie) et une deuxième partie où nous n'avons plus de contrôle sur le déterminant Jacobien, mais qui est de volume petit relativement au volume total de la variété (Lemma 3.3.5); dans la section 3.3.2, nous rappelons un théorème de comparaison général et simple (Lemme 3.3.9, déjà présent dans [**Aub2**]*), basé sur le Principe du Minimax, qui nous permettra, dans la section 3.3.3, d'achever la preuve de la Proposition "technique" 3.1.4 en établissant des estimés des normes L^2 (sur Y) de $f \circ F$ et de $d(f \circ F)$ en fonction des normes L^2 et L^∞ (sur X) de f et de df (pour toute fonction $f \in \mathcal{A}_X(\lambda)$) et en utilisant les estimées du rapport entre norme L^∞ et norme L^2 de f et de df démontrés dans la section 3.2.*
Dans la section 3.4.1, nous donnons une preuve du Théorème 3.1.2 : c'est une conséquence presque immédiate de la méthode du barycentre (Chapitre 1) et du résultat de comparaison technique (Proposition 3.1.4).
Enfin, dans la section 3.4.2, nous donnons des exemples qui illustrent la faiblesse des hypothèses du Théorème 3.1.2, comparées[5] aux hypothèses du résultat de convergence de J. Cheeger et T. Colding ([**Ch-Co**]*) : en particulier, une première série d'exemples prouve que, sous nos hypothèses, il est impossible d'obtenir une minoration des valeurs propres $\lambda_i(Y, g)$ du Laplacien de (Y, g) (Example 3.4.3 et Proposition 3.4.4), alors qu'une telle minoration est assurée sous les hypothèses choisies par J. Cheeger et T. Colding (*[**Ch-Co**]*); une seconde série d'exemples (Example 3.4.1 et Proposition 3.4.2) prouve que les hypothèses que nous faisons dans le Théorème 3.1.2 sont vérifiées (pour au moins une métrique g) sur toute variété Y obtenue par somme connexe de X et de n'importe quelle variété Z de même dimension, ce qui signifie que nos hypothèses n'impliquent pratiquement aucune restriction sur la topologie de Y, alors que, sous les hypothèses choisies par J. Cheeger et T. Colding, Y est forcément difféomorphe à X.*

Prospetto del capitolo 3: *In questo capitolo presenteremo un' applicazione del metodo del baricentro nell'ambito della Geometria Spettrale. In modo semplificato il nostro scopo è quello di paragonare lo spettro del Laplaciano di una varietà Riemanniana* ***qualunque*** *a quello di una varietà Riemanniana di riferimento fissata quando queste due varietà sono ε-vicine (nel senso della distanza di Gromov-Hausdorff e per un valore fissato della costante $\varepsilon > 0$).*
Più precisamente, supponiamo che sia assegnata una varietà Riemanniana di riferimento (X^n, g_0) compatta, connessa di geometria limitata[6], consideriamo una qualunque varietà Riemanniana (Y, g) della stessa dimensione che ammetta una ε-approssimazione di Gromov-Hausdorff[7] continua e di grado assoluto non nullo, su (X^n, g_0). Supporremo inoltre il volume di (Y, g) quasi inferiore al volume di (X, g_0).
Nel Teorema principale 3.1.2 mostriamo che ogni autovalore $\lambda_i(Y, g)$ del Laplaciano di

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[5] La comparaison illustre la différence entre notre choix et celui fait par J. Cheeger et T. Colding: nous avons choisi de prendre des hypothèses plus faibles que celles de J. Cheeger et T. Colding et d'obtenir des conclusions également plus faibles.

[6] Quando parliamo di "geometria limitata" questo significa che il diametro ed il valore assoluto delle curvature sezionali sono limitati superiormente mentre il raggio di iniettività è limitato inferiormente. La condizione di "geometria limitata" non impone alcuna restrizione alla geometria di (X^n, g_0), poiché ogni varietà compatta ammette tali limiti; questa "condizione" indica semplicemente che tutte le costanti universali che compaiono nel risultato sono calcolate in funzione di questi limiti sulla geometria di (X^n, g_0)

[7] Benché non sia necessario scegliere ε piccolo, è comunque necessario che sa inferiore ad un valore critico che può essere calcolato in funzione dei limiti della geometria della varietà di riferimento (X, g_0), affinché i risultati che otteniamo non siano vuoti.

(Y, g) è quasi inferiore all'autovalore corrispondente di (X^n, g_0), più precisamente abbiamo $\lambda_i(Y, g) \leq (1+\eta(\varepsilon))\,\lambda_i(X, g_0)$, dove la funzione $\eta(\varepsilon)$ è esplicita (data nell'enunciato del Teorema 3.1.2) e tende a zero quando ε tende a zero.

Nella sezione 3.1, dopo un breve richiamo storico sul problema della stima degli autovalori dello spettro del Laplaciano nel contesto delle varietà Riemanniane e dopo aver presentato e discusso l'enunciato del Teorema principale 3.1.2, faremo (nella sezione 3.2) dei richiami sulle disuguaglianze di Sobolev quantitative (precisando il modo in cui le costanti che vi intervengono dipendano dai limiti sulla geometria globale, vedere la sezione 3.2.1) e , grazie al metodo di iterazione di Moser, mostreremo come dedurne dei limiti superiori per il rapporto tra le norme L^∞ ed L^2 di f e df per ogni funzione f nello spazio vettoriale $\mathcal{A}_X(\lambda)$ generato dalle autofunzioni del Laplaciano di (X, g_0) associate ad autovalori $\lambda_i(X, g_0) \leq \lambda$ (vedere la sezione[8] 3.2.2). Nella sezione 3.3 dimostreremo un risultato di paragone "tecnico" (la Proposizione 3.1.4) che ci permetterà, grazie al metodo del baricentro, di dimostrare il Teorema 3.1.2. Questo risultato permette di paragonare lo spettro di una varietà Riemanniana qualunque *(Y, g) a quello di una varietà (X, g_0) di "geometria debolmente limitata[9]" quando queste due varietà sono ε-vicine rispetto alla distanza di Gromov-Hausdorff (per un valore fissato della costante $\varepsilon > 0$) e quando esiste una applicazione Lipschitziana $F : (Y, g) \to (X, g_0)$ (di grado assoluto non banale), la cui energia puntuale è quasi inferiore a n (laddove il valore n è critico, poiché esso è il valore dell'energia di una isometria).*
La dimostrazione della Proposizione 3.1.4 è ottenuta in tre passi: nella sezione 3.3.1 mostriamo che la varietà Y può essere suddivisa in due parti: una prima parte dove il determinante Jacobiano di F è quasi uguale ad 1 (il che implica che l'applicazione F è quasi isometrica su tale parte) ed una seconda sulla quale non abbiamo più controllo sul determinante Jacobiano, e che è di volume piccolo rispetto al volume totale della varietà (Lemma 3.3.5); nella sezione 3.3.2, richiamiamo un teorema di paragone generale e semplice (Lemma 3.3.9, già presente in [**Aub2**]*) basato sul Principio del Minimax, che ci permetterà, nella sezione 3.3.3, di concludere la dimostrazione della Proposizione "tecnica" 3.1.4, stabilendo delle stime delle norme L^2 (su Y) di $f \circ F$ e di $d(f \circ F)$ in funzione delle norme L^2 ed L^∞ (su X) di f e df (per ogni funzione $f \in \mathcal{A}_X(\lambda)$) ed utilizzando le stime del rapporto tra le norme L^∞ ed L^2 di f e df dimostrate nella sezione 3.2.*
Nella sezione 3.4.1 forniamo una dimostrazione del Teorema 3.1.2: si tratta di una conseguenza quasi immediata del metodo del baricentro (Capitolo 1) e del risultato di paragone tecnico (Proposizione 3.1.4).
Infine nella sezione 3.4.2 presenteremo degli esempi che illustrano la debolezza delle ipotesi del Teorema 3.1.2 paragonate [10] alle ipotesi del risultato di convergenza di J. Cheeger e T. Colding ([**Ch-Co**]*); una seconda serie di esempi (Esempio 3.4.1 e Proposizione 3.4.2) provano che le ipotesi da noi fatte nel Teorema 3.1.2 sono verificate (per almeno una metrica g) su ogni varietà Y otteuta per somma connessa da X e da una qualunque varietà Z della stessa dimensione, il che significa che le nostre ipotesi non implicano praticamente alcuna restrizione sulla topologia di Y, mentre, sotto le ipotesi scelte da J. Cheeger e T. Colding, Y è necessariamente diffeomorfa a X.*

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[8] I risultati di questa sezione costituiscono una referenza dettagliata per i risultati dimostrati da S. Gallot in due note pubblicate nei Comptes Rendus de l'Académie de Sciences ([**Ga1**], [**Ga2**]). Osserviamo un vincolo importante: per ogni funzione $f \in \mathcal{A}_X(\lambda)$ i limiti che noi otterremo per i rapporti $\frac{\|f\|_\infty}{\|f\|_2}$ e $\frac{\|df\|_\infty}{\|df\|_2}$ (che devono evidentemente dipendere da λ) non decvono dipendere dalla dimensione di $\mathcal{A}_X(\lambda)$.

[9] Nella Proposizione 3.1.4 l'ipotesi di "geometria limitata" fatta sulla varietà (X, g_0) può essere indebolita in una ipotesi di "geometria debolmente limitata", il che consiste nell'imporre unicamente un limite inferiore alla curvatura di Ricci di (X, g_0) ed uno superiore sul suo diametro.

[10] Il paragone illustra la differenza tra la nostra scelta e quella fatta da J. Cheeger e T. Colding: abbiamo scelto di prendere ipotesi più deboli di quelle di J. Cheeger e T. Colding e di ottenere quindi conclusioni più deboli.

Sketch of the chapter 3: *In this chapter we shall present an application of the barycenter method to the Spectral Geometry. Naively our aim is to compare the Laplace spectrum of* ***any*** *Riemannian manifold to the one of a fixed Riemannian manifold, when the Gromov-Hausdorff distance between the two manifolds is smaller than* ε *(here the proximity is intended for a fixed value of the constant* $\varepsilon > 0$ *).*
More precisely, given a compact, connected Riemannian manifold (X^n, g_0) *of bounded geometry*[11]*, let us consider any Riemannian manifold* (Y^n, g) *which admits a continuous Gromov-Hausdorff* ε*-approximation*[12] *of non zero absolute degree onto* (X^n, g_0) *; we shall assume that the volume of* (Y^n, g) *is almost smaller than the volume of* (X^n, g_0) *.*
In the Theorem 3.1.2 we shall prove that any eigenvalue $\lambda_i(Y^n, g)$ *of the Laplacian of* (Y^n, g) *is almost smaller than the corresponding eigenvalue of* (X^n, g_0) *, more precisely we have* $\lambda_i(Y^n, g) \leq (1 + \eta(\varepsilon))\, \lambda_i(X^n, g_0)$ *, where the function* $\eta(\varepsilon)$ *is explicit (it is given in the statement of Theorem 3.1.2) and goes to zero if* ε *goes to zero.*

In the section 3.1, after a brief history of previous estimates on the eigenvalues of the Laplace spectrum in the context of Riemannian manifolds and after the statement and discussion of the Main Theorem (Theorem 3.1.2) we shall recall (in section 3.2) some facts about quantitative Sobolev inequalities (precising how the constants which are involved in these inequalities depend on the bounds on the global geometry of the manifolds, see the section 3.2.1) and, thanks to the Moser's iteration method, we shall show how to deduce some upper bounds for the ratio between the L^∞ *and* L^2 *norms of* f *and* df *for any function* f *in the vector space* $\mathcal{A}_X(\lambda)$ *generated by the eigenfunctions of the Laplacian of* (X^n, g_0) *associated to the eigenvalues* $\lambda_i(X^n, g_0) \leq \lambda$ *(see section*[13] *3.2.2). In the section 3.3 we shall prove a "technical" comparison result (the Proposition 3.1.4), that we shall need, in combination with the barycenter method, in order to prove our Main Theorem 3.1.2. This result compares the spectrum of* any *Riemannian manifold* (Y^n, g) *to the one of a reference manifold* (X^n, g_0) *of "weakly bounded geometry*[14]*" when the Gromov-Hausdorff distance between the two manifolds is smaller than* ε *(for a fixed value of the constant* $\varepsilon > 0$ *) and when there exists a Lipschitz map* $F : (Y^n, g) \to (X^n, g_0)$ *(of non zero absolute degree) whose pointwise energy is almost smaller than* n *(where the value* n *is critical since it is the value of the pointwise energy of an isometry).*
The proof of the Proposition 3.1.4 is obtained in three steps: in the section 3.3.1 we show that the manifold Y *can be divided in two subsets: the first one where the Jacobian determinant of* F *is almost equal to* 1 *(this implies that the map* F *is quasi isometric on this subset), and the second one (where we do not have such a control on the Jacobian determinant) whose volume is small with respect to the total volume of the manifold (Lemma 3.3.5); in the section 3.3.2 we recall a general and simple comparison theorem (Lemma 3.3.9, see also* [**Aub2**]*) based on the Minimax Principle, which allows, in section 3.3.3, to conclude the proof of the "technical" Proposition 3.1.4 by proving some estimates for the norms* L^2 *(on* Y *) of* $f \circ F$ *and* $d(f \circ F)$ *in terms of the* L^2 *and* L^∞ *norms (on* X *) of* f *and* df *(for any function* $f \in \mathcal{A}_X(\lambda)$ *) and by using the estimates for the ratios between the* L^∞ *and* L^2 *norms of* f *and* df *obtained in section 3.2.*

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[11] A Riemannian manifold has "bounded geometry" when its diameter and the absolute value of its sectional curvature are bounded above, and when its injectivity radius is bounded below. Saying that (X^n, g_0) has "bounded geometry" (i. e. that there exists bounds for the geometry of (X^n, g_0)) does not imply any restriction on the geometry of (X^n, g_0) , since any compact manifold admits such bounds (however, when we fix the values of these bounds, it implies some restriction on the possible geometries of (X^n, g_0)); this "condition" only says that all the universal constants which appear in the result are computed in terms of these bounds on the geometry of (X^n, g_0) .

[12] Though ε is not supposed arbitrarily small, it must however be smaller than a critical value (which will be computed in terms of the bounds of the geometry of the reference manifold (X, g_0)).

[13] This subsection provides a detailed reference for the results proved by S. Gallot in two short notes at the Comptes Rendus de l'Académie de Sciences ([**Ga1**], [**Ga2**]). We underline an important constraint: for any function $f \in \mathcal{A}_X(\lambda)$ we do not want the bounds that we shall obtain for the ratios $\frac{\|f\|_\infty}{\|f\|_2}$ and $\frac{\|df\|_\infty}{\|df\|_2}$ (which shall obviously depend on λ) to depend on the dimension of $\mathcal{A}_X(\lambda)$.

[14] In the Proposition 3.1.4, the usual assumption "bounded geometry" can be weakened into the assumption "weakly bounded geometry", which means that we only assume a lower bound on the Ricci curvature of (X^n, g_0) and an upper bound on the diameter.

In section 3.4.1, we give the proof of the main result, Theorem 3.1.2: it is a quasi immediate consequence of the barycenter method (chapter 1) and of the technical comparison result (Proposition 3.1.4).
Finally, in section 3.4.2, we shall present some examples which make evident the weakness of the assumptions of our Main Theorem 3.1.2 compared[15] *to the assumptions of the convergence result of J. Cheeger and T. Colding (*[**Ch-Co**]*); in particular, a first series of examples proves that, under the assumptions of our Main Theorem 3.1.2, it is impossible to obtain lower bounds of the eigenvalues* $\lambda_i(Y^n, g)$ *of the Laplacian of* (Y^n, g) *(Example 3.4.3 and Proposition 3.4.4), while such lower bounds are automatic under the assumptions chosen by J. Cheeger et T. Colding in* [**Ch-Co**]. *A second series of examples (Example 3.4.1 and Proposition 3.4.2) proves that, for every Riemannian manifold* (X^n, g_0)*, any manifold* (Y^n, g_ε) *obtained by connected sum of* (X^n, g_0) *with any Riemannian manifold* (Z^n, h_ε) *of diameter smaller than* $C.\varepsilon$ *satisfies the assumptions of our Main Theorem 3.1.2; this proves that our assumptions do not imply any topological restriction on the manifold* Y^n*, whereas, under the assumptions made by J. Cheeger and T. Colding,* Y^n *is necessarily diffeomorphic to* X^n*.*

3.1. Introduction

The aim of this chapter is to compare the spectra of two Riemannian manifolds (Y, g) and (X, g_0) and to bound the gap in terms of the Gromov-Hausdorff distance between these two spaces when this distance is smaller than some universal constant (see Theorem 3.1.2).

Estimates from above and from below for the eigenvalues of the Laplace-Beltrami operator of manifolds satisfying a lower bound of the Ricci curvature and an upper bound of the diameter were derived in the decade from 1975 to 1985. Namely, following S. Y. Cheng [**Cheng**], and P. Li and S. T. Yau, [**Li-Yau**], we know that, when (Y, g) is a compact Riemannian n-manifold of diameter $\mathrm{Diam}(Y, g) \leq D$, whose Ricci curvature, Ricci_g, satisfies the bound $\mathrm{Ricci}_g \geq -(n-1)\kappa^2$, then the eigenvalues of the Laplace-Beltrami operator admit the following upper bound:

$$\lambda_k(Y, g) \leq \frac{C(\alpha)}{\mathrm{Vol}_g(Y)^{\frac{2}{n}}} \cdot k^{\frac{2}{n}} \tag{3.69}$$

where $\alpha = \kappa D$. On the other hand under the same assumptions we have the following inferior bound for the eigenvalues of the Laplace-Beltrami operator of (Y, g) (see P. Li and S. T. Yau [**Li-Yau**], M. Gromov [**Gro2**] and S. Gallot, [**Ga1**], [**Ga2**], [**Ga3**]):

$$\lambda_k(Y, g) \geq \frac{\Gamma(\alpha)}{\mathrm{Diam}(Y, g)^2} \cdot k^{\frac{2}{n}}. \tag{3.70}$$

Explicit values for the constants $C(\alpha)$ and $\Gamma(\alpha)$ can be found in [**Li-Yau**], [**Ga1**], [**Ga2**], [**Ga3**] or [**BBG**].

Remark 3.1.1. As long as we are only concerned by their dependence with respect to the index k, the inequalities (3.69) and (3.70) agree with the well known Weyl asymptotic formula which says that the sequence of the eigenvalues of the Laplace-Beltrami operator of a compact Riemannian n-manifold (Y, g) behaves asymptotically like

$$\lambda_k \sim \frac{(2\pi)^2}{(\mathrm{Vol}(\mathrm{B}^n) \cdot \mathrm{Vol}_g(M))^{\frac{2}{n}}} \cdot k^{\frac{2}{n}} \quad .$$

For the same reason, assume that we have two compact Riemannian n-manifolds (Y, g), (X, g_0), whose Ricci curvatures satisfy the same lower bound

$$\mathrm{Ricci}_g \geq -(n-1)\kappa^2,\ \mathrm{Ricci}_{g_0} \geq -(n-1)\kappa^2$$

[15] Comparing our assumptions with Cheeger-Colding's, we have chosen to take weaker assumptions and, consequently, to obtain weaker conclusions.

and such that $\mathrm{Diam}(Y,g), \mathrm{Diam}(X,g_0) \le D$. From inequalities (3.69) and (3.70) we obtain:

$$\lambda_k(Y,g) \le \frac{C(\alpha)}{\Gamma(\alpha)} \cdot \left(\frac{\mathrm{Diam}(X,g_0)^n}{\mathrm{Vol}_g(Y)} \right)^{\frac{2}{n}} \cdot \lambda_k(X,g_0)$$

and, exchanging the roles of (Y,g) and (X,g_0):

$$\lambda_k(X,g_0) \le \frac{C(\alpha)}{\Gamma(\alpha)} \cdot \left(\frac{\mathrm{Diam}(Y,g)^n}{\mathrm{Vol}_{g_0}(X)} \right)^{\frac{2}{n}} \cdot \lambda_k(Y,g)$$

However, since the quantity $C(\alpha)/\Gamma(\alpha)$ is considerably greater than 1 (although the constants involved are sharp!) and as the ratio $\frac{(\mathrm{Diam})^n}{\mathrm{Vol}}$ can be arbitrarily large, even if we suppose that the diameters and the volumes of (Y,g) and (X,g_0) are almost the same, we cannot infer an equality or deduce some sharp pinching result between the k^{th} eigenvalues.

In [**Ch-Co**], Theorem 7.11, J. Cheeger and T. Colding gave a convergence result for the eigenvalues of the Laplace operators of a sequence of n-dimensional manifolds (Y_k, g_k) whose Ricci curvatures are bounded from below by $-(n-1)$ and which converge with respect to the Gromov-Hausdorff distance to a given smooth manifold (X,g_0) of the same dimension (notice that, (X,g_0) being fixed, its Ricci curvature is automatically bounded from below). Namely, they prove that, for any fixed $j \in \mathbb{N}^*$, the j^{th} eigenvalue $\lambda_j(Y_k,g_k)$ of the Laplace-Beltrami operator of (Y_k,g_k) converges to $\lambda_j(X,g_0)$ as $k \to +\infty$.
Notice that the Gromov-Hausdorff convergence is not, in itself, a strong assumption (in particular it gives no informations on the local geometries of the (Y_k,g_k)), but that becomes quite a strong one when it is combined with a uniform lower bound on the Ricci curvature of the Riemannian manifolds (Y_k,g_k). Assuming together these two properties one obtains that, for large values of k, the local geometries of (Y_k,g_k) are almost the same (J. Cheeger, T. Colding [**Ch-Co**] and T. Colding [**Col**]), in particular, for every $\varepsilon > 0$, if $y_k \to x$, the volume of the geodesic ball $B(y_k,\varepsilon)$ of (Y_k,g_k) converges to the volume of the geodesic ball $B(x,\varepsilon)$ of (X,g_0). Moreover Y_k is diffeomorphic to (X,g_0) for large values of k ([**Ch-Co**]), and $\mathrm{Vol}_{g_k}(Y_k) \to \mathrm{Vol}_{g_0}(X)$ as $k \to +\infty$ ([**Col**]). The fact that $\mathrm{Diam}(Y_k,g_k)$ converges to $\mathrm{Diam}(X,g_0)$ is an immediate consequence of the Gromov-Hausdorff convergence.

We remark that the eigenvalues approximation methods shows that we have convergence of $\lambda_i(Y_k,g_k) \to \lambda_i(X,g_0)$ when (Y_k,g_k) is a sequence of polyedral approximations converging to (X,g_0) (see [**Dod-Pat**], §3). However the result of Dodziuk and Patodi does not provide an upper bound of the "error" $|\lambda_i(Y_k,g_k) - \lambda_i(X,g_0)|$.

Comparing with the aforementioned results, the comparison between the spectra of two manifolds (Y,g) and (X,g_0) that we aim must obey to quite different rules: namely we are authorized to assume that the geometry of (X,g_0) is bounded. On the contrary, on (Y,g), any assumption which implies a control on the local topology or geometry is prohibited. Let us denote by σ_0 the sectional curvature of (X,g_0) and by $\mathrm{inj}(X,g_0)$ its injectivity radius, the main result in this direction is the following:

THEOREM 3.1.2. *Let (X^n,g_0) be a compact, connected, Riemannian manifold satisfying the assumptions:*

$$\mathrm{Diam}(X,g_0) \le D, \quad \mathrm{inj}(X,g_0) \ge i_0, \quad |\sigma_0| \le \kappa^2,$$

where D, i_0, κ are arbitrary positive constants.
Let (Y^n,g) be any compact, connected Riemannian manifold such that there exists a continuous Gromov-Hausdorff ε-approximation $f : (Y,g) \to (X,g_0)$ of non zero absolute degree, where

$$\varepsilon < \varepsilon_1(n,i_0,\kappa) = \frac{1}{\kappa} \cdot \min\left\{ \left[\frac{\min\{\,1\,;\,\kappa\,\mathrm{inj}(X,g_0)\}}{2^{8n}\,(n+1)^8} \right]^4 ; \left(\frac{\left(\frac{10}{9}\right)^{\frac{2}{n}} - 1}{20} \right)^4 \right\} \tag{3.71}$$

If we assume that

$$[1 - 10\,n\,(\kappa\varepsilon)^{\frac{1}{4}}] \cdot \mathrm{Vol}_g(Y) < \mathrm{Vol}_{g_0}(X)$$

then, for every $i \in \mathbb{N}$*, we have*

$$\lambda_i(Y,g) \leq \left(1 + C_1(n)(\kappa\varepsilon)^{\frac{1}{16}}\right) \cdot \left(1 + C_2(n,\, \kappa\, D,\, D^2 \cdot \lambda_i(X,g_0))(\kappa\varepsilon)^{\frac{1}{8}}\right) \cdot \lambda_i(X,g_0) \tag{3.72}$$

where

$$C_1(n) = 14(n-1)\sqrt[4]{n}$$

$$C_2(n,\, \alpha, \Lambda) = 4\sqrt{n}\,\left[(2n+1)\, e^n \left[1 + B(\alpha)\sqrt{\Lambda + (n-1)\alpha^2}\right]^n + 2\right]$$

where $B(\alpha)$ *is the isoperimetric constant defined in the Proposition 3.2.4 and where the right hand side of (3.71) goes to* $\lambda_i(X,g_0)$ *when* $\varepsilon \to 0_+$ *.*

REMARK 3.1.3. Let us point out the following facts:

(1) The inequality (3.71) given by the Theorem 3.1.2 is sharp: in fact it provides an upper bound of $\frac{\lambda_i(Y,g)}{\lambda_i(X,g_0)}$ which goes to 1 as $\varepsilon \to 0^+$.

(2) Notice that the only assumptions that we make on (Y,g) in the Theorem 3.1.2 are:

  (i) the existence of a continuous Gromov-Hausdorff approximation of nonzero absolute degree from (Y,g) to (X,g_0) ;

  (ii) the assumption that the volume of (Y,g) is almost smaller than the volume of (X,g_0) .

  Notice that there is no assumption on the curvature of (Y,g) .
  The weakness of these assumptions on (Y,g) is first illustrated by the fact that they give no information on the local topology of (Y,g) or on the local topology of Y . In fact, in the Example 3.4.1, for every (X^n,g_0) we construct a family of pairwise non homotopic Riemannian manifolds $(Y_\varepsilon,g_\varepsilon)$ which satisfy the assumptions (i) and (ii) above (with $\varepsilon \to 0_+$) and thus converge to (X,g_0) as $\varepsilon \to 0_+$.

(3) Another illustration of the weakness of the assumptions made on (Y,g) is the fact that it is impossible to get a lower bound of $\frac{\lambda_i(Y,g)}{\lambda_i(X,g_0)}$ under these assumptions. In fact, in the Example 3.4.3 we construct, for any fixed Riemannian manifold (X,g_0), a sequence of Riemannian manifolds (Y_k,g_k) (diffeomorphic to (X,g_0)), which satisfy assumptions (i) and (ii) above, and such that $\frac{\lambda_1(Y_k,g_k)}{\lambda_1(X,g_0)} \to 0$ when $k \to +\infty$. Let us stress the fact that the counter-examples mentioned above in the previous Remark satisfy all the assumptions of the Theorem 3.1.2 for arbitrarily small values of ε .

(4) The Theorem 3.1.2 is not only a convergence result: in fact, it is valid for non small values of ε (*i.e.* for every $\varepsilon < \varepsilon_1(n,i_0,\kappa)$). For every $\varepsilon < \varepsilon_1(n,i_0,\kappa)$, it provides an explicit upper bound for the "error term" $\left(\frac{\lambda_i(Y,g)-\lambda_i(X,g_0)}{\lambda_i(X,g_0)}\right)$.

(5) The Theorem 3.1.2 also works when g is not a smooth Riemannian metric (for example if g is piecewise C^1). It thus provides a sharp estimate of $\lambda_i(X,g_0)$ by the corresponding eigenvalue of a polyedral ε-approximation and a bound of the error (in one sense).

The Theorem 3.1.2 is a consequence of the Theorem 1.4.1 and of the following technical result:

PROPOSITION 3.1.4. *Let* $\kappa, D > 0$. *Let* (X,g_0) *be a connected, compact Riemannian manifold which satisfy* $\mathrm{Ricci}_{g_0} \geq -(n-1)\kappa^2$ *and* $\mathrm{Diam}(X,g_0) \leq D$ *. Let* (Y,g) *be another compact, connected Riemannian manifold such that* $(1-\eta)\,\mathrm{Vol}_g(Y) \leq \mathrm{Vol}_{g_0}(X)$ *(where* $0 < \eta \leq \frac{1}{9}$ *) and such that there exists a Lipschitz map* $F : (Y,g) \to (X,g_0)$

of non zero absolute degree which verifies the following bound on the pointwise energy: $e_y(F) \leq n(1+\eta)^{2/n}$ *a.e.. Then*

$$\lambda_i(Y,g) \leq (1 + 7(n-1)\,\eta^{\frac{1}{4}}) \cdot (1 + C(n, D^2\lambda_i(X,g_0), \alpha)\,\eta^{\frac{1}{2}}) \cdot \lambda_i(X,g_0)$$

where $\alpha = \kappa \cdot D$ *and where*

$$C(n, D^2\lambda_i, \alpha) = (2n+1) \cdot e^n \cdot \left(1 + B(\alpha)\sqrt{\lambda_i D^2 + (n-1)\alpha^2}\right)^n + 2\,.$$

Remark 3.1.5. Let us point out two facts:

(i) It is natural that the constants C and C_2, which appear in the estimates of the Theorem 3.1.2 and of the Proposition 3.1.4, increase when we try to estimate great eigenvalues λ_i. However, we observe that the interplay of η and $\lambda_i\,D^2$ makes sure that the error term $C(n,\, D^2\,\lambda_i(X,g_0)^2,\, \alpha)\,\eta^{\frac{1}{2}}$ remains small when η is small with respect to $(\lambda_i\,D^2)^{-n}$.

(ii) All the geometric quantities from which the constant C depends are invariant by homotheties.

Remark 3.1.6 (Dimension $n = 2$). In order to simplify the notations we give the proofs only for dimensions $n > 2$. However, the same arguments hold in dimension $n = 2$, provided some slight modifications. Just observe that:

(a) Lemma 3.2.1, (i) is valid for $n = 2$;

(b) Lemma 3.2.1 (ii) is valid if we replace n by p where $p = n$ for $n > 2$ and $p > n$ for $n = 2$ (it is sufficient to replace, in the definition of h, the function $f^{\frac{2(n-1)}{n-2}}$ by $f^{\frac{2(p-1)}{p-2}}$);

all the arguments then works, substituting n by p, included the Sobolev inequality (which is not sharp in dimension 2) which says that there exists a constant C such that

$$\|\, f\,\|_{\frac{2p}{p-2}} \leq C \cdot \|\, df\,\|_2 + \|\, f\,\|_2$$

Moser's iteration method then works with $\beta = \frac{p}{p-2}$. We are only missing the sharpness in the inequality of Corollary 3.2.9.

3.2. Geometric-analytic tools

3.2.1. Quantitative Sobolev inequalities. The results of this section are due to S. Gallot. However as S. Gallot's original results were published in a short note in the *Comptes Rendus de l'Académie des Sciences* (see [**Ga1**]), the original proofs are rather dense and we found useful to give more explanations about the method and more detailed proofs.

Let us consider any compact Riemannian manifold (M,g) (without boundary), whose volume will be denoted by $\mathrm{Vol}_g(M)$ or by V according to the context, and whose diameter will be denoted by $\mathrm{Diam}(M,g)$.
Let us consider the Cheeger's isoperimetric constant, h, and the usual isoperimetric constant, C, defined by

$$h = \inf_{\Omega} \frac{\mathrm{Vol}_g(\partial\Omega)}{\mathrm{Vol}_g(\Omega)}\,, \quad C = \inf_{\Omega} \frac{\mathrm{Vol}_g(\partial\Omega)}{\mathrm{Vol}_g(\Omega)^{\frac{n-1}{n}}} \tag{3.73}$$

where Ω runs over all domains in M (with piecewise regular boundary) whose volume satisfies $\mathrm{Vol}_g(\Omega) \leq \frac{1}{2}\,\mathrm{Vol}_g(M)$ [16].

In the euclidean space $(\mathbb{R}^n, \mathrm{can})$, the isoperimetric constant is

$$C_* = \frac{\mathrm{Vol}_{\mathrm{can}}(\mathbb{S}^{n-1})}{(\mathrm{Vol}_{\mathrm{can}}(\mathrm{B}^n(1)))^{\frac{n-1}{n}}} = \frac{\mathrm{Vol}_{\mathrm{can}}(\partial\,\mathrm{B}^n(R))}{(\mathrm{Vol}_{\mathrm{can}}(\mathrm{B}^n(R)))^{\frac{n-1}{n}}}$$

[16] This restriction is necessary, because otherwise the infima of $\frac{\mathrm{Vol}_g(\partial\Omega)}{\mathrm{Vol}_g(\Omega)}$ and of $\frac{\mathrm{Vol}_g(\partial\Omega)}{\mathrm{Vol}_g(\Omega)^{\frac{n-1}{n}}}$ are zero (just make the choice of $\Omega = M \setminus B(x_0,\varepsilon)$ and let $\varepsilon \to 0$).

where $\mathrm{B}^n(R)$ denotes the euclidean ball of radius R the classical isoperimetric inequality says that, for every domain Ω' in $\mathbb{R}^n$ (with piecewise regular boundary),

$$\frac{\mathrm{Vol}_{\mathrm{can}}(\partial\Omega')}{(\mathrm{Vol}_{\mathrm{can}}(\Omega'))^{\frac{n-1}{n}}} \geq C_*$$

(the equality being attained if and only if Ω^* is a ball).
For any open subset U in $\mathbb{R}^n$ it is classical that this isoperimetric inequality is equivalent to the Sobolev inequality:

$$C_* \cdot \left(\int_U |f|^{\frac{n}{n-1}}\, dv_{\mathrm{can}}\right)^{\frac{n-1}{n}} \leq \int_U |\nabla f|\, dv_{\mathrm{can}} \tag{3.74}$$

which is valid for any $f :\in C_c^\infty(U)$ (where $C_c^\infty(U)$ is the space of smooth functions on U whose support is compact in U). Denoting by $H^1_{1,\,c}(U,\mathrm{can})$ the completion of $C_c^\infty(U)$ for the norm $\parallel f \parallel_1^1 = \int_U |\nabla f|\, dv_{\mathrm{can}}$, the inequality (3.74) remains valid for every $f \in H^1_{1,\,c}(U)$, moreover, in the inequality (3.74), the equality is attained when f is the characteristic function of some euclidean ball included in U. Thus the inequality (3.74) is sharp. Using the symmetrization method we shall prove analogous results for domains in Riemannian manifolds, and we shall deduce from these estimates some Sobolev inequalities for the whole manifold. In the sequel, we define the L^p-norms on (M,g) by $\parallel u \parallel_p = \left(\frac{1}{\mathrm{Vol}_g(M)}\int_M |u|^p\, dv_g\right)^{\frac{1}{p}}$, and the space $H_1^p(M,g)$, $p \geq 1$, as the completion of $C^\infty(M)$ with respect to the norm $\parallel f \parallel_{H_1^p} = \parallel f \parallel_p + \parallel \nabla f \parallel_p$.

Lemma 3.2.1. *Let (M,g) be a compact Riemannian manifold of dimension n. For every domain Ω in M such that $\mathrm{Vol}_g(\Omega) \leq \frac{1}{2}\mathrm{Vol}_g(M)$, and for any regular function $f \geq 0$ over Ω, such that $f|_{\partial\Omega} = 0$ we have:*

(i) $\int_\Omega |\nabla f|\, dv_g \geq C \cdot \left(\int_\Omega f^{\frac{n}{n-1}}\, dv_g\right)^{\frac{n-1}{n}}$;

(ii) $\left(\int_\Omega |\nabla f|^2\, dv_g\right)^{\frac{1}{2}} \geq \frac{n-2}{2(n-1)} \cdot C \cdot \left(\int_\Omega f^{\frac{2n}{n-2}}\, dv_g\right)^{\frac{n-2}{2n}}$.

Proof. We start proving (i). Let $\Omega_t = \{x \in \Omega |\, f(x) > t\}$ and let Ω_t^* be the open ball in $\mathbb{R}^n$ (centered at the origin), whose radius is determined by $\mathrm{Vol}_g(\Omega_t) = \mathrm{Vol}_{\mathrm{can}}(\Omega_t^*)$. We denote by Ω_* the open ball in $\mathbb{R}^n$ (centered at the origin) such that $\mathrm{Vol}_g(\Omega) = \mathrm{Vol}_{\mathrm{can}}(\Omega_*)$. We will define $A(t) = \mathrm{Vol}_g(\Omega_t)$ and $A^*(t) = \mathrm{Vol}_{\mathrm{can}}(\Omega_t^*)$, where Ω_t^* is the euclidean ball (centered at the origin) such that $\mathrm{Vol}(\Omega_t^*) = \mathrm{Vol}(\Omega_t)$. We construct the function $f^* : \Omega_* \to \mathbb{R}$ such that

$$f^*(x) = \begin{cases} t & \text{when } x \in \partial\Omega_t^*; \\ \in (t-\varepsilon, t] & \text{when } x \in \Omega_{t-\varepsilon}^* \setminus \Omega_t^*; \end{cases}$$

(*i.e.* if $\bigcap_{\varepsilon>0} \Omega_{t-\varepsilon}^* \setminus \overline{\Omega_t^*} \neq \varnothing$, the function f^* is constant and equal to t on this set). We remark that $\sup(f) = \sup(f^*)$ since $\sup(f)$ is the infimum of the values t such that $A(t) = 0$ (or the sup of the values t such that $A(t) > 0$). Consider now a partition $0 = t_0 < \cdots < t_N = \sup(f)$ of the interval $[0, \sup(f)]$ such that $t_{i+1} - t_i = \frac{\sup(f)}{N}$. Since $A(t)$ is a strictly decreasing function of t we have:

$$S_N^- = \sum_{i=1}^{N-1} (t_i)^{\frac{n}{n-1}} \left(A(t_i) - A(t_{i+1})\right) \leq \int_\Omega f^{\frac{n}{n-1}}\, dv_g \leq$$

$$\leq \sum_{i=1}^{N-1} (t_{i+1})^{\frac{n}{n-1}} \left(A(t_i) - A(t_{i+1})\right) = S_N^+$$

It is easy to see that S_N^+, S_N^- converge to $\int_\Omega f^{\frac{n}{n-1}}\, dv_g$. By definition we have that $f^* > t$ on Ω_t^* and $f^* \leq t$ on the complement of this set. It follows that for $x \in \Omega_{t_i}^* \setminus \Omega_{t_{i+1}}^*$, $t_i < f^*(x) \leq t_{i+1}$, so that:

$$\sum_{i=1}^{N-1} (t_i)^{\frac{n}{n-1}} \left(A^*(t_i) - A^*(t_{i+1})\right) \leq \int_{\Omega_*} (f^*)^{\frac{n}{n-1}}\, dv_{\mathrm{can}} \leq$$

$$\leq \sum_{i=1}^{N-1} (t_{i+1})^{\frac{n}{n-1}} \left(A^*(t_i) - A^*(t_{i+1})\right)$$

By definition $A^*(t) = A(t)$ hence, we find:

$$S_N^- \leq \int_{\Omega_*} (f^*)^{\frac{n}{n-1}} \, dv_{\mathrm{can}} \leq S_N^+$$

Taking the limit for $N \to \infty$ we get the equality

$$\int_\Omega f^{\frac{n}{n-1}} \, dv_g = \int_{\Omega_*} (f^*)^{\frac{n}{n-1}} \, dv_{\mathrm{can}}$$

On the other hand, using the coarea formula ([**Bu-Za**], Theorem 13.4.2) we obtain:

$$\int_\Omega |\nabla f| \, dv_g = \int_0^{\sup(f)} \mathrm{Vol}_g(\{f = t\}) \, dt = \int_0^{\sup(f)} \mathrm{Vol}_g(\partial \Omega_t) \, dt \geq$$

$$\geq \int_0^{\sup(f)} C \cdot A(t)^{\frac{n-1}{n}} \, dt = \frac{C}{C_*} \int_0^{\sup(f^*)} C_* A^*(t)^{\frac{n-1}{n}} \, dt =$$

$$= \frac{C}{C_*} \int_0^{\sup(f^*)} \mathrm{Vol}_{\mathrm{can}}(\partial \Omega_t^*) \, dt$$

where the last equality comes from the fact that we are in the equality-case for the isoperimetric inequality in $\mathbb{R}^n$ and where we intend $\int_0^{\sup(f)}$ as the integral on the set $[0, \sup(f)] \setminus \mathcal{S}_f$ where $\mathcal{S}_f$ is the set of singular values of f which has measure zero by Sard's theorem. It follows that,

$$\int_\Omega |\nabla f| \, dv_g \geq \frac{C}{C_*} \int_0^{\sup(f^*)} \mathrm{Vol}_{\mathrm{can}}(\{f^* = t\}) \, dt = \frac{C}{C_*} \int_{\Omega_*} |\nabla f^*| \, dv_{\mathrm{can}}$$

because the symmetrization method certify that f^* is Lipschitz, and thus we can apply the coarea formula ([**Bu-Za**], Theorem 13.4.2) to f^*. From this and from inequality (3.74), we get

$$\int_\Omega |\nabla f| \, dv_g \geq \frac{C}{C_*} \int_{\Omega_*} |\nabla f^*| \, dv_{\mathrm{can}} \geq C \cdot \left(\int_{\Omega_*} (f^*)^{\frac{n}{n-1}} \, dv_{\mathrm{can}} \right)^{\frac{n-1}{n}} =$$

$$= C \cdot \left(\int_\Omega f^{\frac{n}{n-1}} \, dv_g \right)^{\frac{n-1}{n}}$$

This ends the proof of (i).

Next we prove (ii). Let $h = f^{\frac{2(n-1)}{n-2}}$; since f is a regular function and $f \geq 0$, and since $x \to x^{\frac{2(n-1)}{(n-2)}}$ is Lipschitz on $[0, \sup(f)]$, the function h is Lipschitz on (M, g) (with bounded Lipschitz constant), so it is *a.e.*-differentiable, and $h \in H_1^1(M, g)$. We have:

$$|\nabla h| = \frac{2(n-1)}{n-2} \cdot f^{\frac{n}{n-2}} \cdot |\nabla f|$$

By the Lemma 3.2.1 (i) we know that

$$C \cdot \left(\int_\Omega h^{\frac{n}{n-1}} \, dv_g \right)^{\frac{n-1}{n}} \leq \int_\Omega |\nabla h| \, dv_g$$

so that,

$$C \cdot \left(\int_\Omega f^{\frac{2n}{n-2}} \, dv_g \right)^{\frac{n-1}{n}} \leq \frac{2(n-1)}{n-2} \int_\Omega f^{\frac{n}{n-2}} |\nabla f| \, dv_g \leq$$

$$\leq \frac{2(n-1)}{n-2} \left(\int_\Omega f^{\frac{2n}{n-2}} \, dv_g \right)^{\frac{1}{2}} \cdot \left(\int_\Omega |\nabla f|^2 \, dv_g \right)^{\frac{1}{2}}$$

which implies,

$$\frac{n-2}{2(n-1)} \cdot C \cdot \left(\int_\Omega f^{\frac{2n}{n-2}} \, dv_g \right)^{\frac{n-2}{2n}} \leq \left(\int_\Omega |\nabla f|^2 \, dv_g \right)^{\frac{1}{2}}$$

and this proves the Lemma 3.2.1 (ii). □

LEMMA 3.2.2. *Let (M,g) be a compact Riemannian manifold of dimension n. Let $f\in C^\infty(M)$ and let $a\in\mathbb{R}$ be such that $\Omega_a^+=\{f>a\}$ and $\Omega_a^-=\{f<a\}$ have volume less or equal to $\frac{\mathrm{Vol}_g(M)}{2}$. Then if $V=\mathrm{Vol}_g(M)$ we have:*

(i) $\left(\frac{1}{V}\int_M |f-a|^{\frac{2n}{n-2}}\,dv_g\right)^{\frac{n-2}{2n}} \le \frac{2(n-1)}{(n-2)\,C\,V^{-\frac{1}{n}}}\cdot\left(\frac{1}{V}\int_M|\nabla f|^2\,dv_g\right)^{\frac12}$

(ii) *Moreover if $\int_M f\,dv_g=0$, we have, $a\le\frac{1}{V}\int_M|f|\,dv_g$, and*
$\left(\frac{1}{V}\int_M |f|^{\frac{2n}{n-2}}\,dv_g\right)^{\frac{n-2}{2n}} \le \left[\frac{2(n-1)}{(n-2)\,C\,V^{-\frac{1}{n}}}+\frac{2}{h}\right]\cdot\left(\frac{1}{V}\int_M|\nabla f|^2\,dv_g\right)^{\frac12}$

Proof. First we prove (i). From the Lemma 3.2.1 (ii) we deduce that

$$\left(\int_{\Omega_a^\pm}|f-a|^{\frac{2n}{n-2}}\,dv_g\right)^{\frac{n-2}{n}}\le\left(\frac{2(n-1)}{(n-2)C}\right)^2\cdot\int_{\Omega_a^\pm}|\nabla f|^2\,dv_g$$

if we sum the two inequalities obtained in this way for $\Omega_a^\pm$ we obtain:

$$\left(\int_{\Omega_a^\pm}|f-a|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{4}}\le\left(\int_{\Omega_a^+}(f-a)^{\frac{2n}{n-2}}\,dv_g\right)^{\frac{n-2}{n}}+\left(\int_{\Omega_a^-}|f-a|^{\frac{2n}{n-2}}\,dv_g\right)^{\frac{n-2}{n}}\le$$
$$\le\left(\frac{2(n-1)}{(n-2)C}\right)^2\cdot\int_M|\nabla f|^2\,dv_g$$

Since $(\alpha^{\frac{n-2}{n}}+\beta^{\frac{n-2}{n}})\ge(\alpha+\beta)^{\frac{n-2}{n}}$ when $\alpha,\beta\ge0$, we get

$$\left(\int_M|f-a|^{\frac{2n}{n-2}}\,dv_g\right)^{\frac{n-2}{n}}\le\left(\frac{2(n-1)}{(n-2)C}\right)^2\cdot\int_M|\nabla f|^2\,dv_g$$

and this proves (i). Changing eventually f in $(-f)$ we can suppose that $a\ge0$. Let us now remark that, as $\int_M f\,dv_g=0$ and $\int_{M\setminus\Omega_a^-}(f-a)\,dv_g\ge0$,

$$0\le\int_{M\setminus\Omega_a^-}(f-a)\,dv_g=\left(\int_{M\setminus\Omega_a^-}f\,dv_g\right)-a\cdot\mathrm{Vol}_g(M\setminus\Omega_a^-)$$

hence using the fact that $\mathrm{Vol}(M\setminus\Omega_a^-)\ge\frac{V}{2}$ we see that $a\cdot\frac{V}{2}\le\int_{M\setminus\Omega_a^-}f dv_g$. On the other hand

$$\int_M f\,dv_g=0\quad\Rightarrow\quad\int_{\Omega_a^-}(-f)\,dv_g=\int_{M\setminus\Omega_a^-}f\,dv_g$$

so, if $\int_M f\,dv_g=0$ we have:

$$a\,\mathrm{Vol}_g(M)\le2\int_{M\setminus\Omega_a^-}f\,dv_g=\int_{\Omega_a^-}(-f)\,dv_g+\int_{M\setminus\Omega_a^-}f\,dv_g\le$$
$$\le\int_{\Omega_a^-}|f|\,dv_g+\int_{M\setminus\Omega_a^-}|f|\,dv_g=\int_M|f|\,dv_g$$

which proves that, $a\le\frac{1}{V}\int_M|f|\,dv_g\le\left(\frac{1}{V}\int_M|f|^2\,dv_g\right)^{\frac12}$. From this, from the triangle inequality and from (i) we deduce, when $\int_M f\,dv_g=0$,

$$\left(\frac{1}{V}\int_M|f|^{\frac{2n}{n-2}}\,dv_g\right)^{\frac{n-2}{2n}}\le\left(\frac{1}{V}\int_M|f-a|^{\frac{2n}{n-2}}\,dv_g\right)^{\frac{n-2}{2n}}+a\le$$
$$\le\left(\frac{2(n-1)}{(n-2)CV^{-\frac1n}}\right)\cdot\left(\frac{1}{V}\int_M|\nabla f|^2\,dv_g\right)^{\frac12}+\left(\frac{1}{V}\int_M f^2\,dv_g\right)^{\frac12}\le$$
$$\le\left(\frac{2(n-1)}{(n-2)CV^{-\frac1n}}+\frac{2}{h}\right)\cdot\left(\frac{1}{V}\int_M|\nabla f|^2\,dv_g\right)^{\frac12}$$

where the last inequality comes from the inequality $\frac{\int_M|\nabla f|^2\,dv_g}{\int_M f^2\,dv_g}\ge\lambda_1(M,g)\ge h^2/4$ (here $\lambda_1(M,g)$ stands for the first nonzero eigenvalue of the Laplace-Beltrami operator of (M,g)) proved by J. Cheeger in [**Cheeger2**], from which follows that, for any function $f\in H_1^1(M,g)$ such that $\int_M f\,dv_g=0$, we have

$$\frac{h^2}{4}\int_M f^2\,dv_g\le\int_M|\nabla f|^2\,dv_g.$$

This ends the proof of (ii). □

DEFINITION 3.2.3. Let (M,g) be a Riemannian manifold. Let us denote by Ricci_g the Ricci curvature of (M,g). We define the invariant $r_{\min}$ as the infimum of Ricci_g viewed as function on the unit tangent bundle U_gM.

PROPOSITION 3.2.4 (Sobolev inequality, [**Ga1**]). *Let (M,g) be a compact Riemannian manifold of dimension n, such that $\mathrm{Diam}(M,g) \le D$ and $r_{\min} \cdot D^2 \ge -(n-1)\alpha^2$. For every function $f : M \to \mathbb{R}$ in $H_1^2(M,g)$ we have:*

$$\text{(i)} \quad \left(\frac{1}{V}\int_M |f-\overline{f}|^{\frac{2n}{n-2}}\, dv_g\right)^{\frac{n-2}{2n}} \le \left[\frac{2(n-1)}{(n-2)\Gamma(\alpha)} + \frac{2}{H(\alpha)}\right]\cdot D \cdot \left(\frac{1}{V}\int_M |\nabla f|^2\, dv_g\right)^{\frac{1}{2}}$$

$$\text{(ii)} \quad \left(\frac{1}{V}\int_M f^{\frac{2n}{n-2}}\, dv_g\right)^{\frac{n-2}{2n}} \le$$

$$\le \left(\frac{2(n-1)}{(n-2)\Gamma(\alpha)} + \frac{2}{H(\alpha)}\right)\cdot D \cdot \left(\frac{1}{V}\int_M |\nabla f|^2\, dv_g\right)^{\frac{1}{2}} + \left(\frac{1}{V}\int_M f^2\, dv_g\right)^{\frac{1}{2}}$$

where we denote by $\bar{f}$ the mean value of f, i.e. $\bar{f} = \frac{1}{V}\int_M f\, dv_g$, where

$$H(\alpha) = \alpha\left(\int_0^{\alpha/2} (\cosh(t))^{n-1} dt\right)^{-1}$$

and where

$$\Gamma(\alpha) = \alpha\left(\int_0^{\alpha}\left(\frac{\alpha}{H(\alpha)}\cosh(t) + \frac{1}{n}\sinh(t)\right)^{n-1} dt\right)^{-\frac{1}{n}}$$

We will use the notation $B(\alpha)$ to refer to the quantity $\left(\frac{2(n-1)}{(n-2)\Gamma(\alpha)} + \frac{2}{H(\alpha)}\right)$.

REMARK 3.2.5. We discuss here the behaviour of the invariants of the Proposition 3.2.4 under the action of the homotheties of (M,g) (*i.e.* we look at what happens to these invariants when we change the metric g into the metric $\lambda^2 g$):

(i) $(r_{\min}\,\mathrm{Diam}(M,g)^2)$ is invariant by the action of the homotheties, hence the same holds for α, $H(\alpha)$ and $\Gamma(\alpha)$;

(ii) the Sobolev constant

$$\mathrm{Sob}(g) = \inf_{f\in H_1^2(M,g),\, f \text{ not const}} \frac{\left(\frac{1}{\mathrm{Vol}_g(M)}\int_M |\nabla f|^2\, dv_g\right)^{\frac{1}{2}}}{\left(\frac{1}{\mathrm{Vol}_g(M)}\int_M |f-\overline{f}|^{\frac{2n}{n-2}}\, dv_g\right)^{\frac{n-2}{2n}}}$$

verifies the equation $\mathrm{Sob}(\lambda^2 g) = \frac{1}{\lambda}\mathrm{Sob}(g)$, so it is impossible to bound from below $\mathrm{Sob}(g)$ by a function depending only on α. On the other hand $\mathrm{Sob}(g)\cdot \mathrm{Diam}(M,g)$ is invariant by homotheties hence it is reasonable (but not easy!) to look for a lower bound of this constant in terms of α.

Proof. The triangle inequality

$$\| f \|_{\frac{2n}{n-2}} \le \| f-\overline{f} \|_{\frac{2n}{n-2}} + |\overline{f}| \le \| f-\overline{f} \|_{\frac{2n}{n-2}} + \| f \|_2$$

easily shows that (i) ⇒ (ii). Hence it is sufficient to prove (i). We admit the following isoperimetric inequalities proved by S. Gallot ([**Ga1**], [**Ga3**]), valid for every compact manifold (M,g) such that $r_{\min}\cdot \mathrm{Diam}(M,g)^2 \ge -(n-1)\alpha^2$ and for any domain Ω with regular boundary and volume at most $\frac{\mathrm{Vol}_g(M)}{2}$:

$$\text{(3.75)} \qquad \frac{\mathrm{Vol}_g(\partial\Omega)}{\mathrm{Vol}_g(\Omega)} \ge \frac{H(\alpha)}{D}, \qquad \frac{\mathrm{Vol}_g(\partial\Omega)}{(\mathrm{Vol}_g(\Omega))^{\frac{n-1}{n}}\,\mathrm{Vol}_g(M)^{\frac{1}{n}}} \ge \frac{\Gamma(\alpha)}{D}$$

hence, passing to the infimum with respect to Ω:

$$h \geq \frac{H(\alpha)}{D}, \quad C\,V^{-\frac{1}{n}} \geq \frac{\Gamma(\alpha)}{D} \tag{3.76}$$

Now we apply the Lemma 3.2.2 (ii) to the function $(f-\overline{f})$ and we obtain:

$$\left(\frac{1}{V}\int_M |f-\overline{f}|^{\frac{2n}{n-2}}\,dv_g\right)^{\frac{n-2}{2n}} \leq \left[\frac{2(n-1)}{(n-2)CV^{-\frac{1}{n}}}+\frac{2}{h}\right]\cdot\left(\frac{1}{V}\int_M |\nabla f|^2\,dv_g\right)^{\frac{1}{2}} \leq$$

$$\leq \left(\frac{2(n-1)}{(n-2)\Gamma(\alpha)}+\frac{2}{H(\alpha)}\right)\cdot D\cdot\left(\frac{1}{V}\int_M |\nabla f|^2\,dv_g\right)^{\frac{1}{2}}$$

where the second inequality comes from the inequalities (3.76). □

3.2.2. Moser's iteration method. The method that we are going to use in this section has been introduced by J. Moser in 1961 (see [**Mos**]) in order to prove a Harnack's inequality for solutions of second order, uniformly elliptic partial differential equation of selfadjoint form. The original method allows to derive L^∞ estimates, for eigenfunctions of a differential operator of the prescribed type, in terms of the geometric data of the domain under consideration, this is achieved by means of an iterated use of a Sobolev inequality (see [**Mos**], §4).

Moser's iteration method has been widely used in Spectral Geometry to obtain eigenvalues estimates for the Laplace-Beltrami operator, the Hodge-de Rham Laplacian and the p-Laplacian under appropriate geometric assumptions, see [**Li**], [**Ga1**], [**Ga2**] and, for a more recent application, [**ACGR**].

We shall use the same symbol Δ_g to denote on one hand the usual Laplace-Beltrami operator on functions and, on the other hand, the Hodge-de Rham Laplacian viewed as an operator acting on the space $d[C^\infty(M)]$ of exact differential forms of degree 1 (the discrimination between these two cases will be given by the context). As Δ_g commutes with the exterior derivative d, it comes that the Hodge-de Rham Laplacian maps $d[C^\infty(M)]$ onto $d[C^\infty(M)]$ and that d maps each eigenspace of the Laplace operator on the eigenspace of the Hodge-de Rham Laplacian (restricted to $d[C^\infty(M)]$) corresponding to the same eigenvalue.
Let us define $\mathcal{A}(\lambda)$ as the direct sum of the eigenspaces of the Laplace-Beltrami operator (acting on functions) corresponding to the eigenvalues $\lambda_i \leq \lambda$, the above commutation implies that $\mathcal{A}_1(\lambda) := d[\mathcal{A}(\lambda)]$ is also a direct sum of the eigenspaces of the Hodge-de Rham Laplacian (acting on $d[C^\infty(M)]$) corresponding to the eigenvalues $\lambda_i \leq \lambda$.
In order to obtain a clearer statement for the next Proposition, let $\beta = \frac{n}{n-2}$ and let us define the function:

$$\xi(x) = \prod_{i=0}^{\infty}\left(1+\frac{\beta^i}{\sqrt{2\beta^i-1}}\cdot x\right)^{\beta^{-i}}$$

REMARK 3.2.6. We remark here that the infinite product defining ξ is convergent, as proved in Appendix B.

PROPOSITION 3.2.7 (revisiting [**Ga2**]). *Let (M,g) be a compact Riemannian manifold such that*

$$r_{\min}\cdot \mathrm{Diam}(M,g)^2 \geq -(n-1)\alpha^2$$

and such that $\mathrm{Diam}(M,g) \leq D$. *For any function $f \in \mathcal{A}(\lambda)$, we have:*

(i) $\| f \|_\infty \leq \xi(B(\alpha)D\sqrt{\lambda})\cdot \| f \|_2 \leq \exp\left(\frac{n}{2}\cdot\frac{B(\alpha)D\sqrt{\lambda}}{1+B(\alpha)D\sqrt{\lambda}}\right)\cdot\left(1+B(\alpha)D\sqrt{\lambda}\right)^{\frac{n}{2}}\cdot \| f \|_2$

$$\text{(ii) } \| df \|_\infty \leq \xi\left(B(\alpha)\sqrt{\lambda D^2+(n-1)\alpha^2}\right)\cdot \| df \|_2 \leq$$

$$\leq \exp\left(\frac{n}{2}\cdot\frac{B(\alpha)\sqrt{\lambda D^2+(n-1)\alpha^2}}{1+B(\alpha)\sqrt{\lambda D^2+(n-1)\alpha^2}}\right)\cdot\left(1+B(\alpha)\sqrt{\lambda D^2+(n-1)\alpha^2}\right)^{\frac{n}{2}}\cdot \| df \|_2$$

where $B(\alpha)$ is the Sobolev constant that we defined in the statement of the Proposition 3.2.4.

REMARK 3.2.8. The quantities α and λD^2 in the previous statement are invariant under homotheties.

Proof of the Proposition 3.2.7. Let S denote any element of $\mathcal{A}_1(\lambda)$ (resp. of $\mathcal{A}(\lambda)$), such that $S = df$ (resp. $S = f$) for some C^∞ function f. As we have supposed that $r_{\min} \geq -(n-1)\kappa^2$ (where $\kappa = \alpha/\operatorname{Diam}(M,g)$), we may introduce a new constant κ_0, which allows to handle both cases (*i.e.* the case where $S \in \mathcal{A}_1(\lambda)$ and the case where $S \in \mathcal{A}(\lambda)$) in a unique computation: we thus define κ_0 by:

$$\kappa_0 = \begin{cases} \kappa \text{ when } \Delta_g \text{ is the Hodge-de Rham Laplacian and when } S \in \mathcal{A}_1(\lambda); \\ 0 \text{ when } \Delta_g \text{ is the Laplace-Beltrami operator and when } S \in \mathcal{A}(\lambda). \end{cases}$$

We precise that we use the notation $|S|$ or $|S|(x)$ to denote the pointwise norm of S, whereas we use the notation $\| S \|_p$ when we consider the global L^p norm of S.
By the Böchner Formula (resp. by definition of the Laplace-Beltrami operator) we have:

$$\langle \nabla^* \nabla S, S \rangle = \langle \Delta_g S, S \rangle - \operatorname{Ricci}_g(\nabla f, \nabla f) \leq |\Delta_g S| \cdot |S| + (n-1)\,\kappa^2\,|S|^2$$

when $S \in \mathcal{A}_1(\lambda)$ (resp. $\langle \nabla^* \nabla S, S \rangle = \Delta_g S \cdot S$, when $S \in \mathcal{A}(\lambda)$), which gives the following formula:

$$\langle \nabla^* \nabla S, S \rangle \leq \big(|\Delta_g S| + (n-1)\,\kappa_0^2\,|S| \big)\,|S| \tag{3.77}$$

which is valid in both cases (the case $S \in \mathcal{A}_1(\lambda)$ and the case $S \in \mathcal{A}(\lambda)$) by definition of κ_0. We define (in the case $S \in \mathcal{A}_1(\lambda)$ as in the case $S \in \mathcal{A}(\lambda)$), the function s_ε by $s_\varepsilon = \sqrt{|S|^2 + \varepsilon^2}$. A direct computation leads to $|ds_\varepsilon|^2 \leq |\nabla S|^2$ and $\Delta_g(s_\varepsilon^2) = \Delta_g(|S|^2)$, so that:

$$s_\varepsilon\, \Delta_g s_\varepsilon = \frac{1}{2}\Delta_g(s_\varepsilon^2) + |ds_\varepsilon|^2 \leq \frac{1}{2}\Delta_g(|S|^2) + |\nabla S|^2 = \langle \nabla^* \nabla S, S \rangle \leq$$
$$\leq (|\Delta_g S| + (n-1)\,\kappa_0^2\,|S|)|S| \leq (|\Delta_g S| + (n-1)\,\kappa_0^2\,|S|)s_\varepsilon$$

For every $k > \frac{1}{2}$, we deduce that

$$\int_M |d(s_\varepsilon^k)|^2\, dv_g = \frac{k^2}{2k-1} \int_M \langle ds_\varepsilon, d(s_\varepsilon^{2k-1}) \rangle\, dv_g = \frac{k^2}{2k-1} \int_M s_\varepsilon^{2k-1} (\Delta_g s_\varepsilon)\, dv_g \leq$$
$$\leq \frac{k^2}{2k-1} \left[\int_M |\Delta_g S| \cdot s_\varepsilon^{2k-1}\, dv_g + (n-1)\,\kappa_0^2 \int_M |S| \cdot s_\varepsilon^{2k-1}\, dv_g \right] \leq$$
$$\leq \frac{k^2 V}{2k-1} \left[\| \Delta_g S \|_{2k} \cdot \| s_\varepsilon \|_{2k}^{2k-1} + (n-1)\,\kappa_0^2\, \| S \|_{2k} \cdot \| s_\varepsilon \|_{2k}^{2k-1} \right]$$

where, in the last inequality, we just applied Hölder's inequality. It follows that:

$$\left(\frac{1}{V} \int_M |d(s_\varepsilon^k)|^2\, dv_g \right)^{\frac{1}{2}} \leq \frac{k}{\sqrt{2k-1}} \| s_\varepsilon \|_{2k}^{k-\frac{1}{2}} \cdot \big(\| \Delta_g S \|_{2k} + (n-1)\kappa_0^2 \| S \|_{2k} \big)^{\frac{1}{2}} \tag{3.78}$$

On the other hand the Sobolev inequality of the Proposition 3.2.4 (ii) gives:

$$\left(\frac{1}{V} \int_M |d(s_\varepsilon^k)|^2\, dv_g \right)^{\frac{1}{2}} \geq \frac{1}{B(\alpha)D} \left[\| s_\varepsilon^k \|_{\frac{2n}{n-2}} - \| s_\varepsilon^k \|_2 \right]$$

hence, putting together this estimate with inequality (3.78), we get:

$$\| s_\varepsilon \|_{\frac{2kn}{n-2}}^k - \| s_\varepsilon \|_{2k}^k \leq$$

$$\leq \frac{[B(\alpha)D] \cdot k}{\sqrt{2k-1}} \cdot \| s_\varepsilon \|_{2k}^{k-\frac{1}{2}} \cdot \big[\| \Delta_g S \|_{2k} + (n-1)\kappa_0^2 \| S \|_{2k} \big]^{\frac{1}{2}} \tag{3.79}$$

Observe that, by the triangle inequality:

$$\| S \|_{2p}^2 \leq \| (|S|^2 + \varepsilon^2) \|_p = \| s_\varepsilon \|_{2p}^2 \leq \| (|S|^2) \|_p + \varepsilon^2 \leq \| S \|_{2p}^2 + \varepsilon^2$$

so that, for every $p > \frac{1}{2}$, $\| s_\varepsilon \|_{2p} \to \| S \|_{2p}$ when $\varepsilon \to 0_+$; thus, when $\varepsilon \to 0_+$, inequality (3.79) becomes:

$$\| S \|_{\frac{2kn}{n-2}}^k \leq \frac{[B(\alpha)D] \cdot k}{\sqrt{2k-1}} \cdot$$

$$\cdot \big(\| \Delta_g S \|_{2k} + (n-1)\kappa_0^2 \| S \|_{2k} \big)^{\frac{1}{2}} \| S \|_{2k}^{k-\frac{1}{2}} + \| S \|_{2k}^k \tag{3.80}$$

As we shall show in the Appendix C, it is impossible to give an upper bound to $\| \Delta_g S \|_{2k}$ in terms of $\lambda \| S \|_{2k}$. Nevertheless, we go beyond this difficulty using the following argument: we observe, as we are in the case where $S \in \mathcal{A}_1(\lambda)$ (resp. $S \in \mathcal{A}(\lambda)$), that S and $\Delta_g S$ can be written as

$$S = \sum_{i \ s.t. \ \lambda_i \leq \lambda} \alpha_i S_i \ , \quad \Delta_g S = \sum_{i \ s.t. \ \lambda_i \leq \lambda} \alpha_i \lambda_i S_i,$$

where $\{S_i\}$ is a L^2-orthonormal basis of eigenvectors for Δ_g , which implies that $\Delta_g S \in \mathcal{A}_1(\lambda)$ (resp. $\mathcal{A}(\lambda)$), and, if A_p is the supremum of the ratio $\frac{\|\phi\|_p}{\|\phi\|_2}$ when ϕ runs in $\mathcal{A}_1(\lambda) \setminus \{0\}$ (resp. in $\mathcal{A}(\lambda) \setminus \{0\}$), we have:

$$\frac{\| \Delta_g S \|_{2k}}{\| \Delta_g S \|_2} \leq A_{2k} \tag{3.81}$$

On the other hand the decomposition of S in terms of the S_i's and the Parseval identity give:

$$\| \Delta_g S \|_2^2 = \sum_{i \ s.t. \ \lambda_i \leq \lambda} \lambda_i^2 \alpha_i^2 \leq \lambda^2 \sum_{i \ s.t. \ \lambda_i \leq \lambda} \alpha_i^2 = \lambda^2 \| S \|_2^2$$

This inequality and inequality (3.81) give

$$\frac{\| \Delta_g S \|_{2k}}{\| S \|_2} = \frac{\| \Delta_g S \|_{2k}}{\| \Delta_g S \|_2} \cdot \frac{\| \Delta_g S \|_2}{\| S \|_2} \leq A_{2k} \, \lambda$$

Now bearing this estimate in equation (3.80), we obtain:

$$A^k_{\frac{2kn}{n-2}} \leq \frac{[B(\alpha)D] \cdot k}{\sqrt{2k-1}} \cdot \left[A_{2k} \, \lambda + (n-1) \, \kappa_0^2 \, A_{2k} \right]^{\frac{1}{2}} \cdot A_{2k}^{k-\frac{1}{2}} + A_{2k}^k$$

and thus

$$A_{\frac{2kn}{n-2}} \leq \left[1 + \frac{[B(\alpha)D] \cdot k}{\sqrt{2k-1}} \left(\lambda + (n-1)\kappa_0^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{k}} \cdot A_{2k}$$

Now let us replace k by β^i (we recall that $\beta = \frac{n}{n-2}$) we see that

$$A_{2\beta^m} = \prod_{i=0}^{m-1} \frac{A_{2\beta^{i+1}}}{A_{2\beta^i}} \leq \prod_{i=0}^{m-1} \left[1 + [B(\alpha)D] \cdot \frac{\beta^i}{\sqrt{2\beta^i - 1}} (\lambda + (n-1)\kappa_0^2)^{\frac{1}{2}} \right]^{\frac{1}{\beta^i}}$$

Now, letting m go to infinity, we obtain

$$A_\infty \leq \prod_{i=0}^{\infty} \left[1 + [B(\alpha)D] \cdot \frac{\beta^i}{\sqrt{2\beta^i - 1}} (\lambda + (n-1)\kappa_0^2)^{\frac{1}{2}} \right]^{\frac{1}{\beta^i}}$$

and we deduce that:

$$\frac{\| S \|_\infty}{\| S \|_2} \leq \xi \left([B(\alpha)D] \cdot (\lambda + (n-1)\kappa_0^2)^{\frac{1}{2}} \right).$$

When $S = f \in \mathcal{A}(\lambda)$ this becomes

$$\| f \|_\infty \leq \xi \left(B(\alpha) D \sqrt{\lambda} \right) \| f \|_2,$$

which proves the first inequality of (i); when $S = df \in \mathcal{A}_1(\lambda) = d[\mathcal{A}(\lambda)]$ we obtain

$$\| df \|_\infty \leq \xi \left(B(\alpha) D \sqrt{\lambda + (n-1)\kappa^2} \right) \| df \|_2$$

which proves the first inequality of (ii). We conclude the proof by noticing that $\xi(x) \leq \exp\left(\frac{n}{2} \cdot \frac{x}{1+x} \right) (1+x)^{\frac{n}{2}}$, as proved in Appendix B. □

The sharpness of the upper bound of $\frac{\|f\|_\infty}{\|f\|_2}$ given by the Proposition 3.2.7 is a consequence of the following corollary, and will be debated in the successive remark.

Corollary 3.2.9. *Let (M, g) be a compact Riemannian n-manifold such that $r_{\min} \cdot \mathrm{Diam}(M, g)^2 \geq -(n-1)\alpha^2$ and $\mathrm{Diam}(M, g) \leq D$. Let us define the function $N(\lambda) = \#\{i \mid \lambda_i \leq \lambda\}$, where the λ_i's are the eigenvalues of the Laplace-Beltrami operator, then*

$$N(\lambda) \leq [\xi^2(B(\alpha) \, D \sqrt{\lambda})] \leq \left[\exp\left(n \cdot \frac{B(\alpha) D \sqrt{\lambda}}{1 + B(\alpha) D \sqrt{\lambda}} \right) (1 + B(\alpha) D \sqrt{\lambda})^n \right]$$

(here $[\cdot]$ stands for the integer part of a real number).

REMARK 3.2.10 (About the sharpness of the estimate given by the Corollary 3.2.9). A consequence of the Corollary 3.2.9 (and of the upper bounds of ξ given in the Appendix B, Lemma B.2) is that:

(a) $N(\lambda) \le \left[(1+B(\alpha)\,D\,\sqrt{\lambda})^{2n}\right] \le 1$ when $\lambda < \left(\frac{2^{\frac{1}{2n}}-1}{B(\alpha)\,D}\right)^2$;

(b) $N(\lambda) \le (4\,e)^{\frac{n}{2}}\,(B(\alpha)\,D)^n\,\lambda^{\frac{n}{2}}$ when $\lambda \ge (B(\alpha)^2\,D^2)^{-1}$.

As $N(\lambda) \ge 1$ for every $\lambda \in [0,+\infty)$ (because $\lambda_0 = 0$ is an eigenvalue of the Laplace-Beltrami operator, the corresponding eigenfunctions being the constant ones), the inequality (a) is sharp for every $\lambda \in \left[0, \left(\frac{2^{\frac{1}{2n}}-1}{B(\alpha)\,D}\right)^2\right]$.

As Weyl's asymptotic formula says that $N(\lambda) \sim C(n)\cdot V\cdot\lambda^{\frac{n}{2}}$ when $\lambda \to +\infty$ (where $C(n)$ is a constant depending on the dimension), the inequality (b) is sharp (up to multiplicative universal constant) when $\lambda \to +\infty$. As a consequence the upper bound of the ratio $\| f \|_\infty / \| f \|_2$ (when f runs in $\mathcal{A}(\lambda)$), given by the Proposition 3.2.7 (i) is sharp when λ is small and when $\lambda \to +\infty$.

Proof of the Corollary 3.2.9. Let $m \in M$. Consider the following quadratic form, defined over the vector space $\mathcal{A}(\lambda)$, $Q_m : f \to f(m)^2$. For any L^2-orthonormal basis $\{f_1, ..., f_{N(\lambda)}\}$ of $\mathcal{A}(\lambda)$ we have:

$$\text{Tr}(Q_m) = \sum_{i=1}^{N(\lambda)} Q_m(f_i) = \sum_{i=1}^{N(\lambda)} f_i(m)^2 \,, \tag{3.82}$$

where the Trace of Q_m is, by definition, the trace of the corresponding symmetric bilinear form on $\mathcal{A}(\lambda)\times\mathcal{A}(\lambda)$ with respect to the L^2-scalar product on $\mathcal{A}(\lambda)$. Since Q_m has rank equal to 1, only one of the eigenvalues of Q_m (with respect to the L^2 scalar product) is different from zero. Hence there exists a L^2-orthonormal basis of $\mathcal{A}(\lambda)$ $\{g_1^m, ..., g_{N(\lambda)}^m\}$ such that $\forall i \ge 2$, $Q_m(g_i^m) = 0$, which implies (by equation (3.82) and by the Proposition 3.2.7 (i)) that

$$\begin{aligned}\text{Tr}(Q_m) = Q_m(g_1^m) = g_1^m(m)^2 \le \| g_1^m \|_\infty^2 \le \\ \le \xi^2(B(\alpha)\,D\,\sqrt{\lambda})\,\| g_1^m \|_2^2 = \xi^2(B(\alpha)D\sqrt{\lambda})\end{aligned}$$

From equation (3.82) we deduce

$$N(\lambda) = \sum_{i=1}^{N(\lambda)} \frac{1}{V}\int_M f_i(m)^2 dv_g = \frac{1}{V}\int_M \text{Tr}(Q_m) dv_g \le \xi^2(B(\alpha)D\sqrt{\lambda})\,.$$

This proves the first inequality of the Corollary 3.2.9. The second inequality is nothing but the upper bound of ξ given by the Lemma B.1 of the Appendix B. □

3.3. Spectral comparison between different manifolds in the presence of a map with bounded energy

We find useful to introduce the following definitions:

DEFINITION 3.3.1. Let (Y,g), (X,g_0) be two compact, connected Riemannian manifold. Let $F : (Y,g) \to (X,g_0)$ be a Lipschitz map. In every point $y \in Y$ where F is differentiable (thus in almost every point of Y) we can define the pointwise energy of the map F at y as

$$e_y(F) = \sum_1^n g_0(d_yF(e_i), d_yF(e_i))$$

where $\{e_i\}$ is any g-orthonormal basis of T_yY. Hence the global energy of the map F is given by integration: $\text{E}(F) = \int_Y e_y(F)\,dv_g(y)$. We remark that this notion makes sense because $e_y(F)$ is almost everywhere defined.

Throughout the paper we shall use the notion of *absolute degree* of a continuous map f between two n-dimensional compact manifolds ($\mathrm{Adeg}(f)$) which is a homotopy invariant. Instead of defining here the absolute degree (the definition can be found, for example, in [**Epst**], §1) we shall give the notion of *geometric degree* (more suitable for our purposes) and we remark that in [**Epst**] Epstein proved that they are actually equal. Moreover, they coincide with the absolute value of the usual cohomological degree in case f is a map between orientable manifolds.

Definition 3.3.2 (Geometric degree). Given a continuous map between two compact manifolds of dimension n, $f : Y \to X$, the geometric degree of f is defined as

$$\mathcal{G}_{\deg}(f) = \inf\{G(h) \mid h : Y \to X \text{ is properly homotopic to } f\}$$

where $G(h)$ denotes the minimum number of connected components of
$h^{-1}(D)$, where D varies among the top dimensional n-cells of X such that $h : h^{-1}(D) \to D$ is a covering (if such a disk does not exist we say that $G(h) = \infty$).

Remark 3.3.3. First we observe that what Epstein proved in [**Epst**] is that, for every continuous map f between two n-dimensional compact manifolds, there exists a map g homotopic to f which satisfies $\mathrm{Adeg}(g) = G(g)$ (whether in general just the inequality $\mathrm{Adeg}(f) \leq G(f)$ holds). In other terms: $\mathrm{Adeg}(f) = \mathcal{G}_{\deg}(f)$.
Then we recall that, as pointed out in [**Epst**], in contrast with the topological degree, the absolute degree is only submultiplicative.

Remark 3.3.4. Let $f : Y \to X$ be a continuous map between two n-dimensional compact manifolds. It follows from the definition of geometric degree and the discussion in the Remark 3.3.3 that we have: $\mathrm{Adeg}(f) \leq \# f^{-1}(\{x\})$, $\forall x \in X$.

3.3.1. A Bienaymé-Čebyšëv inequality. This subsection is devoted to the proof of a Bienaymé-Čebyšëv-like inequality, that we will use to derive crucial estimates in the next subsection. We will state and prove this inequality, and we will enlighten the intrinsic interest of the result:

Lemma 3.3.5. *Let (Y,g) and (X,g_0) be two connected, compact Riemannian manifolds of the same dimension, satisfying the following inequality between volumes:*

$$\mathrm{Vol}_g(Y) \cdot (1-\eta) \leq \mathrm{Vol}_{g_0}(X), \quad \textit{for some } \eta \in [0,1),$$

and let $F : (Y,g) \to (X,g_0)$ be a Lipschitz map with non-zero absolute degree, such that $|\operatorname{Jac}(F)(y)| \leq (1+\eta)$, in every point y where F is differentiable. Let us define the set: $Y_\eta^F = \{y \in Y \mid |\operatorname{Jac}(F)(y)| \leq (1-\sqrt{\eta})\}$. Then we have:

$$\frac{\mathrm{Vol}_g(Y_\eta^F)}{\mathrm{Vol}_g(Y)} \leq 2 \cdot \sqrt{\eta}.$$

Proof. Since the map F is Lipschitz, it is differentiable almost everywhere, so the bounds that we gave in the statement are valid almost everywhere; we can apply the coarea formula ([**Bu-Za**], Theorem 13.4.2) and, using the Remark 3.3.4, we get:

$$\mathrm{Adeg}(F) \cdot \mathrm{Vol}_{g_0}(X) \leq \int_X \#(F^{-1}(\{x\}))\, dv_{g_0}(x) =$$

$$= \int_Y |\operatorname{Jac}(F)(y)|\, dv_g(y) \leq (1-\sqrt{\eta}) \cdot \mathrm{Vol}_g(Y_\eta^F) + (1+\eta) \cdot \mathrm{Vol}_g(Y \setminus Y_\eta^F)$$

Since we are assuming $\mathrm{Adeg}(F) \neq 0$ and because of the inequality between the volumes of (Y,g) and (X,g_0) we obtain:

$$(1-\eta) \cdot \mathrm{Vol}_g(Y) \leq (1-\sqrt{\eta}) \cdot \mathrm{Vol}_g(Y_\eta^F) + (1+\eta) \cdot (\mathrm{Vol}_g(Y) - \mathrm{Vol}_g(Y_\eta^F))$$

thus we infer, $\frac{\mathrm{Vol}_g(Y_\eta^F)}{\mathrm{Vol}_g(Y)} \leq 2 \cdot \sqrt{\eta}$ which is the required inequality. □

Let us consider a Lipschitz map $F : (Y,g) \to (X,g_0)$ between two compact, connected Riemannian manifolds, which satisfy the inequality between volumes $\mathrm{Vol}_g(Y) \cdot (1-\eta) \leq \mathrm{Vol}_{g_0}(X)$. If we assume that $\mathrm{Adeg}(F) \neq 0$ and that the pointwise energy of the map F satisfies the upper bound $e_y(F) \leq n \cdot (1+\eta)^{2/n}$, at almost every point y, then F

satisfies the assumptions of the Lemma 3.3.5: in fact the geometric-arithmetic inequality $\prod_{i=1}^{n} \lambda_i^2 \leq \left(\frac{1}{n}\lambda_i^2\right)^n$ implies that

$$|\det(d_yF)| = \left(\det[(d_yF)^t \circ (d_yF)]\right)^{\frac{1}{2}} \leq \left(\frac{1}{n}\,\mathrm{Tr}((d_yF)^t \circ (d_yF))\right)^{\frac{n}{2}} =$$

$$= \left[\frac{1}{n}\, e_y(F)\right]^{\frac{n}{2}} \leq 1+\eta \tag{3.83}$$

Under these new assumptions we obtain the following

LEMMA 3.3.6. *Let* $0 < \eta \leq \frac{1}{4}$. *Let* (X, g_0), (Y, g) *be two compact, connected Riemannian manifolds which satisfy the inequality* $\mathrm{Vol}_g(Y)(1-\eta) \leq \mathrm{Vol}_{g_0}(X)$. *In any point* $y \in Y$ *such that* $|\,\mathrm{Jac}(F)(y)| \geq (1-\sqrt{\eta})$ *and* $e_y(F) \leq n(1+\eta)^{\frac{2}{n}}$, d_yF *is a quasi-isometry; more precisely we have* $\forall u \in T_yY$

$$(1 - 5(n-1)\eta^{\frac{1}{4}})\ \| u \|_g^2 \leq \| d_yF(u) \|_{g_0}^2 \leq (1 + 5(n-1)\eta^{\frac{1}{4}})\ \| u \|_g^2$$

Proof. Let us consider the bilinear symmetric form given by

$$(u,v) \to g_0(d_yF(u), d_yF(v)) = g((d_yF)^t \circ (d_yF)(u), v)$$

defined on $T_yY \times T_yY$. We denote by A the matrix associated to the endomorphism $(d_yF)^t \circ (d_yF)$ in a g-orthonormal basis of (T_yY, g_y); this matrix being symmetric and non-negative with determinant greater or equal to $(1-\sqrt{\eta})^2$ by assumption and with trace equal to $e_y(F)$ (hence, by assumption, less or equal to $n\,(1+\eta)^{\frac{2}{n}}$), we can apply the Proposition A.1 in Appendix A, which gives, $\forall u \in T_yY$,

$$|\ \| d_yF(u) \|_{g_0}^2 - \| u \|_g^2\ | = |g((A - \mathrm{Id})u, u)| \leq$$

$$\leq 2(n-1)\,\eta^{\frac{1}{4}} \left(1 + \frac{n+10}{2n}\sqrt{\eta}\right)^{\frac{1}{2}} \| u \|_g^2 \leq\ 5(n-1)\eta^{\frac{1}{4}}\ \| u \|_g^2$$

where the last inequality comes from the fact that $\eta \leq \frac{1}{4}$. □

3.3.2. A general comparison Lemma.

Let us start with some definitions:

DEFINITION 3.3.7. Let f be a function in $H_1^2(M,g) \setminus \{0\}$ where (M,g) is a fixed Riemannian manifold; the Rayleigh quotient of f is defined as the positive real number:

$$\mathcal{R}_g(f) = \frac{\int_M |df|^2\, dv_g}{\int_M |f|^2\, dv_g}.$$

DEFINITION 3.3.8. We will denote by $\lambda_i(X, g_0)$ and $\lambda_i(Y, g)$ the eigenvalues of the Laplace-Beltrami operators of (X, g_0) and (Y, g) respectively, indexed in increasing order and counted with their multiplicity from zero to infinity. We recall that, on compact manifolds, $\lambda_0 = 0$ and that the multiplicity of λ_0 is equal to 1.

We denote by $\mathcal{A}_X(\lambda)$ the direct sum of the eigenspaces of the Laplace-Beltrami operator of (X, g_0) corresponding to the eigenvalues which are less or equal to λ. We remark that the eigenvalues of the Laplacian Δ_g of (Y, g), are also the eigenvalues of the quadratic form $u \to \int_Y |du|^2\, dv_g$ with respect to the L^2-scalar product. Let us prove the following general comparison lemma:

LEMMA 3.3.9. *If there exists a linear map* $\phi : \mathcal{A}_X(\lambda) \to H_1^2(Y)$ *such that* $\forall u \in \mathcal{A}_X(\lambda)$

(i) $\| \phi(u) \|_2^2 \geq (1-\delta)\, \| u \|_2^2$ *where* $\delta \in [0,1)$;
(ii) $\frac{1}{\mathrm{Vol}_g(Y)} \int_Y |d(\phi(u))|^2\, dv_g \ \leq\ (1+\varepsilon) \frac{1}{\mathrm{Vol}_{g_0}(X)} \int_X |du|^2\, dv_{g_0}$;

then, for every $i \in \mathbb{N}$ *such that* $\lambda_i(X, g_0) \leq \lambda$, *we have*

$$\lambda_i(Y, g) \leq \left(\frac{1+\varepsilon}{1-\delta}\right) \lambda_i(X, g_0).$$

Proof. First we remark that the condition (i) implies that ϕ is injective (in particular the dimension of its image is equal to the dimension of $\mathcal{A}_X(\lambda)$). Now let us recall the Minimax Principle

Minimax Principle. *Let* (M,g) *be a closed, connected Riemannian manifold and let* $\mathrm{Spec}(M,g) = \{0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots\}$ *; for any* $(k+1)$*-dimensional subspace* $V \subseteq H_1^2(M,g)$ *if we denote by* $m_g(V)$ *the supremum of* $\mathcal{R}_g(f)$ *when* f *runs in* $V \setminus \{0\}$ *we have* $\lambda_k(M,g) \le m_g(V)$ *; equality holds in particular when* V *is the subspace generated by the first* $(k+1)$ *eigenfunctions of* Δ_g *.*

It is now clear how to prove the Lemma 3.3.9: for any i such that $\lambda_i(X,g_0) \le \lambda$ consider the linear subspace $E_i \subset \mathcal{A}_X(\lambda)$ generated by the first $i+1$ eigenfunctions. By injectivity of ϕ the dimension of E_i equals the dimension of its image by the map ϕ. By the estimates (i), (ii) we have $\forall u \in E_i$

$$\mathcal{R}_g(\phi(u)) \le \left(\frac{1+\varepsilon}{1-\delta}\right) \cdot \mathcal{R}_{g_0}(u)\ .$$

It comes that $\mathcal{R}_g(f) \le \left(\frac{1+\varepsilon}{1-\delta}\right) \lambda_i(X,g_0)$ for every $f \in \phi(E_i)$. As $\dim[\phi(E_i)] = i+1$, the result follows by applying the Minimax Principle to the space $\phi(E_i)$. □

3.3.3. Proof of the Proposition 3.1.4.

We start with the following lemma:

Lemma 3.3.10. *Under the assumptions of the Proposition 3.1.4, for any function* $f : X \to \mathbb{R}$ *such that* $f \in \mathcal{A}_X(\lambda)$ *, if* $\eta \le \frac{1}{9}$ *we have:*

$$\text{(i)}\ \left(\frac{1-\eta}{1+\eta}\right) \cdot \frac{1}{\mathrm{Vol}_{g_0}(X)} \int_X f^2\, dv_{g_0} \le \frac{1}{\mathrm{Vol}_g(Y)} \int_Y (f \circ F)^2\, dv_g \le$$

$$\le \left[1 + 3\sqrt{\eta}\left(1 + \xi^2(B(\alpha)D\sqrt{\lambda})\right)\right] \cdot \frac{1}{\mathrm{Vol}_{g_0}(X)} \int_X f^2\, dv_{g_0}$$

$$\text{(ii)}\ \frac{1}{\mathrm{Vol}_g(Y)} \int_Y |d(f \circ F)|^2\, dv_g \le (1 + 5(n-1)\eta^{\frac{1}{4}})\cdot$$

$$\cdot \left(1 + \left[2 + (2n+1)\xi^2(B(\alpha)\sqrt{\lambda D^2 + (n-1)\alpha^2})\right]\eta^{\frac{1}{2}}\right) \cdot \frac{1}{\mathrm{Vol}_{g_0}(X)} \int_X |df|^2\, dv_{g_0}$$

Proof. We start with the proof of the first inequality of the property (i). As discussed in section 3.1 (see equation (3.83)), the bound on the pointwise energy of F implies that $|\,\mathrm{Jac}(F)|$ is bounded above by $(1+\eta)$. Hence, using the assumptions and the coarea formula ([**Bu-Za**], Theorem 13.4.2), we find:

$$\frac{1+\eta}{\mathrm{Vol}_g(Y)} \cdot \int_Y |(f \circ F)(y)|^2\, dv_g(y) \ge$$

$$\ge \frac{1}{\mathrm{Vol}_g(Y)} \cdot \int_Y |(f \circ F)(y)|^2\, |\,\mathrm{Jac}(F)(y)|\, dv_g(y) \ge$$

$$\ge \frac{1-\eta}{\mathrm{Vol}_{g_0}(X)} \cdot \int_X |f(x)|^2\, \#(F^{-1}(\{x\}))\, dv_{g_0}(x) \ge \frac{1-\eta}{\mathrm{Vol}_{g_0}(X)} \cdot \int_X |f(x)|^2\, dv_{g_0}(x)$$

which proves the first inequality of (i). We remark that the second inequality of (i) is not necessary in order to prove the Proposition 3.1.4. However we shall provide a proof of this inequality for the sake of completeness. We notice that we are under the assumptions of the Lemma 3.3.5 and thus $\frac{\mathrm{Vol}_g(Y_\eta^F)}{\mathrm{Vol}_g(Y)} \le 2\sqrt{\eta}$ where $Y_\eta^F = \{y \in Y \ |\ |\,\mathrm{Jac}(F)|(y) < 1 - \sqrt{\eta}\}$. As the absolute degree is not trivial, F is surjective and thus $\#(F^{-1}(\{x\})) \ge 1$ for every $x \in X$. Thus we can get a first estimate for the L^2-norm of $F^*(f) = f \circ F$ in terms of the L^∞-norm and the L^2-norm of f; in fact using the coarea formula ([**Bu-Za**], Theorem 13.4.2) we find on $Y \setminus Y_\eta^F$

$$(1-\sqrt{\eta}) \cdot \int_{Y \setminus Y_\eta^F} |f \circ F|^2\, dv_g \le \int_Y |f \circ F|^2 |\,\mathrm{Jac}(F)(y)|\, dv_g =$$

$$= \int_X \#(F^{-1}(\{x\})) |f(x)|^2\, dv_{g_0}(x) \le$$

$$\le \mathrm{Vol}_{g_0}(X) \parallel f \parallel_2^2 + \parallel f \parallel_\infty^2 \int_X (\#(F^{-1}(\{x\})) - 1)\, dv_{g_0} \tag{3.84}$$

where the last equality comes from the fact that $\#(F^{-1}(\{x\})) \geq 1$ for every $x \in X$. On the other hand we know that $\mathrm{Vol}_g(Y)(1-\eta) < \mathrm{Vol}_{g_0}(X)$ and from the coarea formula and the upper bound on the Jacobian of F we deduce:

$$0 \leq \int_X (\#(F^{-1}(\{x\})) - 1)\, dv_{g_0} \leq$$

$$\leq \int_Y |\operatorname{Jac}(F)(y)|\, dv_g(y) - (1-\eta)\,\mathrm{Vol}_g(Y) \leq 2\eta \cdot \mathrm{Vol}_g(Y) \tag{3.85}$$

using the fact that $|\operatorname{Jac}(F)| \leq 1+\eta$ *a.e.* and that F is surjective and applying the coarea formula, we get

$$\mathrm{Vol}_{g_0}(X) \leq \int_X \#(F^{-1}(\{x\}))\, dv_{g_0}(x) = \int_Y |\operatorname{Jac}(F)(y)|\, dv_g(y) \leq (1+\eta)\,\mathrm{Vol}_g(Y) \tag{3.86}$$

hence we obtain, from (3.84), (3.85) and this last estimate:

$$\int_{Y \setminus Y_\eta^F} |f \circ F|^2\, dv_g \leq \left(\frac{1+\eta}{1-\sqrt{\eta}} \parallel f \parallel_2^2 + \frac{2\eta}{1-\sqrt{\eta}} \parallel f \parallel_\infty^2 \right) \cdot \mathrm{Vol}_g(Y)$$

whereas on Y_η^F, using the Lemma 3.3.5, we infer

$$\int_{Y_\eta^F} |f \circ F|^2\, dv_g \leq \parallel f \parallel_\infty^2 \cdot \mathrm{Vol}_g(Y_\eta^F) \leq 2\sqrt{\eta} \cdot \parallel f \parallel_\infty^2 \cdot \mathrm{Vol}_g(Y)$$

we sum these two inequalities, and we divide both sides by $\mathrm{Vol}_g(Y)$:

$$\parallel f \circ F \parallel_2^2 \leq \left(\frac{1+\eta}{1-\sqrt{\eta}} \right) \cdot \left(\parallel f \parallel_2^2 + \frac{2\sqrt{\eta}}{1+\eta} \parallel f \parallel_\infty^2 \right)$$

Now we can use the Proposition 3.2.7 (i), which tells us that, for every $f \in \mathcal{A}_X(\lambda)$,

$$\frac{\parallel f \parallel_\infty^2}{\parallel f \parallel_2^2} \leq \xi^2(B(\alpha) D \sqrt{\lambda})$$

where $\alpha = \kappa D$. Thus we obtain the estimate:

$$\parallel f \circ F \parallel_{L^2(Y)}^2 \leq \left(\frac{1+\eta}{1-\sqrt{\eta}} \right) \left[1 + \frac{2\sqrt{\eta}}{1+\eta} \cdot \xi^2(B(\alpha) D \sqrt{\lambda}) \right] \cdot \parallel f \parallel_{L^2(X)}^2$$

which can be simplified, thanks to the assumption $\eta \leq \frac{1}{9}$, in

$$\parallel f \circ F \parallel_{L^2(Y)}^2 \leq \left[1 + 3\sqrt{\eta} \left(1 + \xi^2(B(\alpha) D \sqrt{\lambda}) \right) \right] \cdot \parallel f \parallel_{L^2(X)}^2$$

This ends the proof of inequalities (i) of the Lemma 3.3.10.
Now we shall prove the inequality (ii). For every $y \in Y$ we have

$$|d_y(f \circ F)|^2 \leq \sup_{u \in T_y Y \setminus \{0_y\}} \left(\frac{|d_{F(y)} f(d_y F(u))|^2}{|d_y F(u)|^2} \cdot \frac{|d_y F(u)|^2}{|u|^2} \right)$$

and thus

$$|d_y(f \circ F)|^2 \leq |d_{F(y)} f|^2 \cdot \parallel\!| d_y F |\!\parallel^2$$

where $\parallel\!| d_y F |\!\parallel$ denotes the operator norm of $d_y F$. By the Lemma 3.3.6 we get for every y in $Y \setminus Y_\eta^F$

$$|d_y(f \circ F)|^2 \leq (1 + 5(n-1)\eta^{\frac{1}{4}}) \cdot |d_{F(y)} f|^2 \tag{3.87}$$

whereas for every $y \in Y_\eta^F$ we have $\parallel\!| d_y F |\!\parallel^2 \leq e_y(F) \leq n(1+\eta)^{\frac{2}{n}}$, which gives $\forall y \in Y_\eta^F$

$$|d_y(f \circ F)|^2 \leq n(1+\eta)^{\frac{2}{n}} \cdot |d_{F(y)} f|^2 \leq n(1+\eta)^{\frac{2}{n}} \parallel df \parallel_\infty^2 \tag{3.88}$$

From equation (3.88) we deduce that:

$$\frac{1}{\mathrm{Vol}_g(Y)} \int_{Y_\eta^F} |d(f \circ F)|^2\, dv_g \leq$$

$$\leq \frac{\mathrm{Vol}_g(Y_\eta^F)}{\mathrm{Vol}_g(Y)}\, n\,(1+\eta)^{\frac{2}{n}} \parallel df \parallel_\infty^2 \leq 2n\,\sqrt{\eta}\,(1+\eta)^{\frac{2}{n}} \parallel df \parallel_\infty^2, \tag{3.89}$$

where the last inequality deduces from the Lemma 3.3.5. On the other hand, from equation (3.87) and from the definition of Y_η^F , we deduce that

$$\frac{1}{\mathrm{Vol}_g(Y)} \int_{Y\setminus Y_\eta^F} |d(f\circ F)|^2\, dv_g \leq$$

$$\leq \frac{(1+5(n-1)\eta^{\frac14})}{\mathrm{Vol}_g(Y)(1-\sqrt{\eta})} \int_{Y\setminus Y_\eta^F} |d_{F(y)} f|^2 |\,\mathrm{Jac}(F)(y)|\, dv_g(y) \leq$$

$$\leq \frac{(1+5(n-1)\eta^{\frac14})}{\mathrm{Vol}_g(Y)(1-\sqrt{\eta})} \int_X |d_x f|^2 (\#(F^{-1}(\{x\})))\, dv_{g_0}(x) \leq$$

$$\leq \frac{(1+5(n-1)\eta^{\frac14})}{\mathrm{Vol}_g(Y)(1-\sqrt{\eta})} \left(\int_X |df|^2\, dv_{g_0} + \parallel df \parallel_\infty^2 \int_X (\# F^{-1}(\{x\}) - 1)\, dv_{g_0}(x) \right),$$

from this inequality and from inequalities (3.85) and (3.86), we have

$$\frac{1}{\mathrm{Vol}_g(Y)} \int_{Y\setminus Y_\eta^F} |d(f\circ F)|^2\, dv_g \leq \frac{(1+5(n-1)\eta^{\frac14})}{1-\sqrt{\eta}} \cdot \left[\frac{\mathrm{Vol}_{g_0}(X)}{\mathrm{Vol}_g(Y)} \parallel df \parallel_2^2 + 2\eta \parallel df \parallel_\infty^2 \right]$$

Now we sum the last inequality with equation (3.89); since

$$\mathrm{Vol}_{g_0}(X) \leq \mathrm{Adeg}(F) \cdot \mathrm{Vol}_{g_0(X)} \leq (1+\eta)\,\mathrm{Vol}_g(Y)$$

we obtain:

$$\parallel d(f\circ F) \parallel_2^2 \leq \frac{(1+5(n-1)\eta^{\frac14})}{(1-\sqrt{\eta})} \left[(1+\eta) \parallel df \parallel_2^2 + 2\eta \parallel df \parallel_\infty^2 \right] + 2\, n \sqrt{\eta}\, (1+\eta)^{\frac{2}{n}} \parallel df \parallel_\infty^2$$

Now using the fact that for $\eta \leq \frac19$ the following inequalities hold,

- $(1+\eta)^{\frac{2}{n}} \leq 1 + 5(n-1)\eta^{\frac14}$;
- $\frac{1}{1-\sqrt{\eta}} \leq 1 + \frac32 \sqrt{\eta}$;
- $\frac{2\eta}{1-\sqrt{\eta}} \leq \sqrt{\eta}$

we get the estimate:

$$\parallel d(f\circ F) \parallel_2^2 \leq (1+5(n-1)\eta^{\frac14}) \left[(1+2\sqrt{\eta}) \parallel df \parallel_2^2 + (2n+1)\sqrt{\eta} \parallel df \parallel_\infty^2 \right]$$

To conclude it is sufficient to apply the Proposition 3.2.7 (ii) which gives

$$\parallel df \parallel_\infty^2 \leq \xi^2 (B(\alpha)\sqrt{\lambda D^2 + (n-1)\alpha^2}) \parallel df \parallel_2^2$$

and thus achieves the proof of inequality (ii) of the Lemma 3.3.10. □

End of the Proof of the Proposition 3.1.4. We just apply the Lemma 3.3.9 to the linear map $F^* : f \to f\circ F$. This map is linear and sends $\mathcal{A}_X(\lambda)$ onto a subspace of $H_1^2(Y,g)$: in fact, as f is C^∞, $f\circ F$ is continuous and Lipschitz, thus $f\circ F$ and $|d(f\circ F)|$ are bounded and have finite L^2-norms, this proves that $f\circ F \in H_1^2(Y,g)$ and that $F^*[A_X(\lambda)]$ is included in $H_1^2(Y,g)$. By the Lemma 3.3.10 the assumptions of the Lemma 3.3.9 are verified for every $f \in \mathcal{A}_X(\lambda)$ and applying the Lemma 3.3.9, we obtain

$$\lambda_i(Y,g) \leq \left(\frac{(1+5(n-1)\eta^{\frac14})(1+\eta)}{1-\eta} \right) \cdot$$

$$\cdot \left(1 + \left[2 + (2n+1)\xi^2 \left(B(\alpha)\sqrt{\lambda D^2 + (n-1)\alpha^2} \right) \right] \eta^{\frac12} \right) \lambda_i(X, g_0).$$

Using the estimate of ξ computed in Appendix B we see that:

$$\lambda_i(Y,g) \leq (1+7(n-1)\eta^{\frac14}) \cdot$$

$$\cdot \left(1 + \left[2 + (2n+1)e^n \left(1 + B(\alpha)\sqrt{\lambda D^2 + (n-1)\alpha^2} \right)^n \right] \eta^{\frac12} \right) \cdot \lambda_i(X, g_0)$$

We conclude by taking $\lambda = \lambda_i(X, g_0)$ in the last inequality. □

Corollary 3.3.11. *Let* (Y,g), (X,g_0) *and* $F:Y\to X$ *be as stated in the Proposition 3.1.4. Let us denote by* $N_{g_0}(\lambda)$ *(resp.* $N_g(\lambda)$ *) the dimension of the vector space generated by the eigenfunctions corresponding to the eigenvalues* λ_i *of* Δ_{g_0} *(resp. of* Δ_g *) such that* $\lambda_i\le\lambda$. *If we define the function* $E(\lambda)=E_{n,\kappa,D,\eta}(\lambda)$ *by*

$$E_{n,\kappa,D,\eta}(\lambda)=(1+7(n-1)\eta^{\frac14})(1+C(n,D^2\lambda,\alpha)\eta^{\frac12})-1$$

(where C *is the constant defined in the Proposition 3.1.4 and* $\alpha=\kappa D$ *) we have:*

$$N_g(\lambda)\ge N_{g_0}\left(\frac{\lambda}{E(\lambda)+1}\right)$$

Notice that $E_{n,\kappa,D,\eta}(\lambda)$ *goes to zero when* $\eta\to0^+$.

3.4. Spectral comparison between manifolds in terms of their Gromov-Hausdorff distance

The main purpose of this subsection is to present the link between the spectra comparison theorem which we proved in the previous subsection and the barycenter method by Besson, Courtois and Gallot (see [**BCG1**],[**BCG2**]). More precisely we will use a recent developement of this technique by L. Sabatini, [**Saba1**]. The main feature of this last version of the barycenter method is that, on one hand, no assumption is made on the sign of the sectional curvature of the "known" manifold (X,g_0) (only its boundedness is required), on the other hand, no condition is assumed on the geometry of the "unknown" manifold (Y,g), except for the fact that the Gromov-Hausdorff distance between (Y,g) and (X,g_0) is supposed to be smaller than some universal constant, which is precised in Chapter 1, Theorem 1.4.1. This technique, combined with the Proposition 3.1.4, will provide a spectra comparison theorem between manifolds satisfying weak assumptions.

3.4.1. Proof of the Theorem 3.1.2. We showed in Chapter 1 (Theorem 1.4.1) the more recent version of the Barycenter Method of G. Besson, G. Courtois and S. Gallot. We shall show how Sabatini's result can be applied in our context.

End of the proof of the Theorem 3.1.2. Any pair of Riemannian manifolds which satisfies the assumptions of the Theorem 3.1.2 also satisfies the assumption of Chapter 1, Theorem 1.4.1. Applying Chapter 1, Theorem 1.4.1 (2), we obtain the existence of a C^1 map $F:(Y,g)\to(X,g_0)$ homotopic to f and thus of non zero absolute degree which (by Chapter 1, Theorem 1.4.1 (1)) satisfies, at every point $y\in Y$,

$$e_y(F)\le n\,(1+\eta(\varepsilon))^{\frac2n}\ ,$$

where $\eta(\varepsilon)$ is defined by

$$\eta(\varepsilon)=\left[1+20\,(\kappa\,\varepsilon)^{\frac14}\right]^{\frac n2}-1\,.$$

The assumption $\varepsilon<\frac1\kappa\left(\frac{(\frac{10}{9})^{\frac2n}-1}{20}\right)^4$ immediately implies that $\eta(\varepsilon)<\frac19$. Finally, the assumption:

$$\frac{\mathrm{Vol}_{g_0}(X)}{\mathrm{Vol}_g(Y)}\ge1-10\,n\,(\kappa\,\varepsilon)^{\frac14}$$

implies that $\frac{\mathrm{Vol}_{g_0}(X)}{\mathrm{Vol}_g(Y)}>1-\eta(\varepsilon)$ because $(1+x)^{\frac n2}-1\ge\frac n2\,x$, $\forall x\in\mathbb{R}^+$. We may thus apply the Proposition 3.1.4 which proves that

$$\frac{\lambda_i(Y,g)}{\lambda_i(X,g_0)}\le\left[1+7\,(n-1)\,\eta(\varepsilon)^{\frac14}\right]\cdot\left[1+C(n,\,D^2\,\lambda_i(X,g_0),\,\kappa\,D)\,\eta(\varepsilon)^{\frac12}\right]$$

we conclude the proof when noticing that, $\forall x\in\mathbb{R}^+$ we have $(1+x)^{\frac n2}-1\le\frac n2\,(1+x)^{\frac n2}\,x$, and thus

$$\eta(\varepsilon)\le10\,n\,(1+\eta(\varepsilon))\,(\kappa\,\varepsilon)^{\frac14}\le\frac{100}{9}\,n\,(\kappa\,\varepsilon)^{\frac14}$$

which leads to

$$7\,(n-1)\,\eta(\varepsilon)^{\frac14}\ <\ 14\,(n-1)\,\sqrt[4]{n}\,(\kappa\varepsilon)^{\frac1{16}}=C_1(n)\,(\kappa\varepsilon)^{\frac1{16}}\ ,$$

$$C(n,\, D^2\,\lambda_i(X,g_0),\,\kappa\, D)\,\eta(\varepsilon)^{\frac{1}{2}} \;\le\; \frac{10}{3}\,\sqrt{n}\,(\kappa\,\varepsilon)^{\frac{1}{8}}\cdot$$
$$\cdot\Big(2+(2n+1)\,e^n\,\Big[1+B(\kappa\, D)\,D\,\sqrt{\lambda_i(X,g_0)+(n-1)\kappa^2}\Big]^n\Big) \le$$
$$\le C_2(n,\,\kappa\, D,\, D^2\,\lambda_i(X,g_0))\cdot(\kappa\,\varepsilon)^{\frac{1}{8}}$$

this concludes the proof of the Theorem 3.1.2. □

3.4.2. Examples. This subsection is devoted to the construction of examples and counterexamples regarding the Theorem 3.1.2.

Let us consider a triple $[(X^n,g_0),\,(Y^n,g),\,f]$ where (X^n,g_0) and (Y^n,g) are two Riemannian manifolds and $f\,:Y^n\to X^n$ is a continuous map. We shall say that *the triple* $[(X^n,g_0),\,(Y^n,g),\,f]$ *satisfies the assumptions of the Theorem* 3.1.2 *for some value* $\varepsilon>0$ if there exist three positive real numbers $D,\, i_0,\, \kappa$ such that

(i) $\mathrm{Diam}(X^n,g_0)\le D,\quad \mathrm{inj}(X^n,g_0)\ge i_0,\quad |\sigma_0|\le\kappa^2$ and $\varepsilon<\varepsilon_1(n,\,i_0,\,\kappa)$, where σ_0 denotes the sectional curvature of g_0 and where the value of the universal constant $\varepsilon_1(n,\,i_0,\,\kappa)$ is given in the Theorem 3.1.2;
(ii) $(1-10\,n\,(n+1)\,(\kappa\,\varepsilon)^{\frac{1}{4}})\ \mathrm{Vol}_g(Y^n)<\mathrm{Vol}_{g_0}(X^n)$;
(iii) f is a Gromov-Hausdorff ε-approximation of non zero absolute degree from (Y^n,g) to (X^n,g_0).

Example 3.4.1. The following Proposition proves that *the fact that some triple* $[(X^n,g_0),\,(Y^n,g),\,f]$ *satisfies the assumptions of the Theorem 3.1.2 does not imply any similarity between the topologies or the homotopies of* X^n *and* Y^n *or between the local geometries of* (Y^n,g) *and* (X^n,g_0). This underlines the weakness of the assumptions of the Theorem 3.1.2.
It shows as well that the assumptions of the Theorem 3.1.2 have very different implications from the "convergence assumption with Ricci curvature bounded below" of J. Cheeger and T. Colding ([**Ch-Co**]), because Cheeger-Colding's assumptions imply that, when the Gromov-Hausdorff distance is small enough, then the manifolds (Y^n,g) and (X^n,g_0) are diffeomorphic and that their small balls have almost the same volume.

Proposition 3.4.2. *For any pair of* $n-$*dimensional Riemannian manifolds* (X,g) *and* (Z,h) *and for any* $\varepsilon>0$ *sufficiently small, there exists, on the connected sum* $X\#Z$, *a metric* g_ε *and a continuous map*

$$f_\varepsilon : X\#Z \longrightarrow X$$

such that the triple $[(X,g),\,(X\#Z,g_\varepsilon),\,f_\varepsilon]$ *satisfies the assumptions of the Theorem 3.1.2 for this value of* ε.

Before giving the proof of the Proposition 3.4.2, we shall provide another example; in fact, the proofs of the Proposition 3.4.2 and of the Proposition 3.4.4 are based on similar constructions by Riemannian surgery.

Example 3.4.3 (Mushrooms). For any $i\in\mathbb{N}^*$, for any ε such that $0<\varepsilon<\varepsilon_1(n,\,i_0,\,\kappa)$ (where the value of the universal constant $\varepsilon_1(n,\,i_0,\,\kappa)$ is given in the Theorem 3.1.2) and for any triple $[(X^n,g_0),\,(Y^n,g),\,f]$ which satisfies the assumptions of the Theorem 3.1.2 for this value of ε, the ratio $\dfrac{\lambda_i(Y^n,g)}{\lambda_i(X^n,g_0)}$ admits an explicit upper bound (it is the result stated in the Theorem 3.1.2). On the other hand *it is impossible to obtain, under the same assumptions, a lower bound for the same ratio*, as we shall show in the following

Proposition 3.4.4. *For any compact Riemannian manifold* (X^n,g_0), *and for any* $i\in\mathbb{N}^*$ *there exists a family of Riemannian metrics* $\{g_\varepsilon\}_{\varepsilon>0}$, *a family of continuous maps* $\{f_\varepsilon\}_{\varepsilon>0}$ *from* X^n *to itself and a constant* $\varepsilon_2(n,i_0,\kappa,D,i)>0$ *such that:*

(i) *the triple* $[(X^n,g_0),\,(X^n,g_\varepsilon),\,f_\varepsilon]$ *satisfies the assumptions of the Theorem 3.1.2 for any* ε *such that* $0<\varepsilon<\varepsilon_2(n,i_0,\kappa,D,i)$;

(ii) *for any* $j \in \{1, ..., i\}$, $\dfrac{\lambda_j(X^n, g_\varepsilon)}{\lambda_j(X^n, g_0)} \to 0$ *when* $\varepsilon \to 0$.

3.4.2.1. *Proofs of the two propositions 3.4.2 and 3.4.4.* As the arguments of the first part of the proof are rather classical (and in order to avoid technicalities), we shall only give a sketch of the proofs in the main text. However some of the parameters, which are a priori small (or controlled), must be precisely computed in order that they remain small (or controlled) after the contractions or deformations of the metrics due to the gluings and to the fact that we want the Gromov-Hausdorff distance to be small, this is the reason why we shall precise these technical arguments in footnotes.

The connected sum of two manifolds:
Let (X, g) and (Z, h) be two compact, oriented, $n-$dimensional manifolds and let us fix two points $x_0 \in X$ and $z_0 \in Z$. Let S_δ^{n-1} be the euclidean sphere of radius δ and let $B_g(x_0, \delta)$ (resp. $B_h(z_0, \delta)$) be the geodesic ball of (X, g) (resp. of (Z, h)) of radius δ centered at x_0 (resp. at z_0); we construct the new manifold Y_ε by gluing $X \setminus B_g(x_0, \delta)$ and $Z \setminus B_h(z_0, \delta)$ at the two ends of the euclidean cylinder $S_\delta^{n-1} \times [0, \delta]$ by means of the identifications of $\partial B_g(x_0, \delta)$ and $\partial B_h(z_0, \delta)$ respectively with $S_\delta^{n-1} \times \{0\}$ and $S_\delta^{n-1} \times \{\delta\}$. This construction[17] provides a new smooth manifold Y_ε, which is the connected sum of X and Z, denoted by $X\#Z$.

Construction of the glued metric g_ε on the connected sum Y_ε:
There are two a priori problems :

- First we cannot identify the Riemannian metrics of $\partial B_g(x_0, \delta)$ and $\partial B_h(z_0, \delta)$ respectively with the canonical metrics of $S_\delta^{n-1} \times \{0\}$ and $S_\delta^{n-1} \times \{\delta\}$, because these spaces are not isometric. We solve this problem by replacing the metric g on X by a metric g_0 which is given by the

  LEMMA 3.4.5. *Let κ_X be an upper bound of $|\sigma_X|^{\frac{1}{2}}$, where σ_X is the sectional curvature of (X, g); for any δ such that $0 < \delta < \frac{1}{4} \min\left(\mathrm{inj}(X, g), \frac{1}{\kappa_X}\right)$ there exists another Riemannian metric[18] g_0 on X such that*
  – $B_{g_0}(x_0, 2\delta) = B_g(x_0, 2\delta)$ as sets;
  – $B_{g_0}(x_0, 2\delta)$ is isometric to the euclidean ball of the same radius and thus $\partial B_{g_0}(x_0, \delta)$ is isometric to $S_\delta^{n-1} \times \{0\}$;
  – Id_X is a Gromov-Hausdorff $(12\, \kappa_X^2\, \delta^3)$-approximation from (X, g_0) to (X, g), where κ_X^2 is a upper bound of the absolute value $|\sigma|$ of the sectional curvature of (X, g).

  We could do the same change of metrics on (Z, h), but we previously have to solve the following second problem:

- The second problem comes from the fact that we want the Gromov-Hausdorff distance between $(X\#Z, g_\varepsilon)$ and (X, g) to be smaller than ε, and thus we want the diameter of Z to be much smaller than ε. More precisely, let D be greater than the diameter of (Z, h) and let κ_Z be an upper bound of $|\sigma_Z|^{\frac{1}{2}}$ (where σ_Z is the sectional curvature of (Z, h)), for any $\varepsilon > 0$ such that $\varepsilon < \min\left(\mathrm{inj}(Z, h), \frac{1}{\kappa_Z}\right)$, we define $\alpha(\varepsilon) = \frac{\varepsilon}{4D}$ and we replace the initial metric h

[17] Since this point we only need $\partial B_g(x_0, \delta)$, $\partial B_h(z_0, \delta)$ and S_δ^{n-1} to be diffeomorphic, condition which is automatically verified if δ is smaller than the injectivity radii of (X, g) and (Z, h). There is a choice of the orientation of Z such that $X\#Z$ admits an orientation which coincides with the initial orientation of X on $X \setminus B_g(x_0, \delta)$.

[18] More precisely, let $\exp_{x_0}^g$ be the exponential map associated to the metric g and centered at the point x_0 and let $\mathrm{inj}_{x_0}(X, g)$ be the injectivity radius of $\exp_{x_0}^g$. Then the value g_{x_0} of the metric g on the tangent space at the point x_0 is an euclidean metric on $T_{x_0}X$ and its pushed-forward $\bar{g}_0 := (\exp_{x_0}^g)_* g_{x_0}$ is a flat metric on the ball $B_g(x_0, \mathrm{inj}_{x_0}(X, g))$ such that $d_{\bar{g}_0}(x_0, x) = d_g(x_0, x)$ for every $x \in B_g(x_0, \mathrm{inj}_{x_0}(X, g))$. For any $0 < \delta < \frac{\mathrm{inj}_{x_0}(X,g)}{4}$, let φ be a smooth function with compact support in $B_g(x_0, 3\,\delta)$ such that $0 \le \varphi \le 1$ and $\varphi = 1$ on $B_g(x_0, 2\,\delta)$, it is then sufficient to define the new Riemannian metric g_0 on X by $g_0 = (1 - \varphi) \cdot g + \varphi \cdot \bar{g}_0$. We then bound the Gromov-Hausdorff distance between (X, g_0) and (X, g) by means of Rauch's comparison Theorem which compares the metrics g and $(\exp_{x_0}^g)_* g_{x_0}$.

on Z by the metric $h_\varepsilon = \alpha(\varepsilon)^2 h$. For every $\delta > 0$ such that $\delta < \frac{\varepsilon^2}{16D^2}$ we can then apply[19] the Lemma 3.4.5 to the initial manifold (Z, h_ε) and get a new metric h_ε^0 on Z such that $\mathrm{Diam}(Z, h_\varepsilon^0) \leq \frac{\varepsilon}{4}$ and $\partial B_{h_\varepsilon^0}(z_0, \delta)$ is isometric to $S_\delta^{n-1} \times \{\delta\}$.

Let us denote by Y_0 , Y_1 and Y_2 the respective images of $(X \setminus B_{g_0}(x_0, \delta))$, $S_\delta^{n-1} \times [0, \delta]$ and $(Z \setminus B_{h_\varepsilon^0}(z_0, \delta))$ by the quotient map

$$(X \setminus B_{g_0}(x_0, \delta)) \cup \left(S_\delta^{n-1} \times [0, \delta]\right) \cup \Big(Z \setminus B_{h_\varepsilon^0}(z_0, \delta)\Big) \longrightarrow Y_\varepsilon$$

which identifies (isometrically) $\partial B_{g_0}(x_0, \delta)$ and $\partial B_{h_\varepsilon^0}(z_0, \delta)$ respectively with $S_\delta^{n-1} \times \{0\}$ and $S_\delta^{n-1} \times \{\delta\}$. We construct the Riemannian metric g_ε on Y_ε by setting

$$g_\varepsilon = g_0 \ \text{ on } \ Y_0 \ , \quad g_\varepsilon = (dr)^2 \oplus g_{S_\delta^{n-1}} \ \text{ on } \ Y_1 \ , \quad g_\varepsilon = h_\varepsilon^0 \ \text{ on } \ Y_2$$

(where $g_{S_\delta^{n-1}}$ is the canonical metric of the euclidean sphere of radius δ); this metric g_ε is continuous and moreover piecewise C^1 , but we can find a smooth metric $\hat{g}_\varepsilon$ arbitrarily near to g_ε (in the C^0 sense, hence with respect to the Gromov-Hausdorff distance).

The Gromov-Hausdorff $\varepsilon-$approximation:
It follows from the

LEMMA 3.4.6. *If ε, $\delta > 0$ moreover satisfy the assumptions*

$$\varepsilon < \left(\frac{\mathrm{Vol}(Z, h)}{\mathrm{Vol}(\mathrm{B}^n, \mathrm{can})}\right)^{\frac{1}{n}} \quad , \quad \delta < \frac{\varepsilon^2}{16D^2} \quad ,$$

the manifold $(Y_\varepsilon, g_\varepsilon)$ verifies:

(i) $\mathrm{Vol}(Y_\varepsilon, g_\varepsilon) \leq \mathrm{Vol}(X, g) \cdot \left[1 + \frac{\mathrm{Vol}(Z,h)}{2^n \, \mathrm{Vol}(X,g)} \cdot \left(\frac{\varepsilon}{\mathrm{Diam}(Z,h)}\right)^n\right]$;

(ii) *The map $f_\varepsilon : Y_\varepsilon \to X$ defined by:*

$$f_\varepsilon(y) = \begin{cases} = & y & \text{when} & y \in Y_0 \, ; \\ = & \exp_{x_0}^{g_0}\left(\left[1 - \frac{t}{\delta}\right] \cdot v\right) & \text{when} & y = (v, t) \in S_\delta^{n-1} \times [0, \delta] \simeq Y_1 \, ; \\ = & x_0 & \text{when} & y \in Y_2 \, . \end{cases}$$

is a Gromov-Hausdorff $\left(\frac{\varepsilon}{2} + \pi\,\delta\right)$-approximation from $(Y_\varepsilon, g_\varepsilon)$ to (X, g) which is continuous of degree 1.

REMARK 3.4.7. By the Lemma 3.4.5 it follows that the map f_ε defined in the Lemma 3.4.6 is a Gromov-Hausdorff $\left(\frac{\varepsilon}{2} + \pi\delta + 12\kappa_X^2 \delta^3\right)$-approximation from $(Y_\varepsilon, g_\varepsilon)$ to (X, g).

Proof of the Lemma. We obtain the inequality (i) by summing the g_ε -volumes of Y_0 , Y_1 , Y_2 [20].
The continuity of f_ε follows directly from the definition of the map. The fact that f_ε has degree one comes from the choice of the orientation of Y_ε and from the fact that $f_\varepsilon^{-1}(\{x\})$ is a single point when $x \in X \setminus B_{g_0}(x_0, \delta)$.

[19] In fact the assumption of the Lemma 3.4.5 is satisfied because, as $\mathrm{inj}(Z, h_\varepsilon) = \frac{\varepsilon}{4D} \, \mathrm{inj}(Z, h)$, we get

$$\delta < \frac{\varepsilon^2}{16D^2} < \frac{1}{4} \min\left(\mathrm{inj}(Z, h_\varepsilon) \ , \ \frac{1}{\kappa_{(Z,h_\varepsilon)}}\right) \quad .$$

[20] More precisely, summing the g_ε -volumes of Y_0 , Y_1 , Y_2 , we obtain:

$$\begin{aligned} \mathrm{Vol}(Y_\varepsilon, g_\varepsilon) &< \mathrm{Vol}(X, g_0) + \mathrm{Vol}(Z, h_\varepsilon^0) + \delta \ \mathrm{Vol}(S_\delta^{n-1}, \mathrm{can}) \\ &< \mathrm{Vol}(X, g) + \mathrm{Vol}(Z, h_\varepsilon) + 2 \ \mathrm{Vol}(\mathrm{B}^n, \mathrm{can}) \, (3\delta)^n + n \ \mathrm{Vol}(\mathrm{B}^n, \mathrm{can}) \, \delta^n \\ &< \mathrm{Vol}(X, g) + \frac{\varepsilon^n}{4^n D^n} \mathrm{Vol}(Z, h) + \left(2 \cdot 3^n + n\right) \ \mathrm{Vol}(\mathrm{B}^n, \mathrm{can}) \frac{\varepsilon^{2n}}{16^n D^{2n}} \end{aligned}$$

and the inequality (i) then follows from the assumptions satisfied by ε and δ.

We can verify directly that the map f_ε is a contraction[21] from $(Y_\varepsilon, g_\varepsilon)$ to (X, g_0).
We now have to prove that $d_{g_0}(f_\varepsilon(y), f_\varepsilon(y')) \geq d_{g_\varepsilon}(y, y') - \pi\,\delta - \frac{\varepsilon}{2}$:
Let us observe that[22]

$$\forall y \in Y_1 \cup Y_2 \qquad d_{g_\varepsilon}(y, Y_0) = d_{g_\varepsilon}(y, S_\delta^{n-1} \times \{0\}) \leq \frac{\varepsilon}{4} \ . \tag{3.90}$$

From this and from the triangle inequality, we deduce that, for every $y \in Y_\varepsilon$ and every $y' \in Y_1 \cup Y_2$, one has

$$\begin{aligned} d_{g_\varepsilon}(y, y') &\leq d_{g_\varepsilon}(y, \partial B_{g_0}(x_0, \delta)) + d_{g_\varepsilon}(y', Y_0) + \mathrm{Diam}(S_\delta^{n-1} \times \{0\}) \\ &\leq d_{g_\varepsilon}(y, \partial B_{g_0}(x_0, \delta)) + \frac{\varepsilon}{4} + \pi\delta \leq d_{g_0}(f(y), f(y')) + \left(\frac{\varepsilon}{2} + \pi\delta\right) , \end{aligned}$$

where the last inequality deduces from the fact that $d_{g_\varepsilon}(y, \partial B_{g_0}(x_0, \delta))$ is bounded above by $\frac{\varepsilon}{4}$ when $y \in Y_1 \cup Y_2$ and by $d_{g_0}(f(y), f(y'))$ when $y \in Y_0$ (because, in this last case, $f(y') \in B_{g_0}(x_0, \delta)$ and $f(y) = y \in X \setminus B_{g_0}(x_0, \delta)$).
Finally when $y, y' \in Y_0$ we have two distinct cases: either the minimizing g_0-geodesic c which joins y to y' in X does not meet the ball $\partial B_{g_0}(x_0, \delta)$, and this implies that

$$d_{g_\varepsilon}(y, y') \leq \ell_{g_\varepsilon}(c) = \ell_{g_0}(c) = d_{g_0}(y, y') = d_{g_0}(f(y), f(y')) ,$$

or c meets $\partial B_{g_0}(x_0, \delta)$ and then:

$$d_{g_0}(y, y') \geq d_{g_0}(y, \partial B_{g_0}(x_0, \delta)) + d_{g_0}(y', \partial B_{g_0}(x_0, \delta))$$

and thus, by the triangle inequality,

$$\begin{aligned} d_{g_\varepsilon}(y, y') &\leq d_{g_0}(y, \partial B_{g_0}(x_0, \delta)) + \mathrm{Diam}(\partial B_{g_0}(x_0, \delta), g_\varepsilon) + d_{g_0}(y', \partial B_{g_0}(x_0, \delta)) \\ &\leq d_{g_0}(y, y') + \pi\,\delta \leq d_{g_0}(f(y), f(y')) + \pi\,\delta\ . \end{aligned}$$

This concludes the proof of the Lemma 3.4.6. □

3.4.2.2. *End of the proof of the Proposition 3.4.2.* Consider any pair of Riemannian manifolds (X^n, g) and (Z^n, h), and let us denote by i_0, D_0 and κ three fixed numbers such that $\mathrm{Diam}(X^n, g) \leq D_0$, $\mathrm{inj}(X^n, g) \geq i_0$, $|\sigma_X| \leq \kappa_X^2$ (where σ_X denotes the sectional curvature of (X, g)); let us denote by D and κ_Z two fixed numbers such that $\mathrm{Diam}(Z^n, h) \leq D$, $|\sigma_Z| \leq \kappa_Z^2$ (where σ_Z denotes the sectional curvature of (Z, h)). Let us still call $\varepsilon_1(n, i_0, \kappa)$ the universal constant defined in the Theorem 3.1.2. For any ε, $\delta > 0$ such that

$$\varepsilon < \min \left[\varepsilon_1(n, i_0, \kappa) \ ; \ \mathrm{inj}(Z, h) \ ; \ \tfrac{1}{\kappa_Z} \ ; \ \left(\tfrac{\mathrm{Vol}(Z,h)}{\mathrm{Vol}(\mathrm{B}^n, \mathrm{can})} \right)^{\frac{1}{n}} \ ; \ \tfrac{1}{\kappa_X} \left(\tfrac{\mathrm{Vol}(X,g)}{\mathrm{Vol}(Z,h)} \right)^{\frac{2}{n}} \left(\tfrac{\mathrm{Diam}(Z,h)}{\mathrm{Diam}(X,g)} \right)^2 \right]$$

$$\delta < \frac{1}{4} \min \left[\frac{\varepsilon^2}{4D^2} \ ; \ \mathrm{inj}(X, g) \ ; \ \frac{1}{\kappa_X} \right] \ ,$$

we have constructed above, on the connected sum $Y_\varepsilon = X^n \# Z^n$, a Riemannian metric g_ε whose volume satisfies, by the Lemma 3.4.6 (i),

$$\begin{aligned} \mathrm{Vol}(X \# Z, g_\varepsilon) &\leq \mathrm{Vol}(X, g) \cdot \left[1 + \frac{\mathrm{Vol}(Z, h)}{2^n \ \mathrm{Vol}(X, g)} \cdot \left(\frac{\varepsilon}{\mathrm{Diam}(Z, h)} \right)^n \right] \\ &\leq \left(1 - 10\, n \, (\kappa\, \varepsilon)^{\frac{1}{4}} \right)^{-1} \mathrm{Vol}(X, g) \end{aligned}$$

(where the last inequality can be deduced from the choice of ε). Moreover the Lemma 3.4.6 (ii) constructs a Gromov-Hausdorff $\left(\frac{\varepsilon}{2} + \pi\,\delta\right)$-approximation f_ε from $(Y_\varepsilon, g_\varepsilon)$ to (X, g)

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[21] In order to prove that f_ε is a contraction, it is sufficient to prove that $(f_\varepsilon^* g_0)_{|y} \leq g_{\varepsilon|y}$ for every $y \in Y_\varepsilon$. By the definition of the map f_ε and of the metric g_ε, we have $(f_\varepsilon^* g_0)_{|y} = g_{\varepsilon|y}$ when $y \in Y_0$ and $(f_\varepsilon^* g_0)_{|y} = 0 \leq g_{\varepsilon|y}$ when $y \in Y_2$. Since, by the Lemma 3.4.5, $\exp_{x_0}^{g_0}$ is an isometry from the euclidean ball of radius δ (in the tangent space $T_{x_0}X$) to $B_{g_0}(x_0, \delta)$ and since $(v, t) \mapsto (v, \delta - t)$ is an isometry of the cylinder $S_\delta^{n-1} \times [0, \delta]$, the definition of f_ε and the fact that $(v, t) \mapsto t \cdot \frac{v}{\delta}$ is a contraction from the cylinder $S_\delta^{n-1} \times [0, \delta]$ to the euclidean ball of radius δ prove that f_ε is a contraction when restricted to Y_1, and this ends the proof of the fact that f_ε is a contraction.

[22] The inequation 3.90 follows from the fact that, for every $y \in Y_1$, we have $d_{g_\varepsilon}(y, Y_0) \leq \delta < \frac{\varepsilon}{4}$ and from the fact that, when $y \in Y_2$, we have:

$$d_{g_\varepsilon}(y, Y_0) \leq d_{g_\varepsilon}(y, \partial B_{h_\varepsilon}(z_0, \delta)) + \delta \leq \mathrm{Diam}(Z, h_\varepsilon) \leq \frac{\varepsilon}{4} \ .$$

which is continuous of degree 1. By the assumptions on δ, f_ε is a Gromov-Hausdorff ε-approximation and thus the triple $[(X,g),\ (X\#Z,g_\varepsilon),\ f_\varepsilon]$ satisfies the assumptions of the Theorem 3.1.2 for any ε satisfying the above upper bound. □

3.4.2.3. *End of the proof of the Proposition 3.4.4.* Let (X^n,g_0) be any compact, connected, oriented Riemannian manifold, and let us denote $i_0 = \mathrm{inj}(X^n,g_0)$, $D = \mathrm{Diam}(X^n,g_0)$ and $|\sigma_X| \le \kappa_X^2$ (where σ_X is the sectional curvature of the metric g_0). We shall construct a Riemannian manifold by gluing (in the way exhibited in the beginning of the general proof above and in the Lemma 3.4.6) to (X^n,g_0) a sphere S^n endowed with a Riemannian metric $h_{i,r}$ defined in the following way: we start from the standard sphere (S^n,can) and we fix a point $z_0 \in S^n$, we consider a great circle (a geodesic) c of length 2π lying in the "equatorial sphere" *i.e.* in the set $\{z \in S^n \mid d_{\mathrm{can}}(z,z_0) = \frac{\pi}{2}\}$. Let us choose, on the great circle c, $(i+1)$ points x_0, ..., x_i such that $d_{\mathrm{can}}(x_j,x_{j+1}) = \frac{2\pi}{i+1}$, $\forall j = 0,..,i-1$, and $d_{\mathrm{can}}(x_0,x_i) = \frac{2\pi}{i+1}$. Then we have for any $r \in [0,\frac{1}{4(i+1)}]$ and for any $j,\ k \in \{0,..,i\}$ such that $j \ne k$, $B_{\mathrm{can}}(x_j,4r) \cap B_{\mathrm{can}}(x_k,4r) = \varnothing$.
Let us consider $S^n_{cut} = S^n \setminus \bigcup_{j=0}^{i} B_{\mathrm{can}}(x_j,r)$. We consider $i+1$ copies C_0, ..., C_i of the cylinder $[0,r^{1/3}] \times S^{n-1}_{\sin r}$ (where $S^{n-1}_{\sin r}$ denotes the standard euclidean sphere of radius $\sin r$) and $i+1$ copies (denoted by Z_0,.., Z_i) of $S^n \setminus B(z_0',r)$, where z_0' is a fixed point on S^n. We glue the cylinders C_0, ..., C_i to S^n_{cut} in the following way: for each value of $j \in \{0,1,..,i\}$, we glue the cylinder C_j to S^n_{cut} by means of an identification of the geodesic sphere $\partial B(x_j,r)$ with the submanifold $\{0\} \times S^{n-1}_{\sin r} \subset C_j$ (notice that this submanifold is isometric to $\partial B(x_j,r)$). At the other end of the cylinder C_j, we glue the sphere (with one hole) Z_j, identifying $\{r^{1/3}\} \times S^{n-1}_{\sin r}$ with $\partial B(z_0,r)$. We thus obtain a new Riemannian manifold $(Z,h_{i,r})$ which is a sphere with $i+1$ "mushrooms" and such that the point $z_0 \in S^n_{cut} \subset Z$ is far from each "mushroom". This new Riemannian manifold is diffeomorphic to S^n and verifies:

$$\mathrm{Diam}(Z,h_{i,r}) \le 3(\pi+1) \tag{3.91}$$

$$\mathrm{inj}_{z_0}(Z,h_{i,r}) \ge 1 \tag{3.92}$$

Now we proceed as in the Lemma 3.4.6[23]: for any ε such that $0 < \varepsilon < \frac{1}{2\pi}$ (hence $\varepsilon < \mathrm{inj}_{z_0}(Z,h_{i,r})$) we put $\alpha(\varepsilon) = \frac{\varepsilon}{12(\pi+1)}$ (in particular we have $\alpha(\varepsilon) \le \frac{\varepsilon}{4\,\mathrm{Diam}(Z,h_{i,r})}$) and we choose δ such that

$$0 < \delta < \frac{1}{4} \cdot \min\left\{ \frac{1}{\kappa_X}\ ;\ i_0\ ;\ \frac{\varepsilon^2}{6(\pi+1)} \right\}.$$

Hence, as we did in the Lemma 3.4.6 we do the connected sum $(Y_\varepsilon,g_\varepsilon)$, starting from (X^n,g_0) and $(Z^n,h_{i,r})$. We deduce that:

$$\mathrm{Vol}_{g_\varepsilon}(Y_\varepsilon) \le \mathrm{Vol}_{g_0}(X) + \frac{\mathrm{Vol}_{\mathrm{can}}(S^n)\,(i+1)}{12^n(\pi+1)^n} \cdot \varepsilon^n + n \cdot \mathrm{Vol}_{\mathrm{can}}(\mathrm{B}^n)\,\delta^n \le$$

$$\le \mathrm{Vol}_{g_0}(X) \cdot \left[1 + \frac{\mathrm{Vol}_{\mathrm{can}}(S^n)\,(i+2)}{12^n\,(\pi+1)^n\ \mathrm{Vol}_{g_0}(X)}\,\varepsilon^n \right]$$

Moreover, in the Lemma 3.4.6 we constructed a Gromov-Hausdorff $\left(\frac{\varepsilon}{2} + \pi\delta + 12\kappa_X^2\delta^3\right)$-approximation f_ε from $(Y_\varepsilon,g_\varepsilon)$ to (X,g_0). The choices made for ε and δ imply the existence of a $\varepsilon_2(n,i_0,\kappa_X,D,i)$ such that the triple $[(X^n,g_0),\ (Y^n_\varepsilon,g_\varepsilon),\ f_\varepsilon]$ verifies the assumptions of the Theorem 3.1.2.

[23]We have to remark at this point that the whole construction of the Lemma 3.4.6 is still valid if we fix *a priori* the two points $x_0 \in X$ and $z_0 \in Z$ where we perform the gluing, and if we replace $\mathrm{inj}(X,g)$, $\mathrm{inj}(Z,h)$ with $\mathrm{inj}_{x_0}(X,g)$ and $\mathrm{inj}_{z_0}(Z,h)$ respectively. In fact it is a local construction and we do not need an assumption on the global injectivity radius: it is sufficient to know the injectivity radii of the exponential maps of the metrics g and h at the center of the balls where we perform the gluing.

Let us denote by $f_0, ..., f_i$ the following functions:

$$f_j(y) = \begin{cases} 1 & \text{if } y \in Z_j \subset Z = Y_3 \ ; \\ \frac{t}{\delta} & \text{if } y = (t, v) \in C_j \subset Z = Y_3 \ ; \\ 0 & \text{if } y \in Z \setminus (C_j \cup Z_j) \ ; \\ 0 & \text{if } y \in Y_0 \cup Y_2 \ ; \end{cases}$$

Hence, taking a point $y \in C_j$, since $g_\varepsilon = \alpha(\varepsilon)^2 h_{i,r}$ in y, we have

$$\| \, df_j \, \|_{g_\varepsilon} (y) = \frac{1}{\alpha(\varepsilon)} \cdot \| \, df_j \, \|_{h_{i,r}} (y) = \frac{1}{\alpha(\varepsilon) \, r^{1/3}}$$

and

$$\int_{Y_\varepsilon} \| \, df_j \, \|^2_{g_\varepsilon} (y) \, dv_{g_\varepsilon}(y) = \frac{\alpha(\varepsilon)^{n-2}}{r^{\frac{2}{3}}} \ \mathrm{Vol}_{h_{i,r}}(C_j) = \frac{\alpha(\varepsilon)^{n-2}}{r^{\frac{1}{3}}} \cdot \omega_{n-1} \cdot (\sin r)^{n-1} \leq$$

$$\leq \alpha(\varepsilon)^{n-2} \cdot \omega_{n-1} \cdot r^{n - \frac{4}{3}}$$

On the other hand we have

$$\int_{Y_\varepsilon} f_j^2(y) \, dv_{g_\varepsilon}(y) \geq \alpha(\varepsilon)^n \ \mathrm{Vol}_{h_{i,r}}(Z_j) > \alpha(\varepsilon)^n \cdot \mathrm{Vol}_{\mathrm{can}}(\mathrm{B}^n) = \frac{\alpha(\varepsilon)^n}{n} \cdot \omega_{n-1}$$

hence the Rayleigh quotient can be written as follows:

$$R_{g_\varepsilon}(f_j) = \frac{\int_{Y_\varepsilon} \| \, df_j \, \|^2_{g_\varepsilon} \ dv_{g_\varepsilon}}{\int_{Y_\varepsilon} f_j^2 \, dv_{g_\varepsilon}} < \frac{n \, r^{n - \frac{4}{3}}}{\alpha(\varepsilon)^2}$$

For any ε we choose $r = r(\varepsilon)$ such that

$$r < \frac{\varepsilon^6}{(12 \, (\pi + 1) \, \sqrt{n})^3}$$

and we obtain that for all $j = 0, ..., i$, $R_{g_\varepsilon}(f_j) < \varepsilon^2$. Since the f_j's have disjoint supports we have that the same inequality is true for every function $f = \sum_{j=0}^{i} \alpha_j \, f_j$ of the $(i+1)-$dimensional subspace spanned by $f_0, \ldots, f_i$. By the Minimax Principle this proves that $\lambda_j(Y_\varepsilon, g_\varepsilon) \leq \varepsilon^2$ for any $j \leq i$. □

Appendix A. Stability of the geometric-arithmetic inequality.

Proposition A. 1. *Let A be a symmetric, non-negative matrix with real entries which satisfies the conditions:*

$$\det(A) \geq (1 - \sqrt{\eta})^2, \quad \mathrm{Tr}(A) \leq n(1 + \eta)^{\frac{2}{n}}$$

where $0 < \eta \leq \frac{1}{4}$. Then,

$$\| \, A - \mathrm{Id} \, \|^2 \leq 4(n-1)^2 \sqrt{\eta} \cdot \left(1 + \frac{n + 10}{n} \sqrt{\eta} \right)$$

Before giving the proof of the previous Proposition we state and prove the following Lemmas:

Lemma A. 2. *Let $(x_1, .., x_n) \in \mathbb{R}^n$ be such that $-1 < x_1 \leq x_2 \leq \cdots \leq x_n$ and such that $\sum_1^n x_i = 0$, then*

$$\prod_{i=1}^{n} (1 + x_i) \leq 1 - \frac{n}{2(n-1)} x_1^2 \leq 1 - \frac{\sum_1^n x_i^2}{2(n-1)^2}$$

Proof of the Lemma A.2. By assumption we have $x_1 \leq 0$ and $\sum_{i=2}^{n} x_i = -x_1 = |x_1|$; so the geometric-arithmetic inequality gives:

$$\prod_{i=2}^{n} (1 + x_i) \leq \left[\frac{1}{n-1} \sum_{i=2}^{n} (1 + x_i) \right]^{n-1} = \left(1 + \frac{|x_1|}{n-1} \right)^{n-1}$$

Hence we obtain the inequality:

$$\prod_{i=1}^{n} (1 + x_i) \leq \left(1 + \frac{|x_1|}{n-1} \right)^{n-1} (1 - |x_1|) \tag{3.93}$$

We now study the sign of

$$f(x) = \log \left[\frac{(1+x)^{n-1}(1-(n-1)x)}{\left(1 - \frac{n(n-1)}{2} x^2\right)} \right]$$

for $x \in \left[0, \frac{1}{n-1}\right)$. We have $f(0) = 0$ and, for $x \in \left(0, \frac{1}{n-1}\right)$,

$$\frac{1}{n(n-1)x} f'(x) = \frac{1}{nx} \cdot \left[\frac{1}{1+x} - \frac{1}{1-(n-1)x} + \frac{nx}{1 - \frac{n(n-1)}{2} x^2} \right] =$$

$$= \frac{1}{1 - \frac{n(n-1)}{2} x^2} - \frac{1}{(1+x)(1-(n-1)x)} =$$

$$= \frac{(n-2)(n-1)}{2} x \left(x - \frac{2}{n-1} \right) \frac{1}{\left(1 - \frac{n(n-1)}{2} x^2\right)(1+x)(1-(n-1)x)} \leq 0$$

it follows that $f(x) \leq 0$ for $x \in \left[0, \frac{1}{n-1}\right)$. If we replace x by $\frac{|x_1|}{(n-1)}$ we obtain:

$$\left(1 + \frac{|x_1|}{n-1}\right)^{n-1} (1 - |x_1|) \leq 1 - \frac{n(n-1)}{2} \left(\frac{|x_1|}{n-1} \right)^2$$

and putting this estimate in (3.93) we end the proof of the first inequality. To prove the second inequality let $x_1 = -a$; the problem is to find the maximum of $\sum_{i=1}^n x_i^2$ over the set

$$\left\{ (x_1, .., x_n) \in \mathbb{R}^n |\ x_1 = -a,\ \forall i\ x_i \geq -a,\ \sum x_i = 0 \right\}$$

We make the change of variables $y_i = x_i + a$, so that we have to look for the maximum of the function $h(y) = \sum_{i=1}^n (y_i - a)^2$ over the $(n-2)$-dimensional simplex

$$D = \left\{ y \,|\, \forall i\, y_i \geq 0,\ y_1 = 0,\ \sum y_i = na \right\}$$

Since h is convex the maximum must be attained at one vertex, so when all entries are zero except a single one, which means that there exists $i \in \{2, .., n\}$ such that;

- $\forall j \neq i,\ y_j = 0$;
- $y_i = na$

and we get:

$$\max_{y \in D} h(y) = (n-1)a^2 + (n-1)^2 a^2 = n(n-1)x_1^2$$

so that $\sum x_i^2 = h(y) \leq n(n-1)x_1^2$, and this proves the second inequality. □

Lemma A. 3 (Stability of the function $A \to \frac{\det(A)}{(\frac{1}{n}\operatorname{Tr}(A))^n}$ near its maximum). *For any real, symmetric, non-negative $(n \times n)$-matrix A we have $\forall \eta' \in (0,1)$*

$$1 - \eta' \leq \frac{\det(A)}{\left(\frac{1}{n}\operatorname{Tr}(A)\right)^n} \Rightarrow \| A - \frac{1}{n}\operatorname{Tr}(A) \cdot \mathrm{Id} \|^2 \leq 2(n-1)^2 \eta' \left(\frac{1}{n} \operatorname{Tr}(A) \right)^2$$

Proof of the Lemma A.3. Since $\det(A) \neq 0$ all the eigenvalues λ_i of A are strictly positive. Let $\bar{\lambda} = \frac{1}{n}\operatorname{Tr}(A) = \frac{1}{n}\sum_1^n \lambda_i$. As, by assumption, $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, we find

$$-1 \leq \frac{\lambda_1 - \bar{\lambda}}{\bar{\lambda}} \leq \frac{\lambda_2 - \bar{\lambda}}{\bar{\lambda}} \leq \cdots \leq \frac{\lambda_n - \bar{\lambda}}{\bar{\lambda}}$$

we can thus apply the Lemma A.2 which gives

$$(1 - \eta') \leq \prod_{i=1}^n \left(1 + \frac{\lambda_i - \bar{\lambda}}{\bar{\lambda}} \right) \leq 1 - \frac{1}{2(n-1)^2} \sum_{i=1}^n \frac{(\lambda_i - \bar{\lambda})^2}{\bar{\lambda}^2}$$

and thus $\sum_1^n (\lambda_i - \bar{\lambda})^2 \leq 2(n-1)^2 \eta' \cdot \bar{\lambda}^2$; but the $(\lambda_i - \bar{\lambda})$ are the eigenvalues of $A - \bar{\lambda}\,\mathrm{Id}$ thus, for any orthonormal basis $\{e_i\}$, we obtain:

$$\| A - \bar{\lambda}\,\mathrm{Id} \|^2 = \sum_{i=1}^n \langle (A - \bar{\lambda}\,\mathrm{Id})e_i, (A - \bar{\lambda}\,\mathrm{Id})e_i \rangle$$

where the previous inequality is independent from the particular orthonormal basis $\{e_i\}$; hence, choosing a orthonormal basis composed of eigenvectors, we obtain:

$$\| A - \bar{\lambda}\,\mathrm{Id} \|^2 = \sum_{i=1}^{n} (\lambda_i - \bar{\lambda})^2$$

which proves the Lemma. □

Proof of the Proposition A.1. Let $\bar{\lambda} = \frac{1}{n}\,\mathrm{Tr}(A)$; by the assumptions and the geometric-arithmetic inequality we have:

$$(1 - \sqrt{\eta})^{\frac{2}{n}} \leq [\det(A)]^{\frac{1}{n}} \leq \bar{\lambda} \leq (1 + \eta)^{\frac{2}{n}} \tag{3.94}$$

so that

$$\| (\bar{\lambda} - 1)\,\mathrm{Id} \|^2 = n(\bar{\lambda} - 1)^2 \leq n \left[1 - (1 - \sqrt{\eta})^{\frac{2}{n}} \right]^2 .$$

Since the function $f(x) = (1 - x)^{\frac{2}{n}}$ is concave, and by the mean value theorem, we have $f(0) - f(x) \leq |f'(x)|\,x$ and we deduce

$$\| \bar{\lambda}\,\mathrm{Id} - \mathrm{Id} \|^2 \leq \frac{4}{n}\,\eta\,(1 + 2\sqrt{\eta})^2 \leq \frac{16}{n}\eta \tag{3.95}$$

when $\eta \leq \frac{1}{4}$. By the assumptions of the Proposition A.1 we get the following inequalities:

$$\frac{\det(A)}{\left(\frac{1}{n}\,\mathrm{Tr}(A)\right)^n} \geq \frac{(1 - \sqrt{\eta})^2}{(1 + \eta)^2} = 1 - \eta' ,$$

where $\eta' = \sqrt{\eta} \cdot \left[\frac{2 - \sqrt{\eta} + \eta}{(1+\eta)^2} \right] \cdot (1 + \sqrt{\eta})$; by the Lemma A.3 we deduce that:

$$\| A - \bar{\lambda}\,\mathrm{Id} \|^2 \leq 2(n-1)^2\,\eta'\,\bar{\lambda}^2 \leq 4(n-1)^2 \sqrt{\eta}\,(1 + \sqrt{\eta})\,\bar{\lambda}^2 .$$

because $\frac{2 - \sqrt{\eta} + \eta}{(1+\eta)^2} \leq 2$. Using equation (3.94) we see that

$$\| A - \bar{\lambda}\,\mathrm{Id} \|^2 \leq 4(n-1)^2 \sqrt{\eta}\,(1 + \sqrt{\eta})\,(1 + \eta)^{\frac{4}{n}}$$

which implies, when $\eta \leq \frac{1}{4}$,

$$\| A - \bar{\lambda}\,\mathrm{Id} \|^2 \leq 4(n-1)^2 \sqrt{\eta} \left[1 + \left(\frac{n+2}{n} \right)^2 \sqrt{\eta} \right] \tag{3.96}$$

Since Id and $A - \bar{\lambda}$ are orthogonal we can use Pythagoras Theorem, and use the estimates (3.95), (3.96) and we obtain:

$$\| A - \mathrm{Id} \|^2 = \| A - \bar{\lambda}\,\mathrm{Id} \|^2 + \| \bar{\lambda}\,\mathrm{Id} - \mathrm{Id} \|^2 \leq 4(n-1)^2 \sqrt{\eta} \left(1 + \frac{n+10}{n} \sqrt{\eta} \right) . \;\square$$

Appendix B. Estimates for the function ξ.

LEMMA B. 1. *The infinite product* $\prod_{i=0}^{\infty} \left(1 + \frac{\beta^i}{\sqrt{2\beta^i - 1}}\, x \right)^{\beta^{-i}}$ *is converging for every* $x \in \mathbb{R}^+$ *and* $\beta = \frac{n}{n-2}$, *to a continuous function* $\xi(x)$ *which satisfies:*

$$\xi(x) \leq e^{\frac{n}{2}\left(\frac{x}{1+x}\right)} (1 + x)^{\frac{n}{2}}, \quad \forall x \geq 0.$$

Proof. We apply the equality $(1 + ax) = (1 + x)\left(1 + (a-1)\frac{x}{x+1} \right)$, which gives the following estimate:

$$(1 + ax) \leq (1 + x) \cdot e^{(a-1)\,\frac{x}{1+x}} \tag{3.97}$$

If we apply the estimate (3.97) for $a = \beta^{\frac{i}{2}}$ we obtain

$$1 + \frac{\beta^i}{\sqrt{2\beta^i - 1}} x \leq (1 + \beta^{\frac{i}{2}} x) \leq (1 + x) e^{\frac{x}{1+x}\,(\beta^{\frac{i}{2}} - 1)}$$

so we get for the infinite product:

$$\prod_{i=0}^{\infty}\left(1+\frac{\beta^i}{\sqrt{2\beta^i-1}}x\right)^{\beta^{-i}} \leq \left(\prod_{i=0}^{\infty}\left[(1+x)\,e^{-\frac{x}{1+x}}\right]^{\beta^{-i}}\right)\cdot\left(\prod_{i=0}^{\infty}e^{\frac{x}{1+x}\beta^{-\frac{i}{2}}}\right)=$$

$$=\left[(1+x)e^{-\frac{x}{1+x}}\right]^{\sum_{i=0}^{\infty}\frac{1}{\beta^i}}\cdot\exp\left(\frac{x}{1+x}\cdot\sum_{i=0}^{\infty}\frac{1}{\beta^{\frac{i}{2}}}\right)$$

since $\sum_0^\infty \frac{1}{\beta^i}=\frac{1}{1-\frac{1}{\beta}}=\frac{n}{2}$ (since we have chosen $\beta=\frac{n}{n-2}$) and

$$\sum_{i=0}^{\infty}\frac{1}{\beta^{\frac{i}{2}}}=\frac{1}{1-\frac{1}{\sqrt{\beta}}}=\frac{1+\frac{1}{\sqrt{\beta}}}{1-\frac{1}{\beta}}=\frac{n}{2}\left(1+\sqrt{\frac{n-2}{n}}\right)\leq n$$

we deduce that $\xi(x)\leq e^{\frac{n}{2}\left(\frac{x}{1+x}\right)}(1+x)^{\frac{n}{2}}$. □

The following estimates are useful in order to prove the sharpness of the upper bound of $N(\lambda)$ given in Corollary 3.2.9:

Lemma B. 2. (i) *for every* $x\in\mathbb{R}^+$, $\xi(x)\leq(1+x)^n$;
(ii) *for every* $x\in[1,+\infty)$, $\xi(x)\leq(4\,e)^{\frac{n}{4}}\,x^{\frac{n}{2}}$.

Proof of (i). Bounding from above the derivative of the exponential function, one obtains, for every $t\in\mathbb{R}^+$, $e^t-e^0\leq t\,e^t$ and thus:

$$e^{\frac{x}{1+x}}-1\leq\frac{1}{1+x}\,e^{\frac{x}{1+x}},$$

which leads to $e^{\frac{x}{1+x}}\leq(1+x)$. From this and from the Lemma B.1, we deduce that

$$\xi(x)\leq e^{\frac{n}{2}\left(\frac{x}{1+x}\right)}(1+x)^{\frac{n}{2}}\leq(1+x)^n.\quad\square$$

Proof of (ii). As $t\to\frac{e^t}{t}$ is decreasing on $[\frac{1}{2},1]$ it comes that

$$\frac{e^t}{t}\leq\frac{e^{\frac{1}{2}}}{\frac{1}{2}}=2\sqrt{e}\,.$$

When $x\geq 1$, then $\frac{x}{x+1}\in[\frac{1}{2},1)$ and thus

$$e^{\frac{x}{1+x}}\left(\frac{1+x}{x}\right)\leq 2\sqrt{e}.$$

A direct consequence is the estimate:

$$\xi(x)\leq e^{\frac{n}{2}\left(\frac{x}{1+x}\right)}(1+x)^{\frac{n}{2}}=\left[e^{\left(\frac{x}{1+x}\right)}\left(\frac{1+x}{x}\right)\right]^{\frac{n}{2}}x^{\frac{n}{2}}\leq(2\sqrt{e})^{\frac{n}{2}}\,x^{\frac{n}{2}}.\quad\square$$

Appendix C. A counterexample.

Lemma C. 1. *In general it is not possible to find a bound of the type:*

$$\sup_{f\in\mathcal{A}(\lambda)\setminus\{0\}}\left(\frac{\|\,\Delta f\,\|_p}{\|\,f\,\|_p}\right)\leq\lambda.$$

Proof. Let us consider, for instance, the case where $(M,g)=(\mathbb{S}^n,\mathrm{can})$. We will denote by Δ_{can} the corresponding Laplace-Beltrami operator. Let $\lambda=2(n+1)$, then $\mathcal{A}(\lambda)$ is the direct sum of the eigenspaces corresponding to of first three eigenvalues of Δ_{can} , *i.e.* $\mathcal{A}(\lambda)=E_0\oplus E_1\oplus E_2$ where:

- E_0 is the set of the constant functions, it is the eigenspace relative to the eigenvalue $\lambda_0=0$ and $\dim(E_0)=1$;
- E_1 is the space generated by $f_1,..,f_{n+1}$, where $f_i(x)=x_i$ (here x are the cartesian coordinates for $\mathbb{S}^n$). E_1 is the eigenspace corresponding to $\lambda_1=n$ and its dimension is $\dim(E_1)=n+1$;

- E_2 is the space generated by the functions of the form:
$$f : \mathbb{S}^n \to \mathbb{R}, \quad f(x) = Q(x)$$
where Q is a quadratic form with trace equal to zero. E_2 is the eigenspace corresponding to $\lambda_2 = 2(n+1)$ and
$$\dim(E_2) = \frac{(n+2)(n+1)}{2} - 1$$
($i.e.$ the dimension of the $(n+1) \times (n+1)$ symmetric matrices with trace equal to zero).

We can consider the functions:
$$\varphi_0(x) = -\left(\frac{n-1}{2(n+1)}\right), \ \varphi_0 \in E_0$$
$$\varphi_2(x) = \frac{1}{n+1}\left[nx_1^2 - x_2^2 - ... - x_{n+1}^2\right] = \frac{1}{(n+1)}\left[(n+1)x_1^2 - \sum_{i=1}^{n+1} x_i^2\right]$$
(hence, since we are restricted to $\mathbb{S}^n$, $\varphi_2(x) = x_1^2 - \frac{1}{(n+1)}$). We remark that $\varphi_2 \in E_2$ and that $\| \varphi_2 \|_\infty = \frac{n}{n+1}$. We define $u = \varphi_0 + \varphi_2$; then $u \in \mathcal{A}(\lambda)$ and
$$u(x) = x_1^2 - \frac{1}{(n+1)} - \frac{n-1}{2(n+1)} = x_1^2 - \frac{1}{2},$$
hence $\| u \|_\infty = \frac{1}{2}$. On the other hand $\Delta_{\text{can}} u = \Delta \varphi_2 = 2(n+1)\varphi_2$, so
$$\| \Delta_{\text{can}} u \|_\infty = 2(n+1) \| \varphi_2 \|_\infty = 2n$$
so we have:
$$\frac{\| \Delta_{\text{can}} u \|_\infty}{\| u \|_\infty} = 4n > \lambda = 2(n+1)$$
Since the ratios $\frac{\|\Delta_{\text{can}} u\|_{2k}}{\|u\|_{2k}}$ converge to $\frac{\|\Delta_{\text{can}} u\|_\infty}{\|u\|_\infty}$ when $k \to \infty$, there are infinite values of k for which $\frac{\|\Delta_{\text{can}} u\|_{2k}}{\|u\|_{2k}} > \lambda$. $\square$

Part 2

Margulis Lemma without curvature assumptions

CHAPTER 4

Margulis Lemma without curvature assumptions

Aperçu du chapitre 4 : *Dans ce chapitre, nous nous intéressons à un thème classique de la Géométrie Riemannienne: le célèbre Lemme de Margulis.*
Rappelons en effet que la "systole" (ou "systole globale") $\operatorname{sys}\pi_1(X)$ *d'une variété riemannienne quelconque* (X, g) *est définie comme l'infimum des longueurs de toutes les courbes fermées non homotopes à zéro sur* X *; introduisons la notion de "systole ponctuelle"* $\operatorname{syst}(x)$ *en un point* $x \in X$ *, définie comme l'infimum des longueurs des lacets non homotopes à zéro de point-base* x *.*
L'énoncé classique le plus simple du "Lemme" de Margulis (voir par exemple [**Bu-Za**]*) dit qu'on peut déterminer a priori des constantes universelles* $\varepsilon_0(n, K) > 0$ *et* $\varepsilon_1(n, K) > 0$ *telles que toute variété Riemannienne compacte* (X^n, g_0) *, dont la courbure sectionnelle* σ *vérifie* $-K^2 \leq \sigma < 0$ *, soit de volume minoré par* $\varepsilon_0(n, K) > 0$ *et vérifie*[1] $\operatorname{syst}(x) \geq \varepsilon_1(n, K)$ *en* **au moins un point** $x \in X$ *.*
Une autre version de ce Lemme (due à M. Gromov, voir [**Bu-Ka**]*) calcule une constante universelle* $\varepsilon_2(n, K, D) > 0$ *telle que toute variété Riemannienne compacte* (X^n, g_0) *dont la courbure sectionnelle vérifie* $-K^2 \leq \sigma < 0$ *et dont le diamètre est majoré par* D *soit de systole globale minorée*[2] *par* $\varepsilon_2(n, K, D)$ *.*

Ce résultat a été récemment généralisé par G. Besson, G. Courtois et S. Gallot ([**BCG3**]*): ils ont montré que les minorations de la systole obtenues dans les deux versions ci-dessus du Lemme de Margulis ne sont pas des propriétés dues à la géométrie de la variété* (X^n, g_0) *, mais des propriétés algébriques de son groupe fondamental, à tel point que ces minorations de la systole restent valables pour toute autre métrique riemannienne* g *sur la même variété* X *et, plus généralement, pour toute autre variété* Y *dont le groupe fondamental est isomorphe à un sous-groupe non abélien du groupe fondamental de* X *et pour toute métrique* g *sur* Y *dont l'entropie volumique est majorée*[3]*. L'hypothèse "entropie volumique majorée" s'avère beaucoup plus faible que l'hypothèse "courbure sectionnelle négative bornée" du Lemme de Margulis classique*[4]*, c'est pourquoi nous l'avons adoptée ici.*

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[1] En fait, formellement, l'énoncé classique du Lemme de Margulis dit que c'est le rayon d'injectivité $\operatorname{inj}(x)$ qui est minoré, en au moins un point x, par $\frac{\varepsilon_1(n,K)}{2}$, mais ceci est équivalent à l'énoncé donné ici car, en courbure négative ou nulle, $\operatorname{syst}(x) = 2 \operatorname{inj}(x)$.

[2] La conclusion du Lemme de Margulis classique est donc que $\sup_{x\in X} \operatorname{syst}(x) \geq \varepsilon_1(n, K)$, tandis que la conclusion de la version due à M. Gromov est que $\inf_{x\in X} \operatorname{syst}(x) \geq \varepsilon_2(n, K, D)$.

[3] En effet, par "minorations de la systole", il faut entendre que la minoration vaut après normalisation de la systole ponctuelle : cette normalisation est indispensable car, la systole étant multipliée par λ quand la métrique est multipliée par λ^2, il est impossible d'en trouver un minorant non trivial. Le Lemme de Margulis classique normalise la systole par la borne de courbure en minorant l'invariant $\max(|\sigma|^{\frac{1}{2}}).\operatorname{syst}(x)$. Dans [**BCG3**], G. Besson, G. Courtois et S. Gallot normalisent la systole par l'entropie volumique de la variété (Y, g) considérée et minorent l'invariant $\operatorname{Ent}(Y, g).\operatorname{syst}(x)$; c'est pour assurer cette normalisation que la majoration de l'entropie volumique est une hypothèse indispensable. Nous suivrons cet exemple en normalisant nous aussi par l'entropie volumique.

[4] En effet la connaissance d'un majorant de l'entropie volumique ne donne aucune information sur la géométrie locale, comme le prouve l'exemple 3.4.3, où l'adjonction de petits "mushrooms" a fortement perturbé la géométrie locale alors qu'elle a très peu perturbé l'entropie volumique, de même la connaissance d'un majorant de l'entropie volumique ne permet pas de limiter la topologie, comme le prouve l'exemple 3.4.1, qui montre que l'adjonction à X (par sommes connexes) de petits champignons $Z_1, \ldots, Z_k$, tous difféomorphes à un projectif complexe, perturbe fortement la topologie alors qu'elle perturbe très peu l'entropie volumique.

Dans ce chapitre, après avoir normalisé les deux notions de systole en les multipliant par un majorant de l'entropie volumique, nous allons établir une minoration universelle de la systole ponctuelle (en au moins un point, Théorème 4.1.1) et une minoration universelle de la systole globale (Théorème 4.1.2), valables pour toutes les variétés Riemanniennes connexes, dont le groupe fondamental est un produit libre non trivial et n'admet pas d'éléments de torsion d'ordre deux[5].
La principale différence entre nos résultats et les résultats antérieurs de G. Besson, G. Courtois et S. Gallot est que, chez eux, les propriétés algébriques du groupe fondamental de Y *qui assurent la minoration des systoles de toute métrique* g *d'entropie bornée sur* Y *sont héritées d'une représentation injective du groupe fondamental de* Y *dans le groupe fondamental d'une* **autre** *variété compacte de courbure négative et de systole globale minorée alors que, dans notre résultat, la propriété algébrique du groupe fondamental de* Y *qui assure la minoration des systoles de toute métrique* g *d'entropie bornée sur* Y *s'exprime directement sur le groupe fondamental de* Y *.*

Dans la section 4.1 nous introduisons le Lemme de Margulis et la généralisation proposée par G. Besson, G. Courtois et S. Gallot. Nous y présentons aussi nos propres résultats énoncés ci-dessus, ainsi que deux conséquences presque directes de notre minoration de la systole globale : un théorème de précompacité et de finitude et une minoration du volume sans hypothèse de courbure.
Dans la section 4.2 nous introduisons rapidement différentes notions d'entropie dont nous avons besoin dans ce chapitre, et nous présentons leurs propriétés.
Dans la section 4.3 nous donnons une preuve du Théorème 4.1.1; la démonstration procède par l'absurde en utilisant un argument de connexité; un rôle important est joué par le Théorème de Kurosh (un théorème de structure pour les sous groupes finiment engendrés d'un produit libre) et par l'existence d'un minorant universel de l'entropie algébrique des produits libres, à l'exception de $\mathbb{Z}_2 * \mathbb{Z}_2$.
Dans la section 4.4 nous donnons une démonstration du Théorème 4.1.2. Cette preuve est basée sur le calcul de l'entropie d'un groupe libre à deux générateurs doté d'une distance algébrique avec des longueurs d'arêtes inégales. Nous présenterons également les preuves des résultats de compacité, de finitude et de minoration du Volume obtenus comme applications du Théorème 4.1.2.
La section 4.5 est consacrée aux exemples et contrexemples qui prouvent l'optimalité de nos hypothèses.

Prospetto del capitolo 4: *In questo capitolo ci interessiamo ad un tema classico di Geometria Riemanniana: il celebre Lemma di Margulis.*
Ricordiamo che la "sistole" (o "sistole globale") $\operatorname{sys}\pi_1(X)$ *di una varietà Riemanniana qualunque* (X,g) *è definita come l'estremo inferiore delle lunghezze di tutte le curve chiuse non omotope a zero su* X *; intorduciamo la nozione di "sistole puntuale",* $\operatorname{syst}(x)$ *in un punto* $x \in X$ *, definito come l'estremo inferiore delle lunghezze dei lacci non nullomotopi di punto base* x *.*
L'enunciato classico e più semplice del "Lemma" di Margulis (vedere per esempio [**Bu-Za**]*) afferma che è possibile determinare a priori delle costanti universali* $\varepsilon_0(n,K) > 0$ *e* $\varepsilon_1(n,K) > 0$ *tali che ogni varietà Riemanniana compatta* (X^n, g_0) *, la cui curvatura sezionale* σ *verifichi* $-K^2 \le \sigma < 0$ *, abbia volume limitato inferiormente da* $\varepsilon_0(n,K)$ *e verifichi*[6] $\operatorname{syst}(x) \ge \varepsilon_1(n,K)$ **in almeno un punto** $x \in X$ *.*
Un'altra versione di questo Lemma (dovuta a M. Gromov, vedere [**Bu-Ka**]*) calcola una*

[5]Plus précisément, sur toute variété riemannienne (Y,g) (d'entropie volumique notée $\operatorname{Ent}(Y,g)$) dont le groupe fondamental est un produit libre non trivial et n'admet pas d'éléments de torsion d'ordre 2, ces deux résultats s'écrivent

$$\operatorname{Ent}(Y,g).\sup_{y\in Y}\operatorname{syst}(y) \ge \frac{\log(3)}{6} \quad \text{et} \quad \operatorname{Ent}(Y,g).\inf_{y\in Y}\operatorname{syst}(y) \ge \log\left(1 + \frac{4}{e^{2\,D.Ent(Y,g)} - 1}\right) ,$$

où la seconde inégalité exige deux hypothèses supplémentaires : le diamètre de la variété doit être majoré par D et le groupe fondamental doit être sans torsion.

[6]Formalmente l'enunciato classico del Lemma di Margulis fornisce una minorazione per il raggio di iniettività $\operatorname{inj}(x)$, in almeno un punto x , uguale a $\frac{\varepsilon_1(n,K)}{2}$, ma questo è equivalente all'enunciato da noi fornito, poiché in curvatura negativa o nulla vale l'uguaglianza $\operatorname{syst}(x) = 2 \cdot \operatorname{inj}(x)$.

costante universale $\varepsilon_2(n, K, D) > 0$ *tale che ogni varietà Riemanniana compatta* (X^n, g_0) *la cui curvatura sezionale verifica* $-K^2 \leq \sigma < 0$ *ed il cui diametro è maggiorato da* D *ha sistole globale limitata inferiormente*[7] *da* $\varepsilon_2(n, K, D)$ *.*

Questo risultato è stato recentemente generalizzato da G. Besson, G. Courtois e S. Gallot ([**BCG3**]*): essi hanno mostrato che i limiti inferiori per la sistole ottenuti nelle due versioni succitate del Lemma di Margulis non sono dovuti alla geometria della varietà* (X^n, g_0) *, bensí a delle proprietà algebriche del suo gruppo fondamentale, al punto che tali minorazioni restano valide per ogni altra metrica* g *su tale varietà* X *, e più in generale per ogni altra varietà* Y *il cui gruppo fondamentale ìsomorfo ad un sottogruppo non abeliano del gruppo fondamentale di* X *e per ogni metrica* g *su* Y *la cui entropia volumica è limitata superiormente*[8]*. L'ipotesi "entropia volumica maggiorata" si rivela essere decisamente più debole dell'ipotesi "curvatura sezionale negativa limitata" del Lemma di Margulis classico*[9]*, ed è per questo che abbiamo preferito tale ipotesi.*

In questo capitolo, dopo aver normalizzato le due nozioni di sistole attraverso la moltiplicazione per l'entropia volumica, stabiliremo una minorazione universale della sistole puntuale (in almeno un punto, Teorema 4.1.1) ed una minorazione universale della sistole globale (Teorema 4.1.2), valida per ogni varietà Riemanniana connessa il cui gruppo fondamentale è un prodotto libero non banale e non ammette elementi di torsione di ordine due[10]*.*
La principale differenza tra i nostri risultati e i risultati precedenti di G. Besson, G. Courtois e S. Gallot è che, nel loro caso, le proprietà algebriche del gruppo fondamentale che garantivano la minorazione della sisstole per ogni metrica g *su* Y *di entropia limitata sono eredità di una rappresentazione iniettiva del gruppo fondamentale di* Y *nel gruppo fondamentale* **di un'altra** *varietà compatta di curvatura negativa e di sistole globale limitata inferiormente, mentre, nel nostro risultato, la proprietà algebrica del gruppo fondamentale di* Y *che assicura la minorazione delle sistoli per ogni metrica* g *di entropia volumica limitata su* Y *si esprime direttamente sul gruppo fondamentale di* Y *.*

Nella sezione 4.1 introduciamo il Lemma di Margulis e la generalizzazione proposta da G. Besson, G. Courtois e S. Gallot. Presenteremo anche i nostri risultati sopra enunciati, e due conseguenze quasi dirette della minorazione della sistole globale: un teorema di precompattezza e finitezza ed una minorazione del volume senza ipotesi di curvatura.

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[7]La conclusione del Lemma di Margulis classico è dunque che $\sup_{x\in X} \text{syst}(x) \geq \varepsilon_1(n, K)$ mentre la versione dovuta a M. Gromov dimostra che $\inf_{x\in X} \text{syst}(x) \geq \varepsilon_2(n, K, D)$.

[8]In realtà, per "dare un minorante della sistole", bisogna intendere la minorazione della sistole puntuale come valida dopo la normalizzazione: in effetti questa normalizzazione è necessaria perché, dato che quando la metrica è moltiplicata per λ^2 la sistole puntuale è moltiplicata per λ , è impossibile trovare un minorante diverso da zero. Il Lemma di Margulis classico normalizza la sistole con un limite sulla curvatura, fornendo una minorazione per l'invariante $\max(|\sigma|^{\frac{1}{2}}) \cdot \text{syst}(x)$. In [**BCG3**], G. Besson, G. Courtois e S. Gallot normalizzano la sistole attraverso l'entropia volumica della varietà (Y, g) considerata e forniscono un minorante per l'invariante $\text{Ent}(Y, g) \cdot \text{syst}(y)$: è per assicurare questa normalizzazione che è necessario fare questa ipotesi di maggiorazione dell'entropia. Seguiremo questo esempio nei nostri risultati, assumendo l'entropia maggiorata da una costante.

[9]La conoscenza di un maggiorante per l'entropia volumica non fornisce alcuna informazione sulla geometria locale, come dimostra l'esempio 3.4.3 dove l'aggiunta di piccoli "mushrooms" perturba sensibilmente la geometria locale senza produrre cambiamenti rilevanti a livello dell'entropia volumica, allo stesso modo l'esistenza di una tale maggiorazione non impone dei vincoli a livello dellla topologia, come si può osservare attraverso l'esempio 3.4.1; tale esempio infatti mostra come aggiungendo ad X (attraverso somme connesse) dei piccoli funghi Z_1, ..., Z_k diffeomorfi ad un proiettivo complesso, perturba molto la topologia senza cambiare l'entropia volumica.

[10]Più precisamente, su ogni varietà Riemanniana (Y, g) (di entropia volumica denotata $\text{Ent}(Y, g)$) il cui gruppo fondamentale è un prodotto libero non banale privo di elementi di trosione di ordine due, questi due risultati si scrivono come segue:

$$\text{Ent}(Y, g) \cdot \sup_{y\in Y} \text{syst}(y) \geq \frac{\log(3)}{6} \quad \text{e} \quad \text{Ent}(Y, g) \cdot \inf_{y\in Y} \text{syst}(y) \geq \log\left(1 + \frac{4}{e^{2\,D\,\text{Ent}(Y,g)} - 1}\right)$$

dove la seconda disuguaglianza richiede due ipotesi supplementari: il diametro della varietà deve essere limitato superiormente da D e il gruppo fondamentale deve essere privo di torsione.

Nella sezione 4.2 introduciamo rapidamente differenti nozioni di entropia delle quali necessitiamo nel corso del capitolo, e presenteremo le loro proprietà.
Nella sezione 4.3 forniamo una dimostrazione del Teorema 4.1.1; la dimostrazione procede per assurdo sfruttando un argomento di connessione; un ruolo rilevante è giocato dal Teorema di Kurosh (un teorema di struttura per i sottogruppi finitamente generati di un prodotto libero) e dall'esistenza di una minorazione universale dell'entropia algebrica dei prodotti liberi, ad eccezione di $\mathbb{Z}_2 * \mathbb{Z}_2$.
Nella sezione 4.4 forniamo una dimostrazione del Teorema 4.1.2. Questa dimostrazione è basata sul calcolo dell'entropia di un gruppo libero a due generatori, dotato di una distanza algebrica con delle lunghezze per i lati diseguali. Presenteremo anche le dimostrazioni dei risultati di compattezza, finitezza e di minorazione del volume, ottenuti come conseguenze del Teorema 4.1.2.
La sezione 4.5 è dedicata agli esempi ed ai controesempi che dimostrano l'ottimalità delle nostre ipotesi.

Sketch of the chapter 4: *In this chapter we shall be concerned with a classical theme in Riemannian Geometry: the celebrated Margulis Lemma.*
We recall that the "systole" (or "global systole") $\mathrm{sys}\,\pi_1(X)$ *of any Riemannian manifold* (X, g), *is defined as the infimum of the lengths of the homotopically non trivial closed paths in* X; *we introduce the notion of "pointwise systole",* $\mathrm{syst}(x)$, *at the point* $x \in X$, *defined as the infimum of the lengths of the homotopically non trivial loops based at* x.
The classical (and simplest) statement of the Margulis Lemma (see for example [**Bu-Za**]*) says that it is possible to determine a priori two universal constants* $\varepsilon_0(n, K) > 0$ *and* $\varepsilon_1(n, K) > 0$ *such that any compact Riemannian manifold* (X^n, g_0), *whose sectional curvature* σ *verifies* $-K^2 \le \sigma < 0$, *has volume bounded below by* $\varepsilon_0(n, K)$ *and satisfies*[11] $\mathrm{syst}(x) \ge \varepsilon_1(n, K)$ **in at least one point** $x \in X$.
Another version of this Lemma (due to M. Gromov, see [**Bu-Ka**]*) gives a universal constant* $\varepsilon_2(n, K, D) > 0$ *such that any compact Riemannian manifold* (X^n, g_0) *whose sectional curvature verifies* $-K^2 \le \sigma < 0$ *and whose diameter is bounded above by* D *has global systole bounded below*[12] *by* $\varepsilon_2(n, K, D)$.

This result has been recently generalized by G. Besson, G. Courtois and S. Gallot ([**BCG3**]*): they have shown that the lower bounds for the systole obtained in the two aforementioned versions of the Margulis Lemma do not come from the geometry of the manifold* (X^n, g_0), *but from the algebraic properties of its fundamental group; in particular these lower bounds remain valid for any other metric* g *(of bounded volume entropy) on* X; *they also remain valid, on any other manifold* Y *whose fundamental group is isomorphic to a non abelian subgroup of the fundamental group of* X, *for any metric* g *on* Y *with bounded volume entropy*[13]. *The assumption "bounded volume entropy" is considerably weaker than the assumption "negative and bounded sectional curvature" of the classical Margulis Lemma*[14]

[11] To be more precise the classical statement of the Margulis Lemma gives a lower bound of the injectivity radius $\mathrm{inj}(x)$ (at at least one point $x \in X$), this lower bound being equal to $\frac{\varepsilon_1(n,K)}{2}$; however, this is equivalent to the statement that we give here, since every Riemannian manifold of non positive curvature verifies the equality: $\forall x \;\; \mathrm{syst}(x) = 2 \cdot \mathrm{inj}(x)$.

[12] The conclusion of the classical Margulis Lemma is thus that $\sup_{x\in X} \mathrm{syst}(x) \ge \varepsilon_1(n, K)$ whereas M. Gromov's version proves that $\inf_{x\in X} \mathrm{syst}(x) \ge \varepsilon_2(n, K, D)$.

[13] In fact, by "getting a lower bound of the systole" we mean that we provide such a bound after having **normalized the systole**: this normalization is necessary because, as $\mathrm{syst}(x)$ is multiplied by λ when the Riemannian metric is multiplied by λ^2, the infimum of $\mathrm{syst}(x)$ is zero in the absence of some normalization. On any Riemannian manifold (Y, g), the classical Margulis Lemma normalizes the systole by the sectional curvature of (Y, g), providing a lower bound for the invariant $\max(|\sigma|^{\frac{1}{2}}) \cdot \mathrm{syst}(y)$; in [**BCG3**] G. Besson, G. Courtois and S. Gallot normalize the systole by the volume entropy of (Y, g), i. e. they give a lower bound for the invariant $\mathrm{Ent}(Y, g) \cdot \mathrm{syst}(y)$; an equivalent normalization is obtained by assuming the entropy to be bounded. We shall choose the same normalization in our results.

[14] The weakness of the assumption "bounded volume entropy" is stressed by the fact that the assumption $\mathrm{Ent}(Y, g) \le 1$ (even when normalized by another assumption like "$Vol(Y, g)$ fixed" or "$\mathrm{Diam}(Y, g)$ fixed", or "$\mathrm{sys}\,\pi_1(Y)$ fixed") does not give any information about the local geometry of (Y, g), as it is shown in the Example 3.4.3, where we add small "mushrooms" to a fixed manifold: this

and this is the reason why we prefer this normalization (by an upper bound of the entropy) in our results.

In this chapter, after having normalized the two notions of systole by multiplication by the volume entropy, we shall establish a universal lower bound of the pointwise systole (in at least one point, Theorem 4.1.1) and a universal lower bound of the global systole (Theorem 4.1.2), which are valid for any connected Riemannian manifold whose fundamental group is a non trivial free product and does not admit torsion elements of order two[15]
The main difference between our results and the ones of G. Besson, G. Courtois and S. Gallot is that, in their case, the algebraic properties of the fundamental group which guarantee the existence of the lower bounds of the systoles (for any metric g on Y with bounded volume entropy) are inherited from an injective representation of the fundamental group of Y into the fundamental group of **another** *compact manifold, which admits a Riemannian metric with sectional curvature bounded above (by a strictly negative constant) and global systole bounded from below, whereas, in our result, the algebraic property of the fundamental group of Y (which guarantee that any metric g of bounded volume entropy on Y has its systoles bounded from below by an explicit constant) is expressed directly in terms of the fundamental group of Y.*

In section 4.1 we shall introduce briefly the classical Margulis Lemma and the generalization given by G. Besson, G. Courtois and S. Gallot. We shall state our aforementioned results and two quasi immediate consequences of the existence of a lower bound for the global systole: a precompactness and a finiteness theorem and a volume estimate without curvature assumptions.
In section 4.2 we introduce the notions of entropy that we shall need during this chapter and we shall present some of their properties.
In section 4.3 we shall give a proof of the minoration of the pointwise systole at at least one point (Theorem 4.1.1); the proof is by contradiction and it uses a connectedness argument; a key role is played by the Kurosh' Theorem (a structure theorem for finitely generated subgroups of a free product) and by the existence of a uniform lower bound for the algebraic entropy of the free products different from $\mathbb{Z}_2 * \mathbb{Z}_2$.
In section 4.4 we shall provide a proof of the minoration of the global systole (Theorem 4.1.2). This proof is based on the computation of the entropy of a free group with 2 generators endowed with different weights. We shall give as well the proofs of the compactness, finiteness results and of the volume estimate from below, which are obtained as consequences of Theorem 4.1.2.
Section 4.5 is devoted to examples and counterexamples which prove the sharpness of our assumptions.

4.1. Introduction

The celebrated *Margulis Lemma*, can be stated as follows:

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changes drastically the local geometry of the manifold without relevant effect on the values of the volume entropy, of the Volume, of the Diameter and of the systole. Similarly the assumption $\mathrm{Ent}(Y, g) \leq 1$ (even when normalized by another assumption like "$Vol(Y, g)$ fixed", or "$\mathrm{Diam}(Y, g)$ fixed", or "$\mathrm{sys}\, \pi_1(Y)$ fixed") does not imply any topological restriction on Y, as can be noticed by looking at the Example 3.4.1: this example shows that we can add, to a fixed manifold X, by connected sum, small mushrooms $Z_1, .., Z_k$, each of them diffeomorphic (for example) to a complex projective space, changing the topology without inducing great perturbations of the volume entropy, of the Volume, of the Diameter and of the systole.

[15]More precisely, on any Riemannian manifold (Y, g) (with volume entropy $\mathrm{Ent}(Y, g)$) whose fundamental group is a non trivial free product without torsion elements of order two, these results can be written as follows:

$$\mathrm{Ent}(Y, g) \cdot \sup_{y \in Y} \mathrm{syst}(y) \geq \frac{\log(3)}{6} \quad \text{and} \quad \mathrm{Ent}(Y, g) \cdot \inf_{y \in Y} \mathrm{syst}(y) \geq \log\left(1 + \frac{4}{e^{2\, D\, \mathrm{Ent}(Y,g)} - 1}\right)$$

where the second inequality requires two additional assumptions: the diameter of the manifold must be smaller than D and the fundamental group must be torsionless.

Margulis Lemma. *Let X be any compact Riemannian manifold of dimension $n \geq 2$, whose sectional curvature $\sigma(X)$ satisfies $-K^2 \leq \sigma(X) < 0$. Then:*

$$\sup_{x \in X} \mathrm{inj}_x(X) \geq \frac{C_2(n)}{K}, \quad \mathrm{Vol}(X) \geq \frac{C_1(n)}{K^n}$$

where $\mathrm{inj}_x(X)$ denotes the injectivity radius at x, and C_1, C_2 are two universal constants depending only on the dimension n.

G. Besson, G. Courtois and S. Gallot gave in [**BCG3**] a more general version of Margulis' result: they replaced the strong assumption on the sectional curvature, by an algebraic hypothesis on the fundamental group (δ-*non abélianité*) together with an upper bound of the volume-entropy to obtain a lower bound for $l_x(X)$, the length of the shortest non-nullhomotopic geodesic loop based at some point x. We call this invariant the *diastole* of X, $\mathrm{dias}(X) = \sup_x l_x(X)$ for easier reference throughout the chapter:

Corollaire 0.5 in [**BCG3**]. *Let $\delta, H > 0$. If X is any Riemannian manifold whose fundamental group Γ is δ-nonabelian, and such that the commutation relation is transitive on $\Gamma \setminus \{\mathrm{id}\}$, with $\mathrm{Ent}_{\mathrm{vol}}(X) \leq H$ we have:*

$$\mathrm{dias}(X) \geq \frac{\delta \log(2)}{4+\delta} \cdot \frac{1}{H}.$$

Replacing $\sup_{x \in X} \mathrm{inj}_x(X)$ with $\mathrm{dias}(X)$ is the price to pay for dropping the negative curvature assumption.

We will denote by $\mathrm{sys}\,\pi_1(X)$ the homotopy systole of a compact Riemannian manifold X, *i.e.* the length of the shortest non-contractible loop in X. We remark that if $\tilde{X}$ is the Riemannian universal covering of X and $\tilde{d}$ its distance function we have $\mathrm{sys}\,\pi_1(X) = \inf_{\tilde{x} \in \tilde{X}} \inf_{\gamma \in \pi_1(X) \setminus \{\mathrm{id}\}} \tilde{d}(\tilde{x}, \gamma\,\tilde{x})$ and $\mathrm{dias}(X) = \sup_{\tilde{x} \in \tilde{X}} \inf_{\gamma \in \pi_1(X) \setminus \{\mathrm{id}\}} \tilde{d}(\tilde{x}, \gamma\,\tilde{x})$
Following [**Wall**] we say that a fundamental group is decomposable if it is isomorphic to a non trivial free product. We will say that a discrete group Γ is without 2-torsion (or 2-torsionless) if there is no element $\gamma \in \Gamma$ such that $\gamma^2 = \mathrm{id}$.

The main results in this chapter are the following:

Theorem 4.1.1. *Let $H > 0$ and let X be a connected Riemannian n-manifold such that $\mathrm{Ent}(X) \leq H$, whose fundamental group is decomposable, without 2-torsion. Then:*

$$\mathrm{dias}(X) \geq \frac{\log(3)}{6H}.$$

Theorem 4.1.2. *Let $H, D > 0$. Let X be a compact Riemannian manifold such that $\mathrm{Ent}(X) \leq H$, $\mathrm{Diam}(X) \leq D$, whose fundamental group is decomposable and torsion-free. Then we have:*

$$\mathrm{sys}\,\pi_1(X) \geq \frac{1}{H} \cdot \log\left(1 + \frac{4}{e^{2DH} - 1}\right).$$

The first theorem is based on the *Kurosh subgroup theorem* (see [**Wall**], Theorem 3.1, pg. 151), the investigation of subgroups generated by "small elements" of $\pi_1(X)$ and a connectedness argument. The second one is a consequence of the *Kurosh subgroup theorem* and of an estimate of the entropy of the Cayley graph of a free group generated by two elements when the corresponding edges have two different lengths. We remark that in [**BCG3**] there is an analogous statement which is valid only for δ-thick groups (see §5). As a byproduct of the Theorem 4.1.2, using an argument of S. Sabourau ([**Sbr**], proof of the Theorem A), under some extra geometric assumption, we obtain a precompactness and homotopy finiteness theorem:

Precompactness Theorem. *Let $\mathfrak{M}_n^{dec}(D, V, H; l)$ denote the family of compact, Riemannian n-manifolds whose fundamental groups are decomposable and torsion-free, whose diameter, volume and volume-entropy are smaller than D, V, H respectively, and such that the length of the shortest geodesic loop in the universal covering is greater than l. This family is precompact with respect to the Gromov-Hausdorff topology.*
Moreover $\mathfrak{M}_n^{dec}(D, V, H; l)$ is finite up to:

(i) *homotopy, for all $n \in \mathbb{N}$;*

(ii) *homeomorphism, for* $n = 4$ *;*
(iii) *diffeomorphism, for* $n \geq 5$ *.*

Moreover, combining the Theorem 4.1.2 and the celebrated Isosystolic Inequality of Michael Gromov ([**Gro5**], Theorem 0.1.A) we will prove a volume estimate, without curvature assumptions, for a certain class of Riemannian manifolds:

VOLUME ESTIMATE. *For any connected and compact, 1-essential Riemannian n-manifold, X with decomposable, torsion-free fundamental group and whose volume-entropy and diameter are bounded above by* $H, D > 0$ *respectively, we have the following estimate:*

$$\mathrm{Vol}(X) \geq \frac{C_n}{H^n} \cdot \log\left(1 + \frac{4}{e^{2DH} - 1}\right)^n$$

where $C_n > 0$ *is a universal constant depending only on the dimension n (an explicit -although not optimal- upper bound to* C_n *can be found in* [**Gro5**]*, Theorem 0.1.A).*

In section 4.3 we recall some basic facts about entropy. Section 4.4 is devoted to the proof of the Theorem 4.1.1, while in section 4.5 we give the proof of the Theorem 4.1.2, with some applications. In section 4.6 we give examples showing that the class of manifolds covered by the Theorem 4.1.1 is distinct from the class considered in [**BCG3**], Corollaire 0.5 (and in particular is orthogonal to the class of δ-thick groups, cf. section 5). We also produce counterexamples showing that the torsion-free assumption in the Theorem 4.1.2 cannot be dropped: namely, we construct a manifold X with $\pi_1(X) = \mathbb{Z}_p * G$, for a non trivial group G, and a sequence of metrics with diameter and volume entropy bounded from above, whose homotopy systole tends to zero.

4.2. Notations and background

DEFINITION 4.2.1. Let (Γ, d) be a metric discrete group, *i.e.* a group Γ endowed with a left invariant distance such that $\#\{\gamma \mid d(\gamma, \mathrm{id}) < R\} < +\infty$, $\forall R > 0$ (we call such a distance an admissible distance). We define the entropy of the metric discrete group (Γ, d), $\mathrm{Ent}(\Gamma, d)$ as:

$$\mathrm{Ent}(\Gamma, d) = \lim_{R \to \infty} \frac{1}{R} \log(\#\{\delta \mid d(\gamma, \delta) < R\})$$

This limit exists and does not depend on the element γ.

REMARK 4.2.2. We are interested in two different kinds of admissible distances on Γ. If Γ is a finitely generated group and Σ is a finite generating set we denote by d_Σ the algebraic distance on Γ associated to Σ. If Γ is the fundamental group of a Riemannian manifold X, for any point $\tilde{x}$ in the Riemannian universal covering $\tilde{X}$, we define the admissible distance d_{geo} on Γ by: $d_{geo}(\gamma, \delta) = \tilde{d}(\gamma\,\tilde{x}, \delta\,\tilde{x})$ where $\tilde{d}$ is the Riemannian distance on $\tilde{X}$.

DEFINITION 4.2.3. Let Γ be a discrete, finitely generated group. The algebraic entropy of Γ is $\mathrm{Ent}_{\mathrm{alg}}(\Gamma) = \inf_\Sigma \mathrm{Ent}(\Gamma, d_\Sigma)$, where the inf is taken over the finite generating sets of Γ.

DEFINITION 4.2.4. Let X be any Riemannian manifold. Its entropy is defined as:

$$\mathrm{Ent}(X) = \liminf_{R \to \infty} \frac{1}{R} \log(\mathrm{Vol}(B(\tilde{x}, R)))$$

where $B(\tilde{x}, R)$ denotes the geodesic ball of radius R and centered at $\tilde{x}$ in the Riemannian universal covering $\tilde{X}$ (this definition does not depend on the choice of the point $\tilde{x}$). When X is compact, the above inf lim is a full limit, and this limit is (by definition) the volume-entropy of the Riemannian manifold X.

NOTATION 1. When we need to stress the dependence of $\mathrm{Ent}(X)$ from the Riemannian metric g on X we use the notation $\mathrm{Ent}(X, g)$ (or $\mathrm{Ent}_{\mathrm{vol}}(X, g)$, in the compact case).

We shall use the following basic properties of the entropy:

(0) When X is a Riemannian manifold and Γ its fundamental group, then: $\mathrm{Ent}(X) \geq \mathrm{Ent}(\Gamma, d_{geo})$ ([**BCG3**], Lemma 2.3). Equality holds when X is compact (see [**Rob**], Proposition 1.4.7).
(1) Let $d_1 \leq d_2$ be two admissible distances on Γ, then we have: $\mathrm{Ent}(\Gamma, d_1) \geq \mathrm{Ent}(\Gamma, d_2)$.
(2) Let d be an admissible distance on Γ and let $\lambda > 0$, then we have: $\mathrm{Ent}(\Gamma, \lambda d) = \frac{1}{\lambda}\,\mathrm{Ent}(\Gamma, d)$.

4.3. Proof of the Theorem 4.1.1

The proof of the Theorem is by contradiction and essentially relies on the following results:

i) A structure theorem for finitely generated subgroups of free products (the well known *Kurosh subgroup theorem*).
ii) The existence of a universal lower bound for the algebraic entropy of nontrivial free products $\neq \mathbb{Z}_2 * \mathbb{Z}_2$ ([**delaH**], §VII.18):

$$\mathrm{Ent}_{\mathrm{alg}}(A * B) \geq \frac{\log(3)}{6} \tag{4.98}$$

iii) The comparison between entropies of X and $(\pi_1(X), d_{geo})$ (see property (0) of the entropy of a metric discrete group).

We recall that $\pi_1(X)$ is decomposable and 2-torsionless. Let $l_0 = \frac{\log(3)}{6H}$ and define the following family of sets:

$$\mathcal{I}(\tilde{x}, l_0) = \{\gamma \in A * B \setminus \{\mathrm{id}\} \mid \tilde{d}(\tilde{x}, \gamma\tilde{x}) < l_0\}, \quad \forall \tilde{x} \in \tilde{X}.$$

Since $A * B$ acts by isometries on $\tilde{X}$ and the action is free and properly discontinuous, all these sets are finite. Moreover we underline the fact that they are symmetric (*i.e.* if $\gamma \in \mathcal{I}(\tilde{x}, l_0)$, then $\gamma^{-1} \in \mathcal{I}(\tilde{x}, l_0)$).

4.3.1. Three Lemmas. We will resume in the following three Lemmas the principal properties of the sets $\mathcal{I}(\tilde{x}, l_0)$:

LEMMA A. *For any* $\tilde{x} \in \tilde{X}$, *then* $\mathcal{I}(\tilde{x}, l_0)$

(i) *either is included in* $\gamma_{\tilde{x}} A \gamma_{\tilde{x}}^{-1}$, *for at least one* $\gamma_{\tilde{x}} \in A * B$;
(ii) *or is included in* $\gamma_{\tilde{x}} B \gamma_{\tilde{x}}^{-1}$, *for at least one* $\gamma_{\tilde{x}} \in A * B$;
(iii) *or* $\langle \mathcal{I}(\tilde{x}, l_0)\rangle \cong \mathbb{Z}$ *and does not satisfy* (i) *or* (ii).

Proof. Assume that conditions (i), (ii), (iii) are not verified; by the Kurosh Subgroup Theorem we know that the subgroup generated by $\mathcal{I}(\tilde{x}, l_0)$ writes

$$\langle \mathcal{I}(\tilde{x}, l_0(H))\rangle = C_1 * \cdots * C_k * \gamma_1 A_1 \gamma_1^{-1} * \cdots * \gamma_r A_r \gamma_r^{-1} * \delta_1 B_1 \delta_1^{-1} * \cdots * \delta_s B_s \delta_s^{-1}$$

where the C_i's are infinite cyclic subgroups of $A * B$ which are not contained in any conjugate of A or B, where $\gamma_i \neq \gamma_j$ and $\delta_i \neq \delta_j$ for $i \neq j$, and where A_j, B_i are respectively subgroups of A, B. Since conditions (i), (ii), (iii) are not verified there should be at least two factors giving a nontrivial free product $\neq \mathbb{Z}_2 * \mathbb{Z}_2$, hence by estimate 4.98 we should have $\mathrm{Ent}_{\mathrm{alg}}(\langle \mathcal{I}(\tilde{x}, l_0)\rangle) \geq \log(3)/6$. On the other hand, by the triangle inequality, we have the inequality $d_{\mathcal{I}(\tilde{x}, l_0)} \cdot l_0 > d_{geo}$, which is valid on $\langle \mathcal{I}(\tilde{x}, l_0)\rangle$; hence using properties (0), (1) and (2) of the entropy and the upper bound on the entropy of X we prove that

$$\mathrm{Ent}(\langle \mathcal{I}(\tilde{x}, l_0)\rangle, d_{\mathcal{I}(\tilde{x}, l_0)}) < \mathrm{Ent}\left(\langle \mathcal{I}(\tilde{x}, l_0)\rangle, \frac{1}{l_0} \cdot d_{geo}\right) =$$

$$= l_0 \cdot \mathrm{Ent}(\langle \mathcal{I}(\tilde{x}, l_0)\rangle, d_{geo}) \leq H \cdot l_0 = \frac{\log(3)}{6}$$

which contradicts estimate (4.98). □

LEMMA B. *For all* $\tilde{x} \in \tilde{X}$ *there exists an* $\varepsilon = \varepsilon(\tilde{x})$ *such that, for any* $\tilde{x}'$, *if* $\tilde{d}(\tilde{x}', \tilde{x}) < \varepsilon$ *the following inclusion holds:* $\mathcal{I}(\tilde{x}, l_0) \subseteq \mathcal{I}(\tilde{x}', l_0)$.

Proof. Let us fix $\varepsilon < \frac{1}{2}\left[l_0 - \sup_{\gamma\in\mathcal{I}(\tilde{x},l_0)} \tilde{d}(\tilde{x},\gamma\tilde{x})\right]$ then by the triangular inequality we get the inclusion. □

LEMMA C. *For all* $\tilde{x} \in \tilde{X}$ *and for all* $\gamma \in \Gamma$ *the following equality holds:*

$$\gamma\mathcal{I}(\tilde{x},l_0)\gamma^{-1} = \mathcal{I}(\gamma(\tilde{x}),l_0).$$

Proof. Let $\delta \in \mathcal{I}(\tilde{x},l_0)$; then $\gamma\delta\gamma^{-1}$ satisfies the inequality:

$$\tilde{d}(\gamma\tilde{x},\gamma\delta\gamma^{-1}\cdot\gamma\tilde{x}) = \tilde{d}(\gamma\tilde{x},\gamma\delta\tilde{x}) = \tilde{d}(\tilde{x},\delta\tilde{x}) < l_0$$

hence $\gamma\delta\gamma^{-1} \in \mathcal{I}(\gamma\tilde{x},l_0)$. To obtain the reverse inclusion suppose to have $\sigma \in \mathcal{I}(\gamma\tilde{x},l_0)$, proceeding as before we obtain $\gamma^{-1}\sigma\gamma \in \mathcal{I}(\tilde{x},l_0)$; hence σ is a γ-conjugate of an element in $\mathcal{I}(\tilde{x},l_0)$. □

4.3.2. End of the proof. *We assume that* $\mathcal{I}(\tilde{x},l_0) \neq \varnothing$, for all $\tilde{x} \in \tilde{X}$ and we shall show that this leads to a contradiction. Let us now define the following sets:

- $\tilde{X}_1 = \{\tilde{x} \in \tilde{X} \mid \exists\gamma \in A*B \text{ such that } \mathcal{I}(\tilde{x},l_0) \subseteq \gamma A\gamma^{-1}\}$;
- $\tilde{X}_2 = \{\tilde{x} \in \tilde{X} \mid \exists\gamma \in A*B \text{ such that } \mathcal{I}(\tilde{x},l_0) \subseteq \gamma B\gamma^{-1}\}$;
- $\tilde{X}_3 = \{\tilde{x} \in \tilde{X} \setminus (\tilde{X}_1 \cup \tilde{X}_2) \mid \exists\tau \in A*B \text{ such that } \langle\mathcal{I}(\tilde{x},l_0)\rangle = \langle\tau\rangle\}$;

the next lemma enlightens some key properties of these sets:

LEMMA 4.3.1. *The* $\tilde{X}_i$*'s are open and disjoint.*

Proof. $\tilde{X}_3$ is disjoint from $\tilde{X}_1$ and $\tilde{X}_2$ by definition, whereas $\tilde{X}_1$ and $\tilde{X}_2$ are disjoint since $\mathcal{I}(\tilde{x},l_0) \neq \varnothing$, $\mathrm{id} \notin \mathcal{I}(\tilde{x},l_0)$ and

$$\gamma A\gamma^{-1} \cap \delta B\delta^{-1} = \{\mathrm{id}\} \ \ \forall\gamma,\delta \in A*B.$$

Now we will prove that $\tilde{X}_i$ is open. Let us take a point $\tilde{x}$ in $\tilde{X}_i$; by the Lemma B, for $\tilde{x}'$ in an open neighbourhood of $\tilde{x}$ we have the inclusion, $\mathcal{I}(\tilde{x},l_0) \subseteq \mathcal{I}(\tilde{x}',l_0)$. Now, by the Lemma A, for $\tilde{x}'$ one condition between (i), (ii) and (iii) should hold, *i.e.* $\exists j \in \{1, 2, 3\}$ such that $\tilde{x}' \in \tilde{X}_j$. As the $\tilde{X}_i$ are disjoint, by the inclusion above it follows that if $\tilde{x} \in \tilde{X}_i$, then also $\tilde{x}' \in \tilde{X}_i$. Hence the $\tilde{X}_i$'s are open. □

Since the $\tilde{X}_i$'s are open and disjoint subsets of $\tilde{X}$ and $\tilde{X}$ is connected one of the following conditions should hold:

(1) $\tilde{X} = \tilde{X}_1$;
(2) $\tilde{X} = \tilde{X}_2$;
(3) $\tilde{X} = \tilde{X}_3$;

we will now show that each of these conditions leads to a contradiction.

CASE (1). We shall prove that there exists γ_0, independent from $\tilde{x}$ such that all the sets $\mathcal{I}(\tilde{x},l_0)$ belong to the same conjugate $\gamma_0 A\gamma_0^{-1}$ of A in $A*B$. For each fixed $\hat{\gamma} \in (A*B)/A$ we define the subset of $\tilde{X}_1$:

$$\tilde{X}_1(\hat{\gamma}) = \{\tilde{x} \in \tilde{X} \mid \exists\gamma \in \hat{\gamma} \text{ such that } \mathcal{I}(\tilde{x},l_0) \subseteq \gamma A\gamma^{-1}\}$$

and we remark that since $\tilde{X} = \tilde{X}_1$, we have $\tilde{X} = \cup_{\hat{\gamma}}\tilde{X}_1(\hat{\gamma})$. The sets $\tilde{X}_1(\hat{\gamma})$ are disjoint: the proof is analogous to the proof of the Lemma 4.3.1. Moreover every $\tilde{X}_1(\hat{\gamma})$ is open: let $\tilde{x} \in \tilde{X}_1(\hat{\gamma})$ and consider a $\tilde{x}'$ at distance $\tilde{d}(\tilde{x},\tilde{x}') < \varepsilon$, where ε is chosen as in the Lemma B. By the Lemma B we know that $\mathcal{I}(\tilde{x},l_0) \subseteq \mathcal{I}(\tilde{x}',l_0)$ so if $\tilde{x} \in \tilde{X}_1(\hat{\gamma})$ since by assumption $\mathcal{I}(\tilde{x},l_0) \neq \varnothing$ then also $\tilde{x}' \in \tilde{X}_1(\hat{\gamma})$, because $\tilde{X}_1(\hat{\gamma})$ and $\tilde{X}_1(\hat{\gamma}')$ are disjoint if $\hat{\gamma} \neq \hat{\gamma}'$. Hence $\tilde{X}$ is covered by the family of disjoint, open sets $\tilde{X}_1(\hat{\gamma})_{\hat{\gamma}\in(A*B)/A}$, and by connectedness of $\tilde{X}$, there exists one $\hat{\gamma}_0 \in (A*B)/A$ such that $\tilde{X} = \tilde{X}_1(\hat{\gamma}_0)$. So there exists $\gamma_0 \in A*B$ such that every subset $\mathcal{I}(\tilde{x},l_0)$ is included in some $\gamma A\gamma^{-1}$ for $\gamma \in \hat{\gamma}_0 = \gamma_0 A$ (*i.e.* in $\gamma_0 A\gamma_0^{-1}$). It follows that the whole subgroup $G = \langle\mathcal{I}(\tilde{x},l_0)\rangle_{\tilde{x}\in\tilde{X}}$ is included in $\gamma_0 A\gamma_0^{-1}$. By construction of G and by the Lemma C, G should be a normal subgroup of $A*B$, and this is a contradiction, as no normal subgroup of a nontrivial free product is included in a conjugate of one factor.

Case (2). Proof of Case (2) is analogous to Case (1).

Case (3). Let $T \subset A*B$ be the subset of primitive elements of $A*B$ [16] with infinite order, which are not contained in any conjugate of A or B. Similarly to (1) we will show that there exists $\tau_0 \in T$ such that $\langle \mathcal{I}(\tilde{x}, l_0)\rangle \subset \langle \tau_0 \rangle$ for all $\tilde{x} \in \tilde{X}$. For each $\tau \in T$ let

$$\tilde{X}_3(\tau) = \tilde{X}_3(\tau^{-1}) = \{\tilde{x} \mid \exists k \in \mathbb{Z} \text{ such that } \langle \mathcal{I}(\tilde{x}, l_0)\rangle = \langle \tau^k \rangle\}$$

We want to show that $\tilde{X} = \bigcup_{\tau \in T} \tilde{X}_3(\tau)$; by assumption every subgroup $\langle \mathcal{I}(\tilde{x}, l_0)\rangle$ is isomorphic to an infinite cyclic subrgroup $\langle \gamma \rangle$ hence it is sufficient to show that for any element γ in $A*B$ there exists an element $\tau \in T$ and $k \in \mathbb{Z}$ such that $\gamma = \tau^k$. We argue by contradiction: assume that there is an element γ, which cannot be written as a power of a primitive element $\tau \in T$; by definition there exists a sequence of elements $\{\gamma_n\}_{n\in\mathbb{N}}$ and a sequence of integers $\{p_n\}_{n\in\mathbb{N}}$ such that $\gamma_0 = \gamma$ and $\gamma_i = (\gamma_{i+1})^{p_{i+1}}$ (with $|\prod_1^i p_j| \to \infty$). Let Σ be any generating system and $d_0 = d_\Sigma(\gamma, \mathrm{id})$; for any $i \in \mathbb{N}$ consider δ_i a cyclically reduced word associated to γ_i and let $N_1(i) = l_\Sigma(\gamma_i) - l_\Sigma(\delta_i)$, $N_2(i) = l_\Sigma(\delta_i)$. Observe that since $\gamma \neq \mathrm{id}$ we shall have $N_2(i) \geq 1$. Then $d_\Sigma(\gamma_i^{p_i}, \mathrm{id}) = N_1(i) + |p_i| \cdot N_2(i)$ so that:

$$d_0 = d_\Sigma(\gamma_i^{\prod_1^i p_j}, \mathrm{id}) \geq |\prod_1^i p_j| \cdot N_2(i) \geq |\prod_1^i p_j|$$

which gives a contradiction for $i \to \infty$.
Let us show that the sets $\tilde{X}_3(\tau)$ are disjoint. Actually assume that $\tilde{x} \in \tilde{X}_3(\tau) \cap \tilde{X}_3(\tau')$, for $\tau' \neq \tau^{\pm 1}$; then there exists k, k' such that $\sigma = \tau^k = (\tau')^{k'}$ generates $\langle \mathcal{I}(\tilde{x}, l_0)\rangle$. The following lemma shows that one among τ and τ' is not primitive, a contradiction:

Lemma 4.3.2 (Primitive powers in free products). *Let $\gamma, \gamma' \in A*B \setminus \{\mathrm{id}\}$ be such that $\gamma^s = (\gamma')^{s'}$ ($s, s' \in \mathbb{N}$), and assume that they are not contained in any conjugate of A or B. Then there exists an element $\tau \in A*B$ and $q, q' \in \mathbb{N}$ such that $\gamma = \tau^q$, $\gamma' = \tau^{q'}$. In particular if γ, γ' are primitive elements then $q, q' \in \{-1, 1\}$ and $\gamma = (\gamma')^{\pm 1}$.*

The proof of this Lemma is rather simple but tedious and will be given in the Appendix. We now prove that the sets $\tilde{X}_3(\tau)$ are open. Let $\tilde{x} \in \tilde{X}_3(\tau)$: we know by the Lemma B that, for all $\tilde{x}'$ sufficiently close to $\tilde{x}$ we have $\mathcal{I}(\tilde{x}, l_0) \subseteq \mathcal{I}(\tilde{x}', l_0)$; if $\tilde{x}' \notin \tilde{X}_3(\tau)$ then

$$\langle (\tau')^{k'} \rangle = \langle \mathcal{I}(\tilde{x}', l_0)\rangle \supset \langle \mathcal{I}(\tilde{x}, l_0)\rangle = \langle \tau^k \rangle$$

for some $k, k' \in \mathbb{Z}$ and $\tau' \in T$ different from $\tau^{\pm 1}$. Again the Lemma 4.3.2 implies that τ' or τ is not primitive, a contradiction. It follows, by connectedness, that $\tilde{X} = \tilde{X}_3(\tau)$ for some fixed $\tau \in T$. Therefore, the group $\langle \tau \rangle$ (that contains $\langle \mathcal{I}(\tilde{x}, l_0)\rangle$, for any $\tilde{x} \in \tilde{X}$) is a normal subgroup: in fact, for any $\gamma \in A*B$ there exists k, k' such that $\langle \mathcal{I}(\gamma\,\tilde{x}, l_0)\rangle = \langle \tau^k \rangle = \langle \gamma \tau^{k'} \gamma^{-1} \rangle$; since τ and $\gamma\tau\gamma^{-1}$ are both primitive elements it follows that $\gamma\tau\gamma^{-1} = \tau^{\pm 1}$. Hence $\langle \tau \rangle$ is an infinite cyclic subgroup of $A*B$, which is normal. This is not possible since no free product different from $\mathbb{Z}_2 * \mathbb{Z}_2$ admits an infinite cyclic normal subgroup[17]. This excludes also Case (3).

Therefore $\mathcal{I}(\tilde{x}, l_0) = \varnothing$ for some $\tilde{x} \in \tilde{X}$, which proves the Theorem 4.1.1. □

4.4. Proof of the Theorem 4.1.2 and Applications

4.4.1. Proof of the Theorem 4.1.2. Let X be a compact Riemannian manifold with decomposable, torsion free fundamental group and assume the bounds $\mathrm{Ent}(X) \leq H$, $\mathrm{Diam}(X) \leq D$. The proof relies on the following Lemma:

Lemma 4.4.1. *Let $\mathcal{C}(\Gamma, \{\gamma_1, \gamma_2\})$ be the Cayley graph of a free group with two generators γ_1, γ_2. Let d_l be the left invariant distance on $\mathcal{C}(\Gamma, \{\gamma_1, \gamma_2\})$, defined by the conditions $d_l(\mathrm{id}, \gamma_1) = l(\gamma_1)$ and $d_l(\mathrm{id}, \gamma_2) = l(\gamma_2)$. Then $h = \mathrm{Ent}(\Gamma, d_l)$ solves the equation:*

$$(e^{h \cdot l(\gamma_1)} - 1)(e^{h \cdot l(\gamma_2)} - 1) = 4 \tag{4.99}$$

[16] *i.e.* elements which cannot be written as powers of any other element in $A*B$.
[17] In fact by [**Wall**], Theorem 3.11, p. 160, every normal subgroup in $A*B$ must have finite index, and an infinite cyclic group in $A*B$ can have finite index if and only if $A*B = \mathbb{Z}_2 * \mathbb{Z}_2$.

Proof. Let $I(c) = \sum_{\gamma\in\Gamma} e^{-c\, d_l(\mathrm{id},\gamma)}$. Since $l(\gamma_1)$ and $l(\gamma_2)$ are strictly positive, the entropy of (Γ, d_l) is finite (but not necessarily bounded independently from $l(\gamma_1)$, $l(\gamma_2)$), $\{c > 0 \mid I(c) < +\infty\} \neq \varnothing$ and $\mathrm{Ent}(\Gamma, d_l) = \inf\{c > 0 \mid I(c) < +\infty\}$. Let us define the sets $S_{\gamma_1^{\pm 1}}$, $S_{\gamma_2^{\pm 1}}$ as the sets of elements of Γ whose reduced writing starts by $\gamma_1^{\pm 1}$, $\gamma_2^{\pm 1}$ (respectively). We define $I_s(c) = \sum_{\gamma\in S_s} e^{-c\, d_l(\mathrm{id},\gamma)}$ where $s \in \{\gamma_1, \gamma_1^{-1}, \gamma_2, \gamma_2^{-1}\}$. By definition we have $I(c) = 1 + I_{\gamma_1}(c) + I_{\gamma_1^{-1}}(c) + I_{\gamma_2}(c) + I_{\gamma_2^{-1}}(c)$ that is:

$$I(c) = 1 + 2(I_{\gamma_1}(c) + I_{\gamma_2}(c)) \tag{4.100}$$

Moreover, since $S_{\gamma_1} = \gamma_1 \cdot (S_{\gamma_1} \cup S_{\gamma_2} \cup S_{\gamma_2^{-1}})$ we have:

$$I_{\gamma_1}(c) = e^{-c\, l(\gamma_1)} \cdot (I_{\gamma_1}(c) + I_{\gamma_2}(c) + I_{\gamma_2^{-1}}(c)).$$

Hence we have: $I_{\gamma_1}(c) + e^{-c\, l(\gamma_1)}\, I_{\gamma_1^{-1}}(c) = e^{-c\, l(\gamma_1)} \cdot I(c)$ and since $I_{\gamma_1}(c) = I_{\gamma_1^{-1}}(c)$ we get:

$$I_{\gamma_1}(c) = \frac{I(c)}{(e^{c\, l(\gamma_1)} + 1)} \tag{4.101}$$

Analogously one has:

$$I_{\gamma_2}(c) = \frac{I(c)}{(e^{c\, l(\gamma_2)} + 1)} \tag{4.102}$$

Now we plug equations (4.101) and (4.102) into equation (4.100):

$$I(c) = 1 + 2 \cdot \left[\frac{1}{e^{c\, l(\gamma_1)} + 1} + \frac{1}{e^{c\, l(\gamma_2)} + 1}\right] \cdot I(c)$$

and since $I(c) \to +\infty$ as $c \to h_+$ we see that equation (4.99) holds. □

End of the proof of Theorem 4.1.2. Let us now fix a point $\tilde{x} \in \tilde{X}$; let $\sigma_1, \sigma_2 \in A * B$ be two elements such that $\langle \sigma_1, \sigma_2 \rangle \simeq \mathbb{F}_2$ is a nontrivial free product (hence a free group since $A * B$ is torsion free). Let us denote $l(\sigma_1) = \tilde{d}(\sigma_1\tilde{x}, \tilde{x})$, $l(\sigma_2) = \tilde{d}(\sigma_2\tilde{x}, \tilde{x})$. Using the Lemma 4.4.1, we obtain

$$H \geq \mathrm{Ent}(\pi_1(X), d_{geo}) \geq \mathrm{Ent}(\langle \sigma_1, \sigma_2 \rangle, d_{geo}) \geq \mathrm{Ent}(\langle \sigma_1, \sigma_2 \rangle, d_l) \geq$$

$$\geq \frac{1}{l(\sigma_1)} \cdot \log\left(1 + \frac{4}{e^{\mathrm{Ent}(\langle \sigma_1, \sigma_2 \rangle, d_l)\, l(\sigma_2)} - 1}\right) \geq \frac{1}{l(\sigma_1)} \cdot \log\left(1 + \frac{4}{e^{H\, l(\sigma_2)} - 1}\right)$$

from which we deduce:

$$\tilde{d}(\tilde{x}, \sigma_1\tilde{x}) \geq \frac{1}{H} \cdot \log\left(1 + \frac{4}{e^{H\, \tilde{d}(\tilde{x}, \sigma_2\tilde{x})} - 1}\right) \tag{4.103}$$

Let σ be a geodesic loop realizing $\mathrm{sys}\,\pi_1(X)$ and let $\tilde{x}$ belong to σ. Let $\Sigma = \{\tau_i\}$ be a finite generating set such that $\tilde{d}(\tau_i\tilde{x}, \tilde{x}) \leq 2D$ ([**Gro6**], Proposition 5.28). There exists at least one $\tau_i \in \Sigma$ such that $\langle \tau_i, \sigma \rangle \simeq \mathbb{F}_2$ is a free product (hence a free group), since Σ is a generating set and $A * B$ is a free product. As $\tilde{d}(\tilde{x}, \tau_i\tilde{x}) \leq 2D$, the inequality (4.103) applied to σ, τ_i gives:

$$\mathrm{sys}\,\pi_1(X) = \tilde{d}(\sigma\tilde{x}, \tilde{x}) \geq \frac{1}{H} \cdot \log\left(1 + \frac{4}{e^{2DH} - 1}\right)$$

This ends the proof of the Theorem 4.1.2. □

Remark 4.4.2. Other lower bounds for the homotopy systole, with upper bounds for the volume entropy and the diameter (in addition to some algebraic assumption on $\pi_1(X)$) have been proved in [**BCG3**]. However in the next section we shall show a quite large class of examples where our estimate can be applied but not those of [**BCG3**].

4.4.2. Applications. Let Y be a complete Riemannian manifold; we will denote by $\mathrm{sgl}(Y)$ the length of the shortest (possibly homotopically trivial) geodesic loop in Y.

THEOREM 4.4.3. *Let X be a simply connected, Riemannian manifold. The family $\mathfrak{M}_X^{dec}(D,V,H)$ of compact, Riemannian quotients of X with torsionless, decomposable fundamental group such that diameter, volume and volume-entropy are bounded by D,V,H, respectively, is finite up to homotopy (for all $n \in \mathbb{N}$), homeomorphism ($n=4$), diffeomorphism ($n \geq 5$).*

Proof. It is a direct consequence of the Precompactness Theorem.□

Proof of the Precompactness Theorem. We follow the proof of the Proposition 4.3 of [**BCG3**]. Let $X \in \mathfrak{M}_n^{dec}(D,V,H;l)$; and let $\tilde{X}$ be its Riemannian universal covering. For any $x \in X$, the distance between distinct points $\tilde{x}_1, \tilde{x}_2$ of the x-fiber in $\tilde{X}$ is greater or equal to $\mathrm{sys}\,\pi_1(X)$, so $B(x, \frac{\mathrm{sys}\,\pi_1(X)}{2})$ is isometric to $\tilde{B}(\tilde{x}, \frac{\mathrm{sys}\,\pi_1(X)}{2})$, for $\tilde{x}$ in the x-fiber in $\tilde{X}$. Since in $B(x, \frac{\mathrm{sys}\,\pi_1(X)}{2})$ we do not have geodesic loops of length less than $\mathrm{sgl}(\tilde{X})$ (by definition of $\mathrm{sgl}(\tilde{X})$), it follows that $\mathrm{sgl}(X) = \min\{\mathrm{sys}\,\pi_1(X), \mathrm{sgl}(\tilde{X})\}$. Hence by the Theorem 4.1.2 and by the assumption we made on $\mathrm{sgl}(\tilde{X})$ it follows that $\mathrm{sgl}(X) \geq l_0 = \min\{l, \frac{1}{H}\cdot\log(1+\frac{4}{e^{2DH}-1})\}$. A theorem of Sabourau ([**Sbr**], Theorem A) states that if M is a complete Riemannian manifold of dimension n there exists a constant C_n, depending only on the dimension of M such that $\mathrm{Vol}(B(x,R)) \geq C_n R^n$, for every ball of radius $R \leq \frac{1}{2}\mathrm{sgl}(M)$. This means that we can bound the maximum number $N(X,\varepsilon)$ of disjoint geodesic balls in X of radius ε by the function $V/(C_n\varepsilon^n)$, and the estimate holds for any manifold in $\mathfrak{M}_n^{dec}(D,V,H;l)$ (obviously for $\varepsilon \leq l_0/2$). Then using the Gromov's packing argument as shown in [**Gro6**] §5.1-5.3 and, for example, in [**Fuk**], Lemma 2.4, we get the precompactness of the family $\mathfrak{M}_n^{dec}(D,V,H;l)$. For what concerns the finiteness results for $\mathfrak{M}_n^{dec}(D,V,H;l)$ we proved that manifolds $X \in \mathfrak{M}_n^{dec}(D,V,H;l)$ satisfy the condition $\mathrm{sgl}(X) \geq l_0 = \min\{l, \frac{1}{H}\cdot\log(1+\frac{4}{e^{2DH}-1})\}$. Let $\alpha_n = \frac{1}{(4.3)^{n-1}-1}$. S. Sabourau shows in the proof of the Theorem A in [**Sbr**] that the function $\rho : [0, \alpha_n\,\mathrm{sgl}(X)) \to \mathbb{R}^+$, $\rho(r) = (4.3)^n\, r$ is a *local geometric contractibility function*[18] for X; in particular $\rho|_{[0,\alpha_n\, l_0)} \to \mathbb{R}^+$ is a local geometric contractibility function for the family $\mathfrak{M}_n^{dec}(D,V,H;l)$. Hence the family $\mathfrak{M}_n^{dec}(D,V,H;l)$ is contained in $\mathcal{C}(\rho,V,n)$, *i.e.* the family of compact Riemannian n-manifolds whose volume is bounded above by V, which admit $\rho : [0,\alpha_n\, l_0) \to \mathbb{R}^+$, $\rho(r) = (4.3)^n\, r$ as local geometric contractibility function. We apply the Theorem 2 in [**GP**] and we obtain the finiteness statements. □

REMARK 4.4.4. We want to compare our precompactness and finiteness theorem with the classical ones by J. Cheeger and M. Gromov ([**Cheeger1**], [**Gro5**] §8.20, [**Gro1**]) and with more recent results by I. Belegradek ([**Bel**]). The first finiteness result has been obtain combining the results of Cheeger and Gromov (see [**Fuk**], Theorem 14.1): they considered the class of n-manifolds with bounded sectional curvature $|\sigma| \leq 1$, volume bounded below by a universal constant $v > 0$ and diameter bounded above by a constant $D > 0$ and they proved the finiteness of diffeomorphism classes (the proof given by Gromov uses the Lipschitz precompactness of the family and his rigidity theorem, see also [**Kat**], [**Fuk**]). Observe that the assumptions of this result implies our geometric assumptions: in fact the first assumption (on curvature) implies the boundedness of the volume-entropy, while the three assumptions both imply the boundedness of the volume and a lower bound of sgl.

Another finiteness theorem has been proved by Gromov in [**Gro1**]: he assumes to have the following bounds $-1 \leq \sigma < 0$ on the sectional curvature and an upper bound for the volume, V, and establishes the finiteness of diffeomorphism types for the class of Riemannian manifolds (of dimension $n \neq 3$) satisfying these bounds. The result is a consequence of the Theorem 1.2 in [**Gro1**], which gives an upper bound for the diameter of a negatively curved Riemannian manifold of sectional curvature $-1 \leq \sigma < 0$ in terms of its volume, combined with Cheeger's finiteness theorem and with Margulis' Lemma.

[18]See the introduction of [**GP**] for the definition.

We observe that also in this case since the Riemannian universal covering $\tilde{X}$ satisfies $\mathrm{sgl}(\tilde{X}) = +\infty$, the assumptions made by M. Gromov imply our geometric assumptions; moreover, the prescribed sign and the boundedness of the sectional curvature impose algebraic restrictions on the possible fundamental groups.

More recently I. Belegradek showed that once we fix a group Γ for any $b \in [-1, 0)$ there exist at most finitely many nondiffeomorphic closed Riemannian manifolds satisfying $-1 \leq \sigma \leq b < 0$ and whose fundamental group is isomorphic to Γ ([**Bel**], Corollary 1.4). Here the isomorphism class of the fundamental group is prescribed, but no assumption has been made on the volume and the diameter (whereas boundedness of sectional curvature implies the boundedness of the volume-entropy).

Another application of the Theorem 4.1.2 is a volume estimate for 1-essential, compact, Riemannian n-manifolds with decomposable torsion free fundamental groups. We recall that a manifold X is said to be 1-essential whenever it admits a map f into a $K(\pi, 1)$-space K, such that the induced homomorphism $H_n(X, \mathbb{Z}) \to H_n(K, \mathbb{Z})$ does not vanish.

Proof of the Volume estimate. Just combine the estimate for the homotopy systole in the Theorem 4.1.2 with the inequality $\mathrm{sys}\,\pi_1(X)^n C_n \leq \mathrm{Vol}(X)$ proved in Theorem 0.1.A in [**Gro5**]. $\square$

4.5. Examples and Counterexamples

In [**Zud**] the following class of groups is defined: a N-nonabelian group is a group Γ without nontrivial normal, abelian subgroups, such that the commutation relation is transitive on $\Gamma \setminus \{\mathrm{id}\}$ and such that $\forall \gamma_1, \gamma_2 \in \Gamma$ that do not commute, there exist two elements in $B(\mathrm{id}, N) \subset (\langle \gamma_1, \gamma_2 \rangle, d_{alg})$ (here d_{alg} denotes the algebraic distance $d_{\{\gamma_1, \gamma_2\}}$), which generate a free semi-group (we call this last property the FSG(N) property). This notion is inspired by the one of δ-nonabelian group, introduced in [**BCG3**]. We remark that in general a δ-nonabelian group (in the sense of [**BCG3**]) is not N-nonabelian (in the sense of [**Zud**]), however δ-nonabelian groups whose commutation relation is transitive are always $[\frac{4}{\delta}]$-nonabelian. Simple examples of N-nonabelian groups are:

- δ-thick groups in the terminology of [**BCG3**] (*i.e.* fundamental groups of Riemannian manifolds with sectional curvature less or equal to -1 and injectivity radius greater than δ) are $[\frac{4}{\delta}]$-nonabelian.
- Free products of δ-thick groups, and free products of δ-thick groups with abelian groups are $[\frac{4}{\delta}]$-nonabelian.
- More generally free products of N-nonabelian groups and free products of N-nonabelian groups with abelian groups are N-nonabelian (this is an easy corollary of the Proposition 1.3 in [**Zud**]).
- $\pi_1(X) * \pi_1(Y)$, the free product of the fundamental groups of two compact Riemannian manifolds X, Y with sectional curvature less or equal to -1 is N-nonabelian for $N \geq 4 \cdot \max\{\frac{1}{\mathrm{inj}(X)}, \frac{1}{\mathrm{inj}(Y)}\}$ (see [**Zud**], §1.4).
- $\mathbb{Z}_n * \mathbb{Z}$, for every odd integer n, is N-nonabelian for $N = 4$ (again in [**Zud**], §1.4). This example is important since it shows that there are N-nonabelian groups which are not δ-nonabelian.

Following Zuddas we remark that δ-nonabelian groups in the sense of [**BCG3**] satisfy a strictly stronger condition than the FSG(N) property. We give some examples of manifolds which satisfy the assumptions of our Theorem 4.1.2, whose fundamental groups are not N-nonabelian or δ-nonabelian:

Example 4.5.1 (Connected sums with flat manifolds). Consider $Y = X \# Z$ the connected sum of a quotient X of $\mathbb{E}^n$ by the action of a discrete, nonabelian, torsion free and cocompact subgroup of $\mathrm{Is}(\mathbb{E}^n)$, with a compact manifold Z whose fundamental group is torsion free and non trivial. Let $A = \pi_1(X)$ and $B = \pi_1(Z)$ then, if $n \geq 3$, $\pi_1(X \# Z) = A * B$. Then, the group $A * B$ does not possess the FSG(N)-property. As A is nonabelian there exist two elements a_1, a_2 which do not commute; since A is a Bieberbach group, it is a group of polynomial growth (it contains a $\mathbb{Z}^n$ with finite index): this means that A does not contain any free semigroup. So A has a couple of elements not commuting and such that $\nexists N \in \mathbb{N}$ for which we can find two elements in $B(\mathrm{id}, N) \subset (\langle a_1, a_2 \rangle, d_{\{a_1,a_2\}})$ that generate a free semigroup.

Example 4.5.2 (Connected sums with infranilmanifolds). More generally the above arguments hold for the connected sums with infranilmanifolds (a infranilmanifold is the quotient of a simply connected nilpotent Lie group by a nonabelian, torsion free, quasi-crystallographic group, see [**Dek**], section 2.2). For instance if X_k is the quotient of the Heisenberg group by

$$\Gamma_k = \langle a, b, c | \ [b, a] = c^k, \ [c, a] = [c, b] = \mathrm{id} \rangle$$

(where a, b, c are the standard generators) and $Y_k = X_k \# M$, where M has non trivial, torsion free fundamental group, we have an infinite number of distinct differentiable manifolds, all non N-nonabelian, to which our Theorem 4.1.2 applies (for *any* choice of a Riemannian metric on Y_k).

Remark 4.5.3. It is well known that connected sums of 1-essential n-manifolds with other n-manifolds are still 1-essential, so the examples above also provide a class of manifolds for which our Volume estimate holds.

Let us do some comments about the Theorem 4.1.1. First of all we considerably enlarge the class of manifolds for which the Margulis Lemma *à la* Besson Courtois Gallot holds: in fact the only free products considered in [**BCG3**] and [**Zud**] were free products of N-nonabelian groups or free proudcts of N-nonabelian groups with certain abelian groups; on the contrary we consider free products without restrictions, except for the 2-torsionless assumption. Finally a remark about the necessity of requiring Γ without 2-torsion: it might be sufficient to ask $\Gamma \neq \mathbb{Z}_2 * \mathbb{Z}_2$ -*i.e.* to exclude the unique case of a free product for which estimate (4.98) does not hold-; however our proof of the Theorem 4.1.1 is not sufficient to conclude even in the case when $\Gamma = A * B$ and B (or A) admits 2-torsion.

Example 4.5.4. We now exhibit a family of manifolds which proves that the 'torsion free' assumption of the Theorem 4.1.2 cannot be dropped. Fix a $p \in \mathbb{N}$ and let X be the connected sum of a lens space $M_p = S^3/\mathbb{Z}_p$ with any non simply connected manifold Y. We can endow X with a family of metrics g_ε, such that for all $\varepsilon \in (0, 1]$:

(1) $\mathrm{Diam}(X, g_\varepsilon) \leq D$ for a suitable $D \in (0, +\infty)$;
(2) $\mathrm{sys}\,\pi_1(X, g_\varepsilon) \leq 2\pi\varepsilon/p$.
(3) $\mathrm{Ent}(X, g_\varepsilon) \leq H$ for a suitable $H \in (0, +\infty)$;

4.5.4.1. *Construction of the metrics g_ε on X.* First we recall that if S^3 is endowed with the canonical metric (which we will denote by h_1 in the sequel) then we have an isometric action of S^1 which can be described as follows:

$$(S^3, can) = \{(u, v) \in \mathbb{C}^2 | |u|^2 + |v|^2 = 1\},$$
$$S^1 \times S^3 \to S^3, \quad (e^{i\theta}, (u, v)) \to (e^{i\theta}u, e^{i\theta}v).$$

Then $M_p = S^3/\langle e^{\frac{2\pi}{p}} \rangle$, so $\pi_1(M_p) = \langle \sigma_p \rangle$ where $\sigma_p = e^{\frac{2\pi}{p}}$. Let γ_p be the shortest non contractible loop of M_p representing g_p (corresponding to an arc $\tilde{\gamma}_p$ on a maximal circle $\tilde{\gamma}$ in S^3). Since the normal bundle of $\tilde{\gamma}$, $N_\delta(\tilde{\gamma})$, is topologically trivial (*i.e.* $N_\delta(\tilde{\gamma}) \simeq \tilde{\gamma} \times D^2_\delta$ where D^2_δ is a euclidean disk of radius δ), in a tubular neighbourhood $N_\delta(\tilde{\gamma})$ we modify the canonical metric of S^3 only in the direction tangent to the S^1-action by a smooth factor $\lambda_\varepsilon(r)$ (where r is the distance from the maximal circle $\tilde{\gamma}$), where $\lambda_\varepsilon \leq 1$ everywhere, $\lambda_\varepsilon(r) \equiv 1$ outside $N_{\frac{2\delta}{3}}(\tilde{\gamma})$ and $\lambda_\varepsilon \equiv \varepsilon^2$ on $N_{\frac{\delta}{3}}(\tilde{\gamma})$. We remark that λ_ε can be constructed in $N_{\frac{2\delta}{3}}(\tilde{\gamma}) \setminus N_{\frac{\delta}{3}}(\tilde{\gamma})$ in order to keep the sectional curvatures bounded below by a negative constant C, independent from ε. We obtain a new metric $\tilde{h}_\varepsilon$. The new metric is still invariant by the action of $\mathbb{Z}_p$ defined before, hence the action of $\mathbb{Z}_p$ on $(S^3, \tilde{h}_\varepsilon)$ is still isometric and induce a metric h_ε on $S^3/\mathbb{Z}_p$. Now, for any fixed metric k on Y we can glue (M_p, h_ε) to (Y, k) by gluing $M_p \setminus B_m$ and $Y \setminus B_y$ on the boundaries of two small balls B_m and B_y of (M_p, h_ε) and (Y, k) (respectively) such that B_m lies outside $N_\delta(\gamma)$. We will call (X, g_ε) the manifolds obtained in this way.

4.5.4.2. *Proof of* (1), (2), (3). As the metric g_ε is equal to g_1 except for the tubular neighbourhood of $\tilde{\gamma}$ and as λ_ε is bounded above by 1 we remark that $0 < d_0 < \mathrm{Diam}(X, g_\varepsilon) \leq \mathrm{Diam}(X, g_1) = D$ for constants d_0 and D independent from ε. Moreover, with respect to this metric $l_{g_\varepsilon}(\gamma) = \tilde{d}_\varepsilon(\tilde{x}, \sigma_p \tilde{x}) = \frac{2\pi\varepsilon}{p}$ when $\tilde{x} \in \tilde{\gamma}$, which proves (2). Let us now prove (3). We choose a point $x_0 \in X$ such that $\mathrm{inj}(x, g_\varepsilon) \geq i_0 > 0$ (such a point exists since $g_\varepsilon = g_1$ in $Y \setminus B_y$). We define the norms $\| \gamma \|_\varepsilon = \tilde{d}_\varepsilon(\tilde{x}, \gamma(\tilde{x}))$ on $\Gamma = \pi_1(X) = \mathbb{Z}_p * G$, where $\tilde{x}$ is in the x-fiber in $\tilde{X}$. For all ε the sets $\Sigma_\varepsilon = \{\gamma \in \Gamma \mid \ \| \gamma \|_\varepsilon \leq 3D\}$ are generating sets for $\mathbb{Z}_p * G$. So:

$$d_1(\tilde{x}, \gamma\tilde{x}) \leq 3D \cdot \| \gamma \|_{\Sigma_1} \leq 3DS \cdot \| \gamma \|_{\Sigma_\varepsilon} \leq \frac{DS}{d_0} \cdot (3\, d_\varepsilon(\tilde{x}, \gamma\tilde{x}) + 1) \tag{4.104}$$

for some $S < \infty$. Actually the first and the last inequality are well known (see [**Gro6**], 3.22); for the second one let us define $S^\varepsilon = \sup\{\| \gamma \|_{\Sigma_1} \mid \gamma \in \Sigma_\varepsilon\}$ and we show that

$$S = \sup\{S^\varepsilon \mid \varepsilon \in (0, 1]\} < +\infty$$

In fact since the sectional curvatures (and hence the Ricci curvature) of g_ε are bounded below independently from ε, there exists a $N(D, i_0) \in \mathbb{N}$ (independent from ε) bounding the maximum number of disjoint g_ε-balls of radius i_0 in a g_ε-ball of radius $3D$; it follows that $\#\Sigma_\varepsilon \leq N(D, i_0)$, for all $\varepsilon > 0$; moreover $\Sigma_\varepsilon \subseteq \Sigma_{\varepsilon'}$ for $\varepsilon' < \varepsilon$ as $d_{\varepsilon'} \leq d_\varepsilon$. We deduce that the sets Σ_ε are all included in a maximal finite subset Σ, so $S < +\infty$. Then from estimate (4.104) we deduce readily: $\mathrm{Ent}_{\mathrm{vol}}(X, g_\varepsilon) \leq \frac{3DS}{d_0} \mathrm{Ent}_{\mathrm{vol}}(X, g_1) = H$. □

EXAMPLE 4.5.5. This example shows the necessity of the upper bound for the diameter in the Theorem 4.1.2 and the 'sharpness' of the result. Let us denote $M_1^\varepsilon = (S^1 \times S^{n-1}, \varepsilon^2 \cdot g_0)$, $M_2^{\varepsilon'} = (S^1 \times S^{n-1}, \frac{1}{(\varepsilon')^2} \cdot g_0)$ (where g_0 is the canonical product metric of $S^1 \times S^{n-1}$). Let $X = M_1^\varepsilon \# M_2^{\varepsilon'}$ where the metric is constructed as follows: we cut a geodesic ball $B_1 \subset M_1^\varepsilon$ (resp. $B_2 \subset M_2^{\varepsilon'}$) of radius ε^2. Consider the cylinder $C = [0,1] \times S^{n-1}$; we endow $\{0\} \times S^{n-1}$ (resp. $\{1\} \times S^{n-1}$) with a Riemannian metric h_0 (resp. h_1) such that $\{0\} \times S^{n-1}$ is isometric to $(\partial B_1, \varepsilon^2 \cdot g_0)$ (resp. $\{1\} \times S^{n-1}$ is isometric to $(\partial B_2, \frac{1}{(\varepsilon')^2} \cdot g_0)$). Next we define the following metric on C: $h_{\varepsilon,\varepsilon'} = (dr)^2 + h_r$ where h_r is the metric on $\{r\} \times S^{n-1}$ defined by $h_r = (1-r)\, h_0 + r\, h_1$. Finally we construct the connected sum gluing M_1^ε and $M_2^{\varepsilon'}$ at the two boundary component of C, and we construct the metric $g_{\varepsilon,\varepsilon}$ as follows:

$$g_{\varepsilon,\varepsilon'} = \begin{cases} \varepsilon^2 \cdot g_0 & \text{on } M_1^\varepsilon \setminus B_1; \\ h_{\varepsilon,\varepsilon'} & \text{on } C; \\ \frac{1}{(\varepsilon')^2} \cdot g_0 & \text{on } M_2^{\varepsilon'} \setminus B_2; \end{cases}$$

We remark that $g_{\varepsilon,\varepsilon'}$ is not C^∞ but just piecewise C^∞. However it is not difficult to show that we can produce smooth metrics arbitrairely close in the sense C^0 to $g_{\varepsilon,\varepsilon'}$. That is why we are allowed to use the metric $g_{\varepsilon,\varepsilon'}$.

Let a, (resp. b) be the generator of the image of $\pi_1(M_1^\varepsilon)$ (resp. $\pi_1(M_2^{\varepsilon'})$) in the free product. By construction $\mathrm{sys}\,\pi_1(X, g_{\varepsilon,\varepsilon'})$ is the length of the periodic geodesic freely homotopic to the geodesic loop a, so that:

$$\mathrm{sys}\,\pi_1(X, g_{\varepsilon,\varepsilon'}) = 2\pi\varepsilon \tag{4.105}$$

On the other hand the diameter $D_{\varepsilon,\varepsilon'}$ of $(X, g_{\varepsilon,\varepsilon'})$ satisfies

$$\frac{\pi}{\varepsilon'} + 1 \leq D_{\varepsilon,\varepsilon'} \leq \frac{\pi}{\varepsilon'} + 1 + \pi\varepsilon \tag{4.106}$$

Take $x \in \partial B_2$. Every geodesic loop based at x can be written in $\pi_1(X, x)$ in the form: $\gamma = a^{p_1} b^{q_1} \cdots a^{p_m} b^{q_m}$. Such a decomposition corresponds to a partition of the loop γ by points $x = x_0, y_0, x_1, ..., x_{m-1}, y_{m-1}, x_m = x \in \partial B_2$ such that a^{p_i} (resp. b^{q_i}) is the homotopy class of the loop obtained by composition of α_i (resp. β_i), the portion of the path γ corresponding to $[x_{i-1}, y_{i-1}]$ (resp. $[y_{i-1}, x_i]$), with the minimizing geodesics joining x with x_{i-1} and y_{i-1} (resp. with y_{i-1} and x_i) in ∂B_2, whose lengths are bounded above by $\varepsilon^2 \cdot C$ where $C \leq 2\pi + 1$. Hence we find:

$$l(\gamma) \geq \sum_{i=1}^{m} (l(\alpha_i) - 2C\varepsilon^2) + \sum_{i=1}^{m} (l(\beta_i) - 2C\varepsilon^2)$$

By construction we have $l(\alpha_i) \geq (2\pi\varepsilon)\,|p_i| + 1$, $l(\beta_i) \geq \frac{2\pi}{\varepsilon'}\,|q_i|$, so that taking ε sufficiently small we get: $l(\gamma) \geq \sum_{i=1}^{m} |p_i|\,(2\pi\varepsilon) + \sum_{i=1}^{m} |q_i|\,\frac{2\pi}{\varepsilon'}$; hence if $\tilde{x}$ is in the fiber of x in the Riemannian universal covering $\tilde{X}$ we see that $d_{geo}(\mathrm{id}, \gamma) = d_{g_{\varepsilon,\varepsilon'}}(\tilde{x}, \gamma\tilde{x}) \geq d_l(\mathrm{id}, \gamma)$ where d_l is the distance on $\mathbb{Z} * \mathbb{Z}$ corresponding to the choice of the generating system $\{a,\, b\}$ with lengths $l(a) = 2\pi\varepsilon$, $l(b) = \frac{2\pi}{\varepsilon'}$. Since X is compact we have:

$$\mathrm{Ent}(X, g_{\varepsilon,\varepsilon'}) \leq \mathrm{Ent}(\Gamma, d_{geo}) \leq \mathrm{Ent}(\Gamma, d_l) = h$$

where, by the Lemma 4.4.1, h satisfies the equation:

$$(e^{2\pi h\varepsilon} - 1)(e^{\frac{2\pi h}{\varepsilon'}} - 1) = 4 \qquad (4.107)$$

4.5.5.1. *End of the counterexample.* If $\varepsilon = \varepsilon'$ by the estimates (4.105), (4.106) the systole and the diameter of $(X, g_{\varepsilon,\varepsilon'})$ tends respectively to 0 and $+\infty$. On the other hand equation (4.107) shows that $\mathrm{Ent}(X, g_{\varepsilon,\varepsilon'})$ is bounded above by $\frac{1}{\pi}$. This proves the necessity of the boundedness of the diameter in the Theorem 4.1.2.

4.5.5.2. *Optimality.* By (4.107) we know that $H_{\varepsilon,\varepsilon'} = \mathrm{Ent}(X, g_{\varepsilon,\varepsilon'})$ satisfies

$$2\pi\varepsilon \leq \frac{1}{H_{\varepsilon,\varepsilon'}} \cdot \log\left(1 + \frac{4}{e^{2\cdot H_{\varepsilon,\varepsilon'}\cdot\frac{\pi}{\varepsilon'}} - 1}\right)$$

Since $\frac{\pi}{\varepsilon'} \simeq \mathrm{Diam}(X, g_{\varepsilon,\varepsilon'})$ the estimate given of the Theorem 4.1.2 is optimal.

Appendix

This appendix is devoted to the proof of the Lemma 4.3.2 that we used in the proof of the Theorem 4.1.1. We recall that a word $\gamma = \alpha_1 \cdots \alpha_p$ in $\Gamma = A * B$ is said to be in the reduced form if $\forall i \;\; \alpha_i \in A$ or $\alpha_i \in B$ and if α_i, α_{i+1} do not belong to the same factor in Γ for all $i = 1, .., p-1$; in this case the length of the reduced word is $l(\gamma) = p$. We remark that this corresponds to the 'algebraic length' of Γ only if we consider $A \sqcup B$ as the generator system of $A * B$. Notice that the reduced form is unique. We say that a word γ is cyclically reduced if its reduced form $\gamma = \alpha_1 \cdots \alpha_p$ is such that $\alpha_p \neq \alpha_1^{-1}$.

Proof of the Lemma 4.3.2 Let $\gamma = \alpha_1 \cdots \alpha_p$, and $\gamma' = \alpha'_1 \cdots \alpha'_{p'}$ be the reduced forms for γ, γ'.

- If p is even, then α_1, α_p belong to two different factors, and the reduced form for γ^r is:

$$\gamma^r = (\alpha_1 \cdots \alpha_p) \cdots (\alpha_1 \cdots \alpha_p)$$

  and the initial letter in the reduced form is not the inverse of the final one, *i.e* γ^r is cyclically reduced. It is clear that, in this case, the knowledge of p and γ^r allows us to recover the whole sequence of letters $\alpha_1, .., \alpha_p$.
- If p is odd, then α_1, α_p belong to the same factor and the writing

$$\gamma^r = (\alpha_1 \cdots \alpha_p) \cdots (\alpha_1 \cdots \alpha_p)$$

  can be reduced a first time by grouping together $(\alpha_p\alpha_1)$; we can not reduce further unless $\alpha_p = \alpha_1^{-1}$, and so on until $\alpha_{p-i} \neq (\alpha_{i+1})^{-1}$ this condition beeing realized for some $i \leq [\frac{p}{2}]$. We find

$$\gamma = (\alpha_1 \cdots \alpha_i)(\alpha_{i+1} \cdots \alpha_{p-i})(\alpha_i^{-1} \cdots \alpha_1^{-1})$$

  and we get that $\gamma^r = (\alpha_1 \cdots \alpha_i)(\alpha_{i+1} \cdots \alpha_{p-i})^r(\alpha_i^{-1} \cdots \alpha_1^{-1})$, so that, grouping together α_{p-i} and α_{i+1}, we obtain $l(\gamma^r) \leq pr - (2i+1)(r-1)$. Moreover the initial letter in the reduced form of γ^r is the inverse of the final one. Also in this case the knowledge of γ^r impose the values of $\alpha_1, ..., \alpha_p$.

The same arguments hold for the decomposition of γ', *i.e.* given $(\gamma')^{r'}$, $\alpha'_1, .., \alpha'_{p'}$ are determined. In particular $\gamma^r = (\gamma')^{r'}$ implies that the initial and the final letter in the reduced forms of γ^r, $(\gamma')^{r'}$ are the same. Thus the first (resp. the final) letter of the reduced word corresponding to γ lies in the same subgroup (A or B) of the first (resp. the final) letter of γ'; this implies that p, p' are both even or both odd. So we are led to consider the following cases:

CASE (1). *p, q even.* Let w be the word in the alphabet $A^* \sqcup B^*$ given by the reduced form of $\gamma^r = (\gamma')^{r'}$ above; since w is invariant by the shift of p and p' places, then it is also invariant by the shift of $d = \mathrm{GCD}(p, p')$ places. Therefore, setting $\tau = \alpha_1 \cdots \alpha_d$ we have $\gamma = \tau^q$, $\gamma' = \tau^{q'}$ for $q = p/d$ and $q' = p'/d$.

CASE (2). *p, q odd.* We know that

$$\gamma^r = (\alpha_1 \cdots \alpha_i)(\alpha_{i+1} \cdots \alpha_{p-i})^r (\alpha_1 ... \alpha_i)^{-1}$$
$$(\gamma')^{r'} = (\alpha'_1 \cdots \alpha'_{i'})(\alpha'_{i'+1} \cdots \alpha'_{p'-i'})^{r'} (\alpha'_1 \cdots \alpha'_{i'})^{-1}$$

with $\gamma_1 = (\alpha_{i+1} \cdots \alpha_{p-i})$ and $\gamma'_1 = (\alpha'_{i'+1} \cdots \alpha'_{p'-i'})$ cyclically reduced hence, comparing the two expressions we deduce that $i = i'$ and $\alpha_k = \alpha'_k$ for $k \leq i$. Now consider $\gamma_1^r = (\alpha_{i+1} \cdots \alpha_{p-i})^r$, $(\gamma'_1)^{r'} = (\alpha_{i+1} \cdots \alpha_{p'-i})^{r'}$. We have $\gamma_1^r = (\gamma'_1)^{r'}$. As γ_1, γ'_1 are cyclically reduced, with $l(\gamma_1)$, $l(\gamma'_1)$ odd, the only reduction that we can perform on γ_1^r is to group together α_{p-i} and α_{i+1} (and $\alpha_{p'-i}$, $\alpha_{i'+1}$ in $(\gamma'_1)^{r'}$):

$$\gamma_1^r = \alpha_{i+1} \cdots (\alpha_{p-i}\alpha_{i+1}) \cdots (\alpha_{p-i}\alpha_{i+1})\alpha_{i+2} \cdots \alpha_{p-i}$$
$$(\gamma'_1)^r = \alpha'_{i+1} \cdots (\alpha'_{p'-i}\alpha'_{i+1}) \cdots (\alpha'_{p'-i}\alpha'_{i+1})\alpha'_{i+2} \cdots \alpha'_{p'-i}$$

this implies $\alpha_{i+1} = \alpha'_{i+1}$ and so setting

$$\tilde{\gamma}_1 = \alpha_{i+2} \cdots \alpha_{p-i-1}(\alpha_{p-i}\alpha_{i+1}), \quad \tilde{\gamma}'_1 = \alpha'_{i+1} \cdots \alpha'_{p'-i-1}(\alpha'_{p'-i}\alpha'_{i+1})$$

we have $\tilde{\gamma}_1^r = (\alpha_{i+1})^{-1} \gamma_1^r \alpha_{i+1} = (\alpha_{i+1})^{-1} (\gamma'_1)^{r'} \alpha_{i+1} = (\tilde{\gamma}'_1)^{r'}$ and we are reduced to the case where $l(\tilde{\gamma}_1)$ and $l(\tilde{\gamma}'_1)$ are even, which we treat as before. Therefore we can find a $\tilde{\tau}$ and integers q, q' such that $\tilde{\gamma}_1 = \tilde{\tau}^q$, $\tilde{\gamma}'_1 = (\tilde{\tau}')^{q'}$. Setting $\tau = \alpha\tilde{\tau}\alpha^{-1}$ for $\alpha = \alpha_1 \cdots \alpha_{i+1}$ we finally have: $\gamma = \tau^q$, $\gamma' = \tau^{q'}$. $\square$

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