
TESI DI DOTTORATO

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Higher Deligne groupoids, derived brackets and deformation problems in holomorphic Poisson geometry

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**Higher Deligne groupoids, derived brackets and
deformation problems in holomorphic Poisson
geometry**

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Chapter 0

Introduction

In the beautiful paper [39] Getzler studies the problem of integrating nilpotent dg Lie algebras to ∞ groupoids in a way which generalizes the classical way a nilpotent Lie algebra integrates to its exponential group via the Baker-Campbell-Hausdorff product, where in a loose, homotopical, sense we are using, at least for the moment, ∞ groupoid as a synonym for Kan complex (cf. e.g. [73] for a justification). He notices that there is a “homotopically right” answer given by rational homotopy theory (a la Sullivan [100], cf. [39], Proposition 1.1): namely, the functor $\text{MC}_\infty(-)$ sending a nilpotent dg Lie algebra algebra L to the simplicial set $\text{MC}_\infty(L) := \text{MC}(\Omega(\Delta_\bullet; L))$ of Maurer-Cartan forms on the standard cosimplicial simplex Δ_\bullet with coefficients in L , that is, n -simplices of $\text{MC}_\infty(L)$ are 1-forms $\omega \in \Omega^1(\Delta_n; L) = (\Omega_n \otimes L)^1$, where $\Omega_n = \Omega(\Delta_n)$ is the de Rham-Sullivan algebra [100, 11] of polynomial differential forms on the n -th standard simplex Δ_n , satisfying the Maurer-Cartan equation

$$d\omega + \frac{1}{2}[\omega, \omega] = 0, \tag{0.0.1}$$

where the bracket is induced by the one on L via scalar extension by Ω_n . The functor $\text{MC}_\infty(-)$ had been previously studied by Hinich [45] in the context of deformation theory, more about this later: on the other hand this is bigger than what we wanted, for instance if \mathfrak{g} is an ordinary nilpotent Lie algebra then the nerve $\text{N}(\exp(\mathfrak{g}))$ of its exponential group is only a deformation retract of $\text{MC}_\infty(\mathfrak{g})$. We review Getzler’s solution from a point of view to our knowledge not covered in the literature.

After Whitney [108] and Dupont [30], integration of forms over simplexes induces a simplicial contraction $(\Omega(\Delta_\bullet; L) \xrightarrow[\iota]{\int} C(\Delta_\bullet; L), K)$ from the simplicial dg Lie algebra $\Omega(\Delta_\bullet; L)$ to the simplicial complex $C(\Delta_\bullet; L)$ of non-degenerate cochains on Δ_\bullet with coefficients in L : the standard theorem on homotopical transfer of L_∞ structures says then that there is an induced simplicial nilpotent L_∞ algebra structure on $C(\Delta_\bullet; L)$, for which it makes sense to consider the Maurer-Cartan equation. The solution given in [39] is to consider the simplicial subset $\gamma_\bullet(L) \subset \text{MC}_\infty(L)$ of Maurer-Cartan forms in the kernel of Dupont’s Gauge $K : \Omega(\Delta_\bullet; L) \rightarrow \Omega(\Delta_\bullet; L)[-1]$: by a formal analog of Kuranishi theorem, also due to Getzler (it will be reviewed in Section 2.3, notice that the enunciate given there is a bit more general than the ones we found in the literature, the proof on the other hand is essentially taken from [39]), this is isomorphic to the simplicial set $\text{Del}_\infty(L) := \text{MC}(C(\Delta_\bullet; L))$ of Maurer-Cartan cochains on Δ_\bullet with coefficients in L , where the isomorphism is again induced by integration of forms over simplexes. In fact, for a nilpotent Lie algebra \mathfrak{g} we have this time an isomorphism $\text{Del}_\infty(\mathfrak{g}) \cong \text{N}(\exp(\mathfrak{g}))$, but more is true: if L is a non

negatively graded nilpotent dg Lie algebra¹ then $\text{Del}_\infty(L)$ is isomorphic to the nerve $\text{N}(\text{Del}(L))$ of the Deligne groupoid of L (see e.g. [27, 42, 77, 34]), look at Section 5.2.2 for a proof, explaining the notation and establishing an important bridge towards deformation theory; again, more on this later. On the other hand, one of the main results of [39] is the existence of a natural weak equivalence of simplicial sets $\text{Del}_\infty(L) \rightarrow \text{MC}_\infty(L)$, thus in the complementary case where L is negatively graded $\text{Del}_\infty(L)$ represents the simply connected rational homotopy type associated to L , establishing this time a bridge towards rational homotopy theory.

In Section 5.2 we review some important results by Getzler [39] and Berglund [6] on the structure of the simplicial set $\text{Del}_\infty(L)$. From [39] we review the existence of the natural weak equivalence $\text{Del}_\infty(L) \rightarrow \text{MC}_\infty(L)$ and the proof that $\text{Del}_\infty(L)$ is an ∞ groupoid². From [6] we review the computation of the homotopy groups $\pi_i(\text{Del}_\infty(L), x)$ for all base points $x \in \text{MC}(L) = \text{Del}_\infty(L)_0$ and the following theorem. Given a simplicial set X and a nilpotent dg Lie algebra L we can form, again via homotopy transfer from the dg Lie algebra $\Omega(X; L) = \Omega(X) \otimes L$ along Dupont's contraction, the nilpotent L_∞ algebra $C(X; L)$ of non-degenerate cochains on X with coefficients in L : then the simplicial set $\text{Del}_\infty(C(X; L))$ (this makes sense, cf. below) is naturally weakly equivalent to the simplicial mapping space $\mathbf{SSet}(X, \text{Del}_\infty(L))$. In [6] this latter fact is used to deduce that if X is finite and L is a Lie model for the rational homotopy type of Y , then $C(X; L)$ is an L_∞ model for the rational homotopy type of the mapping space $\mathbf{SSet}(X, Y)$: notice that mapping spaces are usually not connected, so they are beyond the scope of classical rational homotopy theory (much recent work has been devoted to the rational homotopy theory of mapping spaces, cf. [6, 15, 68]).

In Section 5.2.1 we look at the role of the functor $\text{Del}_\infty(-)$ in the Lie approach to disconnected rational homotopy theory developed by Lazarev and Markl [70]. It should be clear by the above discussion that all we need to define $\text{Del}_\infty(-)$ are scalar extension, homotopy transfer and the Maurer-Cartan equation, so more in general $\text{Del}_\infty(L)$ can be defined for any complete (that is, pronilpotent) L_∞ algebra L . We take a pause to recall another construction by Sullivan [102] and Cheng-Getzler [22]. If X is a simplicial set, then via homotopy transfer from the dg commutative algebra $\Omega(X)$ it is induced a C_∞ algebra structure on the complex $C^*(X)$ of non-degenerate cochains on X , that is, a commutative dg algebra structure which is associative only up to system of coherent homotopies. If X is moreover finite it is induced a dual C_∞ coalgebra structure on the complex $C_*(X)$ of non-degenerate chains on X : the latter is by definition a dg Lie algebra structure on the complete free Lie algebra $\widehat{L}(C_*(X)[-1])$. It can be checked that this defines a colimit preserving functor from the category of finite simplicial sets to the category $\widehat{\mathbf{DGLA}}$ of complete dg Lie algebras, which extends uniquely to a colimit preserving functor $L(-) : \mathbf{SSet} \rightarrow \widehat{\mathbf{DGLA}}$: for instance, $L(\Delta_1)$ is the well

¹We work with cohomological gradation.

²We have already said that this should mean in particular that $\text{Del}_\infty(L)$ is a Kan complex, that is, every horn $\Lambda_n^i \rightarrow \text{Del}_\infty(L)$ admits a filling $\alpha \in \text{Del}_\infty(L)_n$, but again more is true: we can select a set of distinguished simplexes $T_n \text{Del}_\infty(L) \subset \text{Del}_\infty(L)_n$ for all $n \geq 1$, the *thin* simplexes (namely, the Maurer-Cartan cochains evaluating to zero on the top dimensional simplex), such that every degenerate simplex is thin and every horn admits a unique thin filling. In the theses of Ashley [1] and Dakin [26] it is studied in great detail the notion of a T -complex. This is a simplicial set X together with subsets of distinguished simplexes $T_n X \subset X_n$, $n \geq 1$, called as before the thin simplexes, satisfying the previous two conditions and moreover the third one: if all the faces of a horn are thin, then the remaining face of the unique thin filling is also thin. For instance, if L is a quantum type dg Lie algebra, that is, L is concentrated in degrees ≥ -1 , then $\text{Del}_\infty(L)$ is a T -complex, cf. [39]. In general only the first two conditions are satisfied, and $\text{Del}_\infty(L)$ is what we call a weak T -complex: in [39] this is simply called an ∞ groupoid. Although we won't really have an use for this fact, we believe that it is important to keep it in mind: weak T -complex, and even more so actual T -complexes, are (the nerve of) ∞ groupoids in a much more precise sense than Kan complexes, cf. [1] and [26] where it is proved that T -complexes are the nerves of crossed complexes in groupoids: in fact, hidden in their definition there is a rich algebraic structure given by filling procedures (cf. again [1] and [26] as well as the discussion in Remark 5.2.23) where the remaining axiom for a T -complex plays the role to impose regularity to this structure.

known [67, 16, 17, 85, 22] Lawrence-Sullivan model of the interval. We show (cf. Proposition 5.2.27: this result appears to be new, at least to our knowledge, of course it is just another appearance of the well established mechanism of Koszul duality and twisting cochains [72, 13, 84]) that $L(-)$ is a left adjoint functor, whose right adjoint is $\text{Del}_\infty(-) : \widehat{\mathbf{DGLA}} \rightarrow \mathbf{SSet}$. For instance, this recovers the observation [17] that to give a morphism of dg Lie algebras $L(\Delta_1) \rightarrow M$ is the same as to give Maurer-Cartan elements $x, y \in \text{MC}(M)$ and a Gauge equivalence $e^a * y = x$. The adjunction $L(-) : \mathbf{SSet} \rightleftarrows \widehat{\mathbf{DGLA}} : \text{Del}_\infty(-)$, which is induced via Koszul duality and homotopy transfer from the adjunction $\Omega(-) : \mathbf{SSet}^{op} \rightleftarrows \mathbf{DGCA} : \langle - \rangle$ usual from Sullivan's approach to rational homotopy theory [100, 11] (in the disconnected version developed in [70], where we do *not* restrict to non negatively graded algebras), should play an analog role in Quillen's Lie theoretical approach. To enforce this point we recover the model category structure on $\widehat{\mathbf{DGLA}}$ introduced in [70], modeling disconnected rational homotopy theory (cf. Theorem D in loc. cit.), by transferring the usual model category structure on \mathbf{SSet} along $L(-) : \mathbf{SSet} \rightleftarrows \widehat{\mathbf{DGLA}} : \text{Del}_\infty(-)$, this also automatically shows that the the model category structure on $\widehat{\mathbf{DGLA}}$ is cofibrantly generated and the adjunction is a Quillen adjunction³.

Now for the deformation theory side of the story: as we said the functor $\text{MC}_\infty(-)$ had been studied by Hinich in this context, the reason was to prove the important property of descent of Deligne groupoids [45]. We make yet another digression to sketch the nowadays standard approach to deformation theory via dg Lie algebras [27, 42, 58, 76, 77, 34, 51, 35]. Given a nilpotent dg Lie algebra L , its degree zero part L^0 integrates via the classical Baker-Campbell-Hausdorff product to the exponential group $\exp(L^0)$, moreover, the latter acts in a natural way on the set $\text{MC}(L)$ of solutions $x \in L^1$ of the Maurer-Cartan equation (0.0.1) via the Gauge action (cf. the previous references or Definition 5.2.33) $* : \exp(L^0) \times \text{MC}(L) \rightarrow \text{MC}(L) : (e^a, x) \rightarrow e^a * x$. The Deligne groupoid $\text{Del}(L)$ is the action groupoid associated to the Gauge action: that is, objects are Maurer-Cartan elements $x, y \in \text{MC}(L)$, while the arrows from x to y are the $a \in L^0$ such that $e^a * x = y$, finally, the composition of arrows is given by the Baker-Campbell-Hausdorff product. In classical situations, a formal moduli problem is encoded in a formal groupoid $M : \mathbf{Art}_\mathbb{K} \rightarrow \mathbf{Grpd}$, where $\mathbf{Art}_\mathbb{K}$ is the category of Artin \mathbb{K} -algebras with field of residues isomorphic to \mathbb{K} (by a *formal* groupoid we mean precisely a functor $F : \mathbf{Art}_\mathbb{K} \rightarrow \mathbf{Grpd}$ such that moreover $F(\mathbb{K})$ is the trivial groupoid): given $A \in \mathbf{Art}_\mathbb{K}$, typically $M(A)$ will be the groupoid whose objects are deformations of the structure we are considering over the fat point $\text{Spec } A$, and whose arrows are isomorphisms of deformations. We say that a dg Lie algebra L controls the deformation theory if there is an equivalence of formal groupoids between M and $\text{Del}_L : \mathbf{Art}_\mathbb{K} \rightarrow \mathbf{Grpd} : A \rightarrow \text{Del}(L \otimes \mathfrak{m}_A)$, where $\mathfrak{m}_A \subset A$ is the maximal ideal and $L \otimes \mathfrak{m}_A$ has the nilpotent dg Lie algebra structure given by scalar extension.

It is usually a hard task to find a dg Lie algebra controlling a given deformation problem: to this end several (homotopical) methods have been developed [45, 77, 51, 34, 53, 54, 2] and, as we try to illustrate in the final chapter, descent of Deligne groupoids is a powerful one. To see a typical situation where it applies, we consider a complex manifold X and the deformations of the complex structure on X , that is, the formal moduli problem $\text{Def}_X : \mathbf{Art}_\mathbb{C} \rightarrow \mathbf{Grpd}$ sending $A \in \mathbf{Art}_\mathbb{C}$ to the groupoid whose objects are deformations of X over $\text{Spec } A$ and whose arrows are isomorphisms of deformations. We denote by Θ_X the tangent sheaf on X , with the standard structure of sheaf of Lie algebras. As a well know consequence of Kodaira-Spencer's theory and

³Although we won't elaborate more on this point, we believe that the utility of this approach should be to get rid of the finite assumptions in [70], Theorem D, which should become unnecessary in a Lie-Quillen version of rational homotopy theory, cf. [89].

Cartan's Theorem B, when X is Stein every deformation of X over A is isomorphic to the trivial one $X \times \text{Spec } A$, so the deformation groupoid $\text{Def}_X(A)$ has (essentially) only one object, and every automorphism of the trivial deformation is of the form $X \times \text{Spec } A \xrightarrow{e^\eta} X \times \text{Spec } A$, where η is a global vector field $\eta \in H^0(X; \Theta_X) \otimes \mathfrak{m}_A$: in other words, there is an equivalence of formal groupoids $\text{Def}_X \simeq \text{Del}_{H^0(X; \Theta_X)}$. In general, if $\mathcal{U} = \{U_i\}_{i \in I}$ is a covering of X by Stein open sets, then every deformation of X over A is isomorphic to one obtained by gluing the trivial deformations $U_i \times \text{Spec } A$ along a family of transition automorphisms $U_{ij} \times \text{Spec } A \xrightarrow{e^{\eta_{ij}}} U_{ij} \times \text{Spec } A$ on double intersections, where $\eta_{ij} \in \Theta_X(U_{ij}) \otimes \mathfrak{m}_A$ and as usual $U_{ij} := U_i \cap U_j$, satisfying the cocycle condition $e^{\eta_{ij}} e^{\eta_{jk}} = e^{\eta_{ik}}$ on the triple intersections U_{ijk} : in a slightly fancier language, cf. [45, 34, 2] and Definition 5.3.10, this says that Def_X is equivalent to the (formal) groupoid of descent data $\text{Tot}(\text{Def}_{\mathcal{U}})$ of the formal semicosimplicial groupoid

$$\text{Def}_{\mathcal{U}} : \prod_i \text{Def}_{U_i} \rightrightarrows \prod_{i,j} \text{Def}_{U_{ij}} \rightrightarrows \prod_{i,j,k} \text{Def}_{U_{ijk}} \cdots$$

where the faces are induced by the restrictions $U_j \rightarrow U_{ij}$, $U_i \rightarrow U_{ij}$, etc., as in the usual Čech construction, cf. [45, 34, 2], and by what we said this is equivalent to the formal semicosimplicial groupoid

$$\text{Del}_{\Theta_X(\mathcal{U})} : \prod_i \text{Del}_{\Theta_X(U_i)} \rightrightarrows \prod_{i,j} \text{Del}_{\Theta_X(U_{ij})} \rightrightarrows \prod_{i,j,k} \text{Del}_{\Theta_X(U_{ijk})} \cdots$$

This should be enough motivation to see the importance of the following theorem by Hinich [45], in the refined form given in [34, 35]. Given a semicosimplicial *non negatively graded* dg Lie algebra

$$L_\bullet : L_0 \rightrightarrows L_1 \rightrightarrows L_2 \cdots$$

there is an L_∞ algebra structure on the total complex $\text{Tot}(L_\bullet)$, the usual Čech totalization $\prod_{n \geq 0} L_n[-n]$ of L_\bullet regarded as a semicosimplicial dg space, and an isomorphism of formal groupoids $\text{Del}_{\text{Tot}(L_\bullet)} \cong \text{Tot}(\text{Del}_{L_\bullet})^4$. In the final chapter of the thesis we apply the previous yoga to study several deformation problems in holomorphic Poisson geometry, more about this in the next paragraph. We remark that it is essential for the validity of the theorem that we are working with non negatively graded dg Lie algebras. In section 5.3, Theorem 5.3.6, we prove the analog descent theorem for the functor $\text{Del}_\infty(-)$, more precisely, we prove that given a semicosimplicial complete L_∞ algebra L_\bullet with no grading restrictions there is a natural weak equivalence of simplicial sets $\text{Del}_\infty(\text{Tot}(L_\bullet)) \simeq \text{Tot}(\text{Del}_\infty(L_\bullet))$, where again $\text{Tot}(L_\bullet)$ in the left hand side is the Čech totalization of L_\bullet with its natural L_∞ algebra structure, while this time $\text{Tot}(-)$ in the right hand side is the Bousfield-Kan totalization [9]⁵. We recover the descent theorem from [45, 34, 35] for the ordinary Deligne groupoid in the sequent Section 5.3.1.

In Chapter 6, to illustrate the utility of Hinich's theorem on descent, we consider several deformation problems in holomorphic Poisson geometry: this is the content of the paper [2] by

⁴To make sense of the formal groupoid $\text{Del}_{\text{Tot}(L_\bullet)}$, we remark that for any *non negatively graded* nilpotent L_∞ algebra L it is easy to see that $\text{Del}_\infty(L)$ is the nerve of a groupoid (as every horn $\Lambda_n^i \rightarrow \text{Del}_\infty(L)$, $n \geq 2$, admits a unique filling), so in this case we can define $\text{Del}(L)$ as (with our conventions, the opposite of) this groupoid, and according to Theorem 5.2.37 this is consistent with the usual definition in the dg Lie algebra case.

⁵Possible applications of this more general result, which we hope to give elsewhere, should include descent of Deligne 2-groupoids, cf. [38, 110], where in the second reference the result is used to study deformation quantization of algebraic Poisson varieties. What we are missing is a rigorous comparison between the Deligne 2-groupoid as defined in [38, 110] and the one implicitly encoded in the structure of $\text{Del}_\infty(L)$, where L is a dg Lie algebra in degrees ≥ -1 . To this regard cf. also [110], Remark 8.11.

the author and M. Manetti. Recall that a holomorphic Poisson manifold is a complex manifold X equipped with a Poisson bivector π , that is, a global section $\pi \in H^0(X; \bigwedge^2 \Theta_X)$ satisfying the integrability condition $[\pi, \pi] = 0$, where the bracket is the Schouten-Nijenhuis bracket, and that a closed submanifold $Z \subset X$ is coisotropic if π is in the kernel of the natural projection $\bigwedge^2 \Theta_X \rightarrow \bigwedge^2 \mathcal{N}_{Z|X}$, where $\mathcal{N}_{Z|X}$ is the normal sheaf of Z in X . In particular the Lichnerowicz-Poisson differential $d_\pi = [\pi, \cdot]$ (Schouten-Nijenhuis bracket) induces a dg structure on the Gerstenhaber algebra $\bigwedge \Theta_X$ of holomorphic polyvector fields on X : if $Z \subset X$ is coisotropic this factors to a dg structure on the graded algebra $\bigwedge \mathcal{N}_{Z|X}$. We study deformations of holomorphic Poisson manifolds, of a pair (Poisson manifold, coisotropic submanifold) and finally embedded coisotropic deformations, and using descent of Deligne groupoids in all cases we determine controlling dg Lie algebras, see Theorems 6.2.4, 6.3.3 and 6.3.6. As a first application of these results, we show in Corollary 6.3.4 an analog of Kodaira stability theorem for coisotropic submanifolds. Other applications are given in Section 6.4. Recall that the anchor map $\pi^\# : \Omega_X \rightarrow \bigwedge \Theta_X$ is the morphism of sheaves of dg \mathcal{O}_X -algebras, where Ω_X is the sheaf of holomorphic differential forms, uniquely defined so that $\pi^\#(f) = f$ for all $f \in \mathcal{O}_X$ ⁶: if $Z \subset X$ is coisotropic this factors to a morphism of sheaves of dg \mathcal{O}_Z -algebras $\pi^\# : \Omega_Z \rightarrow \bigwedge \mathcal{N}_{Z|X}$. In [47] Hitchin proves that, if (X, π) is a compact Kähler Poisson manifold, then every element in the image of $\pi^\# : H^1(X, \Omega_X^1) \rightarrow H^1(X, \Theta_X)$ is the Kodaira-Spencer class of a deformation of the pair (X, π) over a germ of smooth curve. We recover and extend this result in Theorem 6.4.11, following the method of [33], and we consider the analog situation for embedded coisotropic deformations. We show in Theorem 6.4.10 that under some mild additional assumption (namely, if the Hodge to de Rham spectral sequence of Z degenerates at E^1 , in particular if Z is compact Kähler) then every element in the image of the anchor map $\pi^\# : H^0(Z; \Omega_Z^1) \rightarrow H^0(Z; \mathcal{N}_{Z|X})$ is the Kodaira-Spencer class of a coisotropic embedded deformation of Z in (X, π) over a germ of smooth curve. Such result applied to a compact Kähler Lagrangian submanifold Z of a holomorphic symplectic manifold X shows that every small deformation in X of Z is Lagrangian and the Hilbert scheme of X is smooth at Z ; when X is compact Kähler we recover in this way a classical result by Voisin and McLean [82, 104].

Up to now we have been trying to talk about dg Lie algebras when possible, mainly for simplicity, but already at some points we couldn't avoid to use the language of L_∞ algebras. It seems a bit late now to recall the definition, but here it goes: an L_∞ algebra structure on a space L is equivalently the datum of a dg coalgebra structure on the reduced symmetric coalgebra $\overline{S}(L[1]) = \bigoplus_{n \geq 1} L[1]^{\odot n}$ (where $L[1]^{\odot n}$ is the n -th symmetric power, and we remark that we work with coinvariants). We will denote by $\overline{\text{CE}}(L[1])$ the graded Lie algebra of coderivations of $\overline{S}(L[1])$, thus an L_∞ structure on L is the datum of $Q \in \overline{\text{CE}}(L[1])$ such that Q has degree one and squares to zero. As the coalgebra $\overline{S}(L[1])$ is cofree, the projection $p : \overline{S}(L[1]) \rightarrow L[1]$ induces an isomorphism $\overline{\text{CE}}(L[1]) = \prod_{n \geq 1} \text{Hom}(L[1]^{\odot n}, L) : Q \rightarrow (q_1, \dots, q_n, \dots)$ of graded space, thus $Q \in \overline{\text{CE}}(L[1])$ is determined by the sequence of Taylor coefficients $q_n \in \text{Hom}^1(L[1]^{\odot n}, L[1])$. To see the link with dg Lie algebras we shift again the gradation, then the q_n go into a sequence of higher brackets $l_n \in \text{Hom}^{2-n}(L^{\wedge n}, L)$ (where $L^{\wedge n}$ is the n -th exterior power), and the identity $[Q, Q] = 0$ translates into a series of identities for these brackets: the first few say that $l_1 : L^i \rightarrow L^{i+1}$ is a differential on L satisfying the Leibniz identity with respect to the bracket $l_2 : L^i \otimes L^j \rightarrow L^{i+j}$, and the latter satisfies the Jacobi identity up to the homotopy $l_3 : L^i \otimes L^j \otimes L^k \rightarrow L^{i+j+k-1}$. In particular, a dg Lie algebra structure on L is the same as an L_∞ structure $Q \in \overline{\text{CE}}(L[1])$ such that $q_n = 0$ for all $n \geq 3$. The utility of dealing with L_∞ algebras rather than dg Lie algebras is that while they have the same homotopy category (once we have defined the appropriate notions of L_∞ morphism and weak equivalence, cf. [46]) L_∞ algebras are better behaved with respect to homotopical

⁶This determines $\pi^\#$ on $\Omega_X^{\leq 1}$, since we must have $\pi^\#(df) = [\pi, f]$ for all $f \in \mathcal{O}_X$, and thus $\pi^\#$, since $\Omega_X^{\leq 1}$ is a set of generators of the algebra Ω_X .

constructions. For instance two L_∞ algebras are isomorphic in the homotopy category if and only if there are quasi-inverses weak equivalences between them. As another example, if $M \subset L$ is a quasi-isomorphic subcomplex and L carries a dg Lie algebra structure, in general there is no induced dg Lie algebra structure on M : on the other hand every L_∞ algebra structure on L transfers naturally to a weakly equivalent L_∞ algebra structure on M , and moreover we have explicit formulas for the transferred structure and the weak equivalence. This is the content of the already mentioned homotopy transfer theorem, which will be reviewed in Section 2.2, following the proof we learned from the arXiv version of [31]⁷: as homotopy transfer will be a fundamental tool throughout the thesis, we spend some time proving some technical necessary lemmas.

The price we paid to obtain this more flexible theory is of course that we complicated the structure of dg Lie algebra by adding an infinite number of higher brackets and an infinite number of higher Jacobi relations they must satisfy, which makes it hard to exhibit explicit L_∞ algebra structures on a given space: another part of this thesis deals with certain explicit constructions of higher brackets and L_∞ structures and the study of their homotopical properties.

In Section 3.3 we study and generalize Fiorenza-Manetti's construction of an L_∞ structure on the mapping cocone $\text{co}C(f)$ of a morphism of dg Lie algebras $f : L \rightarrow M$ [31]. More in general, we study homotopy equalizers of a pair of ∞ morphisms between two ∞ algebras, by which we mean either A_∞ (dg algebras associative up to a system of coherent homotopies), C_∞ (their commutative version) or L_∞ algebras: we prove existence of the homotopy equalizer (Theorem 3.3.1), which is not trivial since the category of ∞ algebras and ∞ morphisms is not complete. Together with the computations in [31], cf. also [51, 22], we can deduce explicit formulas when the target ∞ algebra is a dg (resp.: associative, commutative, Lie) algebra: as an example, we give explicit formulas for the mapping cocone of an L_∞ morphism of dg Lie algebras, generalizing the ones from [31].

In Section 4.1 we study Th. Voronov's construction(s) of L_∞ algebra structures via higher derived brackets [105, 106], for instance this has been successfully applied to the study of coisotropic deformations in differentiable Poisson geometry⁸ [19, 20, 36, 91, 92] and simultaneous deformations of algebraic structures [37]. The algebraic setup requires a graded Lie algebra M together with graded Lie subalgebras $L, A \subset M$, with A abelian, such that $M = L \oplus A$ as a graded space; we denote by $P : M \rightarrow A$ the projection with kernel L . In these hypotheses, let $D \in \text{Der}(M)$ a derivation of the Lie algebra structure such that $D(L) \subset L$, in [106] Voronov defines a sequence of higher derived brackets $\Phi(D)_n : A^{\odot n} \rightarrow A$, $n \geq 1$ on A associated to D ⁹ by the formula $\Phi(D)_n(a_1 \odot \cdots \odot a_n) = P[\cdots [Da_1, a_2] \cdots, a_n]$ (graded symmetry follows from the hypothesis that A is abelian, cf. loc. cit.). We denote by $\text{Der}(M, L) \subset \text{Der}(M)$ the graded Lie subalgebra of derivations D such that $D(L) \subset L$, thus higher derived brackets define a morphism of graded spaces $\Phi : \text{Der}(M, L) \rightarrow \overline{\text{CE}}(A) : D \rightarrow (\Phi(D)_1, \dots, \Phi(D)_n, \dots)$. This construction of higher derived brackets is similar but slightly different to another construction also due to Voronov [105]: in the above setup, this time we associate a sequence of higher derived brackets $\Phi(m)_n : A^{\odot n} \rightarrow A$, $n \geq 0$, to every $m \in M$, always by the formula $\Phi(m)_n(a_1 \odot \cdots \odot a_n) = P[\cdots [[m, a_1], a_2] \cdots, a_n]$ when $n \geq 1$, with moreover the 0-th bracket $\Phi(m)_0 : A^{\odot 0} = \mathbb{K} \rightarrow A : 1 \rightarrow Pm$. The difference between the two constructions is that this time we get a morphism of graded spaces $\Phi : M \rightarrow \text{CE}(A)$, where $\text{CE}(A)$ is the graded Lie algebra of coderivations of the *non reduced* symmetric coalgebra $S(A) = \bigoplus_{n \geq 0} A^{\odot n}$ over A (the difference will become more apparent in the non-abelian case). In [106] the following two facts are proven, which are the key to our approach:

⁷The only thing we add is the extension of the argument to see homotopy transfer for C_∞ algebra structures.

⁸The method fails in the holomorphic setting since it depends essentially on the choice of an identification of the normal bundle $N_{Z|X}$ and a tubular neighborhood of Z in X . See Corollary 6.3.8 for a comparison with our methods in the (rare) case that such a choice is nonetheless possible.

⁹Notice that we use a different notation than the one in [105, 106].

1. In Theorem 3 of loc. cit. it is proved that that the correspondences $\Phi : \text{Der}(M, L) \rightarrow \overline{\text{CE}}(A)$ and $\Phi : M \rightarrow \text{CE}(A)$ are morphism of graded Lie algebras: in particular, this tells us that if D has degree one and $D^2 = 0$ (in most applications $D = [l, -]$ for some degree one $l \in L$ such that $[l, l] = 0$) then $\Phi(D)$ is an L_∞ structure on $A[-1]$.
2. Moreover, in Section 4 of loc. cit. it is proved that in this case the L_∞ algebra $(A[-1], \Phi(D))$ is a homotopy fiber of the inclusion of dg Lie algebras $i : (L, D, [\cdot, \cdot]) \rightarrow (M, D, [\cdot, \cdot])$ ¹⁰, in other words, it is weakly equivalent to Fiorenza-Manetti's mapping cocone $\text{coC}(i)$.

As it is almost immediate to exhibit an explicit contraction from $\text{coC}(i)$ to $A[-1]$, this suggests that $\Phi(D)$ should be induced via homotopy transfer along this contraction, and in fact this is the case as will follow from our results, but the interesting fact here is that the existence of the contraction does not depend on the hypothesis that $A \subset M$ is an abelian Lie subalgebra, showing a possible way to generalize Voronov's construction when we drop this hypothesis. This is what we do in Section 4.1, following the paper [3] by the author. We maintain the assumption that $A \subset M$ is a graded Lie subalgebra but we drop the one that it is abelian¹¹. Following a more refined version of the sketched argument via homotopy transfer, depending also in an essential way by the classification of L_∞ extensions made in [24, 83, 69] (this will be briefly reviewed in Section 1.3.3), we define correspondences $\Phi : \text{Der}(M, L) \rightarrow \overline{\text{CE}}(A)$ and $\Phi : M \rightarrow \text{CE}(A)$ in this more general setup, reducing to the ones by Voronov when A is abelian and such that the above items 1 and 2 remain true (this is shown in Theorem 4.1.6 and Theorem 4.1.7: as for item 1 we prove something more, that the correspondence $\text{Der}(M, L) \times M \rightarrow \text{CE}(A) : (D, m) \rightarrow \Phi(D) + \Phi(m)$ is a morphism of graded Lie algebras, where $\text{Der}(M, L) \times M$ is the semi-direct product). See Definition 4.1.3 for explicit formulas, these involve Bernoulli numbers. Theorem 4.1.6 remains interesting also in the case $L = 0$, where it clarifies some results from [7], Section 4. As a first application of these theorems, we recover an L_∞ generalization of the adjoint morphism of a dg Lie algebra and a geometrically appealing criterion for homotopy abelianity due to Chuang and Lazarev [23], cf. Example 4.1.18, as well as the construction by Getzler [40] of an L_∞ structure on the suspension of the negatively graded part of any dg Lie algebra, generalizing the well known construction of a Lie algebra structure on the degree minus one part of a quantum type dg Lie algebra, cf. Example 4.1.23. Other applications are given in the sequent sections.

In Section 4.2 we study the classical [64] construction of the Koszul brackets $\mathcal{K}(\Delta)_n : A^{\odot n} \rightarrow A$, $n \geq 1$, associated to an operator $\Delta \in \text{End}(A)$ on a graded commutative algebra A . As observed in [105, 106] these can be recovered in a natural way as higher derived brackets: as an application of our non-abelian construction we introduce a similar sequence of higher brackets $\mathcal{K}(Q)_n : L^{\odot n} \rightarrow L$, $n \geq 0$, associated this time to a coderivation $Q \in \text{CE}(L)$ on the symmetric coalgebra $S(L)$ over a graded left pre-Lie algebra L ¹². We recover Koszul's construction when $L = A$ is a graded commutative algebra and $Q = \Delta$ is a linear coderivation. This works as follows: for all $x \in L$ we denote by $\sigma_x \in \text{Hom}(L^{\odot 0}, L) \subset \text{CE}(L)$ the constant coderivation $\sigma_x : L^{\odot 0} = \mathbb{K} \rightarrow L : 1 \rightarrow x$, and by $\nabla_x \in \text{End}(L) \subset \text{CE}(L)$ the left adjoint $\nabla_x : L \rightarrow L : y \rightarrow x \triangleright y$, where \triangleright is the left pre-Lie product. It is well known that to say that \triangleright is a left pre-Lie product on L is equivalent as to say that $\sigma^\nabla : L \rightarrow \text{CE}(L) : x \rightarrow (\sigma_x, \nabla_x, 0, \dots, 0, \dots)$ is a morphism of graded Lie algebras (where the graded Lie algebra structure on L is given by the commutator), this sends L isomorphically onto

¹⁰This explains the utility of higher derived brackets in deformation theory, as homotopy fibers are naturally associated to semi-trivial deformation problems, see e.g. [77].

¹¹The main theorems remain true even if we assume that A is just a complement of L in M , but the explicit formulas in Definition 4.1.3 don't hold anymore, cf. Remark 4.1.14

¹²Recall that a product $\triangleright : L \otimes L \rightarrow L$ is left (resp.: right) pre-Lie if the associator $A(x, y, z) = (x \triangleright y) \triangleright z - x \triangleright (y \triangleright z)$ is graded symmetric in the first (resp.: last) two arguments: then the commutator $[x, y] = x \triangleright y - (-1)^{|x||y|} y \triangleright x$ is a Lie bracket on L . It is a trivial matter to translate the following discussion for graded right pre-Lie algebras, cf. Definition 4.2.1.

its image $L^\nabla := \sigma^\nabla(L)$. We have a decomposition $\mathrm{CE}(L) = \overline{\mathrm{CE}}(L) \oplus L^\nabla$ as in the hypotheses of Theorem 4.1.6, thus a morphism of graded Lie algebras $\mathcal{K} : \mathrm{CE}(L) \rightarrow \mathrm{CE}(L^\nabla) \xrightarrow{\cong} \mathrm{CE}(L)$, where the first arrow is given by (non-abelian, if \triangleright is not graded commutative) higher derived brackets; this is the desired \mathcal{K} . In Proposition 4.2.6 we prove that \mathcal{K} is an automorphism of the graded Lie algebra $\mathrm{CE}(L)$: the inverse $\mathcal{K}^{-1} : \mathrm{CE}(L) \rightarrow \mathrm{CE}(L)$ generalizes the construction of higher brackets on pre-Lie algebras given by the author in the paper [4], where they are called Kapranov brackets. The reason for this is the following particularly interesting example of such brackets. Given a compact Kähler manifold X , there is a pre-Lie algebra structure on the complex $\mathcal{A}^{0,*}(T_X)$ of Dolbeault forms with coefficients in the tangent bundle T_X , induced by the usual Chern connection on T_X . In this case the Kapranov brackets $\mathcal{K}^{-1}(\bar{\partial})$, where $\bar{\partial}$ is Dolbeault differential on $\mathcal{A}^{0,*}(T_X)$, recover the L_∞ algebra structure on $\mathcal{A}^{0,*}(T_X)[-1]$ introduced by Kapranov in [56], which has recently attracted much attention [18, 21, 25, 44, 66] due to its role in derived geometry. As an application of our results, we prove the expected fact that Kapranov's L_∞ algebra structure on $\mathcal{A}^{0,*}(T_X)[-1]$ is homotopy abelian over the field \mathbb{C} of complex numbers: cf. Corollary 4.2.10. Coming back to Koszul brackets, in the case when $L = A$ is a commutative graded algebra, we recover as a byproduct of our analysis a recent result by Markl [79, 80], namely, that the Koszul brackets $\mathcal{K} : \mathrm{CE}(A) \rightarrow \mathrm{CE}(A)$ are the twisting $\mathcal{K} = F - F^{-1}$ by a natural automorphism $F : \bar{S}(A) \rightarrow \bar{S}(A)$. As a final application, in Section 4.2.1 we prove an interesting result also proved by Braun and Lazarev [12] with different methods. This regards commutative BV_∞ algebras, that is, the homotopy version of Batalin-Vilkovisky (BV) algebras introduced in [65]: BV algebras and homotopy BV algebras are important in algebraic topology, differential geometry and mathematical physics. Associated to every BV algebra there is a dg Lie algebra, and similarly associated to a commutative BV_∞ algebra there is an L_∞ algebra. In the paper [96] the authors prove that if a BV algebra satisfies a certain degeneration property (examples include the de Rham algebra of a symplectic manifold or the Dolbeault algebra of a Calabi-Yau manifold) then the associated dg Lie algebra is homotopy abelian, and in [12] this result is extended to commutative BV_∞ algebras satisfying an appropriate analog degeneration property and the associated L_∞ algebra: we give an alternative proof of this latter fact as an application of Theorem 4.1.6 and a criterion for homotopy abelianity by M. Manetti (cf. [53, 54] and Theorem 3.3.5), following the method of proof of the original formality result in [96] we learned from the paper [52].

Warnings: We will always work over a field \mathbb{K} of characteristic zero. Graded spaces are cohomologically \mathbb{Z} -graded.

Notation 0.0.1. Given a category \mathbf{C} and objects X, Y in \mathbf{C} , we will denote by $\mathbf{C}(X, Y)$ the set of morphisms in \mathbf{C} from X to Y .

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Chapter 1

Review of ∞ algebras

Roughly speaking, an algebraic structure is homotopy invariant if it can be transferred along homotopy equivalences. The idea to consider these kind of structures goes back to the 60s in the seminal work of Jim Stasheff [97, 98, 99], who introduced A_∞ algebras and gave the first spectacular application of these ideas by proving that a space admits an A_∞ algebra structure if and only if it has the weak homotopy type of an associative monoid. In the following decades homotopy invariant algebraic structures were studied mostly in algebraic topology, notably in the works of Boardman-Vogt [8], Kadeishvili [55] (C_∞ algebras), Schlessinger-Stasheff [94] (L_∞ algebras) among others, until the 90s when they started to find applications in other areas of geometry and mathematical physics as well, cf. for instance the beautiful papers by Kontsevich on deformation quantization [58] and homological mirror symmetry [59]: nowadays, as the relevance of higher (homotopical, categorical) algebra in mathematics has been widely recognized, they are extensively studied and an important tool in many situations.

In this thesis we deal mainly with L_∞ algebras, that is, dg Lie algebras where the Jacobi identity has been relaxed up to a system of coherent homotopies, but occasionally we will also consider A_∞ algebras (algebras associative up to a system of coherent homotopies) and C_∞ algebras (their commutative version). In Sections 1.1, 1.2 and 1.3 we review the basic definitions. We will spend more time on L_∞ algebras: in Section 1.3.1 we introduce complete L_∞ algebras and the Maurer-Cartan equation, in Section 1.3.2 we review convolution L_∞ algebras and finally in Section 1.3.3 we review the classification of L_∞ extensions from [24, 83, 69].

1.1 Review of graded spaces

We work over a field \mathbb{K} of characteristic zero; graded means \mathbb{Z} -graded. We denote by \mathbf{G} the category of graded \mathbb{K} -vector spaces $V = \bigoplus_{i \in \mathbb{Z}} V^i$ and degree preserving morphisms. An element $v \in V$ will usually be a homogeneous one and its degree will be denoted by $|v|$.

\mathbf{G} is a symmetric monoidal category (cf. [74]) via the usual tensor product (with the gradation $(V \otimes W)^i = \bigoplus_{j+k=i} (V^j \otimes W^k)$), the symmetry given by the Koszul isomorphism $V \otimes W \xrightarrow{\cong} W \otimes V : v \otimes w \rightarrow (-1)^{|v||w|} w \otimes v$ and the unit \mathbb{K} , seen as a graded space concentrated in degree zero. It is defined an internal $\text{Hom}(-, -) : \mathbf{G}^{op} \times \mathbf{G} \rightarrow \mathbf{G}$ functor, sending spaces V and W to $\text{Hom}(V, W) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}^i(V, W)$, where $\text{Hom}(V, W)$ is the space of all linear maps from V to W (forgetting the gradation) and $\text{Hom}^i(V, W)$ is the space of those f such that $f(V^k) \subset W^{k+i}$, $\forall k \in \mathbb{Z}$: $\text{Hom}(-, -)$

and \otimes are related by the usual exponential law, saying that for each graded space V the functors $-\otimes V : \mathbf{G} \rightleftarrows \mathbf{G} : \text{Hom}(V, -)$ form an adjoint pair. We denote by $\text{End}(V) := \text{Hom}(V, V)$.

We denote by \mathbf{DG} the category of differential graded (dg) spaces $(V, d_V) - d_V \in \text{End}^1(V)$, $d_V^2 = 0$ - and dg morphisms $f : (V, d_V) \rightarrow (W, d_W) - fd_V = d_W f$ - between them: it has a symmetric monoidal category structure with tensor product $(V, d_V) \otimes (W, d_W) = (V \otimes W, d_V \otimes \text{id}_W + \text{id}_V \otimes d_W)$ and an internal $\text{Hom}(-, -) : \mathbf{DG}^{op} \times \mathbf{DG} \rightarrow \mathbf{DG}$ functor defined by $\text{Hom}((V, d_V), (W, d_W)) = (\text{Hom}(V, W), d_{\text{Hom}(V, W)}(f) = d_W \circ f - (-1)^{|f|} f \circ d_V)$, again the two are related by the exponential law.

For $n \geq 0$ we denote by $V^{\otimes n}$ the n -th tensor power over V , and by $V^{\odot n}$ and $V^{\wedge n}$ the n -th symmetric power and the n -th exterior power over V respectively, which are the space of *coinvariants* of $V^{\otimes n}$ under the natural, resp. alternate, action of the symmetric group S_n (according to the usual Koszul rule for twisting signs); $V^{\odot 0} = V^{\wedge 0} = V^{\otimes 0} := \mathbb{K}$.

For a graded space V and an integer k we denote by $V[k]$ the shifted space $V[k]^i = V^{k+i}$, and by $s^{-k} \in \text{Hom}^{-k}(V, V[k])$ the shift map. We follow the convention according to which degrees are shifted on the left, that is, we will always identify $V[i]$ with $\mathbb{K}[i] \otimes V$: for instance this implies that the canonical isomorphism $V[1]^{\otimes n} \cong V^{\otimes n}[n]$ sends $s^{-1}v_1 \otimes \cdots \otimes s^{-1}v_n$ to $(-1)^{\sum_{j=1}^n (n-j)|v_j|} s^{-n}(v_1 \otimes \cdots \otimes v_n)$. There is also a canonical isomorphism $\text{Hom}(V, W)[-i+j] \cong \text{Hom}(V[i], W[j])$, sending f (more precisely $s^{i-j}f$) to the map $V[i] \rightarrow W[j] : s^{-i}v \rightarrow (-1)^{|f|} s^{-j}f(v)$.

Definition 1.1.1. Décalage is the composite isomorphism

$$\text{déc} : \text{Hom}(V^{\otimes n}, W) \xrightarrow{\cong} \text{Hom}(V^{\otimes n}[n], W[1][n-1]) \xrightarrow{\cong} \text{Hom}(V[1]^{\otimes n}, W[1][n-1]), \quad (1.1.1)$$

explicitly given by

$$\text{déc}(f)(s^{-1}v_1 \otimes \cdots \otimes s^{-1}v_n) = (-1)^{n|f| + \sum_{j=1}^n (n-j)|v_j|} s^{-1}f(v_1 \otimes \cdots \otimes v_n).$$

This restricts to an isomorphism

$$\text{déc} : \text{Hom}(V^{\wedge n}, W) \xrightarrow{\cong} \text{Hom}(V[1]^{\odot n}, W[1][n-1]). \quad (1.1.2)$$

We denote by \mathbf{GA} and \mathbf{GA}_{au} the categories of graded associative algebras and graded associative algebras with a unit $v : \mathbb{K} \rightarrow A : 1 \rightarrow 1_A$ and an augmentation $\varepsilon : A \rightarrow \mathbb{K}$ respectively: recall the reduced algebra functor $\mathbf{GA}_{au} \rightarrow \mathbf{GA}$ sending A to $\bar{A} = \text{Ker } \varepsilon$. We denote by $\mathbf{GCA} \subset \mathbf{GA}$ and $\mathbf{GCA}_{au} \subset \mathbf{GA}_{au}$ the full subcategories of graded commutative algebras (**Warning:** in particular, when we talk about a commutative graded algebra we will always tacitly assume that it is also associative). We denote by \mathbf{GC} and \mathbf{GC}_{au} the categories of graded coassociative coalgebras and graded coassociative coalgebras with a counit $\mu : C \rightarrow \mathbb{K}$ and a coaugmentation $\epsilon : \mathbb{K} \rightarrow C$ respectively, and by $\mathbf{GCC} \subset \mathbf{GC}$ and $\mathbf{GCC}_{au} \subset \mathbf{GC}_{au}$ the full subcategories of graded cocommutative coalgebras (with a similar warning as before). Let $\Delta : C \rightarrow C^{\otimes 2}$ be the coproduct, $1_C := \epsilon(1)$ and $\bar{C} := \text{Ker } \mu$ (so that there is a splitting $C = 1_C \mathbb{K} \oplus \bar{C}$), and finally $\bar{\Delta} : \bar{C} \rightarrow \bar{C}^{\otimes 2} : c \rightarrow \Delta(c) - 1_C \otimes c - c \otimes 1_C$: then the reduced coalgebra functor $\mathbf{GC}_{au} \rightarrow \mathbf{GC}$ sends $(C, \Delta, \mu, \epsilon)$ to $(\bar{C}, \bar{\Delta})$. We denote by $\mathbf{GNC} \subset \mathbf{GC}$ the full subcategory of coalgebras (C, Δ) which are locally conilpotent, by which we mean that $C = \bigcup_{n \geq 1} \text{Ker } \Delta^n$, where $\Delta^n : C \rightarrow C^{\otimes n+1}$ is the n -th iterated coproduct, and we denote by $\mathbf{GNC}_{au} \subset \mathbf{GC}_{au}$ the preimage of \mathbf{GNC} under the reduced coalgebra functor; similarly for the full subcategories $\mathbf{GNCC} \subset \mathbf{GCC}$ and $\mathbf{GNCC}_{au} \subset \mathbf{GCC}_{au}$. Recall that a graded bialgebra (B, m, Δ) is a graded space B with a graded associative algebra structure $m : B^{\otimes 2} \rightarrow B$ and a graded coassociative coalgebra structure $\Delta : B \rightarrow B^{\otimes 2}$ such that Δ is a morphism of algebras and m is a morphism of coalgebras, where we consider $B^{\otimes 2}$ with the

induced (co)algebra structure. (B, m, Δ) is commutative if such is (B, m) and cocommutative if such is (B, Δ) . Graded bialgebras form a category \mathbf{GB} , the category \mathbf{GB}_{au} consists of bialgebras with a unit $\mathbb{K} \rightarrow B$ and a counit $B \rightarrow \mathbb{K}$ such that the first one is also a coaugmentation and the second one is also an augmentation. The reduced bialgebra functor $\mathbf{GB}_{au} \rightarrow \mathbf{GB}$ is defined by combining the algebra and coalgebra cases. Finally, we denote by \mathbf{GLA} the category of graded Lie algebras. All of the above categories have a dg analogous for which we use the same notation preceded by a \mathbf{D} , for instance \mathbf{DGLA} will be the category of dg Lie algebras, \mathbf{DGC} the category of dg coassociative coalgebras, and so on: in this case the differential has to be a derivation (resp.: coderivation, biderivation) of the corresponding algebraic structure. The reader unfamiliar with these categories is referred to the first chapter of [101].

The (resp.: reduced) tensor space over V is the graded space $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ (resp.: $\bar{T}(V) = \bigoplus_{n \geq 1} V^{\otimes n}$), the (resp.: reduced) symmetric space over V is the graded space $S(V) = \bigoplus_{n \geq 0} V^{\odot n}$ (resp.: $\bar{S}(V) = \bigoplus_{n \geq 1} V^{\odot n}$). These spaces carry several algebraic structures as we now recall. The tensor space $T(V)$ carries two standard bialgebra structures, one commutative and the other cocommutative, both augmented with unit $\mathbb{K} = V^{\otimes 0} \rightarrow T(V)$ and counit $T(V) \rightarrow V^{\otimes 0} = \mathbb{K}$ the natural inclusion and projection. We first recall the commutative bialgebra structure on $T(V)$: its reduced bialgebra is given by the deconcatenation coproduct $\bar{\Delta} : \bar{T}(V) \rightarrow \bar{T}(V) \otimes \bar{T}(V)$

$$\bar{\Delta} : v_1 \otimes \cdots \otimes v_n \rightarrow \sum_{i=1}^{n-1} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n), \quad (1.1.3)$$

and the shuffle product $\otimes : \bar{T}(V) \otimes \bar{T}(V) \rightarrow \bar{T}(V)$

$$\otimes : (v_1 \otimes \cdots \otimes v_p) \otimes (v_{p+1} \otimes \cdots \otimes v_{p+q}) \rightarrow \sum_{\sigma \in S(p,q)} \varepsilon(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(p+q)}, \quad (1.1.4)$$

where we denote by $S(p, q)$ the set of (p, q) -unshuffles, i.e., those permutations $\sigma \in S_{p+q}$ such that $\sigma(k) < \sigma(k+1)$ for $k \neq p$, and by $\varepsilon(\sigma) = \varepsilon(\sigma; v_1, \dots, v_n)$ the Koszul sign. As for the cocommutative bialgebra structure on $T(V)$, its reduced bialgebra is given by the concatenation product $\otimes : (v_1 \otimes \cdots \otimes v_p) \otimes (v_{p+1} \otimes \cdots \otimes v_{p+q}) \rightarrow v_1 \otimes \cdots \otimes v_{p+q}$ and the unshuffle coproduct $\bar{\Delta}_{sh} : v_1 \otimes \cdots \otimes v_n \rightarrow \sum_{i=1}^{n-1} \sum_{\sigma \in S(i, n-i)} \varepsilon(\sigma) (v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)}) \otimes (v_{\sigma(i+1)} \otimes \cdots \otimes v_{\sigma(n)})$, we will have less use for it. Finally $S(V)$ carries a bialgebra structure which is both commutative and cocommutative, and augmented as before by $\mathbb{K} = V^{\odot 0} \rightarrow S(V) \rightarrow V^{\odot 0} = \mathbb{K}$, whose reduced bialgebra is given by the concatenation product $\odot : \bar{S}(V) \otimes \bar{S}(V) \rightarrow \bar{S}(V)$

$$\odot : (v_1 \odot \cdots \odot v_p) \otimes (v_{p+1} \odot \cdots \odot v_{p+q}) \rightarrow v_1 \odot \cdots \odot v_{p+q} \quad (1.1.5)$$

and the unshuffle coproduct $\bar{\Delta}_{sh} : \bar{S}(V) \rightarrow \bar{S}(V) \otimes \bar{S}(V)$

$$\bar{\Delta}_{sh} : v_1 \odot \cdots \odot v_n \rightarrow \sum_{i=1}^{n-1} \sum_{\sigma \in S(i, n-i)} \varepsilon(\sigma) (v_{\sigma(1)} \odot \cdots \odot v_{\sigma(i)}) \otimes (v_{\sigma(i+1)} \odot \cdots \odot v_{\sigma(n)}). \quad (1.1.6)$$

Remark 1.1.2. Symmetrization

$$\text{sym} : \bar{S}(V) \rightarrow \bar{T}(V) : v_1 \odot \cdots \odot v_n \rightarrow \sum_{\sigma \in S_n} \varepsilon(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

is a morphism of bialgebras from $\bar{S}(V)$ with the (concatenation product, unshuffle coproduct) structure to $\bar{T}(V)$ with the (shuffle product, deconcatenation coproduct) structure.

Definition 1.1.3. The graded coalgebra $(\overline{T}(V), \overline{\Delta})$, where $\overline{\Delta}$ is the deconcatenation coproduct, is called the reduced tensor coalgebra over V : this is the cofree locally conilpotent coassociative coalgebra over V , that is, the functor $(\overline{T}(-), \overline{\Delta}) : \mathbf{G} \rightarrow \mathbf{GNC}$ is right adjoint to the forgetful functor (we remark that this is no longer true if we consider $(\overline{T}(-), \overline{\Delta})$ as a functor into the larger category \mathbf{GC}). The graded coalgebra $(T(V), \Delta)$ equipped with the deconcatenation coproduct is called the tensor coalgebra over V , the functor $(T(-), \Delta) : \mathbf{G} \rightarrow \mathbf{GNC}_{au}$ is right adjoint to the composition of the reduced coalgebra functor and the forgetful functor.

The graded cocommutative coalgebra $(S(V), \overline{\Delta}_{sh})$ (resp.: $(\overline{S}(V), \overline{\Delta}_{sh})$), where Δ_{sh} is the unshuffle coproduct, is called the (resp.: reduced) symmetric coalgebra over V : the functor $(\overline{S}(-), \overline{\Delta}_{sh}) : \mathbf{G} \rightarrow \mathbf{GNCC}$ (resp.: $(S(-), \Delta_{sh}) : \mathbf{G} \rightarrow \mathbf{GNCC}_{au}$) is right adjoint to the forgetful functor (resp.: composed with the reduced coalgebra functor).

The graded algebras $(T(V), \otimes)$, $(T(V), \otimes)$ and $(S(V), \odot)$ (resp.: $(\overline{T}(V), \otimes)$, $(\overline{T}(V), \otimes)$ and $(\overline{S}(V), \odot)$) are called respectively the (resp.: reduced) tensor algebra, shuffle algebra and symmetric algebra over V . The functors $(\overline{T}(-), \otimes) : \mathbf{G} \rightarrow \mathbf{GA}$, $(\overline{S}(-), \odot) : \mathbf{G} \rightarrow \mathbf{GCA}$ are left adjoint to the forgetful functor, the functors $(T(-), \otimes) : \mathbf{G} \rightarrow \mathbf{GA}_{au}$, $(S(-), \odot) : \mathbf{G} \rightarrow \mathbf{GCA}_{au}$ are left adjoint to the composition of of the reduced algebra functor and the forgetful functor.

Finally, the graded coalgebra $(T(V), \overline{\Delta}_{sh})$ (resp.: $(\overline{T}(V), \overline{\Delta}_{sh})$), where Δ_{sh} is the unshuffle coproduct, is called the (resp.: reduced) unshuffle coalgebra over V .

Notation 1.1.4. Given a linear map $F : \overline{T}(V) \rightarrow \overline{T}(W)$ (resp.: $F : \overline{S}(V) \rightarrow \overline{S}(W)$) we denote by $F_n^k : V^{\otimes n} \rightarrow W^{\otimes k}$ (resp.: $F_n^k : V^{\odot n} \rightarrow W^{\odot k}$) the composition $V^{\otimes n} \hookrightarrow \overline{T}(V) \xrightarrow{F} \overline{T}(W) \rightarrow W^{\otimes k}$ (resp.: $V^{\odot n} \hookrightarrow \overline{S}(V) \xrightarrow{F} \overline{S}(W) \rightarrow W^{\odot k}$), where the last arrow is the natural projection.

Given a graded associative algebra A let $m_n : A^{\otimes n} \rightarrow A$, $n \geq 2$, be the $(n-1)$ -th iterated product.

Definition 1.1.5. For graded spaces V, W and an integer $n \geq 2$ extension of scalars by A is the morphism $(-)_A : \text{Hom}(V^{\otimes n}, W) \rightarrow \text{Hom}((A \otimes V)^{\otimes n}, A \otimes W)$ defined by

$$(-)_A : f \rightarrow \{f_A : (A \otimes V)^{\otimes n} \xrightarrow{\cong} A^{\otimes n} \otimes V^{\otimes n} \xrightarrow{m_n \otimes f} A \otimes W\}.$$

More explicitly $f_A((a_1 \otimes v_1) \otimes \cdots \otimes (a_n \otimes v_n)) = (-1)^{\sum_{i=1}^n |a_i|(|f| + \sum_{j=1}^{i-1} |v_j|)} a_1 \cdots a_n \otimes f(v_1 \otimes \cdots \otimes v_n)$. If A is graded commutative then extension of scalars by A restricts to morphisms

$$(-)_A : \text{Hom}(V^{\odot n}, W) \rightarrow \text{Hom}((A \otimes V)^{\odot n}, A \otimes W), \quad (-)_A : \text{Hom}(V^{\wedge n}, W) \rightarrow \text{Hom}((A \otimes V)^{\wedge n}, A \otimes W).$$

Lemma 1.1.6. *Extensions of scalars by A commutes with the décalage isomorphism (1.1.1), if moreover A is graded commutative it commutes with the décalage isomorphism (1.1.2).*

Proof. This is a direct and straightforward verification: notice that we have to consider the isomorphism $(A \otimes V)[1] \xrightarrow{\cong} A \otimes V[1] : s^{-1}(a \otimes v) \rightarrow (-1)^{|a|} a \otimes s^{-1}v$ to make the statement of the lemma precise. \square

1.2 A_∞ and C_∞ algebras

For a graded space V , we denote by $\text{Hoch}(V)$ the graded Lie algebra $\text{Coder}(T(V))$ of coderivations of the tensor coalgebra $T(V)$ over V , then as a consequence of cofreeness corestriction induces an

isomorphism of graded spaces

$$\text{Hoch}(V) = \text{Coder}(T(V)) \cong \text{Hom}(T(V), V) = \prod_{n \geq 0} \text{Hom}(V^{\otimes n}, V) : Q \rightarrow pQ = (q_0, q_1, \dots, q_n, \dots),$$

where $p : T(V) \rightarrow V^{\otimes 1} = V$ is the natural projection. We call the n -th component $q_n : V^{\otimes n} \rightarrow V$ of Q under corestriction its n -th Taylor coefficient: in particular we have the constant Taylor coefficient $q_0 : \mathbb{K} \rightarrow V$ and the linear Taylor coefficient $q_1 : V \rightarrow V$, a coderivation is linear (resp.: constant) if all Taylor coefficients but the linear (resp.: constant) one vanish. The inverse to corestriction sends $(q_0, q_1, \dots, q_n, \dots)$ to the coderivation

$$Q(v_1 \otimes \dots \otimes v_n) = \sum_{i=0}^n \sum_{j=0}^{n-i} (-1)^{|Q| \sum_{k \leq j} |v_k|} v_1 \otimes \dots \otimes q_i(v_{j+1} \otimes \dots \otimes v_{j+i}) \otimes \dots \otimes v_n, \quad (1.2.1)$$

with the understanding $q_0(\emptyset) := q_0(1)$ (so that for instance, for $n = 0, 1$, Equation (1.2.1) reads $Q(1) = q_0(1)$, $Q(v) = q_0(1) \otimes v + (-1)^{|Q||v|} v \otimes q_0(1) + q_1(v)$).

We call the natural commutator bracket on $\text{Hoch}(V)$, as well as the induced bracket on $\text{Hom}(T(V), V)$, the Gerstenhaber bracket. It is induced by a right pre-Lie product (cf. Definition 4.2.1) which we call the Gerstenhaber product and denote by \circ , sending coderivations Q and R to the only coderivation $Q \circ R$ which corestricts to pQR . More explicitly: if $f \in \text{Hom}(V^{\otimes i}, V)$ and $g \in \text{Hom}(V^{\otimes j}, V)$, then $f \circ g \in \text{Hom}(V^{\otimes i+j-1}, V)$ is given by

$$f \circ g(v_1 \otimes \dots \otimes v_{i+j-1}) = \sum_{k=0}^{i-1} (-1)^{|g| \sum_{p \leq k} |v_p|} f(v_1 \otimes \dots \otimes g(v_{k+1} \otimes \dots \otimes v_{k+j}) \otimes \dots \otimes v_{i+j-1}), \quad (1.2.2)$$

with the same understanding as for Equation (1.2.1) if either i or j equals 0 (in particular, if $i = 0$ then $f \circ g = 0$).

We denote by $\overline{\text{Hoch}}(V)$ the graded Lie algebra of coderivations of the reduced tensor coalgebra $\overline{T}(V)$ over V , then again corestriction induces an isomorphism of graded spaces

$$\overline{\text{Hoch}}(V) \cong \text{Hom}(\overline{T}(V), V) = \prod_{n \geq 1} \text{Hom}(V^{\otimes n}, V) : Q \rightarrow pQ = (q_1, \dots, q_n, \dots).$$

The natural inclusion

$$\text{Hom}(\overline{T}(V), V) \rightarrow \text{Hom}(T(V), V) : (q_1, \dots, q_n, \dots) \rightarrow (0, q_1, \dots, q_n, \dots)$$

identifies $\overline{\text{Hoch}}(V) \subset \text{Hoch}(V)$ with the graded Lie subalgebra of coderivations $Q \in \text{Hoch}(V)$ such that $Q(1) = 0$.

Remark 1.2.1. For $v \in V$ we denote by τ_v the constant coderivation $p\tau_v = (j_v, 0, \dots, 0, \dots)$, where $j_v : \mathbb{K} \rightarrow V : 1 \rightarrow v$. The above formula shows that τ_v is explicitly $\tau_v = v \otimes - : T(V) \rightarrow T(V)$, where $- \otimes -$ is the shuffle product (1.1.4). Constant coderivations span an abelian Lie subalgebra $\text{Hoch}_0(V) \subset \overline{\text{Hoch}}(V)$ such that as graded spaces $\text{Hoch}_0(V) \cong V$ and $\text{Hoch}(V)$ splits in a direct sum $\text{Hoch}(V) = \overline{\text{Hoch}}(V) \oplus \text{Hoch}_0(V)$.

Finally, given graded spaces V, W , then by cofreeness of $\overline{T}(W)$ corestriction induces an isomorphism

$$\mathbf{GNC}(\overline{T}(V), \overline{T}(W)) \cong \mathbf{G}(\overline{T}(V), W) = \prod_{n \geq 1} \text{Hom}^0(V^{\otimes n}, W) : F \rightarrow pF = (f_1, \dots, f_n, \dots).$$

Again, we call the f_n the n -th Taylor coefficient of F , which is called linear if all Taylor coefficients but the linear one f_1 vanish. The inverse to corestriction sends (f_1, \dots, f_n, \dots) to the morphism of graded coalgebras $F : \overline{T}(V) \rightarrow \overline{T}(W)$

$$F(v_1 \otimes \cdots \otimes v_n) = \sum_{k=1}^n \sum_{i_1 + \cdots + i_k = n} f_{i_1}(v_1 \otimes \cdots \otimes v_{i_1}) \otimes \cdots \otimes f_{i_k}(v_{n-i_k+1} \otimes \cdots \otimes v_n). \quad (1.2.3)$$

Remark 1.2.2. A morphism of graded coalgebras $F : \overline{T}(V) \rightarrow \overline{T}(W)$ is an isomorphism (resp.: monomorphism, epimorphism) if and only if such is its linear part $f_1 : V \rightarrow W$ (cf. [62]).

Definition 1.2.3. An $A_\infty[1]$ algebra structure on a graded space V is a dg coalgebra structure on $\overline{T}(V)$, i.e., is the datum of a degree one $Q \in \overline{\text{Hoch}}^1(V)$ such that $Q^2 = Q \circ Q = \frac{1}{2}[Q, Q] = 0$. An $A_\infty[1]$ morphism between $A_\infty[1]$ algebras (V, Q) and (W, R) is a morphism $F : \overline{T}(V) \rightarrow \overline{T}(W)$ of dg coalgebras, that is, $FQ - RF = 0$. A linear $A_\infty[1]$ morphism is called strict. $A_\infty[1]$ algebras and $A_\infty[1]$ morphisms between them form the category $\mathcal{A}_\infty[1]$, $A_\infty[1]$ algebras and strict morphisms between them form the subcategory $\mathbf{A}_\infty[1] \subset \mathcal{A}_\infty[1]$.

Example 1.2.4. Given an associative dg algebra (A, d, \cdot) , there is an induced $A_\infty[1]$ structure Q on the desuspension $A[1]$, given in Taylor coefficients by $q_1 = \text{déc}(d) : A[1] \rightarrow A[1] : s^{-1}a \rightarrow -s^{-1}da$, $q_2 = \text{déc}(\cdot) : A[1]^{\otimes 2} \rightarrow A[1] : s^{-1}a \otimes s^{-1}b \rightarrow (-1)^{|a|}s^{-1}(ab)$ and $q_k = 0$ for $k \geq 3$.

This motivates the following definition.

Definition 1.2.5. An A_∞ algebra structure $(A, m_1, \dots, m_n, \dots)$ on a graded space A is a hierarchy of maps $m_n \in \text{Hom}^{2-n}(A^{\otimes n}, A)$, $n \geq 1$, such that $(\text{déc}(m_1), \dots, \text{déc}(m_n), \dots)$ are the Taylor coefficients of an $A_\infty[1]$ -structure structure on $A[1]$. A strict morphism between A_∞ algebras $(A, m_2, \dots, m_n, \dots)$ and $(A', m'_1, \dots, m'_n, \dots)$ is a morphism $f : A \rightarrow A'$ of graded spaces such that $m'_n f^{\otimes n} = f m_n$, $\forall n \geq 1$. An A_∞ morphism between A and A' is a hierarchy of maps $f_n \in \text{Hom}^{1-n}(A^{\otimes n}, A')$, $n \geq 1$, such that $(\text{déc}(f_1), \dots, \text{déc}(f_n), \dots)$ are the Taylor coefficients of an $A_\infty[1]$ morphism $(A[1], \text{déc}(m_1), \dots, \text{déc}(m_n), \dots) \rightarrow (A'[1], \text{déc}(m'_1), \dots, \text{déc}(m'_n), \dots)$. A_∞ algebras and A_∞ morphisms between them form the category \mathcal{A}_∞ , A_∞ algebras and strict morphisms between them form the subcategory $\mathbf{A}_\infty \subset \mathcal{A}_\infty$.

Remark 1.2.6. By construction décalage induces isomorphisms of categories $\text{déc} : \mathbf{A}_\infty \rightarrow \mathbf{A}_\infty[1]$, $\text{déc} : \mathcal{A}_\infty \rightarrow \mathcal{A}_\infty[1]$.

Informally speaking, A_∞ algebras are dg algebras which are associative only up to a coherent system of higher homotopies. To explain this last assertion: $(A, m_1, \dots, m_n, \dots)$ is an A_∞ algebra if and only if the hierarchy of identities

$$\sum_{i=1}^n \text{déc}(m_{n-i+1}) \circ \text{déc}(m_i) = 0, \quad n \geq 1,$$

(where \circ is the Gerstenhaber product) is satisfied. For instance, for $n = 1$ this says that m_1 is a differential on A , for $n = 2$ it says that m_1 satisfies the Leibnitz rule with respect to the product $m_2 : A^{\otimes 2} \rightarrow A$ and for $n = 3$ it says that m_2 is associative up to a homotopy given by m_3 : in particular m_2 induces an associative product on $H(A, m_1)$. Similarly given a family $f_n \in \text{Hom}^{1-n}(A^{\otimes n}, A')$ this defines an $A_\infty[1]$ morphism $F = (f_1, \dots, f_n, \dots) : (A, m_1, \dots, m_n, \dots) \rightarrow (A', m'_1, \dots, m'_n, \dots)$ if and only if a hierarchy of identities obtained by corestricting $\text{déc}(M')\text{déc}(F) = \text{déc}(F)\text{déc}(M) : \overline{T}(A[1]) \rightarrow \overline{T}(A'[1])$ is satisfied: the first identity tells us that $f_1 : (A, m_1) \rightarrow (A', m'_1)$ is a dg morphism, the second identity tells us that f_1 commutes with the products m_2, m'_2 up to a homotopy given by f_2 : in particular $H(f_1) : H(A, m_1) \rightarrow H(A', m'_1)$ is a morphism of graded associative algebras.

Definition 1.2.7. The tangent complex of an A_∞ algebra $(A, m_1, \dots, m_n, \dots)$ is the dg space (A, m_1) . The tangent cohomology of A is $H(A) = H(A, m_1)$ with the structure of graded associative algebra induced by m_2 : tangent cohomology is a functor $H(-) : \mathcal{A}_\infty \rightarrow \mathbf{GA}$ sending an A_∞ morphism $F = (f_1, \dots, f_n, \dots) : A \rightarrow A'$ to $H(F) := H(f_1) : H(A) \rightarrow H(A')$.

Definition 1.2.8. An A_∞ morphism F between A_∞ algebras A and A' is a weak equivalence if $H(F) : H(A) \rightarrow H(A')$ is an isomorphism.

Next we consider C_∞ algebras, which are, informally speaking, strictly commutative A_∞ algebras. We denote by $\overline{\text{Harr}}(V) \subset \overline{\text{Hoch}}(V)$ the graded Lie subalgebra of biderivations of $\overline{T}(V)$ with respect to the (shuffle product, deconcatenation coproduct) bialgebra structure.

Lemma 1.2.9. *Given $Q \in \overline{\text{Hoch}}(V)$, then $Q \in \overline{\text{Harr}}(V)$ if and only if the corestriction pQ vanishes on the image of the shuffle product $\otimes : \overline{T}(V) \rightarrow \overline{T}(V)$. Similarly, a morphism between the tensor coalgebras $F : \overline{T}(V) \rightarrow \overline{T}(W)$ is a morphism of the (shuffle product, deconcatenation coproduct) bialgebra structures if and only if the corestriction pF vanishes on the image of the shuffle product.*

Proof. Recall that given a morphism $F : (C, \Delta_C) \rightarrow (\overline{T}(V), \overline{\Delta})$ of locally conilpotent coalgebras, an F -coderivation $R : C \rightarrow \overline{T}(V)$ is a linear map such that $\overline{\Delta}R = (R \otimes F + F \otimes R)\Delta_C$: we need the fact that every F -coderivation as above is uniquely determined by its corestriction, hence the local conilpotence hypothesis. Let $Q \in \overline{\text{Hoch}}(V)$, we want to know if Q is a derivation with respect to the shuffle product (1.1.4) $Q\otimes = \otimes(Q \otimes \text{id}_{\overline{T}(V)} + \text{id}_{\overline{T}(V)} \otimes Q)$: but \otimes is a morphism of locally conilpotent coalgebras and both $Q\otimes$ and $\otimes(Q \otimes \text{id}_{\overline{T}(V)} + \text{id}_{\overline{T}(V)} \otimes Q)$ are \otimes -coderivations, thus $Q \in \overline{\text{Harr}}(V)$ if and only if $pQ\otimes = p \otimes(Q \otimes \text{id}_{\overline{T}(V)} + \text{id}_{\overline{T}(V)} \otimes Q) = 0$, as $p\otimes = 0$.

A similar argument shows the second claim of the lemma. The coalgebra morphism F is a morphism of bialgebras if and only if $F\otimes = \otimes F^{\otimes 2} : \overline{T}(V)^{\otimes 2} \rightarrow \overline{T}(W)$: since both $F\otimes$ and $\otimes F^{\otimes 2}$ are morphisms of locally conilpotent coalgebras and $\overline{T}(W)$ is cofree, this happens if and only if $pF\otimes = p \otimes F^{\otimes 2} = 0$. \square

Definition 1.2.10. A $C_\infty[1]$ algebra structure on a graded space V is a dg bialgebra structure on $\overline{T}(V)$ with its (shuffle product, deconcatenation coproduct) bialgebra structure, that is, it is the datum of $Q \in \overline{\text{Harr}}^1(V)$ such that $Q^2 = 0$. $C_\infty[1]$ algebras span a full subcategory $\mathbf{C}_\infty[1] \subset \mathbf{A}_\infty[1]$. A $C_\infty[1]$ morphism between $C_\infty[1]$ algebras (V, Q) and (W, R) is a dg bialgebra morphism $F : (\overline{T}(V), Q) \rightarrow (\overline{T}(W), R)$: $C_\infty[1]$ algebras and $C_\infty[1]$ morphisms between span a subcategory (not full) $\mathcal{C}_\infty[1] \hookrightarrow \mathcal{A}_\infty[1]$. The categories $\mathbf{C}_\infty \subset \mathbf{A}_\infty$, $\mathcal{C}_\infty \subset \mathcal{A}_\infty$, are the preimage of $\mathbf{C}_\infty[1]$, $\mathcal{C}_\infty[1]$, under décalage.

Example 1.2.11. Let $(A, d, \cdot, 0, \dots, 0, \dots)$ be a dg associative algebra, seen as an A_∞ algebra as in Example 1.2.4: it is a C_∞ algebra if and only if the product \cdot is graded commutative.

Definition 1.2.12. Let (A, d, \cdot) be an associative dg algebra: given an $A_\infty[1]$ algebra $(V, Q) = (V, q_1, \dots, q_n, \dots)$ extensions of scalars by A (cf. Definition 1.1.5) induces an $A_\infty[1]$ -structure Q_A on $A \otimes V$ given by

$$pQ_A = (d \otimes \text{id}_V + \text{id}_A \otimes q_1, (q_2)_A, \dots, (q_n)_A, \dots).$$

It is a $C_\infty[1]$ structure if such is Q and moreover A is graded commutative.

We close this subsection by recalling the dual definition of A_∞ and C_∞ coalgebras. Recall that the free Lie algebra $L(V)$ over a graded space V is the smallest Lie subalgebra of $\overline{T}(V)$, with the

commutator bracket induced by the concatenation product, containing V : this is also the space of primitives in its universal enveloping (bi)algebra, which is $\overline{T}(V)$ with the (concatenation product, unshuffle coproduct) bialgebra structure, cf. [101], namely, $L(V) = \text{Ker } \overline{\Delta}_{sh} : \overline{T}(V) \rightarrow \overline{T}(V)^{\otimes 2}$.

The reduced tensor algebra $\overline{T}(V)$ over V is filtered by $F^p \overline{T}(V) = \bigoplus_{n \geq p} V^{\otimes n}$: the reduced complete tensor algebra over V is the completion $\widehat{T}(V) = \lim \overline{T}(V)/F^p \overline{T}(V)$, thus as a graded space $\widehat{T}(V) = \prod_{n \geq 1} V^{\otimes n}$. Let $V^* = \text{Hom}(V, \mathbb{K})$ be the dual of V , then transposition induces a morphism of graded Lie algebras $-^t : \text{Der}(\widehat{T}(V)) \rightarrow \overline{\text{Hoch}}(V^*)$: in fact we can identify $\text{Der}(\widehat{T}(V)) = \text{Hom}(V, \widehat{T}(V)) = \prod_{n \geq 1} \text{Hom}(V, V^{\otimes n})$ and for each $n \geq 1$ transposition and pullback by the canonical $(V^*)^{\otimes n} \rightarrow (V^{\otimes n})^*$ induce a map $-^t : \text{Hom}(V, V^{\otimes n}) \rightarrow \text{Hom}((V^{\otimes n})^*, V^*) \rightarrow \text{Hom}((V^*)^{\otimes n}, V^*)$, finally it is not hard to verify that

$$-^t : \text{Der}(\widehat{T}(V)) \rightarrow \overline{\text{Hoch}}(V^*) : (q_1, \dots, q_n, \dots) \rightarrow (-q_1^t, \dots, -q_n^t, \dots)$$

is a morphism of graded Lie algebras (notice that we have to take the minus sign since otherwise we would get an *antihomomorphism* of graded Lie algebras).

The free Lie algebra $L(V)$ over V is filtered by $F^p L(V) = L(V) \cap F^p \overline{T}(V)$, the completion $\widehat{L}(V) = \lim L(V)/F^p L(V)$ is the complete free Lie algebra over V . As a graded space $\text{Der}(\widehat{L}(V)) = \text{Hom}(V, \widehat{L}(V))$, the pushforward $\text{Der}(\widehat{L}(V)) = \text{Hom}(V, \widehat{L}(V)) \rightarrow \text{Hom}(V, \widehat{T}(V)) = \text{Der}(\widehat{T}(V))$ identifies the graded Lie algebra $\text{Der}(\widehat{L}(V))$ with a graded Lie subalgebra of $\text{Der}(\widehat{T}(V))$: moreover, given $Q = (q_1, \dots, q_n, \dots) \in \text{Der}(\widehat{L}(V))$, since the composition $V \xrightarrow{q_n} V^{\otimes n} \xrightarrow{\overline{\Delta}_{sh}} \overline{T}(V) \otimes \overline{T}(V)$ vanishes, dually we also see that the transpose q_n^t vanishes on the image of the shuffle product, thus by Lemma 1.2.9 transposition restrict to a morphism of graded Lie algebras $-^t : \text{Der}(\widehat{L}(V)) \rightarrow \overline{\text{Harr}}(V^*)$.

If V is finite dimensional (that is, each V^k is finite dimensional and $V^k = 0$ for $|k| \gg 0$), with a preferred choice of basis, likewise transposition defines isomorphisms of graded Lie algebras $\overline{\text{Hoch}}(V) \rightarrow \text{Der}(\widehat{T}(V^*))$, $\overline{\text{Harr}}(V) \rightarrow \text{Der}(\widehat{L}(V^*))$.

Definition 1.2.13. An A_∞ coalgebra structure on a graded space V is a dg algebra structure on the completed reduced tensor algebra $\widehat{T}(V[-1])$. A C_∞ coalgebra structure on a graded space V is a dg Lie algebra structure on the completed free Lie algebra $\widehat{L}(V[-1])$.

Remark 1.2.14. Taking into account the natural $(V[-1])^* \cong V^*[1]$, the previous discussion shows that transposition (together with décalage) sends A_∞ (resp.: C_∞) coalgebra structures on a graded space to A_∞ (resp.: C_∞) algebra structures on its dual, as well as A_∞ (resp.: C_∞) algebra structures on a finite dimensional graded space with a basis to A_∞ (resp.: C_∞) coalgebra structures on its dual.

1.3 L_∞ algebras

We denote by $\text{CE}(V)$ the graded Lie algebra of coderivations of the symmetric coalgebra $S(V)$ over V , corestriction induces an isomorphism $\text{CE}(V) \cong \prod_{n \geq 0} \text{Hom}(V^{\odot n}, V)$, we call the components of $pQ = (q_0, q_1, \dots, q_n, \dots)$ the Taylor coefficients of the coderivation Q : we call q_1 and q_0 respectively the linear and the constant Taylor coefficient of Q , we say that Q is a linear (resp.: constant) coderivation if all Taylor coefficients but the linear (resp.: constant) one vanish. The inverse to corestriction sends $(q_0, q_1, \dots, q_n, \dots)$ to

$$Q(v_1 \odot \dots \odot v_n) = \sum_{i=0}^n \sum_{\sigma \in S(i, n-i)} \varepsilon(\sigma) q_i(v_{\sigma(1)} \odot \dots \odot v_{\sigma(i)} \odot \dots \odot v_{\sigma(n)}), \quad (1.3.1)$$

(with the same understanding as for Equation (1.2.1), for instance for $n = 0, 1$ Equation (1.3.1) reads $Q(1) = q_0(1)$, $Q(v) = q_0(1) \odot v + q_1(v)$).

We call the commutator bracket on $\text{CE}(V)$, as well as the induced bracket on $\text{Hom}(S(V), V)$, the Nijenhuis-Richardson bracket. It is induced by a right pre-Lie product which we call the Nijenhuis-Richardson product and denote by \bullet , where again $Q \bullet R$ corestricts to pQR . Explicitly: if $f \in \text{Hom}(V^{\odot i}, V)$ and $g \in \text{Hom}(V^{\odot j}, V)$, then $f \bullet g \in \text{Hom}(V^{\odot i+j-1}, V)$ is given by

$$f \bullet g(v_1 \odot \cdots \odot v_{i+j-1}) = \sum_{\sigma \in S(j, i-1)} \varepsilon(\sigma) f(g(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(j)})) \odot \cdots \odot v_{\sigma(i+j-1)}. \quad (1.3.2)$$

The graded Lie algebra $\overline{\text{CE}}(V) = \text{Coder}(\overline{S}(V)) \cong \prod_{n \geq 1} \text{Hom}(V^{\odot n}, V)$ identifies, via the natural embedding $(q_1, \dots, q_n, \dots) \rightarrow (0, q_1, \dots, q_n, \dots)$, with the graded Lie subalgebra of coderivations $Q \in \text{CE}(V)$ such that $Q(1) = 0$.

Remark 1.3.1. Let $v \in V$, denote by σ_v the constant coderivation $p\sigma_v = (j_v, 0, \dots, 0, \dots)$, where $j_v : \mathbb{K} \rightarrow V : 1 \rightarrow v$. Formula (1.3.1) shows that σ_v is explicitly $\sigma_v = v \odot - : S(V) \rightarrow S(V)$, where $- \odot -$ is the concatenation product (1.1.4), and formula (1.3.2) shows that $\sigma_v \bullet Q = 0$ for all $Q = (q_0, \dots, q_n, \dots) \in \text{CE}(V)$, while $Q \bullet \sigma_v = [Q, \sigma_v] = ([q_1, \sigma_v], \dots, [q_{n+1}, \sigma_v], \dots)$ is given in Taylor coefficients by

$$[q_1, \sigma_v](1) = q_1(v), \quad [q_{n+1}, \sigma_v](v_1 \odot \cdots \odot v_n) = q_{n+1}(v \odot v_1 \cdots \odot v_n). \quad (1.3.3)$$

Constant coderivations span an abelian Lie subalgebra $\text{CE}_0(V) \subset \text{CE}(V)$, such that as graded spaces $\text{CE}_0(V) \cong V$, and $\text{CE}(V)$ splits in a direct sum $\text{CE}(V) = \overline{\text{CE}}(V) \oplus \text{CE}_0(V)$.

Finally, given graded spaces V, W , then corestriction induces an isomorphism

$$\mathbf{GNCC}(\overline{S}(V), \overline{S}(W)) \cong \mathbf{G}(\overline{S}(V), W) = \prod_{n \geq 1} \text{Hom}^0(V^{\odot n}, W) : F \rightarrow pF = (f_1, \dots, f_n, \dots).$$

The f_n are the Taylor coefficients of F , which is linear if all Taylor coefficients but the linear one f_1 vanish. The inverse to corestriction sends (f_1, \dots, f_n, \dots) to the morphism $F : \overline{S}(V) \rightarrow \overline{S}(W)$

$$F(v_1 \odot \cdots \odot v_n) = \sum_{k=1}^n \frac{1}{k!} \sum_{i_1 + \cdots + i_k = n} \sum_{\sigma \in S(i_1, \dots, i_k)} \varepsilon(\sigma) f_{i_1}(v_{\sigma(1)} \odot \cdots) \odot \cdots \odot f_{i_k}(\cdots \odot v_{\sigma(n)}) \quad (1.3.4)$$

Remark 1.3.2. A morphism of graded coalgebras $F : \overline{S}(V) \rightarrow \overline{S}(W)$ is an isomorphism (resp.: monomorphism, epimorphism) if and only if such is its linear part $f_1 : V \rightarrow W$ (cf. [62]).

Definition 1.3.3. A $L_\infty[1]$ algebra structure on a graded space V is a dg coalgebra structure on $\overline{S}(V)$, that is, the datum of $Q \in \overline{\text{CE}}^1(V)$ such that $Q^2 = Q \bullet Q = \frac{1}{2}[Q, Q] = 0$. A $L_\infty[1]$ morphism between $L_\infty[1]$ algebras (V, Q) and (W, R) is a morphism $F : (\overline{S}(V), Q) \rightarrow (\overline{S}(W), R)$ of dg coalgebras: a linear $L_\infty[1]$ morphism is called strict. $L_\infty[1]$ algebras and $L_\infty[1]$ morphisms between them form the category $\mathcal{L}_\infty[1]$, $L_\infty[1]$ algebras and strict morphisms between them form the subcategory $\mathbf{L}_\infty[1] \subset \mathcal{L}_\infty[1]$.

Example 1.3.4. If $(L, d, [\cdot, \cdot])$ is a dg Lie algebra then there is a $L_\infty[1]$ structure Q on the desuspension $L[1]$ given by $q_1 = \text{dec}(d) : L[1] \rightarrow L[1] : s^{-1}l \rightarrow -s^{-1}dl$, the graded symmetric bracket $q_2 = \text{dec}([\cdot, \cdot]) : L[1]^{\odot 2} \rightarrow L[1] : s^{-1}l \odot s^{-1}m \rightarrow (-1)^{|l|}s^{-1}[l, m]$ and $q_k = 0$ for $k \geq 3$.

Definition 1.3.5. A L_∞ algebra structure on a graded space L is the data of a hierarchy of maps $l_n \in \text{Hom}^{2-n}(L^{\wedge n}, L)$, $n \geq 1$, such that the coderivation given in Taylor coefficients by $(\text{déc}(l_1), \dots, \text{déc}(l_n), \dots)$ is an $L_\infty[1]$ structure on $L[1]$. A strict morphism between L_∞ algebras $(L, l_1, \dots, l_n, \dots)$ and $(L', l'_1, \dots, l'_n, \dots)$, is a morphism of graded spaces $f : L \rightarrow L'$ such that $l'_n f^{\wedge n} = f l_n$, $\forall n \geq 1$. A L_∞ morphism between L and L' is a hierarchy $f_n \in \text{Hom}^{1-n}(L^{\wedge n}, L')$ such that $(\text{déc}(f_1), \dots, \text{déc}(f_n), \dots)$ are the Taylor coefficients of an $L_\infty[1]$ morphism $L[1] \rightarrow L'[1]$. L_∞ algebras and L_∞ (resp.: strict) morphisms between them form the category \mathbf{L}_∞ (resp.: the subcategory $\mathbf{L}_\infty \subset \mathcal{L}_\infty$).

Remark 1.3.6. By construction décalage induces isomorphisms of categories $\text{déc} : \mathbf{L}_\infty \rightarrow \mathbf{L}_\infty[1]$, $\text{déc} : \mathcal{L}_\infty \rightarrow \mathcal{L}_\infty[1]$.

Informally speaking, L_∞ algebras are dg Lie algebras where the Jacobi identity has been relaxed up to a coherent system of higher homotopies. For instance, expanding the first few identities $\sum_{i=1}^n \text{déc}(l_{n-i+1}) \bullet \text{déc}(l_i)$, $n \geq 1$, we see that l_1 is a differential on L satisfying the Leibnitz identity with respect to the bracket $l_2 : L^{\wedge 2} \rightarrow L$, and the last one satisfy the Jacobi identity up to the homotopy l_3 . If (f_1, \dots, f_n, \dots) is an L_∞ morphism, then $f_1 : L \rightarrow L'$ is a dg morphism, preserving the brackets up to the homotopy f_2 .

Definition 1.3.7. Given an L_∞ algebra $(L, l_1, \dots, l_n, \dots)$, its tangent complex and its tangent cohomology are (L, l_1) and $H(L) = H(L, l_1)$ respectively: the last one has a graded Lie algebra structure induced by l_2 . Tangent cohomology is a functor $H(-) : \mathcal{L}_\infty \rightarrow \mathbf{GLA}$, by putting $H(F) := H(f_1) : H(L) \rightarrow H(L')$.

Definition 1.3.8. A L_∞ morphism $F = (f_1, \dots, f_n, \dots) : (L, l_1, \dots, l_n, \dots) \rightarrow (L', l'_1, \dots, l'_n, \dots)$ is a weak equivalence between L and L' if $H(F) : H(L) \rightarrow H(L')$ is an isomorphism.

Definition 1.3.9. Given $L_\infty[1]$ algebra $(V, Q) = (V, q_1, \dots, q_n, \dots)$ and a commutative dg algebra (A, d, \cdot) via extensions of scalars by A there is an $L_\infty[1]$ structure Q_A on the tensor product $A \otimes V$, given (as in the $A_\infty[1]$ case, Definition 1.2.12) by

$$pQ_A = (d \otimes \text{id}_V + \text{id}_A \otimes q_1, (q_2)_A, \dots, (q_n)_A, \dots).$$

Finally, we recall the following definition: homotopy abelian L_∞ algebras play an important role in deformation theory, cf. Remark 6.4.4.

Definition 1.3.10. An L_∞ algebra $(L, l_1, \dots, l_n, \dots)$ is abelian if $l_n = 0$ for $n \geq 2$, it is homotopy abelian if it is weakly equivalent to an abelian one.

Remark 1.3.11. For graded spaces V, W , we still denote by $\text{sym} : \text{Hom}(\overline{T}(V), W) \rightarrow \text{Hom}(\overline{S}(V), W)$ the pullback by the symmetrization $\text{sym} : \overline{S}(V) \rightarrow \overline{T}(V)$. It is well known and easy to prove directly (cf. also Remark 4.1.19) that when $W = V$ the resulting

$$\text{sym} : \overline{\text{Hoch}}(V) \rightarrow \overline{\text{CE}}(V) : (q_1, \dots, q_n, \dots) \rightarrow (\text{sym}(q_1), \dots, \text{sym}(q_n), \dots),$$

$$\text{sym}(q_n)(v_1 \odot \dots \odot v_n) = \sum_{\sigma \in S_n} \varepsilon(\sigma) q_n(v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}),$$

is a morphism of graded Lie algebras, in particular it sends $A_\infty[1]$ structures on V to $L_\infty[1]$ structures on V . It is also not hard to show that if $F = (f_1, \dots, f_n, \dots)$ is an $A_\infty[1]$ morphism $F : (V, q_1, \dots, q_n, \dots) \rightarrow (W, r_1, \dots, r_n, \dots)$, then $\text{sym}(F) = (\text{sym}(f_1), \dots, \text{sym}(f_n), \dots)$ is an $L_\infty[1]$ morphism $\text{sym}(F) : (V, \text{sym}(q_1), \dots, \text{sym}(q_n), \dots) \rightarrow (W, \text{sym}(r_1), \dots, \text{sym}(r_n), \dots)$, thus symmetrization is a functor $\text{sym} : \mathcal{A}_\infty[1] \rightarrow \mathcal{L}_\infty[1]$. Given $Q \in \overline{\text{Harr}}(V)$, since $\text{sym} : \overline{S}(V) \rightarrow \overline{T}(V)$

is a morphism from the symmetric algebra over V to the shuffle algebra over V , Lemma 1.2.9 implies that the corestriction $p\text{sym}(Q)$ has to vanish on the image of the concatenation product and since the latter is $\sum_{n \geq 2} V^{\odot n}$ this means that $\text{sym}(Q)$ is a linear coderivation: thus $C_\infty[1]$ structures symmetrize to abelian $L_\infty[1]$ structures - that is, complexes - and in the same way we see that $C_\infty[1]$ morphisms symmetrize to strict $L_\infty[1]$ morphisms - that is, dg morphisms.

1.3.1 Complete L_∞ algebras

A complete graded space $(V, F^\bullet V)$ is a graded space V equipped with a decreasing filtration $V = F^1 V \supset \dots \supset F^p V \supset \dots$ such that V is complete in the induced topology, i.e., the natural morphism $V \rightarrow \lim V/F^p V$ is an isomorphism. Continuous morphisms of complete graded spaces are the ones compatible with the filtrations. We denote by $\widehat{\mathbf{G}}$ the category of complete graded spaces and continuous morphisms between them. Likewise we can define the categories $\widehat{\mathbf{DG}}$, $\widehat{\mathbf{DGA}}$, $\widehat{\mathbf{DGLA}}$ by requiring the differentials and the respective algebraic structures to be compatible with the filtration in the usual way.

Definition 1.3.12. A complete $L_\infty[1]$ algebra is a complete graded space $V = \lim V/F^p V$ with an $L_\infty[1]$ structure $(V, q_1, \dots, q_n, \dots)$ which are compatible $q_n(F^{i_1} V \odot \dots \odot F^{i_n} V) \subset F^{i_1 + \dots + i_n} V$, $\forall n, i_1, \dots, i_n \geq 1$. A $L_\infty[1]$ morphism $F = (f_1, \dots, f_n, \dots) : V \rightarrow W$ between complete $L_\infty[1]$ algebras is continuous if $f_n(F^{i_1} V \odot \dots \odot F^{i_n} V) \subset F^{i_1 + \dots + i_n} W$ is satisfied for every $n, i_1, \dots, i_n \geq 1$. We denote by $\widehat{\mathcal{L}}_\infty[1]$ (resp.: $\widehat{\mathbf{L}}_\infty[1]$) the category of complete $L_\infty[1]$ algebras and continuous $L_\infty[1]$ (resp.: strict) morphisms between them.

Remark 1.3.13. Let $(V, F^\bullet V, q_1, \dots, q_n, \dots)$ be a complete $L_\infty[1]$ algebra, then for all $q \geq 1$ there is an induced complete $L_\infty[1]$ algebra structure on $V/F^q V$ such that the natural $V \rightarrow \lim V/F^p V$ is an isomorphism in the category $\widehat{\mathbf{L}}_\infty[1]$: the filtration on $V/F^q V$ is the induced one $F^p(V/F^q V) = F^p V/F^q V$ if $1 \leq p \leq q$ and $F^p(V/F^q V) = 0$ if $p > q$.

Definition 1.3.14. The curvature of a complete $L_\infty[1]$ algebra $(V, q_1, \dots, q_n, \dots)$ is the function

$$\mathcal{R} : V^0 \rightarrow V^1 : v \rightarrow \sum_{n \geq 1} \frac{1}{n!} q_n(v^{\odot n}).$$

Sometimes we also denote it by \mathcal{R}_V or $\mathcal{R}_{(V, Q)}$. An $x \in V^0$ is called a Maurer-Cartan element of (V, Q) if $\mathcal{R}(x) = 0$, we denote the set of Maurer-Cartan elements of V by $\text{MC}(V)$, or sometimes $\text{MC}(V, Q)$. The Maurer-Cartan functor $\text{MC}(-) : \widehat{\mathcal{L}}_\infty[1] \rightarrow \mathbf{Set}$ sends the $L_\infty[1]$ morphism $F = (f_1, \dots, f_n, \dots) : V \rightarrow W$ to

$$\text{MC}(F) : \text{MC}(V) \rightarrow \text{MC}(W) : x \rightarrow \sum_{n \geq 1} \frac{1}{n!} f_n(x^{\odot n})$$

Remark 1.3.15. From the previous definition it is not clear that $\text{MC}(F)$ sends Maurer-Cartan elements to Maurer-Cartan elements, so we take a moment to show this well known fact. First of all we consider more in general the forgetful functor $-^{0\#} : \widehat{\mathcal{L}}_\infty[1] \rightarrow \mathbf{Set}$ sending V to V^0 and a continuous $L_\infty[1]$ morphism $F = (f_1, \dots, f_n, \dots) : (V, Q) \rightarrow (W, R)$ to

$$F^{0\#} : V^0 \rightarrow W^0 : v \rightarrow \sum_{n \geq 1} \frac{1}{n!} f_n(v^{\odot n}).$$

Functoriality is easy and left to the reader. We have to prove that $F^{0\#}$ sends Maurer-Cartan elements to Maurer-Cartan elements: let $v \in V^0$ and $w := F^{0\#}(v) \in W^0$, $\mathcal{R}_V : V^0 \rightarrow V^1$ and

$\mathcal{R}_W : W^0 \rightarrow W^1$ the respective curvature functions, this follows from

$$\begin{aligned}
\mathcal{R}_W(w) &= r_1(w) + \sum_{n \geq 2} \frac{1}{n!} r_n (F^{0\#}(v)^{\odot n}) = \\
&= r_1(w) + \sum_{n \geq 2} \frac{1}{n!} r_n \left(\sum_{j_1, \dots, j_n \geq 1} \frac{f_{j_1}(v^{\odot j_1})}{j_1!} \odot \dots \odot \frac{f_{j_n}(v^{\odot j_n})}{j_n!} \right) = \\
&= r_1(w) + \sum_{n \geq 2} \sum_{k \geq 0} \frac{1}{(n+k)!} r_n \left(\frac{1}{n!} \sum_{j_1 + \dots + j_n = n+k} \frac{(n+k)!}{j_1! \dots j_n!} f_{j_1}(v^{\odot j_1}) \odot \dots \odot f_{j_n}(v^{\odot j_n}) \right) = \\
&= r_1(w) + \sum_{n \geq 2} \sum_{k \geq 0} \frac{1}{(n+k)!} r_n F_{n+k}^n (v^{\odot n+k}) = \sum_{n \geq 1} \frac{1}{n!} r_1 f_n (v^{\odot n}) + \sum_{n \geq 2} \frac{1}{n!} \sum_{k=2}^n r_k F_n^k (v^{\odot n}) = \\
&= \sum_{n \geq 1} \frac{1}{n!} pRF(v^{\odot n}) = \sum_{n \geq 1} \frac{1}{n!} pFQ(v^{\odot n}) = \sum_{n \geq 1} \frac{1}{n!} \sum_{k=1}^n \frac{n!}{k!(n-k)!} f_{n-k+1}(q_k(v^{\odot k}) \odot v^{\odot n-k}) = \\
&= \sum_{n \geq 1} \frac{1}{(n-1)!} f_n(\mathcal{R}_V(v) \odot v^{\odot n-1}).
\end{aligned}$$

Remark 1.3.16. It is immediate to see that $\text{MC}(-)$ seen as a functor $\widehat{\mathcal{L}}_\infty[1] \rightarrow \mathbf{Set}$ commutes with small limits, as it commutes with both equalizers and arbitrary products. The category $\widehat{\mathcal{L}}_\infty[1]$ is not complete, since in general equalizers don't exist, but one could still ask if $\text{MC}(-)$ as a functor $\widehat{\mathcal{L}}_\infty \rightarrow \mathbf{Set}$ preserves small limits whenever these exist. This is not the case, as the following easy counterexample shows: let $V = W = \mathbb{K}$ with the trivial $L_\infty[1]$ structure, and consider the pair of $L_\infty[1]$ morphisms $F = (f_1, \dots, f_n, \dots)$ and $G = (g_1, \dots, g_n, \dots)$ from V to W given by $f_1 = \text{id}_{\mathbb{K}}$, $f_k = 0$ for $k \neq 1$, $g_2 : \mathbb{K}^{\odot 2} \rightarrow \mathbb{K}$ is the product and $g_k = 0$ for $k \neq 2$. In this case $\text{MC}(V) = \text{MC}(W) = \mathbb{K}$ and $\text{MC}(F) : \mathbb{K} \rightarrow \mathbb{K} : t \rightarrow t$, $\text{MC}(G) : \mathbb{K} \rightarrow \mathbb{K} : t \rightarrow \frac{1}{2}t^2$, thus the equalizer of $\text{MC}(F)$ and $\text{MC}(G)$ is the inclusion of the two points set $\{0, 2\} \rightarrow \mathbb{K}$: on the other hand it is not hard to see that $0 \rightarrow V$ is an equalizer for F and G in the category $\widehat{\mathcal{L}}_\infty[1]$. This also implies that $\text{MC}(-)$ is not representable as a functor $\widehat{\mathcal{L}}_\infty[1] \rightarrow \mathbf{Set}$ (it is representable as a functor $\widehat{\mathbf{DGLA}} \rightarrow \mathbf{Set}$ by the complete dg Lie algebra $L(\Delta_0)$, cf. Section 5.2.1: this is the complete free Lie algebra $\widehat{L}(x)$ over a single generator x in degree one and with differential $dx = -\frac{1}{2}[x, x]$).

It is possible to twist $L_\infty[1]$ structures by Maurer-Cartan elements, cf. for instance [39, 109]. Let $(V, F^\bullet V)$ be a complete graded space, we denote by $\text{CE}_c(V) \subset \text{CE}(V)$ the right pre-Lie subalgebra

$$\begin{aligned}
\text{CE}_c(V) &= \{Q = (q_0, \dots, q_n, \dots) \in \text{CE}(V) \text{ s.t.} \\
&\quad q_n (F^{i_1} V \odot \dots \odot F^{i_n} V) \subset F^{i_1 + \dots + i_n} V, \forall n, i_1, \dots, i_n \geq 1\}.
\end{aligned}$$

The fact that $\text{CE}_c(V)$ is closed with respect to the Nijenhuis-Richardson product follows easily from (1.3.2). Similarly we denote by $\text{CE}_c(V) \subset \text{CE}_c(V)$ the right pre-Lie subalgebra of coderivations with vanishing constant Taylor coefficient. The graded space $\text{CE}_c(V)$ is complete with respect to the filtration $F^0 \text{CE}_c(V) = \text{CE}_c(V)$, and for $p \geq 1$

$$\begin{aligned}
F^p \text{CE}_c(V) &= \{Q = (q_0, \dots, q_n, \dots) \in \text{CE}(V) \text{ s.t.} \\
&\quad q_n(1) \in F^p V, q_n (F^{i_1} V \odot \dots \odot F^{i_n} V) \subset F^{i_1 + \dots + i_n + p} V, \forall n, i_1, \dots, i_n \geq 1\},
\end{aligned}$$

and moreover this filtration is compatible with the Nijenhuis-Richardson bracket in the usual way. As in Remark 1.3.1, for all $x \in V$ we denote by $\sigma_x \in F^1 \text{CE}_c(V)$ the constant coderivation with constant Taylor coefficient $1 \rightarrow x$. Since the inner derivation $[\sigma_x, -] : \text{CE}_c(V) \rightarrow \text{CE}_c(V)$ sends $F^p \text{CE}_c(V)$ to $F^{p+1} \text{CE}_c(V)$, it is well defined the automorphism of graded Lie algebras $e^{[-, \sigma_x]} : \text{CE}_c(V) \rightarrow \text{CE}_c(V) : Q \rightarrow Q_x$, which we call the twisting by x . Remark 1.3.1 shows that $Q_x = (q_{x,0}, \dots, q_{x,n}, \dots)$ is given explicitly by

$$q_{x,0}(1) = \sum_{k \geq 0} \frac{1}{k!} q_k(x^{\odot k}), \quad q_{x,n}(v_1 \odot \dots \odot v_n) = \sum_{k \geq 0} \frac{1}{k!} q_{n+k}(x^{\odot k} \odot v_1 \odot \dots \odot v_n).$$

In particular, if $Q \in \overline{\text{CE}}_c(V)$ is a complete $L_\infty[1]$ structure on $(V, F^\bullet V)$ also $[Q_x, Q_x] = 0$, thus Q_x is a new complete $L_\infty[1]$ structure on V if moreover $q_{x,0}(1) = 0$, that is, by the above, if $x \in \text{MC}(V, Q)$.

Proposition 1.3.17. *Given a complete $L_\infty[1]$ algebra $(V, q_1, \dots, q_n, \dots)$ and a Maurer-Cartan element $x \in \text{MC}(V)$, there is a new complete $L_\infty[1]$ algebra structure $Q_x = (q_{x,1}, \dots, q_{x,n}, \dots)$ on V , given by*

$$q_{x,n}(v_1 \odot \dots \odot v_n) = \sum_{k \geq 0} \frac{1}{i!} q_{n+k}(x^{\odot k} \odot v_1 \odot \dots \odot v_n).$$

Proof. Given above. □

Lemma 1.3.18. $\text{MC}(V, Q_x) = \{x' \in V^0 \text{ s.t. } x + x' \in \text{MC}(V, Q)\}$.

Proof. An easy computation shows that $\mathcal{R}_{(V, Q_x)}(x') = \mathcal{R}_{(V, Q)}(x + x')$ for all $x' \in V^0$. □

There is also a relative version of the previous proposition: given a continuous $L_\infty[1]$ morphism $F = (f_1, \dots, f_n, \dots) : (V, Q) \rightarrow (W, R)$ of complete $L_\infty[1]$ algebras and a Maurer-Cartan element $x \in \text{MC}(V)$, let $F_x : \overline{S}(V) \rightarrow \overline{S}(W)$ be the morphism of coalgebras given in Taylor coefficients $F_x = (f_{x,1}, \dots, f_{x,n}, \dots)$ by

$$f_{x,n}(v_1 \odot \dots \odot v_n) = \sum_{i \geq 0} \frac{1}{i!} f_{n+i}(x^{\odot i} \odot v_1 \odot \dots \odot v_n).$$

Proposition 1.3.19. $F_x : (V, Q_x) \rightarrow (W, R_{\text{MC}(F)(x)})$ is a continuous $L_\infty[1]$ morphism.

Proof. We prefer to omit a detailed proof: let $y := \text{MC}(F)(x) \in \text{MC}(W, R)$, the required identity $F_x Q_x = R_y F_x$ can be proved either by a direct computation, along the lines (but more involved) of the one in Remark 1.3.15, or by making sense of the equalities $Q_x = e^{-\sigma_x} Q e^{\sigma_x}$, $R_y = e^{-\sigma_y} R e^{\sigma_y}$, $F_x = e^{-\sigma_y} F e^{\sigma_x}$ (cf. [109], the problem in the latter case, of course, is that the coalgebra automorphisms $e^{\sigma_x} : \overline{S}(V) \rightarrow \overline{S}(V)$, $e^{\sigma_y} : \overline{S}(W) \rightarrow \overline{S}(W)$ are not well defined). □

Remark 1.3.20. Under some additional hypothesis Proposition 1.3.17 and Proposition 1.3.19 also hold in the non complete setting, for instance if $q_n = r_n = f_n = 0$ for $n \gg 0$: the following remark is a bit pedantic but we will need it in section 4.1. Consider the following situation where (V, Q) , (W, R) are $L_\infty[1]$ algebras (not complete ones) and $F : (V, Q) \rightarrow (W, R)$ is an $L_\infty[1]$ morphism: a given $x \in V^0$ satisfies $0 = q_{k+n}(x^{\odot k} \odot -) : V^{\odot n} \rightarrow V$ and $0 = f_{k+n}(x^{\odot k} \odot -) : V^{\odot n} \rightarrow W$ for all $k \geq 0$ and $n \gg 0$, likewise $\sum_{n \geq 1} \frac{1}{n!} f_n(x^{\odot n}) := y \in W^0$ satisfies $0 = r_{k+n}(y^{\odot k} \odot -) : W^{\odot n} \rightarrow W$ for all $k \geq 0$ and $n \gg 0$. It makes sense to say that x is Maurer-Cartan if $0 = \sum_{n \geq 1} \frac{1}{n!} q_n(x^{\odot n})$, moreover in this case the computation in Remark 1.3.15 shows that also y is Maurer-Cartan. By

the hypotheses the previous formulas well define coderivations $Q_x \in \overline{\text{CE}}(V)$, $R_y \in \overline{\text{CE}}(W)$ and a morphism of coalgebras $F_x : \overline{S}(V) \rightarrow \overline{S}(W)$. **Claim:** in the previous hypotheses the conclusions of Proposition 1.3.17 and Proposition 1.3.19 still hold (in fact both could be proved by a cumbersome but direct computation which continue to make sense).

We close this subsection with an useful proposition. Since for a complete $L_\infty[1]$ algebra $V = \lim V/F^p V$ we have $\text{MC}(V) = \lim \text{MC}(V/F^p V)$, it can be useful to study inductively the sets $\text{MC}(V/F^p V)$, and this can be done using the following result.

Proposition 1.3.21. *Let $\varepsilon : 0 \rightarrow I \rightarrow V \rightarrow W \rightarrow 0$ be a central extension of complete $L_\infty[1]$ algebras, as in Definition 1.3.31, then there is an obstruction map $o : \text{MC}(W) \rightarrow H^1(I)$ with the property $o(x) = 0$ if and only if x lifts to a Maurer-Cartan element of $\text{MC}(V)$. If the set of Maurer-Cartan liftings of x is not empty it has the structure of an affine space over $Z^0(I)$: more precisely given a Maurer-Cartan lifting $\tilde{x} \in \text{MC}(V)$ of x the set of all Maurer-Cartan liftings of x is in bijective correspondence with $Z^0(I)$ via $Z^0(I) \rightarrow \text{MC}(V) : \tilde{z} \rightarrow \tilde{x} + \tilde{z}$.*

Proof. We denote by $q_n : V^{\odot n} \rightarrow V$ the $L_\infty[1]$ structure operations. Given $x \in \text{MC}(W)$ let $\tilde{x} \in V^0$ be an arbitrary lifting of x , then $\mathcal{R}(\tilde{x}) \in Z^1(I)$: in fact it is clear that $\mathcal{R}(\tilde{x}) \in I$, and since I is an abelian ideal we also see that

$$\begin{aligned} q_1(\mathcal{R}(\tilde{x})) &= \sum_{n \geq 2} \frac{1}{n!} q_1 q_n(\tilde{x}^{\odot n}) = - \sum_{n \geq 2} \frac{1}{n!} \left(\sum_{i=2}^n \frac{n!}{(n-1)!(n-i+1)!} q_i(q_{n-i+1}(\tilde{x}^{\odot n-i+1}) \odot \tilde{x}^{\odot i-1}) \right) = \\ &= - \sum_{i \geq 2} \frac{1}{(i-1)!} q_i(\mathcal{R}(\tilde{x}) \odot \tilde{x}^{\odot i-1}) = 0 \end{aligned}$$

If \bar{x} is another lifting of x , $n \geq 2$, then $q_n(\tilde{x}^{\odot n}) - q_n(\bar{x}^{\odot n}) = \sum_{i=0}^{n-1} q_n(\tilde{x}^{\odot i} \odot (\tilde{x} - \bar{x}) \odot \bar{x}^{\odot n-i-1}) = 0$, as $\tilde{x} - \bar{x} \in I$ and I is abelian, thus

$$\mathcal{R}(\tilde{x}) - \mathcal{R}(\bar{x}) = q_1(\tilde{x} - \bar{x}) \tag{1.3.5}$$

Thus it is well defined o sending x to the cohomology class of $\mathcal{R}(\tilde{x})$, for \tilde{x} an arbitrary lifting of x . If $\mathcal{R}(\tilde{x}) = q_1(\tilde{z})$, with $\tilde{z} \in I^0$, then (1.3.5) implies that $\mathcal{R}(\tilde{x} - \tilde{z}) = 0$, thus x admits a Maurer-Cartan lifting and the converse is obvious. Finally, the last statement also follows immediately from Equation (1.3.5). \square

Remark 1.3.22. The obstruction map defined in the previous proposition is natural with respect to strict morphisms between central extensions of $L_\infty[1]$ algebras, that is strict morphisms between the bases, the fibers and the total spaces of two given central extensions making the obvious diagram commutative: this is immediate by construction.

1.3.2 Convolution L_∞ algebras

In this subsection we associate to every pair of $L_\infty[1]$ algebras (V, Q) and (W, R) an $L_\infty[1]$ structure on the graded space $\overline{\text{CE}}(V, W) := \text{Hom}(\overline{S}(V), W) = \prod_{n \geq 1} \text{Hom}(V^{\odot n}, W)$, called the *convolution $L_\infty[1]$ structure*. In fact this correspondence is induced by a morphism graded Lie algebras $\overline{\text{CE}}(V) \times \overline{\text{CE}}(W) \rightarrow \overline{\text{CE}}(\overline{\text{CE}}(V, W))$ (where the graded Lie algebras structures are given by the commutator bracket): this sends $Q \in \overline{\text{CE}}(V)$ to the linear coderivation $Q^* = (Q^*, 0, \dots, 0, \dots) \in \overline{\text{CE}}(\overline{\text{CE}}(V, W))$, with $Q^*(F) = -(-1)^{|Q||F|} FQ$ (here we are considering Q as a coderivation

$\overline{S}(V) \rightarrow \overline{S}(V)$, and $F, Q^*(F)$ as graded morphisms $\overline{S}(V) \rightarrow W$, and sends $R \in \overline{\text{CE}}(W)$ to the coderivation $R_* = (r_{*,1}, \dots, r_{*,n}, \dots) \in \overline{\text{CE}}(\overline{\text{CE}}(V, W))$ defined by

$$r_{*,n}(F_1 \odot \dots \odot F_n) : \overline{S}(V) \xrightarrow{\overline{\Delta}^{n-1}} \overline{S}(V)^{\otimes n} \xrightarrow{F_1 \otimes \dots \otimes F_n} W^{\otimes n} \rightarrow W^{\odot n} \xrightarrow{r_n} W,$$

where: $F_1, \dots, F_n \in \overline{\text{CE}}(V, W)$, $\overline{S}(V) \xrightarrow{\overline{\Delta}^{n-1}} \overline{S}(V)^{\otimes n}$ is the iterated coproduct and $W^{\otimes n} \rightarrow W^{\odot n}$ the natural projection. This is graded symmetric since so is r_n and the coproduct on $\overline{S}(V)$ is cocommutative.

Lemma 1.3.23. *The correspondence $\overline{\text{CE}}(W) \rightarrow \overline{\text{CE}}(\overline{\text{CE}}(V, W)) : R \rightarrow R_*$ is a morphism of graded right pre-Lie algebras.*

Proof. We have to prove that $R_* \bullet R'_* = (R \bullet R')_*$ for all $R, R' \in \text{CE}(W)$, where \bullet is the Nijenhuis-Richardson product: it suffices to consider $R = r_i \in \text{Hom}(W^{\odot i}, W)$ and $R' = r'_j \in \text{Hom}(W^{\odot j}, W)$, then $r_{*,i} \bullet r'_{*,j}$ is given, for $F_1, \dots, F_{i+j-1} \in \overline{\text{CE}}(V, W)$, by

$$\begin{aligned} r_{*,i} \bullet r'_{*,j}(F_1 \odot \dots \odot F_{i+j-1}) &= \sum_{\sigma \in S(j, i-1)} \varepsilon(\sigma) r_{*,i} (r'_{*,j}(F_{\sigma(1)} \odot \dots \odot F_{\sigma(j)} \odot \dots \odot F_{\sigma(i+j-1)}) = \\ &= \sum_{\sigma \in S(j, i-1)} \varepsilon(\sigma) r_i \left(r'_j(F_{\sigma(1)} \otimes \dots \otimes F_{\sigma(j)} \overline{\Delta}^{j-1} \otimes \dots \otimes F_{\sigma(i+j-1)}) \overline{\Delta}^{i-1} = \right. \\ &= \sum_{\sigma \in S(j, i-1)} \varepsilon(\sigma) r_i (r'_j(F_{\sigma(1)} \otimes \dots \otimes F_{\sigma(j)} \otimes \dots \otimes F_{\sigma(i+j-1)}) \overline{\Delta}^{i+j-2} = \\ &= (r_i \bullet r'_j)(F_1 \otimes \dots \otimes F_{i+j-1}) \overline{\Delta}^{i+j-2} = (r_i \bullet r'_j)_*(F_1 \odot \dots \odot F_{i+j-1}). \end{aligned}$$

□

Lemma 1.3.24. $[Q^*, R_*] = 0$ for all $Q \in \overline{\text{CE}}(V)$, $R \in \overline{\text{CE}}(W)$.

Proof. For $n = 1$

$$\begin{aligned} [Q^*, r_{*,1}](F) &= Q^*(r_1 F) - (-1)^{|Q||R|} r_{*,1} \left((-1)^{|Q||F|} F Q \right) = \\ &= -(-1)^{|Q|(|R|+|F|)} r_1 F Q + (-1)^{|Q|(|R|+|F|)} r_1 F Q = 0, \end{aligned}$$

and, since $Q : \overline{S}(V) \rightarrow \overline{S}(V)$ is a coalgebra coderivation, for $n \geq 2$

$$\begin{aligned} Q^*(r_{*,n}(F_1 \odot \dots \odot F_n)) &= -(-1)^{|Q|(|R|+\sum_{j=1}^n |F_j|)} r_n(F_1 \otimes \dots \otimes F_n) \overline{\Delta}^{n-1} Q = \\ &= \sum_{i=1}^n -(-1)^{|Q|(|R|+\sum_{j=1}^n |F_j|)} r_n(F_1 \otimes \dots \otimes F_n) \left(\text{id}_{\overline{S}(V)}^{\otimes i-1} \otimes Q \otimes \text{id}_{\overline{S}(V)}^{\otimes n-i} \right) \overline{\Delta}^{n-1} = \\ &= (-1)^{|Q||R|} \sum_{i=1}^n (-1)^{\sum_{j=1}^{i-1} |Q||F_j|} r_n(F_1 \otimes \dots \otimes (-1)^{|Q||F_i|} F_i Q \otimes \dots \otimes F_n) \overline{\Delta}^{n-1} = \\ &= (-1)^{|Q||R|} \sum_{i=1}^n (-1)^{\sum_{j=1}^{i-1} |Q||F_j|} r_{n,*}(F_1 \odot \dots \odot Q^*(F_i) \odot \dots \odot F_n). \end{aligned}$$

□

Lemma 1.3.25. *The correspondence $\overline{\text{CE}}(V) \rightarrow \overline{\text{CE}}(\overline{\text{CE}}(V, W)) : Q \rightarrow Q^*$ is a morphism of graded Lie algebras.*

Proof. Clear. □

Finally, putting all the lemmas together we see that as claimed

Proposition 1.3.26. *The correspondence $\overline{\text{CE}}(V) \times \overline{\text{CE}}(W) \rightarrow \overline{\text{CE}}(\overline{\text{CE}}(V, W)) : (Q, R) \rightarrow Q^* + R_*$ is a morphism of graded Lie algebras.*

Definition 1.3.27. Given a pair (V, Q) and (W, R) of $L_\infty[1]$ algebras, by the previous proposition $Q^* + R_*$ is an $L_\infty[1]$ structure on $\overline{\text{CE}}(V, W)$. We call the $L_\infty[1]$ algebra $(\overline{\text{CE}}(V, W), Q^* + R_*)$ the *convolution $L_\infty[1]$ algebra* of (V, Q) and (W, R) .

The descending filtration

$$F^p \overline{\text{CE}}(V, W) = \{F = (f_1, \dots, f_n, \dots) \in \text{Hom}(\overline{S}(V), W) \text{ s.t. } f_i = 0 \forall i < p\},$$

turns $\overline{\text{CE}}(V, W)$ into a complete space, in fact

$$\begin{aligned} \lim \overline{\text{CE}}(V, W) / F^p \overline{\text{CE}}(V, W) &= \lim \text{Hom}(\oplus_{i=1}^{p-1} V^{\odot i}, W) = \text{Hom}(\text{colim } \oplus_{i=1}^{p-1} V^{\odot i}, W) = \\ &= \text{Hom}(\overline{S}(V), W) = \overline{\text{CE}}(V, W). \end{aligned}$$

Working out the definitions, given $F_1 = (f_{1,1}, \dots, f_{1,n}, \dots), \dots, F_j = (f_{j,1}, \dots, f_{j,n}, \dots) \in \overline{\text{CE}}(V, W)$, $r_j \in \text{Hom}(W^{\odot j}, W)$, $j \geq 1$, we have

$$\begin{aligned} r_{*,j}(F_1 \odot \dots \odot F_j)(v_1 \odot \dots \odot v_n) &= \\ &= \sum_{k_1 + \dots + k_j = n} \sum_{\sigma \in S(k_1, \dots, k_j)} \varepsilon(\sigma) r_j(f_{1,k_1}(v_{\sigma(1)} \odot \dots) \odot \dots \odot f_{j,k_j}(\dots \odot v_{\sigma(n)})), \end{aligned}$$

which shows that $r_{*,j}$ is continuous. Moreover, given $F = (f_1, \dots, f_n, \dots) \in \overline{\text{CE}}(V, W)$ and $Q = (q_1, \dots, q_n, \dots) \in \overline{\text{CE}}(V)$ we have

$$Q^*(F)(v_1 \odot \dots \odot v_n) = \sum_{i=1}^n \sum_{\sigma \in S(i, n-i)} \varepsilon(\sigma) f_{n-i+1}(q_i(v_{\sigma(1)} \odot \dots \odot v_{\sigma(i)}) \odot \dots \odot v_{\sigma(n)}),$$

which shows that $Q^* : \overline{\text{CE}}(V, W) \rightarrow \overline{\text{CE}}(V, W)$ is continuous as well. Thus the filtration $F^\bullet \overline{\text{CE}}(V, W)$ turns the convolution $L_\infty[1]$ algebra $(\overline{\text{CE}}(V, W), Q^* + R_*)$ from the previous definition into a complete $L_\infty[1]$ algebra, in particular it makes sense to consider the Maurer-Cartan equation.

Proposition 1.3.28. *A graded morphism $F \in \text{Hom}^0(\overline{S}(V), W)$ is a Maurer-Cartan element of $(\overline{\text{CE}}(V, W), Q^* + R_*)$ if and only if it is the corestriction of an $L_\infty[1]$ morphism $F : (V, Q) \rightarrow (W, R)$.*

Proof. By the previous computations, F is Maurer-Cartan if and only if for all $v_1, \dots, v_n \in V$, $n \geq 1$, we have

$$\begin{aligned} \sum_{j=1}^n \frac{1}{j!} \sum_{k_1 + \dots + k_j = n} \sum_{\sigma \in S(k_1, \dots, k_j)} \varepsilon(\sigma) r_j(f_{k_1}(v_{\sigma(1)} \odot \dots) \odot \dots \odot f_{k_j}(\dots \odot v_{\sigma(n)})) &= \\ &= \sum_{i=1}^n \sum_{\sigma \in S(i, n-i)} \varepsilon(\sigma) f_{n-i+1}(q_i(v_{\sigma(1)} \odot \dots \odot v_{\sigma(i)}) \odot \dots \odot v_{\sigma(n)}), \end{aligned}$$

which is exactly the equation for (f_1, \dots, f_n, \dots) to be the corestriction of an $L_\infty[1]$ morphism $F : (V, Q) \rightarrow (W, R)$. \square

We close this subsection by sketching a different construction of convolution L_∞ algebras, this time associated to a C_∞ coalgebra C and a dg Lie algebra M . We will need the following result in Section 5.2.1.

Proposition 1.3.29. *Given a C_∞ coalgebra structure on a space C , that is, a dg Lie algebra structure on the complete free Lie algebra $\widehat{L}(C[-1])$, and a complete dg Lie algebra $(M, F^\bullet M)$ there is a complete L_∞ algebra structure on $\text{Hom}(C, M)$ with the induced filtration $F^p \text{Hom}(C, M) = \text{Hom}(C, F^p M)$. The natural identification*

$$\text{Hom}^1(C, M) = \text{Hom}^0(C[-1], M) = \mathbf{G}(C[-1], M) = \widehat{\mathbf{GLA}}(\widehat{L}(C[-1]), M)$$

restricts to a natural identification $\text{MC}(\text{Hom}(C, M)) = \widehat{\mathbf{DGLA}}(\widehat{L}(C[-1]), M)$.

Proof. (sketch, cf. [84] for details, also notice that this is just another appearance of the well established mechanism of twisting cochains and Koszul duality [72, 13]) We only recall the construction of the $L_\infty[1]$ algebra structure on $\text{Hom}(C, M)[1] = \text{Hom}(C[-1], M)$. The transpose of the C_∞ coalgebra structure on C gives a $C_\infty[1]$ algebra structure on the space $C^*[1]$, hence via extensions of scalars by the universal enveloping algebra $U(M)$ there is an $A_\infty[1]$ algebra structure on the space $C^*[1] \otimes U(M) = \text{Hom}(C[-1], U(M))$, thus via symmetrization also an $L_\infty[1]$ algebra structure: finally one verifies that $\text{Hom}(C[-1], M) \subset \text{Hom}(C[-1], U(M))$ is an $L_\infty[1]$ subalgebra, cf. [84], Lemma 3.3. The last assertion follows by unwinding the definitions, cf. [84], Section 3.2. \square

1.3.3 L_∞ extensions

The aim of this subsection is to review the classification of $L_\infty[1]$ extensions from [24, 83, 69].

Definition 1.3.30. A $L_\infty[1]$ ideal of an $L_\infty[1]$ algebra $(V, q_1, \dots, q_n, \dots)$ is a graded subspace $I \subset V$ such that $q_{n+1}(I \otimes V^{\otimes n}) \subset I$, for all $n \geq 0$: then Q restrict to an $L_\infty[1]$ structure on I . There is an induced $L_\infty[1]$ algebra structure $(V/I, r_1, \dots, r_n, \dots)$ on the quotient V/I : if $[v] \in V/I$ denotes the class of $v \in V$, this is given by $r_n; (V/I)^{\otimes n} \rightarrow V/I : [v_1] \odot \dots \odot [v_n] \rightarrow [q_n(v_1 \odot \dots \odot v_n)]$. A $L_\infty[1]$ ideal I of V is abelian if moreover $q_{n+1}(I \otimes V^{\otimes n}) = 0$ for all $n \geq 1$.

Definition 1.3.31. An exact sequence $\varepsilon : 0 \rightarrow (I, Q) \rightarrow (V, S) \rightarrow (W, R) \rightarrow 0$ of $L_\infty[1]$ algebras and strict morphism is called a (resp.: central) extension of $L_\infty[1]$ algebras of base (W, R) and fibre (I, Q) if the image of the first arrow is an (resp.: abelian) $L_\infty[1]$ ideal in (V, S) .

Given an extension ε of $L_\infty[1]$ algebras as in the previous definition there is a unique isomorphism of graded spaces $V \xrightarrow{\cong} W \times I$ making the diagram

$$\begin{array}{ccc} & & V \\ & \nearrow & \searrow \\ I & & W \\ & \searrow & \nearrow \\ & 0 \times \text{id}_I & p_W \\ & & W \times I \end{array}$$

commutative, and correspondingly an induced $L_\infty[1]$ structure Q_ε on $W \times I$ making the sequence $0 \rightarrow (I, Q) \xrightarrow{0 \times \text{id}_I} (W \times I, Q_\varepsilon) \xrightarrow{p_W} (W, R) \rightarrow 0$ into an extension of $L_\infty[1]$ algebras: conversely, each isomorphism class of $L_\infty[1]$ extensions with a fixed base (W, R) and a fixed fiber (I, Q) (where we consider isomorphisms given by diagrams as the one above) contains a unique representative of this form.

Remark 1.3.32. It is convenient to decompose the symmetric powers of $W \times I$ into types $(W \times I)^{\odot i} \cong \bigoplus_{j=0}^i W^{\odot i-j} \otimes I^{\odot j}$. For any other graded space V , in particular for $V = W \times I$, we have corresponding isomorphisms $\text{Hom}(\overline{S}(W \times I), V) = \prod_{i \geq 1} \text{Hom}((W \times I)^{\odot i}, V) = \prod_{j+k \geq 1} \text{Hom}(W^{\odot j} \otimes I^{\odot k}, V)$.

The coderivation $Q_\varepsilon \in \overline{CE}(W \times I)$ lies in the subspace

$$\text{Hom}(\overline{S}(W), W) \times \text{Hom}(\overline{S}(I), I) \times \text{Hom}(\overline{S}(W) \otimes S(I), I) \subset \text{Hom}(\overline{S}(W \times I), W \times I), \quad (1.3.6)$$

and moreover the $\text{Hom}(\overline{S}(W), W)$ component has to be R and the $\text{Hom}(\overline{S}(I), I)$ component has to be Q , so the coderivation Q_ε is determined by its component

$$F_\varepsilon \in \text{Hom}^1(\overline{S}(W) \otimes S(I), I) = \text{Hom}^0(\overline{S}(W), \text{Hom}(S(I), I)[1]) = \overline{CE}^0(W, \text{CE}(I)[1]).$$

Remark 1.3.33. Explicitly $Q_\varepsilon = (q_{\varepsilon,1}, \dots, q_{\varepsilon,n}, \dots)$ and $F_\varepsilon = (f_{\varepsilon,1}, \dots, f_{\varepsilon,n}, \dots) : W \rightarrow \text{CE}(I)[1]$ determine each other via the formulas

$$\begin{aligned} q_{\varepsilon,i}(w_1 \odot \dots \odot w_i) &= (r_i(w_1 \odot \dots \odot w_i), sf_{\varepsilon,i}(w_1 \odot \dots \odot w_i)_0(1)), \\ q_{\varepsilon,j}(v_1 \odot \dots \odot v_j) &= (0, q_j(v_1 \odot \dots \odot v_j)), \\ q_{\varepsilon,i+j}(w_1 \odot \dots \odot w_i \otimes v_1 \odot \dots \odot v_j) &= (0, sf_{\varepsilon,i}(w_1 \odot \dots \odot w_i)_j(v_1 \odot \dots \odot v_j)) \quad \text{if } i, j \geq 1, \end{aligned}$$

where we denote by $sf_{\varepsilon,i}$ the composition $W^{\odot i} \xrightarrow{f_{\varepsilon,i}} \text{CE}(I)[1] \xrightarrow{s} \text{CE}(I)$.

Proposition 1.3.34. *The set of isomorphism classes of $L_\infty[1]$ extensions of base (I, Q) and fiber (W, R) is in bijective correspondence with the set of $L_\infty[1]$ morphisms from (W, R) to the dg Lie algebra $(\text{CE}(I), [Q, \cdot], [\cdot, \cdot])$, seen as an $L_\infty[1]$ algebra via décalage. This correspondence is given explicitly as in Remark 1.3.33.*

Proof. We omit to give a detailed proof. By the results of the previous section the graded Lie algebra structure on $\text{CE}(I)$ given by the Nijenhuis-Richardson bracket $[-, -]$ induces by convolution a graded Lie algebra structure $[-, -]_*$ on $\overline{CE}(W, \text{CE}(I))$, moreover we have a morphism of graded Lie algebras

$$\overline{CE}(W) \times \overline{CE}(I) \rightarrow \text{Der}(\overline{CE}(W, \text{CE}(I))) : (R, Q) \rightarrow R^* + [Q, -]_*,$$

hence we can form the semidirect product $(\overline{CE}(W) \times \overline{CE}(I)) \rtimes \overline{CE}(W, \text{CE}(I))$. We claim that the inclusion (1.3.6), which we rewrite as

$$\overline{CE}(W) \times \overline{CE}(I) \times \overline{CE}(W, \text{CE}(I)) \subset \overline{CE}(W \times I),$$

identifies this semidirect product with a graded Lie subalgebra of $\overline{CE}(W \times I)$ with the Nijenhuis-Richardson bracket: this could be seen by a direct and rather unpleasant computation¹. In particular Q_ε is an $L_\infty[1]$ structure, that is, $[Q_\varepsilon, Q_\varepsilon] = 0$, if and only if $R^*(F_\varepsilon) + [Q, -]_*(F_\varepsilon) + \frac{1}{2}[F_\varepsilon, F_\varepsilon]_* =$

¹Or more easily by passing to the dual picture where we consider $L_\infty[1]$ algebra structure on a graded space as dg commutative algebra structures on the completed symmetric algebra over its dual, cf. [24, 83, 69].

0, that is, according to Proposition 1.3.28, if and only if $F_\varepsilon \in \overline{\text{CE}}^1(W, \text{CE}(I)) = \overline{\text{CE}}^0(W, \text{CE}(I)[1])$ is the corestriction of an $L_\infty[1]$ morphism from (W, R) to the dg Lie algebra $(\text{CE}(I), [Q, -], [-, -])$, seen as an $L_\infty[1]$ algebra via décalage. \square

Notation 1.3.35. Given an $L_\infty[1]$ morphism $F_\varepsilon : W \rightarrow \text{CE}(I)[1]$ as in the previous proposition we denote the $L_\infty[1]$ algebra $(W \times I, Q_\varepsilon)$ also by $W \times_{F_\varepsilon} I$. We say that $(W \times I, Q_\varepsilon)$ is a semidirect product of (W, R) and (I, Q) if F_ε factors through the inclusion $\overline{\text{CE}}(I)[1] \rightarrow \text{CE}(I)[1]$, and in this case we also denote it by $W \rtimes_{F_\varepsilon} I$.

Lemma 1.3.36. *Given an $L_\infty[1]$ extension $0 \rightarrow I \rightarrow W \times_\Phi I \rightarrow W \rightarrow 0$ and an $L_\infty[1]$ morphism $F : W' \rightarrow W$ we denote by $F^*\Phi$ the pullback $F^*\Phi : W' \xrightarrow{F} W \xrightarrow{\Phi} \text{CE}(I)[1]$: there is a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & W' \times_{F^*\Phi} I & \longrightarrow & W' \longrightarrow 0 \\ & & \parallel & & \downarrow \tilde{F} & & \downarrow F \\ 0 & \longrightarrow & I & \longrightarrow & W \times_\Phi I & \longrightarrow & W \longrightarrow 0 \end{array}$$

where the $L_\infty[1]$ morphism \tilde{F} is given by $\tilde{f}_n((w'_1, i_1) \odot \cdots \odot (w'_n, i_n)) = (f_n(w'_1 \odot \cdots \odot w'_n), 0)$ for $n \geq 2$, conversely if an $L_\infty[1]$ extension $0 \rightarrow I \rightarrow W' \times_\Psi I \rightarrow W' \rightarrow 0$ fits into a diagram as the one above then $\Psi = F^*\Phi$. Given an $L_\infty[1]$ isomorphism $G : I \xrightarrow{\cong} I'$ we denote by $G_*\Phi : W \rightarrow \text{CE}(I')[1]$ the pushforward of Φ by the isomorphism of graded Lie algebras $G - G^{-1} : \text{CE}(I) \rightarrow \text{CE}(I')$: there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & W \times_\Phi I & \longrightarrow & W \longrightarrow 0 \\ & & \cong \downarrow G & & \downarrow \tilde{G} & & \parallel \\ 0 & \longrightarrow & I' & \longrightarrow & W \times_{G_*\Phi} I' & \longrightarrow & W \longrightarrow 0 \end{array}$$

where the $L_\infty[1]$ morphism \tilde{G} is given by $\tilde{g}_n((w_1, i_1) \odot \cdots \odot (w_n, i_n)) = (0, g_n(i_1 \odot \cdots \odot i_n))$ for $n \geq 2$, conversely if an $L_\infty[1]$ extension $0 \rightarrow I' \rightarrow W \times_\Psi I' \rightarrow W \rightarrow 0$ fits into a diagram as the one above then $\Psi = G_*\Phi$.

Proof. Omitted, this could be seen by verifying through a direct computation that the $L_\infty[1]$ identities for the morphisms \tilde{F} and \tilde{G} exactly translates into commutativity of the diagrams

$$\begin{array}{ccc} W & \xrightarrow{F} & W' \\ & \searrow \Phi & \swarrow F^*\Phi \\ & \text{CE}(I)[1] & \end{array} \qquad \begin{array}{ccc} & W & \\ \Phi \swarrow & & \searrow G_*\Phi \\ \text{CE}(I)[1] & \xrightarrow{G-G^{-1}} & \text{CE}(I')[1] \end{array}$$

\square

1.3.4 O_∞ algebras

To avoid to keep using constructs such as “ A_∞ (resp.: C_∞, L_∞)”, especially in Chapters 2 and 3, we establish the following

Convention: We denote by \mathbf{O}_∞ (resp.: \mathcal{O}_∞) a category which could be either one of the categories $\mathbf{A}_\infty, \mathbf{C}_\infty, \mathbf{L}_\infty$ (resp.: $\mathcal{A}_\infty, \mathcal{C}_\infty, \mathcal{L}_\infty$), and correspondingly we also talk about O_∞ algebra and strict (resp.: \mathcal{O}_∞) morphisms between them.

It is clear how some of the definitions we gave in this section for $L_\infty[1]$ algebras also apply immediately, mutatis mutandis, to $O_\infty[1]$ algebras: for instance, we say that an $O_\infty[1]$ algebra $(V, q_1, \dots, q_n, \dots)$ is homotopy abelian if $q_n = 0$ for $n \geq 2$. It is also clear how to define the category $\widehat{O}_\infty[1]$ (resp.: $\widehat{O}_\infty[1]$) of complete $O_\infty[1]$ algebras and continuous strict (resp.: $O_\infty[1]$) morphisms between them.

Chapter 2

Homotopy transfer of ∞ structures

As already remarked in several occasions, one of the most useful features of ∞ structures is that they can be transferred (unlike, for instance, dg algebra structures) along homotopy retractions. In Section 2.2 we prove this fundamental fact in the A_∞ , C_∞ and L_∞ cases, following an argument we learned from the arXiv version of [31], together with a series of technical lemmas we will need in the sequel. In the L_∞ case, if the homotopy retraction satisfies some side conditions (then we call it a contraction) a formal analog of classical Kuranishi's theorem hold, due to Getzler, specifying the behavior of Maurer-Cartan sets under homotopy transfer: this is reviewed in Section 2.3, the proof is a slight generalization of an argument from [39].

2.1 Homotopy retractions and contractions

Definition 2.1.1. A homotopy retraction of dg spaces is the data $(V \xrightleftharpoons[i]{p} W, K)$ of a pair of dg morphism $p : V \rightarrow W$ and $i : W \rightarrow V$ such that p is a left inverse to i - $pi = \text{id}_W$ - and a homotopy $K \in \text{Hom}^{-1}(V, V)$ between ip and $\text{id}_V - d_V K + K d_V = ip - \text{id}_V$: in particular i and p are quasi-isomorphisms. Homotopy retractions form a category **Horet**, where a morphism

$$f : (V \xrightleftharpoons[i]{p} W, K) \rightarrow (V' \xrightleftharpoons[i']{p'} W', K')$$

is the datum of a dg morphism $\text{pr}_1(f) : V \rightarrow V'$ commuting with the homotopy operators - $\text{pr}_1(f)K = K' \text{pr}_1(f)$. A homotopy retraction $(V \xrightleftharpoons[i]{p} W, K)$ is a contraction if it satisfies the side conditions $K^2 = pK = 0$: contractions span a full subcategory **Contr** \subset **Horet**.

Remark 2.1.2. Given a contraction $(V \xrightleftharpoons[i]{p} W, K)$ then

$$(Ki)p = K(d_V K + K d_V + \text{id}_V) = K d_V K + K = (d_V K + K d_V + \text{id}_V)K = i(pk) = 0,$$

thus $Ki = 0$, as p is epi.

Remark 2.1.3. There are natural projection functors $\text{pr}_i : \mathbf{Horet} \rightarrow \mathbf{DG}$, $i = 1, 2$: pr_1 sends

$$f : (V \xrightleftharpoons[i]{p} W, K) \rightarrow (V' \xrightleftharpoons[i']{p'} W', K')$$

to the corresponding dg morphism $\text{pr}_1(f) : V \rightarrow V'$

(cf. the previous definition), $\text{pr}_2(f) : W \rightarrow W'$ is the composition $W \xrightarrow{i} V \xrightarrow{\text{pr}_1(f)} V' \xrightarrow{p'} W'$. To see that pr_2 is a functor we notice the identities $\text{pr}_2(f)p = p' \text{pr}_2(f)$ and $\text{pr}_1(f)i = i' \text{pr}_2(f)$: we show the first one, the second one is proved similarly

$$\text{pr}_2(f)p = p \text{pr}_1(f)i p = p' \text{pr}_1(f)(d_V K + K d_V + \text{id}_V) = p'(d_{V'} K' + K' d_{V'} + \text{id}_{V'}) \text{pr}_1(f) = p' \text{pr}_2(f).$$

Thus we also see $\text{pr}_1(f)i p = i' p' \text{pr}_1(f)$. Given $g : (V' \xrightleftharpoons[i']{p'} W', K') \rightarrow (V'' \xrightleftharpoons[i'']{p''} W'', K'')$ now it is easy

$$\text{pr}_2(g) \text{pr}_2(f) = p'' \text{pr}_1(g) i' p' \text{pr}_1(f) i = (p'' i'') p'' \text{pr}_1(g) \text{pr}_1(f) i = \text{pr}_2(gf),$$

thus pr_2 is indeed a functor and moreover $p : \text{pr}_1 \rightarrow \text{pr}_2$ and $i : \text{pr}_2 \rightarrow \text{pr}_1$ are natural transformations.

The categories **Horet** and **Contr** are complete. The existence of products is clear, so we consider equalizers: let $f, g : (V \xrightleftharpoons[i]{p} W, K) \rightarrow (V' \xrightleftharpoons[i']{p'} W', K')$ be a pair of morphisms in **Horet**, and let $\bar{V} \subset V$ and $\bar{W} \subset W$ be the equalizers in **DG** of $\text{pr}_1(f), \text{pr}_1(g)$ and $\text{pr}_2(f), \text{pr}_2(g)$ respectively. We put $\bar{i} = i|_{\bar{V}}$, $\bar{p} = p|_{\bar{V}}$, thus by naturality of $p : \text{pr}_1 \rightarrow \text{pr}_2$ and $i : \text{pr}_2 \rightarrow \text{pr}_1$ we see that $\bar{i}(\bar{W}) \subset \bar{V}$ and $\bar{p}(\bar{V}) \subset \bar{W}$. If $v \in \bar{V}$ also $K(v) \in \bar{V}$, in fact $\text{pr}_1(f)K(v) = K' \text{pr}_1(f)(v) = K' \text{pr}_1(g)(v) = \text{pr}_1(g)K(v)$: we put $\bar{K} = K|_{\bar{V}}$ and we have just seen that $K(\bar{V}) \subset \bar{V}$, but now it is clear that $(\bar{V} \xrightleftharpoons[\bar{i}]{\bar{p}} \bar{W}, \bar{K})$ is a homotopy retraction and an equalizer of f and g in **Horet**,

moreover it is a contraction if such is $(V_1 \xrightleftharpoons[i_1]{p_1} W_1, K_1)$. This shows that **Horet** and **Contr** are complete: in fact more precisely it shows that the functor $\text{pr}_1 \times \text{pr}_2 : \mathbf{Horet} \rightarrow \mathbf{DG} \times \mathbf{DG}$ creates small limits (cf. [74]).

Homotopy retractions and contractions can be composed as in the following

Definition 2.1.4. Given a pair of homotopy retractions $(V \xrightleftharpoons[i]{p} W, K)$ and $(W \xrightleftharpoons[l]{\pi} Z, H)$ their composition is

$$(V \xrightleftharpoons[i]{p} W, K) \circ (W \xrightleftharpoons[l]{\pi} Z, H) := (V \xrightleftharpoons[iu]{\pi p} Z, K + iHp)$$

Notice that the composite of two contractions remains a contraction.

Finally we need the fact that complete graded space structures on V transfer along homotopy retractions if some compatibility condition with the filtration is satisfied.

Definition 2.1.5. A complete homotopy retraction is a homotopy retraction $(V \xrightleftharpoons[i]{p} W, K)$ and a complete dg space structure $(V, F^\bullet V)$ on $V = \lim V/F^p V$ such that $K : V \rightarrow V$ and $ip : V \rightarrow V$ are continuous. In these hypotheses W is complete with respect to the induced filtration $F^\bullet W = i^{-1}(F^\bullet V)$ (notice that this is preserved by d_W) and i, p are continuous morphisms of complete dg spaces: in fact by definition $i(F^p W) = i(W) \cap F^p V$ and since ip is continuous also $p(F^p V) = F^p W$, the remaining claim follows easily from the other two (if $\{w_n\}$ is a Cauchy

sequence in W , then $\{v_n\} := \{i(w_n)\}$ is a Cauchy sequence in V , and $w := p(v) := p(\lim v_n)$ is the limit of $\{w_n\}$ - in fact we only used continuity of ip , continuity of K will be used in Section 2.3).

A morphism f of homotopy retractions is continuous if such is $\text{pr}_1(f)$ (and then so is $\text{pr}_2(f)$): complete homotopy retractions and continuous morphisms form a category $\widehat{\mathbf{Horet}}$, and similarly one defines the full subcategory $\widehat{\mathbf{Contr}} \subset \widehat{\mathbf{Horet}}$ of complete contractions.

2.2 Homotopy transfer theorems

$O_\infty[1]$ algebra structures can be transferred along homotopy retractions as in the following theorem.

Theorem 2.2.1. *Given an $O_\infty[1]$ algebra $(V, q_1, \dots, q_n, \dots)$ together with a homotopy retraction $((V, q_1) \xrightleftharpoons[f_1]{g_1} (W, r_1), K)$ there is a transferred $O_\infty[1]$ structure R on W with linear part r_1 and an $O_\infty[1]$ weak equivalence $F : (W, R) \rightarrow (V, Q)$ with linear part f_1 . The higher Taylor coefficients of F are determined recursively by the equation*

$$pF = f = f_1 + Kq_+F, \quad (2.2.1)$$

where we consider Q as a perturbation of the abelian $O_\infty[1]$ structure q_1 on V and we denote by $Q_+ = Q - q_1$ the perturbation and by $q_+ = pQ_+ = (0, q_2, \dots, q_n, \dots)$ its corestriction. The higher Taylor coefficients of R are determined recursively by

$$pR = r = r_1 + g_1q_+F. \quad (2.2.2)$$

Proof. We give the proof first in the $L_\infty[1]$ case, which is the one we are most concerned with. Equation (2.2.1) determines the Taylor coefficients f_n , $n \geq 2$, recursively: in fact once expanded it says

$$f_n(v_1 \odot \dots \odot v_n) = \sum_{k=2}^n \frac{1}{k!} \sum_{i_1 + \dots + i_k = n} \sum_{\sigma \in S(i_1, \dots, i_k)} \varepsilon(\sigma) Kq_k(f_{i_1}(v_{\sigma(1)} \odot \dots) \odot \dots \odot f_{i_k}(\dots \odot v_{\sigma(n)})).$$

In order to prove $H := QF - FR = 0$ it suffices to show $pH = 0$ (since $H : \overline{S}(W) \rightarrow \overline{S}(V)$ is an F -coderivation, cf. the proof of Lemma 1.2.9).

Since $0 = pQ^2 = qQ = (q_1 + q_+)Q = q_1(q_1 + q_+) + q_+Q$ we see that $q_1q_+ = -q_+Q : \overline{S}(V) \rightarrow V$. We use this fact in the following computation

$$\begin{aligned} pH &= p(QF - FR) = q_1(f_1 + Kq_+F) + q_+F - (f_1 + Kq_+F)R = \\ &= q_1f_1 + (f_1g_1 - \text{id}_V - Kq_1)q_+F + q_+F - f_1(r_1 + g_1q_+F) - Kq_+FR = \\ &= (q_1f_1 - f_1r_1) - Kq_1q_+F - Kq_+FR = Kq_+(QF - FR) = Kq_+H. \end{aligned}$$

We can reconstruct the F -coderivation H explicitly from its corestriction $pH = (h_1, \dots, h_n, \dots)$ as

$$H_n^i(v_1 \odot \dots \odot v_n) = \sum_{k=1}^{n-i+1} \sum_{\sigma \in S(k, n-k)} \varepsilon(\sigma) h_k(v_{\sigma(1)} \odot \dots \odot v_{\sigma(k)}) \odot F_{n-k}^{i-1}(v_{\sigma(k+1)} \odot \dots \odot v_{\sigma(n)}),$$

for instance

$$H(w) = h_1(w), \quad H(w_1 \odot w_2) = h_2(w_1 \odot w_2) + h_1(w_1) \odot f_1(w_2) + (-1)^{|w_1|} f_1(w_1) \odot h_1(w_2),$$

and so on. Thus $pH = Kq_+H$ implies inductively $H = 0$: in fact $h_1(w) = pH(w) = Kq_+h_1(w) = 0 \forall w \in W$, since $q_+h_1 = 0$, and if we know inductively that $h_1 = \dots = h_{n-1} = 0$ then by the above also $H_n^2 = \dots = H_n^n = 0$, thus $h_n = \sum_{k=2}^n Kq_k H_n^k = 0$. This proves $QF = FR$, it remains to show that $R^2 = 0$: by the above

$$pR^2 = (r_1 + g_1q_+F)R = r_1(r_1 + g_1q_+F) + g_1q_+FR = r_1g_1q_+F + g_1q_+QF = g_1(q_1q_+ + q_+Q)F = 0.$$

The $A_\infty[1]$ case is treated in the exact same way. In the $C_\infty[1]$ case, by Lemma 1.2.9 it only remains to show that for all $p, q \geq 1$, $v_1, \dots, v_{p+q} \in V$, the following hold

$$f_{p+q}((v_1 \otimes \dots \otimes v_p) \otimes (v_{p+1} \otimes \dots \otimes v_{p+q})) = r_{p+q}((v_1 \otimes \dots \otimes v_p) \otimes (v_{p+1} \otimes \dots \otimes v_{p+q})) = 0.$$

Suppose we have proven inductively the above for $p+q < n$, the $n = 1$ case being empty. Consider the morphism of coalgebras $F_{<n} : \overline{T}(W) \rightarrow \overline{T}(V)$ defined by $pF_{<n} = (f_1, \dots, f_{n-1}, 0, \dots, 0, \dots)$: by the inductive hypothesis and Lemma 1.2.9 $F_{<n}$ is a morphism of the (shuffle product, deconcatenation coproduct) bialgebra structures. Suppose $p+q = n$, then

$$\begin{aligned} f_{p+q}((v_1 \otimes \dots \otimes v_p) \otimes (v_{p+1} \otimes \dots \otimes v_{p+q})) &= pF((v_1 \otimes \dots \otimes v_p) \otimes (v_{p+1} \otimes \dots \otimes v_{p+q})) = \\ &= Kq_+F((v_1 \otimes \dots \otimes v_p) \otimes (v_{p+1} \otimes \dots \otimes v_{p+q})) = Kq_+F_{<n}((v_1 \otimes \dots \otimes v_p) \otimes (v_{p+1} \otimes \dots \otimes v_{p+q})) = \\ &= Kq_+(F_{<n}(v_1 \otimes \dots \otimes v_p) \otimes F_{<n}(v_{p+1} \otimes \dots \otimes v_{p+q})) = 0, \end{aligned}$$

since Q is supposed to be a $C_\infty[1]$ structure. $r_{p+q}((v_1 \otimes \dots \otimes v_p) \otimes (v_{p+1} \otimes \dots \otimes v_{p+q})) = 0$ is proved in the same way. \square

Remark 2.2.2. It is not hard to see that (2.2.1) and (2.2.2) are just compact forms of the tree summation formulas by Kontsevich and Soibelman, cf. [60].

We need a series of technical lemmas on homotopy transfer.

Lemma 2.2.3. *Given $O_\infty[1]$ algebras $(V, q_1, \dots, q_n, \dots)$, $(V', q'_1, \dots, q'_n, \dots)$ and homotopy retractions $(V \xrightleftharpoons[f_1]{g_1} W, K)$, $(V' \xrightleftharpoons[f'_1]{g'_1} W', K')$ let $F : (W, R) \rightarrow (V, Q)$ and $F' : (W', R') \rightarrow (V', Q')$ as in Theorem 2.2.1. If*

$$h : (V \xrightleftharpoons[f_1]{g_1} W, K) \rightarrow (V' \xrightleftharpoons[f'_1]{g'_1} W', K')$$

is a morphism of homotopy retractions such that $\text{pr}_1(h) : V \rightarrow V'$ is a strict $O_\infty[1]$ morphism, then $\text{pr}_2(h) : W \rightarrow W'$ (cf. Remark 2.1.3) is a strict $O_\infty[1]$ morphism between the transferred $O_\infty[1]$ structures R and R' : moreover $F' \text{pr}_2(h) = \text{pr}_1(h)F : (W, R) \rightarrow (V', Q')$.

Proof. Consider for instance the $A_\infty[1]$ case, the others are treated similarly. We have to prove the identities $\text{pr}_1(h)f_i = f'_i \text{pr}_2(h)^{\otimes i}$ and $\text{pr}_2(h)r_i = r'_i \text{pr}_2(h)^{\otimes i}$ for all $i \geq 1$. We begin with the first identity, suppose inductively we have proven it for $i < n$, then by (1.2.3) also $\text{pr}_1(h)^{\otimes k} F_n^k = F_n'^k \text{pr}_2(h)^{\otimes n}$ for $2 \leq k \leq n$, but then (cf. Remark 2.1.3)

$$\text{pr}_1(h)f_n = \sum_{k=2}^n \text{pr}_1(h)Kq_k F_n^k = \sum_{k=2}^n K'q'_k \text{pr}_1(h)q_k F_n^k = \sum_{k=2}^n K'q'_k F_n'^k \text{pr}_2(h)^{\otimes n} = f'_n \text{pr}_2(h)^{\otimes n}.$$

Similarly

$$\mathrm{pr}_2(h)r_n = \sum_{k=2}^n \mathrm{pr}_2(h)g_1q_kF_n^k = \sum_{k=2}^n g'_1 \mathrm{pr}_1(h)q_kF_n^k = \sum_{k=2}^n g'_1q'_kF_n'^k \mathrm{pr}_2(h)^{\otimes n} = r'_n \mathrm{pr}_2(h)^{\otimes n}.$$

□

By the lemma homotopy transfer can be seen as a functor $\mathbf{O}_\infty[1] \times_{\mathbf{DG}} \mathbf{Horet} \rightarrow \mathbf{O}_\infty[1]$, where the fibred product is taken over the projection functor $\mathrm{pr}_1 : \mathbf{Horet} \rightarrow \mathbf{DG}$ and the tangent complex functor $\mathbf{O}_\infty[1] \rightarrow \mathbf{DG} : (V, q_1, \dots, q_n, \dots) \rightarrow (V, q_1)$.

Remark 2.2.4. The lemma also tells us that F from Theorem 2.2.1 is a natural transformation of functors $\mathbf{O}_\infty[1] \times_{\mathbf{DG}} \mathbf{Horet} \rightarrow \mathcal{O}_\infty[1]$ from homotopy transfer to the projection onto the first factor (both followed by the inclusion $\mathbf{O}_\infty[1] \hookrightarrow \mathcal{O}_\infty[1]$). In the proof of Theorem 5.2.16 we will also need a natural transformation in the opposite direction. Unfortunately, the previous proof of Theorem 2.2.1 does not give an $\mathcal{O}_\infty[1]$ morphism $G : V \rightarrow W$ right inverse to F and with linear part g_1 : although it is not hard to prove indirectly that such a G exists it is more difficult to show that we can choose G with the required naturality properties. In the $A_\infty[1]$ case there is a well known proof of Theorem 2.2.1 based on the ordinary perturbation lemma and what is known as the *tensor trick* (cf. [5]): this has the advantage to give also a possible choice for $G : V \rightarrow W$ with the desired properties. In the hypothesis of Theorem 2.2.1 let $\bar{K} : \bar{T}(V) \rightarrow \bar{T}(V)$ be the degree minus one linear map defined by $0 = \bar{K}_n^i : V^{\circ n} \rightarrow V^{\circ i}$ if $i \neq n$ (cf. Notation 1.1.4), $\bar{K}_1^1 = K : V \rightarrow V$ and

$$\bar{K}_n^n(v_1 \otimes \dots \otimes v_n) = \sum_{j=0}^{n-1} (-1)^{\sum_{k=1}^j |v_k|} f_1 g_1(v_1) \otimes \dots \otimes f_1 g_1(v_j) \otimes K(v_{j+1}) \otimes v_{j+2} \otimes \dots \otimes v_n.$$

A possible choice of $G : V \rightarrow W$ is given by the recursion $pG = g = g_1 + gQ_+\bar{K}$ (in other words, $g_n(v_1 \otimes \dots \otimes v_n) = \sum_{i=1}^{n-1} g_i Q_n^i \bar{K}_n^n(v_1 \otimes \dots \otimes v_n)$ for $n \geq 2$). Then a similar argument as in the proof of the previous lemma shows that this choice of G defines a natural transformation of functors $\mathbf{O}_\infty[1] \times_{\mathbf{DG}} \mathbf{Horet} \rightarrow \mathcal{O}_\infty[1]$ from the projection onto the first factor to homotopy transfer (both followed by the inclusion $\mathbf{O}_\infty[1] \hookrightarrow \mathcal{O}_\infty[1]$).

Actually we need this result in the $L_\infty[1]$ case, where the tensor trick breaks down: the solution is due to Berglund [5]. First we consider the symmetrization $\bar{K}^{\mathrm{sym}} : \bar{T}(V) \rightarrow \bar{T}(V)$ of the operator \bar{K} : explicitly $(\bar{K}^{\mathrm{sym}})_n^i = 0$ if $i \neq n$ and $(\bar{K}^{\mathrm{sym}})_n^n = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma^{-1} \bar{K}_n^n \sigma$ (where $\sigma^{-1} \bar{K}_n^n \sigma$ is the composition $V^{\otimes n} \xrightarrow{\sigma} V^{\otimes n} \xrightarrow{\bar{K}_n^n} V^{\otimes n} \xrightarrow{\sigma^{-1}} V^{\otimes n}$). The operator \bar{K}^{sym} preserves S_n -invariants, so it pulls back via $\mathrm{sym} : \bar{S}(V) \rightarrow \bar{T}(V)$ to a degree minus one operator which we denote again by $\bar{K} : \bar{S}(V) \rightarrow \bar{S}(V)$: then a possible choice of $G : \bar{S}(V) \rightarrow \bar{S}(W)$ is defined as before by the recursion $pG = g = g_1 + gQ_+\bar{K}$, cf. [5]. Again we see that this choice of G has the required naturality properties.

We notice that

Lemma 2.2.5. *As a functor $\mathbf{O}_\infty[1] \times_{\mathbf{DG}} \mathbf{Horet} \rightarrow \mathbf{O}_\infty[1]$ homotopy transfer commutes with small limits.*

Proof. As it is immediate that it commutes with arbitrary products and equalizers, cf. Remark 2.1.3. □

Lemma 2.2.6. *In the hypotheses of Theorem 2.2.1 let $I \subset V$ be an (resp.: abelian) $O_\infty[1]$ ideal such that $K(I) \subset I$ and $f_1 g_1(I) \subset I$, then $J := g_1(I)$ is an (resp.: abelian) $O_\infty[1]$ ideal of W with the transferred $O_\infty[1]$ structure.*

Proof. The hypotheses say that the homotopy retraction in the claim of Theorem 2.2.1 restricts to a homotopy retraction $(I \xrightleftharpoons[f_1]{g_1} J, K)$, by the previous Lemma J with the transferred $O_\infty[1]$ structure is an $O_\infty[1]$ subalgebra of W with the transferred $O_\infty[1]$ structure. To fix the ideas consider the $L_\infty[1]$ case, the others are treated in the same way. If we suppose to have shown inductively that $r_i(J \otimes W^{\odot i-1}) \subset J$ and $f_i(J \otimes W^{\odot i-1}) \subset I$ for all $i < n$, then we also see that $q_i F_n^i(J \otimes W^{\odot n-1}) \subset q_i(I \otimes V^{\odot i-1}) \subset I$ for all $2 \leq i \leq n$ by the inductive hypothesis and since I is an $O_\infty[1]$ ideal of (V, Q) , and thus $r_n(J \otimes W^{\odot n-1}) = \sum_{i=2}^n g_1 q_i F_n^i(J \otimes W^{\odot n-1}) \subset g_1(I) = J$ and $f_n(J \otimes W^{\odot n-1}) = \sum_{i=2}^n K q_i F_n^i(J \otimes W^{\odot n-1}) \subset K(I) = I$ and we can go on with the induction. If I is an abelian $O_\infty[1]$ ideal we see by the same inductive argument that $r_n(J \otimes W^{\odot n-1}) = 0$ and $f_n(J \otimes W^{\odot n-1}) = 0$ for all $n \geq 2$. \square

Lemma 2.2.7. *Let $g_1 : (V, q_1, \dots, q_n, \dots) \rightarrow (W, r_1, \dots, r_n, \dots)$ be a strict morphism of $O_\infty[1]$ algebras fitting into a contraction $(V \xrightleftharpoons[f_1]{g_1} W, K)$ from (V, q_1) to (W, r_1) . The transferred $O_\infty[1]$ algebra structure on W is again $(W, r_1, \dots, r_n, \dots)$, moreover the $O_\infty[1]$ morphism $g_1 : V \rightarrow W$ is left inverse to $F : W \rightarrow V$ from Theorem 2.2.1.*

Proof. The first statement follows from Lemma 2.2.3 applied to the morphism of contractions

$$h : (V \xrightleftharpoons[f_1]{g_1} W, K) \rightarrow (W \xrightleftharpoons[\text{id}_W]{\text{id}_W} W, 0)$$

such that $\text{pr}_1(h) = g_1$, $\text{pr}_2(h) = \text{id}_W$. For the second statement we have $g_1 f_1 = \text{id}_W$ and $g_1 K = 0$ by hypothesis, thus also $g_1 f_n = \sum_{i=2}^n g_1 K q_i F_n^i = 0$ for all $n \geq 2$ and the thesis follows. \square

Lemma 2.2.8. *Homotopy transfer commutes with composition of homotopy retractions.*

Proof. Let $(V, q_1, \dots, q_n, \dots)$ be a $O_\infty[1]$ algebra, (W, r_1) and (X, s_1) dg spaces,

$$(V \xrightleftharpoons[f_1]{g_1} W, K), \quad (W \xrightleftharpoons[f'_1]{g'_1} X, H) \quad \text{homotopy retractions.}$$

Let $(W, r_1, \dots, r_n, \dots)$ and $F : W \rightarrow V$ induced via homotopy transfer along the first homotopy retraction as in Theorem 2.2.1, likewise let $(X, s_1, \dots, s_n, \dots)$ and $F' : X \rightarrow W$ induced via homotopy transfer from $(W, r_1, \dots, r_n, \dots)$ along the second homotopy retraction. We show that $FF' : X \rightarrow V$ is the $O_\infty[1]$ morphism induced via homotopy transfer along the composite homotopy retraction

$$(V \xrightleftharpoons[f_1 f'_1]{g'_1 g_1} X, K + f_1 H g_1).$$

In fact

$$\begin{aligned} pFF' &= (f_1 + Kq_+F)F' = f_1(f'_1 + Hr_+F') + Kq_+FF' = \\ &= f_1 f'_1 + f_1 H(g_1 q_+ F)F' + Kq_+FF' = f_1 f'_1 + (K + f_1 H g_1)q_+ FF', \end{aligned}$$

which is exactly Equation (2.2.1). We show that S is the $O_\infty[1]$ structure on X induced via homotopy transfer along the composite homotopy retraction: in fact

$$pS = s_1 + g'_1 r_+ F' = s_1 + g'_1 g_1 q_+ F F',$$

which by the above is exactly Equation (2.2.2). \square

Finally, homotopy transfer is compatible with scalar extension and symmetrization.

Lemma 2.2.9. *Given an $O_\infty[1]$ algebra $(V, q_1, \dots, q_n, \dots)$, a homotopy retraction $(V \xrightleftharpoons[f_1]{g_1} W, K)$ and a commutative dg algebra (A, d, \cdot) , then the two $O_\infty[1]$ structures on $A \otimes W$, the one induced first by homotopy transfer and then by extension of scalars by A and the one induced first by extension of scalars of V by A and then by homotopy transfer along $(A \otimes V \xrightleftharpoons[\text{id}_A \otimes f_1]{\text{id}_A \otimes g_1} A \otimes W, \text{id}_A \otimes K)$, coincide. In the $A_\infty[1]$ case, this remains true even if A is only dg associative.*

Proof. Straightforward. \square

Lemma 2.2.10. *Given an $A_\infty[1]$ algebra $(V, q_1, \dots, q_n, \dots)$ together with a homotopy retraction $(V \xrightleftharpoons[f_1]{g_1} W, K)$, then the two $L_\infty[1]$ structures on W , the one induced first by homotopy transfer and then by symmetrization and the one induced first by symmetrization and then by homotopy transfer, coincide.*

Proof. Given $F : \bar{T}(W) \rightarrow \bar{T}(V)$ satisfying the recursion (2.2.1) we have to show that its symmetrization $\text{sym}(F) : \bar{S}(W) \rightarrow \bar{S}(V)$ satisfies the same recursion: this follows by writing down the formulas explicitly, details are left to the reader. Similarly, given $R = r_1 + g_1 q_+ F$ the transferred $A_\infty[1]$ structure on W then one checks that $p \text{sym}(R) = r_1 + g_1 \text{sym}(q_+) \text{sym}(F)$, and then by the first part $\text{sym}(R)$ is induced by $\text{sym}(Q)$ via homotopy transfer. \square

We close this section by recalling probably the most important consequences of Theorem 2.2.1, namely the existence of the minimal model.

Definition 2.2.11. An $O_\infty[1]$ algebra (V, Q) is called minimal if Q has vanishing linear part, that is, $q_1 = 0$: unlike other properties, for instance the one of being abelian or nilpotent, the property of an $O_\infty[1]$ algebra of being minimal is preserved by $O_\infty[1]$ isomorphisms. A minimal model of an $O_\infty[1]$ algebra (V, Q) is the datum of a minimal $O_\infty[1]$ algebra (W, R) and a weak equivalence $F : (W, R) \rightarrow (V, Q)$.

Since a weak equivalence of minimal $O_\infty[1]$ algebras has to be an isomorphism, a minimal model as in the previous definition is determined up to an isomorphism over (V, Q) . Existence of a minimal model is ensured by Theorem 2.2.1 since over a field of characteristic zero every complex (V, d) retracts onto its homology $(H(V, d), 0)$.

Theorem 2.2.12. *Given an $O_\infty[1]$ algebra $(V, q_1, \dots, q_n, \dots)$ there is a minimal $O_\infty[1]$ algebra structure $(H(V), 0, r_2, \dots, r_n, \dots)$ on the tangent cohomology $H(V) = (V, q_1)$ together with a weak equivalence $H(V) \rightarrow V$ of $O_\infty[1]$ algebras.*

In fact with some more work (cf. [62]) it can be proved that every $O_\infty[1]$ algebra is isomorphic to the direct product of a minimal model and an acyclic complex (considered as an abelian $O_\infty[1]$ algebra). We will need the following lemma.

Lemma 2.2.13. *An $O_\infty[1]$ algebra is homotopy abelian if and only if the $O_\infty[1]$ structure of some (and then all) minimal model is trivial. Given an $O_\infty[1]$ morphism $F : (V, Q) \rightarrow (W, R)$: if $H(F) = H(f_1) : H(V) \rightarrow H(W)$ is injective and (W, R) is homotopy abelian so is (V, Q) ; similarly, if $H(F)$ is surjective and (V, Q) is homotopy abelian so is (W, R) .*

Proof. If $(W, 0)$ is a minimal model of (V, Q) , then (V, Q) is homotopy abelian by definition: conversely, given an abelian $O_\infty[1]$ algebra $(V, q_1, 0, \dots, 0, \dots)$ it is clear that $(H(V), 0)$ is a minimal model and since weakly equivalent $O_\infty[1]$ algebras have isomorphic minimal models the first statement follows.

Given an $O_\infty[1]$ morphism $F : (V, Q) \rightarrow (W, R)$ there is an $O_\infty[1]$ morphism $H(V) \rightarrow H(W)$ between the minimal models with linear part $H(f_1)$. If $H(f_1)$ is surjective so is the $O_\infty[1]$ morphism $H(V) \rightarrow H(W)$ (that is, the associated morphism of graded coalgebras is surjective), thus if the $O_\infty[1]$ structure on $H(W)$ is trivial so has to be the one on $H(V)$ and by the first part of the lemma (V, Q) is homotopy abelian. The other case is proved similarly. \square

2.3 Formal Kuranishi theorem

Homotopy transfer can be enhanced to a functor $\widehat{\mathbf{O}}_\infty[1] \times_{\widehat{\mathbf{DG}}} \widehat{\mathbf{Horet}} \rightarrow \widehat{\mathbf{O}}_\infty[1]$.

Proposition 2.3.1. *In the hypotheses of Theorem 2.2.1, suppose that V is a complete $O_\infty[1]$ algebra and that the homotopy retraction is complete with respect to the same filtration $F^\bullet V$ on V . Then W with the transferred $O_\infty[1]$ structure and the induced filtration $F^\bullet W = i^{-1}(F^\bullet V)$ is a complete $O_\infty[1]$ algebra and $F : W \rightarrow V$ from Theorem 2.2.1 (as well as $G : V \rightarrow W$ from Remark 2.2.4 in the $A_\infty[1]$ and $L_\infty[1]$ cases) is a continuous $O_\infty[1]$ morphism.*

Proof. As in Definition 2.1.5 we see that $(W, F^\bullet W, r_1)$ is a complete dg space and f_1, g_1 are continuous morphisms. Continuity of f_n and r_n for $n \geq 2$ follows inductively from the continuity of $K, g_1, f_1, q_k \forall k \geq 2$, and from $f_n = \sum_{k=2}^n K q_k F_n^k, r_n = \sum_{k=2}^n g_1 q_k F_n^k$. Continuity of G is proved similarly. \square

Remark 2.3.2. All the various lemmas from the previous section remain valid in the complete setting, details are left to the reader.

The aim of this section is to prove the following formal analog of Kuranishi's theorem, essentially due to Getzler [39] (cf. the proofs of Lemma 4.6 and Lemma 5.3 in loc. cit.).

Theorem 2.3.3. *In the same hypotheses as in Proposition 2.3.1, in the $L_\infty[1]$ case, suppose moreover that $(V \xrightleftharpoons[f_1]{g_1} W, K)$ is a complete contraction. Let $G = (g_1, \dots, g_n, \dots) : V \rightarrow W$ be a continuous $L_\infty[1]$ left inverse to $F : W \rightarrow V$ (not necessarily the particular choice of such a G made in Remark 2.2.4): the correspondence*

$$\rho : \text{MC}(V) \rightarrow \text{MC}(W) \times \left(\text{Im } K \cap V^{-1} \right) : x \rightarrow (\text{MC}(G)(x), K(x))$$

is bijective. The inverse ρ^{-1} admits the following recursive construction: given $y \in \text{MC}(W)$ and $K(v) \in \text{Im } K \cap V^{-1}$ we define a succession $x_i \in V^0$, $i \geq -1$, by $x_{-1} = 0$ and for $i + 1 \geq 0$ by

$$x_{i+1} = f_1(y) - q_1 K(v) + \sum_{n \geq 2} \frac{1}{n!} (Kq_n - f_1 g_n) (x_i^{\odot n}), \quad (2.3.1)$$

this succession is convergent (with respect to the complete topology induced by the filtration on V) and $\rho^{-1}(y, K(v)) = x := \lim x_i$. We have $\rho^{-1}(-, 0) = \text{MC}(F) : \text{MC}(W) \rightarrow \text{MC}(V)$ and together with the restriction of g_1 they are inverses bijective correspondences between the sets $\text{MC}(W)$ and $\text{Ker } K \cap \text{MC}(V)$.

Proof. We proceed as in [39], moreover we use the notations and computations from Remark 1.3.15. First of all if $x \in \text{MC}(V)$ then

$$x = f_1 g_1(x) - q_1 K(x) - Kq_1(x) = f_1 G^{0\#}(x) - q_1 K(x) + \sum_{n \geq 2} \frac{1}{n!} (Kq_n - f_1 g_n) (x^{\odot n}) \quad (2.3.2)$$

Equation (2.3.2) implies injectivity of ρ as follows: if $y \in \text{MC}(V)$ is such that $G^{0\#}(x) = G^{0\#}(y)$, $K(x) = K(y)$, then subtracting the respective equations (2.3.2) for x and y we obtain

$$x - y = \sum_{n \geq 2} \frac{1}{n!} \sum_{j=0}^{n-1} (Kq_n - f_1 g_n) (x^{\odot j} \odot (x - y) \odot y^{\odot n-j-1})$$

The above shows $x - y \in F^p V \Rightarrow x - y \in F^{p+1} V$, thus inductively $x - y \in \bigcap_{p \geq 1} F^p V = 0$.

Now consider $y \in \text{MC}(W)$, $K(v) \in \text{Im } K \cap V^{-1}$ and the sequence $x_i \in V^0$ defined by the recursion (2.3.1): we show that the limit $x = \lim x_i$ is well defined. We suppose inductively, starting with $x_0 - x_{-1} \in F^1 V = V$, that $x_i - x_{i-1} \in F^{i+1} V$, and deduce

$$x_{i+1} - x_i = \sum_{n \geq 2} \sum_{j=0}^{n-1} \frac{1}{n!} (Kq_n - f_1 g_n) (x_i^{\odot j} \odot (x_i - x_{i-1}) \odot x_{i-1}^{\odot n-j-1}) \in F^{i+2} V.$$

By completeness the infinite sum $\sum_{i \geq 0} (x_i - x_{i-1})$ converges to a well defined $x \in V^0$.

By construction x satisfies

$$x = f_1(y) - q_1 K(v) + \sum_{n \geq 2} \frac{1}{n!} (Kq_n - f_1 g_n) (x^{\odot n})$$

Applying K , since $Kf_1 = K^2 = 0 = g_1 K = 0$,

$$K(x) = -Kq_1 K(v) = (q_1 K + \text{id}_V - f_1 g_1) K(v) = K(v).$$

Applying g_1 , since moreover $g_1 f_1 = \text{id}_W$,

$$g_1(x) = y - \sum_{n \geq 2} \frac{1}{n!} g_n(x^{\odot n}) \quad \implies \quad G^{0\#}(x) = y.$$

Applying q_1 , we get

$$q_1(x) = q_1 f_1(y) + \sum_{n \geq 2} \frac{1}{n!} (f_1 g_1 - \text{id}_V - Kq_1) q_n(x^{\odot n}) - \sum_{n \geq 2} \frac{1}{n!} q_1 f_1 g_n(x^{\odot n}).$$

We notice that

$$q_1 f_1(y) - \sum_{n \geq 2} \frac{1}{n!} q_1 f_1 g_n(x^{\odot n}) = q_1 f_1(y) + q_1 f_1 g_1(x) - q_1 f_1 G^{0\#}(x) = q_1 f_1 g_1(x) = f_1 g_1 q_1(x),$$

hence

$$\mathcal{R}_V(x) = f_1 g_1 \mathcal{R}_V(x) - \sum_{n \geq 2} \frac{1}{n!} K q_1 q_n(x^{\odot n}).$$

We denote by $fG_k = \sum_{i=1}^k f_i G_k^i$ the k -th Taylor coefficient of the continuous $L_\infty[1]$ morphism $FG : V \rightarrow V$. By the computation in Remark 1.3.15, and since $y \in \text{MC}(W)$, we see that

$$0 = \mathcal{R}_V F^{0\#}(y) = \mathcal{R}_V F^{0\#} G^{0\#}(x) = f_1 g_1 \mathcal{R}_V(x) + \sum_{k \geq 2} \frac{1}{(k-1)!} fG_k(\mathcal{R}_V(x) \odot x^{\odot k-1}).$$

On the other hand

$$\begin{aligned} - \sum_{n \geq 2} \frac{1}{n!} K q_1 q_n(x^{\odot n}) &= \sum_{n \geq 2} \frac{1}{n!} \sum_{j=1}^{n-1} \frac{n!}{j!(n-j)!} K q_{n-j+1}(q_j(x^{\odot j}) \odot x^{\odot n-j}) = \\ &= \sum_{k \geq 2} \frac{1}{(k-1)!} K q_k(\mathcal{R}_V(x) \odot x^{\odot k-1}). \end{aligned}$$

Finally, putting everything together

$$\mathcal{R}_V(x) = \sum_{k \geq 2} \frac{1}{(k-1)!} (K q_k - fG_k)(\mathcal{R}_V(x) \odot x^{\odot k-1}),$$

which shows $\mathcal{R}_V(x) \in F^p V \Rightarrow \mathcal{R}_V(x) \in F^{p+1} V$, thus inductively $\mathcal{R}_V(x) = 0$. We have constructed $x \in \text{MC}(V)$ such that $\text{MC}(G)(x) = y$ and $K(x) = K(v)$, thus ρ is surjective.

Since $f_n = \sum_{k=2}^n K q_k F_n^k$ for $n \geq 2$, we also see that $g_1 f_n = K f_n = 0$ for $n \geq 2$, hence the identities

$$g_1 F^{0\#} = \text{id}_W, \quad K F^{0\#} = 0.$$

Remark 2.3.4. In fact applying respectively g_1 and K to the identity

$$\mathcal{R}_V F^{0\#}(w) = \sum_{n \geq 1} \frac{1}{(n-1)!} f_n(\mathcal{R}_W(x) \odot x^{\odot n-1})$$

we also see that

$$g_1 \mathcal{R}_V F^{0\#} = \mathcal{R}_W, \quad K \mathcal{R}_V F^{0\#} = 0.$$

Given $x \in \text{MC}(W)$ we have $\text{MC}(G)(\text{MC}(F)(x)) = x$ and by the above also $K(\text{MC}(F)(x)) = 0$, thus $\text{MC}(F)(x) = \rho^{-1}(x, 0)$. It is now clear that $\text{MC}(F) = \rho^{-1}(-, 0) : \text{MC}(W) \rightarrow \text{Ker } K \cap \text{MC}(V)$ is a bijective correspondence and we have already observed that $g_1 \text{MC}(F) = \text{id}_{\text{MC}(W)}$. \square

Chapter 3

∞ structures on cochain complexes

In Section 3.1 we review the construction by Whitney [108] and Dupont [30] of a simplicial contraction $(\Omega(\Delta_\bullet) \xrightleftharpoons[f]{\iota} C(\Delta_\bullet), K)$ from the simplicial dg algebra of polynomial forms on the standard cosimplicial simplex Δ_\bullet to the simplicial complex of non-degenerate cochains on Δ_\bullet . As in the paper [22], in Section 3.2 we use this contraction together with scalar extension and homotopy transfer to put a natural $O_\infty[1]$ algebra structure on the complex $C(X; V)$ of non-degenerate cochains on a simplicial set X with coefficients in an $O_\infty[1]$ algebra V . In the final section we compute explicitly the $O_\infty[1]$ structure on $C(\Delta_1; V)$ when V is a dg (resp.: associative, commutative, Lie) algebra, seen as an $O_\infty[1]$ algebra via décalage. Moreover we prove the existence of certain homotopy limits in the category $\mathcal{O}_\infty[1]$, namely homotopy equalizers, and we give explicit formulas when the target $O_\infty[1]$ algebra is a dg (resp.: associative, commutative, Lie) algebra, generalizing formulas by Fiorenza and Manetti [31].

3.1 Dupont's contraction

By an abuse of notation we denote by the same symbol the standard n -th simplex $\Delta_n \in \mathbf{SSet}$ and the standard affine n -th simplex $\Delta_n = \{(t_0, \dots, t_n) \in \mathbb{K}^{n+1} \text{ s.t. } t_0 + \dots + t_n = 1\}$: we denote by (Ω_n, d, \wedge) the dg commutative algebra of polynomial forms on Δ_n [11, 100], formally

$$\Omega_n = \frac{S(t_0, \dots, t_n, dt_0, \dots, dt_n)}{(t_0 + \dots + t_n - 1, dt_0 + \dots + dt_n)}$$

is the symmetric algebra over variables t_0, \dots, t_n in degree 0, dt_0, \dots, dt_n in degree 1 and with differential induced by $t_i \rightarrow dt_i$, $i = 0, \dots, n$, modulo the dg ideal generated by $t_0 + \dots + t_n - 1$ and $dt_0 + \dots + dt_n$. $\Omega_\bullet = \{\Omega_n\}_{n \geq 0}$ has a natural structure of simplicial dg commutative algebra, where the faces and the degeneracies are induced by the usual cosimplicial structure on Δ_\bullet via pullback of forms: for $i = 0, \dots, n$

$$\begin{aligned} \partial_i : \Omega_n &\rightarrow \Omega_{n-1} : \omega(t_0, \dots, t_i, \dots, t_n, dt_0, \dots, dt_i, \dots, dt_n) \rightarrow \\ &\rightarrow \omega(t_0, \dots, 0, \dots, t_{n-1}, dt_0, \dots, 0, \dots, dt_{n-1}) \quad \text{and} \end{aligned}$$

$$\begin{aligned} s_i : \Omega_n &\rightarrow \Omega_{n+1} : \omega(t_0, \dots, t_i, \dots, t_n, dt_0, \dots, dt_i, \dots, dt_n) \rightarrow \\ &\rightarrow \omega(t_0, \dots, t_i + t_{i+1}, \dots, t_{n+1}, dt_0, \dots, dt_i + dt_{i+1}, \dots, dt_{n+1}). \end{aligned}$$

For each $k \geq 0$, $\Omega_{\bullet}^k = \{\Omega_n^k\}_{n \geq 0}$ is the simplicial \mathbb{K} -vector space of polynomial k -forms on the standard cosimplicial simplex Δ_{\bullet} . Given a simplicial set X we denote by $\Omega^k(X) := \mathbf{SSet}(X, \Omega_{\bullet}^k)$ the space of polynomial k -forms on X , it has a natural structure of \mathbb{K} -vector space defined pointwise. We denote by $\omega(\sigma) \in \Omega_n^k$ the value of a k -form $\omega \in \Omega^k(X)$ on an n -simplex $\sigma : \Delta_n \rightarrow X$. The Sullivan-de Rham algebra $\Omega(X) := \bigoplus_{k \geq 0} \Omega^k(X)$ of polynomial forms on X has a natural structure of dg commutative algebra defined pointwise by $(\omega \wedge \phi)(\sigma) = \omega(\sigma) \wedge \phi(\sigma)$ and $d\omega(\sigma) = d(\omega(\sigma))$: we notice that $\Omega(\Delta_n) = \Omega_n$. The Sullivan-de Rham algebra functor $\Omega(-) : \mathbf{SSet}^{op} \rightarrow \mathbf{DGCA}$ sends $f : X \rightarrow Y$ to the pullback $\Omega(f) =: f^* : \Omega(Y) \rightarrow \Omega(X)$ defined pointwise by $f^*\omega(\sigma) = \omega(f\sigma)$.

We denote by $C(X) := C(X; \mathbb{K})$ the complex of non-degenerate simplicial \mathbb{K} -cochains on X , that is, $C(X) = \bigoplus_{k \geq 0} C^k(X)$ where $C^k(X)$ is the space of \mathbb{K} -valued k -cochains $\alpha : X_k \rightarrow \mathbb{K} : \sigma \rightarrow \alpha_{\sigma}$ on X vanishing on degenerate simplices. As usual we equip $C(X)$ with the differential $d : C^k(X) \rightarrow C^{k+1}(X) : \alpha \rightarrow (d\alpha)_{\sigma} = \sum_{i=0}^{k+1} (-1)^i \alpha_{\partial_i \sigma}$, where $\partial_i : X_{k+1} \rightarrow X_k$, $i = 0, \dots, k+1$, are the face maps: the functor $C(-) : \mathbf{SSet}^{op} \rightarrow \mathbf{DG}$ sends $f : X \rightarrow Y$ to the pullback $C(f) =: f^* : C(Y) \rightarrow C(X)$ defined pointwise by $f^*\alpha_{\sigma} = \alpha_{f\sigma}$. The aim of this section is to review the construction of a natural contraction from $\Omega(X)$ to $C(X)$, following Whitney [108] and Dupont [30].

Definition 3.1.1. We denote by $\omega_{i_0 \dots i_k} \in \Omega_n^k$, where $0 \leq i_0 < \dots < i_k \leq n$, the Whitney elementary form

$$\Omega_n^k \ni \omega_{i_0 \dots i_k} := k! \sum_{j=0}^k (-1)^j t_{i_j} dt_{i_0} \wedge \dots \wedge \widehat{dt_{i_j}} \wedge \dots \wedge dt_{i_k}.$$

We denote by $\sigma_{i_0 \dots i_k} : \Delta_k \rightarrow \Delta_n : [0 \dots k] \rightarrow [i_0 \dots i_k]$ considered as an element of $(\Delta_n)_k$ (the k -simplices of the standard n -th simplex), and given a cochain $\alpha \in C^k(\Delta_n)$ we denote by $\alpha_{i_0 \dots i_k} := \alpha_{\sigma_{i_0 \dots i_k}} \in \mathbb{K}$. Following [108] we define a dg embedding $\iota : C(\Delta_n) \rightarrow \Omega_n$ by

$$\iota : C^k(\Delta_n) \rightarrow \Omega_n^k : \alpha \rightarrow \sum_{0 \leq i_0 < \dots < i_k \leq n} \alpha_{i_0 \dots i_k} \omega_{i_0 \dots i_k},$$

it sends $C(\Delta_n)$ isomorphically onto the subcomplex of Ω_n spanned by Whitney's elementary forms.

In [108] Whitney also defines a dg left inverse $\int : \Omega_n \rightarrow C(\Delta_n)$ to ι by integrating k -forms over k -simplices: explicitly, for each $k \geq 0$ let

$$\int_{\Delta_k} : \Omega_k^k \rightarrow \mathbb{K} : t_1^{i_1} \dots t_k^{i_k} dt_1 \wedge \dots \wedge dt_k \rightarrow \frac{i_1! \dots i_k!}{(i_1 + \dots + i_k + k)!},$$

then $\int : \Omega_n \rightarrow C(\Delta_n)$ is defined by

$$\int : \Omega_n^k = \Omega^k(\Delta_n) \rightarrow C^k(\Delta_n) : \omega \rightarrow \left(\int \omega \right)_{i_0 \dots i_k} = \int_{\Delta^k} \omega(\sigma_{i_0 \dots i_k}).$$

This is a dg morphism by the classical Stokes' formula. By construction the above assemble to a simplicial dg morphism $\iota : C(\Delta_{\bullet}) \rightarrow \Omega_{\bullet}$ of simplicial dg spaces and a simplicial dg left inverse $\int : \Omega_{\bullet} \rightarrow C(\Delta_{\bullet})$.

We recall the construction of a simplicial homotopy between ιf and $\text{id}_{\Omega_\bullet}$ due to Dupont [30], cf. also Getzler's paper [39]. For $i = 0, \dots, n$, we denote by $\vec{e}_i \in \Delta_n$ the i -th vertex and we put

$$\varphi_i : [0, 1] \times \Delta_n \rightarrow \Delta_n : (u, \vec{t}) \rightarrow u \cdot \vec{t} + (1 - u) \cdot \vec{e}_i.$$

(here we are considering Δ_n as the affine n -simplex). We define $h^i : \Omega_n^k \rightarrow \Omega_n^{k-1}$ as the composition of pullback by φ_i and integration along the fibres of the projection $[0, 1] \times \Delta_n \rightarrow \Delta_n$. Explicitly

$$\begin{aligned} h^i : f(t_0, \dots, t_n) dt_{i_1} \wedge \dots \wedge dt_{i_k} \rightarrow \\ \rightarrow \left(\sum_{j=1}^k (-1)^{j-1} (t_{i_j} - \delta_{i_j}^i) dt_{i_1} \wedge \dots \wedge \widehat{dt_{i_j}} \wedge \dots \wedge dt_{i_k} \right) \int_0^1 u^{k-1} (f \circ \varphi_i) du, \end{aligned} \quad (3.1.1)$$

where $\delta_{i_j}^i$ is Kronecker's delta. Finally, following Dupont¹ we put

$$K : \Omega_n^k \rightarrow \Omega_n^{k-1} : \omega \rightarrow \sum_{j=0}^{k-1} (-1)^{j+1} \sum_{0 \leq i_0 < \dots < i_j \leq n} \omega_{i_0 \dots i_j} \wedge h^{i_j} \dots h^{i_0}(\omega) \quad (3.1.2)$$

This assemble to a simplicial $K : \Omega_\bullet^k \rightarrow \Omega_\bullet^{k-1}$.

Theorem 3.1.2. *With the previous definitions $(\Omega_\bullet \xrightleftharpoons[\iota]{f_1} C(\Delta_\bullet), K)$ is a simplicial contraction.*

We refer to [30] for a proof that it is a simplicial homotopy retraction, where in the process the following lemma is also proved (cf. the footnote).

Lemma 3.1.3. *Given $\omega \in \Omega_n^k$ and $0 \leq i_0 < \dots < i_k \leq n$, then $(f\omega)_{i_0 \dots i_k} = \varepsilon^{i_k} h^{i_{k-1}} \dots h^{i_0}(\omega)$, where $\varepsilon^i : \Omega_n^0 \rightarrow \mathbb{K}$ is evaluation at the vertex \vec{e}_i , $i = 0, \dots, n$.*

We refer to [39] for a proof that $(\Omega_\bullet \xrightleftharpoons[\iota]{f_1} C(\Delta_\bullet), K)$ is moreover a simplicial contraction, where in the process it is also proved that

Lemma 3.1.4. $h^i h^j + h^j h^i = 0, \forall 0 \leq i, j \leq n$.

Definition 3.1.5. The Dupont's contraction $(\Omega(X) \xrightleftharpoons[\iota_X]{f_X} C(X), K_X)$ associated to a simplicial set X is defined on $\omega \in \Omega^k(X)$ and $\alpha \in C^k(X)$ by

$\iota_X(\omega) \in C^k(X)$ is the k -form sending an n -simplex $\sigma : \Delta_n \rightarrow X$ to $\iota_X(\omega)(\sigma) = \iota(\sigma^* \omega)$, where $\sigma^* \omega \in C^k(\Delta_n)$ is the pullback of ω by σ and $\iota : C^k(\Delta_n) \rightarrow \Omega_n^k$ is as in Definition 3.1.1.

$\int_X \omega \in C^k(X)$ is the (non-degenerate) k -cochain evaluating to $(\int_X \omega)_\sigma = \int_{\Delta_k} \omega(\sigma) \in \mathbb{K}$ on a k -simplex $\sigma : \Delta_k \rightarrow X$.

$K_X(\omega) \in \Omega^{k-1}(X)$ is the $(k-1)$ -form sending an n -simplex $\sigma : \Delta_n \rightarrow X$ to $K_X(\omega)(\sigma) = K(\omega(\sigma))$, where $K : \Omega_n^k \rightarrow \Omega_n^{k-1}$ is as in (3.1.2).

¹In the formulas from [Dupont] and [Getzler] the sign $(-1)^{j+1}$ doesn't appear, on the other hand Dupont had defined φ_i as $(u, \vec{t}) \rightarrow u \cdot \vec{e}_i + (1 - u) \cdot \vec{t}$, with the effect of changing the formula (3.1.1) for h^i by a sign (-1) , thus our formula coincides with Dupont's original one. It can be checked that given $\omega_0 \wedge \omega_{012} \in \Omega_2$, without the sign $(-1)^{j+1}$ in the formula (3.1.2) one would obtain $K(\omega_0 \wedge \omega_{012}) = 0$, contradicting $Kd + dK = f_1 \int_1 - \text{id}_{\Omega_2}$.

In the sequel, if there is no risk of confusion, we will omit the explicit reference to X in the notation and we will write $(\Omega(X) \xrightleftharpoons[\iota]{f} C(X), K)$. It is easy to see that it is defined a functor $\text{Dup} : \mathbf{SSet}^{op} \rightarrow \mathbf{Contr}$ sending X to the associated Dupont's contraction.

3.2 ∞ structures on cochain complexes

Given an $O_\infty[1]$ algebra $(V, q_1, \dots, q_n, \dots)$ and a simplicial set X it is defined, by extension of scalars (Definitions 1.2.12 and 1.3.9), an $O_\infty[1]$ structure on the space $\Omega(X; V) := \Omega(X) \otimes V$ of polynomial forms on X with coefficients in V : as pullback by a morphism $f : X \rightarrow Y$ and pushforward by a strict $O_\infty[1]$ morphism $g : V \rightarrow W$ commute with each other, that is, the diagram

$$\begin{array}{ccc} \Omega(Y; V) & \xrightarrow{f^*} & \Omega(X; V) \\ g_* \downarrow & & \downarrow g_* \\ \Omega(Y; W) & \xrightarrow{f^*} & \Omega(X; W) \end{array}$$

is commutative, it is defined the functor $\Omega(-; -) : \mathbf{SSet}^{op} \times \mathbf{O}_\infty[1] \rightarrow \mathbf{O}_\infty[1]$.

Let $C(X; V)$ be the complex of cochains on X with coefficients in the dg space (V, q_1) : as a space $C(X; V) = C(X) \otimes V$, the differential is $d_{C(X)} \otimes \text{id}_V + \text{id}_{C(X)} \otimes q_1$. Dupont's contraction (Definition 3.1.5) induces a contraction

$$(\Omega(X; V) \xrightleftharpoons[\iota \otimes \text{id}_V]{f \otimes \text{id}_V} C(X; V), K \otimes \text{id}_V) \quad (3.2.1)$$

from $\Omega(X; V)$ to $C(X; V)$, hence by homotopy transfer a $O_\infty[1]$ algebra structure on $C(X; V)$. By Lemma 2.2.3, the pullback $f^* : C(Y; V) \rightarrow C(X; V)$ by $f : X \rightarrow Y$ is a strict morphism of $O_\infty[1]$ algebras, and so is the pushforward $g_* : C(X; V) \rightarrow C(X; W)$ by a strict $O_\infty[1]$ morphism $g : V \rightarrow W$: these commute with each other, in the sense that the following diagram commutes.

$$\begin{array}{ccc} C(Y; V) & \xrightarrow{f^*} & C(X; V) \\ g_* \downarrow & & \downarrow g_* \\ C(Y; W) & \xrightarrow{f^*} & C(X; W) \end{array}$$

Definition 3.2.1. The functor $C(-; -) : \mathbf{SSet}^{op} \times \mathbf{O}_\infty[1] \rightarrow \mathbf{O}_\infty[1]$ is defined as above, by applying homotopy transfer to $\Omega(-; -) : \mathbf{SSet}^{op} \times \mathbf{O}_\infty[1] \rightarrow \mathbf{O}_\infty[1]$ along Dupont's contraction $\text{Dup} : \mathbf{SSet}^{op} \rightarrow \mathbf{Contr}$.

Remark 3.2.2. As a particular case of the previous construction we see that the non-degenerate cochains functor $C(-) : \mathbf{SSet} \rightarrow \mathbf{DG}$ has a natural enhancement to a functor $C(-) : \mathbf{SSet} \rightarrow \mathbf{C}_\infty$, by applying homotopy transfer to $\Omega(-) : \mathbf{SSet} \rightarrow \mathbf{DGCA}$ along Dupont's contraction, cf. [22].

Given a simplicial set X the functor $C(X; -) : \mathbf{O}_\infty[1] \rightarrow \mathbf{O}_\infty[1]$ has an important technical advantage over the functor $\Omega(X; -) : \mathbf{O}_\infty[1] \rightarrow \mathbf{O}_\infty[1]$.

Lemma 3.2.3. *For all $X \in \mathbf{SSet}$ the functor $C(X; -) : \mathbf{O}_\infty[1] \rightarrow \mathbf{O}_\infty[1]$ commutes with small limits.*

Proof. Let $\mathcal{C} \rightarrow \mathbf{O}_\infty[1] : c \rightarrow V_c$ be a functor from a small category \mathcal{C} . Since the forgetful functor $-\# : \mathbf{O}_\infty[1] \rightarrow \mathbf{DG} : (V, q_1, \dots, q_n, \dots) \rightarrow (V, q_1)$ commutes with small limits, reflects isomorphisms and sends $C(X; V)$ to $C(X; V)^\# = \text{Hom}(C_*(X)/D_*(X), V^\#)$, where $C_*(X)$ is the complex of simplicial \mathbb{K} -chains on X (according to our cohomological convention, we take it concentrated in degrees ≤ 0) and $D_*(X)$ is the subcomplex spanned by the degenerate simplices, and since moreover $\text{Hom}(C_*(X)/D_*(X), -) : \mathbf{DG} \rightarrow \mathbf{DG}$ commutes with small limits, the lemma follows by looking at the diagram

$$\begin{array}{ccc} C(X; \lim_{c \in \mathcal{C}} V_c) & \xrightarrow{\quad\quad\quad} & \lim_{c \in \mathcal{C}} C(X; V_c) \\ \# \downarrow & & \downarrow \# \\ \text{Hom}(C_*(X)/D_*(X), \lim_{c \in \mathcal{C}} V_c^\#) & \xrightarrow{\cong} & \lim_{c \in \mathcal{C}} \text{Hom}(C_*(X)/D_*(X), V_c^\#) \end{array}$$

□

Remark 3.2.4. On the other hand, the functor $\Omega(X; -) : \mathbf{O}_\infty[1] \rightarrow \mathbf{O}_\infty[1]$ in general only commutes with finite limits.

Given a dg space W and a complete dg space $(V, F^\bullet V)$ the usual tensor product $W \otimes V$ is not in general complete with respect to the induced filtration $F^p(W \otimes V) = W \otimes F^p V$, so the definition of $\Omega(-; -)$ has to be modified slightly in the complete case. We introduce the completed tensor product $(W \widehat{\otimes} V, F^\bullet(W \widehat{\otimes} V))$: namely, this is the completion

$$W \widehat{\otimes} V = \lim (W \otimes V / W \otimes F^p V) = \lim W \otimes (V / F^p V),$$

with the induced filtration $F^p(W \widehat{\otimes} V) = \text{Ker}(W \widehat{\otimes} V \rightarrow W \otimes (V / F^p V))$. If $(V, F^\bullet V, q_1, \dots, q_n, \dots)$ is a complete $O_\infty[1]$ algebra and (A, d, \cdot) is a dg commutative algebra there is an induced $O_\infty[1]$ algebra structure on $A \otimes (V / F^p V)$ for all $p \geq 1$, hence also an induced $O_\infty[1]$ algebra structure on the completion $A \widehat{\otimes} V$ and this is compatible with the filtration. Finally, we define the functor $\Omega(-; -) : \mathbf{SSet}^{op} \times \widehat{\mathbf{O}}_\infty[1] \rightarrow \widehat{\mathbf{O}}_\infty[1]$ by $(\Omega(X; V), F^\bullet \Omega(X; V)) := (\Omega(X) \widehat{\otimes} V, F^\bullet(\Omega(X) \widehat{\otimes} V))$.

As for the functor $C(-; -)$ we see that for all complete spaces $(V, F^\bullet V)$ and simplicial sets X the space $C(X; V)$ is always complete with respect to the filtration $F^\bullet C(X; V) := C(X; F^\bullet V)$. There are two ways to put an $O_\infty[1]$ structure on $C(X; V)$, one by homotopy transfer from $\Omega(X) \otimes V$ with its ordinary (not complete) $O_\infty[1]$ structure along Dupont's contraction (3.2.1), the other by homotopy transfer from $\Omega(X; V)$ along the limit of the contractions $\Omega(X; V / F^p V) \xleftarrow{\cong} C(X; V / F^p V)$ (recall - Remark 2.1.3 - that the category \mathbf{Contr} is complete): in fact by applying Lemma 2.2.5 to the natural $\Omega(X) \otimes V \rightarrow \Omega(X) \widehat{\otimes} V$ we see that both ways induce the same $O_\infty[1]$ structure on $C(X; V)$, moreover the second construction shows that $(C(X; V), F^\bullet C(X; V))$ with the transferred $O_\infty[1]$ structure is a complete $O_\infty[1]$ algebra and the induced $C(X; V) \rightarrow \Omega(X; V)$ is a continuous $O_\infty[1]$ morphism. .

Definition 3.2.5. The functor $C(-; -) : \mathbf{SSet}^{op} \times \widehat{\mathbf{O}}_\infty[1] \rightarrow \widehat{\mathbf{O}}_\infty[1]$ is defined as in the previous discussion by sending a simplicial set X and a complete $O_\infty[1]$ algebra $(V, F^\bullet V, q_1, \dots, q_n, \dots)$ to the complete $O_\infty[1]$ algebra structure on $(C(X; V), F^\bullet C(X; V) = C(X; F^\bullet V))$ induced from $\Omega(X; V)$ (equivalently, from $\Omega(X) \otimes V$) via homotopy transfer along Dupont's contraction.

Lemma 3.2.3 remains true in the complete case.

Lemma 3.2.6. For each $X \in \mathbf{SSet}$, the functor $C(X; -) : \widehat{\mathbf{O}}_\infty[1] \rightarrow \widehat{\mathbf{O}}_\infty[1]$ commutes with small limits.

Proof. The family of functors $F^p - : \widehat{\mathbf{O}}_\infty[1] \rightarrow \mathbf{O}_\infty[1] : (V, F^\bullet V) \rightarrow F^p V$, $p \geq 0$, commutes with small limits and has the property that a morphism $f \in \widehat{\mathbf{O}}_\infty[1]$ is an isomorphism if and only if such is $F^p(f)$, $\forall p \geq 0$. Given a functor $\mathcal{C} \rightarrow \widehat{\mathbf{O}}_\infty[1] : c \rightarrow V_c$ from a small category, since $F^p C(X; -) = C(X; F^p -) : \widehat{\mathbf{O}}_\infty[1] \rightarrow \mathbf{O}_\infty[1]$, by looking at

$$\begin{array}{ccc} C(X; \lim_{c \in \mathcal{C}} V_c) & \longrightarrow & \lim_{c \in \mathcal{C}} C(X; V_c) \\ \downarrow F^p - & & \downarrow F^p - \\ C(X; \lim_{c \in \mathcal{C}} F^p V_c) & \longrightarrow & \lim_{c \in \mathcal{C}} C(X; F^p V_c) \end{array}$$

the lemma follows from Lemma 3.2.3. \square

The following simple lemma says that the $O_\infty[1]$ structure on $C(X; V)$ satisfies a certain locality condition over X .

Lemma 3.2.7. *If $\overline{X} \subset X$ is a sub-simplicial set then the space $C(X, \overline{X}; V) \subset C(X; V)$ of cochains vanishing on \overline{X} is an $O_\infty[1]$ ideal.*

Proof. This follows immediately from Lemma 2.2.6. \square

We close this section by remarking an open problem.

Remark 3.2.8. The functor $\Omega(-; -) : \mathbf{SSet}^{op} \times \mathbf{O}_\infty[1] \rightarrow \mathbf{O}_\infty[1]$ (as well as its complete version) admits a natural enhancement to a functor $\Omega(-; -) : \mathbf{SSet}^{op} \times \mathcal{O}_\infty[1] \rightarrow \mathcal{O}_\infty[1]$, as extension of scalars by a commutative dg algebra is a functor $\mathcal{O}_\infty[1] \rightarrow \mathcal{O}_\infty[1]$. It would be highly desirable to have a similar enhancement of the functor $C(-; -)$: unfortunately, this doesn't seem an easy matter, cf. the related problem of constructing an A_∞ algebra structure on the tensor product of two A_∞ algebras [90, 78, 71]. Let us notice, given an $O_\infty[1]$ morphism $H : V \rightarrow W$, that this can't be given by the recipe $C(X; H) : C(X; V) \xrightarrow{F} \Omega(X; V) \xrightarrow{\Omega(X; H)} \Omega(X; W) \xrightarrow{G} C(X; W)$, where F is induced by homotopy transfer and G is a left inverse as in Remark 2.2.4, since this fails to be functorial if the morphisms are not strict (consider the case of abelian $O_\infty[1]$ algebras and $O_\infty[1]$ morphisms between them). In other words, Lemma 2.2.3 and the considerations in Remark 2.2.4 do not extend if we work with general $O_\infty[1]$ morphisms².

3.3 Mapping cocones

Let (V, Q_V) and (W, Q_W) be $O_\infty[1]$ algebras, and let $(C(\Delta_1; W), Q_{C(\Delta_1; W)})$ be the $O_\infty[1]$ algebra of non-degenerate cochains on the 1-simplex with coefficients in W , as in the previous section. We denote by $j^* : C(\Delta_1; W) \rightarrow C(\partial\Delta_1; W)$ the pullback by the inclusion $j : \partial\Delta_1 \rightarrow \Delta_1$, notice the isomorphism $C(\partial\Delta_1; W) \cong W \times W$ in $\mathbf{O}_\infty[1]$.

²the point here is that $\Omega(X; H)FG = FG\Omega(X; H)$ if $H = h$ is strict, where FG in the left and right hand side denotes the respective $FG : \Omega(X; W) \rightarrow \Omega(X; W)$ and $FG : \Omega(X; V) \rightarrow \Omega(X; V)$, but not in general, cf. with the situation in Remark 2.1.3.

Theorem 3.3.1. *Let $F = (f_1, \dots, f_n, \dots) : V \rightarrow W$ and $G = (g_1, \dots, g_n, \dots) : V \rightarrow W$ be a pair of $\mathcal{O}_\infty[1]$ morphisms: there is an $\mathcal{O}_\infty[1]$ algebra $E^h(F, G)$ and a cartesian diagram*

$$\begin{array}{ccc} E^h(F, G) & \longrightarrow & C(\Delta_1; W) \\ \downarrow & & \downarrow j^* \\ V & \xrightarrow{F \times G} & C(\partial\Delta_1; W) \end{array}$$

in $\mathcal{O}_\infty[1]$.

Remark 3.3.2. This result is not trivial since the category $\mathcal{O}_\infty[1]$ is not complete, and in particular there are pair of morphisms which do not admit equalizers.

Proof. We start by considering the $A_\infty[1]$ case. As a graded space $C(\Delta_1; W) = W \times W \times W[-1]$, the underlying space of $E^h(F, G)$ will be $E^h(F, G) = V \times W[-1]$. In order to simplify the notations, and hoping this will not cause too much confusion, in the course of the proof we denote by the same symbol $p_{W[-1]}$ the projections $E^h(F, G) \rightarrow W[-1]$ and $C(\Delta_1; W) \rightarrow W[-1]$, as well as $\bar{T}(E^h(F, G)) \xrightarrow{p} E^h(F, G) \xrightarrow{p_{W[-1]}} W[-1]$ and $\bar{T}(C(\Delta_1; W)) \xrightarrow{p} C(\Delta_1; W) \xrightarrow{p_{W[-1]}} W[-1]$, leaving to the context to make it clear to which one we are referring to. Similarly, we denote by the same symbol p_V the projections $E^h(F, G) \rightarrow V$, $\bar{T}(E^h(F, G)) \xrightarrow{p} E^h(F, G) \xrightarrow{p_V} V$ and $\bar{T}(V) \xrightarrow{p} V$.

Our strategy will be roughly to guess the universal pair of morphisms $P : E^h(F, G) \rightarrow V$ and $I : E^h(F, G) \rightarrow C(\Delta_1; W)$, and then show that these determine the $A_\infty[1]$ structure on $E^h(F, G)$ and everything works properly. We take P to be the linear morphism of coalgebras $P = (p_V, 0 \dots, 0 \dots)$ and we define $I = (i_1, \dots, i_n, \dots)$ by

$$i_1(v, sw) = (f_1(v), g_1(v), sw),$$

$$i_n((v_1, sw_1) \otimes \dots \otimes (v_n, sw_n)) = (f_n(v_1 \otimes \dots \otimes v_n), g_n(v_1 \otimes \dots \otimes v_n), 0) \quad \text{for } n \geq 2.$$

The identity $(F \times G)P = j^*I$ follows immediately.

We claim that there is a unique coderivation $Q_{E^h(F, G)} \in \overline{\text{Hoch}}(E^h(F, G))$ such that the identities $PQ_{E^h(F, G)} = Q_V P$ and $IQ_{E^h(F, G)} = Q_{C(\Delta_1; W)} I$ are satisfied. In fact, the corestriction $pQ_{E^h(F, G)}$ decomposes as $pQ_{E^h(F, G)} = p_V Q_{E^h(F, G)} \times p_{W[-1]} Q_{E^h(F, G)} : \bar{T}(E^h(F, G)) \rightarrow E^h(F, G) = V \times W[-1]$. It is clear that (with the abuse of notation explained in the beginning) $\bar{T}(E^h(F, G)) \xrightarrow{p_V} V$ and $\bar{T}(E^h(F, G)) \xrightarrow{p} \bar{T}(V) \xrightarrow{p_V} V$ coincide - that is, $p_V = p_V P : \bar{T}(E^h(F, G)) \rightarrow V$ - thus the identity $PQ_{E^h(F, G)} = Q_V P$ implies that $p_V Q_{E^h(F, G)} = p_V P Q_{E^h(F, G)} = p_V Q_V P$: in other words the component $p_V Q_{E^h(F, G)}$ of the corestriction $pQ_{E^h(F, G)}$ has to be

$$p_V Q_{E^h(F, G)} : \bar{T}(E^h(F, G)) \xrightarrow{p} \bar{T}(V) \xrightarrow{Q_V} \bar{T}(V) \xrightarrow{p_V} V.$$

Similarly, the identity $IQ_{E^h(F, G)} = Q_{C(\Delta_1; W)} I$ and the observation that

$$p_{W[-1]} = p_{W[-1]} I : \bar{T}(E^h(F, G)) \rightarrow W[-1]$$

(by construction of I) imply that $p_{W[-1]} Q_{E^h(F, G)} = p_{W[-1]} I Q_{E^h(F, G)} = p_{W[-1]} Q_{C(\Delta_1; W)} I$: in other words $p_{W[-1]} Q_{E^h(F, G)}$ has to be

$$p_{W[-1]} Q_{E^h(F, G)} : \bar{T}(E^h(F, G)) \xrightarrow{I} \bar{T}(C(\Delta_1; W)) \xrightarrow{Q_{C(\Delta_1; W)}} \bar{T}(C(\Delta_1; W)) \xrightarrow{p_{W[-1]}} W[-1].$$

This shows uniqueness. Conversely, if $pQ_{E^h(F,G)} = p_V Q_{E^h(F,G)} \times p_{W[-1]} Q_{E^h(F,G)}$ is defined as above by construction it is immediate that $PQ_{E^h(F,G)} = Q_V P$ and moreover by what we have already seen $p_{W[-1]} I Q_{E^h(F,G)} = p_{W[-1]} Q_{E^h(F,G)} = p_{W[-1]} Q_{C(\Delta_1; W)} I$ (the first identity since $p_{W[-1]} = p_{W[-1]} I$, the second by definition), thus to prove to prove $I Q_{E^h(F,G)} = Q_{C(\Delta_1; W)} I$ it only remains to show that $p_{W \times W} I Q_{E^h(F,G)} = p_{W \times W} Q_{C(\Delta_1; W)} I$: notice that $p_{W \times W} = j^*$ is pullback by the inclusion $j : \partial \Delta_1 \rightarrow \Delta_1$, so we see

$$\begin{aligned} j^* I Q_{E^h(F,G)} &= (F \times G) P Q_{E^h(F,G)} = (F \times G) Q_V P = \\ &= (Q_W \times Q_W)(F \times G) P = (Q_W \times Q_W) j^* I = j^* Q_{C(\Delta_1; W)} I. \end{aligned}$$

Next we claim that $Q_{E^h(F,G)}$ defines an $A_\infty[1]$ algebra structure on $E^h(F,G)$, but this is easy since $p_V (Q_{E^h(F,G)})^2 = (p_V Q_{E^h(F,G)}) Q_{E^h(F,G)} = (p_V Q_V P) Q_{E^h(F,G)} = p_V (Q_V)^2 P = 0$ and similarly

$$\begin{aligned} p_{W[-1]} (Q_{E^h(F,G)})^2 &= (p_{W[-1]} Q_{E^h(F,G)}) Q_{E^h(F,G)} = \\ &= (p_{W[-1]} Q_{C(\Delta_1; W)} I) Q_{E^h(F,G)} = p_{W[-1]} (Q_{C(\Delta_1; W)})^2 I = 0. \end{aligned}$$

Given an $A_\infty[1]$ algebra (X, Q_X) together with a pair of $A_\infty[1]$ morphisms $J : X \rightarrow V$ and $K : X \rightarrow C(\Delta_1; W)$ such that $(F \times G)J = j^* K : X \rightarrow C(\partial \Delta_1; W)$ we have to show that there is a unique $A_\infty[1]$ morphism $H : X \rightarrow E^h(F,G)$ such that $J = HP$ and $K = HI$. In fact uniqueness is proved as before and $pH = p_V H \times p_{W[-1]} H : \bar{T}(X) \rightarrow E^h(F,G) = V \times W[-1]$ has to be given by $p_V H : \bar{T}(X) \xrightarrow{J} \bar{T}(V) \xrightarrow{p_V} V$ and $p_{W[-1]} H : \bar{T}(X) \xrightarrow{K} \bar{T}(C(\Delta_1; W)) \xrightarrow{p_{W[-1]}} W[-1]$. It is straightforward that $PH = J$ so we have to check $IH = K$: this follows as before from $p_{W[-1]} IH = p_{W[-1]} H := p_{W[-1]} K$ and from $j^* IH = (F \times G)PH = (F \times G)J = j^* K$. It only remains to show that H is an $A_\infty[1]$ morphism - that is, $Q_{E^h(F,G)} H = H Q_X$: on the one hand we see that $p_V Q_{E^h(F,G)} H = p_V Q_V P H = p_V Q_V J = p_V J Q_X = p_V H Q_X$, and on the other

$$p_{W[-1]} Q_{E^h(F,G)} H = p_{W[-1]} Q_{C(\Delta_1; W)} I H = p_{W[-1]} Q_{C(\Delta_1; W)} K = p_{W[-1]} K Q_X = p_{W[-1]} H Q_X.$$

In the $C_\infty[1]$ case it is clear that $P : \bar{T}(E^h(F,G)) \rightarrow \bar{T}(V)$ and $I : \bar{T}(E^h(F,G)) \rightarrow \bar{T}(C(\Delta_1; W))$ constructed as before satisfy the assumption of Lemma 1.2.9, thus are morphisms of the (shuffle product, deconcatenation coproduct) bialgebra structures.

For $p, q \geq 1$ we see that

$$\begin{aligned} p_V Q_{E^h(F,G)} (((v_1, sw_1) \otimes \cdots \otimes (v_p, sw_p)) \otimes ((v_{p+1}, sw_{p+1}) \otimes \cdots \otimes (v_{p+q}, sw_{p+q}))) &= \\ = p_V Q_V P (((v_1, sw_1) \otimes \cdots \otimes (v_p, sw_p)) \otimes ((v_{p+1}, sw_{p+1}) \otimes \cdots \otimes (v_{p+q}, sw_{p+q}))) &= \\ = p_V Q_V ((v_1 \otimes \cdots \otimes v_p) \otimes (v_{p+1} \otimes \cdots \otimes v_{p+q})) &= 0 \\ p_{W[-1]} Q_{E^h(F,G)} (((v_1, sw_1) \otimes \cdots \otimes (v_p, sw_p)) \otimes ((v_{p+1}, sw_{p+1}) \otimes \cdots \otimes (v_{p+q}, sw_{p+q}))) &= \\ = p_{W[-1]} Q_{C(\Delta_1; W)} I (((v_1, sw_1) \otimes \cdots \otimes (v_p, sw_p)) \otimes ((v_{p+1}, sw_{p+1}) \otimes \cdots \otimes (v_{p+q}, sw_{p+q}))) &= \\ = p_{W[-1]} Q_{C(\Delta_1; W)} (I((v_1, sw_1) \otimes \cdots \otimes (v_p, sw_p)) \otimes I((v_{p+1}, sw_{p+1}) \otimes \cdots \otimes (v_{p+q}, sw_{p+q}))) &= 0 \end{aligned}$$

This says that the corestriction $pQ_{E^h(F,G)}$ vanishes on the image of the shuffle product and thus by Lemma 1.2.9 $Q_{E^h(F,G)} \in \overline{\text{Harr}}(E^h(F,G))$. A similar argument - this time using the second part of Lemma 1.2.9 - shows that $H : X \rightarrow E^h(F,G)$ is a $C_\infty[1]$ morphism if X is a $C_\infty[1]$ algebra and J, K are $C_\infty[1]$ morphisms.

Finally, the $L_\infty[1]$ case is treated exactly as the $A_\infty[1]$ one, simply replacing $\bar{T}(-)$ by $\bar{S}(-)$, $\overline{\text{Hoch}}(-)$ by $\overline{\text{CE}}(-)$ and \otimes by \odot . \square

Definition 3.3.3. We call the $O_\infty[1]$ algebra $E^h(F, G)$ as in the previous theorem the homotopy equalizer of the $O_\infty[1]$ morphisms F and G . In particular we call the homotopy equalizer of 0 and F the mapping cocone of F and we denote it by $\text{coC}(F) := E^h(0, F)$. Given $O_\infty[1]$ morphisms $F : V \rightarrow W$ and $G : V' \rightarrow W$ we call the homotopy equalizer of the morphisms $V \times V' \xrightarrow{p_V} V \xrightarrow{F} W$ and $V \times V' \xrightarrow{p_{V'}} V' \xrightarrow{G} W$ the homotopy fiber product of V and V' along F and G and we denote it by $V \times_W^h V'$. In particular we call the homotopy fiber product $W \times_W^h V$ along $\text{id}_W : W \rightarrow W$ and $F : V \rightarrow W$ the mapping cocylinder of F and we denote it by $\text{Cyl}(F)$.

Definition 3.3.4. We say that a sequence $X \rightarrow Y \rightarrow Z$ of $O_\infty[1]$ algebras and $O_\infty[1]$ morphisms is a homotopy fiber sequence if it is weakly equivalent to one of the form $\text{coC}(F) \xrightarrow{p_V} V \xrightarrow{F} W$, and in this case we say that X is a homotopy fiber of $Y \rightarrow Z$.

The following theorem is due to M. Manetti when both F and G are strict morphisms of dg Lie algebras, we merely observe that the same proof also shows more in general that

Theorem 3.3.5. *In the hypotheses of Theorem 3.3.1, if $H(F) - H(G) : H(V) \rightarrow H(W)$ is injective then the homotopy equalizer $E^h(F, G)$ is homotopy abelian.*

Proof. The $O_\infty[1]$ subalgebra $C(\Delta_1, \partial\Delta_1; W) \subset C(\Delta_1; W)$ is abelian (not an abelian ideal!) as follows easily from Corollary 2.2.6: in fact it is (strictly) isomorphic to $(W[-1], -q_1, 0, \dots, 0, \dots)$, where we denote by $(W, q_1, \dots, q_n, \dots)$ the $O_\infty[1]$ algebra structure on W . The commutative diagram

$$\begin{array}{ccc} C(\Delta_1, \partial\Delta_1; W) & \hookrightarrow & C(\Delta_1; W) \\ \downarrow 0 & & \downarrow j^* \\ V & \xrightarrow{F \times G} & C(\partial\Delta_1; W) \end{array}$$

together with the proof of the previous theorem shows that

$$i : C(\Delta_1, \partial\Delta_1; W) = W[-1] \rightarrow E^h(F, G) = V \times W[-1] : sw \rightarrow (0, sw)$$

is a strict $O_\infty[1]$ morphism. The exact sequence of complexes $0 \rightarrow W[-1] \xrightarrow{i} E^h(F, G) \xrightarrow{p_V} V \rightarrow 0$ induces a long exact sequence in cohomology

$$\dots \rightarrow H^k(V) \xrightarrow{H(f_1 - g_1)} H^k(W) \xrightarrow{H(i)} H^{k+1}(E^h(F, G)) \xrightarrow{H(p_V)} H^{k+1}(V) \rightarrow \dots,$$

in particular the hypothesis $H(F) - H(G) = H(f_1 - g_1)$ injective implies that $H(j)$ is surjective, so the thesis follows from Lemma 2.2.13. \square

In the final part of the chapter, at last, we want to give some explicit formulas in some particular hypotheses. This requires the explicit computation of the $O_\infty[1]$ structure on $C(\Delta_1; W)$.

We denote by $\{B_i\}_{i \geq 0}$ the sequence of Bernoulli numbers, i.e., the sequence of (rational) numbers defined by the power series expansion $\frac{t}{e^t - 1} = \sum_{i \geq 0} \frac{B_i}{i!} t^i = 1 - \frac{1}{2}t + \frac{1}{24}t^2 - \frac{1}{480}t^4 + \dots$. For $n \geq 2$ the n -th Bernoulli number B_n can be calculated by the recursion

$$\sum_{k=1}^n \binom{n}{k} B_{n-k} = 0, \quad \forall n \geq 2, \quad (3.3.1)$$

it is well known, and in any case easy to prove, that $B_{2n+1} = 0$ for all $2n+1 \geq 3$. The Bernoulli polynomials $B_n(t) \in \mathbb{Q}[t]$, $n \geq 0$, are defined by $B_n(t) = \sum_{k=0}^n \binom{n}{k} B_{n-k} t^k$, in particular $B_n(0) = B_n$ and for $n \geq 2$ also $B_n(1) = B_n$: since $B_1(1) = \frac{1}{2} = -B_1$ and $B_{2n+1} = 0$ for $n \geq 1$ in all cases we have $B_n(1) = (-1)^n B_n$.

Proposition 3.3.6. *Let (A, d, \cdot) be a dg associative (resp.: commutative) algebra. The induced $A_\infty[1]$ (resp.: $C_\infty[1]$) structure $Q = (q_1, \dots, q_k, \dots)$ on $C(\Delta_1; A[1]) = s^{-1}A \times s^{-1}A \times A$ is explicitly given by*

$$q_1(s^{-1}a, s^{-1}b, c) = (-s^{-1}da, -s^{-1}db, -b + a + dc)$$

$$\begin{aligned} q_2((s^{-1}a_1, s^{-1}b_1, c_1) \otimes (s^{-1}a_2, s^{-1}b_2, c_2)) &= \\ &= \left((-1)^{|a_1|} s^{-1}(a_1 a_2), (-1)^{|b_1|} s^{-1}(b_1 b_2), \frac{1}{2} \left((a_1 + b_1)c_2 + (-1)^{1+|c_1|} c_1(a_2 + b_2) \right) \right) \end{aligned}$$

$$\begin{aligned} q_k((s^{-1}a_1, s^{-1}b_1, c_1) \otimes \dots \otimes (s^{-1}a_k, s^{-1}b_k, c_k)) &= \\ &= \left(0, 0, \sum_{i=1}^k (-1)^{k-i+\sum_{j<i} |c_j|} \frac{B_{k-1}}{(i-1)!(k-i)!} c_1 \dots (a_i - b_i) \dots c_k \right) \end{aligned}$$

for $k \geq 3$. Let $(L, d, [\cdot, \cdot])$ be a dg Lie algebra: the induced $L_\infty[1]$ structure $Q = (q_1, \dots, q_k, \dots)$ on $C(\Delta_1; L[1])$ is given by

$$q_1(s^{-1}l, s^{-1}m, n) = (-s^{-1}dl, -s^{-1}dm, -m + l + dn)$$

$$\begin{aligned} q_2((s^{-1}l_1, s^{-1}m_1, n_1) \odot (s^{-1}l_2, s^{-1}m_2, n_2)) &= \\ &= \left((-1)^{|l_1|} s^{-1}[l_1, l_2], (-1)^{|m_1|} s^{-1}[m_1, m_2], \frac{1}{2} \left([l_1 + m_1, n_2] + (-1)^{|n_1|+1} [n_1, l_2 + m_2] \right) \right) \end{aligned}$$

$$\begin{aligned} q_k((s^{-1}l_1, s^{-1}m_1, n_1) \odot \dots \odot (s^{-1}l_k, s^{-1}m_k, n_k)) &= \\ &= \left(0, 0, \frac{B_{k-1}}{(k-1)!} \sum_{\sigma \in S_k} \varepsilon(\sigma) [\dots [l_{\sigma(1)} - m_{\sigma(1)}, n_{\sigma(2)}] \dots, n_{\sigma(k)}] \right) \end{aligned}$$

for $k \geq 3$.

Proof. The computation is due to Fiorenza and Manetti [31], for the $C_\infty[1]$ case cf. also [22]. we identify $\Omega_1 = S(t_0, t_1, dt_0, dt_1)/(t_0 + t_1 - 1, dt_0 + dt_1)$ with $S(t_1, dt_1)$ - by eliminating t_0 - and we denote it (as in [31]) by $\mathbb{K}[t, dt]$. We denote the space $\mathbb{K}[t, dt] \otimes A[1] = (\mathbb{K}[t, dt] \otimes A)[1]$, with the $A_\infty[1]$ structure induced by extension of scalars by $\mathbb{K}[t, dt]$ (recall - cf. Lemma 1.1.6 - that this commutes with décalage), by $(A[t, dt][1], m_1, m_2, 0, \dots, 0, \dots)$. Dupont's contraction is given by: $\int : A[t, dt][1] \rightarrow C(\Delta_1; A[1]) : s^{-1}\omega(t) \rightarrow (s^{-1}\omega(0), s^{-1}\omega(1), \int_0^1 \omega(t))$ where $\int_0^1 : A[t, dt][1] \rightarrow A$ is formal integration in dt , namely $\int_0^1 : s^{-1}(t^n \cdot a) \rightarrow 0$ and $\int_0^1 : s^{-1}(t^n dt \cdot a) \rightarrow \frac{1}{n+1}a$ for all $n \geq 1$; the homotopy retraction K is given by $K : s^{-1}(t^n \cdot a) \rightarrow 0$ and

$$K : s^{-1}(t^n dt \cdot a) \rightarrow s^{-1} \left(\left(t \int_0^1 s^n ds - \int_0^t s^n ds \right) \cdot a \right) = s^{-1} \left(\frac{t - t^{n+1}}{n+1} \cdot a \right);$$

finally, we denote by $f_1 : C(\Delta_1; A[1]) \rightarrow A[t, dt][1] : (s^{-1}a, s^{-1}b, c) \rightarrow s^{-1}((1-t) \cdot a + t \cdot b + dt \cdot c)$ the remaining morphism, in stead of ι as in the previous sections. We consider the $A_\infty[1]$ case

first: in the course of the proof we will also prove inductively that the $A_\infty[1]$ morphism induced by homotopy transfer $F = (f_1, \dots, f_n, \dots) : C(\Delta_1; A[1]) \rightarrow A[t, dt][1]$ is explicitly - where $k \geq 2$ -

$$\begin{aligned} f_k((s^{-1}a_1, s^{-1}b_1, c_1) \otimes \dots \otimes (s^{-1}a_k, s^{-1}b_k, c_k)) &= \\ &= s^{-1} \left(\frac{B_k(t) - B_k}{k!} \cdot \sum_{i=1}^k \binom{k-1}{i-1} (-1)^{\sum_{j<i} (|c_j|+1)} c_1 \dots (b_i - a_i) \dots c_k \right). \end{aligned}$$

For $k = 2$

$$\begin{aligned} m_2(f_1(s^{-1}a_1, s^{-1}b_1, c_1) \otimes f_1(s^{-1}a_2, s^{-1}b_2, c_2)) &= \\ &= m_2(s^{-1}((1-t) \cdot a_1 + t \cdot b_1 + dt \cdot c_1) \otimes s^{-1}((1-t) \cdot a_2 + t \cdot b_2 + dt \cdot c_2)) = \\ &= (-1)^{|a_1|} s^{-1} \left((1-t)^2 \cdot a_1 a_2 + t(1-t) \cdot (a_1 b_2 + b_1 a_2) + t^2 \cdot b_1 b_2 + \right. \\ &\quad \left. + dt \left((-1)^{|a_1|} (1-t) \cdot a_1 c_2 + (-1)^{|a_1|} t \cdot b_1 c_2 + (1-t) \cdot c_1 a_2 + t \cdot c_1 b_2 \right) \right) \end{aligned}$$

Moreover,

$$\begin{aligned} \int_0^1 (1-s) ds &= \int_0^1 s ds = \frac{1}{2} \\ t \int_0^1 (1-s) ds - \int_0^t (1-s) ds &= - \left(t \int_0^1 s ds - \int_0^t s ds \right) = \frac{t^2 - t}{2} = \frac{B_2(t) - B_2}{2!}, \end{aligned}$$

thus $q_2 = \int m_2 f_1^{\otimes 2}$ and $f_2 = K m_2 f_1^{\otimes 2}$ are given by (notice that $K s^{-1} = -s^{-1} K$)

$$\begin{aligned} q_2((s^{-1}a_1, s^{-1}b_1, c_1) \otimes (s^{-1}a_2, s^{-1}b_2, c_2)) &= \\ &= \left((-1)^{|a_1|} s^{-1}(a_1 a_2), (-1)^{|b_1|} s^{-1}(b_1 b_2), \frac{1}{2} \left((a_1 + b_1) c_2 + (-1)^{|c_1|+1} c_1 (a_2 + b_2) \right) \right) \end{aligned}$$

$$\begin{aligned} f_2((s^{-1}a_1, s^{-1}b_1, c_1) \otimes (s^{-1}a_2, s^{-1}b_2, c_2)) &= \\ &= s^{-1} \left(\frac{B_2(t) - B_2}{2!} \cdot \left((b_1 - a_1) c_2 + (-1)^{|c_1|+1} c_1 (b_2 - a_2) \right) \right) \end{aligned}$$

For $k > 2$

$$\begin{aligned} m_2 F_k^2((s^{-1}a_1, s^{-1}b_1, c_1) \otimes \dots \otimes (s^{-1}a_k, s^{-1}b_k, c_k)) &= \\ &= m_2(f_1(s^{-1}a_1, s^{-1}b_1, c_1) \otimes f_{k-1}((s^{-1}a_2, s^{-1}b_2, c_2) \otimes \dots \otimes (s^{-1}a_k, s^{-1}b_k, c_k))) + \\ &\quad + m_2(f_{k-1}((s^{-1}a_1, s^{-1}b_1, c_1) \otimes \dots \otimes (s^{-1}a_{k-1}, s^{-1}b_{k-1}, c_{k-1})) \otimes f_1(s^{-1}a_k, s^{-1}b_k, c_k)) + \\ &\quad + \text{terms in Ker } K \cap \text{Ker } f, \end{aligned}$$

since the inductive hypothesis implies $m_2(f_j(\dots) \otimes f_{k-j}(\dots)) \in A[t][1]$ for $1 < j < k-1$ (also notice that $B_{k-1}(0) - B_{k-1} = B_{k-1}(1) - B_{k-1} = 0$ for $k \geq 3$). Modulo terms in $\text{Ker } K \cap \text{Ker } f$ this is

$$s^{-1} \left(\frac{B_{k-1}(t) - B_{k-1}}{(k-1)!} dt \cdot \sum_{i=1}^k \left(\binom{k-2}{i-1} + \binom{k-2}{i-1} \right) (-1)^{\sum_{j<i} (|c_j|+1)} c_1 \dots (b_i - a_i) \dots c_k \right) =$$

$$= s^{-1} \left(\frac{B_{k-1}(t) - B_{k-1}}{(k-1)!} dt \cdot \sum_{i=1}^k \binom{k-1}{i-1} (-1)^{\sum_{j<i}(|c_j|+1)} c_1 \cdots (b_i - a_i) \cdots c_k \right),$$

always by the inductive hypothesis. The formulas claimed for $f_k = Km_2F_k^2$ and $q_k = \int m_2F_k^2$ follow if we show that

$$\int_0^1 \frac{B_{k-1}(t) - B_{k-1}}{(k-1)!} dt = -\frac{B_{k-1}}{(k-1)!} \quad \text{and} \quad K \left(\frac{B_{k-1}(t) - B_{k-1}}{(k-1)!} dt \right) = -\frac{B_k(t) - B_k}{k!}.$$

Both identities follow easily from the following ($k \geq 3$)

$$\begin{aligned} \int_0^t \frac{B_{k-1}(s) - B_{k-1}}{(k-1)!} ds &= \sum_{i=1}^{k-1} \int_0^t \frac{B_{k-i-1}}{i!(k-i-1)!} s^i ds = \\ &= \sum_{i=2}^k \frac{B_{k-i}}{i!(k-i)!} s^i = \frac{B_k(t) - kB_{k-1}t - B_k}{k!}. \end{aligned}$$

Finally, notice that the above would give the sign $(-1)^{\sum_{j<i}(|c_j|+1)} = (-1)^{i-1+\sum_{j<i}|c_j|}$ in the formula for q_k , $k \geq 3$: to get the signs right we observe that either $k-1$ is even or $B_{k-1} = 0$ (for $k \geq 3$), in any case we always have $(-1)^{i-1}B_{k-1} = (-1)^{k-i}B_{k-1}$.

This concludes the proof in the $A_\infty[1]$ case, thus also in the $C_\infty[1]$ case. Finally, the $L_\infty[1]$ case can be treated by a completely similar computation, details are left to the reader, in particular the $L_\infty[1]$ morphism $F = (f_1, \dots, f_n, \dots) : C(\Delta_1; L[1]) \rightarrow L[t, dt][1]$ is given - for $k \geq 2$ - by

$$\begin{aligned} f_k((s^{-1}l_1, s^{-1}m_1, n_1) \odot \cdots \odot (s^{-1}l_k, s^{-1}m_k, n_k)) = \\ = s^{-1} \left(\frac{B_k(t) - B_k}{k!} \sum_{\sigma \in S_k} \varepsilon(\sigma) [\cdots [m_{\sigma(1)} - l_{\sigma(1)}, n_{\sigma(2)}] \cdots, n_{\sigma(k)}] \right). \end{aligned}$$

We refer to [31] for a proof using tree summation formulas. \square

From this we also obtain explicit formulas for the homotopy equalizer $E^h(F, G)$ when the target $A_\infty[1]$ (resp.: $C_\infty[1]$, $L_\infty[1]$) algebra is associated to a dg associative (resp.: commutative, Lie) algebra via décalage. As an example we describe explicitly the $L_\infty[1]$ structure on the mapping cocone $\text{coC}(F) = V \times M$ of an $L_\infty[1]$ morphism $F = (f_1, \dots, f_n, \dots) : V \rightarrow s^{-1}M$ from an $L_\infty[1]$ algebra $(V, q_1, \dots, q_n, \dots)$ to a dg Lie algebra $(M, d, [\cdot, \cdot])$ the last one seen as an $L_\infty[1]$ algebra via décalage. When F is a strict morphism of dg Lie algebras we recover the $L_\infty[1]$ structure from Fiorenza and Manetti [31].

Corollary 3.3.7. *In the above set up the $L_\infty[1]$ structure $R = (r_1, \dots, r_n, \dots)$ on the mapping cocone $\text{coC}(F)$ is given explicitly by $r_1(v, m) = (q_1(v), dm - sf_1(v))$ (recall that as a graded space $\text{coC}(F) = V \times M$) and for $n \geq 2$ by*

$$\begin{aligned} r_n((v_1, m_1) \odot \cdots \odot (v_n, m_n)) = \\ = \left(q_n(v_1 \odot \cdots \odot v_n), - \sum_{k=1}^n \frac{B_{n-k}}{k!(n-k)!} \sum_{\sigma \in S_n} \varepsilon(\sigma) [\cdots [sf_k(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k)}), m_{\sigma(k+1)}] \cdots, m_{\sigma(n)}] \right) \end{aligned}$$

where we denote by sf_k the composition $V^{\odot k} \xrightarrow{f_k} s^{-1}M \xrightarrow{s} M$.

Proof. As in the proof of Theorem 3.3.1 we write

$$pR = p_V R \times p_M R : \overline{S}(\text{coC}(F)) = \overline{S}(V \times M) \rightarrow V \times M.$$

According to the same proof we have to show that $p_V R$ is the composition - with R as in the claim of the proposition and P and I as in the proof of Theorem 3.3.1 (and of course with $(0, F)$ in stead of (F, G)) -

$$\overline{S}(\text{coC}(F)) \xrightarrow{P} \overline{S}(V) \xrightarrow{pQ} V,$$

which is clear, and that $p_M R$ is the composition

$$\overline{S}(\text{coC}(F)) \xrightarrow{I} \overline{S}(C(\Delta_1; M[1])) \xrightarrow{pQ_{C(\Delta_1; M[1])}} C(\Delta_1; M[1]) \xrightarrow{pM} M,$$

which follows readily from the explicit formulas for $pQ_{C(\Delta_1; M[1])}$ we gave in the previous proposition. \square

Chapter 4

Higher derived brackets

This section is devoted to the study of various explicit constructions of higher brackets and L_∞ algebra structures. In Section 4.1 we propose a non-abelian version of Voronov's construction of higher derived brackets [105, 106], as applications we recover some constructions by Bering [7] and Getzler [40] and some results by Chuang and Lazarev [23]. As another application, in Section 4.2 we propose a possible generalization to graded pre-Lie algebras of Koszul's construction of higher brackets on a graded commutative algebra [64]: we recover from a different perspective some results on Koszul brackets by M. Markl [79, 80]. There is also an inverse (in a precise sense) construction of brackets on a graded pre-Lie algebra which we call the Kapranov brackets: in fact, as a particular example of the latter we recover in Section 4.2.2 Kapranov's construction of an L_∞ algebra structure on the suspended Dolbeault complex of a Kähler manifold [56]. As a final application of the results of this chapter, in Section 4.2.1 we give an alternative proof of the result by Braun and Lazarev [12] that the L_∞ algebra associated to a commutative BV_∞ algebra satisfying a degeneration property analog to the one from [96] is homotopy abelian.

4.1 Nonabelian higher derived brackets

Notation 4.1.1. Let M be a graded Lie algebra and $L \subset M$ a graded Lie subalgebra. We denote by $\text{Der}(M)$ the graded Lie algebra of derivations of M and by $\text{Der}(M, L) \subset \text{Der}(M)$ the graded Lie subalgebra of derivations D such that $D(L) \subset L$.

Recall the following construction, due to Th. Voronov [105, 106]. Let M be a graded Lie algebra, together with complementary graded Lie subalgebras $L, A \subset M$ such that A is abelian, that is, we require that as a graded space M is the direct sum $M = L \oplus A$; we denote by $P : M \rightarrow A$ the projection with kernel L . In these hypotheses, we have the following easy lemma.

Lemma 4.1.2. *For all $D \in \text{Der}(M)$ the linear maps $A^{\otimes i} \rightarrow M : a_1 \otimes \cdots \otimes a_i \rightarrow [\cdots [Da_1, a_2] \cdots, a_i]$, $i \geq 2$, are graded symmetric.*

Proof. Easy induction, as in [106]: this follows since $A \subset M$ is supposed to be abelian. For $i = 2$ we have $[Da_1, a_2] - (-1)^{|a_1||a_2|}[Da_2, a_1] = D([a_1, a_2]) = 0$ by the Leibnitz identity. For $i \geq 3$ we have by induction that the given $A^{\otimes i} \rightarrow M$ is graded symmetric in the first $(i - 1)$ -arguments, so

we have to show graded symmetry in the last two: by Jacobi

$$\begin{aligned} [[\cdots [Da_1, a_2] \cdots, a_{i-1}], a_i] - (-1)^{|a_{i-1}||a_i|} [[\cdots [Da_1, a_2] \cdots, a_i], a_{i-1}] &= \\ &= [[\cdots [Da_1, a_2] \cdots], [a_{i-1}, a_i]] = 0. \end{aligned}$$

□

Following [105], in the previous setup we define the hierarchy of higher derived brackets on A associated to $m \in M$ by $\Phi(m)_i : A^{\odot i} \rightarrow A : a_1 \odot \cdots \odot a_i \rightarrow P[\cdots [m, a_1] \cdots, a_i]$ when $i \geq 1$, with moreover the 0-th bracket $\Phi(m)_0 : A^{\odot 0} \rightarrow A : 1 \rightarrow Pm$. Graded symmetry follows from the above lemma. The main result of [105] says that when $m \in L$, $|m| = 1$ and $[m, m] = 0$, the higher derived brackets $\Phi(m)_i$ define an $L_\infty[1]$ algebra structure on A (when $m \notin L$ but the remaining conditions are satisfied, they define what is known as a *curved* $L_\infty[1]$ algebra structure on A , cf. [20]).

The construction considered in [106] is similar but slightly different (the difference will become more apparent in the non-abelian case). With $M = L \oplus A$ and $P : M \rightarrow A$ as in the previous paragraph, this time we associate to all $D \in \text{Der}(M, L)$ the hierarchy of higher derived brackets $\Phi(D)_i(a_1 \odot \cdots \odot a_i) = P[\cdots [Da_1, a_2] \cdots, a_i]$, $i \geq 1$, on A , with no 0-th bracket: then again if $|D| = 1$, $[D, D] = 0$, this defines an $L_\infty[1]$ algebra structure on A . In fact in [106], Theorem 3, it is proved something more: that the correspondence

$$\Phi : \text{Der}(M, L) \rightarrow \overline{\text{CE}}(A) : D \rightarrow (\Phi(D)_1, \dots, \Phi(D)_i, \dots)$$

is a morphism of graded Lie algebras. Finally, in [106], Section 4, it is established a link with homotopy theory which is the key to our approach: there it is proved that $A[-1]$ with the induced L_∞ algebra structure is a homotopy fiber (as in Definition 3.3.4) of the inclusion of dg Lie algebras $i : (L, D, [\cdot, \cdot]) \rightarrow (M, D, [\cdot, \cdot])$. As we argued in the introduction, this leads the way to a possible non-abelian generalization of Voronov's construction (similar results were also obtained very recently by M. Bordemann with different methods).

We drop the assumption that the graded Lie subalgebra $A \subset M$ is abelian. To say it in other words, we suppose we are given a graded Lie algebra M together with a linear idempotent $P : M \rightarrow M$, that is, $P^2 = P$, such that both $L = \text{Ker } P$ and $A = \text{Im } P$ are graded Lie subalgebras of M . We denote by $P^\perp = \text{id}_M - P$ and notice that $D \in \text{Der}(M, L)$, that is $D(L) \subset L$, if and only if $PDP^\perp = 0$, that is,

$$PDP = PD. \quad (4.1.1)$$

The aim of this section is to generalize the previous constructions of higher derived brackets in this setup.

Definition 4.1.3. For $m \in M$ the higher derived brackets on A associated to m are the graded symmetric maps $\Phi(m)_i : A^{\odot i} \rightarrow A$, $i \geq 0$, defined by

$$\Phi(m)_i(a_1 \odot \cdots \odot a_i) = \sum_{\sigma \in S_i} \varepsilon(\sigma) \sum_{k=0}^i \frac{B_{i-k}}{k!(i-k)!} \overbrace{[\cdots [P([\cdots [m, a_{\sigma(1)}] \cdots, a_{\sigma(k)}]), a_{\sigma(k+1)}] \cdots, a_{\sigma(i)}}^{i-k}$$

for $i \geq 1$, and $\Phi(m)_0(1) = Pm$. We denote by $(\Phi(m)_0, \dots, \Phi(m)_i, \dots) = \Phi(m) \in \text{CE}(A)$ the corresponding coderivation.

For $D \in \text{Der}(M, L)$ the higher derived brackets on A associated to D are the graded symmetric maps $\Phi(D)_i : A^{\odot i} \rightarrow A$, $i \geq 1$, defined by

$$\Phi(D)_i(a_1 \odot \cdots \odot a_i) = \sum_{\sigma \in S_i} \varepsilon(\sigma) \sum_{k=1}^i \frac{B_{i-k}}{k!(i-k)!} \overbrace{[\cdots [P([\cdots [Da_{\sigma(1)}, a_{\sigma(2)}] \cdots, a_{\sigma(k)}]), a_{\sigma(k+1)}] \cdots, a_{\sigma(i)}}^{i-k}$$

We denote by $(\Phi(D)_1, \dots, \Phi(D)_i, \dots) = \Phi(D) \in \overline{\text{CE}}(A)$ the corresponding coderivation.

The first few brackets are given by

$$\begin{aligned}\Phi(D)_1(a) &= PDa, & \Phi(m)_1(a) &= P[m, a] - \frac{1}{2}[Pm, a], \\ \Phi(D)_2(a_1 \odot a_2) &= \sum_{\sigma \in S_2} \varepsilon(\sigma) \left(\frac{1}{2}P[Da_{\sigma(1)}, a_{\sigma(2)}] - \frac{1}{2}[PDa_{\sigma(1)}, a_{\sigma(2)}] \right),\end{aligned}$$

$$\Phi(m)_2(a_1 \odot a_2) = \sum_{\sigma \in S_2} \varepsilon(\sigma) \left(\frac{1}{2}P[[m, a_{\sigma(1)}], a_{\sigma(2)}] - \frac{1}{2}[P[m, a_{\sigma(1)}], a_{\sigma(2)}] + \frac{1}{12}[[Pm, a_{\sigma(1)}], a_{\sigma(2)}] \right).$$

Remark 4.1.4. If $l \in L$, then the inner derivation $[l, \cdot]$ is in $\text{Der}(M, L)$, and in this case we have $\Phi([l, \cdot]) = \Phi(l)$. However, if $[m, \cdot] \in \text{Der}(M, L)$, that is, m is in the normalizer of L in M , but $m \notin L$, then $\Phi([m, \cdot]) \neq \Phi(m)$, as the two differ by the terms involving Pm , so the two constructions shouldn't be confused in general.

Proposition 4.1.5. *If $A \subset M$ is an abelian Lie subalgebra, given $m \in M$, $D \in \text{Der}(M, L)$, $a_1, \dots, a_i \in A$, $i \geq 1$, the previous brackets become*

$$\Phi(m)_i(a_1 \odot \dots \odot a_i) = P[\dots [m, a_1] \dots, a_i], \quad \Phi(D)_i(a_1 \odot \dots \odot a_i) = P[\dots [Da_1, a_2] \dots, a_i],$$

that is, they are the same as the ones from [105, 106].

Proof. If $A \subset M$ is an abelian Lie subalgebra only the $k = i$ term in the summation for the brackets remains, thus the thesis follows immediately from Lemma 4.1.2. \square

Our main results are the following two theorems, whose proof will take most of the section.

Theorem 4.1.6. *In the set up of Definition 4.1.3, for every $D, D_k \in \text{Der}(M, L)$, $m, m_k \in M$, $k = 1, 2$, the following identities hold:*

$$[\Phi(m_1), \Phi(m_2)] = \Phi([m_1, m_2]), \quad (4.1.2)$$

$$[\Phi(D_1), \Phi(D_2)] = \Phi([D_1, D_2]), \quad (4.1.3)$$

$$[\Phi(D), \Phi(m)] = \Phi(Dm), \quad (4.1.4)$$

where the bracket in the left hand side is the usual (Nijenhuis-Richardson) bracket of coderivations.

Theorem 4.1.7. *If $D \in \text{Der}(M, L)$, $|D| = 1$, $[D, D] = 0$, then the L_∞ algebra $(A[-1], \Phi(D))$ is a homotopy fiber (Definition 3.3.4) of the inclusion of dg Lie algebras $i : (L, D, [\cdot, \cdot]) \rightarrow (M, D, [\cdot, \cdot])$. More precisely, the sequence $A \rightarrow s^{-1}L \rightarrow s^{-1}M$ of $L_\infty[1]$ algebras and $L_\infty[1]$ morphisms, where the first arrow is given by*

$$A^{\odot n} \rightarrow s^{-1}L : a_1 \odot \dots \odot a_n \rightarrow s^{-1} \left(\frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) P^\perp[\dots [Da_{\sigma(1)}, a_{\sigma(2)}] \dots, a_{\sigma(n)}] \right), \quad n \geq 1,$$

and the second arrow is the inclusion i , is a homotopy fiber sequence.

Theorem 4.1.6 will follow from the classification of $L_\infty[1]$ extensions (reviewed in Section 1.3.3) and the following proposition, which is proved via homotopy transfer.

Proposition 4.1.8. *The coderivation $pR = (r_1, \dots, r_n, \dots) \in \overline{\text{CE}}(s^{-1}\text{Der}(M, L) \times s^{-1}M \times A)$, given in Taylor coefficients by (cf. Remark 1.3.32)*

$$\begin{aligned} r_1(s^{-1}D, s^{-1}m, a) &= (0, 0, Pm), \\ r_2(s^{-1}m_1 \odot s^{-1}m_2) &= (-1)^{|m_1|} s^{-1}[m_1, m_2], \\ r_2(s^{-1}D_1 \odot s^{-1}D_2) &= (-1)^{|D_1|} s^{-1}[D_1, D_2], \\ r_2(s^{-1}D \otimes s^{-1}m) &= (-1)^{|D|} s^{-1}Dm, \\ r_{n+1}(s^{-1}D \otimes a_1 \odot \dots \odot a_n) &= \Phi(D)_n(a_1 \odot \dots \odot a_n), \\ r_{n+1}(s^{-1}m \otimes a_1 \odot \dots \odot a_n) &= \Phi(m)_n(a_1 \odot \dots \odot a_n), \end{aligned}$$

for $n \geq 1$, and $R = 0$ otherwise, is an $L_\infty[1]$ structure on $s^{-1}\text{Der}(M, L) \times s^{-1}M \times A$.

Proof. As usual we consider $s^{-1}L, s^{-1}M, s^{-1}\text{Der}(M, L)$ with the induced $L_\infty[1]$ algebra structure via décalage. Let $\text{Cyl}(i)$ be the mapping cocylinder of the inclusion $i : s^{-1}L \rightarrow s^{-1}M$ as in Definition 3.3.3: the underlying space is $s^{-1}M \times s^{-1}L \times M$, the $L_\infty[1]$ structure is the restriction of the one on $C(\Delta_1; s^{-1}M) = s^{-1}M \times s^{-1}M \times M$, cf. Proposition 3.3.6 for explicit formulas, we denote it by $Q' = (q'_1, \dots, q'_n, \dots)$. We need a lemma.

Lemma 4.1.9. *The correspondence*

$$\begin{aligned} \Psi : (\text{Der}(M, L), 0, [\cdot, \cdot]) &\rightarrow (\overline{\text{CE}}(\text{Cyl}(i)), [Q', \cdot], [\cdot, \cdot]) : D \rightarrow (\Psi(D)_1, 0, \dots, 0, \dots), \\ \Psi(D)_1(s^{-1}m, s^{-1}l, n) &= ((-1)^{|D|} s^{-1}Dm, (-1)^{|D|} s^{-1}Dl, Dn), \end{aligned}$$

is a morphism of dg Lie algebras.

Proof. It is clear that $[\Psi(D_1), \Psi(D_2)] = \Psi([D_1, D_2])$, it remains to show that $[q'_n, \Psi(D)_1] = 0$ for all $D \in \text{Der}(M, L)$ and $n \geq 1$: since the formula for q'_n involves nested brackets, this follows easily from D being a derivation.

We sketch a different approach, anticipating the one we will use to prove Theorem 4.1.6. We consider the dg Lie algebra

$$H_i = \{(m, l, m(t, dt)) \in M \times L \times M[t, dt] \text{ s.t. } m(0, 0) = m, m(1, 0) = l\},$$

It is clear from the definitions and by Lemma 2.2.5 that there is a contraction $s^{-1}H_i \xleftarrow{\cong} \text{Cyl}(i)$, induced by Dupont's contraction, such that the $L_\infty[1]$ structure on $\text{Cyl}(i)$ is induced from the $L_\infty[1]$ structure on $s^{-1}H_i$ via homotopy transfer. The graded Lie algebra $\text{Der}(M, L)$ acts on H_i by derivations in an obvious way, hence we can take the semidirect product $\text{Der}(M, L) \rtimes H_i$ and the contraction $s^{-1}(\text{Der}(M, L) \rtimes H_i) \xleftarrow{\cong} s^{-1}\text{Der}(M, L) \times \text{Cyl}(i)$ induced in an obvious way. According to Proposition 1.3.34 Ψ is a morphism of graded Lie algebras if and only if the corresponding coderivation Q on $s^{-1}\text{Der}(M, L) \times \text{Cyl}(i)$, determined by Ψ as in Remark 1.3.33, is an $L_\infty[1]$ structure: but in fact it is not hard to see, by repeating the computations in the proof of Proposition 3.3.6, that Q is precisely the $L_\infty[1]$ structure induced via homotopy transfer along the contraction $s^{-1}(\text{Der}(M, L) \rtimes H_i) \xleftarrow{\cong} s^{-1}\text{Der}(M, L) \times \text{Cyl}(i)$. \square

As in the proof of the lemma, we can form the semidirect product $(s^{-1}\text{Der}(M, L) \rtimes_\Psi \text{Cyl}(i), Q)$, cf. Proposition 1.3.34 and Notation 1.3.35: this is an $L_\infty[1]$ algebra whose underlying graded space we denote for simplicity by

$$V := s^{-1}\text{Der}(M, L) \times (s^{-1}M \times s^{-1}L \times M).$$

We know the $L_\infty[1]$ structure $Q = (q_1, \dots, q_n, \dots) \in \overline{\text{CE}}(V)$ explicitly: first of all the linear bracket is $q_1(s^{-1}D, (s^{-1}m, s^{-1}l, n)) = (0, (0, 0, m-l))$, moreover, the restriction of Q to the $L_\infty[1]$ subalgebra $\text{Cyl}(i) \subset V$ is given as in the claim of Proposition 3.3.6¹, finally, the only remaining non-vanishing bracket is $q_2(s^{-1}D \otimes (s^{-1}m, s^{-1}l, n)) = (0, ((-1)^{|D|}s^{-1}Dm, (-1)^{|D|}s^{-1}Dl, Dn))$, compare the lemma and Remark 1.3.33.

We consider the following contraction from (V, q_1) to $(s^{-1}\text{Der}(M, L) \times s^{-1}M \times A, r_1)$, with r_1 as in the claim of the proposition:

$$\begin{aligned} g_1 : V &\rightarrow s^{-1}\text{Der}(M, L) \times s^{-1}M \times A : \\ &: (s^{-1}D, (s^{-1}m, s^{-1}l, n)) \longrightarrow (s^{-1}D, s^{-1}m, Pn) \end{aligned} \quad (4.1.5)$$

$$\begin{aligned} f_1 : s^{-1}\text{Der}(M, L) \times s^{-1}M \times A &\rightarrow V : \\ &: (s^{-1}D, s^{-1}m, a) \longrightarrow (s^{-1}D, (s^{-1}m, s^{-1}P^\perp m, a)) \end{aligned} \quad (4.1.6)$$

$$K : V \rightarrow V : (s^{-1}D, (s^{-1}m, s^{-1}l, n)) \longrightarrow (0, (0, s^{-1}P^\perp n, 0)) \quad (4.1.7)$$

We are going to prove that the transferred $L_\infty[1]$ structure R on $s^{-1}\text{Der}(M, L) \times s^{-1}M \times A$ is as in the claim of the proposition.

Let $F = (f_1, \dots, f_n, \dots) : s^{-1}\text{Der}(M, L) \times s^{-1}M \times A \rightarrow V$ be the $L_\infty[1]$ morphism as in Theorem 2.2.1: we claim that that f_{n+1} vanishes everywhere for all $n \geq 1$ but on mixed terms of type $s^{-1}m \otimes a_1 \odot \dots \odot a_n$ and $s^{-1}D \otimes a_1 \odot \dots \odot a_n$, and moreover it factors through the inclusion $s^{-1}L \rightarrow V : s^{-1}l \rightarrow (0, (0, s^{-1}l, 0))$. It is easy to check the claim directly for $f_2 = Kq_2f_1^{\odot 2}$, so we suppose inductively to have proven it for f_2, \dots, f_n . By definition

$$f_{n+1}(\dots) = \sum_{k=2}^{n+1} \frac{1}{k!} \sum_{i_1 + \dots + i_k = n+1} \sum_{\sigma \in S(i_1, \dots, i_k)} \varepsilon(\sigma) Kq_k(f_{i_1}(\dots) \odot \dots \odot f_{i_k}(\dots))$$

Since K factors through the inclusion $s^{-1}L \rightarrow V$, so does f_{n+1} . Since Kq_{k+1} vanishes everywhere but on terms of type $s^{-1}m \otimes m_1 \odot \dots \odot m_k$, $s^{-1}l \otimes m_1 \odot \dots \odot m_k$ and $s^{-1}D \otimes m_1 \odot \dots \odot m_k$ (notice that in the latter case $k = 1$), the above formula reduces to

$$f_{n+1}(\dots) = \sum_{k=1}^n \frac{1}{k!} \sum_{\sigma \in S(n-k+1, 1, \dots, 1)} \varepsilon(\sigma) Kq_{k+1}(f_{n-k+1}(\dots) \odot f_1(\dots) \odot \dots \odot f_1(\dots))$$

and the claim follows easily from this using the inductive hypothesis.

A similar reasoning shows that for $n \geq 2$ the $(n+1)$ -th Taylor coefficient r_{n+1} of the transferred $L_\infty[1]$ structure

$$r_{n+1}(\dots) = \sum_{k=2}^{n+1} \frac{1}{k!} \sum_{i_1 + \dots + i_k = n+1} \sum_{\sigma \in S(i_1, \dots, i_k)} \varepsilon(\sigma) g_1 q_k(f_{i_1}(\dots) \odot \dots \odot f_{i_k}(\dots))$$

reduces to

$$r_{n+1}(\dots) = \sum_{k=1}^n \frac{1}{k!} \sum_{\sigma \in S(n-k+1, 1, \dots, 1)} \varepsilon(\sigma) g_1 q_{k+1}(f_{n-k+1}(\dots) \odot f_1(\dots) \odot \dots \odot f_1(\dots))$$

¹Since i is an inclusion, we can identify $\text{Cyl}(i)$ with an $L_\infty[1]$ subalgebra

$$\text{Cyl}(i) = s^{-1}M \times s^{-1}L \times M \subset s^{-1}M \times s^{-1}M \times M = C(\Delta_1; M).$$

and vanish everywhere but on terms of type $s^{-1}m \otimes a_1 \odot \cdots \odot a_n$ and $s^{-1}D \otimes a_1 \odot \cdots \odot a_n$.

Remark 4.1.10. For future reference (cf. Remark 4.1.14) we remark that up to this point we never used the hypothesis that $A \subset M$ is a graded Lie subalgebra, nor we used the explicit formulas in Definition 4.1.3. The hypothesis that A is $[\cdot, \cdot]$ -closed will be essential in the following computation.

We show that f_{n+1} is explicitly given, for $n \geq 1$, by

$$\begin{aligned} f_{n+1}(s^{-1}D \otimes a_1 \odot \cdots \odot a_n) &= \\ &= \left(0, \left(0, s^{-1} \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) P^\perp[\cdots [Da_{\sigma(1)}, a_{\sigma(2)}] \cdots, a_{\sigma(n)}], 0 \right) \right), \end{aligned} \quad (4.1.8)$$

$$\begin{aligned} f_{n+1}(s^{-1}m \otimes a_1 \odot \cdots \odot a_n) &= \\ &= \left(0, \left(0, s^{-1} \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) P^\perp[\cdots [m, a_{\sigma(1)}] \cdots, a_{\sigma(n)}], 0 \right) \right), \end{aligned} \quad (4.1.9)$$

and $f_{n+1} = 0$ otherwise.

We leave to the reader to check the case $n = 1$, using $f_2 = Kq_2 f_1^{\odot 2}$ and recalling (4.1.1). Next for all $n \geq 2$ we have to prove $f_{n+1} = \sum_{j=1}^n Kq_{j+1} F_{n+1}^{j+1}$. To simplify the computation we notice that we are only interested in keeping track of $p_M q_{j+1} F_{n+1}^{j+1}$, where we denote by p_M the projection $V \rightarrow M$: in fact $Kq_{j+1} F_{n+1}^{j+1} = \left(0, \left(0, s^{-1} P^\perp(p_M q_{j+1} F_{n+1}^{j+1}), 0 \right) \right)$.

The considerations preceding Remark 4.1.10 imply that for $1 \leq j \leq n-1$

$$\begin{aligned} & p_M q_{j+1} F_{n+1}^{j+1}(s^{-1}D \otimes a_1 \odot \cdots \odot a_n) = \\ &= \sum_{\sigma \in S(n-j, j)} \varepsilon(\sigma) p_M q_{j+1} (f_{n-j+1}(s^{-1}D \otimes a_{\sigma(1)} \odot \cdots \odot a_{\sigma(n-j)}) \otimes a_{\sigma(n-j+1)} \odot \cdots \odot a_{\sigma(n)}) = \\ &= -\frac{B_j}{j!(n-j)!} \sum_{\sigma \in S_n} \varepsilon(\sigma) \overbrace{[\cdots [P^\perp([\cdots [Da_{\sigma(1)}, a_{\sigma(2)}] \cdots], a_{\sigma(n-j+1)}] \cdots, a_{\sigma(n)}]}^j \end{aligned}$$

where in the first identity we used graded symmetry of q_{j+1} and f_{n-j+1} , and in the second we substituted the explicit formulas we knew from Proposition 3.3.6 and the inductive hypothesis. In the same way, for $1 \leq j \leq n-1$

$$\begin{aligned} & p_M q_{j+1} F_{n+1}^{j+1}(s^{-1}m \otimes a_1 \odot \cdots \odot a_n) = \\ &= -\frac{B_j}{j!(n-j)!} \sum_{\sigma \in S_n} \varepsilon(\sigma) \overbrace{[\cdots [P^\perp([\cdots [m, a_{\sigma(1)}] \cdots], a_{\sigma(n-j+1)}] \cdots, a_{\sigma(n)}]}^j \end{aligned}$$

The remaining terms to consider are

$$p_M q_{n+1} F_{n+1}^{n+1}(s^{-1}D \otimes a_1 \odot \cdots \odot a_n) = 0 \quad \text{for } n \geq 2,$$

and

$$p_M q_{n+1} F_{n+1}^{n+1}(s^{-1}m \otimes a_1 \odot \cdots \odot a_n) = p_M q_{n+1} ((s^{-1}m, s^{-1}P^\perp m, 0) \otimes a_1 \odot \cdots \odot a_n) =$$

$$= \frac{B_n}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) [\cdots [m - P^\perp m, a_{\sigma(1)}] \cdots, a_{\sigma(n)}] = \frac{B_n}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) [\cdots [Pm, a_{\sigma(1)}] \cdots, a_{\sigma(n)}]$$

Summing over j we see that

$$\begin{aligned} \sum_{j=1}^n p_M q_{j+1} F_{n+1}^{j+1} (s^{-1} D \otimes a_1 \odot \cdots \odot a_n) &= \left(- \sum_{j=1}^{n-1} \frac{B_j}{j!(n-j)!} \right) \sum_{\sigma \in S_n} \varepsilon(\sigma) [\cdots [Da_{\sigma(1)}, a_{\sigma(2)}] \cdots, a_{\sigma(n)}] + \\ &+ \sum_{j=1}^{n-1} \frac{B_j}{j!(n-j)!} \sum_{\sigma \in S_n} \varepsilon(\sigma) \overbrace{[\cdots [P([\cdots [Da_{\sigma(1)}, a_{\sigma(2)}] \cdots]), a_{\sigma(n-j+1)}] \cdots, a_{\sigma(n)}]}^j \end{aligned}$$

Finally, by the identity (3.3.1) on Bernoulli numbers and with a change of variable $k = n - j$ we obtain

$$\begin{aligned} \sum_{j=1}^n p_M q_{j+1} F_{n+1}^{j+1} (s^{-1} D \otimes a_1 \odot \cdots \odot a_n) &= \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) [\cdots [Da_{\sigma(1)}, a_{\sigma(2)}] \cdots, a_{\sigma(n)}] + \\ &+ \sum_{k=1}^{n-1} \frac{B_{n-k}}{k!(n-k)!} \sum_{\sigma \in S_n} \varepsilon(\sigma) \overbrace{[\cdots [P([\cdots [Da_{\sigma(1)}, a_{\sigma(2)}] \cdots]), a_{\sigma(k+1)}] \cdots, a_{\sigma(n)}]}^{n-k} \quad (4.1.10) \end{aligned}$$

In a similar way

$$\begin{aligned} \sum_{j=1}^n p_M q_{j+1} F_{n+1}^{j+1} (s^{-1} m \otimes a_1 \odot \cdots \odot a_n) &= \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) [\cdots [m, a_{\sigma(1)}] \cdots, a_{\sigma(n)}] + \\ &+ \sum_{k=0}^{n-1} \frac{B_{n-k}}{k!(n-k)!} \sum_{\sigma \in S_n} \varepsilon(\sigma) \overbrace{[\cdots [P([\cdots [m, a_{\sigma(1)}] \cdots]), a_{\sigma(k+1)}] \cdots, a_{\sigma(i)}]}^{n-k} \quad (4.1.11) \end{aligned}$$

Notice that in both the identities (4.1.10) and (4.1.11) the bottom line lies in A , and this is the only (essential) passage where it is used the hypothesis that A is $[\cdot, \cdot]$ -closed. Applying K , it is now clear that $f_{n+1} = \sum_{j=1}^n K q_{j+1} F_{n+1}^{j+1}$ is given as in Equations (4.1.8) and (4.1.9).

It remains to prove that r_{n+1} is as in the claim of the proposition: first of all, we leave to the reader to check directly that so is $r_2 = g_1 q_2 f_1^{\odot 2}$. For $n \geq 2$ we already observed that

$$r_{n+1} = \sum_{j=1}^n g_1 q_{j+1} F_{n+1}^{j+1} = \sum_{j=1}^n \left(p_{s^{-1} \text{Der}(M,L)} q_{j+1} F_{n+1}^{j+1}, p_{s^{-1} M} q_{j+1} F_{n+1}^{j+1}, P \left(p_M q_{j+1} F_{n+1}^{j+1} \right) \right)$$

vanishes everywhere but on mixed terms of type $s^{-1} m \otimes a_1 \odot \cdots \odot a_n$ and $s^{-1} D \otimes a_1 \odot \cdots \odot a_n$. Comparing equations (4.1.10) and (4.1.11) with the definition of the brackets in 4.1.3, the thesis follows if we show that $p_{s^{-1} \text{Der}(M,L)} q_{j+1} F_{n+1}^{j+1} = 0 = p_{s^{-1} M} q_{j+1} F_{n+1}^{j+1}$ for all $n \geq 2$ and $1 \leq j \leq n$. We consider the second identity, the first one is treated similarly: $p_{s^{-1} M} q_{j+1} F_{n+1}^{j+1}$ is zero for $j > 1$ since in this case $p_{s^{-1} M} q_{j+1} = 0$, while in the $j = 1$ case

$$p_{s^{-1} M} q_2 F_{n+1}^2 = q_2 p_{s^{-1} M}^{\odot 2} F_{n+1}^2 = \frac{1}{2} \sum_{i=1}^n \sum_{\sigma \in S(i, n-i+1)} \varepsilon(\sigma) q_2 (p_{s^{-1} M} f_i(\cdots) \odot p_{s^{-1} M} f_{n-i+1}(\cdots)) = 0$$

since $p_{s^{-1} M} f_k = 0$ whenever $k \geq 2$. □

Proof. (of Theorem 4.1.6) We have to prove that

$$\Phi : \text{Der}(M, L) \rtimes M \rightarrow \text{CE}(A) : (D, m) \rightarrow \Phi(D) + \Phi(m)$$

is a morphism of graded Lie algebras. In the previous proposition we constructed an $L_\infty[1]$ algebra fitting into an $L_\infty[1]$ extension $0 \rightarrow A \rightarrow s^{-1} \text{Der}(M, L) \times s^{-1} M \times A \rightarrow s^{-1}(\text{Der}(M, L) \rtimes M) \rightarrow 0$ of fibre $(A, 0)$ and base $\text{Der}(M, L) \rtimes M$, seen as an $L_\infty[1]$ algebra via décalage: this is classified by an L_∞ morphism $\text{Der}(M, L) \rtimes M \rightarrow \text{CE}(A)$ as in Proposition 1.3.34, and by comparing with Remark 1.3.33 we see immediately that this L_∞ morphism is $(\Phi, 0, \dots, 0, \dots)$, thus Φ is a morphism of graded Lie algebras. \square

Remark 4.1.11. Theorem 4.1.6 is interesting even in the case $L = 0$, where it says that given a graded Lie algebra $(M, [\cdot, \cdot])$ the correspondence $\Phi : M \rightarrow \text{CE}(M) : m \rightarrow (\Phi(m)_0, \dots, \Phi(m)_n, \dots)$

$$\Phi(m)_0(1) = m, \quad \Phi(m)_n(m_1 \odot \dots \odot m_n) = \frac{(-1)^n B_n}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) [\dots [m, m_{\sigma(1)}] \dots, m_{\sigma(n)}] \text{ for } n \geq 1,$$

is a morphism of graded Lie algebras: this seems to clarify some computations from [7], Section 4. On the other hand, in this case the higher derived brackets associated to derivations vanish after the linear one, that is, we have $\Phi : \text{Der}(M) \rightarrow \text{CE}(M) : D \rightarrow (D, 0, \dots, 0, \dots)$.

We prove Theorem 4.1.7 as a particular case of a more general result. Consider the $L_\infty[1]$ algebra $s^{-1} \text{Der}(M/L) \rtimes_{\Psi} \text{Cyl}(i) = (V, Q)$ as in the proof of Proposition 4.1.8: then in the same proof we constructed an $L_\infty[1]$ morphism

$$F : (s^{-1} \text{Der}(M, L) \times s^{-1} M \times A, R) \rightarrow (V, Q),$$

in fact a weak equivalence, cf. Equations (4.1.8) and (4.1.9). We notice that if $D \in \text{Der}(M, L)$ satisfies $|D| = 1$, $[D, D] = 0$, then $(s^{-1} D, 0, 0) \in s^{-1} \text{Der}(M, L) \times s^{-1} M \times A$ satisfies the assumptions of Remark 1.3.20, thus we can twist everything by D to get a new $L_\infty[1]$ morphism

$$F_D : (s^{-1} \text{Der}(M, L) \times s^{-1} M \times A, R_D) \rightarrow (V, Q_D).$$

Let $j : (N, D, [\cdot, \cdot]) \rightarrow (M, D, [\cdot, \cdot])$ be the inclusion of a dg Lie subalgebra, then R_D restricts to an $L_\infty[1]$ algebra structure on $s^{-1} N \times A$, which we continue to denote by $R_D = (r_{D,1}, \dots, r_{D,n}, \dots)$, explicitly given by

$$r_{D,1}(s^{-1} n, a) = (-s^{-1} Dn, P(Da + n)), \quad r_{D,2}(s^{-1} n_1 \odot s^{-1} n_2) = (-1)^{|n_1|} s^{-1} [n_1, n_2], \quad (4.1.12)$$

$$r_{D,i+1}(s^{-1} n \otimes a_1 \odot \dots \odot a_i) = \Phi(n)_i(a_1 \odot \dots \odot a_i), \quad (4.1.13)$$

$$r_{D,i}(a_1 \odot \dots \odot a_i) = \Phi(D)_i(a_1 \odot \dots \odot a_i), \quad (4.1.14)$$

and $R_D = 0$ otherwise. Similarly, Q_D restricts to an $L_\infty[1]$ algebra structure on the subspace $s^{-1} N \times s^{-1} L \times M \subset V$, which we continue to denote by Q_D , which is exactly the homotopy fiber product (Definition 3.3.3) $s^{-1} N \times_{s^{-1} M}^h s^{-1} L$ along the inclusions $j : (N, D, [\cdot, \cdot]) \rightarrow (M, D, [\cdot, \cdot])$ and $i : (L, D, [\cdot, \cdot]) \rightarrow (M, D, [\cdot, \cdot])$ (as usual, via décalage). The $L_\infty[1]$ morphism F_D restricts to an $L_\infty[1]$ morphism $F_D : (s^{-1} N \times A, R_D) \rightarrow (s^{-1} N \times_{s^{-1} M}^h s^{-1} L, Q_D)$, explicitly

$$f_{D,1}(s^{-1} n, a) = (s^{-1} n, s^{-1} P^\perp(n + Da), a), \quad (4.1.15)$$

$$f_{D,i+1}(s^{-1} n \otimes a_1 \odot \dots \odot a_i) = \left(0, s^{-1} \frac{1}{i!} \sum_{\sigma \in S_i} \varepsilon(\sigma) P^\perp[\dots [n, a_{\sigma(1)}] \dots, a_{\sigma(i)}], 0 \right), \quad (4.1.16)$$

$$f_{D,i}(a_1 \odot \cdots \odot a_i) = \left(0, s^{-1} \frac{1}{i!} \sum_{\sigma \in S_i} \varepsilon(\sigma) P^\perp [\cdots [Da_{\sigma(1)}, a_{\sigma(2)}] \cdots, a_{\sigma(i)}], 0 \right), \quad (4.1.17)$$

and $F_D = 0$ otherwise. Finally, we notice that F_D is a weak equivalence of $L_\infty[1]$ algebras: in fact the restrictions $g_1 : s^{-1}N \times_{s^{-1}M}^h s^{-1}L \rightarrow s^{-1}N \times A : (s^{-1}n, s^{-1}l, m) \rightarrow (s^{-1}n, Pm)$ of (4.1.5) and $K : s^{-1}N \times_{s^{-1}M}^h s^{-1}L \rightarrow s^{-1}N \times_{s^{-1}M}^h s^{-1}L : (s^{-1}n, s^{-1}l, m) \rightarrow (0, s^{-1}P^\perp m, 0)$ of (4.1.7) are respectively a dg left inverse to $f_{D,1}$ and a homotopy between $f_{D,1}g_1$ and $\text{id}_{s^{-1}N \times_{s^{-1}M}^h s^{-1}L}$ (recall (4.1.1)). This proves

Proposition 4.1.12. *The $L_\infty[1]$ algebra $(s^{-1}N \times A, r_{D,1}, \dots, r_{D,n}, \dots)$ as in (4.1.12)-(4.1.14) is weakly equivalent to the homotopy fiber product $(s^{-1}N \times_{s^{-1}M}^h s^{-1}L, Q_D)$.*

Remark 4.1.13. In fact, with the previous notations, it can be proved that R_D is the $L_\infty[1]$ structure induced from Q_D via homotopy transfer along the contraction $f_{D,1}, g_1, K$: this can be done by adapting the computations in Proposition 4.1.8.

Proof. (of Theorem 4.1.7) The first claim is the particular case of the previous proposition when $N = 0$. The second claim follows by the commutative diagram of $L_\infty[1]$ algebras and $L_\infty[1]$ morphisms

$$\begin{array}{ccccc} A & \longrightarrow & s^{-1}L & \xrightarrow{i} & s^{-1}M \\ \downarrow F_D & & \parallel & & \parallel \\ \text{coC}(i) & \longrightarrow & s^{-1}L & \xrightarrow{i} & s^{-1}M \end{array}$$

where the top sequence is as in the claim of the theorem and F_D is as in (4.1.15)-(4.1.17). \square

Remark 4.1.14. In this remark we consider what happens when we further remove the assumption that $A \subset M$ is a graded Lie subalgebra and we just suppose that A is a complement to L in M . We sketch a proof that in this case there is still a correspondence $\Phi : \text{Der}(M, L) \rtimes M \rightarrow \text{CE}(A)$ such that both Theorem 4.1.6 and Theorem 4.1.7 hold, but explicit formulas for Φ will have to be more involved than those in Definition 4.1.3. To construct Φ it is sufficient to construct the corresponding $L_\infty[1]$ extension and this can be done using homotopy transfer, following the proof of Proposition 4.1.8 step by step up to Remark 4.1.10. There is still an induced $L_\infty[1]$ structure R on $s^{-1}\text{Der}(M, L) \times s^{-1}M \times A$ and again this fits into an L_∞ extension of base $\text{Der}(M, L) \rtimes M$ and fiber $(A[-1], 0)$: finally, the classifying L_∞ morphism $\text{Der}(M, L) \rtimes M \rightarrow \text{CE}(A)$ is again a strict morphism, thus a morphism of graded Lie algebras. To see that explicit formulas for Φ will have to be more involved than those in Definition 4.1.3, we notice that if A is not $[\cdot, \cdot]$ -closed then there is no guarantee that it will be closed with respect to the brackets in 4.1.3: alternatively one could try to compute the first brackets directly via homotopy transfer. Proposition 4.1.12 and Theorem 4.1.7 can be proved as above, except that the various explicit formulas do not longer hold.

As a corollary to Theorem 4.1.7 and Theorem 3.3.5 we have the following

Corollary 4.1.15. *In the hypotheses of Theorem 4.1.7, if $H(i) : H(L, D) \rightarrow H(M, D)$ is injective (equivalently, if $H(P) : H(M, D) \rightarrow H(A, PD)$ is surjective) then the $L_\infty[1]$ algebra $(A, \Phi(D))$ is homotopy abelian.*

We will need two lemmas: the first Lemma gives a certain functoriality property of higher derived brackets (the problem in general is that $\text{CE}(-)$ is not a functor), the second a certain invariance property (this should be confronted with the results of [20]).

Lemma 4.1.16. *Given $M_k = L_k \oplus A_k$, $k = 1, 2$, as in the assumptions of Definition 4.1.3 together with a commutative diagram*

$$\begin{array}{ccccc} L_1 & \longrightarrow & M_1 & \longleftarrow & A_1 \\ \downarrow & & \downarrow & & \downarrow \cong \\ L_2 & \longrightarrow & M_2 & \longleftarrow & A_2 \end{array}$$

of morphisms of graded Lie algebras such that the right vertical arrow is an isomorphism (thus it induces an isomorphism $\text{CE}(A_1) \rightarrow \text{CE}(A_2)$ of graded Lie algebras), the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{\Phi} & \text{CE}(A_1) \\ \downarrow & & \downarrow \\ M_2 & \xrightarrow{\Phi} & \text{CE}(A_2) \end{array}$$

is also commutative, where $\Phi : M_k \rightarrow \text{CE}(A_k)$ is given by higher derived brackets.

Proof. This follows immediately from the definitions. \square

Lemma 4.1.17. *Given $M = L \oplus A_k$, $k = 1, 2$, as in the assumptions of Definition 4.1.3, we denote by $\Phi_k : \text{Der}(M, L) \rightarrow \overline{\text{CE}}(A_k)$ the respective constructions of higher derived brackets: if D is as in the hypotheses of Theorem 4.1.7 then the $L_\infty[1]$ algebras $(A_1, \Phi_1(D))$ and $(A_2, \Phi_2(D))$ are isomorphic (in the category $\mathcal{L}_\infty[1]$).*

Proof. Let $(\text{coC}(i), Q_D)$ be the mapping cocone of the inclusion $i : s^{-1}L \rightarrow s^{-1}M$, according to Remark 4.1.13 homotopy transfer induces $L_\infty[1]$ morphisms $(A_1, \Phi_1(D)) \rightarrow (\text{coC}(i), Q_D)$ and $(\text{coC}(i), Q_D) \rightarrow (A_2, \Phi_2(D))$: the composite $L_\infty[1]$ morphism $(A_1, \Phi_1(D)) \rightarrow (A_2, \Phi_2(D))$ has linear Taylor coefficient the dg isomorphism $(A_2, P_1D) \rightarrow (A_2, P_2D) : a \rightarrow P_2a$, where we denote by $P_k : M \rightarrow A_k$ the projection with kernel L , hence it is an $L_\infty[1]$ isomorphism. \square

We close with two examples, other examples and applications will be given in the following sections.

Example 4.1.18. As in Remark 1.3.1 we denote by $\text{CE}_0(V) \subset \text{CE}(V)$ the abelian Lie subalgebra spanned by constant coderivations, then $\text{CE}(V) = \overline{\text{CE}}(V) \oplus \text{CE}_0(V)$ satisfies the hypotheses of Voronov's construction [105] of higher derived brackets. Notice that $\text{CE}_0(V)$ is isomorphic to V via $\sigma : V \rightarrow \text{CE}_0(V) : v \rightarrow \sigma_v$ (cf. Remark 1.3.1), thus there is induced a morphism of graded Lie algebras $\text{CE}(V) \xrightarrow{\Phi} \text{CE}(\text{CE}_0(V)) \xrightarrow{\cong} \text{CE}(V)$: we claim that this is just the identity. We denote by $P : \text{CE}(V) \rightarrow \text{CE}_0(V)$ the projection with kernel $\overline{\text{CE}}(V)$: the claim follows since Equation (1.3.3) implies immediately that for all $Q = (q_0, \dots, q_n, \dots) \in \text{CE}(V)$ and $n \geq 1$ we have $P[\dots[Q, \sigma_{v_1}] \dots, \sigma_{v_n}] = \sigma_{q_n(v_1 \odot \dots \odot v_n)}$. In particular, if $Q \in \overline{\text{CE}}(V)$ is an $L_\infty[1]$ structure on V then $(V, Q) = (V, \Phi(Q)) = (V, \Phi([Q, \cdot]))$.

Remark 4.1.19. As noticed in [37], in the similar situation $\text{Hoch}(V) = \overline{\text{Hoch}}(V) \oplus \text{Hoch}_0(V)$ considered in Remark 1.2.1, the induced morphism $\text{Hoch}(V) \xrightarrow{\Phi} \text{CE}(\text{Hoch}_0(V)) \xrightarrow{\cong} \text{CE}(V)$ is symmetrization $\text{sym} : \text{Hoch}(V) \rightarrow \text{CE}(V)$, in fact given $Q = (q_0, \dots, q_n, \dots) \in \text{Hoch}(V)$ we have (with the notations of Remark 1.2.1)

$$P[\dots[Q, \tau_{v_1}] \dots, \tau_{v_n}] = \tau_{q_n(v_1 \otimes \dots \otimes v_n)} = \tau_{q_n(\text{sym}(v_1 \odot \dots \odot v_n))} = \tau_{\text{sym}(q_n)(v_1 \odot \dots \odot v_n)}.$$

As an application of Theorem 4.1.7 we recover the following result of Chuang and Lazarev [23].

Theorem 4.1.20. *Every $L_\infty[1]$ algebra (V, Q) is a homotopy fiber (Definition 3.3.4), more precisely, (V, Q) is weakly equivalent to the mapping cocone of the inclusion of dg Lie algebras $i : (\overline{\text{CE}}(V), [Q, \cdot], [\cdot, \cdot]) \rightarrow (\text{CE}(V), [Q, \cdot], [\cdot, \cdot])$ and $V \xrightarrow{\text{Ad}} s^{-1}\overline{\text{CE}}(V) \xrightarrow{i} s^{-1}\text{CE}(V)$ is a homotopy fiber sequence, where the $L_\infty[1]$ morphism*

$$\text{Ad} = (\text{Ad}_1, \dots, \text{Ad}_n, \dots) : V \rightarrow s^{-1}\overline{\text{CE}}(V),$$

is explicitly given by $s \text{Ad}_n(v_1 \odot \dots \odot v_n)_k(v_{n+1} \odot \dots \odot v_{n+k}) = q_{n+k}(v_1 \odot \dots \odot v_{n+k})$ for all $n, k \geq 1$ (we denote by $s \text{Ad}_n$ the composition $V^{\odot n} \xrightarrow{\text{Ad}_n} s^{-1}\overline{\text{CE}}(V) \xrightarrow{s} \overline{\text{CE}}(V)$).

Remark 4.1.21. The $L_\infty[1]$ morphism $\text{Ad} : V \rightarrow s^{-1}\overline{\text{CE}}(V)$ in an $L_\infty[1]$ generalization of the adjoint morphism $\text{ad} : L \rightarrow \text{End}(L) : l \rightarrow [l, \cdot]$ of a graded Lie algebra L .

Finally, as a consequence of Corollary 4.1.15 we obtain the following necessary and sufficient condition for an $L_\infty[1]$ algebra (V, Q) to be homotopy abelian. This is more nicely stated in the language of formal pointed dg manifolds by Kontsevich and Soibelman [62]. Recall that a formal pointed dg manifold is a dg coalgebra which is cofree as a coalgebra, thus we may think of an L_∞ algebra (V, Q) as a formal pointed dg manifold with a choice of coordinates. We may also think of the graded Lie algebra $\text{CE}(V)$ as the Lie algebra of vector fields on the underlying dg manifold and of the dg morphism $(\text{CE}(V), [Q, \cdot]) \rightarrow (V, q_1) : Q \rightarrow q_0(1)$ as evaluation of vector fields on the tangent complex at the marked point. When the latter admits a dg right inverse (this does not depend on the choice of coordinates) we say that the formal pointed dg manifold has the *splitting property*: according to Corollary 4.1.15 and the proof of the following proposition this is equivalent to say that the formal pointed dg manifold admits an abelian $L_\infty[1]$ system of coordinates.

Proposition 4.1.22. *An $L_\infty[1]$ algebra (V, Q) is homotopy abelian if and only if the dg morphism $(\text{CE}(V), [Q, \cdot]) \rightarrow (V, q_1) : Q \rightarrow q_0(1)$ admits a dg right inverse.*

Proof. The if part follows from Corollary 4.1.15. For the only if part, recall [62] that every $L_\infty[1]$ algebra is isomorphic to the direct product of a minimal model and an abelian $L_\infty[1]$ algebra. If (V, Q) is homotopy abelian, then the $L_\infty[1]$ structure on the minimal model is trivial, thus there is an $L_\infty[1]$ isomorphism between (V, Q) and an abelian $L_\infty[1]$ algebra: since we can always compose such an $L_\infty[1]$ isomorphism with the inverse of its linear part, there is an $L_\infty[1]$ isomorphism $F : (V, q_1, q_2, \dots, q_n, \dots) \rightarrow (V, q_1, 0, \dots, 0, \dots)$ with linear part the identity. We extend F to an automorphism of graded coalgebras $F : S(V) \rightarrow S(V)$ by putting $F(1) = 1$. The required right inverse is given by $V \rightarrow \text{CE}(V) : v \rightarrow F^{-1}\sigma_v F$, where $\sigma_v \in \text{CE}(V)$ is the constant derivation as in Remark 1.3.1. This is a right inverse to $(\text{CE}(V), [Q, \cdot]) \rightarrow (V, q_1) : Q \rightarrow q_0(1) = Q(1)$, in fact $F^{-1}\sigma_v F(1) = F^{-1}\sigma_v(1) = F^{-1}(v) = v$, since F^{-1} has linear part the identity. This is a dg morphism, in fact $\text{CE}(V) \rightarrow \text{CE}(V) : Q \rightarrow F^{-1}QF$ is clearly bracket preserving, thus $[Q, F^{-1}\sigma_v F] = [F^{-1}q_1 F, F^{-1}\sigma_v F] = F^{-1}[q_1, \sigma_v]F = F^{-1}\sigma_{q_1(v)}F$. \square

Example 4.1.23. Given a dg Lie algebra $(L, D, [\cdot, \cdot])$, we denote by $L^{\geq 0} = \bigoplus_{i \geq 0} L^i$ the non negatively graded part and by $L^{< 0} = \bigoplus_{i < 0} L^i$ the negatively graded part of L respectively, these are graded Lie subalgebras. Clearly $L = L^{\geq 0} \oplus L^{< 0}$ and $D \in \text{Der}(L, L^{\geq 0})$, thus we are in the setup of Theorem 4.1.7 and there is an induced $L_\infty[1]$ algebra structure $\Phi(D)$ on $L^{< 0}$ by higher derived bracket. These are explicitly given, for $i \geq 2$, by

$$\Phi(D)_i(l_1 \odot \dots \odot l_i) = \sum_{\sigma \in S_i} \varepsilon(\sigma) \sum_{k=1}^i \frac{B_{i-k}}{k!(i-k)!} \overbrace{[\dots [P([\dots [Dl_{\sigma(1)}, l_{\sigma(2)}] \dots)], l_{\sigma(k+1)}] \dots, l_{\sigma(i)}}^{i-k} =$$

$$\begin{aligned}
&= \sum_{\sigma \in \mathcal{S}_i} \varepsilon(\sigma) \left(\frac{B_{i-1}}{(i-1)!} [\cdots [PDl_{\sigma(1)}, l_{\sigma(2)}] \cdots, l_{\sigma(i)}] + \left(\sum_{k=2}^i \frac{B_{i-k}}{k!(i-k)!} \right) [\cdots [Dl_{\sigma(1)}, l_{\sigma(2)}] \cdots, l_{\sigma(i)}] \right) = \\
&\quad = -\frac{B_{i-1}}{(i-1)!} \sum_{\sigma \in \mathcal{S}_i} \varepsilon(\sigma) [\cdots [P^\perp Dl_{\sigma(1)}, l_{\sigma(2)}] \cdots, l_{\sigma(i)}]
\end{aligned}$$

where in the second identity we used the fact that for $k > 1$ the P becomes irrelevant, since it applies to an element already in $L^{<0}$, and in the third one we used the identity (3.3.1). The linear bracket is $\Phi(D)_1 = PD$, which is 0 on L^{-1} and D on $L^{<-1}$, moreover, $P^\perp D$ acts on $L^{<0}$ as D on L^{-1} and 0 elsewhere: we see that these are essentially the same brackets as those introduced by Getzler in [40]. When L is concentrated in degrees ≥ -1 , then we say that L is a quantum type dg Lie algebra, $L^{<0} = L^{-1} \subset L$ is an abelian Lie subalgebra and the brackets reduce to $\Phi(D)_2(l_1 \odot l_2) = [Dl_1, l_2]$ for $l_1, l_2 \in L^{-1}$ and $\Phi(D)_n = 0$ for $n \neq 2$: thus we see that the previous construction generalizes the well known construction [38, 110, 105] of a Lie algebra structure on the degree minus one part of a quantum type dg Lie algebra. Finally, by Theorem 4.1.7 we see that the strict $L_\infty[1]$ morphism $L^{<0} \rightarrow s^{-1}L^{\geq 0} : l \rightarrow s^{-1}P^\perp Dl$ fits into a homotopy fiber sequence $L^{<0} \rightarrow s^{-1}L^{\geq 0} \rightarrow s^{-1}L$ of $L_\infty[1]$ algebras.

4.2 Koszul brackets and Kapranov brackets

In [64] Koszul defined for any graded commutative algebra (A, \cdot) with a unit $1_A \in A$ a morphism of graded Lie algebras $\mathcal{K}_{1_A} : \text{End}(A) \rightarrow \text{CE}(A) : f \rightarrow \mathcal{K}_{1_A}(f) = (\mathcal{K}_{1_A}(f)_0, \dots, \mathcal{K}_{1_A}(f)_n, \dots)$, where it is usual to call the Taylor coefficients $\mathcal{K}_{1_A}(f)_n : A^{\odot n} \rightarrow A$ the Koszul brackets on A associated to f . These are defined by $\mathcal{K}_{1_A}(f)_0(1) = f(1_A)$, $\mathcal{K}_{1_A}(f)_1(x) = f(x) - f(1_A)x$ and then for $n \geq 2$ recursively by

$$\begin{aligned}
&\mathcal{K}_{1_A}(f)_n(x_1 \odot \cdots \odot x_n) = \mathcal{K}_{1_A}(f)_{n-1}(x_1 \odot \cdots \odot x_{n-2} \odot x_{n-1}x_n) - \\
&- \mathcal{K}_{1_A}(f)_{n-1}(x_1 \odot \cdots \odot x_{n-2} \odot x_{n-1})x_n - (-1)^{|x_{n-1}||x_n|} \mathcal{K}_{1_A}(f)_{n-1}(x_1 \odot \cdots \odot x_{n-1} \odot x_n)x_{n-1}.
\end{aligned}$$

We are going to generalize the construction of Koszul brackets in several directions: as a first step we recall how they arise as higher derived brackets. Let (A, \cdot) be a graded commutative algebra (without need to assume the existence of a unit) and define the graded Lie algebra $\text{Aff}(A)$ as the semidirect product $\text{Aff}(A) := \text{End}(A) \rtimes A$, where we consider A with the abelian Lie structure: explicitly, given $(f, x), (g, y) \in \text{Aff}(A)$ their bracket is $[(f, x), (g, y)] = ([f, g], f(y) - (-1)^{|g||x|}g(x))$. We denote by $\nabla_- : A \rightarrow \text{End}(A) : x \rightarrow \{\nabla_x : y \rightarrow xy\}$ the left adjoint morphism and we denote by $A^\nabla \subset \text{Aff}(A)$ the abelian Lie subalgebra spanned by elements of the form (∇_x, x) , then $\text{Aff}(A) = \text{End}(A) \oplus A^\nabla$ and we are in the setup of Voronov's construction of higher derived brackets, thus it is defined a morphism of graded Lie algebras $\Phi : \text{End}(A) \rightarrow \overline{\text{CE}}(A^\nabla)$ as in Theorem 4.1.6, and by further composing this with the isomorphism $\overline{\text{CE}}(A^\nabla) \xrightarrow{\cong} \overline{\text{CE}}(A)$ it is defined a morphism of graded Lie algebras $\mathcal{K} : \text{End}(A) \rightarrow \overline{\text{CE}}(A) : f \rightarrow (\mathcal{K}(f)_1, \dots, \mathcal{K}(f)_n, \dots)$. We want to compare this construction with the one by Koszul when a unit $1_A \in A$ exists: we claim that with the previous definitions

$$\mathcal{K}(f)_n(x_1 \odot \cdots \odot x_n) = \mathcal{K}_{1_A}(f)_n(x_1 \odot \cdots \odot x_n) + (-1)^n f(1_A)x_1 \cdots x_n. \quad (4.2.1)$$

We introduce the graded symmetric maps

$$\mathcal{K}(f)_n^1 : A^{\odot n} \rightarrow \text{End}(A) : x_1 \odot \cdots \odot x_n \rightarrow [\cdots [f, \nabla_{x_1}] \cdots, \nabla_{x_n}],$$

where graded symmetry follows since the ∇_x span an abelian Lie subalgebra of $\text{End}(A)$: in this way we have $[\cdots [(f, 0), (\nabla_{x_1}, x_1)] \cdots, (\nabla_{x_n}, x_n)] = (\mathcal{K}(f)_n^1(x_1 \odot \cdots \odot x_n), \mathcal{K}(f)_n(x_1 \odot \cdots \odot x_n))$, and thus

$$\begin{aligned} (\mathcal{K}(f)_n^1(x_1 \odot \cdots \odot x_n), \mathcal{K}(f)_n(x_1 \odot \cdots \odot x_n)) &= \\ &= [(\mathcal{K}(f)_{n-1}^1(x_1 \odot \cdots \odot x_{n-1}), \mathcal{K}(f)_{n-1}(x_1 \odot \cdots \odot x_{n-1})), (\nabla_{x_n}, x_n)], \end{aligned}$$

which implies the equations $\mathcal{K}_n^1(x_1 \odot \cdots \odot x_n) = [\mathcal{K}_{n-1}^1(x_1 \odot \cdots \odot x_{n-1}), \nabla_{x_n}]$ and

$$\mathcal{K}(f)_n(x_1 \odot \cdots \odot x_n) = \mathcal{K}(f)_{n-1}^1(x_1 \odot \cdots \odot x_{n-1})(x_n) - \mathcal{K}(f)_{n-1}(x_1 \odot \cdots \odot x_{n-1})x_n, \quad (4.2.2)$$

that is, $\mathcal{K}(f)_n(x_1 \odot \cdots \odot x_{n-1} \odot -) = \mathcal{K}(f)_{n-1}^1(x_1 \odot \cdots \odot x_{n-1}) - \nabla_{\mathcal{K}(f)_{n-1}(x_1 \odot \cdots \odot x_{n-1})}$. From this we deduce that the $\mathcal{K}(f)_n$ satisfy the same recursive relation as the brackets $\mathcal{K}_{1A}(f)_n$, in fact

$$\begin{aligned} \mathcal{K}(f)_n(x_1 \odot \cdots \odot x_n) &= [\mathcal{K}(f)_{n-2}^1(x_1 \odot \cdots \odot x_{n-2}), \nabla_{x_{n-1}}](x_n) - \mathcal{K}(f)_{n-1}(x_1 \odot \cdots \odot x_{n-1})x_n = \\ &= [\mathcal{K}(f)_{n-1}(x_1 \odot \cdots \odot x_{n-2} \odot -) - \nabla_{\mathcal{K}(f)_{n-2}(x_1 \odot \cdots \odot x_{n-2})}, \nabla_{x_{n-1}}](x_n) - \mathcal{K}(f)_{n-1}(x_1 \odot \cdots \odot x_{n-1})x_n = \\ &= [\mathcal{K}(f)_{n-1}(x_1 \odot \cdots \odot x_{n-2} \odot -), \nabla_{x_{n-1}}](x_n) - \mathcal{K}(f)_{n-1}(x_1 \odot \cdots \odot x_{n-1})x_n = \\ &= \mathcal{K}(f)_{n-1}(x_1 \odot \cdots \odot x_{n-2} \odot x_{n-1})x_n - \mathcal{K}(f)_{n-1}(x_1 \odot \cdots \odot x_{n-2} \odot x_{n-1})x_n - \\ &\quad - (-1)^{|x_{n-1}||x_n|} \mathcal{K}(f)_{n-1}(x_1 \odot \cdots \odot x_{n-1} \odot x_n)x_{n-1}. \end{aligned}$$

But then both the left and the right hand side of Equation (4.2.1) satisfy the same recursive relation, so the claim follows inductively by the case $n = 1$.

Next we observe that the previous construction makes sense for left pre-Lie algebras.

Definition 4.2.1. A bilinear product $\diamond : V^{\otimes 2} \rightarrow V$ on a graded space V is Lie admissible if the associated commutator $[x, y] := x \diamond y - (-1)^{|x||y|} y \diamond x$ satisfies the Jacobi identity.

A graded left pre-Lie algebra (L, \triangleright) is a graded space L with a bilinear product $\triangleright : L^{\otimes 2} \rightarrow L$, such that the associator, defined by

$$A : L^{\otimes 3} \rightarrow L : x \otimes y \otimes z \rightarrow A(x, y, z) = (x \triangleright y) \triangleright z - x \triangleright (y \triangleright z),$$

is graded symmetric in the first two arguments, that is, $A(x, y, z) = (-1)^{|x||y|} A(y, x, z)$, $\forall x, y, z$. In terms of the left adjoint morphism $\nabla : L \rightarrow \text{End}(L) : x \rightarrow \{\nabla_x : y \rightarrow x \triangleright y\}$ and the associated commutator $[\cdot, \cdot] := L^{\wedge 2} \rightarrow L$ the left pre-Lie identity becomes

$$[\nabla_x, \nabla_y] = \nabla_{[x, y]} \quad \forall x, y \in L, \quad (4.2.3)$$

In particular \triangleright is Lie admissible (cf. the proof of the next proposition). A graded right pre-Lie algebra (L, \triangleleft) is a graded space L together with a bilinear product $\triangleleft : L^{\otimes 2} \rightarrow L$ such that the associator is graded symmetric in the last two arguments: we remark that this is true if and only if $x \triangleright y := -(-1)^{|x||y|} y \triangleleft x$ is a left pre-Lie product on L , in particular since the associated commutator of \triangleright and \triangleleft is the same right pre-Lie products are also Lie admissible.

Proposition 4.2.2. The set of left pre-Lie products on L is in bijective correspondence with the set of graded Lie subalgebras $L^\nabla \subset \text{Aff}(L)$ such that $\text{Aff}(L) = \text{End}(L) \oplus L^\nabla$.

Proof. Given a graded Lie subalgebra $L^\nabla \subset \text{Aff}(L)$ as in the claim of the proposition, the composition of the inclusion $L^\nabla \subset \text{Aff}(L)$ and the projection $p_L : \text{Aff}(L) \rightarrow L : (f, x) \rightarrow x$ is an

isomorphism of graded spaces $L^\nabla \rightarrow L$: let $\sigma^\nabla : L \rightarrow L^\nabla$ be the inverse, thus $\sigma^\nabla(x) = (\nabla_x, x)$ for some $\nabla_x \in \text{End}(L)$ depending linearly on x , and since $L^\nabla \subset \text{Aff}(A)$ is a Lie subalgebra

$$\begin{aligned} [\sigma^\nabla(x), \sigma^\nabla(y)] &= [(\nabla_x, x), (\nabla_y, y)] = ([\nabla_x, \nabla_y], \nabla_x(y) - (-1)^{|x||y|}\nabla_y(x)) = \\ &= \sigma^\nabla p_L([\sigma^\nabla(x), \sigma^\nabla(y)]) = (\nabla_{\nabla_x(y) - (-1)^{|x||y|}\nabla_y(x)}, \nabla_x(y) - (-1)^{|x||y|}\nabla_y(x)), \end{aligned}$$

that is, the product $x \triangleright y := \nabla_x(y)$ satisfies the left pre-Lie identity (4.2.3). Conversely, if \triangleright is a left pre-Lie product then the graded morphism $\sigma^\nabla : L \rightarrow \text{Aff}(L) : x \rightarrow (\nabla_x, x)$ sends L isomorphically onto a graded Lie subalgebra $L^\nabla \subset \text{Aff}(L)$ satisfying the assumptions of the proposition. \square

By the proposition given a left pre-Lie product \triangleright on L we are in the setup of Theorem 4.1.6, thus by (nonabelian) higher derived brackets it is defined a morphism $\mathcal{K} : \text{End}(L) \rightarrow \overline{\text{CE}}(L)$ of graded Lie algebras, moreover, by the previous discussion when the left pre-Lie product is an associative and graded commutative product this is essentially the usual construction of Koszul brackets. In fact this can be generalized further: we have the embedding of graded Lie algebras $\text{Aff}(L) \hookrightarrow \text{CE}(L) : (f, x) \rightarrow (\sigma_x, f, 0, \dots, 0, \dots)$, if L^∇ satisfies the assumptions of the proposition and we denote by the same symbol its image in $\text{CE}(L)$ now we have $\text{CE}(L) = \overline{\text{CE}}(L) \oplus L^\nabla$ still as in the assumptions of Theorem 4.1.6, thus by higher derived brackets an endomorphism $\mathcal{K} : \text{CE}(L) \rightarrow \text{CE}(L)$ sending the graded Lie subalgebra $\overline{\text{CE}}(L) \subset \text{CE}(L)$ into itself (notice that when the product \triangleright is trivial we recover Example 4.1.18, thus in this case \mathcal{K} is the identity). Lemma 4.1.16 implies that the morphism \mathcal{K} restricts to the previously defined one on $\text{End}(L) \subset \overline{\text{CE}}(L)$.

Definition 4.2.3. Given a graded Left pre-Lie algebra (L, \triangleright) , we call the morphism of graded Lie algebras $\mathcal{K} : \text{CE}(L) \rightarrow \text{CE}(L) : Q \rightarrow \mathcal{K}(Q) = (\mathcal{K}(Q)_0, \dots, \mathcal{K}(Q)_n, \dots)$ as in the previous discussion the *Koszul transform* associated to the pre-Lie product \triangleright , and we call the Taylor coefficients $\mathcal{K}(Q)_n$ the *Koszul brackets* on (L, \triangleright) associated to Q .

The following easy lemma has an interesting consequence, cf. Proposition 4.2.6.

Lemma 4.2.4. *Given a graded left pre-Lie algebra (L, \triangleright) together with $q_n : L^{\odot n} \rightarrow L$, $n \geq 0$, the associated Koszul brackets satisfy $\mathcal{K}(q_n)_i = 0$ for all $0 \leq i < n$ and $\mathcal{K}(q_n)_n = q_n$.*

Proof. The fact that $\mathcal{K}(q_n)_i = 0$ for $i < n$ follows from the definition via higher derived brackets and the observation that all terms in the summation of Definition 4.1.3 vanish, since for $j \leq i < n$ and $x_1, \dots, x_j \in L$ we have $[\dots [q_n, \sigma_{x_1} + \nabla_{x_1}] \dots, \sigma_{x_j} + \nabla_{x_j}] \in \overline{\text{CE}}(L)$, moreover by the same observation the only non-vanishing contribute to $\mathcal{K}(q_n)_n(x_1 \odot \dots \odot x_n)$ is given by

$$\begin{aligned} \mathcal{K}(q_n)_n(x_1 \odot \dots \odot x_n) &= \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) [\dots [q_n, \sigma_{x_{\sigma(1)}} + \nabla_{x_{\sigma(1)}}] \dots, \sigma_{x_{\sigma(n)}} + \nabla_{x_{\sigma(n)}}](1) = \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) [\dots [q_n, \sigma_{x_{\sigma(1)}}] \dots, \sigma_{x_{\sigma(n)}}](1) = \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) q_n(x_{\sigma(1)} \odot \dots \odot x_{\sigma(n)}) = q_n(x_1 \odot \dots \odot x_n) \end{aligned}$$

\square

Example 4.2.5. The explicit computation of the Koszul brackets on a general pre-Lie algebra seems quite intricate. For a constant coderivation σ_x we obtain the following formula

$$\begin{aligned} \mathcal{K}(\sigma_x)_n(x_1 \odot \dots \odot x_n) &= \\ &= \sum_{k=0}^n \frac{B_{n-k}}{k!(n-k)!} \sum_{\sigma \in S_n} \varepsilon(\sigma) (-1)^{k+|x| \sum_{j \leq k} |x_j|} [\dots [x_{\sigma(1)} \triangleright (\dots \triangleright (x_{\sigma(k)} \triangleright x) \dots), x_{\sigma(k+1)}] \dots, x_{\sigma(n)}] \end{aligned}$$

For instance if we denote by $*$: $L^{\odot 2} \rightarrow L$ the symmetrized product $x*y = \frac{1}{2}(x \triangleright y + (-1)^{|x||y|}y \triangleright x)$ we have $\mathcal{K}(\sigma_x)_1(x_1) = -\frac{1}{2}[x, x_1] - (-1)^{|x||x_1|}x_1 \triangleright x = -x * x_1$. If \triangleright is an associative and graded commutative product the previous formula reduces to $\mathcal{K}(\sigma_x)_n(x_1 \odot \cdots \odot x_n) = (-1)^n x x_1 \cdots x_n$. Given $f \in \text{End}(L)$ by the lemma $\mathcal{K}(f)_0 = 0$ and $\mathcal{K}(f)_1 = f$, the quadratic Koszul bracket $\mathcal{K}(f)_2$ is given by

$$\mathcal{K}(f)_2(x \odot y) = f(x * y) - f(x) * y - (-1)^{|f||x|}x * f(y).$$

We have already observed how the higher Koszul brackets are (essentially) the usual ones when \triangleright is an associative and graded commutative product, if \triangleright is only associative we should recover the hierarchy of higher Koszul brackets considered by Bering in [7]. As a last example we consider a graded commutative algebra (A, \cdot) and an antisymmetric bracket $[\cdot, \cdot] : A[-1]^{\wedge 2} \rightarrow A[-1]$: let $\{\cdot, \cdot\} : A^{\odot 2} \rightarrow A$ be the symmetric bracket associated via décalage, the first Koszul bracket not given in Lemma 4.2.4 is

$$\begin{aligned} \mathcal{K}(\{\cdot, \cdot\})_3(x \odot y \odot z) &= [[[\{\cdot, \cdot\}, \sigma_x + \nabla_x], \sigma_y + \nabla_y], \sigma_z + \nabla_z](1) = \\ &= [[[\{\cdot, \cdot\}, \nabla_x], \sigma_y], \sigma_z](1) + [[[\{\cdot, \cdot\}, \sigma_x], \nabla_y], \sigma_z](1) + [[[\{\cdot, \cdot\}, \sigma_x], \sigma_y], \nabla_z](1) = \\ &= [\{\cdot, \cdot\}, \nabla_x](y \odot z) + [\{x, -\}, \nabla_y](z) - \{x, y\}z \end{aligned}$$

The term $[\{x, -\}, \nabla_y](z) - \{x, y\}z = \{x, yz\} - \{x, y\}z - (-1)^{(|x|+1)|y|}y\{x, z\}$ measures how far is $[sx, -]$ to be a derivation with respect to the product \cdot on A , while the term $[\{\cdot, \cdot\}, \nabla_x](y \odot z)$ measures how far is ∇_x to be a derivation with respect to the bracket $[\cdot, \cdot]$ on $A[-1]$, in particular, if $(A, \cdot, [\cdot, \cdot])$ is a graded Poisson algebra of degree (-1) (Definition 6.1.1) only the latter remains.

Proposition 4.2.6. *The Koszul transform $\mathcal{K} : \text{CE}(L) \rightarrow \text{CE}(L)$ associated to a left pre-Lie product on the space L is an isomorphism of graded Lie algebras.*

Proof. On the one hand Lemma 4.2.4 tells us that the first non-vanishing Taylor coefficient of Q and $\mathcal{K}(Q)$ is the same, thus \mathcal{K} is injective; on the other hand given $R = (r_0, \dots, r_n, \dots)$ we put $q_0 = r_0$ and then recursively $q_n = r_n - \mathcal{K}(q_1, \dots, q_{n-1}, 0, \dots, 0, \dots)_n$, then the lemma shows that $Q = (q_0, \dots, q_n, \dots)$ is such that $\mathcal{K}(Q) = R$, thus \mathcal{K} is surjective. In fact

$$\mathcal{K}(Q)_n = \mathcal{K}(q_1, \dots, q_n, 0, \dots, 0, \dots)_n = q_n + \mathcal{K}(q_1, \dots, q_{n-1}, 0, \dots, 0, \dots)_n = r_n.$$

□

Definition 4.2.7. Given a graded Left pre-Lie algebra (L, \triangleright) we call the automorphism of graded Lie algebras $\mathcal{K}^{-1} : \text{CE}(L) \rightarrow \text{CE}(L) : Q \rightarrow \mathcal{K}^{-1}(Q) = (\mathcal{K}^{-1}(Q)_0, \dots, \mathcal{K}^{-1}(Q)_n, \dots)$ the *Kapranov transform* associated to \triangleright , and we call the Taylor coefficients $\mathcal{K}^{-1}(Q)_n$ the *Kapranov brackets* on (L, \triangleright) associated to Q .

Remark 4.2.8. In this remark we show how both \mathcal{K} and \mathcal{K}^{-1} can be obtained via an abelian higher derived brackets construction: let $\text{CE}_0(L)$ be the abelian Lie algebra spanned by constant coderivations, then $\mathcal{K}(\text{CE}_0(L)) \subset \text{CE}(L)$ is an abelian Lie subalgebra such that we continue to have $\text{CE}(L) = \overline{\text{CE}}(L) \oplus \mathcal{K}(\text{CE}_0(L))$ as in the the assumptions of Voronov's construction of higher derived brackets: the induced morphism $\Phi : \text{CE}(L) \rightarrow \text{CE}(L)$ is given by $\Phi(Q)_0(1) = q_0(1)$

$$\begin{aligned} \Phi(Q)_n(x_1 \odot \cdots \odot x_n) &= [\cdots [Q, \mathcal{K}(\sigma_{x_1})] \cdots, \mathcal{K}(\sigma_{x_n})](1) = \mathcal{K}([\cdots [\mathcal{K}^{-1}(Q), \sigma_{x_1}] \cdots, \sigma_{x_n}]) (1) = \\ &= [\cdots [\mathcal{K}^{-1}(Q), \sigma_{x_1}] \cdots, \sigma_{x_n}](1) = \mathcal{K}^{-1}(Q)_n(x_1 \odot \cdots \odot x_n), \end{aligned}$$

that is, in this case the morphism Φ induced by higher derived brackets coincides with \mathcal{K}^{-1} . More in general, by the same argument we see that given $j \in \mathbb{Z}$ and the morphism $\mathcal{K}^j : \text{CE}(L) \rightarrow \text{CE}(L)$ of graded Lie algebras, this is induced by an abelian higher derived brackets construction applied to $\text{CE}(L) = \overline{\text{CE}}(L) \oplus \mathcal{K}^{-j}(\text{CE}_0(L))$.

By the previous remark and Lemma 4.1.17 we conclude that

Proposition 4.2.9. *Given an $L_\infty[1]$ algebra structure $Q \in \overline{\text{CE}}(L)$ on L , the three $L_\infty[1]$ algebras (L, Q) , $(L, \mathcal{K}(Q))$ and $(L, \mathcal{K}^{-1}(Q))$ are all isomorphic to each other.*

Corollary 4.2.10. *Given $d \in \text{End}(L)$ such that $|d| = 1$ and $d^2 = 0$, the $L_\infty[1]$ algebras $(L, \mathcal{K}(d))$ and $(L, \mathcal{K}^{-1}(d))$ are homotopy abelian.*

In the next proposition we find an interesting characterization of Koszul and Kapranov brackets, cf. at the end of the section for an interpretation in terms of L_∞ extensions. Let (L, \triangleright) be a graded left pre-Lie algebra, $\sigma^\nabla : L \rightarrow \text{CE}(L)$ the embedding $L \xrightarrow{\cong} L^\nabla \hookrightarrow \text{CE}(L)$ and $\Phi : L \rightarrow \text{CE}(L)$ the morphism associated to the graded Lie algebra structure $(L, [\cdot, \cdot])$ on L via higher derived brackets as in Remark 4.1.11.

Proposition 4.2.11. *The Koszul transform $\mathcal{K} : \text{CE}(L) \rightarrow \text{CE}(L)$ is the only morphism of graded Lie algebras sending the graded Lie subalgebra $\overline{\text{CE}}(L) \subset \text{CE}(L)$ into itself and making the diagram on the left commutative*

$$\begin{array}{ccc} \text{CE}(L) & \xrightarrow{\mathcal{K}} & \text{CE}(L) \\ & \swarrow \sigma^\nabla & \nearrow \Phi \\ & L & \end{array} \qquad \begin{array}{ccc} \text{CE}(L) & \xleftarrow{\mathcal{K}^{-1}} & \text{CE}(L) \\ & \swarrow \sigma^\nabla & \nearrow \Phi \\ & L & \end{array}$$

Similarly, the Kapranov transform $\mathcal{K}^{-1} : \text{CE}(L) \rightarrow \text{CE}(L)$ is the only morphism of graded Lie algebras sending the graded Lie subalgebra $\overline{\text{CE}}(L) \subset \text{CE}(L)$ into itself and making the diagram on the right commutative.

Proof. The fact that \mathcal{K} (and thus also \mathcal{K}^{-1}) makes the previous diagram commutative follows by Lemma 4.1.16 applied to

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xlongequal{\quad} & L \\ \downarrow & & \downarrow \sigma^\nabla & & \downarrow \sigma^\nabla \\ \overline{\text{CE}}(L) & \longrightarrow & \text{CE}(L) & \longleftarrow & L^\nabla \end{array}$$

which shows that

$$\begin{array}{ccc} L & \xrightarrow{\Phi} & \text{CE}(L) \\ \downarrow \sigma^\nabla & & \downarrow \cong \\ \text{CE}(L) & \xrightarrow{\Phi} & \text{CE}(L^\nabla) \end{array}$$

is commutative: but by definition \mathcal{K} is the composition of the bottom arrow and the inverse of the right vertical arrow.

The idea behind uniqueness is that the identities $\mathcal{K}(Q)_0 = q_0$, $\mathcal{K}([Q, \sigma_x + \nabla_x]) = [\mathcal{K}(Q), \Phi(x)]$ for all $x \in L$ determine the Taylor coefficients $\mathcal{K}(Q)_n$ recursively and similarly for \mathcal{K}^{-1} , cf. the proof of the next proposition. We give a more indirect argument: the decomposition $\text{CE}(L) = \overline{\text{CE}}(L) \oplus \Phi(L)$ satisfies the assumptions of Theorem 4.1.6, thus induces via higher derived brackets a morphism of graded Lie algebras $\mathcal{K}^\Phi : \text{CE}(L) \rightarrow \text{CE}(\Phi(L)) \xrightarrow{\cong} \text{CE}(L)$. If $\varphi : \text{CE}(L) \rightarrow \text{CE}(L)$ is such that $\sigma^\nabla = \varphi \circ \Phi : L \rightarrow \text{CE}(L)$ and moreover φ sends $\overline{\text{CE}}(L)$ into itself we can form a commutative

diagram

$$\begin{array}{ccccc}
 \overline{\text{CE}}(L) & \longrightarrow & \text{CE}(L) & \longleftarrow & \Phi(L) \\
 \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi \\
 \overline{\text{CE}}(L) & \longrightarrow & \text{CE}(L) & \longleftarrow & L^\nabla
 \end{array}
 \begin{array}{c}
 \swarrow \Phi \\
 L \\
 \searrow \sigma^\nabla
 \end{array}$$

and then Lemma 4.1.16 implies that the diagram

$$\begin{array}{ccc}
 \text{CE}(L) & & \\
 \downarrow \varphi & \searrow \mathcal{K}^\Phi & \\
 & & \text{CE}(L) \\
 \uparrow \mathcal{K} & & \\
 \text{CE}(L) & &
 \end{array}$$

is commutative. In particular in the previous discussion we can take $\varphi = \mathcal{K}^{-1}$, which shows that $\mathcal{K}^\Phi = \text{id}_{\text{CE}(L)}$: but then the uniqueness claim for \mathcal{K}^{-1} also follows by the previous diagram, since any φ as above has to be a left inverse to the isomorphism \mathcal{K} , thus it has to be \mathcal{K}^{-1} . To prove uniqueness for \mathcal{K} we consider $\varphi : \text{CE}(L) \rightarrow \text{CE}(L)$ filling a similar diagram as the previous one with the direction of the vertical arrows reversed, then again by Lemma 4.1.16 we see that $\mathcal{K} = \mathcal{K}^\Phi \circ \varphi = \varphi$. \square

The previous proposition allows to compare Definition 4.2.7 of Kapranov brackets with the one we gave in [4].

Proposition 4.2.12. *Let (L, \triangleright) be a graded left pre-Lie algebra. For all $d \in \text{Der}(L, [\cdot, \cdot])$ the Kapranov's brackets $\mathcal{K}^{-1}(d)_n : L^{\odot n} \rightarrow L$, $n \geq 1$, admit the following recursive definition*

$$\begin{cases} \mathcal{K}^{-1}(d)_1 = d \\ \mathcal{K}^{-1}(d)_2(x \odot y) = \nabla_{dx}(y) - [d, \nabla_x](y) \\ \mathcal{K}^{-1}(d)_{n+1}(x \odot y_1 \odot \cdots \odot y_n) = -[\mathcal{K}^{-1}(d)_n, \nabla_x](y_1 \odot \cdots \odot y_n) \end{cases} \quad \text{for } n \geq 2. \quad (4.2.4)$$

Proof. Given $d \in \text{Der}(L, [\cdot, \cdot])$, by Lemma 4.2.4 we have $\mathcal{K}^{-1}(d)_0 = 0$ and $\mathcal{K}^{-1}(d)_1 = d$: according to Theorem 4.1.6 and Remark 4.1.11 we see that $[d, \Phi(x)] = \Phi(dx)$ for all $x \in L$, hence by the previous proposition we also see that $[\mathcal{K}^{-1}(d), \sigma_x + \nabla_x] = \mathcal{K}^{-1}([d, \Phi(x)]) = \mathcal{K}^{-1}(\Phi(dx)) = \sigma_{dx} + \nabla_{dx}$ for all $x \in L$, and thus

$$[\mathcal{K}^{-1}(d)_2, \sigma_x] + [\mathcal{K}^{-1}(d)_1, \nabla_x] = \nabla_{dx} \quad \text{and for } n \geq 2 \quad [\mathcal{K}^{-1}(d)_{n+1}, \sigma_x] + [\mathcal{K}^{-1}(d)_n, \nabla_x] = 0.$$

It is clear that the above is equivalent to the recursion in the claim of the proposition. \square

Remark 4.2.13. We notice how it is not obvious that the above recursion is well defined, since it is not a priori obvious with the previous definition that the $(n+1)$ -bracket is graded symmetric given graded symmetry of the first n . We give a direct proof of this fact following [4], Proposition 4.3, which also illustrates nicely the role played by the left pre-Lie identity (4.2.3).

We rewrite $\mathcal{K}^{-1}(d)_2$ as

$$\mathcal{K}^{-1}(d)_2(x \odot y) = \nabla_{dx}(y) - [d, \nabla_x](y) = dx \triangleright y + (-1)^{|x||d|} x \triangleright dy - d(x \triangleright y).$$

In other words, $\mathcal{K}^{-1}(d)_2$ measures how far is d from satisfying the Leibniz rule with respect to the pre-Lie product \triangleright . Thus a straightforward computation shows

$$\mathcal{K}^{-1}(d)_2(x \odot y) - (-1)^{|x||y|} \mathcal{K}^{-1}(d)_2(y \odot x) = [dx, y] + (-1)^{|x||d|} [x, dy] - d[x, y] = 0,$$

since $d \in \text{Der}(L, [\cdot, \cdot])$. The recursive definition implies that $\mathcal{K}^{-1}(d)_3$ is graded symmetric in the last two arguments, so it suffices to show that it is also graded symmetric in the first two. We notice that

$$\mathcal{K}^{-1}(d)_3(x \odot y \odot z) = -[\mathcal{K}^{-1}(d)_2, \nabla_x](y \odot z) = -[[\mathcal{K}^{-1}(d)_2, \nabla_x], \sigma_y](z),$$

hence graded symmetry of $\mathcal{K}^{-1}(d)_3$ follows from the following computation

$$\begin{aligned} & [[\mathcal{K}^{-1}(d)_2, \nabla_x], \sigma_y] - (-1)^{|x||y|} [[\mathcal{K}^{-1}(d)_2, \nabla_y], \sigma_x] = \\ & = [\mathcal{K}^{-1}(d)_2, [\nabla_x, \sigma_y]] + (-1)^{|x||y|} [[\mathcal{K}^{-1}(d)_2, \sigma_y], \nabla_x] - (-1)^{|x||y|} [\mathcal{K}^{-1}(d)_2, [\nabla_y, \sigma_x]] - [[\mathcal{K}^{-1}(d)_2, \sigma_x], \nabla_y] = \\ & = [\mathcal{K}^{-1}(d)_2, \sigma_{\nabla_x(y)}] + (-1)^{|x||y|} [\nabla_{dy} - [d, \nabla_y], \nabla_x] - (-1)^{|x||y|} [\mathcal{K}^{-1}(d)_2, \sigma_{\nabla_y(x)}] - [\nabla_{dx} - [d, \nabla_x], \nabla_y] = \\ & = \nabla_{d\nabla_x(y)} - (-1)^{|x||y|} \nabla_{d\nabla_y(x)} - [\nabla_{dx}, \nabla_y] + (-1)^{|x||y|} [\nabla_{dy}, \nabla_x] + \\ & + [[d, \nabla_x], \nabla_y] - (-1)^{|x||y|} [[d, \nabla_y], \nabla_x] - [d, \nabla_{\nabla_x(y)}] + (-1)^{|x||y|} [d, \nabla_{\nabla_y(x)}] = \\ & = \nabla_{d[x, y]} - \nabla_{[dx, y]} - (-1)^{|x||d|} \nabla_{[x, dy]} + [d, [\nabla_x, \nabla_y] - \nabla_{[x, y]}] = 0, \end{aligned}$$

Given $n \geq 3$, we suppose inductively to have showed graded symmetry of $\mathcal{K}^{-1}(d)_i$ for all $2 \leq i \leq n$. To show graded symmetry of $\mathcal{K}^{-1}(d)_{n+1}$ we only have to show it for the last two arguments, where it follows by a similar (and actually a little simpler) computation as the one before

$$[[\mathcal{K}^{-1}(d)_n, \nabla_x], \sigma_y] - (-1)^{|x||y|} [[\mathcal{K}^{-1}(d)_n, \nabla_y], \sigma_x] = [\mathcal{K}^{-1}(d)_{n-1}, [\nabla_x, \nabla_y] - \nabla_{[x, y]}] = 0 \quad \forall x, y \in L.$$

The morphisms of graded Lie algebras $\Phi : L \rightarrow \text{CE}(L)$ and $\sigma^\nabla : L \rightarrow \text{CE}(L)$ classify, as in Proposition 1.3.34, cf. also Notation 1.3.35, a pair of $L_\infty[1]$ extensions of fiber $(L, 0)$ and base the graded Lie algebra $s^{-1}L$ (seen as usual via décalage). In fact these are two very natural $L_\infty[1]$ extension associated to the graded pre-Lie algebra (L, \triangleright) : the $L_\infty[1]$ extension classified by Φ is strictly isomorphic to $0 \rightarrow C(\Delta_1, \partial\Delta_1; s^{-1}L) \rightarrow C(\Delta_1, e_1; s^{-1}L) \rightarrow C(e_1; s^{-1}L) \rightarrow 0$, cf. Proposition 3.3.6, where e_1 denotes the vertex $\Delta_0 \rightarrow \Delta_1 : [0] \rightarrow [1]$, that is, $s^{-1}L \times_\Phi L$ is essentially the mapping cocone $\text{coC}(\text{id}_{s^{-1}L})$ of the identity $\text{id}_{s^{-1}L} : s^{-1}L \rightarrow s^{-1}L$, as in Definition 3.3.3²; as for the morphism $\sigma^\nabla : L \rightarrow \text{CE}(L)$ this classifies the extension $0 \rightarrow L \rightarrow s^{-1}L_\triangleright \rightarrow s^{-1}L \rightarrow 0$, where L_\triangleright is the dg Lie algebra as in the following proposition.

Proposition 4.2.14. *There is a bijective correspondence between the set of left pre-Lie products \triangleright on L and the set of dg Lie algebra structures $L_\triangleright = (L \times sL, \delta, [\cdot, \cdot])$ on the space $L \times sL$ with the differential $\delta(x, sy) = (0, -sx)$ and such that $[L, L] \subset L$, $[L, sL] \subset sL$, $[sL, sL] = 0$.*

Proof. It is easy to see that a left pre-Lie product \triangleright on L and a bracket with the required properties on $L_\triangleright = sL \times L$ determine each others via $[x, sy] = (-1)^{|x|} s(x \triangleright y)$, $[x, y] = x \triangleright y - (-1)^{|x||y|} y \triangleright x$. \square

²We say essentially since we defined the mapping cocone of a morphism F as the homotopy equalizer $E^h(0, F)$, so to be precise the mapping cocone $\text{coC}(\text{id}_{s^{-1}L})$ would be the $L_\infty[1]$ algebra $C(\Delta_1, e_0; s^{-1}L)$.

Comparing with Proposition 4.2.11 and Lemma 1.3.36, it is natural to conjecture that there should be an $L_\infty[1]$ automorphism $F : (L, 0) \rightarrow (L, 0)$ of the fiber such that, with the notations of Lemma 1.3.36, we have $F_*\sigma^\nabla = \Phi$, $\mathcal{K} = F - F^{-1} : \text{CE}(L) \rightarrow \text{CE}(L)$. Such an F would sit in a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \longrightarrow & s^{-1}L_\triangleright & \longrightarrow & s^{-1}L \longrightarrow 0 \\ & & \cong \downarrow F & & \downarrow \tilde{F} & & \parallel \\ 0 & \longrightarrow & L & \longrightarrow & C(\Delta_1, e_1; s^{-1}L) & \longrightarrow & s^{-1}L \longrightarrow 0 \end{array}$$

of $L_\infty[1]$ algebras and $L_\infty[1]$ morphisms: once we identify $s^{-1}L_\triangleright = C(\Delta_1, e_1; s^{-1}L) = s^{-1}L \times L$ as graded spaces, the $L_\infty[1]$ morphism \tilde{F} is determined by F according to $\tilde{f}_1(s^{-1}m, l) = (s^{-1}m, f_1(l))$ and $\tilde{f}_n((s^{-1}m_1, l_1) \odot \cdots \odot (s^{-1}m_n, l_n)) = (0, f_n(l_1 \odot \cdots \odot l_n))$ for $n \geq 2$. Since not only the underlying spaces, but also the underlying complexes of $s^{-1}L_\triangleright$ and $C(\Delta_1, e_1; s^{-1}L)$ identify, we may moreover require that $f_1 = \text{id}_L$. Then we claim that if an $F : (L, 0) \rightarrow (L, 0)$ with the desired properties exists, it can be uniquely determined by the fact that it makes the corresponding \tilde{F} an $L_\infty[1]$ morphism. This is because, denoting by $R = (r_1, \dots, r_n, \dots)$ and $Q = (q_1, q_2, 0, \dots, 0, \dots)$ the $L_\infty[1]$ structures on $C(\Delta_1, e_1; s^{-1}L)$ and $s^{-1}L_\triangleright$ respectively, we would have

$$\begin{aligned} (0, f_n(l_1 \odot \cdots \odot l_n)) &= \tilde{f}_n q_1((s^{-1}l_1, 0) \odot (0, l_2) \odot \cdots \odot (0, l_n)) = \\ &= (\tilde{f}_n q_1 - r_1 \tilde{f}_n)((s^{-1}l_1, 0) \odot (0, l_2) \odot \cdots \odot (0, l_n)) = \\ &= (-\tilde{f}_{n-1} \bullet q_2 + \sum_{j=2}^n r_j \tilde{F}_n^j)((s^{-1}l_1, 0) \odot (0, l_2) \odot \cdots \odot (0, l_n)) \end{aligned} \quad (4.2.5)$$

where the last term only depends on f_2, \dots, f_{n-1} . Thus we obtain formulas specifying recursively the Taylor coefficients f_n for all $n \geq 2$: the point would be to show that this recursion is well defined, that is, graded symmetry of f_2, \dots, f_{n-1} also implies graded symmetry of f_n . We notice that for $n = 2$ the above becomes $f_2(l_1 \odot l_2) = -l_1 \triangleright l_2 + \frac{1}{2}[l_1, l_2] = -l_1 * l_2$ (where $*$ is as in Example 4.2.5), thus we get in fact a graded symmetric f_2 .

From this point on we consider the case when $L = A$ is a graded commutative algebra, and hope to treat the general pre-Lie case somewhere else. In this case the graded Lie algebra structure on A is abelian, and so is the $L_\infty[1]$ algebra structure on $C(\Delta_1, e_1; s^{-1}A)$, that is, $r_n = 0$ for all $n \geq 2$: thus the recursion (4.2.5) simplifies to

$$f_n(x_1 \odot \cdots \odot x_n) = - \sum_{\sigma \in S(2, n-2)} \varepsilon(\sigma) f_{n-1}(x_{\sigma(1)} x_{\sigma(2)} \odot x_{\sigma(3)} \odot \cdots \odot x_{\sigma(n)}).$$

For $n = 2$ this becomes $f_2(x \odot y) = -xy$, which we already saw, and for $n \geq 2$ we see inductively that it becomes $f_n(x_1 \odot \cdots \odot x_n) = (n-1)!(-1)^{n-1} x_1 \cdots x_n$, $n \geq 2$, which is clearly graded symmetric. In other words $f_n = (n-1)!(-1)^{n-1} m_n$, where $m_n : A^{\odot n} \rightarrow A$ is the iterated multiplication map.

Remark 4.2.15. As in [79], we may introduce the group $\text{Aut}(A)$ of automorphisms $F : \bar{S}(A) \rightarrow \bar{S}(A)$ of the graded coalgebra structures of the form $F = (\text{id}_A, k_2 m_2, \dots, k_n m_n, \dots)$, where $k_n \in \mathbb{K}$ are scalars and the m_n the iterated multiplications. We can associate to $F = (\text{id}_A, k_2 m_2, \dots, k_n m_n, \dots)$ its generating series $1 + t + \frac{k_2}{2!} t^2 + \cdots + \frac{k_n}{n!} t^n + \cdots \in \mathbb{K}[[t]]$, then it is not hard to verify that the generating series of a composition is the composition of the generating series, cf. [79]. In the case of $F = (\text{id}_A, -m_2, \dots, (n-1)!(-1)^{n-1} m_n, \dots)$ as in the previous paragraph, the associated series is the logarithmic series $1 + t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots$.

Summing up, given $F = (\text{id}_A, -m_2, \dots, (n-1)!(-1)^{n-1}m_n, \dots)$ as above, it is easy to see that in fact the associated $\tilde{F} : s^{-1}L_{\triangleright} = s^{-1}L \times L \rightarrow s^{-1}L \times L = C(\Delta_1, e_1; s^{-1}L)$ is an $L_{\infty}[1]$ morphism: the necessary identities follow by construction on mixed terms of type $s^{-1}x \otimes y_1 \odot \dots \odot y_n$ and trivially in the remaining cases, details are left to the reader. Comparing with Lemma 1.3.36 and Proposition 4.2.11, this shows the following result by Markl [79, 80].

Proposition 4.2.16. *Given a graded commutative algebra (A, \cdot) , the associated Koszul transform is given by $\mathcal{K} = F - F^{-1} : \text{CE}(A) \rightarrow \text{CE}(A)$, where $F : S(A) \rightarrow S(A)$ is the natural automorphism $F = (\text{id}_A, -m_2, \dots, (n-1)!(-1)^{n-1}m_n, \dots)$.*

4.2.1 Commutative BV_{∞} algebras

We give another interesting application of the results of Section 4.1. First we recall the definition of differential operators on graded commutative algebras and modules.

Definition 4.2.17. Given a graded commutative algebra (A, \cdot) , the space $\text{Diff}_0(A) \subset \text{End}(A)$ of differential operators of order ≤ 0 on A is the abelian Lie subalgebra spanned by the adjoint operators $\nabla_x : A \rightarrow A : y \rightarrow xy$. For $i \geq 1$ the space $\text{Diff}_i(A) \subset \text{End}(A)$ of differential operators of order $\leq i$ on A is defined recursively by $\text{Diff}_i(A) = \{f \in \text{End}(A) \text{ s.t. } [f, \text{Diff}_0(A)] \subset \text{Diff}_{i-1}(A)\}$.

Remark 4.2.18. An easy and well known inductive argument shows that for all $i, j \geq 0$ we have (where we put $\text{Diff}_{-1}(A) := 0$) $[\text{Diff}_i(A), \text{Diff}_j(A)] \subset \text{Diff}_{i+j-1}(A)$, moreover if we denote by \circ the composition product on $\text{End}(A)$ we also have $\text{Diff}_i(A) \circ \text{Diff}_j(A) \subset \text{Diff}_{i+j}(A)$: in particular $\text{Diff}(A) = \bigcup_{i \geq 0} \text{Diff}_i(A)$ is a graded Poisson subalgebra of the graded Poisson algebra - cf. Definition 6.1.1 - $(\text{End}(A), \circ, [\cdot, \cdot])$.

Recall the two constructions of Koszul brackets explained at the beginning of the previous section (namely, $\mathcal{K} : \text{End}(A) \rightarrow \overline{\text{CE}}(A)$ and $\mathcal{K}_{1_A} : \text{End}(A) \rightarrow \text{CE}(A)$).

Lemma 4.2.19. *Given a graded commutative algebra (A, \cdot) with a unit $1_A \in A$ together with an operator $f \in \text{End}(A)$ the following are equivalent conditions*

$$\begin{aligned} \mathcal{K}_{1_A}(f)_{i+1} = 0 & \quad \Leftrightarrow \quad \mathcal{K}_{1_A}(f)_n = 0 \quad \forall n > i & \quad \Leftrightarrow \quad f \in \text{Diff}_i(A), \\ \mathcal{K}(f)_{i+1} = 0 & \quad \Leftrightarrow \quad \mathcal{K}(f)_n = 0 \quad \forall n > i & \quad \Leftrightarrow \quad f \in \text{Diff}_i(A) \text{ and } f(1_A) = 0. \end{aligned}$$

Proof. The fact that $\mathcal{K}(f)_{i+1} = 0$ if and only if $\mathcal{K}(f)_n = 0$ for all $n > i$ follows from the recursive formula defining the Koszul brackets, and similarly for $\mathcal{K}_{1_A}(f)_{i+1} = 0$ if and only if $\mathcal{K}_{1_A}(f)_n = 0$ for all $n > i$. We have $f \in \text{Diff}_0(A)$ if and only if $f = \nabla_{f(1_A)}$ if and only if $\mathcal{K}_{1_A}(f)_1 = 0$, moreover $f \in \text{Diff}_0(A)$ and $f(1_A) = 0$ is equivalent to $0 = f = \mathcal{K}(f)_1$. For $x \in A$ we have that $\mathcal{K}_{1_A}(\nabla_x) = \sigma_x$ is a constant coderivation, moreover by definition $f \in \text{Diff}_i(A)$ if and only if $[f, \nabla_x] \in \text{Diff}_{i-1}(A)$ for all $x \in A$, and we see inductively that this is true if and only if the Taylor coefficient $\mathcal{K}_{1_A}([f, \nabla_x])_i = [\mathcal{K}_{1_A}(f)_{i+1}, \sigma_x]$ vanishes for all $x \in A$, that is, if and only if $\mathcal{K}_{1_A}(f)_{i+1} = 0$. In light of equation (4.2.1) it only remains to prove $\mathcal{K}(f)_{i+1} = 0 \Rightarrow f(1_A) = 0$: an easy induction using the recursive definition of the brackets shows $\mathcal{K}_{1_A}(f)_{n+1}(x_1 \odot \dots \odot x_n \odot 1_A) = 0$ for all $n \geq 0$ and $x_1, \dots, x_n \in A$, in particular if $\mathcal{K}(f)_{i+1} = 0$ equation (4.2.1) with $n = i + 1$ and $x_1 = \dots = x_{i+1} = 1_A$ implies that $f(1_A) = 0$. \square

We recall the definition of commutative BV_{∞} algebras due to O. Kravshenko (this is a particular case of the notion of BV_{∞} algebras one obtains by resolving the operad of BV algebras via the Koszul duality machinery [72]).

Definition 4.2.20. A commutative BV_∞ algebra $(A, d = \Delta_0, \Delta_1, \dots, \Delta_i, \dots)$ of degree k , where k is an odd integer, consists of a commutative dg algebra (A, d, \cdot) with unit 1_A and for all $i \geq 1$ an operator $\Delta_i \in \text{End}^{1-i(k+1)}(A)$ on A of degree $1 - i(k+1)$ such that

- (1) For all $i \geq 0$ we have $\mathcal{K}_{i+2}(\Delta_i) = 0$ (by the previous lemma, for $i = 0$ this just says that $d = \Delta_0$ is a derivation vanishing at the identity), and
- (2) if we denote by t a central variable of (even) degree $k+1$, then the degree one operator

$$\Delta = \Delta_0 + t\Delta_1 + \dots + t^i\Delta_i + \dots : A[[t]] \rightarrow A[[t]] : \sum_{j \geq 0} t^j \cdot x_j \rightarrow \sum_{n \geq 0} t^n \left(\sum_{i+j=n} \Delta_i(x_j) \right)$$

on the algebra of formal power series $A[[t]]$ squares to zero.

If $(A, d = \Delta_0, \Delta_1, \dots, \Delta_i, \dots)$ is a commutative BV_∞ algebra of (odd) degree k as in the previous definition there is an associated $L_\infty[1]$ structure on $A[k+1]$ as we now describe. Consider the algebra of formal Laurent series $A((t)) = \bigcup_{j \in \mathbb{Z}} t^j A[[t]]$, we denote by $p_+ : A((t)) \rightarrow A[[t]]$ the projection with kernel $A((t))^- = \bigcup_{j < 0} t^j A[[t]]$ and by $p_- = \text{id}_{A((t))} - p_+ : A((t)) \rightarrow A((t))^-$ the projection with kernel $A[[t]]$. The graded Lie algebra $M = \text{End}(A((t)))$ splits as $M = L \oplus B$, where L is the graded Lie subalgebra $L = \{f \in M \text{ s.t. } f(1_{A((t))}) \in A[[t]]\}$ and $B \subset M$ is the abelian Lie subalgebra of left adjoint operators $\nabla_{x(t)}$, where $x(t)$ varies in the subalgebra $A((t))^- \subset A((t))$. The decomposition $M = L \oplus B$ satisfies the assumptions of Voronov's construction of higher derived brackets, moreover, $\Delta : A[[t]] \rightarrow A[[t]]$ extends by $\mathbb{K}((t))$ -linearity to an operator which we denote by the same symbol $\Delta : A((t)) \rightarrow A((t))$, which is an element of L^1 such that $[\Delta, \Delta] = 0$: by higher derived brackets we have an $L_\infty[1]$ structure $\Phi(\Delta) = \Phi([\Delta, -])$ on B .

Proposition 4.2.21. Consider the linear embedding $\nabla_{t^{-1}, -} : A[k+1] \rightarrow B : s^{-k-1}x \rightarrow \nabla_{t^{-1}, x}$, its image $\nabla_{t^{-1}, -}(A[k+1])$ is an $L_\infty[1]$ subalgebra of $(B, \Phi(\Delta))$. The induced $L_\infty[1]$ algebra structure on $A[k+1]$ is given in Taylor coefficients by $(\Delta_0, \mathcal{K}(\Delta_1)_2, \dots, \mathcal{K}(\Delta_{n-1})_n, \dots)$.

Proof. Item (1) in the previous definition implies $\Delta_j(1_A) = 0$ for all $j \geq 0$, therefore as in the proof of the previous lemma $\mathcal{K}(\Delta_j)_{n+1}(x_1 \odot \dots \odot x_n \odot 1_A) = 0$ for all $n \geq 0$ and $x_1, \dots, x_n \in A$ and we see from equation (4.2.2) that $\mathcal{K}(\Delta_j)_n(x_1 \odot \dots \odot x_n) = [\dots [\Delta_j, \nabla_{x_1}] \dots, \nabla_{x_n}](1_A)$ for all $x_1, \dots, x_n \in A$. Now a straightforward computation shows

$$\begin{aligned} & [[\Delta_0 + t\Delta_1 + \dots + t^j\Delta_j + \dots, \nabla_{t^{-1}, x_1}] \dots, \nabla_{t^{-1}, x_n}](1_{A((t))}) = \\ & = \sum_{j \geq 0} t^{j-n} \cdot [\dots [\Delta_j, \nabla_{x_1}] \dots, \nabla_{x_n}](1_A) = \sum_{j \geq 0} t^{j-n} \cdot \mathcal{K}(\Delta_j)_n(x_1 \odot \dots \odot x_n), \end{aligned}$$

thus

$$\Phi(\Delta)_n(\nabla_{t^{-1}, x_1} \odot \dots \odot \nabla_{t^{-1}, x_n}) = \nabla_{\sum_{j=0}^{n-1} t^{j-n} \cdot \mathcal{K}(\Delta_j)_n(x_1 \odot \dots \odot x_n)},$$

and since $\mathcal{K}(\Delta_j)_n = 0$ for $j < n-1$ by the definition of commutative BV_∞ algebras and the previous lemma we finally see that

$$\Phi(\Delta)_n(\nabla_{t^{-1}, -}(x_1) \odot \dots \odot \nabla_{t^{-1}, -}(x_n)) = \nabla_{t^{-1}, -}(\mathcal{K}(\Delta_{n-1})_n(x_1 \odot \dots \odot x_n)).$$

□

Definition 4.2.22. A commutative BV_∞ algebra has the degeneration property if the projection $(A[[t]], \Delta) \rightarrow (A, \Delta_0) : x(t) \rightarrow x(0)$ is surjective in homology.

The following theorem, which was proved with different methods by Braun and Lazarev [12], generalizes the formality theorem from [96]. We follow the proof of this last result given in [52], Theorem 6.6.

Theorem 4.2.23. *If a commutative BV_∞ algebra $(A, \Delta_0, \Delta_1, \dots, \Delta_i, \dots)$ of (odd) degree k has the degeneration property, then the $L_\infty[1]$ algebra $(A[k+1], \Delta_0, \mathcal{K}(\Delta_1)_2, \dots, \mathcal{K}(\Delta_{n-1})_n, \dots)$ is homotopy abelian.*

Proof. Consider the decreasing \mathbb{Z} -filtration $F^j A((t)) = t^j A[[t]] \subset A((t))$ by Δ -closed subspaces: then the degeneration property is equivalent to injectivity in homology of $F^1 A((t)) \hookrightarrow F^0 A((t))$, which readily implies injectivity in homology of $F^j A((t)) \hookrightarrow F^{j-1} A((t))$ for all $j \in \mathbb{Z}$ (by $\mathbb{K}((t))$ -linearity of Δ) and then also of $F^j A((t)) \hookrightarrow F^k A((t))$ for all $j > k$. In particular $A[[t]] \hookrightarrow A((t))$ is injective in homology, so $p_- : A((t)) \rightarrow A((t))^-$ is surjective in homology: in turn this also implies that the projection $P : M \rightarrow B$ induced by the splitting $M = L \oplus B$ is surjective in homology, this follows by looking at the commutative diagram of dg spaces

$$\begin{array}{ccc} (M, [\Delta, \cdot]) & \xrightarrow{P} & (B, P[\Delta, \cdot]) \\ \downarrow \text{ev}_{1_{A((t))}} & & \downarrow \text{ev}_{1_{A((t))}} \\ (A((t)), \Delta) & \xrightarrow{p_-} & (A((t))^-, p_- \Delta) \end{array}$$

Since we are working over a field $H(\text{ev}_{1_{A((t))}}) = \text{ev}_{1_{H(A((t))})} : H(M) = \text{End}(H(A((t)))) \rightarrow H(A((t)))$ is surjective, while $\text{ev}_{1_{A((t))}} : B \rightarrow A((t))^-$ is an isomorphism. Thus by Corollary 4.1.15 the $L_\infty[1]$ structure $\Phi(\Delta) = \Phi([\Delta, \cdot])$ on B is homotopy abelian.

The thesis follows from Lemma 2.2.13 if we show that the embedding $\nabla_{t-1,-}$ from the previous proposition is injective in homology. We look at the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F^0 A((t)) & \longrightarrow & F^{-1} A((t)) & \longrightarrow & F^{-1} A((t))/F^0 A((t)) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A[[t]] & \longrightarrow & A((t)) & \longrightarrow & A((t))^- \longrightarrow 0 \end{array}$$

The rows are split exact and the middle vertical arrow is injective in homology, then so must be the right one: but this is isomorphic to $\nabla_{t-1,-} : A[k+1] \rightarrow B$. \square

4.2.2 Kapranov brackets in Kähler geometry

Let X be a hermitian manifold, we denote by \mathcal{A}_X the de Rham algebra of complex valued forms on X , and by $\mathcal{A}(T_X)$ the \mathcal{A}_X -module of smooth forms with coefficients in the tangent bundle T_X . We denote by $D = \nabla + \bar{\partial} : \mathcal{A}^{*,*}(T_X) \rightarrow \mathcal{A}^{*+1,*}(T_X) \oplus \mathcal{A}^{*,*+1}(T_X)$ the Chern connection on $\mathcal{A}(T_X)$ (that is, the only connection compatible with both the metric and the complex structure on T_X , see e.g. [57]). Finally, we denote by (z^1, \dots, z^d) a local system of holomorphic coordinates on some open $U \subset X$, together with the corresponding local frame $(\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^d})$ of T_X .

For $\alpha \in \mathcal{A}^{p,q}(T_X)$, the contraction operator $\mathbf{i}_\alpha \in \text{End}^{p-1,q}(\mathcal{A}(T_X))$ is defined as follows: locally $\alpha = \sum_i \alpha^i \otimes \frac{\partial}{\partial z^i}$ and $\beta = \sum_j \beta^j \otimes \frac{\partial}{\partial z^j}$, then locally $\mathbf{i}_\alpha(\beta) = \sum_j (\sum_i \alpha^i \wedge (\frac{\partial}{\partial z^i} \lrcorner \beta^j)) \otimes \frac{\partial}{\partial z^j}$, where we denote by \lrcorner the contraction of forms with vector fields. An easy computation, left to the reader, shows that $[\bar{\partial}, \mathbf{i}_\alpha] = \mathbf{i}_{\bar{\partial}\alpha}$.

We denote by $D_\alpha := [\mathbf{i}_\alpha, D] \in \text{End}^{p+q}(\mathcal{A}(T_X))$ and by $\nabla_\alpha := [\mathbf{i}_\alpha, \nabla] \in \text{End}^{p,q}(\mathcal{A}(T_X))$. Recall that $\mathcal{A}(T_X)$ carries a natural structure of (bi)graded Lie algebra induced by the bracket of vector fields, cf. for instance [76]: under the additional hypothesis $\mathbf{i}_\alpha(\beta) = \mathbf{i}_\beta(\alpha) = 0$, the usual Cartan identities $[\mathbf{i}_\alpha, \mathbf{i}_\beta] = 0$ and $[D_\alpha, \mathbf{i}_\beta] = \mathbf{i}_{[\alpha, \beta]}$ hold. This hypothesis is verified in particular for $\alpha, \beta \in \mathcal{A}^{0,*}(T_X)$: since $D_\alpha = [\mathbf{i}_\alpha, \nabla + \bar{\partial}] = \nabla_\alpha + (-1)^{|\alpha|} \mathbf{i}_{\bar{\partial}\alpha}$, in this case we also see that

$$[\nabla_\alpha, \mathbf{i}_\beta] = \mathbf{i}_{[\alpha, \beta]} \quad \forall \alpha, \beta \in \mathcal{A}^{0,*}(T_X).$$

We define $\triangleright : \mathcal{A}^{0,*}(T_X) \otimes \mathcal{A}^{0,*}(T_X) \rightarrow \mathcal{A}^{0,*}(T_X)$ by

$$\alpha \triangleright \beta := \nabla_\alpha(\beta) = D_\alpha(\beta) \quad \forall \alpha, \beta \in \mathcal{A}^{0,*}(T_X),$$

then \triangleright is a graded left pre-Lie product on $\mathcal{A}^{0,*}(T_X)$ precisely when the hermitian metric on X is Kähler. This can be seen as follows.

As well known [57], the curvature $D^2 \in \text{End}^2(\mathcal{A}(T_X))$ is \mathcal{A}_X -linear: this implies that if locally $\beta = \sum_j \beta^j \otimes \frac{\partial}{\partial z^j}$, then locally $D^2(\beta) = \sum_i (\sum_j \beta^j \wedge \Omega_j^i) \otimes \frac{\partial}{\partial z^i}$, where the forms $\Omega_j^i \in \mathcal{A}_X^2$ are defined by $D^2(\frac{\partial}{\partial z^j}) = \sum_i \Omega_j^i \otimes \frac{\partial}{\partial z^i}$. For the Chern connection we have moreover that $\Omega_j^i \in \mathcal{A}_X^{1,1}$, $\forall i, j$ [57]: this implies $D^2 = \frac{1}{2}[\nabla + \bar{\partial}, \nabla + \bar{\partial}] \in \text{End}^{1,1}(\mathcal{A}(T_X))$, thus

$$D^2 = [\bar{\partial}, \nabla], \quad [\nabla, \nabla] = 0.$$

By the Jacobi identity also $[\nabla_\alpha, \nabla] = [[\mathbf{i}_\alpha, \nabla], \nabla] = 0$, $\forall \alpha \in \mathcal{A}(T_X)$. For $\alpha, \beta \in \mathcal{A}^{0,*}(T_X)$ we see (by the Jacobi and Cartan identities) that

$$[\nabla_\alpha, \nabla_\beta] = [\nabla_\alpha, [\mathbf{i}_\beta, \nabla]] = [[\nabla_\alpha, \mathbf{i}_\beta], \nabla] = [\mathbf{i}_{[\alpha, \beta]}, \nabla] = \nabla_{[\alpha, \beta]}$$

Then the pre-Lie identity (4.2.3) holds if and only if the bracket on $\mathcal{A}^{0,*}(T_X)$ associated to \triangleright is the natural one induced by the bracket of vector fields, that is, if and only if for all $\forall \alpha, \beta \in \mathcal{A}^{0,*}(T_X)$ we have $[\alpha, \beta] = \nabla_\alpha(\beta) - (-1)^{|\alpha||\beta|} \nabla_\beta(\alpha) = D_\alpha(\beta) - (-1)^{|\alpha||\beta|} D_\beta(\alpha)$: in other words \triangleright is a left pre-Lie product on $\mathcal{A}^{0,*}(T_X)$ if and only if D is torsion free, but as well known [57] this is equivalent to the hermitian metric on X being Kähler.

We assume in the remainder that X is a Kähler manifold. The Dolbeault differential $\bar{\partial}$ induces a dg Lie algebra structure on the graded Lie algebra associated to $(\mathcal{A}^{0,*}(T_X), \triangleright)$: in fact, $(\mathcal{A}^{0,*}(T_X), \bar{\partial}, [\cdot, \cdot])$ is the Kodaira-Spencer dg Lie algebra controlling the infinitesimal deformations of the complex structure on X , cf. [76]. According to Corollary 4.2.10, the Kapranov brackets $\mathcal{K}^{-1}(\bar{\partial})_n$ induce a homotopy abelian $L_\infty[1]$ algebra structure on $\mathcal{A}^{0,*}(T_X)$.

Next we recall the construction of the $L_\infty[1]$ algebra structure on $\mathcal{A}^{0,*}(T_X)$ by Kapranov [56]. We can form the bundles of commutative coalgebras $\bar{S}(T_X) = \bigoplus_{n \geq 1} T_X^{\odot n}$, $S(T_X) = \bigoplus_{n \geq 0} T_X^{\odot n}$ and the bundles of Lie algebras $\overline{\text{CE}}(T_X) = \text{Coder}(\bar{S}(T_X))$, $\text{CE}(T_X) = \text{Coder}(S(T_X))$ over X as in sections 1.1 and 1.3 but working in the symmetric monoidal category of holomorphic vector bundles over X . As a holomorphic vector bundle $\overline{\text{CE}}(T_X) = \prod_{n \geq 1} \text{Hom}_{\mathcal{O}_X}(T_X^{\odot n}, T_X)$, and similarly $\text{CE}(T_X) = \prod_{n \geq 0} \text{Hom}_{\mathcal{O}_X}(T_X^{\odot n}, T_X)$. The bundle of Lie algebras structure on $\overline{\text{CE}}(T_X)$ induces a dg Lie algebra structure on the Dolbeault complex $\mathcal{A}^{0,*}(\overline{\text{CE}}(T_X)) = \prod_{n \geq 1} \mathcal{A}^{0,*}(\text{Hom}_{\mathcal{O}_X}(T_X^{\odot n}, T_X))$. Finally, there is a morphism of dg Lie algebras

$$\Psi : (\mathcal{A}^{0,*}(\overline{\text{CE}}(T_X)), \bar{\partial}, [\cdot, \cdot]) \rightarrow (\overline{\text{CE}}(\mathcal{A}^{0,*}(T_X)), [\bar{\partial}, \cdot], [\cdot, \cdot]),$$

where in the left hand side $\bar{\partial}$ is the Dolbeault differential on $\mathcal{A}^{0,*}(\overline{\text{CE}}(T_X))$ while in the right hand side $\bar{\partial}$ is the Dolbeault differential on $\mathcal{A}^{0,*}(T_X)$, regarded as a linear coderivation on $\bar{S}(\mathcal{A}^{0,*}(T_X))$.

The morphism Ψ sends $R_n \in \mathcal{A}^{0,*}(\text{Hom}(T_X^{\otimes n}, T_X))$ to $\Psi(R_n) : \mathcal{A}^{0,*}(T_X)^{\odot n} \rightarrow \mathcal{A}^{0,*}(T_X)$ given by the composition

$$\Psi(R_n) : \mathcal{A}^{0,*}(T_X)^{\odot n} \xrightarrow{R_n \otimes -} \mathcal{A}^{0,*}(\text{Hom}(T_X^{\otimes n}, T_X)) \otimes \mathcal{A}^{0,*}(T_X^{\otimes n}) \rightarrow \mathcal{A}^{0,*}(T_X),$$

where the second map is the natural contraction, and where in the first one we are also implicitly considering $\mathcal{A}^{0,*}(T_X)^{\odot n} \rightarrow \mathcal{A}^{0,*}(T_X^{\otimes n})$ induced by the wedge product of forms. We leave to the reader the easy verification that Ψ is indeed a morphism of dg Lie algebras. We notice that the brackets $\Psi(R_n)$ are $\mathcal{A}_X^{0,*}$ -multilinear in the following graded sense:

$$\Psi(R_n)(\alpha_1 \odot \cdots \odot (\omega \wedge \alpha_k) \odot \cdots \odot \alpha_n) = (-1)^{|\omega|(|R_n| + \sum_{j=1}^{k-1} |\alpha_j|)} \omega \wedge \Psi(R_n)(\alpha_1 \odot \cdots \odot \alpha_n), \quad (4.2.6)$$

$$\forall \alpha_1, \dots, \alpha_n \in \mathcal{A}^{0,*}(T_X), \omega \in \mathcal{A}_X^{0,*}.$$

Recall that the Chern connection $D = \nabla + \bar{\partial}$ on T_X induces the Chern connection on each one of the associated bundles $\text{Hom}(T_X^{\otimes n}, T_X)$, $n \geq 1$, which we still denote by the same symbol $D = \nabla + \bar{\partial} \in \text{End}^{1,0}(\mathcal{A}(\text{Hom}(T_X^{\otimes n}, T_X))) \oplus \text{End}^{0,1}(\mathcal{A}(\text{Hom}(T_X^{\otimes n}, T_X)))$. Following Kapranov [56], we define recursively a hierarchy of tensors $R_n \in \mathcal{A}^{0,1}(\text{Hom}(T_X^{\otimes n}, T_X))$, $n \geq 2$, starting with the curvature form $R_2 = \Omega = \sum_{i,j} \Omega_j^i dz^j \otimes \frac{\partial}{\partial z^i} \in \mathcal{A}^{1,1}(\text{End}(T_X)) \cong \mathcal{A}^{0,1}(\text{Hom}(T_X^{\otimes 2}, T_X))$, and then by

$$R_{n+1} = \nabla(R_n) \in \mathcal{A}^{1,1}(\text{Hom}(T_X^{\otimes n}, T_X)) \cong \mathcal{A}^{0,1}(\text{Hom}(T_X^{\otimes n+1}, T_X)). \quad (4.2.7)$$

It turns out ([56], it will also follow from Theorem 4.2.24 and Proposition ??), by torsion freeness of D , that the tensors R_n are symmetric in their holomorphic covariant indices: in other words, the above Recursion (4.2.7) actually defines a hierarchy $R_n \in \mathcal{A}^{0,1}(\text{Hom}(T_X^{\otimes n}, T_X))$, $n \geq 2$, which we can assemble to $R = (0, R_2, \dots, R_n, \dots) \in \mathcal{A}^{0,1}(\overline{\text{CE}}(T_X))$. Finally, in [56], Theorem 2.6, it is proved that $\bar{\partial}R + \frac{1}{2}[R, R] = 0$, that is, R is a Maurer-Cartan element of the dg Lie algebra $(\mathcal{A}^{0,*}(\overline{\text{CE}}(T_X)), \bar{\partial}, [\cdot, \cdot])$. It follows that $\bar{\partial} + \Psi(R)$ is an $L_\infty[1]$ structure on $\mathcal{A}^{0,*}(T_X)$, where again we are regarding $\bar{\partial}$ as a linear coderivation on $\overline{\mathcal{S}}(\mathcal{A}^{0,*}(T_X))$, in fact, $\frac{1}{2}[\bar{\partial} + \Psi(R), \bar{\partial} + \Psi(R)] = \Psi(\bar{\partial}R + \frac{1}{2}[R, R]) = 0$: this is the $L_\infty[1]$ structure on $\mathcal{A}^{0,*}(T_X)$ considered in [56]. Conversely, we can deduce that R is Maurer-Cartan by the following theorem, as Ψ is clearly injective.

Theorem 4.2.24. *With the previous notations, $\bar{\partial} + \Psi(R) = \mathcal{K}^{-1}(\bar{\partial})$: in particular, the $L_\infty[1]$ algebra structure on $\mathcal{A}^{0,*}(T_X)$ by Kapranov [56] is homotopy abelian over the field \mathbb{C} of complex numbers.*

Proof. Since $\bar{\partial} \in \text{Der}(\mathcal{A}^{0,*}(T_X), [\cdot, \cdot])$, the brackets $\mathcal{K}^{-1}(\bar{\partial})_n$ can be defined via the recursion in the claim of Proposition 4.2.12. Of course $\bar{\partial} = \mathcal{K}^{-1}(\bar{\partial})_1$, we have to prove $\Psi(R_n) = \mathcal{K}^{-1}(\bar{\partial})_n$, $\forall n \geq 2$. We start by computing $\mathcal{K}^{-1}(\bar{\partial})_2$, which is given by

$$\begin{aligned} \mathcal{K}^{-1}(\bar{\partial})_2(\alpha \odot \beta) &= \nabla_{\bar{\partial}\alpha}(\beta) - [\bar{\partial}, \nabla_\alpha](\beta) = [\mathbf{i}_{\bar{\partial}\alpha}, \nabla](\beta) - [\bar{\partial}, [\mathbf{i}_\alpha, \nabla]](\beta) = \\ &= (-1)^{|\alpha|} [\mathbf{i}_\alpha, [\bar{\partial}, \nabla]](\beta) = (-1)^{|\alpha|} \mathbf{i}_\alpha D^2(\beta). \end{aligned}$$

If locally $\alpha = \sum_i \alpha^i \otimes \frac{\partial}{\partial z^i}$ and $\beta = \sum_j \beta^j \otimes \frac{\partial}{\partial z^j}$, then locally

$$\begin{aligned} \mathcal{K}^{-1}(\bar{\partial})_2(\alpha \odot \beta) &= \sum_k \left(\sum_{i,j} (-1)^{|\alpha|} \alpha^i \wedge \left(\frac{\partial}{\partial z^i} \lrcorner (\beta^j \wedge \Omega_j^k) \right) \right) \otimes \frac{\partial}{\partial z^k} = \\ &= \sum_k \left(\sum_{i,j} (-1)^{|\alpha|+|\beta|} \alpha^i \wedge \beta^j \wedge \left(\frac{\partial}{\partial z^i} \lrcorner \Omega_j^k \right) \right) \otimes \frac{\partial}{\partial z^k}. \end{aligned}$$

Graded symmetry of this expression also follows from the identity $\frac{\partial}{\partial z^i} \lrcorner \Omega_j^k = \frac{\partial}{\partial z^j} \lrcorner \Omega_i^k$, $\forall i, j, k$, which itself is a consequence of torsion freeness of D . This shows that $\Psi(R_2) = \mathcal{K}^{-1}(\bar{\partial})_2$, in particular it implies that $\mathcal{K}^{-1}(\bar{\partial})_2$ is $\mathcal{A}_X^{0,*}$ -bilinear. More in general, every $\mathcal{K}^{-1}(\bar{\partial})_n$, $n \geq 2$, is $\mathcal{A}_X^{0,*}$ -multilinear in the sense of (4.2.6): by graded symmetry it suffices to consider the case $k = 1$ in the formula (4.2.6), which is seen by induction using the recursive definition and the easily established identity $\nabla_{\omega \wedge \alpha}(\beta) = \omega \wedge \nabla_{\alpha}(\beta)$, $\forall \alpha, \beta \in \mathcal{A}^{0,*}(T_X)$, $\omega \in \mathcal{A}_X^{0,*}$. Finally, in order to prove in general that $\Psi(R_n)(\alpha_1 \odot \cdots \odot \alpha_n) = \mathcal{K}^{-1}(\bar{\partial})_n(\alpha_1 \odot \cdots \odot \alpha_n)$, $\forall \alpha_1, \dots, \alpha_n \in \mathcal{A}^{0,*}(T_X)$, we have reduced to the case $\alpha_k = \frac{\partial}{\partial z^{i_k}}$, $k = 1, \dots, n$. Proceeding by induction and using the recursive definition 4.2.4 we see that for all $n \geq 3$

$$\begin{aligned} \Psi(R_n) \left(\frac{\partial}{\partial z^{i_1}} \odot \cdots \odot \frac{\partial}{\partial z^{i_n}} \right) &= \sum_k \left(\sum_h (R_n)_{i_1 \dots i_n \bar{h}}^k d\bar{z}^h \right) \otimes \frac{\partial}{\partial z^k} = \\ &= \sum_k \left(\sum_h (\nabla_{\frac{\partial}{\partial z^{i_1}}} R_{n-1})_{i_2 \dots i_n \bar{h}}^k d\bar{z}^h \right) \otimes \frac{\partial}{\partial z^k} = [\nabla_{\frac{\partial}{\partial z^{i_1}}, \Psi(R_{n-1})] \left(\frac{\partial}{\partial z^{i_2}} \odot \cdots \odot \frac{\partial}{\partial z^{i_n}} \right) = \\ &= -[\mathcal{K}^{-1}(\bar{\partial})_{n-1}, \nabla_{\frac{\partial}{\partial z^{i_1}}}] \left(\frac{\partial}{\partial z^{i_2}} \odot \cdots \odot \frac{\partial}{\partial z^{i_n}} \right) = \mathcal{K}^{-1}(\bar{\partial})_n \left(\frac{\partial}{\partial z^{i_1}} \odot \cdots \odot \frac{\partial}{\partial z^{i_n}} \right) \end{aligned}$$

Together with Corollary 4.2.10 this proves the theorem. \square

Remark 4.2.25. As the brackets $\mathcal{K}^{-1}(\bar{\partial})_n$, $n \geq 1$, are all \mathcal{O}_X -multilinear, $(\mathcal{A}^{0,*}(T_X), \Phi(\bar{\partial}))$ is an \mathcal{O}_X -multilinear $L_{\infty}[1]$ -algebra in the sense, for instance, of [109]: in the claim of the previous theorem we had to specify the field of definition since otherwise the homotopy abelianity part would fail (it should remain true when the Atiyah class α_{T_X} vanishes).

Chapter 5

Higher Deligne groupoids

After some necessary preliminaries on model categories and simplicial sets, in Section 5.2 we introduce the main subject of this chapter, namely, what we call the Deligne-Getzler ∞ groupoid (cf. 5.2.23 as for why it is an ∞ groupoid) $\text{Del}_\infty(L)$ of a complete $L_\infty[1]$ algebra L , after the seminal paper [39], and we study some of its main properties from [39, 6]: in Section 5.2.2 we show that $\text{Del}_\infty(-)$ generalizes the Deligne groupoid of a (nilpotent) dg Lie algebra. In Section 5.2.1 we study the role of the functor $\text{Del}_\infty(-)$ in the Lie approach to disconnected rational homotopy theory developed by Lazarev and Markl [70]. Finally, in the last two sections we study descent of Deligne groupoids: in Section 5.3.1 the original result by Hinich [45], and in Section 5.3 a corresponding descent theorem for the functor $\text{Del}_\infty(-)$.

5.1 Miscellanea on model categories and simplicial sets

In this section we recall the various results from model category theory that we will need in the rest of the chapter. This is an extremely rich subject, and we merely recall the essential facts to make sense of Proposition 5.1.2 and Proposition 5.1.5, which are the results we will actually need. Our references here are [50, 48], the original by Quillen [88], and finally the appendices of [73]: numerous other references can be found in the bibliographies of loc. cit.. In the second part of the section we review the model category structure on the category of simplicial sets, together with another Proposition 5.1.9 and some standard definitions we will need in the sequel: we refer mainly to [50, 88] and [81], but again, possibly even more excellent introductions to this material are available. We close this section by recalling the definition of T -complex, cf. Ashley's and Dakin's theses [1, 26], and their weak versions considered by Getzler [39]. Even if we will have no actual use for these concepts, it will be good to keep them in mind.

Usually, in homotopy theory we are concerned with a category \mathbf{C} and a subcategory \mathcal{W} of weak equivalences that we would like to consider as isomorphisms: there is always a way to localize \mathbf{C} at \mathcal{W} , that is, there is a category $\mathbf{C}[\mathcal{W}^{-1}]$ (called the Gabriel-Zisman localization of \mathbf{C} at \mathcal{W}) with the same class of objects of \mathbf{C} and a functor $\mathbf{C} \rightarrow \mathbf{C}[\mathcal{W}^{-1}]$ which is the identity on objects and universal with the property that it sends arrows in \mathcal{W} to isomorphisms in $\mathbf{C}[\mathcal{W}^{-1}]$. This does not require any assumption on \mathcal{W} , however Gabriel-Zisman localization is hard to work with since in general we have to quotient a large class (not a set!) of morphisms by an equivalence relation, and moreover one can incur in set theoretical difficulties (resolved by working with Grothendieck universes) as

it is not always the case that this quotient will be a set. In a model category $(\mathbf{C}, \mathcal{W})$ is enriched to $(\mathbf{C}, \mathcal{C}, \mathcal{F}, \mathcal{W})$, where \mathcal{C} and \mathcal{F} are two auxiliary subcategories and the data satisfy a series of axioms listed below: even if we are ultimately only interested in the pair $(\mathbf{C}, \mathcal{W})$ the structure of model category on \mathbf{C} provides a neat description of the category $\mathbf{C}[\mathcal{W}^{-1}]$ and guarantees that we do not incur in the aforementioned difficulties.

Recall that given a pair of arrows $f : A \rightarrow B$ and $g : X \rightarrow Y$ in \mathbf{C} then f has the left lifting property with respect to g , and conversely g has the right lifting property with respect to f , if in all commutative diagrams in \mathbf{C}

$$\begin{array}{ccc} A & \longrightarrow & X \\ f \downarrow & \nearrow & \downarrow g \\ B & \longrightarrow & Y \end{array}$$

the dotted arrow can be filled so that the whole diagram remains commutative.

A model category $(\mathbf{C}, \mathcal{C}, \mathcal{F}, \mathcal{W})$ is a category \mathbf{C} together with distinguished subcategories \mathcal{C} , \mathcal{F} and \mathcal{W} whose class of objects is the same as \mathbf{C} and whose arrows are called respectively cofibrations, fibrations and weak equivalences, while the arrows in $\mathcal{C} \cap \mathcal{W}$ and $\mathcal{F} \cap \mathcal{W}$ are called respectively trivial cofibrations and trivial fibrations, such that: \mathbf{C} is complete and cocomplete, weak equivalences satisfy the two out of three properties (namely, if two out of f , g and fg are weak equivalences so is the third) and the subcategories \mathcal{C} , \mathcal{F} and \mathcal{W} are stable under retracts, and moreover

trivial cofibrations have the left lifting property with respect to fibrations and trivial fibrations have the right lifting property with respect to cofibrations;

every arrow admits functorial factorizations as a cofibration followed by a trivial fibration and as a trivial cofibration followed by a fibration.

We refer to [50] for a more detailed discussion of the previous list of axioms, in particular the original definition given in [88] is more general, and instead we have defined what Quillen calls a *closed* model category, where the closed adjective refers to the following important fact: if $(\mathbf{C}, \mathcal{C}, \mathcal{F}, \mathcal{W})$ is a model category as in the previous definition, then \mathcal{C} (resp.: $\mathcal{C} \cap \mathcal{W}$) is precisely the class of arrows with the left lifting property with respect to arrows in $\mathcal{F} \cap \mathcal{W}$ (resp.: \mathcal{F}), and conversely $\mathcal{F} \cap \mathcal{W}$ (resp.: \mathcal{F}) is precisely the class of arrows with the right lifting property with respect to arrows in \mathcal{C} (resp.: $\mathcal{C} \cap \mathcal{W}$). In particular we can recover the whole structure $(\mathbf{C}, \mathcal{C}, \mathcal{F}, \mathcal{W})$ knowing two out of \mathcal{C} , \mathcal{F} , \mathcal{W} , $\mathcal{C} \cap \mathcal{W}$ and $\mathcal{F} \cap \mathcal{W}$: for instance, given $(\mathbf{C}, \mathcal{C}, \mathcal{C} \cap \mathcal{W})$ the subcategory \mathcal{F} (resp.: $\mathcal{F} \cap \mathcal{W}$) consists of arrows with the right lifting property with respect to arrows in $\mathcal{C} \cap \mathcal{W}$ (resp.: \mathcal{C}), while weak equivalences can be characterized as those arrows admitting a factorization as a trivial cofibration followed by a trivial fibration. In many important situations it is possible to reduce the generating data further, from two classes of arrows to two *sets* of arrows. A model category $(\mathbf{C}, \mathcal{C}, \mathcal{F}, \mathcal{W})$ is cofibrantly generated if it admits distinguished *sets* $I \subset \mathcal{C}$ of *generating cofibrations* and $J \subset \mathcal{C} \cap \mathcal{W}$ of *generating trivial cofibrations* such that \mathcal{F} (resp.: $\mathcal{F} \cap \mathcal{W}$) is precisely the subcategory of arrows with the right lifting property with respect to arrows in J (resp.: I) and an additional technical assumption is satisfied, roughly saying that I and J allow Quillen's small objects argument (cf. [50, 48] for details, let us just say that the assumption will be automatically satisfied for the categories we will be interested in, such as \mathbf{SSet} , \mathbf{DG} , \mathbf{DGLA} , $\widehat{\mathbf{DGLA}}$ etc.). Conversely, given a complete and cocomplete category \mathbf{C} together with sets of arrows I and J satisfying the aforementioned technical assumption, there is an useful criterion (attributed to Kan in [48]) to check whether these data generates a (by construction, cofibrantly generated) model category structure on \mathbf{C} , cf. [48], Theorem 11.3.1, and [50], Theorem 2.1.19.

As we said, if $(\mathbf{C}, \mathcal{C}, \mathcal{F}, \mathcal{W})$ is a model category then the localization $\mathbf{C}[\mathcal{W}^{-1}]$, which in this case is called the homotopy category of $(\mathbf{C}, \mathcal{C}, \mathcal{F}, \mathcal{W})$ and denoted by $\text{Ho}(\mathbf{C})$, admits a neat description. The category \mathbf{C} has an initial object \emptyset since it is cocomplete and a final object $*$ since it is complete, and we say that an object X is cofibrant if $\emptyset \rightarrow X$ is a cofibration and it is fibrant if $X \rightarrow *$ is a fibration: when X is cofibrant or Y is fibrant there is a well defined equivalence relation \sim on the set of morphisms $\mathbf{C}(X, Y)$ constructed from the axioms of model category: this is called the homotopy relation. The model category axioms imply that for any object X there are functorially defined weak equivalences $X \xrightarrow{\sim} \overline{X}$ and $\overline{X} \xrightarrow{\sim} X$ with \overline{X} both fibrant and cofibrant: finally, the homotopy category $\text{Ho}(\mathbf{C})$ has the same class of objects as \mathbf{C} and the sets of morphisms are $\text{Ho}(\mathbf{C})(X, Y) = \mathbf{C}(\overline{X}, \overline{Y}) / \sim$, where \overline{X} and \overline{Y} are fibrant and cofibrant models of X and Y respectively and \sim is the homotopy relation. Typically, functors between model categories come in adjoint pairs: an adjunction between model categories $F : \mathbf{C} \rightleftarrows \mathbf{D} : G$ is a Quillen adjunction if F preserves cofibrations and trivial cofibrations and G preserves fibrations and trivial fibrations, the two conditions are not independent and in fact each one of them implies the other, moreover they imply that F preserves cofibrant objects and weak equivalences between them while G preserves fibrant objects and weak equivalences between them. We also say that F is a left Quillen functor and G is a right Quillen functor. If $F : \mathbf{C} \rightleftarrows \mathbf{D} : G$ is a Quillen adjunction there is a derived adjunction between the homotopy categories $\text{Ho}(F) : \text{Ho}(\mathbf{C}) \rightleftarrows \text{Ho}(\mathbf{D}) : \text{Ho}(G)$, if the derived adjunction is an equivalence of categories then $F : \mathbf{C} \rightleftarrows \mathbf{D} : G$ is called a Quillen equivalence, see [88, 48, 50] for necessary and sufficient conditions.

We recall a proposition from [50] which will be used in several occasions in the forthcoming sections. First recall that a small category \mathcal{S} is direct if it admits a functor $\mathcal{S} \rightarrow \omega$ into some ordinal ω , seen as a category via the order, sending non identity arrows into non identity arrows. For instance the semicosimplicial indexing category $\underline{\Delta}$ (Definition 5.1.7) is direct by $\underline{\Delta} \rightarrow \omega_0 : \Delta_n \rightarrow \underline{n}$, where ω_0 is the first limit ordinal. As another example, given a simplicial set X the category $\underline{\Delta}X$ of non degenerate simplices of X (Definition 5.1.8) is direct by $\underline{\Delta}X \rightarrow \omega_0 : \{\sigma : \Delta_n \rightarrow X\} \rightarrow \underline{n}$. Of course every ordinal is a direct category. A small category \mathcal{S} is inverse if \mathcal{S}^{op} is direct. According to [50], Theorem 5.1.3, given a direct category \mathcal{S} and a model category \mathbf{C} there is a model category structure on the category of functors $\mathbf{C}^{\mathcal{S}}$, called the projective model structure, where fibrations and weak equivalences are defined pointwise (that is, given functors $\phi, \psi : \mathcal{S} \rightarrow \mathbf{C}$ and a natural transformation $f : \phi \rightarrow \psi$, then f is a weak equivalence (resp.: fibration) if for all $s \in \mathcal{S}$ so is $f(s) : \phi(s) \rightarrow \psi(s)$) and the cofibrations are the Reedy cofibrations discussed in the following remark. Dually, there is a model category structure on $\mathbf{C}^{\mathcal{S}^{op}}$, called the injective model structure, where cofibrations and weak equivalences are defined pointwise and the fibrations are the Reedy fibrations.

Remark 5.1.1. We recall the definition of Reedy fibrations, Reedy cofibrations are defined in the dual (Eckmann-Hilton duality) way, cf. [48, 50] for more details. Given an inverse category \mathcal{S}^{op} and an object $i \in \mathcal{S}^{op}$, we denote by $\mathcal{S}_{i\uparrow}^{op}$ be the category of arrows in \mathcal{S}^{op} with codomain i , then the matching space functor $M_i - : \mathbf{C}^{\mathcal{S}^{op}} \rightarrow \mathbf{C}$ is defined as the composition of the restriction functor $\mathbf{C}^{\mathcal{S}^{op}} \rightarrow \mathbf{C}^{\mathcal{S}_{i\uparrow}^{op}}$ and the limit functor $\mathbf{C}^{\mathcal{S}_{i\uparrow}^{op}} \rightarrow \mathbf{C}$. Given a functor $X : \mathcal{S}^{op} \rightarrow \mathbf{C} : j \rightarrow X_j$, we denote by $M_i X$ the matching space of X at $i \in \mathcal{S}^{op}$, notice that there is a natural $X_i \rightarrow M_i X$. Let $f : X \rightarrow Y$ be a natural transformation in $\mathbf{C}^{\mathcal{S}^{op}}$, the matching morphism of f at $i \in \mathcal{S}^{op}$ is by definition the induced $X_i \rightarrow Y_i \times_{M_i Y} M_i X$: we say that f is a Reedy fibration if the induced matching morphisms are fibration for all $i \in \mathcal{S}^{op}$. For instance, when $i \in \mathcal{S}^{op} = \omega_0^{op}$, where ω_0 is the first limit ordinal, then a functor $X : \omega_0^{op} \rightarrow \mathbf{C}$ is a tower $\cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0$ in \mathbf{C} , and X is a Reedy fibration if and only if X_0 is fibrant and all the $X_{n+1} \rightarrow X_n$, $n \geq 0$, are fibrations.

In the following proposition we fix a choice of $\text{colim} : \mathbf{C}^{\mathcal{S}} \rightarrow \mathbf{C}$ and $\text{lim} : \mathbf{C}^{\mathcal{S}^{op}} \rightarrow \mathbf{C}$ functors.

Proposition 5.1.2. *Let \mathcal{S} be a direct category and \mathbf{C} a model category. If we equip $\mathbf{C}^{\mathcal{S}}$ with the projective model structure then $\text{colim} : \mathbf{C}^{\mathcal{S}} \rightarrow \mathbf{C}$ is a left Quillen functor. If we equip $\mathbf{C}^{\mathcal{S}^{op}}$ with the injective model structure then $\text{lim} : \mathbf{C}^{\mathcal{S}^{op}} \rightarrow \mathbf{C}$ is a right Quillen functor.*

Remark 5.1.3. In particular $\text{lim} : \mathbf{C}^{\mathcal{S}^{op}} \rightarrow \mathbf{C}$ preserves fibrations, trivial fibrations and weak equivalences between fibrant objects.

Definition 5.1.4. Given a model category $(\mathbf{C}, \mathcal{C}, \mathcal{F}, \mathcal{W})$, a complete and cocomplete category \mathbf{D} and an adjunction $F : \mathbf{C} \rightleftarrows \mathbf{D} : G$ we say that the model category structure on \mathbf{C} transfers along the adjunction if it is well defined - that is, all axioms are satisfied - a model category structure on \mathbf{D} by saying that an arrow f in \mathbf{D} is a weak equivalence or a fibration if such is the arrow $G(f)$ in \mathbf{C} , and then by defining cofibrations via the corresponding lifting property. When this happens $F : \mathbf{C} \rightleftarrows \mathbf{D} : G$ becomes automatically a Quillen adjunction.

If the model category \mathbf{C} is cofibrantly generated, by sets I of generating cofibrations and J of generating trivial cofibrations, there is a general and powerful criterion, which can be found in Hirschhorn's book [48], Theorem 11.3.2, where it is again attributed to Kan, to check whether the model category structure on \mathbf{C} transfers along a given adjunction $F : \mathbf{C} \rightleftarrows \mathbf{D} : G$: when this criterion is met the transferred model category structure on \mathbf{D} is again cofibrantly generated, this time by $F(I)$ and $F(J)$. It is of course important to reduce to a minimum the things to be checked in order for the criterion to apply: in the situation we will be interested we can use the following version of Quillen's path object argument ([88], Chapter II, 4.9) which we learned from the proof of [95], Lemma A.3. Recall that a path space factorization of an object X in \mathbf{C} is a factorization of the diagonal $X \rightarrow X \times X$ as a trivial cofibration followed by a fibration.

Proposition 5.1.5. *Let \mathbf{C} be a cofibrantly generated model category, \mathbf{D} a complete and cocomplete category and $F : \mathbf{C} \rightleftarrows \mathbf{D} : G$ an adjunction. We notice that since G is a right adjoint it preserves limits and in particular final objects. If for every object X of \mathbf{D} we have that $G(X)$ is fibrant in \mathbf{C} and moreover there is a factorization of the diagonal $X \rightarrow Y \rightarrow X \times X$ such that $G(X) \rightarrow G(Y) \rightarrow G(X \times X) = G(X) \times G(X)$ is a path space factorization of $G(X)$, then the cofibrantly generated model category structure on \mathbf{C} transfers along the adjunction $F : \mathbf{C} \rightleftarrows \mathbf{D} : G$.*

For instance the previous proposition applies when \mathbf{C} is the category \mathbf{DG} with the model category structure where fibrations are epimorphisms and weak equivalences quasi-isomorphisms (this is cofibrantly generated, cf [50], Section 2.3), \mathbf{D} is one of the categories \mathbf{DGA} , \mathbf{DGCA} , \mathbf{DGLA} , F is the free algebra functor and G is the forgetful functor.

Remark 5.1.6. In the claim of the previous proposition we are omitting a technical assumption, namely that the sets of arrows $F(I)$ and $F(J)$ - where I and J generate the model category structure on \mathbf{C} - should allow the small object argument: however, this will be automatically satisfied in the aforementioned cases and in the case of Theorem 5.2.28.

Of fundamental importance is the model category of simplicial sets.

Definition 5.1.7. The cosimplicial indexing category is the category Δ whose objects are finite ordinals, which we depict as $\underline{n} = [0 \cdots n]$ and whose morphisms are order preserving maps, which we depict as $\underline{n} \rightarrow \underline{m} : [0 \cdots n] \rightarrow [i_0 \cdots i_n]$, where $0 \leq i_0 \leq \cdots \leq i_n \leq m$. The simplicial indexing category is the category Δ^{op} . The semicosimplicial indexing category is the subcategory $\underline{\Delta} \hookrightarrow \Delta$ with the same set of objects but morphisms the injective order preserving maps, namely, those $\underline{n} \rightarrow \underline{m} : [0 \cdots n] \rightarrow [i_0 \cdots i_n]$ with $0 \leq i_0 < \cdots < i_n \leq m$. The semisimplicial indexing category is the category $\underline{\Delta}^{op}$. Given a category \mathbf{C} , the categories of functors \mathbf{C}^{Δ} , $\mathbf{C}^{\Delta^{op}}$, $\mathbf{C}^{\underline{\Delta}}$ and $\mathbf{C}^{\underline{\Delta}^{op}}$

are called respectively the category of cosimplicial, simplicial, semicosimplicial and semisimplicial objects in \mathbf{C} (e.g. simplicial sets, cosimplicial dg Lie algebras, etc.). Equivalently, objects in the category \mathbf{C}^Δ (resp.: $\mathbf{C}^{\Delta^{op}}$) could be defined as sequences X_n of objects of \mathbf{C} together with faces $\partial_i : X_n \rightarrow X_{n+1}$ (resp.: $d_i : X_{n+1} \rightarrow X_n$) and degeneracies $\sigma_j : X_{n+1} \rightarrow X_n$ (resp.: $s_j : X_n \rightarrow X_{n+1}$) for all $n \geq 0$, $0 \leq i \leq n+1$ and $0 \leq j \leq n$ satisfying a well known list of relations, see e.g. [81]: morphisms are collections $f_n : X_n \rightarrow Y_n$ commuting with faces and degeneracies. The categories $\mathbf{C}^{\hat{\Delta}}$ and $\mathbf{C}^{\hat{\Delta}^{op}}$ can be defined similarly by only considering faces and the relations between them. Finally, we denote the category $\mathbf{Set}^{\Delta^{3p}}$ of simplicial sets by \mathbf{SSet} .

Each ordinal \underline{n} represents a simplicial set $\Delta^{op} \rightarrow \mathbf{Set} : \underline{m} \rightarrow \Delta(\underline{m}, \underline{n})$ which is called the standard n -th simplex and denoted by Δ_n , the cosimplicial simplicial set $\Delta \rightarrow \mathbf{SSet} : \underline{n} \rightarrow \Delta_n$ is called the standard cosimplicial simplex and is denoted by Δ_\bullet . According to Yoneda's lemma for all simplicial sets $X : \Delta \rightarrow \mathbf{Set} : \underline{n} \rightarrow X_n$ and $n \geq 0$ we have a natural identification

$$X_n = \mathbf{SSet}(\Delta_n, X). \quad (5.1.1)$$

In particular we can depict n -simplices $\sigma \in X_n$ as arrows $\sigma : \Delta_n \rightarrow X$: we put $|\sigma| = n$ and call it the order of σ .

Definition 5.1.8. Given a simplicial set X the category ΔX of simplices of X is the small category whose objects are all the simplices $\sigma : \Delta_{|\sigma|} \rightarrow X$ of X (with $|\sigma|$ variable) and whose morphisms are the commutative diagrams

$$\begin{array}{ccc} \Delta_{|\sigma|} & \xrightarrow{\quad} & \Delta_{|\tau|} \\ & \searrow \sigma & \swarrow \tau \\ & X & \end{array}$$

in \mathbf{SSet} . We can reconstruct X from its category of simplices as the colimit of the natural functor $\Delta X \rightarrow \mathbf{SSet} : \sigma \rightarrow \Delta_{|\sigma|}$

$$X = \operatorname{colim}_{\sigma \in \Delta X} \Delta_{|\sigma|}.$$

An n -simplex $\sigma \in X_n$ is degenerate if it is in the image of a degeneracy $s_j : X_{n-1} \rightarrow X_n$ and non-degenerate otherwise. The category of non-degenerate simplices of X is the subcategory $\underline{\Delta} X \hookrightarrow \Delta X$ whose objects are the non degenerate simplices $\sigma : \Delta_{|\sigma|} \rightarrow X$ of X and where morphisms are defined as above, but $\Delta_{|\sigma|} \rightarrow \Delta_{|\tau|}$ is required to be a mono (that is, the image under the functor Δ_\bullet of a morphism in the subcategory $\underline{\Delta} \hookrightarrow \Delta$). As before, we can reconstruct X from the category $\underline{\Delta} X$ as the colimit of $\underline{\Delta} X \rightarrow \mathbf{SSet} : \sigma \rightarrow \Delta_{|\sigma|}$

$$X = \operatorname{colim}_{\sigma \in \underline{\Delta} X} \Delta_{|\sigma|}. \quad (5.1.2)$$

The category \mathbf{SSet} has an internal $\operatorname{Hom}(-, -)$ functor, namely, the simplicial mapping space functor $\underline{\mathbf{SSet}}(-, -) : \mathbf{SSet}^{op} \times \mathbf{SSet} \rightarrow \mathbf{SSet}$ defined on simplicial sets X and Y by

$$\underline{\mathbf{SSet}}(X, Y) : \Delta^{op} \rightarrow \mathbf{Set} : \underline{n} \rightarrow \underline{\mathbf{SSet}}(X, Y)_n = \mathbf{SSet}(X \times \Delta_n, Y)$$

In other words, $\underline{\mathbf{SSet}}(X, Y) = \mathbf{SSet}(X \times \Delta_\bullet, Y)$.

We denote by $\Delta_{\leq n} \subset \Delta$ the full subcategory of ordinals $\leq n$. The natural restriction functor $\mathbf{SSet} = \mathbf{Set}^{\Delta^{op}} \rightarrow \mathbf{Set}^{\Delta_{\leq n}^{op}}$ admits a left adjoint, the composition $\mathbf{SSet} \rightarrow \mathbf{Set}^{\Delta_{\leq n}^{op}} \rightarrow \mathbf{SSet}$ of these two functors is called the n -skeleton and denoted by $\operatorname{sk}_n(-)$, namely, $\operatorname{sk}_n(X)$ is the simplicial set with the same simplices of X up to order n and after that only the degeneracies of simplices

of order $\leq n$. The boundary $\partial\Delta_n$ of the n -simplex is defined by $\partial\Delta_0 = \emptyset$ and for $n \geq 1$ by $\partial\Delta_n = \text{sk}_{n-1}(\Delta_n)$: in other words $\partial\Delta_n$ has the same non-degenerate simplexes as Δ_n minus the top dimensional simplex $\text{id}_{\underline{n}} : \underline{n} \rightarrow \underline{n}$. For $0 \leq i \leq n$ the i -th horn Λ_n^i is the simplicial set with the same non-degenerate simplexes as Δ_n minus the top dimensional simplex $\text{id}_{\underline{n}} : \underline{n} \rightarrow \underline{n}$ and the i -th face $\underline{n-1} \rightarrow \underline{n} : [0 \cdots n-1] \rightarrow [0 \cdots \widehat{i} \cdots n-1]$.

The importance of the category **SSet** lies in the fact that, while being algebraic in nature, it admits a model category structure Quillen equivalent to the one on the category **Top** of topological spaces where the fibrations are the Serre fibrations and the weak equivalences are the weak homotopy equivalences (namely, the $f : X \rightarrow Y$ inducing an isomorphism $\pi_n(f) : \pi_n(X) \xrightarrow{\cong} \pi_n(Y)$ between the homotopy groups for all $n \geq 0$). The model category structure on **SSet** is cofibrantly generated: the generating cofibrations are the boundary inclusions $\partial\Delta_n \rightarrow \Delta_n$ and the generating trivial cofibrations are the horn inclusions $\Lambda_n^i \rightarrow \Delta_n$ for all $n \geq 1$ and $0 \leq i \leq n$. The fibrations in the model category **SSet**, which are called *Kan fibrations*, are thus defined according to the corresponding lifting property with respect to horn inclusions: fibrant object in the model category **SSet** are called *Kan complexes*, the full subcategory of Kan complexes is denoted by **Kan** \subset **SSet**. The standard topological n -th simplex $|\Delta_n|$ is defined as usual as $|\Delta_n| = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \text{ s.t. } t_0 + \dots + t_n = 1\}$, they assemble into the standard cosimplicial topological simplex $|\Delta_\bullet| \in \mathbf{Top}^\Delta$: according to the following Proposition 5.1.9 the cosimplicial object $|\Delta_\bullet|$ in **Top** induces an adjunction $|-| : \mathbf{SSet} \rightleftarrows \mathbf{Top} : C_\bullet(-)$, where n -simplices of the simplicial set $C_\bullet(Y)$ are as usual continuous morphisms $|\Delta_n| \rightarrow Y$, and the faces and degeneracies are defined in the obvious way: the left adjoint $|-| : \mathbf{SSet} \rightarrow \mathbf{Top}$ is called the geometric realization functor, roughly, the topological space $|X|$ is obtained by pasting various copies of $|\Delta_{|\sigma|}|$, one for each simplex $\sigma : \Delta_{|\sigma|} \rightarrow X$ of X , according to the incidence relation prescribed by the faces and the degeneracies. Coming back to the model category structure on **SSet**, the weak equivalences are the morphisms $f : X \rightarrow Y$ going into weak homotopy equivalences $|f| : |X| \xrightarrow{\sim} |Y|$ after geometric realization: there is also a purely combinatorial way to define the homotopy groups $\pi_n(X)$ of a Kan complex X (we refer to [81], this definition of homotopy groups is the one we will use in the proof of Theorem 5.2.20), as well as fibrant replacement functors **SSet** \rightarrow **Kan** (Kan's Ex^∞ functor is a possible choice, cf. [41]), thus we could characterize the weak equivalences $f : X \rightarrow Y$ without any need to leave the category **SSet**, first if both X and Y are Kan complexes by requiring, as for **Top**, that $\pi_n(f) : \pi_n(X) \xrightarrow{\cong} \pi_n(Y)$ is an isomorphism for all $n \geq 0$, and then in general by requiring that the induced morphisms between fibrant replacements (for instance $\text{Ex}^\infty(f) : \text{Ex}^\infty(X) \rightarrow \text{Ex}^\infty(Y)$) is a weak equivalence. Finally, the class of cofibrations in **SSet** is determined by the other two and the corresponding lifting property: cofibrations turn out to be simply the monomorphisms of simplicial sets $X \hookrightarrow Y$. With these definitions the adjunction $|-| : \mathbf{SSet} \rightleftarrows \mathbf{Top} : C_\bullet(-)$ becomes a Quillen equivalence of model categories [89, 50].

We recall the correspondence between cosimplicial objects in a category **C** and adjunctions $F : \mathbf{SSet} \rightleftarrows \mathbf{C} : G$, in combination with Proposition 5.1.5 this gives a powerful way to generate a cofibrantly generated model category structure on **C** from a minimal amount of data, namely, a cosimplicial object of **C**. As in [50], we can define the category with objects the adjunctions $F : \mathbf{SSet} \rightleftarrows \mathbf{C} : G$.

Proposition 5.1.9. *Let **C** be a cocomplete category. There is an equivalence of categories between the category \mathbf{C}^Δ of cosimplicial objects in **C** and the category of adjunctions $F : \mathbf{SSet} \rightleftarrows \mathbf{C} : G$.*

Proof. We refer to [50] Proposition 3.1.5, here we just recall the construction of the correspondence. Given an adjunction $F : \mathbf{SSet} \rightleftarrows \mathbf{C} : G$ the corresponding cosimplicial object in **C** is just the image $F(\Delta_\bullet)$ of the standard cosimplicial simplex. Conversely, given a cosimplicial object

$X : \Delta \rightarrow \mathbf{C} : \underline{n} \rightarrow X_n$ of \mathbf{C} , since \mathbf{C} is cocomplete, it is defined by formal nonsense a colimit preserving $F : \mathbf{SSet} \rightarrow \mathbf{C} : Y \rightarrow F(Y) := \operatorname{colim}_{\sigma \in \Delta Y} X_{|\sigma|}$, the right adjoint G is given by

$$G : \mathbf{C} \rightarrow \mathbf{SSet} : Z \rightarrow \{G(Z) : \Delta^{op} \rightarrow \mathbf{Set} : \underline{n} \rightarrow G(Z)_n = \mathbf{C}(X_n, Z)\}.$$

□

This allows a two steps construction of a cofibrantly generated model category structure on the category \mathbf{C} : first we give a cosimplicial object of \mathbf{C} , then we look at the induced adjunction $F : \mathbf{SSet} \rightleftarrows \mathbf{C} : G$ and try to see if the conditions of Proposition 5.1.9 (or more in general Kan's criterion [48]) are satisfied: when this happens the model category structure on \mathbf{SSet} transfers to a cofibrantly generated model category structure on \mathbf{C} , where the generating cofibrations are the $F(\partial\Delta_n) \rightarrow F(\Delta_n)$ and the generating trivial cofibrations are the $F(\Lambda_n^i) \rightarrow F(\Delta_n)$, for all $n \geq 1$ and $0 \leq i \leq n$. We will follow this strategy in the proof of Theorem 5.2.28.

Example 5.1.10. We apply the previous proposition with the cosimplicial simplicial set $\operatorname{sk}_k(\Delta_\bullet)$. The left adjoint is the k -th skeleton $\operatorname{sk}_k(-)$, as this preserves colimits, while the right adjoint is called the k -th coskeleton functor and denoted by $\operatorname{cosk}_k(-) : \mathbf{SSet} \rightarrow \mathbf{SSet}$. By definition, given a simplicial set X the k -coskeleton is the simplicial set $\operatorname{cosk}_k(X)_n = \mathbf{SSet}(\operatorname{sk}_k(\Delta_n), X)$. A simplicial set is said to be k -coskeletal if it is in the essential image of $\operatorname{cosk}_k(-)$, equivalently, if the natural $X \rightarrow \operatorname{cosk}_k(X)$ given by $X_n = \mathbf{SSet}(\Delta_n, X) \rightarrow \mathbf{SSet}(\operatorname{sk}_k \Delta_n, X) = \operatorname{cosk}_k(X)_n$ is an isomorphism.

Finally, although we won't really need the next definition, we believe it is important to keep it in mind in order to better understand the results of the following sections. Recall that given a horn $\Lambda_n^i \rightarrow X$ and a factorization $\Lambda_n^i \rightarrow \Delta_n \xrightarrow{\sigma} X$, the n -simplex $\sigma : \Delta_n \rightarrow X$ is called a filling of the horn. Kan complexes are by definition those simplicial sets such that every horn admits a (in general not unique) filling.

Definition 5.1.11. A weak T -complex (X, TX) is a simplicial set X equipped with a family of distinguished simplexes $T_n X \subset X_n$, $n \geq 1$, called the *thin* simplexes, such that

1. every degenerate simplex is thin; and
2. every horn $\Lambda_n^i \rightarrow X$ admits a unique thin filling, for all $n \geq 1$ and $0 \leq i \leq n$.

A morphism $f : (X, TX) \rightarrow (Y, TY)$ of weak T -complexes is a morphism $f : X \rightarrow Y$ of simplicial sets such that $f(T_n X) \subset T_n Y$, $\forall n \geq 1$. It is clear, by item 2, that the forgetful functor sending (X, TX) to the simplicial set X factors through the full subcategory $\mathbf{Kan} \subset \mathbf{SSet}$ of Kan complexes. We denote by \mathbf{wTKan} the category of weak T -complexes. We say that a weak T -complex is of rank k if every n -simplex is thin for $n > k$, we denote by $\mathbf{wTKan}_{\leq k} \subset \mathbf{wTKan}$ the full subcategory of weak T -complexes of rank k .

We call a horn $\Lambda_n^i \rightarrow X$ a thin horn if all of its $(n-1)$ -th faces are thin. A weak T -complex (X, TX) is a T -complex if moreover

3. in the unique thin filling of a thin horn, the remaining $(n-1)$ -th face is also thin.

We denote by $\mathbf{TKan} \subset \mathbf{wTKan}$ the full subcategory of T -complexes, and by $\mathbf{TKan}_{\leq k} \subset \mathbf{TKan}$ the full subcategory of T -complexes of rank k .

Remark 5.1.12. In the theses of Ashley [1] and Dakin [26] it is shown that T -complexes are equivalent in a precise sense to crossed complexes over groupoids: in particular the full subcategory $\mathbf{TKan}_{\leq 1} \subset \mathbf{TKan}$ of T -complexes of rank ≤ 1 is equivalent to the category of groupoids via the nerve and fundamental groupoid functors, and we refer to [1, 26] for a detailed proof of the fact that the full subcategory $\mathbf{TKan}_{\leq 2} \subset \mathbf{TKan}$ of T -complexes of rank ≤ 2 is equivalent to the category of crossed modules over groupoids. Objects in the category of weak T -complexes should then be considered as the nerves of weak crossed complexes over groupoids (whatever the latter means!), in fact in [39] they are simply called ∞ groupoids. Morphisms in the category \mathbf{wTKan} should be thought of as *strict* morphisms between ∞ groupoids: accordingly, it would be interesting to have a category $w\mathcal{TKan}$ of weak T -complexes and weak morphisms between them.

5.2 The Deligne-Getzler ∞ groupoid of a complete L_∞ algebra

Let $(V, F^\bullet V, q_1, \dots, q_n, \dots)$ be a complete $L_\infty[1]$ algebra, there is an associated simplicial complete $L_\infty[1]$ algebra $C(\Delta_\bullet; V)$ of non-degenerate cochains on the standard cosimplicial simplex Δ_\bullet with coefficients in V , as in Definition 3.2.5. According to Lemma 2.2.3, given a strict continuous morphism $f : V \rightarrow W$ the various $f_* : C(\Delta_n; V) \rightarrow C(\Delta_n; W)$ fit into a simplicial pushforward $f_* : C(\Delta_\bullet; V) \rightarrow C(\Delta_\bullet; W)$, thus we have a functor $C(\Delta_\bullet; -) : \widehat{\mathbf{L}}_\infty[1] \rightarrow \widehat{\mathbf{L}}_\infty[1]^{\Delta^{op}}$. Of course the Maurer-Cartan functor $\text{MC}(-) : \widehat{\mathbf{L}}_\infty[1] \rightarrow \mathbf{Set}$ also induces a functor $\text{MC}(-) : \widehat{\mathbf{L}}_\infty[1]^{\Delta^{op}} \rightarrow \mathbf{SSet}$.

Definition 5.2.1. The functor $\text{Del}_\infty(-) : \widehat{\mathbf{L}}_\infty[1] \rightarrow \mathbf{SSet}$ is the composition

$$\text{Del}_\infty(-) : \widehat{\mathbf{L}}_\infty[1] \xrightarrow{C(\Delta_\bullet; -)} \widehat{\mathbf{L}}_\infty[1]^{\Delta^{op}} \xrightarrow{\text{MC}(-)} \mathbf{SSet}.$$

Given a complete $L_\infty[1]$ algebra $(V, F^\bullet V, q_1, \dots, q_n, \dots)$ we call the simplicial set $\text{Del}_\infty(V) := \text{MC}(C(\Delta_\bullet; V))$ the Deligne-Getzler ∞ groupoid of V (or sometimes, for simplicity, the higher Deligne groupoid of V).

Remark 5.2.2. We observe that $\text{Del}_\infty(V)$ actually only depends on the $L_\infty[1]$ structure Q on V , and not on the particular filtration $F^\bullet V$ making (V, Q) into a complete $L_\infty[1]$ algebra.

Remark 5.2.3. If we had the enhancement $C(\Delta_n; -) : \widehat{\mathcal{L}}_\infty[1] \rightarrow \widehat{\mathcal{L}}_\infty[1]$ from Remark 3.2.8, we would also have enhancements $C(\Delta_\bullet; -) : \widehat{\mathcal{L}}_\infty[1] \rightarrow \widehat{\mathcal{L}}_\infty[1]^{\Delta^{op}}$ and $\text{Del}_\infty(-) : \widehat{\mathcal{L}}_\infty[1] \rightarrow \mathbf{SSet}$.

We have the following formal consequence of the definition, which anticipates the more profound relationship between $\text{Del}_\infty(-)$ and simplicial mapping spaces in Theorem 5.2.16.

Lemma 5.2.4. *There is a natural isomorphism $\text{MC}(C(-; -)) \xrightarrow{\cong} \mathbf{SSet}(-, \text{Del}_\infty(-))$ of functors $\mathbf{SSet}^{op} \times \widehat{\mathbf{L}}_\infty[1] \rightarrow \mathbf{Set}$.*

Proof. Let V be a complete $L_\infty[1]$ algebra and X a simplicial set, recall that we denote by $\underline{\Delta} X$ the direct category of non degenerate simplices of X (Definition 5.1.8) and the natural isomorphism $X = \text{colim}_{\sigma \in \underline{\Delta} X} \Delta_{|\sigma|}$. The induced natural $C(X; V) \rightarrow \lim_{\sigma \in (\underline{\Delta} X)^{op}} C(\Delta_{|\sigma|}; V)$ is a dg isomorphism, thus also a strict isomorphism of the transferred $L_\infty[1]$ algebra structures, and since $\text{MC}(-)$ commutes with small limits

$$\text{MC}(C(X; V)) = \lim_{\sigma \in (\underline{\Delta} X)^{op}} \text{MC}(C(\Delta_{|\sigma|}; V)) =: \lim_{\sigma \in (\underline{\Delta} X)^{op}} \text{Del}_\infty(V)_{|\sigma|} =$$

$$= \lim_{\sigma \in (\underline{\Delta}^X)^{op}} \mathbf{SSet}(\Delta_{|\sigma|}, \mathrm{Del}_\infty(V)) = \mathbf{SSet}(\mathrm{colim}_{\sigma \in \underline{\Delta}^X} \Delta_{|\sigma|}, \mathrm{Del}_\infty(V)) = \mathbf{SSet}(X, \mathrm{Del}_\infty(V)).$$

That this is natural in X and V is easy and left to the reader. \square

For instance this immediately implies

Corollary 5.2.5. *If for some $k \geq 0$ the complete $L_\infty[1]$ algebra V is concentrated in degrees $> -k$, then the simplicial set $\mathrm{Del}_\infty(V)$ is k -coskeletal (Example 5.1.10).*

Proof. If X is a simplicial set and V is concentrated in degrees $> -k$ then clearly pullback by the inclusion restricts to isomorphisms $C^0(X; V) \xrightarrow{\cong} C^0(\mathrm{sk}_k(X); V)$ and $C^1(X; V) \xrightarrow{\cong} C^1(\mathrm{sk}_k(X); V)$, in particular it sends $\mathrm{MC}(C(X; V))$ isomorphically onto $\mathrm{MC}(C(\mathrm{sk}_k(X); V))$. Thus by the previous lemma the natural

$$\mathrm{Del}_\infty(V)_n = \mathbf{SSet}(\Delta_n, \mathrm{Del}_\infty(V)) \rightarrow \mathrm{cosk}_k(\mathrm{Del}_\infty(V))_n := \mathbf{SSet}(\mathrm{sk}_k(\Delta_n), \mathrm{Del}_\infty(V))$$

is an isomorphism. \square

Another useful lemma is an immediate corollary of Lemma 3.2.6.

Lemma 5.2.6. *The functor $\mathrm{Del}_\infty(-)$ commutes with small limits.*

Recall that an $L_\infty[1]$ algebra $(V, q_1, \dots, q_n, \dots)$ has a canonical compatible filtration $F^\bullet V$, the central descending filtration: this is the smallest compatible filtration on V , it can be defined recursively so that $F^p V$ is the span of $q_k(F^{p_1} V \odot \dots \odot F^{p_k} V)$, $\forall k \geq 2, p_1 + \dots + p_k = p$, and of course $F^1 V = V$. The $L_\infty[1]$ algebra V is nilpotent if $F^p V = 0$ for some $p \gg 0$. A nilpotent $L_\infty[1]$ algebra is complete when equipped with the central descending filtration.

Proposition 5.2.7. *The restriction of $\mathrm{Del}_\infty(-)$ to the full subcategory of nilpotent $L_\infty[1]$ algebras is naturally isomorphic to the functor $\gamma_\bullet(-)$ defined by Getzler in [39].*

Proof. Recall that Getzler defines the simplicial set $\gamma_\bullet(V) = \{\gamma_n(V)\}_{n \geq 0}$ by

$$\gamma_n(V) = \{\omega \in \mathrm{MC}(\Omega_n \otimes V) \text{ s.t. } (K \otimes \mathrm{id}_V)(\omega) = 0\},$$

where K is Dupont's contraction operator (3.1.2). For a nilpotent $L_\infty[1]$ algebra V the completed tensor product $\Omega_n \widehat{\otimes} V =: \Omega(\Delta_n; V)$ and the usual tensor product $\Omega_n \otimes V$, with the filtration $F^p(\Omega_n \otimes V) = \Omega_n \otimes F^p V$, are naturally isomorphic. We apply Theorem 2.3.3 to Dupont's contraction (3.2.1). \square

Remark 5.2.8. Let $(V, F^\bullet V, q_1, \dots, q_n, \dots)$ be a complete $L_\infty[1]$ algebra, then $V = \lim V/F^p V$ in the category $\widehat{\mathbf{L}}_\infty[1]$, where we equip $V/F^p V$ with the induced filtration $F^q(V/F^p V) = F^q V/F^p V$ if $1 \leq q \leq p$, and $F^q(V/F^p V) = 0$ if $q > p$. By Lemma 5.2.6 $\mathrm{Del}_\infty(V) = \lim \mathrm{Del}_\infty(V/F^p V)$, for each $p \geq 1$ the $L_\infty[1]$ algebra $V/F^p V$ is nilpotent, and as we already remarked it doesn't really matter which filtration we are considering on it: by the previous proposition $\mathrm{Del}_\infty(-)$ extends the functor $\gamma_\bullet(-)$ to the category $\widehat{\mathbf{L}}_\infty[1]$ according to $\mathrm{Del}_\infty(V) = \lim \gamma_\bullet(V/F^p V)$, as in [6].

Example 5.2.9. Let $(V, q_1, 0, \dots, 0, \dots)$ be an abelian $L_\infty[1]$ algebra, of course it is a nilpotent $L_\infty[1]$ algebra, and the central descending filtration becomes $F^1 V = V$, $F^p V = 0$, $\forall p \geq 2$. Since extension of scalars and homotopy transfer send abelian $L_\infty[1]$ structures to abelian $L_\infty[1]$ structures, every $C(\Delta_n; V)$ is an abelian $L_\infty[1]$ algebra, and $\mathrm{Del}_\infty(V)$ is the simplicial vector space $\mathrm{Del}_\infty(V)_n = Z^0(C(\Delta_n; V))$, where $Z^0(-) : \mathbf{DG} \rightarrow \mathbf{V}$ is the functor of 0-cocycles. A

direct inspection shows that under the Dold-Kan correspondence $\mathbf{V}^{\Delta^{op}} \rightarrow \mathbf{DG}_{\leq 0}$ (cf. [107]) from simplicial vector spaces to complexes concentrated in non positive degrees, $\text{Del}_{\infty}(V)$ goes into the 0-truncation of the complex (V, q_1)

$$\dots \xrightarrow{q_1} V^{-2} \xrightarrow{q_1} V^{-1} \xrightarrow{q_1} Z^0(V).$$

In particular, by [107], Theorem 8.4.1, we see that $\pi_i(\text{Del}_{\infty}(V), x) \cong H^{-i}(V)$ for all $i \geq 0$ and (when $i \geq 1$) all base points $x \in \text{MC}(V) = Z^0(V)$.

We recall some result from [39], Section 5.

Theorem 5.2.10. *A strict morphism $f : V \rightarrow W$ of complete $L_{\infty}[1]$ algebras induces a Kan fibration $\text{Del}_{\infty}(f) : \text{Del}_{\infty}(V) \rightarrow \text{Del}_{\infty}(W)$ of simplicial sets if and only if it is surjective in degrees < 0 . In particular, the functor $\text{Del}_{\infty}(-)$ factors through the full subcategory $\mathbf{Kan} \subset \mathbf{SSet}$ of Kan complexes.*

Proof. For $n \geq 1$ and $0 \leq i \leq n$ we denote by $\lambda_n^i : \Delta_n^i \rightarrow \Delta_n$ the i -th horn. By Lemma 5.2.4 the set of horns $\Delta_n^i \rightarrow \text{Del}_{\infty}(V)$ is in bijective correspondence with $\text{MC}(C(\Delta_n^i; V))$. We consider the contraction

$$(C(\Delta_n; V) \xrightleftharpoons[f_1]{(\lambda_n^i)^*} C(\Delta_n^i; V), K).$$

The dg morphism f_1 is defined by $f_1(\beta)_{0\dots n} = 0$, $f_1(\beta)_{0\dots\widehat{i}\dots n} = \sum_{j \neq i} (-1)^{i+j+1} \beta_{0\dots\widehat{j}\dots n}$ and $(\lambda_n^i)^* f_1 = \text{id}_{C(\Delta_n^i; V)}$. The operator K is defined by $K(\alpha)_{0\dots\widehat{i}\dots n} = (-1)^{i+1} \alpha_{0\dots n}$ and $K(\alpha)$ evaluates at 0 elsewhere. If the notations are confusing cf. Section 3.1. By Theorem 2.3.3 (cf. also Lemma 2.2.7) there is an identification

$$\text{Del}_{\infty}(V)_n \xrightarrow{\cong} \text{MC}(C(\Delta_n^i; V)) \times V^{-n} : \alpha \rightarrow ((\lambda_n^i)^*(\alpha), \alpha_{0\dots n}).$$

Likewise,

$$\begin{aligned} \mathbf{SSet}(\Delta_n^i, \text{Del}_{\infty}(V)) \times_{\mathbf{SSet}(\Delta_n^i, \text{Del}_{\infty}(W))} \text{Del}_{\infty}(W)_n &\xrightarrow{\cong} \\ &\xrightarrow{\cong} \text{MC}(C(\Delta_n^i; V)) \times_{\text{MC}(C(\Delta_n^i; W))} (\text{MC}(C(\Delta_n^i; W)) \times W^{-n}) = \text{MC}(C(\Delta_n^i; V)) \times W^{-n}. \end{aligned}$$

.

By definition $\text{Del}_{\infty}(f)$ is a Kan fibration if and only if for every horn $\Delta_n^i \rightarrow \Delta_n$ the induced

$$\text{Del}_{\infty}(V)_n \rightarrow \mathbf{SSet}(\Delta_n^i, \text{Del}_{\infty}(V)) \times_{\mathbf{SSet}(\Delta_n^i, \text{Del}_{\infty}(W))} \text{Del}_{\infty}(W)_n$$

is surjective, and this identifies with

$$\text{id} \times f : \text{MC}(C(\Delta_n^i; V)) \times V^{-n} \rightarrow \text{MC}(C(\Delta_n^i; V)) \times W^{-n}.$$

□

Definition 5.2.11. We denote by $\mathcal{MC}(-)$ the functor $\mathcal{MC}(-) := \pi_0(\text{Del}_{\infty}(-)) : \widehat{\mathbf{L}}_{\infty}[1] \rightarrow \mathbf{Set}$. By the previous proposition, if V is a complete $L_{\infty}[1]$ algebra then the set $\mathcal{MC}(V)$ is the quotient of $\text{Del}_{\infty}(V)_0 = \text{MC}(V)$ by an equivalence relation, which by extension of the case of a dg Lie algebra we call the *Gauge equivalence* relation (cf. Definition 5.2.33).

We introduce, for $i = 0, \dots, n$, a degree minus one operator $h^i : C(\Delta_n; V) \rightarrow C(\Delta_n; V)$: with the same notations of Definition 3.1.1

$$h^i : C^{k+1}(\Delta_n; V) \rightarrow C^k(\Delta_n; V) : \\ : \alpha \rightarrow h^i(\alpha)_{i_0 \dots i_k} = \begin{cases} 0 & \text{if } i \in \{i_0, \dots, i_k\} \\ (-1)^j \alpha_{i_0 \dots i_{j-1} i i_j \dots i_k} & \text{if } 0 \leq i_0 < \dots < i_{j-1} < i < i_j < \dots < i_k \leq n \end{cases}$$

We denote by $e_i : \Delta_0 \rightarrow \Delta_n : [0] \rightarrow [i]$ the i -th vertex of Δ_n and by $\pi : \Delta_n \rightarrow \Delta_0$ the final morphism. We leave to the reader to check that the above operator h^i fits into a contraction

$$(C(\Delta_n; V) \xrightleftharpoons[\pi^*]{e_i^*} V, -h^i)$$

satisfying the hypotheses of Lemma 2.2.7. If $\partial_i : \Delta_{n-1} \rightarrow \Delta_n : [0 \dots n-1] \rightarrow [0 \dots \widehat{i} \dots n]$ is the i -th face of the simplex Δ_n , then we notice that ∂_i^* sends $\text{Im } h^i \cap C^{-1}(\Delta_n; V)$ isomorphically onto $C^{-1}(\Delta_{n-1}; V)$. Now Theorem 2.3.3 (cf. also Lemma 2.2.7) implies the following result.

Proposition 5.2.12. *For all $i = 0, \dots, n$ the correspondence*

$$\rho^i : \text{Del}_\infty(V)_n \rightarrow \text{MC}(V) \times C^{-1}(\Delta_{n-1}; V) : \alpha \rightarrow (e_i^*(\alpha), \partial_i^* h^i(\alpha))$$

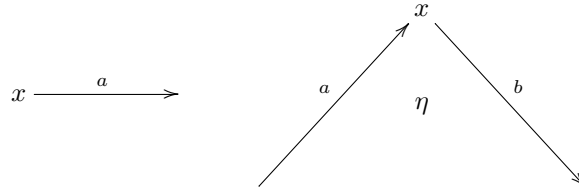
is bijective.

Remark 5.2.13. Lemma 3.1.3 and Lemma 3.1.4 together show that

$$\left(\int \otimes \text{id}_V \right) (h^i \otimes \text{id}_V) = h^i \left(\int \otimes \text{id}_V \right) : \Omega_n \otimes V \rightarrow C(\Delta_n; V),$$

where $\int \otimes \text{id}_V : \Omega_n \otimes V \rightarrow C(\Delta_n; V)$ is integration of forms, $h^i \otimes \text{id}_V : \Omega_n \otimes V \rightarrow \Omega_n \otimes V$ in the left hand side is as in Equation (3.1.1) and finally $h^i : C(\Delta_n; V) \rightarrow C(\Delta_n; V)$ in the right hand side is as above. Keeping this in mind, the reader will recognize that the above proposition is the same as [39], Lemma 5.3.

Remark 5.2.14. We can visualize the thesis of the previous proposition as follows: for each $n \geq 0$ and $0 \leq i \leq n$ we call the open star around the vertex $e_i \in \Delta_n$ the collection of (non-degenerate) simplices which are not in the opposite face $\partial_i \Delta_n$ (notice that an open star is *not* a subsimplicial set), and we assign a Maurer-Cartan element to e_i and arbitrary elements in V^{-j} for each j -simplex of the open star around e_i , as in the following pictures (where $x \in \text{MC}(V)$, $a, b \in V^{-1}$ and $\eta \in V^{-2}$)



The previous proposition tells us that for any such a choice there is a unique Maurer-Cartan cochain in $\text{Del}_\infty(V)_n$ with the assigned restriction on the open star around e_i . Evaluating this cochain on the face $\partial_i \Delta_n$ defines functions

$$\gamma_-^{1,i} : \text{MC}(V) \times V^{-1} \rightarrow \text{MC}(V) : (x, a) \rightarrow \gamma_x^{1,i}(a), \quad i = 0, 1, \text{ and for } n \geq 2$$

$$\gamma_-^{n,i} : \text{MC}(V) \times \prod_{i=i}^n (V^{-i})^{\times \binom{n}{i}} \rightarrow V^{1-n}, \quad 0 \leq i \leq n,$$

as in the following figures

$$\begin{array}{c}
 x \xrightarrow{a} := \gamma_x^{1,0}(a) \\
 \\
 \begin{array}{ccc}
 & x & \\
 a \nearrow & & \searrow b \\
 \gamma_x^{1,1}(a) & \xrightarrow{:= \gamma_x^{2,1}(a,b,\eta)} & \gamma_x^{1,0}(b)
 \end{array}
 \end{array}$$

The functions $\gamma_-^{n,i}$ are called generalized Baker-Campbell-Hausdorff series in [39] (more precisely, Getzler defines these series as the functions $\bar{\gamma}_-^{n,i} = \gamma_-^{n,i}(-, 0) : \text{MC}(V) \times \prod_{i=1}^{n-1} (V^{-i})^{\times \binom{n}{i}} \rightarrow V^{1-n}$): this is because when L is a complete dg Lie algebra then $\bar{\gamma}_0^{2,1}(-, -) = \gamma_0^{2,1}(-, -, 0) : L^0 \times L^0 \rightarrow L^0$ is *precisely* the Baker-Campbell-Hausdorff product on L^0 , cf. Proposition 5.2.36. In this case, moreover, we will see in Proposition 5.2.34 that the functions $\gamma_-^{1,0}, \gamma_-^{1,1} : \text{MC}(L) \times L^0 \rightarrow \text{MC}(L)$ recover the usual Gauge action $* : \exp(L^0) \times \text{MC}(L) \rightarrow \text{MC}(L)$ of the exponential group on Maurer-Cartan elements (the Gauge action will be reviewed, together with the Baker-Campbell-Hausdorff product, in Section 5.3.1), more precisely, we have $\gamma_x^{1,0}(a) = e^{-a} * x$ and $\gamma_x^{1,1}(a) = e^a * x$. We refer to [39], Proposition 5.7, for an explicit computation of $\gamma_-^{1,0}, \gamma_-^{1,1}$ in the case of a general $L_\infty[1]$ algebra. The higher Baker-Campbell-Hausdorff $\gamma_-^{n,i}$ should be intimately related with the structure of ∞ groupoid (in Getzler's sense) on $\text{Del}_\infty(V)$, cf. Proposition 5.2.22 and the following remark.

The main result of [39] is the weak equivalence of the functor $\text{Del}_\infty(-)$ and the classical Hinich-Sullivan functor $\text{MC}_\infty(-) := \text{MC}(\Omega(\Delta_\bullet; -)) : \hat{\mathbf{L}}_\infty[1] \rightarrow \mathbf{SSet}$ [100, 45]. This will be reviewed at the end of the section. Instead we turn our attention to the proof of an important theorem by Berglund [6] and Brown-Szcharba [14] (where in the latter they consider $\text{MC}_\infty(-)$) on the relation between the functor $\text{Del}_\infty(-)$ and simplicial mapping spaces. First we need to prove the following analog of Proposition 1.3.21.

Proposition 5.2.15. *If $0 \rightarrow I \rightarrow V \rightarrow W \rightarrow 0$ in an extension of complete $L_\infty[1]$ algebras, as in Definition 1.3.31, the square in \mathbf{SSet}*

$$\begin{array}{ccc}
 \text{Del}_\infty(I) & \longrightarrow & \text{Del}_\infty(V) \\
 \downarrow & & \downarrow \\
 \Delta_0 & \xrightarrow{0} & \text{Del}_\infty(W)
 \end{array}$$

is cartesian, where we denote by $\Delta_0 \xrightarrow{0} \text{Del}_\infty(W)$ the inclusion of the vertex $0 \in \text{MC}(W)$. If $0 \rightarrow I \rightarrow V \rightarrow W \rightarrow 0$ is a central extension of complete $L_\infty[1]$ algebras, then the simplicial abelian group $\text{Del}_\infty(I)$ acts principally on the right on $\text{Del}_\infty(V)$. Moreover, there is an obstruction map $o : \text{MC}(W) \rightarrow H^1(I)$ such that the kernel of the obstruction coincides with the image of $\text{MC}(V) \rightarrow \text{MC}(W)$. Finally, if we denote by $K(W)$ the Kan subcomplex of $\text{Del}_\infty(W)$ consisting of connected components in $\text{Ker } o$, then $\text{Del}_\infty(V) \rightarrow K(W)$ is isomorphic to the principal fibration associated to the action of $\text{Del}_\infty(I)$, as in [81], § 18.

Proof. The first claim follows from Lemma 5.2.6, as

$$\begin{array}{ccc}
 I & \longrightarrow & V \\
 \downarrow & & \downarrow \\
 0 & \longrightarrow & W
 \end{array}$$

is a cartesian square in $\widehat{\mathbf{L}}_\infty[1]$.

We claim that a Maurer-Cartan cochain $\alpha \in \text{Del}_\infty(W)_n$ lifts to $\tilde{\alpha} \in \text{Del}_\infty(V)_n$ if and only if for some (and then for all) $i = 0, \dots, n$ its evaluation $x := e_i^*(\alpha) \in \text{MC}(W)$ at the i -th vertex $e_i : \Delta_0 \rightarrow \Delta_n : [0] \rightarrow [i]$ lifts to a Maurer-Cartan element $\tilde{x} \in \text{MC}(V)$: in fact, we have a morphism of central extensions of complete $L_\infty[1]$ algebras

$$\begin{array}{ccccccc} 0 & \longrightarrow & C(\Delta_n; I) & \longrightarrow & C(\Delta_n; V) & \longrightarrow & C(\Delta_n; W) \longrightarrow 0 \\ & & e_i^* \downarrow & & e_i^* \downarrow & & e_i^* \downarrow \\ 0 & \longrightarrow & I & \longrightarrow & V & \longrightarrow & W \longrightarrow 0 \end{array}$$

and by Remark 1.3.22 $H(e_i^*)(o(\alpha)) = o(e_i^*(\alpha)) = o(x)$, as $H(e_i^*)$ is an isomorphism the claim follows. By Proposition 1.3.21 the abelian group $\text{Del}_\infty(I)_n = Z^0(C(\Delta_n; I))$ acts on the right on the set of Maurer-Cartan liftings of α , when this is not empty.

Next we claim that the obstruction map $o : \text{MC}(W) \rightarrow H^1(I)$ from Proposition 1.3.21 factors through the projection $\text{MC}(W) \rightarrow \mathcal{MC}(W)$: in fact, we notice that $H(e_0^*) = H(e_1^*)$, as they are both inverses to pullback by the final morphism $H(\pi^*) : H(W) \rightarrow H(C(\Delta_1; W))$, thus, given Maurer-Cartan elements $x, y \in \text{MC}(W)$ and $\alpha \in \text{Del}_\infty(W)_1$ such that $x = e_0^*(\alpha), y = e_1^*(\alpha)$, we see that $o(x) = H(e_0^*)(o(\alpha)) = H(e_1^*)(o(\alpha)) = o(y)$. By the previous claim the resulting $o : \mathcal{MC}(W) \rightarrow H^1(I)$ has the required properties.

Now the rest of the proposition follows easily. \square

Recall that we denote by $\mathbf{SSet}(-, -) = \mathbf{SSet}(\Delta_\bullet \times -, -) : \mathbf{SSet}^{op} \times \mathbf{SSet} \rightarrow \mathbf{SSet}$ the simplicial mapping space functor, as in the previous section. The reader should compare the following theorem with [6], Theorem 5.5, and with [14], Theorem 2.20, in particular, notice that we are not putting any restriction on X or V .

Theorem 5.2.16. *There is a natural weak equivalence $\text{Del}_\infty(C(-; -)) \xrightarrow{\sim} \mathbf{SSet}(-, \text{Del}_\infty(-))$ of functors $\mathbf{SSet}^{op} \times \widehat{\mathbf{L}}_\infty[1] \rightarrow \mathbf{SSet}$.*

Proof. For a complete $L_\infty[1]$ algebra V , a simplicial set X and an integer $n \geq 0$, we define the required $\text{Del}_\infty(C(X; V))_n \rightarrow \mathbf{SSet}(X, \text{Del}_\infty(V))_n$ as the following long composition (at this point the reader should probably take another look at Remark 2.2.4, which was made having the next passage in mind)

$$\begin{aligned} \text{Del}_\infty(C(X; V))_n &= \text{MC}(C(\Delta_n; C(X; V))) \xrightarrow{\text{MC}(F)} \\ &\xrightarrow{\text{MC}(F)} \text{MC}(\Omega(\Delta_n; \Omega(X; V))) \xrightarrow{\text{MC}(p_1^* \wedge p_2^*)} \text{MC}(\Omega(\Delta_n \times X; V)) \xrightarrow{\text{MC}(G)} \\ &\xrightarrow{\text{MC}(G)} \text{MC}(C(\Delta_n \times X; V)) \xrightarrow{\cong} \mathbf{SSet}(\Delta_n \times X, \text{Del}_\infty(V)) = \mathbf{SSet}(X, \text{Del}_\infty(V))_n, \end{aligned}$$

where F is the composite homotopy transfer morphism

$$C(\Delta_n; C(X; V)) \rightarrow \Omega_n \widehat{\otimes} C(X; V) \rightarrow \Omega_n \widehat{\otimes} (\Omega(X) \widehat{\otimes} V) = \Omega(\Delta_n; \Omega(X; V)),$$

cf. Lemma 2.2.8, $G : \Omega(\Delta_n \times X; V) \rightarrow C(\Delta_n \times X; V)$ is induced by homotopy transfer as in Remark 2.2.4, p_1 and p_2 are the projections of $\Delta_n \times X$ onto the first and second factor respectively and finally $p_1^* \wedge p_2^* : \Omega(\Delta_n; \Omega(X; V)) = \Omega_n \widehat{\otimes} (\Omega(X) \widehat{\otimes} V) = (\Omega_n \otimes \Omega(X)) \widehat{\otimes} V \rightarrow \Omega(\Delta_n \times X) \widehat{\otimes} V = \Omega(\Delta_n \times X; V)$ is induced by

$$\Omega_n \otimes \Omega(X) \xrightarrow{p_1^* \otimes p_2^*} \Omega(\Delta_n \times X)^{\otimes 2} \xrightarrow{\wedge} \Omega(\Delta_n \times X).$$

The fact that this defines a morphism of simplicial sets and the fact that this is natural in X and V are both consequences of Lemma 2.2.3 and Remark 2.2.4.

Remark 5.2.17. Notice that we used in an essential way the fact that we are requiring naturality only with respect to *strict* $L_\infty[1]$ morphisms $f : V \rightarrow W$. This is trivially so, since we don't know any way to define an enhancement $C(X; -) : \widehat{\mathcal{L}}_\infty[1] \rightarrow \widehat{\mathcal{L}}_\infty[1]$, but there is another reason more subtle: as already remarked such an enhancement $\Omega(X; -) : \widehat{\mathcal{L}}_\infty[1] \rightarrow \widehat{\mathcal{L}}_\infty[1]$ exists, since scalar extension by a dg commutative algebra is a functor $\widehat{\mathcal{L}}_\infty[1] \rightarrow \widehat{\mathcal{L}}_\infty[1]$. If we look for an enhancement of $C(X; -)$ such that F and G remain natural transformation as in Lemma 2.2.3 and Remark 2.2.4, then the pushforward of an $L_\infty[1]$ morphism $H : V \rightarrow W$ would have to be $H_* : C(X; V) \xrightarrow{F} \Omega(X; V) \xrightarrow{H_*} \Omega(X; W) \xrightarrow{G} C(X; W)$: in fact, when $H = h$ is strict the above composition is precisely $h_* : C(X; V) \rightarrow C(X; W)$. On the other hand this definition fails to be functorial with respect to general $L_\infty[1]$ morphisms, as one can see by looking at the case when V and W are abelian. To sum up, even if we had an enhancement $C(X; -) : \widehat{\mathcal{L}}_\infty[1] \rightarrow \widehat{\mathcal{L}}_\infty[1]$ this can't be obtained by transferring the enhancement $\Omega(X; -) : \widehat{\mathcal{L}}_\infty[1] \rightarrow \widehat{\mathcal{L}}_\infty[1]$. Moreover, to give a natural transformation $\text{Del}_\infty(C(-; -)) \xrightarrow{\sim} \mathbf{SSet}(-, \text{Del}_\infty(-))$ of functors $\mathbf{SSet}^{op} \times \widehat{\mathcal{L}}_\infty[1] \rightarrow \mathbf{SSet}$ we would still need a natural $C(\Delta_n; C(X; -)) \xrightarrow{\sim} C(\Delta_n \times X; -)$ of functors $\widehat{\mathcal{L}}_\infty[1] \rightarrow \widehat{\mathcal{L}}_\infty[1]$, that is, a natural $L_\infty[1]$ enhancement of the classical Eilenberg-Zilber quasi-isomorphism, and this can't be obtained anymore by transferring the natural $\Omega(\Delta_n; \Omega(X; -)) \xrightarrow{\sim} \Omega(\Delta_n \times X; -)$.

Having defined the natural $\text{Del}_\infty(C(X; V)) \rightarrow \mathbf{SSet}(X, \text{Del}_\infty(V))$, we have to check that this is a weak equivalence. Pull back from the terminal morphism $\pi : X \rightarrow \Delta_0$ induces

$$\begin{aligned} \text{Del}_\infty(V) &= \text{Del}_\infty(C(\Delta_0; V)) \rightarrow \text{Del}_\infty(C(X; V)) \quad \text{and} \\ \text{Del}_\infty(V) &= \mathbf{SSet}(\Delta_0, \text{Del}_\infty(V)) \rightarrow \mathbf{SSet}(X, \text{Del}_\infty(V)), \end{aligned}$$

and the following diagram is commutative

$$\begin{array}{ccc} \text{Del}_\infty(C(X; V)) & \xrightarrow{\quad} & \mathbf{SSet}(X, \text{Del}_\infty(V)) \\ & \searrow & \nearrow \\ & \text{Del}_\infty(V) & \end{array}$$

When $X = \Delta_m$ is a simplex, since Δ_m is contractible, that is, $\Delta_m \rightarrow \Delta_0$ is a weak equivalence, and $\text{Del}_\infty(V)$ is a Kan complex, it is well known that $\text{Del}_\infty(V) \rightarrow \mathbf{SSet}(\Delta_m, \text{Del}_\infty(V))$ is a weak equivalence. We claim that $\text{Del}_\infty(V) \rightarrow \text{Del}_\infty(C(\Delta_m; V))$ is a weak equivalence for all $m \geq 0$, thus, by two out of three, so is $\text{Del}_\infty(C(\Delta_m; V)) \rightarrow \mathbf{SSet}(\Delta_m, \text{Del}_\infty(V))$.

When V is abelian with the central descending filtration, so is $C(\Delta_m; V)$ for all $m \geq 0$, and the claim follows from Example 5.2.9. We suppose inductively that the claim has been proved for the nilpotent $L_\infty[1]$ algebra V/F^pV , the filtration defined as in Remark 1.3.13, and wish to prove it for the nilpotent $L_\infty[1]$ algebra $V/F^{p+1}V$. Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C(\Delta_m; F^pV/F^{p+1}V) & \longrightarrow & C(\Delta_m; V/F^{p+1}V) & \longrightarrow & C(\Delta_m; V/F^pV) \longrightarrow 0 \\ & & \uparrow \pi^* & & \uparrow \pi^* & & \uparrow \pi^* \\ 0 & \longrightarrow & F^pV/F^{p+1}V & \longrightarrow & V/F^{p+1}V & \longrightarrow & V/F^pV \longrightarrow 0 \end{array}$$

The rows are central extensions of complete L_∞ algebras, we will use Proposition 5.2.15: recall from its claim the definition of $K(V/F^pV)$, $K(C(\Delta; V/F^pV))$. By the inductive hypothesis

$\mathrm{Del}_\infty(V/F^pV) \rightarrow \mathrm{Del}_\infty(C^*(\Delta_m; V/F^pV))$ is a weak equivalence, and by naturality of the obstructions it restricts to a weak equivalence $K(V/F^pV) \rightarrow K(C(\Delta_m; V/F^pV))$, but then we also see that $\mathrm{Del}_\infty(V/F^{p+1}V) \rightarrow \mathrm{Del}_\infty(C(\Delta_m; V/F^{p+1}V))$ is a morphism of principal fibrations inducing weak equivalences between the bases and the fibres, hence a weak equivalence. This concludes the inductive step, the claim for $V = \lim V/F^pV$ follows since

$$\lim \mathrm{Del}_\infty(V/F^pV) = \mathrm{Del}_\infty(V) \rightarrow \mathrm{Del}_\infty(C(\Delta_m; V)) = \lim \mathrm{Del}_\infty(C(\Delta_m; V/F^pV))$$

is the limit of a weak equivalence between injectively fibrant towers of simplicial sets, cf. Remark 5.1.3 (here the Reedy fibrancy condition explained in Remark 5.1.1 simply says that all the arrows of the tower are fibrations and the bottom is fibrant, which is clear by Theorem 5.2.10).

Finally, the morphism $\mathrm{Del}_\infty(C(X; V)) \rightarrow \underline{\mathbf{SSet}}(X, \mathrm{Del}_\infty(V))$ is a weak equivalence, since it is the limit

$$\lim_{\sigma \in (\underline{\Delta}X)^{op}} \mathrm{Del}_\infty(C(\Delta_{|\sigma|}; V)) \rightarrow \lim_{\sigma \in (\underline{\Delta}X)^{op}} \underline{\mathbf{SSet}}(\Delta_{|\sigma|}, \mathrm{Del}_\infty(V))$$

of a weak equivalence between injectively fibrant diagrams of simplicial sets over the inverse category $(\underline{\Delta}X)^{op}$ of non degenerate simplices of X . For instance, the fact that

$$\mathrm{Del}_\infty(C(\Delta_{|\cdot|}; V)) : (\underline{\Delta}X)^{op} \rightarrow \mathbf{SSet} : \sigma \rightarrow \mathrm{Del}_\infty(C(\Delta_{|\sigma|}; V))$$

where $\sigma : \Delta_{|\sigma|} \rightarrow X$ is a non-degenerate simplex of X , is Reedy fibrant, that is, all matching morphisms

$$\mathrm{Del}_\infty(C(\Delta_{|\sigma|}; V)) \rightarrow M_\sigma \mathrm{Del}_\infty(C(\Delta_{|\cdot|}; V)) := \lim_{\tau \in (\underline{\Delta}X)^{op}, \tau < \sigma} \mathrm{Del}_\infty(C(\Delta_{|\tau|}; V))$$

are fibrations, cf. Remark 5.1.1, where $\tau < \sigma$ means that the non-degenerate simplex τ is a proper face of the non-degenerate simplex σ , follows from the fact that $\mathrm{Del}_\infty(-)$ commutes with small limits and Theorem 5.2.10. \square

In [6] the previous theorem is proved via the explicit computations of the homotopy groups $\pi_i(\mathrm{Del}_\infty(V), x)$, for all $n \geq 1$ and all base points $x \in \mathrm{MC}(V)$, next we want to review this computation

Definition 5.2.18. Given a complete $L_\infty[1]$ algebra V and a Maurer-Cartan element $x \in \mathrm{MC}(V)$, we call $\pi_i(\mathrm{Del}_\infty(V), x)$ the i -th homotopy group of V at x and we denote it by $\pi_i(V, x)$.

Given $x \in \mathrm{MC}(V)$ we denote by V_x the $L_\infty[1]$ algebra obtained after twisting the $L_\infty[1]$ structure on V by x , as in Proposition 1.3.17.

Lemma 5.2.19. *Let V be a complete $L_\infty[1]$ algebra. For all $x \in \mathrm{MC}(V)$ and $i \geq 1$, there is a natural isomorphism of groups $\pi_i(V, x) \cong \pi_i(V_x, 0)$.*

Proof. (cf. [6], Proposition 4.9) Let $\pi : \Delta_n \rightarrow \Delta_0$ be the final morphism, $x \in \mathrm{MC}(V)$ and $\mathrm{MC}(\pi^*)(x) \in \mathrm{Del}_\infty(V)_n$ ($\mathrm{MC}(\pi^*)(x)$ is the 0-cochain which evaluates to x at the vertices of Δ_n and to zero elsewhere). As it is easy to see that $C(\Delta_n; V)_{\mathrm{MC}(\pi^*)(x)} = C(\Delta_n; V_x)$, the thesis follows from Lemma 1.3.18, which implies in fact an isomorphism $(\mathrm{Del}_\infty(V), x) \rightarrow (\mathrm{Del}_\infty(V_x), 0)$ of pointed simplicial sets. \square

Theorem 5.2.20. *Let $(V, F^\bullet V, q_1, \dots, q_n, \dots)$ be a complete $L_\infty[1]$ algebra together with a Maurer-Cartan element $x \in \mathrm{MC}(V)$. For all $i \geq 2$ there is an isomorphism $\pi_i(V, x) \cong H^{-i}(V_x)$ of abelian groups. Moreover, $\pi_1(V, x) \cong H^{-1}(V_x)$, the latter seen as a group via the Baker-Campbell-Hausdorff product.*

Proof. Cf. Section 5.3.1 for the definition of the Baker-Campbell-Hausdorff product. By the previous lemma we are reduced to the case $x = 0$, we may also suppose that $F^\bullet V$ is the central descending filtration, cf. Remark 5.2.2. Recall from [81], Definition 3.6, the following combinatorial description, due to D. M. Kan, of the homotopy groups $\pi_i(V, 0)$. Let $\tilde{\pi}_i(V, 0)$ be the set of Maurer-Cartan cochains $\alpha \in \text{MC}(C(\Delta_i, \partial\Delta_i; V)) \subset \text{MC}(C(\Delta_i; V)) = \text{Del}_\infty(V)_i$, in other words, α evaluates at zero everywhere but on the top dimensional simplex. This set is in bijective correspondence with the set of $(-i)$ -cocycles $z \in Z^{-i}(V)$ via $\alpha \rightarrow \alpha_{0\dots i} = z \in V^{-i}$, in fact, since we immediately see thanks to Lemma 2.2.3 that the $L_\infty[1]$ subalgebra $C(\Delta_i, \partial\Delta_i; V) \subset C(\Delta_i; V)$ is abelian, such an α is Maurer-Cartan if and only if $q_1(z) = 0$. As a set, $\pi_i(V, 0)$ is the quotient of $\tilde{\pi}_i(V, 0)$ by the homotopy relation: recall, [81] Definition 3.1, that given $z, z' \in Z^{-i}(V)$ and the respective $\alpha, \alpha' \in \tilde{\pi}_i(V, 0)$ a homotopy between α and α' is a Maurer-Cartan cochain $\beta \in \text{MC}(C(\Delta_{i+1}; V))$ such that β evaluates at zero everywhere but on the last two faces $\beta_{0\dots\widehat{i}(i+1)} = z$, $\beta_{0\dots i\widehat{(i+1)}} = z'$ and on the top dimensional simplex $\beta_{0\dots i(i+1)} = v \in V^{-i-1}$.

Given $i \geq 1$ we denote by $\tau_{\leq -i}V$ the truncation of V at the degree $(-i)$, that is, the $L_\infty[1]$ subalgebra $\tau_{\leq -i}V \subset V$ such that $(\tau_{\leq -i}V)^j = V^j$ for $j < -i$, $(\tau_{\leq -i}V)^{-i} = Z^{-i}(V)$ and finally $(\tau_{\leq -i}V)^j = 0$ for $j > -i$. By the previous discussion it is clear that the inclusion $\tau_{\leq -i}V \rightarrow V$ induces an isomorphism $\pi_i(\tau_{\leq -i}V, 0) \rightarrow \pi_i(V, 0)$: in fact any two $\alpha, \alpha' \in \tilde{\pi}_i(V, 0)$ and any homotopy between them factor through the inclusion $\text{Del}_\infty(\tau_{\leq -i}V) \rightarrow \text{Del}_\infty(V)$. We consider $H^{-i}(V)$ as an $L_\infty[1]$ algebra concentrated in degree $(-i)$, equipped with the trivial $L_\infty[1]$ structure if $i > 1$, and with the Lie bracket induced by q_2 if $i = 1$. It is straightforward that $\tau_{\leq -i}V \rightarrow H^{-i}(V)$ sending $z \in Z^{-i}(V)$ to its cohomology class $[z] \in H^{-i}(V)$ (and obviously V^j to zero if $j < -i$) is a strict morphism of $L_\infty[1]$ algebras, as such it induces $\pi_i(\tau_{\leq -i}V, 0) \rightarrow \pi_i(H^{-i}(V), 0)$: we claim that this is an isomorphism for all $i \geq 1$. The claim implies the thesis: in fact, we see by Example 5.2.9 that $\pi_i(H^{-i}(V), 0) \cong H^{-i}(V)$ as abelian groups whenever $i > 1$, moreover, it will follow from Proposition 5.2.36 that $\pi_1(H^{-1}(V), 0) \cong H^{-1}(V)$, the latter seen as a group via the Baker-Campbell-Hausdorff product.

Surjectivity of $\pi_i(\tau_{\leq -i}V, 0) \rightarrow \pi_i(H^{-i}(V), 0)$ is clear by the previous explicit description as sets, to show injectivity we notice that the induced

$$Z^{-i}(V) \xrightarrow{\cong} \tilde{\pi}_i(\tau_{\leq -i}V, 0) \rightarrow \pi_i(\tau_{\leq -i}V, 0) \rightarrow \pi_i(H^{-i}(V), 0) \xrightarrow{\cong} H^{-i}(V)$$

sends $z \in Z^{-i}(V)$ to its cohomology class $[z] \in H^{-i}(V)$, so we have to show that if $z = dv$ for some $v \in V^{-i-1}$ then the corresponding Maurer-Cartan cochain $\alpha \in \tilde{\pi}_i(\tau_{\leq -i}V, 0)$ is homotopic to zero. The homotopy is given by $\beta \in \text{MC}(C(\Delta_{i+1}, \Lambda_{i+1}^i; \tau_{\leq -i}V)) \subset \text{MC}(C(\Delta_{i+1}; \tau_{\leq -i}V))$ such that $\beta_{0\dots\widehat{i}(i+1)} = z$ and $\beta_{0\dots i(i+1)} = v$: the fact that β is Maurer-Cartan follows since $C(\Delta_{i+1}, \Lambda_{i+1}^i; \tau_{\leq -i}V)$ is an abelian $L_\infty[1]$ subalgebra of $C(\Delta_{i+1}; \tau_{\leq -i}V)$, again by Lemma 2.2.3. \square

We close this long section by comparing some properties of the functor $\text{Del}_\infty(-)$ and the more classical Hinich-Sullivan functor $\text{MC}(-)_\infty$. We have already observed that the functor of polynomial forms $\Omega(\Delta_\bullet; -) : \widehat{\mathbf{L}}_\infty[1] \rightarrow \widehat{\mathbf{L}}_\infty[1]^{\Delta^{op}}$ on the standard cosimplicial simplex has a natural enhancement to a functor $\Omega(\Delta_\bullet; -) : \widehat{\mathcal{L}}_\infty[1] \rightarrow \widehat{\mathcal{L}}_\infty[1]^{\Delta^{op}}$, since scalar extension by a commutative dg algebra is a functor $\mathcal{L}_\infty[1] \rightarrow \mathcal{L}_\infty[1]$.

Definition 5.2.21. The functor $\text{MC}_\infty(-) : \widehat{\mathcal{L}}_\infty[1] \rightarrow \mathbf{SSet}$ is the composition

$$\text{MC}_\infty(-) : \widehat{\mathcal{L}}_\infty[1] \xrightarrow{\Omega(\Delta_\bullet; -)} \widehat{\mathcal{L}}_\infty[1]^{\Delta^{op}} \xrightarrow{\text{MC}(-)} \mathbf{SSet}$$

Given a complete $L_\infty[1]$ algebra V we call the simplicial set $\mathrm{MC}_\infty(V)$ the Maurer-Cartan ∞ groupoid of V ¹.

Of course one of the advantages of working with the functor $\mathrm{MC}_\infty(-)$ is that it is easily defined on $L_\infty[1]$ morphisms: for instance in the recent preprint [29] it is proven the expected result that given a continuous $L_\infty[1]$ morphism $F : V \rightarrow W$, if F is a weak equivalence then so is $\mathrm{MC}_\infty(F)$. There are some technical advantages when working with the functor $\mathrm{Del}_\infty(-)$, for instance the fact that it commutes with all small limits whereas in general $\mathrm{MC}_\infty(-)$ only commutes with finite limits. From the point of view of homotopy theory we will prove after Getzler [39] that the space $\mathrm{Del}_\infty(V)$ is a deformation retract of the space $\mathrm{MC}_\infty(V)$, in particular they both model the same rational homotopy type. In this context the role of the functor $\mathrm{Del}_\infty(-)$ will become apparent in the next section: it is the right adjoint of a natural Lie model functor on spaces. From another point of view the functor $\mathrm{Del}_\infty(-)$ is more closely related to classical constructions in deformation theory: in particular, when V is associated to a dg Lie algebra concentrated in degrees ≥ 0 via décalage, the space $\mathrm{Del}_\infty(V)$ is precisely the nerve of the Deligne groupoid (Theorem 5.2.37). Although we will not pursue this point of view, the real reason to consider the functor $\mathrm{Del}_\infty(-)$ is that, while both $\mathrm{MC}_\infty(-)$ and $\mathrm{Del}_\infty(-)$ are means to integrate pronilpotent L_∞ algebras to ∞ groupoids in a way which generalizes (homotopically) the way a nilpotent Lie algebra integrates to its exponential group via the Baker-Campbell-Hausdorff product, $\mathrm{MC}_\infty(-)$ only factors through the category **Kan** of Kan complexes, which are ∞ groupoid only in a loose (homotopical) sense, while the functor $\mathrm{Del}_\infty(-)$ factors through the more structured category **wTKan** of weak T -complexes (Definition 5.1.11), which are ∞ groupoids in a more precise (even if not fully understood) sense, cf. also the following Remark 5.2.23: in fact, as already said, for an ordinary nilpotent Lie algebra \mathfrak{g} the space $\mathrm{Del}_\infty(\mathfrak{g})$ is precisely the nerve of $\exp(\mathfrak{g})$, so in this case the generalization is not just by analogy.

Proposition 5.2.22. *The functor $\mathrm{Del}_\infty(-)$ factors through $\mathrm{Del}_\infty(-) : \widehat{\mathbf{L}}_\infty[1] \rightarrow \mathbf{wTKan}$ and the forgetful functor $\mathbf{wTKan} \rightarrow \mathbf{SSet}$.*

Proof. Given a complete $L_\infty[1]$ algebra V , the thin simplices of $\mathrm{Del}_\infty(V)$ are the Maurer-Cartan cochains evaluating to 0 on the top dimensional simplex: for $n \geq 1$

$$T_n \mathrm{Del}_\infty(V) = \{\alpha \in \mathrm{Del}_\infty(V)_n \text{ s.t. } \alpha_{0\dots n} = 0\}.$$

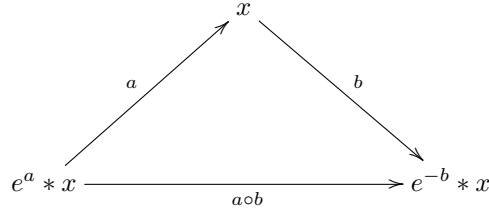
It is easy, by the proof of Theorem 5.2.10, cf. also Remark 5.2.14 that the required conditions (1) and (2) in Definition 5.1.11 are satisfied. \square

Remark 5.2.23. Hidden in the axioms of a weak T -complex there is a rich algebraic structure given by filling procedures, cf. the theses of Ashley [1] and Dakin [26]: to illustrate this point let us again consider the case of a complete dg Lie algebra L (seen as usual via décalage). We will prove in Section 5.2.2 that in this case we can recover the Baker-Campbell-Hausdorff product on L via the following filling procedure: we take a horn $\Lambda_2^1 \rightarrow \mathrm{Del}_\infty(L)$

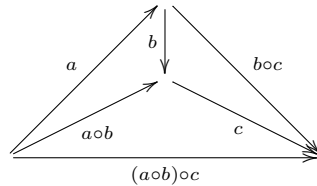
$$\begin{array}{ccc} & x & \\ a \nearrow & & \searrow b \\ e^a * x & & e^{-b} * x \end{array}$$

¹Although this is not a groupoid in the sense of Getzler's paper [39], that is, a weak T -complex as in Definition 5.1.11, but only in the loose sense that it is a Kan complex.

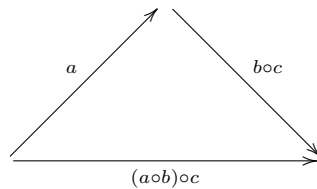
(it doesn't matter the particular choice of $x \in \text{MC}(L)$) and then the unique thin filling is



where we denote by $a \circ b$ the Baker-Campbell-Hausdorff product $L^0 \times L^0 \rightarrow L^0$. With the notations of Remark 5.2.14 we have that $\bar{\gamma}_x^{2,1} = \circ$ for all $x \in \text{MC}(L)$. The fact that when L is concentrated in degrees ≥ 0 then $\text{Del}_\infty(L)$ is the nerve of the ordinary Deligne groupoid follows from the above description of 2-simplices and Lemma 5.2.5. Even without knowing the above facts, it is immediate to notice that when L is concentrated in degrees ≥ 0 the remaining axiom for $\text{Del}_\infty(L)$ to be a T -complex is satisfied, since every n -simplex is automatically thin for $n \geq 2$, and in fact this says more, that $\text{Del}_\infty(L)$ is a T -complex of rank one, thus by the results in [1, 26] the nerve of a groupoid (this is a well known argument, cf. also [73], Chapter 1): notice that this only uses the fact that L is concentrated in degrees ≥ 0 , so L maybe an otherwise arbitrary L_∞ algebra (and it makes sense in this case to talk about the Deligne groupoid of L , cf. Definition 5.2.33). We refer to [39], in particular the proof of Proposition 5.4, to see that when the dg Lie algebra L is of quantum type, that is, concentrated in degrees ≥ -1 , then $\text{Del}_\infty(L)$ is a T -complex of rank two (here the fact that we are working with dg Lie algebras is essential), in particular by [1, 26] it defines a crossed complex in groupoids (and in fact two possible such structures), that is, a 2-groupoid: this should recover the Deligne 2-groupoid introduced by Deligne in private correspondence [27], cf. also [38, 110], but we remark that it would be essential to have some explicit computation to see what's actually going on. It is in general *not* true that if L is a complete dg Lie algebra concentrated in degrees ≥ -2 then $\text{Del}_\infty(L)$ is a T -complex of rank three. For simplicity, let us assume now that $L = \mathfrak{g}$ is an ordinary Lie algebra: even without knowing that it is the Baker-Campbell-Hausdorff product let us sketch a proof, borrowed from [1], of the fact that $\circ := \bar{\gamma}_0^{2,1}$ is an associative product on \mathfrak{g} . With the notations of Remark 5.2.14 consider the following horn $\Lambda_3^2 \rightarrow \text{Del}_\infty(\mathfrak{g})$



We take the unique thin filling of this horn, since \mathfrak{g} has no elements in degree minus one the remaining face is automatically thin, that is,



is thin: but by definition this means that $(a \circ b) \circ c = a \circ (b \circ c)$ (in other words, associativity follows from the uniqueness assumption in axiom (2)). Let us notice here that the above filling

procedure makes sense for an arbitrary $L_\infty[1]$ algebra $(V, q_1, \dots, q_n, \dots)$ concentrated in negative degrees (in Getzler's paper this is called an ∞ Lie algebra, and $\text{Del}_\infty(V)$ an ∞ group): in this case $\text{MC}(V) = 0$ and $\text{Del}_\infty(V)_1 \cong V^{-1}$. We put $\circ := \gamma_0^{1,2} : V^{-1} \times V^{-1} \rightarrow V^{-1}$, by universality of the formulas this will be again the Baker-Campbell-Hausdorff product \circ associated to the bracket q_2 on V^{-1} , but notice that the latter does not satisfy the Jacobi identity anymore, the failure being measured by the bracket $q_3 : (V^{-1})^{\odot 3} \rightarrow V^{-2}$. Now the previous filling procedure followed by evaluation on the remaining face defines a function $V^{-1} \times V^{-1} \times V^{-1} \rightarrow V^{-2}$: this measures the failure of \circ to be associative and should integrate in an appropriate sense the bracket q_3 . More in general it should be possible to integrate concretely (that is, as functions) the higher brackets via similar filling procedures, giving a precise meaning to the slogan that the simplicial set $\text{Del}_\infty(V)$ integrates the $L_\infty[1]$ algebra V . We stop this informal discussion here, and close this section with the promised proof of the weak equivalence between $\text{Del}_\infty(-)$ and $\text{MC}_\infty(-)$.

Theorem 5.2.24. *There is a natural weak equivalence $\text{Del}_\infty(-) \xrightarrow{\sim} \text{MC}_\infty(-)$, where both are seen as functors $\widehat{\mathbf{L}}_\infty[1] \rightarrow \mathbf{SSet}$.*

Proof. We remark that the analogs of Example 5.2.9, Theorem 5.2.10 and Proposition 5.2.15 hold for the functor $\text{MC}_\infty(-)$ with similar proofs, we refer to [39] (cf. also [6] for the latter) for details. For all $n \geq 0$ we have by homotopy transfer an $L_\infty[1]$ morphism $F : C(\Delta_n; V) \rightarrow \Omega(\Delta_n; V)$, hence an induced $\text{MC}(F) : \text{Del}_\infty(V)_n \rightarrow \text{MC}_\infty(V)_n$: this defines a natural $\text{Del}_\infty(-) \rightarrow \text{MC}_\infty(-)$ by Lemma 2.2.7. To prove that this is a weak equivalence we proceed inductively as in the proof of Theorem 5.2.16. If V is an abelian $L_\infty[1]$ algebra with the central descending filtration, then $F = f_1$ is a quasi-isomorphism and the thesis follows from 5.2.9. As in the proof of 5.2.16, we show the thesis inductively for the $L_\infty[1]$ algebras V/F^pV by comparing the central extensions of simplicial complete $L_\infty[1]$ algebras $C(\Delta_\bullet; F^pV/F^{p+1}V) \rightarrow C(\Delta_\bullet; V/F^{p+1}V) \rightarrow C(\Delta_\bullet; V/F^pV)$ and $\Omega(\Delta_\bullet; F^pV/F^{p+1}V) \rightarrow \Omega(\Delta_\bullet; V/F^{p+1}V) \rightarrow \Omega(\Delta_\bullet; V/F^pV)$ via Proposition 5.2.15, then the thesis for $V = \lim V/F^pV$ follows by passing to the limit. \square

5.2.1 Disconnected rational homotopy theory

The aim of this section is to construct a model category structure on the category $\widehat{\mathbf{DGLA}}$ of complete dg Lie algebras and explain briefly its relevance to rational homotopy theory. This model category structure has been defined in [70], via Koszul duality between commutative and Lie algebras and the usual model category structure on the category of (unbounded) commutative dg algebras. We follow a different route, already outlined in Section 5.1: namely, we consider a natural cosimplicial object $L(\Delta_\bullet)$ in $\widehat{\mathbf{DGLA}}$, the Lie-Sullivan model of the standard cosimplicial simplex. By Proposition 5.1.9 this induces an adjunction $\mathbf{SSet} \rightleftarrows \widehat{\mathbf{DGLA}}$, we will see that both the left and the right adjoint are geometrically meaningful, in fact: the right adjoint is just the restriction of $\text{Del}_\infty(-) : \widehat{\mathbf{L}}_\infty \rightarrow \mathbf{SSet}$ to the full subcategory of complete dg Lie algebras, we will see that the left adjoint $L(-) : \mathbf{SSet} \rightarrow \widehat{\mathbf{DGLA}}$ can be interpreted as sending the simplicial set X to a natural C_∞ coalgebra structure on the space $C_*(X)/D_*(X)$ of non-degenerate chains on X (we notice here that the quadratic part will be an approximation of the diagonal, thus we recover a construction considered by Sullivan in the appendix of the paper [102]), moreover, further transposing this C_∞ coalgebra structure to a C_∞ algebra structure on the space $C(X)$ of non-degenerate cochains on X we recover the natural C_∞ enhancement $C(-) : \mathbf{SSet} \rightarrow \mathbf{C}_\infty$ induced, as in the paper [22] and Section 3.2, via homotopy transfer from $\Omega(-) : \mathbf{SSet} \rightarrow \mathbf{DGCA}$ along Dupont's contraction $\text{Dup} : \mathbf{SSet} \rightarrow \mathbf{Contr}$. Finally, we will see that we can transfer the usual model category structure on \mathbf{SSet} to a model category structure on $\widehat{\mathbf{DGLA}}$ along the adjunction

$L(-) : \mathbf{SSet} \rightleftarrows \widehat{\mathbf{DGLA}} : \text{Del}_\infty(-)$, as in Definition 5.1.4, and that we recover this way the model category structure defined by Lazarev and Markl [70], moreover, this also automatically shows that this is cofibrantly generated and the adjunction is a Quillen adjunction.

Via homotopy transfer along Dupont's contraction it is defined the simplicial C_∞ algebra $C(\Delta_\bullet)$ of non-degenerate cochains on the standard cosimplicial simplex. Since the $C(\Delta_n)$ are all finite dimensional, by transposition it is also defined the cosimplicial C_∞ coalgebra of non-degenerate chains on the standard simplex, cf. Remark 1.2.14. By definition, a C_∞ coalgebra structure on the space $C_*(\Delta_n)/D_*(\Delta_n)$ of non-degenerate chains on Δ_n is a dg Lie algebra structure on the complete free Lie algebra $\widehat{L}(C_*(\Delta_n)/D_*(\Delta_n)[-1])$, we denote the dg Lie algebra associated to $C(\Delta_n)$ by $L(\Delta_n)$. Thus, it is defined the cosimplicial complete dg Lie algebra $L(\Delta_\bullet)$: since the category $\widehat{\mathbf{DGLA}}$ is cocomplete, according to Proposition 5.1.9 this induces a left adjoint functor $L(-) : \mathbf{SSet} \rightarrow \widehat{\mathbf{DGLA}}$. We call the complete dg Lie algebra $L(X)$ the Lie-Sullivan model of the simplicial set X , this is defined by $L(X) = \text{colim}_{\sigma \in \Delta X} L(\Delta_{|\sigma|})$: since the functor $\widehat{L}(-)$ preserves colimits we always have that the underlying graded Lie algebra of $L(X)$ is $\widehat{L}(C_*(X)/D_*(X)[-1])$, thus the functor $L(-) : \mathbf{SSet} \rightarrow \widehat{\mathbf{DGLA}}$ is the datum of a natural C_∞ coalgebra structure on the non-degenerate chains on a simplicial set. It remains to check that the C_∞ algebra structure on $C(X)$ associated to $L(X)$ via transposition is again the one induced via homotopy transfer along Dupont's contraction: but this is so by construction when $X = \Delta_n$, thus it follows in general from Lemma 2.2.5.

Example 5.2.25. The only cases where we know explicitly the C_∞ algebra structure on $C(\Delta_n)$ are: trivially when $n = 0$, and less trivially when $n = 1$, cf. Proposition 3.3.6. We determine explicitly the dg Lie algebra structure on $L(\Delta_0)$ and $L(\Delta_1)$ ². When $n = 0$, then $C(\Delta_0) = \mathbb{K}$ with its \mathbb{K} -algebra structure: this dualizes to the coalgebra $\mathbb{K} \rightarrow \mathbb{K}^{\otimes 2} : 1 \rightarrow 1 \otimes 1$. Shifting the degrees by one and taking the opposite sign (recall that we change the sign since transposition is an antihomomorphism of graded Lie algebras, cf. the discussion at the end of Section 1.2), the above tells us that if we regard $C(\Delta_0)$ as an A_∞ algebra the dual A_∞ coalgebra is given by the differential $x \rightarrow -x \otimes x$ on the complete tensor algebra $\widehat{T}(x)$ over a generator x in degree one³. Since $x \otimes x = \frac{1}{2}[x, x]$ in the associated Lie algebra, we conclude that $L(\Delta_0)$ is the graded Lie algebra $\widehat{L}(x)$, $|x| = 1$ (in particular, since $[x, [x, x]] = 0$ by Jacobi, we see that as a graded space $L(\Delta_0) = \mathbb{K}x \oplus \mathbb{K}[x, x]$) with the dg Lie algebra structure given by $dx = -\frac{1}{2}[x, x]$. This makes evident the fact that $L(\Delta_0)$ represents the functor $\text{MC}(-) : \widehat{\mathbf{DGLA}} \rightarrow \mathbf{Set}$: here the Maurer-Cartan equation takes its most classical form

$$dx + \frac{1}{2}[x, x] = 0.$$

In the case $n = 1$, considering $C(\Delta_1)$ as an A_∞ algebra the explicit formulas in 3.3.6 imply that the dual A_∞ coalgebra is given by the differential on the complete tensor algebra $\widehat{T}(x, y, a)$, $|x| = |y| = 1, |a| = 0$, given by $x \rightarrow -x \otimes x$, $y \rightarrow -y \otimes y$ and finally

$$a \rightarrow y - x - \frac{1}{2}((x + y) \otimes a - a \otimes (x + y)) + \sum_{k=3}^{\infty} \frac{B_{k-1}}{(k-1)!} \sum_{i=1}^k \binom{k-1}{i-1} (-1)^{k-i} a^{\otimes i-1} \otimes (y-x) \otimes a^{\otimes k-i}$$

Let $\text{ad}_a = [a, -]$ be the adjoint: by the well known formula, valid more in general in any associative

²It is clear how from $L(\Delta_1)$ we can describe $L(X)$ for all 1-dimensional cell complexes X , cf. [67], the second version (the one linked to in the bibliography), for some examples

³In other words, this is the algebra of formal power series $\mathbb{K}[[x]]$, $|x| = 1$, equipped with the differential $x \rightarrow -x^2$

algebra,

$$(\mathrm{ad}_a)^{k-1}(y-x) = \sum_{i=1}^k \binom{k-1}{i-1} (-1)^{k-i} a^{\otimes i-1} \otimes (y-x) \otimes a^{\otimes k-i}$$

we see that the differential of a in $L(\Delta_1) = \widehat{L}(x, y, a)$ becomes $da = y - x + \frac{1}{2} \mathrm{ad}_a(x + y) + \sum_{k \geq 3} \frac{B_{k-1}}{(k-1)!} (\mathrm{ad}_a)^{k-1}(y-x)$, and finally the dg Lie algebra structure on $L(\Delta_1)$ is given as in

$$dx = -\frac{1}{2}[x, x], \quad dy = -\frac{1}{2}[y, y], \quad da = \mathrm{ad}_a(y) + \sum_{k \geq 0} \frac{B_k}{k!} (\mathrm{ad}_a)^k(y-x). \quad (5.2.1)$$

Thus we recovered the well known [67, 16, 17, 85, 22] Lawrence-Sullivan model of the interval: in particular this answers, after the paper [22], an answer posed by Sullivan in [102], cf. [85] for a different proof of the same result.

By definition the right adjoint $\widehat{\mathbf{DGLA}} \rightarrow \mathbf{SSet}$ sends a complete dg Lie algebra M to the simplicial set $\widehat{\mathbf{DGLA}}(L(\Delta_\bullet), M)$.

Lemma 5.2.26. *There is a natural isomorphism $\widehat{\mathbf{DGLA}}(L(\Delta_n), -) \rightarrow \mathrm{MC}(C(\Delta_n), -)$ of functors $\widehat{\mathbf{DGLA}} \rightarrow \mathbf{Set}$.*

Proof. This follows from Lemma 1.3.29, we should check that given a complete dg Lie algebra M the L_∞ structure on $C(\Delta_n; M) = \mathrm{Hom}(C_*(\Delta_n)/D_*(\Delta_n), M)$ induced via convolution with the C_∞ coalgebra structure on $C_*(\Delta_n)/D_*(\Delta_n)$, as in Lemma 1.3.29, is the same as the usual one, that is, the one induced via homotopy transfer along Dupont's contraction. Recall that the convolution L_∞ structure was defined by restricting the L_∞ algebra structure on $\mathrm{Hom}(C_*(\Delta_n)/D_*(\Delta_n), U(L)) = C(X; U(L)) = C(X) \otimes U(L)$, where $U(L)$ is the universal enveloping algebra, induced by extension of scalars of the C_∞ algebra $C(X)$ by the dg associative algebra $U(L)$ and then by symmetrization. By Lemmas 2.2.9 and 2.2.10 the latter is the same as the L_∞ algebra structure on $C(X; U(L))$ obtained via homotopy transfer from the dg Lie algebra $\Omega(X) \otimes U(L)$, seen as a dg Lie algebra via the commutator, along Dupont's contraction: it is then clear by Lemma 2.2.3 that this restricts to the usual L_∞ algebra structure on $C(X; L)$. \square

Putting the previous lemma and Lemma 5.2.4 together shows the following

Proposition 5.2.27. *The functors $L(-) : \mathbf{SSet} \rightleftarrows \widehat{\mathbf{DGLA}} : \mathrm{Del}_\infty(-)$ form an adjoint pair.*

We are ready to put the model category structure on $\widehat{\mathbf{DGLA}}$.

Theorem 5.2.28. *There is a model category structure on the category $\widehat{\mathbf{DGLA}}$ such that, given a continuous morphism $f : L \rightarrow M$ of complete dg Lie algebras:*

f is a fibration if such is $\mathrm{Del}_\infty(f)$, that is, if it is surjective in degrees ≤ 0 , and

f is a weak equivalence if such is $\mathrm{Del}_\infty(f)$, that is, if $\mathrm{MC}(f) : \mathrm{MC}(L) \rightarrow \mathrm{MC}(M)$ is bijective and for all $x \in \mathrm{MC}(L)$ and $i \leq 0$ the induced $H^{-i}(f) : H^{-i}(L_x) \rightarrow H^{-i}(M_{f(x)})$ is an isomorphism, finally,

f is a cofibration if it has the left lifting property with respect to trivial fibrations.

Moreover, this model category structure is cofibrantly generated, with generating cofibrations the inclusions $L(\partial\Delta_n) \rightarrow L(\Delta_n)$, for all $n \geq 1$, and generating trivial cofibrations the inclusions $L(\Lambda_n^i) \rightarrow L(\Delta_n)$, for all $n \geq 1$ and $0 \leq i \leq n$. Finally, $L(-) : \mathbf{SSet} \rightleftarrows \widehat{\mathbf{DGLA}} : \mathrm{Del}_\infty(-)$ is a Quillen adjunction.

Proof. Keeping in mind Theorem 5.2.10 and Theorem 5.2.20, all the claims follow once we show that we are in the hypotheses of Proposition 5.1.5. Since the functor $\mathrm{Del}_\infty(-)$ factors through the full subcategory \mathbf{Kan} of Kan complexes, we only have to show the existence of path space factorizations. We prove that for a complete dg Lie algebra L the factorization of the diagonal $L \rightarrow \Omega(\Delta_1; L) \rightarrow L \times L$ has the required properties (where the first arrow is pullback by the terminal morphism $\Delta_1 \rightarrow \Delta_0$ and the second arrow is pullback by the inclusion $\partial\Delta_1 \rightarrow \Delta_1$).

Since Theorem 5.2.10 immediately implies that $\mathrm{Del}_\infty(\Omega(\Delta_1; L)) \rightarrow \mathrm{Del}_\infty(L \times L)$ is a fibration, and clearly $\mathrm{Del}_\infty(L) \rightarrow \mathrm{Del}_\infty(\Omega(\Delta_1; L))$ is a cofibration, that is, an inclusion, we have to show that the latter is also a weak equivalence. We proceed inductively as in the proof of Theorem 5.2.16: when L is abelian with the central descending filtration, so is $\Omega(\Delta_1; L)$, and the thesis follows from Example 5.2.9 since $L \rightarrow \Omega(\Delta_1; L)$ is a quasi-isomorphism, then we can prove the claim inductively for the dg Lie algebras $L/F^p L$ by comparing, via Proposition 5.2.15, the central extensions of complete L_∞ algebras $F^p L/F^{p+1} L \rightarrow L/F^{p+1} L \rightarrow L/F^p L$ and $\Omega(\Delta_1; F^p L/F^{p+1} L) \rightarrow \Omega(\Delta_1; L/F^{p+1} L) \rightarrow \Omega(\Delta_1; L/F^p L)$. Finally, we deduce the claim for $L = \lim L/F^p L$ by passing to the limit. \square

In the following remark we explain the relevance of the previous theorem in rational homotopy theory, first let us notice that Theorem 5.2.16 immediately implies the following proposition, which says that $\mathrm{Del}_\infty(-)$ takes values in \mathbb{K} -local spaces [10].

Proposition 5.2.29. *For a morphism $f : X \rightarrow Y$ of simplicial sets the following are equivalent conditions:*

pullback along f is a weak equivalence $\mathbf{SSet}(Y, \mathrm{Del}_\infty(L)) \rightarrow \mathbf{SSet}(X, \mathrm{Del}_\infty(L))$ for every complete dg Lie algebra L ; and

f induces an isomorphism in \mathbb{K} -homology.

Proof. To prove that the second item implies the first it is sufficient, by Theorem 5.2.16, to show that $\mathrm{Del}_\infty(C(Y; L)) \rightarrow \mathrm{Del}_\infty(C(X; L))$ is a weak equivalence for every L . If L is abelian, since $C(Y; L) \rightarrow C(X; L)$ is a quasi-isomorphism by hypothesis, the thesis follows from Example 5.2.9: then the thesis follows in general by the usual inductive argument.

Conversely, if the first item hold then we have that $\mathrm{Del}_\infty(C(Y; \mathbb{K}[i])) \rightarrow \mathrm{Del}_\infty(C(X; \mathbb{K}[i]))$ is a weak equivalence for all i , where we take $\mathbb{K}[i]$ with the trivial L_∞ structure, thus $C(Y) \rightarrow C(X)$ is a quasi-isomorphism. \square

Remark 5.2.30. Notice that the previous model category structure on $\widehat{\mathbf{DGLA}}$ is quite different from the one where the fibrations are all the surjections and the weak equivalences are all the quasi-isomorphisms⁴: however, the two coincide on the full subcategory $\widehat{\mathbf{DGLA}}^{\leq 0}$ of complete dg Lie algebras concentrated in non positive degrees. To see that this is the same model category structure as the one defined by Lazarev and Markl [70] we refer to loc. cit., Section 10.

⁴The latter is more relevant in deformation theory, since it is the homotopy theory of L seen as a derived deformation functor $\mathrm{Del}_\infty(L \otimes \mathfrak{m}_-) : \mathbf{dgArt}_{\leq 0} \rightarrow \mathbf{SSet}$, cf. [86].

Since the seminal work of Quillen [89], we know that dg Lie algebras concentrated in negative degrees model (connected and) simply connected rational homotopy types, and since the seminal work of Sullivan [100], cf. also [39], Proposition 1.1, we know that the homotopy type associated to a negatively graded dg Lie algebra L is represented by the Maurer-Cartan ∞ groupoid $\text{MC}_\infty(L)$ of L , thus according to Theorem 5.2.24 also by the Deligne-Getzler ∞ groupoid $\text{Del}_\infty(L)$. In fact this (very well known) result could be probably proved in the spirit of [100], cf. also [11], using the Lie-Sullivan model $L(-)$ in place of the de Rham-Sullivan model $\Omega(-)$ and the functor $\text{Del}_\infty(-)$ instead of the geometric realization functor (cf. loc. cit.) $\langle - \rangle : \mathbf{DGCA} \rightarrow \mathbf{SSet}$. The aim of the paper [70] was to show how this classical equivalences of categories in fact are restrictions of equivalences between a certain rational homotopy category of spaces not necessarily connected, all of whose connected components are nilpotent and of finite \mathbb{Q} -type, and the homotopy category of a certain subcategory of arbitrarily graded, but satisfying some finite type assumption, complete dg Lie algebras, we refer to [70] (in particular Theorem D) for a precise statement: again the functor $\text{Del}_\infty(-)$ realizes the rational homotopy type associated to a complete dg Lie algebra. In fact, we suspect that the finiteness assumptions are unnecessary, and an undesired byproduct of the method of proof⁵. We mention here that a slightly different (but still closely related with the results discussed in this section) approach to disconnected rational homotopy theory then the one in [70] has been given [17], by U. Buijs and A. Murillo.

Typical examples of non connected spaces we would like to model are mapping spaces [6, 15, 68]. When the space X has finite cohomology and $Y \rightarrow Y_\mathbb{Q}$ is a localization, then, cf. [6], Section 6, the pushforward $\mathbf{SSet}(X, Y) \rightarrow \mathbf{SSet}(X, Y_\mathbb{Q})$ is a localization as well. In these hypotheses we can conclude thanks to Theorem 5.2.16 that if L is a model of Y , that is, there is a weak equivalence $Y_\mathbb{Q} \rightarrow \text{Del}_\infty(L)$, then the L_∞ algebra $C(X; L)$ of non-degenerate cochains on X with coefficients in L is a model (an L_∞ one) of the mapping space $\mathbf{SSet}(X, Y)$. We also recover the result by Brown-Szcharba [14] and Berglund [6] that $\Omega(X; L)$ is a Lie model of $\mathbf{SSet}(X, Y)$.

Remark 5.2.31. It has been pointed out in [16] that the complete dg Lie algebra $L(X)$ actually models the space X_+ , where we have added a disjoint base-point to X : the extra point will correspond to the vertex $0 \in \text{MC}(L(X))$. This can be seen as follows: the functor $\text{Del}_\infty(-)$ factors naturally through the category \mathbf{SSet}_* of pointed simplicial sets, namely, $\text{Del}_\infty(L)$ is pointed by $0 \in \text{MC}(L)$, but since the category $\widehat{\mathbf{DGLA}}$ is pointed more is true, cf. [50], Corollary 3.1.6, namely, that the whole adjunction in Theorem 5.2.28 factors as

$$\mathbf{SSet} \begin{array}{c} \xrightarrow{+} \\ \xleftarrow{\#} \end{array} \mathbf{SSet}_* \begin{array}{c} \xleftarrow{\overline{L}(-)} \\ \xrightarrow{\text{Del}_\infty(-)} \end{array} \widehat{\mathbf{DGLA}}$$

where $\# : \mathbf{SSet}_* \rightarrow \mathbf{SSet}$ is the forgetful functor and $-_+ : \mathbf{SSet} \rightarrow \mathbf{SSet}_* : X \rightarrow (X_+, +)$ adds a disjoint base point to X . The functor $\overline{L}(-) : \mathbf{SSet}_* \rightarrow \widehat{\mathbf{DGLA}} : (X, x) \rightarrow \overline{L}(X, x)$, that we call the reduced Lie-Sullivan model functor, is defined by the cocartesian square

$$\begin{array}{ccc} L(\Delta_0) & \xrightarrow{L(x)} & L(X) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \overline{L}(X, x) \end{array}$$

in $\widehat{\mathbf{DGLA}}$, where $x : \Delta_0 \rightarrow X$ is the inclusion of the vertex x . $\overline{L}(-) : \mathbf{SSet}_* \rightleftarrows \widehat{\mathbf{DGLA}} : \text{Del}_\infty(-)$

⁵Namely, by proving the corresponding fact for commutative dg algebras, where in fact the finite type assumptions are essential, and then by translating via Koszul duality.

is a Quillen adjunction, and in fact we could equivalently have defined the model category structure on **DGLA** via transfer along this adjunction.

Remark 5.2.32. We notice an interesting fact, which will be used in the proof of Proposition 5.2.36, namely, that the generating trivial cofibrations $L(\Lambda_n^i) \rightarrow L(\Delta_n)$ split, where the splitting is induced by the operation of thin filling: in fact, since $T : \mathbf{SSet}(\Lambda_n^i, \text{Del}_\infty(M)) \rightarrow \mathbf{SSet}(\Delta_n, \text{Del}_\infty(M))$ sending a horn to his unique thin filling is obviously natural in M , by Yoneda it is induced by a splitting $L(\Delta_n) \rightarrow L(\Lambda_n^i)$ of $L(\Lambda_n^i) \rightarrow L(\Delta_n)$. The morphism $L(\Delta_n) \rightarrow L(\Lambda_n^i)$ has the property that it sends the generator corresponding to the top dimensional simplex to zero, conversely, this is the unique left inverse to $L(\Lambda_n^i) \rightarrow L(\Delta_n)$ with this property: in fact, any such a morphism would determine a natural filling operation $\mathbf{SSet}(\Lambda_n^i, \text{Del}_\infty(M)) \rightarrow \mathbf{SSet}(\Delta_n, \text{Del}_\infty(M))$, moreover, this must send a horn to a filling evaluating to zero on the top dimensional simplex, but then it must send it to its unique thin filling.

5.2.2 Comparison with the Deligne groupoid

We begin by recalling some classical definitions [43, 27, 42, 76].

Definition 5.2.33. The *Gauge action* on the Maurer-Cartan set of a complete dg Lie algebra $(L, F^\bullet L, d, [\cdot, \cdot])$ is defined by

$$e^- * - : L^0 \times \text{MC}(L) \rightarrow \text{MC}(L) : (a, y) \rightarrow e^a * y := y + \sum_{k \geq 0} \frac{1}{(k+1)!} (\text{ad}_a)^k (\text{ad}_a(y) - da),$$

this is well defined by completeness: it is well known and easy to prove that it sends Maurer-Cartan elements to Maurer-Cartan elements, this follows immediately from the following alternative definition. Since $[d, d] = 0$, the inclusion $\mathbb{K}d \rightarrow \text{Der}(M)$ is a morphism of graded Lie algebras and thus defines a semidirect product $\mathbb{K}d \rtimes M$: it is clear that in the latter graded Lie algebra we have $e^a * x = e^{\text{ad}_a}(x + d) - d$. Since the Maurer-Cartan equation $dx + \frac{1}{2}[x, x]$ becomes the equation $[x + d, x + d] = 0$ in $\mathbb{K}d \rtimes M$, this makes evident that the Gauge action preserves Maurer-Cartan elements. We have to explain yet why the Gauge action is an action: it is an action with respect to the group structure on M^0 induced by the Baker-Campbell-Hausdorff product, see e.g. [43, 76]. This is the group operation $\circ : M^0 \times M^0 \rightarrow M^0$ defined by $e^a e^b = e^{a \circ b}$, where we denote by $\exp(M^0)$ the exponential group integrating the (pronilpotent) Lie algebra $(M^0, [\cdot, \cdot])$ and by $e^- : M^0 \rightarrow \exp(M^0) : a \rightarrow e^a$ the exponential map (which in this case is a natural identification), cf. [76]. It is well known that the Baker-Campbell-Hausdorff product $a \circ b$ can be expressed only in terms of the Lie algebra structure on M : for instance, a recursive definition using Bernoulli numbers goes as follows [43]

$$\xi_0 = b, \quad \xi_{n+1} = \frac{1}{n+1} \sum_{k \geq 0} \frac{B_n}{n!} \sum_{i_1 + \dots + i_k = n} [\xi_{i_1}, \dots, [\xi_{i_k}, a] \dots], \quad a \circ b = \sum_{n \geq 0} \xi_n.$$

In particular this shows that morphisms of graded Lie algebras are compatible with the Baker-Campbell-Hausdorff product. To see why the Gauge action is an action with respect to the Baker-Campbell-Hausdorff product, since $\text{ad} : M^0 \rightarrow \text{End}^0(M)$ is a morphism of Lie algebras

$$e^a * (e^b * x) = e^{\text{ad}_a} (e^{\text{ad}_b}(x + d) - d + d) - d = e^{\text{ad}_a} e^{\text{ad}_b}(x + d) - d = e^{\text{ad}_{a \circ b}}(x + d) - d = e^{a \circ b} * x.$$

The *Deligne groupoid* $\text{Del}(M)$ of M [27, 42] is the action groupoid associated to the Gauge action, namely, the groupoid whose objects are the Maurer-Cartan elements of M and whose arrows are the Gauge equivalences, that is, arrows from x to y are the $a \in M^0$ such that $e^a * x = y$. Clearly this

defines a functor $\text{Del}(-) : \widehat{\text{DGLA}} \rightarrow \mathbf{Grpd}$. The Deligne groupoid is of fundamental importance in deformation theory, see e.g. [42, 45, 76].

We notice that $\sum_{k \geq 0} \frac{1}{(k+1)!} (\text{ad}_a)^k = \frac{e^{\text{ad}_a} - \text{id}}{\text{ad}_a}$ and $\sum_{k \geq 0} \frac{B_k}{k!} (\text{ad}_a)^k = \frac{\text{ad}_a}{e^{\text{ad}_a} - \text{id}}$ are inverse operators, thus in the dg Lie algebra $L(\Delta_1) = \widehat{L}(x, y, a)$, cf. Example 5.2.25, we have

$$e^a * y = y + \frac{e^{\text{ad}_a} - \text{id}}{\text{ad}_a} (\text{ad}_a(y) - da) = y + \frac{e^{\text{ad}_a} - \text{id}}{\text{ad}_a} \left(-\frac{\text{ad}_a}{e^{\text{ad}_a} - \text{id}} (y - x) \right) = x.$$

Let M be a complete dg Lie algebra and $f : \widehat{L}(x, y, a) \rightarrow M$ any morphism of complete graded Lie algebras: this is the datum of arbitrary $f(x), f(y) \in M^1$ and $f(a) \in M^0$. If f has to be a morphism of dg Lie algebras then $f(x)$ and $f(y)$ have to be Maurer-Cartan elements of M and since morphisms of dg Lie algebras obviously preserve the Gauge action we also must have that $e^{f(a)} * f(y) = f(x)$. Conversely, given Maurer-Cartan elements $f(x), f(y) \in \text{MC}(M)$ and a Gauge equivalence $f(a) \in M^0$, $e^{f(a)} * f(y) = f(x)$, between them, then $f : L(\Delta_1) \rightarrow M$ is a morphism of dg Lie algebras: in fact, clearly $df(x) = fd(x)$ and $df(y) = fd(y)$, moreover $f(x) = e^{f(a)} * f(y) = f(y) + \frac{e^{\text{ad}_{f(a)}} - \text{id}}{\text{ad}_{f(a)}} (\text{ad}_{f(a)}(f(y)) - df(a))$ and thus

$$df(a) = \text{ad}_{f(a)}(f(y)) + \frac{\text{ad}_{f(a)}}{e^{\text{ad}_{f(a)}} - \text{id}} (f(y) - f(x)) = f \left(\text{ad}_a(y) + \frac{\text{ad}_a}{e^{\text{ad}_a} - \text{id}} (y - x) \right) = fd(a).$$

This proves the following [31, 17].

Proposition 5.2.34. *The set of morphism $f : L(\Delta_1) \rightarrow M$ of complete dg Lie algebras, equivalently, the set of Maurer-Cartan cochains $f(x) \xrightarrow{f(a)} f(y)$ in $\text{Del}_\infty(M)_1$, is in bijective correspondence with the set $\{(f(x), f(y), f(a)) \in \text{MC}(M) \times \text{MC}(M) \times M^0 \text{ s.t. } e^{f(a)} * f(y) = f(x)\}$.*

Corollary 5.2.35. *The set of simplicial loops $f(x) \xrightarrow{f(a)} f(x)$ in $\text{Del}_\infty(M)$ at $f(x)$ is in bijective correspondence with the set of cocycles $f(a) \in Z^{-1}(M_{f(x)})$.*

Proof. Since by the previous computation $e^{f(a)} * f(x) = f(x)$ if and only if $df(a) = \text{ad}_{f(a)}(f(x))$ if and only if $d_{f(x)}(f(a)) = 0$, where $d_{f(x)} := d + \text{ad}_{f(x)}$ by definition is the twisted differential. \square

The previous proposition tells us what we said without proof in Remark 5.2.14, with the notations used there, that $e^a * y = \bar{\gamma}_y^{1,1}(a)$ and $e^{-a} * x = \bar{\gamma}_x^{1,0}(a)$: in the next proposition we prove the remaining unproved claim, namely, that the function $\bar{\gamma}_x^{2,1} : M^0 \times M^0 \rightarrow M^0$ is the Baker-Campbell-Hausdorff product for all choices of $x \in \text{MC}(L)$.

Proposition 5.2.36. *Given a complete dg Lie algebra $(M, F^\bullet M, d, [\cdot, \cdot])$, $y \in \text{MC}(M)$, $a, b \in M^0$, then the unique thin filling of*

$$\begin{array}{ccc} & y & \\ & \nearrow^a & \searrow^b \\ e^a * y = x & & z = e^{-b} * y \end{array}$$

is

$$\begin{array}{ccc}
 & & y \\
 & \nearrow^a & \\
 e^a * y = x & & z = e^{-b} * y \\
 & \xrightarrow{a \circ b} &
 \end{array}$$

Proof. As noticed in Remark 5.2.32, the thin filling operation is induced by a morphism of dg Lie algebras $L(\Delta_2) \rightarrow L(\Lambda_2^1)$ left inverse to the inclusion $L(\Lambda_2^1) \rightarrow L(\Delta_2)$. We consider the composition $L(\Delta_1) \xrightarrow{L(\partial_1)} L(\Delta_2) \rightarrow L(\Lambda_2^1)$, where ∂_1 is the face $\partial_1 : \Delta_1 \rightarrow \Delta_2 : [01] \rightarrow [02]$, this induces the operation $\tilde{\gamma}_-^{2,1}$. By the previous proposition, if $L(\Delta_1) = \widehat{L}(x', y', a')$ and $L(\Lambda_2^1) = \widehat{L}(x, y, z, a, b)$, then this composition sends a' to a $\eta \in L(\Lambda_2^1)$, $|\eta| = 0$, such that $e^\eta * z = x$. After [67], Theorem 2 in the version of the paper linked to in the bibliography, we notice that a solution to this equation is given by $\eta = a \circ b$, since $x = e^a * y = e^a * (e^b * z) = e^{a \circ b} * z$: the thesis follows if we show that there are no other solutions, and clearly this is equivalent to the fact that the only loop $e^\xi * x = x$ at x is the trivial one $\xi = 0$, thus, by the corollary, to the fact that $\xi = 0$ is the only solution of

$$d(\xi) + \text{ad}_x(\xi) = 0, \quad |\xi| = 0, \quad (5.2.2)$$

in the dg Lie algebra $L(\Lambda_2^1) = \widehat{L}(x, y, z, a, b)$. We notice that the degree zero part of $L(\Lambda_2^1)$ is the Lie algebra $\widehat{L}(a, b)$. Let $\delta : L(\Lambda_2^1) \rightarrow L(\Lambda_2^1)$ be the linear differential $\delta(x) = \delta(y) = \delta(z) = 0$, $\delta(a) = y - x$, $\delta(b) = z - y$. According to [89], Proposition 2.1, the dg Lie algebra $(L(\Lambda_2^1), \delta, [\cdot, \cdot])$ has no cohomology in degree zero, and then the equation $\delta(\xi_p) = 0$, $|\xi_p| = 0$, admits the only solution $\xi_p = 0$. Let $L := \widehat{L}(a, b)$ equipped with the central descending filtration $F^\bullet L$, given $\xi \in L$ satisfying Equation (5.2.2), we complete the proof if we show inductively that $\xi \in F^p L$ for all $p \geq 1$, the case $p = 1$ being trivial. This is easy from the previous observation: if we suppose $\xi \in F^p L$ and we write $\xi = \xi_p + F^{p+1} L$, then Equation (5.2.2) implies that

$$0 = d(\xi) + \text{ad}_x(\xi) = \delta(\xi_p) + F^{p+1} L \quad \Rightarrow \quad \delta(\xi_p) = 0 \quad \Rightarrow \quad \xi_p = 0 \quad \Rightarrow \quad \xi \in F^{p+1} L. \quad \square$$

Together the previous propositions imply the following theorem.

Theorem 5.2.37. *If M is a complete dg Lie algebra, then $\text{Del}_\infty(M^{\geq 0})$ is isomorphic to the nerve of the opposite of the Deligne groupoid $\text{Del}(M)$.*

Proof. This follows from the definition of the Deligne groupoid, the above explicit description of the k -simplices of $\text{Del}_\infty(M^{\geq 0})$ for $k \leq 2$ and the fact that the simplicial set $\text{Del}_\infty(M^{\geq 0})$ is 2-coskeletal, cf. Corollary 5.2.5 and Example 5.1.10. \square

5.3 Descent of higher Deligne groupoids

In the subsection we review the fundamental theorem by Hinich on descent of Deligne groupoids, its role in the approach to deformation theory via dg Lie algebras will be illustrated through specific examples in the following chapter. The aim of this section is to give the analog of Hinich's result,

which we remark only applies when we work with non-negatively graded dg Lie algebras (actually, it suffices if their cohomology is non-negatively graded), for the Deligne-Getzler ∞ groupoid functor on $L_\infty[1]$ algebras with no grading restrictions (thus, also for dg Lie algebras this tells us something new): we will recover Hinich's theorem as a particular case in the next section, using the results from the previous one. We will work from the outset with semicosimplicial objects, since this seems more natural for the applications in deformation theory we have in mind, cf. [34, 2].

Recall that the restricted totalization functor $\text{Tot}(-) : \mathbf{SSet}^{\Delta} \rightarrow \mathbf{SSet}$ (cf. [9]) sends a semicosimplicial simplicial set X_\bullet to the simplicial set $\text{Tot}(X_\bullet)_n := \mathbf{SSet}^{\Delta}(\Delta_n \times \underline{\Delta}_\bullet, X_\bullet)$ with the obvious faces and degeneracies, where $\underline{\Delta}_\bullet$ is the standard semicosimplicial simplex in \mathbf{SSet}^{Δ} . We are going to introduce an analog restricted totalization functor $\text{Tot}(-) : \widehat{\mathbf{L}}_\infty[1]^{\Delta} \rightarrow \widehat{\mathbf{L}}_\infty[1]$ and prove that $\text{Del}_\infty(-)$ commutes with totalization up to homotopy.

We denote the restriction of $X_\bullet \in \mathbf{SSet}^{\Delta}$ to the full subcategory $\underline{\Delta}_{\leq k} \subset \underline{\Delta}$, with objects the \underline{i} with $i \leq k$, by $X_{\leq k} \in \mathbf{SSet}^{\Delta_{\leq k}}$: we remark an ambiguity in the notation, since we denote by the same symbol both the category $\underline{\Delta}_{\leq k}$ and the restriction to $\underline{\Delta}_{\leq k}$ of the standard semicosimplicial simplex $\underline{\Delta}_\bullet$. There is a tower of functors $\text{Tot}_{\leq k}(-) : \mathbf{SSet}^{\Delta} \rightarrow \mathbf{SSet}$ together with a natural isomorphism $\text{Tot}(-) \cong \lim \text{Tot}_{\leq k}(-)$: these are defined as before by $\text{Tot}_{\leq k}(X_\bullet)_n = \mathbf{SSet}^{\Delta_{\leq k}}(\Delta_n \times \underline{\Delta}_{\leq k}, X_{\leq k})$. It is then clear that $\text{Tot}_{\leq 0}(X_\bullet) = \mathbf{SSet}(\Delta_0, X_0) = X_0$, moreover from the definition there are natural maps $\text{Tot}_{\leq k}(X_\bullet) \rightarrow \mathbf{SSet}(\Delta_k, X_k)$ such that $\text{Tot}_{\leq k}(X_\bullet) = \text{Tot}_{\leq k-1}(X_\bullet) \times_{\mathbf{SSet}(\partial\Delta_k, X_k)} \mathbf{SSet}(\Delta_k, X_k)$ for all $k \geq 1$.

Remark 5.3.1. The family of morphisms $\text{Tot}(X_\bullet) = \lim \text{Tot}_{\leq k}(X_\bullet) \rightarrow \mathbf{SSet}(\Delta_k, X_k)$, $k \geq 0$, is universal with the property that for each arrow $\underline{i} \rightarrow \underline{j}$ in $\underline{\Delta}$ the induced diagram

$$\begin{array}{ccc} \text{Tot}(X_\bullet) & \longrightarrow & \mathbf{SSet}(\Delta_j, X_j) \\ \downarrow & & \downarrow \\ \mathbf{SSet}(\Delta_i, X_i) & \longrightarrow & \mathbf{SSet}(\Delta_i, X_j) \end{array}$$

is commutative. As in [45] we define the category $\underline{\mathcal{M}}$ whose objects are the arrows in $\underline{\Delta}$ and whose arrows $\{\underline{i} \rightarrow \underline{j}\} \rightarrow \{\underline{i}' \rightarrow \underline{j}'\}$ are the factorizations $\{\underline{i}' \rightarrow \underline{j}'\} = \{\underline{i}' \rightarrow \underline{i} \rightarrow \underline{j} \rightarrow \underline{j}'\}$ in $\underline{\Delta}$: then the above says that $\text{Tot}(X_\bullet)$ is a limit of $\underline{\mathcal{M}} \rightarrow \mathbf{SSet} : \{\underline{i} \rightarrow \underline{j}\} \rightarrow \mathbf{SSet}(\Delta_i, X_j)$, similarly $\text{Tot}_{\leq k}(X_\bullet)$ is a limit of the restriction of this functor to the full subcategory $\underline{\mathcal{M}}_{\leq k}$ of arrows in $\underline{\Delta}_{\leq k}$.

Keeping in mind Theorem 5.2.16, this suggests how to define the restricted totalization functor $\text{Tot}(-) : \widehat{\mathbf{L}}_\infty[1]^{\Delta} \rightarrow \widehat{\mathbf{L}}_\infty[1]$.

Definition 5.3.2. The restricted totalization $\text{Tot}(-) : \widehat{\mathbf{L}}_\infty[1]^{\Delta} \rightarrow \widehat{\mathbf{L}}_\infty[1]$ sends a semicosimplicial complete $L_\infty[1]$ algebra $V_\bullet \in \widehat{\mathbf{L}}_\infty[1]^{\Delta}$ to the limit of $\underline{\mathcal{M}} \rightarrow \widehat{\mathbf{L}}_\infty[1] : \{\underline{i} \rightarrow \underline{j}\} \rightarrow C(\Delta_i; V_j)$. Restricting to the full subcategory $\underline{\mathcal{M}}_{\leq k} \subset \underline{\mathcal{M}}$ and taking the limit, we similarly define a functor $\text{Tot}_{\leq k}(-) : \widehat{\mathbf{L}}_\infty[1]^{\Delta} \rightarrow \widehat{\mathbf{L}}_\infty[1]$: we see immediately that with these definitions we still have $\text{Tot}_{\leq 0}(V_\bullet) = V_0$, $\text{Tot}_{\leq k}(V_\bullet) = \text{Tot}_{\leq k-1}(V_\bullet) \times_{C(\partial\Delta_k; V_k)} C(\Delta_k; V_k)$ and $\text{Tot}(V_\bullet) = \lim \text{Tot}_{\leq k}(V_\bullet)$.

Remark 5.3.3. Obviously the above also makes sense in the non complete setting and defines functors $\text{Tot}(-) : \mathbf{L}_\infty[1]^{\Delta} \rightarrow \mathbf{L}_\infty[1]$, $\text{Tot}_{\leq k}(-) : \mathbf{L}_\infty[1]^{\Delta} \rightarrow \mathbf{L}_\infty[1]$.

In order to compare the simplicial sets $\text{Del}_\infty(\text{Tot}(V_\bullet))$ and $\text{Tot}(\text{Del}_\infty(V_\bullet))$, first of all we observe that they have the same set of vertices.

Proposition 5.3.4. *There is a natural isomorphism $\text{MC}(\text{Tot}(-)) \xrightarrow{\cong} \text{Tot}(\text{Del}_\infty(-))_0$ of functors $\widehat{\mathbf{L}}_\infty[1]^{\Delta} \rightarrow \mathbf{Set}$.*

Proof. Since $\text{MC}(-)$ commutes with small limits, by Lemma 5.2.4

$$\text{MC}(\text{Tot}(V_\bullet)) = \lim_{\underline{\mathcal{M}}} \text{MC}(C(\Delta_i; V_j)) = \lim_{\underline{\mathcal{M}}} \mathbf{SSet}(\Delta_i, \text{Del}_\infty(V_j)) = \text{Tot}(\text{Del}_\infty(V_\bullet))_0.$$

□

Remark 5.3.5. In the same way there are natural isomorphisms $\text{MC}(\text{Tot}_{\leq k}(-)) \xrightarrow{\cong} \text{Tot}_{\leq k}(\text{Del}_\infty(-))_0$ of functors $\widehat{\mathbf{L}}_\infty[1]^{\Delta} \rightarrow \mathbf{Set}$.

Theorem 5.3.6. *There is a natural weak equivalence $\text{Del}_\infty(\text{Tot}(-)) \xrightarrow{\sim} \text{Tot}(\text{Del}_\infty(-))$ of functors $\widehat{\mathbf{L}}_\infty[1]^{\Delta} \rightarrow \mathbf{SSet}$.*

Proof. We have morphisms $\text{Del}_\infty(\text{Tot}(V_\bullet)) \rightarrow \text{Del}_\infty(C(\Delta_i; V_i)) \rightarrow \mathbf{SSet}(\Delta_i, \text{Del}_\infty(V_i))$, $i \geq 0$, given by Theorem 5.2.16, and for each arrow $\{i \rightarrow j\}$ in $\underline{\Delta}$, in the induced diagram

$$\begin{array}{ccccc} \text{Del}_\infty(\text{Tot}(V_\bullet)) & \longrightarrow & \text{Del}_\infty(C(\Delta_j; V_j)) & \longrightarrow & \mathbf{SSet}(\Delta_j, \text{Del}_\infty(V_j)) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Del}_\infty(C(\Delta_i; V_i)) & \longrightarrow & \text{Del}_\infty(C(\Delta_i; V_j)) & & \\ \downarrow & & \searrow & & \downarrow \\ \mathbf{SSet}(\Delta_i, \text{Del}_\infty(V_i)) & \longrightarrow & & \longrightarrow & \mathbf{SSet}(\Delta_i, \text{Del}_\infty(V_j)) \end{array}$$

the inner squares are commutative, thus the outer square is commutative as well and there is induced a unique natural $\text{Del}_\infty(\text{Tot}(V_\bullet)) \rightarrow \text{Tot}(\text{Del}_\infty(V_\bullet))$ making the diagram

$$\begin{array}{ccc} \text{Del}_\infty(\text{Tot}(V_\bullet)) & \longrightarrow & \text{Del}_\infty(C(\Delta_i; V_i)) \\ \downarrow & & \downarrow \\ \text{Tot}(\text{Del}_\infty(V_\bullet)) & \longrightarrow & \mathbf{SSet}(\Delta_i, \text{Del}_\infty(V_i)) \end{array}$$

commutative for all $i \geq 0$. In the same way for each $k \geq 0$ we define natural transformations $\text{Del}_\infty(\text{Tot}_{\leq k}(-)) \rightarrow \text{Tot}_{\leq k}(\text{Del}_\infty(-))$.

We prove inductively that $\text{Del}_\infty(\text{Tot}_{\leq k}(V_\bullet)) \rightarrow \text{Tot}_{\leq k}(\text{Del}_\infty(V_\bullet))$ is a weak equivalence, the case $k = 0$ being obvious. To continue the induction we look at the commutative diagram

$$\begin{array}{ccccc} \text{Del}_\infty(\text{Tot}_{\leq k-1}(V_\bullet)) & \longrightarrow & \text{Del}_\infty(C(\partial\Delta_k; V_k)) & \longleftarrow & \text{Del}_\infty(C(\Delta_k; V_k)) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Tot}_{\leq k-1}(\text{Del}_\infty(V_\bullet)) & \longrightarrow & \mathbf{SSet}(\partial\Delta_k, \text{Del}_\infty(V_k)) & \longleftarrow & \mathbf{SSet}(\Delta_k, \text{Del}_\infty(V_k)) \end{array}$$

where all spaces are Kan complexes. As the right to left arrows are Kan fibrations, the top one by Theorem 5.2.10 and the bottom one as $\text{Del}_\infty(V_k)$ is a Kan complex and $\partial\Delta_k \rightarrow \Delta_k$ is a cofibration (see e.g. [48]), we see that the fiber products of the rows

$$\text{Del}_\infty(\text{Tot}_{\leq k}(V_\bullet)) = \text{Del}_\infty(\text{Tot}_{\leq k-1}(V_\bullet)) \times_{\text{Del}_\infty(C(\partial\Delta_k; V_k))} \text{Del}_\infty(C(\partial\Delta_k; V_k))$$

$$\mathrm{Tot}_{\leq k}(\mathrm{Del}_{\infty}(V_{\bullet})) = \mathrm{Tot}_{\leq k-1}(\mathrm{Del}_{\infty}(V_{\bullet})) \times_{\mathbf{SSet}(\partial\Delta_k, \mathrm{Del}_{\infty}(V_k))} \mathbf{SSet}(\Delta_k, \mathrm{Del}_{\infty}(V_k))$$

are also homotopy fiber products (cf. [73], Remark A.2.4.5). As the vertical arrows in the diagram are weak equivalences, by the inductive hypothesis and Theorem 5.2.16, this implies that also $\mathrm{Del}_{\infty}(\mathrm{Tot}_{\leq k}(V_{\bullet})) \rightarrow \mathrm{Tot}_{\leq k}(\mathrm{Del}_{\infty}(V_{\bullet}))$ is a weak equivalence and the inductive step is proven. Finally, both $\omega_0^{op} \rightarrow \mathbf{SSet} : \underline{k} \rightarrow \mathrm{Del}_{\infty}(\mathrm{Tot}_{\leq k}(V_{\bullet}))$ and $\omega_0^{op} \rightarrow \mathbf{SSet} : \underline{k} \rightarrow \mathrm{Tot}_{\leq k}(\mathrm{Del}_{\infty}(V_{\bullet}))$ are injectively fibrant towers of simplicial sets, as both $\mathrm{Del}_{\infty}(\mathrm{Tot}_{\leq k}(V_{\bullet})) \rightarrow \mathrm{Del}_{\infty}(\mathrm{Tot}_{\leq k-1}(V_{\bullet}))$ and $\mathrm{Tot}_{\leq k}(\mathrm{Del}_{\infty}(V_{\bullet})) \rightarrow \mathrm{Tot}_{\leq k-1}(\mathrm{Del}_{\infty}(V_{\bullet}))$ are pullbacks of a Kan fibration for all $k \geq 1$ and $\mathrm{Del}_{\infty}(\mathrm{Tot}_{\leq 0}(V_{\bullet})) = \mathrm{Tot}_{\leq 0}(\mathrm{Del}_{\infty}(V_{\bullet})) = \mathrm{Del}_{\infty}(V_0)$ is fibrant, hence we see by Proposition 5.1.2 that also $\mathrm{Del}_{\infty}(\mathrm{Tot}(V_{\bullet})) \rightarrow \mathrm{Tot}(\mathrm{Del}_{\infty}(V_{\bullet})) = \lim_k \left(\mathrm{Del}_{\infty}(\mathrm{Tot}_{\leq k}(V_{\bullet})) \xrightarrow{\sim} \mathrm{Tot}_{\leq k}(\mathrm{Del}_{\infty}(V_{\bullet})) \right)$ is a weak equivalence. \square

We notice that the previous theorem combined with the previous proposition imply the following corollary, where we denote by $\pi_{\leq 1}(-) : \mathbf{Kan} \rightarrow \mathbf{Grpd}$ the functor sending a Kan complex to its fundamental groupoid.

Corollary 5.3.7. *There is a natural isomorphism $\pi_{\leq 1}(\mathrm{Del}_{\infty}(\mathrm{Tot}(-))) \xrightarrow{\cong} \pi_{\leq 1}(\mathrm{Tot}(\mathrm{Del}_{\infty}(-)))$ of functors $\widehat{\mathbf{L}}_{\infty}[1]^{\Delta} \rightarrow \mathbf{Grpd}$.*

Proof. Since an equivalence of groupoids which is an isomorphism on the set of objects has to be an isomorphism. \square

5.3.1 Descent of Deligne groupoids

In this section we see how Theorem 5.3.6 translates when applied to a semicosimplicial dg Lie algebra concentrated in degrees ≥ 0 , thus recovering the important theorem by Hinich on Descent of Deligne groupoids [45] in the enhanced version from the papers [34, 35]. The utility of this theorem in deformation theory will be illustrated in the following chapter through specific examples, cf. Theorems 6.2.4, 6.3.3 and 6.3.6.

We denote by $\mathbf{Art}_{\mathbb{K}}$ the category of Artin \mathbb{K} -algebras A with residue field isomorphic to \mathbb{K} , that is, if we denote by \mathfrak{m}_A the maximal ideal then $A/\mathfrak{m}_A \cong \mathbb{K}$: in particular $A = \mathbb{K} \oplus \mathfrak{m}_A$ and this induces an isomorphism between $\mathbf{Art}_{\mathbb{K}}$ and the category of nilpotent finite dimensional \mathbb{K} -algebras. We will use the following terminology.

Definition 5.3.8. Let \mathbf{C} be a category with finite limits, and in particular a final object $*$. The category \mathbf{fC} of formal objects in \mathbf{C} is the category of functors $\mathbf{Art}_{\mathbb{K}} \rightarrow \mathbf{C}$ such that $F(\mathbb{K}) = *$. For instance, objects in the category \mathbf{fSet} of formal sets are usually just called functors of Artin rings, but we can also talk about formal simplicial sets, formal groupoids, and so on.

Every L_{∞} algebra L can be regarded as a formal nilpotent L_{∞} algebra $L \otimes \mathfrak{m}_{-} : A \rightarrow L \otimes \mathfrak{m}_A$, where $L \otimes \mathfrak{m}_A$ has the L_{∞} structure induced via extension of scalars by the nilpotent commutative algebra \mathfrak{m}_A . Since nilpotent L_{∞} algebras are in particular complete, naturally associated to L are the formal sets $\mathrm{MC}_L := \mathrm{MC}(L \otimes \mathfrak{m}_{-})$, $\mathrm{Def}_L := \mathcal{MC}(L \otimes \mathfrak{m}_{-})$ and the formal simplicial sets $\mathrm{MC}_{\infty, L} := \mathrm{MC}_{\infty}(L \otimes \mathfrak{m}_{-})$, $\mathrm{Del}_{\infty, L} := \mathrm{Del}_{\infty}(L \otimes \mathfrak{m}_{-})$, in fact the latter is a formal ∞ groupoid (in the sense of Getzler). When L is a dg Lie algebra there is moreover associated the formal groupoid $\mathrm{Del}_L := \mathrm{Del}(L \otimes \mathfrak{m}_{-})$: this makes sense also if L is an L_{∞} algebra concentrated in non-negative degrees, in fact in this case $\mathrm{Del}_{\infty}(L \otimes \mathfrak{m}_A)$ is a T -complex of rank 2 and thus the nerve of a groupoid, cf. Remark 5.2.23, and we may define $\mathrm{Del}_L(A)$ as the opposite of this groupoid, which is consistent with the dg Lie algebra case by Theorem 5.2.37. Most of the results in section 5.2

translate straightforwardly for their formal counterparts, for instance there is a natural (in L , with respect to strict L_∞ morphisms) weak equivalence of formal simplicial sets $\text{Del}_{\infty,L} \xrightarrow{\cong} \text{MC}_{\infty,L}$.

After Grothendieck's philosophy and the foundational work of Schlessinger [93], it is usually convenient to regard a formal moduli problem as a formal pointed set $M : \mathbf{Art}_{\mathbb{K}} \rightarrow \mathbf{Set}_*$ satisfying some additional conditions, the *Schlessinger's conditions* [93, 75], or better yet, due to well known problems linked to the existence of non-trivial automorphisms of the trivial deformation, as a formal pointed groupoid $M : \mathbf{Art}_{\mathbb{K}} \rightarrow \mathbf{Grpd}_*$, to which we can associate a formal pointed set via the functor $\pi_0(-) : \mathbf{Grpd}_* \rightarrow \mathbf{Set}_*$. We call M regarded as a formal pointed groupoid the deformation groupoid of the formal moduli problem, namely, objects of $M(A)$ are the deformations over A of the structure we are considering and arrows are isomorphisms of deformations. We call M regarded as a formal pointed set the deformation functor of the formal moduli problem, namely, this sends A to the set of isomorphism classes of deformations over A (in both cases $M(A)$ is pointed by the trivial deformation). Cf. Definitions 6.2.1, 6.2.2 and 6.3.2 for some examples of this kind.

Definition 5.3.9. We say that an L_∞ algebra L governs (or controls) a formal moduli problem, regarded as a formal pointed set $M : \mathbf{Art}_{\mathbb{K}} \rightarrow \mathbf{Set}_*$, if there is an isomorphism of formal pointed sets $\text{Def}_L \cong M$. If L is a dg Lie algebra or an L_∞ algebra concentrated in non negative degrees and M is enhanced to a formal pointed groupoid $M : \mathbf{Art}_{\mathbb{K}} \rightarrow \mathbf{Grpd}_*$, we require an equivalence $\text{Del}_L \xrightarrow{\sim} M$ of pointed groupoids (both Def_L and Del_L are pointed by 0): since $\pi_0(-)$ sends equivalences of groupoids to isomorphisms in this case L also governs M regarded as a formal set.

Let L_\bullet be a semicosimplicial dg Lie algebra concentrated pointwise in non negative degrees, this induces a semicosimplicial formal groupoid Del_{L_\bullet} and a semicosimplicial formal ∞ groupoid (in the sense of Getzler) $\text{Del}_{\infty,L_\bullet}$: by the results of Section 5.2.2 we can recover the one from the other via the functors nerve $\mathbf{N}(-) : \mathbf{Grpd} \rightarrow \mathbf{Kan}$ and fundamental groupoid $\pi_{\leq 1}(-) : \mathbf{Kan} \rightarrow \mathbf{Grpd}$. As observed in [45], cf. the lemma on page 6, the totalization $\text{Tot}(-)$ commutes with the nerve, that is, there is a natural isomorphism $\text{Tot}(\mathbf{N}(-)) \xrightarrow{\cong} \mathbf{N}(\text{Tot}(-))$ of functors $\mathbf{Grpd}^{\Delta} \rightarrow \mathbf{SSet}$, where in the right hand side $\text{Tot}(-) : \mathbf{Grpd}^{\Delta} \rightarrow \mathbf{Grpd}$ is the classical totalization functor on semicosimplicial groupoids via the groupoid of descent data.

Definition 5.3.10. The functor $\text{Tot}(-) : \mathbf{Grpd}^{\Delta} \rightarrow \mathbf{Grpd}$ sends a semicosimplicial groupoid

$$\mathcal{G}_\bullet : \quad \mathcal{G}_0 \rightrightarrows \mathcal{G}_1 \rightrightarrows \mathcal{G}_2 \rightrightarrows \cdots$$

to the groupoid $\text{Tot}(\mathcal{G}_\bullet)$, also called *groupoid of descent data*, defined in the following way [45, 34]:

The objects of $\text{Tot}(\mathcal{G}_\bullet)$ are the pairs (l, m) with l an object in \mathcal{G}_0 and m a morphism in \mathcal{G}_1 between $\partial_0 l$ and $\partial_1 l$, where $\partial_0, \partial_1 : \mathcal{G}_0 \rightarrow \mathcal{G}_1$ are the faces, such that the three images $\partial_i m \in \mathcal{G}_2$, $i = 0, 1, 2$, are the edges of a 2-simplex in the nerve of \mathcal{G}_2 , explicitly

$$(\partial_0 m)(\partial_1 m)^{-1}(\partial_2 m) = 1 \text{ in } \mathcal{G}_2(\partial_2 \partial_0 l, \partial_2 \partial_1 l).$$

The morphisms between (l_0, m_0) and (l_1, m_1) are the morphisms $a \in \mathcal{G}_0(l_0, l_1)$ making the diagram

$$\begin{array}{ccc} \partial_0 l_0 & \xrightarrow{m_0} & \partial_1 l_0 \\ \partial_0 a \downarrow & & \downarrow \partial_1 a \\ \partial_0 l_1 & \xrightarrow{m_1} & \partial_1 l_1 \end{array}$$

commutative in \mathcal{G}_1 .

Remark 5.3.11. Totalization commutes with equivalences: more precisely, if $\gamma: \mathcal{F}_\bullet \rightarrow \mathcal{G}_\bullet$ is a morphism of semicosimplicial groupoids, then $\text{Tot}(\gamma): \text{Tot}(\mathcal{F}_\bullet) \rightarrow \text{Tot}(\mathcal{G}_\bullet)$ is an equivalence of groupoids if such are the various $\gamma_n: \mathcal{F}_n \rightarrow \mathcal{G}_n$. The easiest way to see this is to check it directly, we leave it to the reader.

According to Corollary 5.3.7 we obtain the following enhancement of the main result of [45], already proved in [34, 35].

Theorem 5.3.12. *Given a semicosimplicial dg Lie algebra L_\bullet concentrated in non negative degrees, there is a natural isomorphism of formal groupoids $\text{Del}_{\text{Tot}(L_\bullet)} \xrightarrow{\cong} \text{Tot}(\text{Del}_{L_\bullet})$.*

Remark 5.3.13. Notice that $\text{Del}_{\text{Tot}(L_\bullet)}$ makes sense since the L_∞ algebra $\text{Tot}(L_\bullet)$, as in Definition 5.3.2, is concentrated in non negative degrees. We also remark that the conclusion of the theorem hold more in general, with the same proof, for any semicosimplicial non negatively graded L_∞ algebra L_\bullet .

Proof. Corollary 5.3.7 implies a natural isomorphism $\pi_{\leq 1}(\text{Del}_{\infty, \text{Tot}(L_\bullet)}) \xrightarrow{\cong} \pi_{\leq 1}(\text{Tot}(\text{Del}_{\infty, L_\bullet}))$ of formal groupoids, where the right hand side is by definition $\text{Del}_{\text{Tot}(L_\bullet)}^{\text{op}}$. On the other hand, by theorem 5.2.37 and since totalization commutes with the nerve we also see that

$$\pi_{\leq 1}(\text{Tot}(\text{Del}_{\infty, L_\bullet})) = \pi_{\leq 1}(\text{Tot}(\text{N}(\text{Del}_{L_\bullet}^{\text{op}}))) = \pi_{\leq 1}(\text{N}(\text{Tot}(\text{Del}_{L_\bullet})^{\text{op}})) = \text{Tot}(\text{Del}_{L_\bullet})^{\text{op}}.$$

□

As we show in the following chapter, the previous theorem provides a powerful tool to find an L_∞ algebra governing a given deformation problem. On the other hand, in some situations we may want to stick with dg Lie algebras, and in this case we simply have to replace the totalization $\text{Tot}(-)$ with the Thom-Whitney totalization $\text{Tot}_{\text{TW}}(-)$; as a price we get a slightly weaker result.

Definition 5.3.14. Given a morphism $f: L \rightarrow M$ of dg Lie algebras the homotopy fiber $K(f)$ is defined by the pullback square

$$\begin{array}{ccc} K(f) & \longrightarrow & \Omega(\Delta_1; M), \\ \downarrow & & \downarrow \\ L & \xrightarrow{0 \times f} & M \times M \end{array}$$

in the category **DGLA**, cf. Definition 3.3.3, more explicitly

$$K(f) = \{(l, m(t)) \in L \times M[t, dt] \text{ s.t. } m(0) = 0, m(1) = f(l)\}.$$

The Thom-Whitney totalization functor $\text{Tot}_{\text{TW}}(-): \mathbf{DGLA}^{\Delta} \rightarrow \mathbf{DGLA}$ sends a semicosimplicial dg Lie algebra L_\bullet to the dg Lie algebra $\text{Tot}_{\text{TW}}(L_\bullet) := \lim_{\mathcal{M}} \Omega(\Delta_i; L_j)$, cf. Definition 5.3.2.

It is clear by Lemma 2.2.5 that the homotopy fiber $K(f)$ and the Thom-Whitney totalization $\text{Tot}_{\text{TW}}(L_\bullet)$ are respectively dg Lie algebra models of the mapping cocone $\text{coC}(f)$ (Definition 3.3.3) and the totalization $\text{Tot}(L_\bullet)$ we considered in the previous section. Moreover, weakly equivalent L_∞ algebras have weakly equivalent Deligne-Getzler ∞ groupoids (this can be seen by putting together Theorem 5.2.24 and the main result from [29]), so Theorem 5.3.12 implies the following corollary.

Corollary 5.3.15. *Given a semicosimplicial dg Lie algebra L_\bullet concentrated in non negative degrees, there is an equivalence of formal groupoids $\text{Del}_{\text{Tot}_{\text{TW}}(L_\bullet)} \simeq \text{Tot}(\text{Del}_{L_\bullet})$.*

Chapter 6

Deformation problems in holomorphic Poisson geometry

We study several deformation problems in holomorphic Poisson geometry, namely, deformations of Poisson manifolds, coisotropic deformations of a pair (Poisson manifold coisotropic submanifold) and finally embedded coisotropic deformations: using Hinich's theorem on descent of Deligne groupoids, in all cases we determine controlling dg Lie algebras. In the final section, under some mild additional assumption, we show that the infinitesimal first order deformations induced by the anchor map are unobstructed. Applications of these results include the analog of Kodaira stability theorem or coisotropic deformation (cf. Corollary 6.3.4) and a generalization of McLean-Voisin's theorem about the local moduli space of lagrangian submanifold (cf. Corollary 6.4.12).

6.1 Review of holomorphic Poisson geometry

We recall the definition of graded Poisson algebras and Gerstenhaber algebras, these play a major role in Poisson geometry.

Definition 6.1.1. Let $k \in \mathbb{Z}$ be an integer, we call a graded k -Poisson algebra the data $(A, \cdot, [\cdot, \cdot])$ of a graded space A together with a product $\cdot : A^p \otimes A^q \rightarrow A^{p+q}$ making (A, \cdot) into a graded commutative algebra and a bracket $[\cdot, \cdot] : A^p \otimes A^q \rightarrow A^{p+q+k}$ making $(A[-k], [\cdot, \cdot])$ into a graded Lie algebra, and such that moreover the (odd) Poisson identity

$$[a, bc] = [a, b]c + (-1)^{(|a|+k)|b|} b[a, c], \quad \forall a, b, c \in A, \quad (6.1.1)$$

is satisfied: in other words, the adjoint $[a, -]$ has to be a derivation of the dg algebra structure for all $a \in A$. In particular, a graded 0-Poisson algebra is just called a graded Poisson algebra, and if moreover A is concentrated in degree zero (which will be the only graded Poisson algebras we will consider) just a Poisson algebra. We call a graded (-1)-Poisson algebra a Gerstenhaber algebra¹, of course it only makes sense to consider Gerstenhaber algebras in the graded setting. A differential Gerstenhaber algebra $(A, d, \cdot, [\cdot, \cdot])$ is a Gerstenhaber algebra together with a differential d that is a derivation of both (A, \cdot) and $(A[-k], [\cdot, \cdot])$.

¹Some authors call a Gerstenhaber algebra what we call a graded 1-Poisson algebra, in any case notice that a Gerstenhaber algebra in our sense induces a graded 1-Poisson algebra structure on the space $A^o = \bigoplus_{i \in \mathbb{Z}} A^{-i}$, and conversely.

Remark 6.1.2. Let $(A, \cdot, [\cdot, \cdot])$ be a Gerstenhaber algebra and let $I \subset A$ be an ideal of the underlying commutative graded algebra (A, \cdot) generated by a set of homogeneous elements $S \subset I$. Then the Poisson identity (6.1.1) immediately implies that I is $[\cdot, \cdot]$ -closed if and only if $[S, S] \subset I$.

Let X be a complex manifold, we denote by \mathcal{O}_X the sheaf of holomorphic functions on X , by Θ_X the holomorphic tangent sheaf and by $\bigwedge \Theta_X = \bigoplus_{i \geq 0} \bigwedge_{\mathcal{O}_X}^i \Theta_X[-i]$ the sheaf of holomorphic polyvector fields, moreover, we denote by $(\Omega_X, \partial, \wedge)$ the sheaf of holomorphic differential forms on X , equipped with the usual structure of sheaf of dg commutative algebras.

The sheaf $\bigwedge \Theta_X$ carries a natural structure of sheaf of Gerstenhaber algebras $(\bigwedge \Theta_X, \wedge, [\cdot, \cdot])$ with the exterior product and the Schouten-Nijenhuis bracket

$$[\cdot, \cdot] : \bigwedge^i \Theta_X \otimes \bigwedge^j \Theta_X \rightarrow \bigwedge^{i+j-1} \Theta_X,$$

see e.g. [103]. Recall that this is defined uniquely according to (6.1.1) so that $[\eta, \xi]$ is the usual bracket of vector fields if $\eta, \xi \in \Theta_X$, while $[\eta, f] = \eta(f) = \eta \lrcorner df$ if $\eta \in \Theta_X$ and $f \in \bigwedge^0 \Theta_X = \mathcal{O}_X$ ². The only thing to be checked to see this turns $\bigwedge \Theta_X$ into a sheaf of Gerstenhaber algebras is the (odd) Jacobi identity for $[\cdot, \cdot]$, but using the Poisson identity and the fact that $\bigwedge \Theta_X$ is generated as an algebra by $\bigwedge^{\leq 1} \Theta_X$ we can reduce ourselves to check the Jacobi identity on the latter: in this case it is clear, since it just says that the bracket of vector fields is a Lie bracket and that this bracket is given by the commutator. Notice that for $\eta \in \Theta_X$ the operator $[\eta, -] : \bigwedge \Theta_X \rightarrow \bigwedge \Theta_X$ is the Lie derivative with respect to η .

Given a polyvector field $\eta \in \bigwedge^i \Theta_X$, we denote by

$$\mathbf{i}_\eta : \Omega_X^* \rightarrow \Omega_X^{*-i}, \quad \mathbf{i}_\eta(\alpha) = \eta \lrcorner \alpha,$$

the corresponding contraction operator. In particular, if $\eta \in \bigwedge^0 \Theta_X = \mathcal{O}_X$ is a function then \mathbf{i}_η is just multiplication by this function, while if $\eta \in \bigwedge^1 \Theta_X = \Theta_X$ is a vector field then \mathbf{i}_η is the only degree (-1) derivation of the graded algebra (Ω_X, \wedge) such that $\mathbf{i}_\eta(\partial f) = \eta(f)$ for all $f \in \mathcal{O}_X$: there are two possible conventions to extend this to all of $\bigwedge \Theta_X$, we adopt the one according to which $\mathbf{i}_{\alpha \wedge \beta} = \mathbf{i}_\alpha \circ \mathbf{i}_\beta$, where the product \circ on the right hand side is the composition product. We see that if $\eta \in \bigwedge^i \Theta_X$ then $\mathbf{i}_\eta \in \text{Diff}_i(\Omega_X)$ is a differential operator of order $\leq i$ on the graded commutative algebra (Ω_X, \wedge) , cf. Section 4.2.1, by definition if $i = 0$ or $i = 1$, and then in general since $\text{Diff}_i(\Omega_X) \circ \text{Diff}_j(\Omega_X) \subset \text{Diff}_{i+j}(\Omega_X)$. Given a polyvector field $\eta \in \bigwedge^i \Theta_X$, we denote by $\mathbf{l}_\eta = [\mathbf{i}_\eta, \partial] : \Omega_X^* \rightarrow \Omega_X^{*-i+1}$ the holomorphic Lie derivative on differential forms with respect to η : since $\partial \in \text{Diff}_1(\Omega_X)$ and $[\text{Diff}_i(\Omega_X), \text{Diff}_j(\Omega_X)] \subset \text{Diff}_{i+j-1}(\Omega_X)$, we also see that $\mathbf{l}_\eta \in \text{Diff}_i(\Omega_X)$. The contractions and the Lie derivatives are related by the classical Cartan formulas, see e.g. the abstract formulation in [32],

$$[\mathbf{i}_\eta, \mathbf{i}_\xi] = 0, \quad [\mathbf{i}_\eta, \mathbf{l}_\xi] = \mathbf{i}_{[\eta, \xi]}, \quad (6.1.2)$$

where in the second formula the bracket in the right hand side is the Schouten-Nijenhuis bracket of polyvector fields.

We introduce the main subject of this chapter.

Definition 6.1.3. A holomorphic Poisson bivector on X is a global section $\pi \in H^0(X, \bigwedge^2 \Theta_X)$ satisfying the integrability condition:

$$[\pi, \pi] = 0. \quad (6.1.3)$$

²Here and in the sequel, to alleviate the notations, we allow ourselves to write $f \in \mathcal{O}_X$, $\eta \in \Theta_X$, while clearly we are talking about local sections $f \in \mathcal{O}_X(U)$, $\eta \in \Theta_X(U)$ over some open $U \subset X$.

A holomorphic Poisson manifold is a pair (X, π) consisting of a complex manifold X and a holomorphic Poisson bivector π on X .

Example 6.1.4. If $\dim X = 2$ then every global section of $\bigwedge^2 \Theta_X$ is a holomorphic Poisson bivector. If $\dim X = 3$, via the natural identification

$$\bigwedge^2 \Theta_X = \Omega_X^1 \otimes \bigwedge^3 \Theta_X = \Omega_X^1(K_X^{-1}),$$

a global section $\alpha \in H^0(X, \Omega_X^1(K_X^{-1}))$ corresponds to a holomorphic Poisson bivector if and only if

$$\alpha \wedge \partial \alpha = 0 \in H^0(X, \Omega_X^3(K_X^{-2})) = H^0(X, K_X^{-1}).$$

As another example, if $A \subseteq H^0(X, \Theta_X)$ is an abelian Lie subalgebra, then every element in the image of $\bigwedge^2 A \rightarrow H^0(X, \bigwedge^2 \Theta_X)$ is a Poisson bivector. Finally, every holomorphic symplectic form $\omega \in \Omega_X^2$ induces a Poisson bivector π on X uniquely by the condition

$$\mathbf{i}_\pi(\mathbf{i}_\eta(\omega) \wedge \alpha) = \mathbf{i}_\eta(\alpha), \quad \eta \in \Theta_X, \quad \alpha \in \Omega_X^1.$$

Remark 6.1.5. If π is a holomorphic Poisson bivector on X , by the Jacobi and Cartan identities,

$$[\mathbf{l}_\pi, \partial] = [[\mathbf{i}_\pi, \partial], \partial] = 0, \quad [\mathbf{l}_\pi, \mathbf{l}_\pi] = [[\mathbf{i}_\pi, \partial], \mathbf{l}_\pi] = [[\mathbf{i}_\pi, \mathbf{l}_\pi], \partial] = [\mathbf{i}_{[\pi, \pi]}, \partial] = 0.$$

The datum of a holomorphic Poisson bivector π on X induces several additional structures, for more details we refer to [63, 103]:

1) the *Lichnerowicz-Poisson differential* $d_\pi = [\pi, \cdot]: \bigwedge^* \Theta_X \rightarrow \bigwedge^{*+1} \Theta_X$, inducing on $\bigwedge \Theta_X$ the structure of sheaf of differential Gerstenhaber algebras.

2) the *Poisson bracket* $\{\cdot, \cdot\}_\pi: \mathcal{O}_X \wedge \mathcal{O}_X \rightarrow \mathcal{O}_X$, given by

$$\{f, g\}_\pi = [[\pi, f], g] = [d_\pi f, g] = \mathbf{i}_\pi(\partial f \wedge \partial g).$$

This clearly satisfies the (even) Poisson identity (6.1.1) and it is well known, see e.g. [103], that condition (6.1.3) is equivalent to the Jacobi identity for $\{\cdot, \cdot\}_\pi$. Therefore a holomorphic Poisson manifold could be equivalently defined as a complex manifold X together with a sheaf of Poisson algebras structure on \mathcal{O}_X (cf. [103]). We notice that in a system of local holomorphic coordinates z_1, \dots, z_n we can reconstruct the Poisson bivector from the Poisson bracket by the formula

$$\pi = \sum_{1 \leq i < j \leq n} \pi_{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \quad \text{where} \quad \pi_{ij} = -\{z_i, z_j\}_\pi.$$

3) the *Koszul bracket* $[\cdot, \cdot]_\pi: \Omega_X^i \otimes \Omega_X^j \rightarrow \Omega_X^{i+j-1}$, defined by the formula

$$[\alpha, \beta]_\pi := (-1)^{|\alpha|} (\mathbf{l}_\pi(\alpha \wedge \beta) - \mathbf{l}_\pi(\alpha) \wedge \beta) - \alpha \wedge \mathbf{l}_\pi(\beta), \quad (6.1.4)$$

inducing on $(\Omega_X, \partial, \wedge)$ the structure of a sheaf of differential Gerstenhaber algebras. We take a moment to sketch a proof of this well know fact. We notice that $[\alpha, \beta]_\pi$ coincides with the Koszul bracket $\mathcal{K}(\mathbf{l}_\pi)_2(\alpha \odot \beta)$ up to the sign $(-1)^{|\alpha|}$, cf. Section 4.2, in particular the identity $\mathcal{K}(\mathbf{l}_\pi)_3 = 0$, which follows since $\mathbf{l}_\pi \in \text{Diff}_2(\Omega_X)$, immediately translates into the odd Poisson identity for $[\cdot, \cdot]_\pi$. Similarly, the Jacobi identity translates into the identity $[\mathcal{K}(\mathbf{l}_\pi)_2, \mathcal{K}(\mathbf{l}_\pi)_2] = 0$ in the graded Lie algebra $\overline{\text{CE}}(\Omega_X)$: for the latter, we know from Section 4.2 that \mathcal{K} is a morphism of graded Lie

algebras, thus by Remark 6.1.5 $[\mathcal{K}(\mathbf{l}_\pi)_2, \mathcal{K}(\mathbf{l}_\pi)_2] = \mathcal{K}([\mathbf{l}_\pi, \mathbf{l}_\pi])_3 = 0^3$. Finally, the Leibniz identity translates into $[\partial, \mathcal{K}(\mathbf{l}_\pi)_2] = 0$, then again this follows since $\mathcal{K}(\partial)_2 = 0$ and by the remark we see that $[\partial, \mathcal{K}(\mathbf{l}_\pi)_2] = \mathcal{K}([\partial, \mathbf{l}_\pi])_2 = 0$.

4) the *anchor map*

$$\pi^\# : (\Omega_X, \partial, \wedge, [\cdot, \cdot]_\pi) \rightarrow (\bigwedge \Theta_X, d_\pi, \wedge, [\cdot, \cdot]),$$

which is the morphism of graded sheaves defined for $\alpha \in \Omega_X^1$ by the formula

$$\pi^\#(\alpha)(f) = \mathbf{i}_\pi(\alpha \wedge \partial f), \quad f \in \mathcal{O}_X, \quad (6.1.5)$$

and then uniquely extended to an \mathcal{O}_X -linear morphism of sheaves of graded algebras. It is well known that it is a morphism of sheaves of differential Gerstenhaber algebras. Again, we only sketch the proof. In order to show that $\pi^\#([\alpha, \beta]_\pi) = [\pi^\#(\alpha), \pi^\#(\beta)]$ it is sufficient to check it for $\alpha \in \Omega_X^1$, $\beta \in \Omega_X^0 = \mathcal{O}_X$, where it follows from definition (cf. (6.1.5) and (6.1.4)), or $\beta \in \Omega_X^1$, which is shown for instance in [64, 103]; the proof of $\pi^\#(\partial\alpha) = d_\pi(\pi^\#(\alpha))$ follows again from the Formula (6.1.5) for $\alpha = f \in \Omega_X^0 = \mathcal{O}_X$; in general it is sufficient to check it on forms of the type $\alpha = f \partial g_1 \wedge \cdots \wedge \partial g_k$, with $f, g_1, \dots, g_k \in \mathcal{O}_X$, which is a straightforward direct inspection.

Remark 6.1.6. When the Poisson bivector is induced by a holomorphic symplectic form ω the anchor map $\pi^\# : \Omega_X^1 \rightarrow \Theta_X$ is an isomorphism with inverse $\omega^\flat : \Theta_X \rightarrow \Omega_X^1$, $\omega^\flat(\eta) = \mathbf{i}_\eta(\omega)$.

Remark 6.1.7. Given a differential form $\alpha \in \Omega_X^k$, if we denote by $\alpha \wedge - : \Omega_X^* \rightarrow \Omega_X^{*+k}$ the left multiplication by α , it is not difficult to prove the formula

$$\mathbf{i}_{\pi^\#(\alpha)} = \frac{[\mathbf{i}_\pi, -]^k}{k!}(\alpha \wedge -)$$

in the graded Lie algebra $\text{End}(\Omega_X)$.

Before we close the section, we turn our attention to an important class of submanifolds of a holomorphic Poisson manifold, namely, the coisotropic ones. Recall that a multiplicative ideal I of a Poisson algebra $(A, \cdot, \{\cdot, \cdot\})$ is called coisotropic if it is closed with respect to $\{\cdot, \cdot\}$.

Definition 6.1.8. Let (X, π) be a holomorphic Poisson manifold. A holomorphic closed submanifold $Z \subset X$ is called coisotropic if its ideal sheaf \mathcal{I}_Z is coisotropic in \mathcal{O}_X .

Given a closed submanifold Z of a complex manifold X we denote by $\mathcal{N}_{Z|X}$ the normal sheaf of Z in X and by $\bigwedge \mathcal{N}_{Z|X} := \bigoplus_{i \geq 0} \bigwedge_{\mathcal{O}_Z}^i \mathcal{N}_{Z|X}[-i]$ its graded exterior algebra: by a little abuse of notation we also denote by $\bigwedge \mathcal{N}_{Z|X}$ its direct image under the inclusion $Z \rightarrow X$. Moreover, we denote by $\Theta_X(-\log Z)$ the subsheaf of vector fields $\eta \in \Theta_X$ such that $\eta(\mathcal{I}_Z) \subset \mathcal{I}_Z$. There is a natural epimorphism $\bigwedge \Theta_X \rightarrow \bigwedge \mathcal{N}_{Z|X}$ of sheaves of graded algebras on X : we denote its kernel by \mathcal{L}_Z and we notice that $\mathcal{L}_Z^0 = \mathcal{I}_Z$ and $\mathcal{L}_Z^1 = \Theta_X(-\log Z)$, where the latter is by definition the sheaf of vector fields tangent everywhere to Z , that is, the sheaf of Lie subalgebras $\Theta_X(-\log Z) \subset \Theta_X$ of derivations $\eta \in \Theta_X$ such that $\eta(\mathcal{I}_Z) \subset \mathcal{I}_Z$.

Proposition 6.1.9. *In the notation above, \mathcal{L}_Z is a sheaf of Gerstenhaber subalgebras of $\bigwedge \Theta_X$. Moreover the following conditions are equivalent:*

1. Z is coisotropic;

³In other words, since $\mathcal{K}(\mathbf{l}_\pi) \in \overline{\text{CE}}(\Omega_X)$ is a degree minus one coderivation such that $\mathcal{K}([\mathbf{l}_\pi, \mathbf{l}_\pi]) = 0$, it is what we may call a homological $L_\infty[1]$ structure on Ω_X , whereas we have been considering cohomological $L_\infty[1]$ structures: via décalage this induces a homological dg Lie algebra structure on $\Omega_X[1]$, whose Lie bracket is exactly $[\cdot, \cdot]_\pi$

2. $\pi \in H^0(X; \mathcal{L}_Z^2)$;
3. $d_\pi(\mathcal{L}_Z) \subseteq \mathcal{L}_Z$, i.e., $\mathcal{L}_Z \subset \bigwedge \Theta_X$ is a sheaf of differential Gerstenhaber subalgebras.

Proof. We first prove that \mathcal{L}_Z is generated as a multiplicative ideal of $\bigwedge \Theta_X$ by \mathcal{L}_Z^0 and \mathcal{L}_Z^1 : choosing a system of holomorphic coordinates z_1, \dots, z_n such that $Z = \{z_1 = \dots = z_p = 0\}$, we have $\frac{\partial}{\partial z_i} \in \mathcal{L}_Z^1$ if and only if $i > p$, while a polyvector field

$$\xi = \sum_{1 \leq i_1 < \dots < i_k \leq n} \xi_{i_1 \dots i_k} \frac{\partial}{\partial z_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial z_{i_k}}$$

belongs to \mathcal{L}_Z if and only if $\xi_{i_1 \dots i_k} \in \mathcal{I}_Z$ whenever $1 \leq i_1 < \dots < i_k \leq p$.

The proof of the first claim of the proposition amounts to show that \mathcal{L}_Z is $[\cdot, \cdot]$ -closed: by Remark 6.1.2 this is equivalent to the fact that $\mathcal{L}_Z^{\leq 1}$ is $[\cdot, \cdot]$ -closed, which is clear since by definition $\Theta_X(-\log Z) \subset \Theta_X$ is the sub Lie algebra of derivations sending \mathcal{I}_Z into itself.

If $\pi \in H^0(X, \mathcal{L}_Z^2)$ and $f, g \in \mathcal{I}_Z = \mathcal{L}_Z^0$, also $\{f, g\}_\pi = [[\pi, f], g] \in \mathcal{I}_Z$: thus Z is coisotropic and $\mathcal{L}_Z \subset \bigwedge \Theta_X$ is a differential Gerstenhaber subalgebra. It is clear that item 3 implies that Z is coisotropic, so it remains to show that the latter implies $\pi \in H^0(X; \mathcal{L}_Z^2)$. We write in local coordinates

$$\pi = \sum_{1 \leq i < j \leq n} \pi_{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \quad \pi_{ij} = -\{z_i, z_j\}_\pi.$$

If $Z = \{z_1 = \dots = z_p = 0\}$ is coisotropic then $z_i, z_j \in \mathcal{I}_Z$ for $1 \leq i < j \leq p$ and then also $\pi_{ij} = -\{z_i, z_j\}_\pi \in \mathcal{I}_Z$ for $1 \leq i < j \leq p$: this says that π is a section of \mathcal{L}_Z^2 . \square

6.2 Deformations of holomorphic Poisson manifolds

Let X be a complex manifold.

Definition 6.2.1. A deformation of (the complex structure on) X over $A \in \mathbf{Art}_{\mathbb{C}}$ is a pullback diagram of complex spaces

$$\begin{array}{ccc} X & \xrightarrow{i} & \mathcal{X} \\ \downarrow & & \downarrow p \\ \text{Spec } \mathbb{C} & \longrightarrow & \text{Spec } A \end{array} \quad (6.2.1)$$

with p a smooth morphism. More concretely, we shall identify X with the only closed fiber in \mathcal{X} and we shall look at the structure sheaf $\mathcal{O}_{\mathcal{X}}$ as a sheaf of A -algebras over X together with a morphism of sheaves of A -algebras $\mathcal{O}_{\mathcal{X}} \rightarrow \mathcal{O}_X$, where we regard the sheaf of \mathbb{C} -algebras \mathcal{O}_X as a sheaf of A -algebras via the projection $A \rightarrow A/\mathfrak{m}_A = \mathbb{C}$: then the fact that p is smooth says moreover that $\mathcal{O}_{\mathcal{X}}$ is a sheaf of flat (unitary) A -algebras on X and that in some neighborhood U of any point $x \in X$ the deformation trivializes, that is, there is an isomorphism $\mathcal{O}_{\mathcal{X}}(U) \rightarrow \mathcal{O}_X(U) \otimes A$ over $\mathcal{O}_X(U)$. Equivalences between deformations \mathcal{X}_0 and \mathcal{X}_1 of X over A are morphisms of sheaves of A -algebras $\mathcal{O}_{\mathcal{X}_0} \rightarrow \mathcal{O}_{\mathcal{X}_1}$ over \mathcal{O}_X : we call a deformation equivalent to $X \rightarrow X \times \text{Spec } A \rightarrow \text{Spec } A$ trivial.

Every deformation \mathcal{X} of X over A trivializes globally on a Stein open $U \subset X$, that is, we have $\mathcal{O}_{\mathcal{X}}(U) \xrightarrow{\cong} \mathcal{O}_X(U) \otimes A$ over $\mathcal{O}_X(U)$: as usual, exponential and logarithm induce a bijective correspondence between the graded Lie algebra $\Theta_X(U) \otimes \mathfrak{m}_A$ and the group of automorphisms of A -algebras $\mathcal{O}_X(U) \otimes A \rightarrow \mathcal{O}_X(U) \otimes A$ over $\mathcal{O}_X(U)$, in other words, the group of self-equivalences of the trivial deformation $U \times \text{Spec } A$ is the exponential group $\exp(\Theta_X(U) \otimes \mathfrak{m}_A)$. Thus, given a covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X by Stein open subsets, the whole \mathcal{X} can be reconstructed up to isomorphism by gluing the family of trivial deformations $U_i \times \text{Spec } A$ along the double intersections $U_{ij} = U_i \cap U_j$ via a family of transition automorphisms $e^{\eta_{ij}} : \mathcal{O}_X(U_{ij}) \otimes A \rightarrow \mathcal{O}_X(U_{ij}) \otimes A$, where $\eta_{ij} \in \Theta_X(U_{ij}) \otimes \mathfrak{m}_A$, satisfying the cocycle condition $e^{\eta_{ij}} e^{\eta_{jk}} = e^{\eta_{ik}} : \mathcal{O}_X(U_{ijk}) \otimes A \rightarrow \mathcal{O}_X(U_{ijk}) \otimes A$ on triple intersections, that is, $\eta_{ij} \circ \eta_{jk} = \eta_{ik}$ in the nilpotent Lie algebra $\Theta_X(U_{ijk}) \otimes \mathfrak{m}_A$, where \circ is the Baker-Campbell-Hausdorff product.

In order to deform the datum of a Poisson bivector $\pi \in H^0(X; \bigwedge^2 \Theta_X)$ on X together with the complex structure, given a deformation \mathcal{X} of X over A , we consider the sheaf of $\mathcal{O}_{\mathcal{X}/A}$ -modules $\Theta_{\mathcal{X}/A}$ of A -linear derivations of $\mathcal{O}_{\mathcal{X}}$: as in the previous section the sheaf $\Theta_{\mathcal{X}/A}$ has a natural structure of sheaf of Gerstenhaber algebras given by the Schouten-Nijenhuis bracket, uniquely defined so that the brackets of vector fields is the usual one and the bracket of a vector field and a function is the contraction. We notice that when $U \subset X$ is a Stein open subset, there is an isomorphism $\bigwedge^* \Theta_{\mathcal{X}/A}(U) \xrightarrow{\cong} \bigwedge^* \Theta_X(U) \otimes A$ of Gerstenhaber algebras over $\bigwedge^* \Theta_X(U)$, where the Gerstenhaber algebra structure on the right hand side is given via scalar extension by A . Let as before \mathcal{U} be a covering of X by Stein open sets, we denote by $e^{\eta_{ij}}$ a family of transition automorphisms for $\mathcal{O}_{\mathcal{X}}$ over \mathcal{U} as before. Looking at the $\eta_{ij} \in \Theta_X(U_{ij}) \otimes \mathfrak{m}_A$ as elements of $\bigwedge \Theta_X(U_{ij}) \otimes A \cong \bigwedge \Theta_{\mathcal{X}/A}(U_{ij})$, the adjoints

$$\text{ad}_{\eta_{ij}} = [\eta_{ij}, \cdot]_{SN} : \bigwedge \Theta_{\mathcal{X}/A}(U_{ij}) \rightarrow \bigwedge \Theta_{\mathcal{X}/A}(U_{ij})$$

are degree zero nilpotent Gerstenhaber derivations, that is, derivations both of the algebra (by the Poisson identity) and the Lie algebra (by the Jacobi identity) structure, which shows that their exponentials $e^{\text{ad } \eta_{ij}} : \bigwedge \Theta_{\mathcal{X}/A}(U_{ij}) \rightarrow \bigwedge \Theta_{\mathcal{X}/A}(U_{ij})$ are automorphisms of Gerstenhaber algebras over $\bigwedge \Theta_X(U_{ij})$. As before the sheaf $\bigwedge \Theta_{\mathcal{X}/A}$ can be reconstructed by gluing the local pieces $\bigwedge \Theta_X(U_i) \otimes A$ along the double intersections, we see that the transition automorphisms are exactly the $e^{\text{ad } \eta_{ij}} : \bigwedge \Theta_X(U_{ij}) \otimes A \rightarrow \bigwedge \Theta_X(U_{ij}) \otimes A$: in fact, it suffices to check this on functions $f \in \mathcal{O}_{\mathcal{X}}(U_{ij}) = \bigwedge^0 \Theta_X(U_{ij}) \otimes A$, where $e^{\text{ad } \eta_{ij}}(f) = e^{\eta_{ij}}(f)$, and on vector fields $\xi \in \Theta_{\mathcal{X}/A}(U_{ij})$, where $e^{\text{ad } \eta_{ij}}(\xi) = e^{\eta_{ij}} \circ \xi \circ e^{-\eta_{ij}}$. We can treat equivalences of deformations by the same reasoning: namely, an equivalence between deformations \mathcal{X}_0 and \mathcal{X}_1 of X over A induces an isomorphism $\bigwedge \Theta_{\mathcal{X}_0/A} \rightarrow \bigwedge \Theta_{\mathcal{X}_1/A}$ over $\bigwedge \Theta_X$, if the equivalence is locally given by

$$\mathcal{O}_{\mathcal{X}_0}(U_i) \cong \mathcal{O}_X(U_i) \otimes A \xrightarrow{e^{\eta_i}} \mathcal{O}_X(U_i) \otimes A \cong \mathcal{O}_{\mathcal{X}_1}(U_i) ,$$

where $\eta_i \in \Theta_X(U_i) \otimes \mathfrak{m}_A$, then the induced $\bigwedge \Theta_{\mathcal{X}_0/A} \rightarrow \bigwedge \Theta_{\mathcal{X}_1/A}$ is locally given by

$$\bigwedge \Theta_{\mathcal{X}_0/A}(U_i) \cong \bigwedge \Theta_X(U_i) \otimes A \xrightarrow{e^{\text{ad } \eta_i}} \bigwedge \Theta_X(U_i) \otimes A \cong \bigwedge \Theta_{\mathcal{X}_1/A}(U_i) .$$

It is clear now how to define infinitesimal deformations of (X, π) over A .

Definition 6.2.2. A deformation of a holomorphic Poisson manifold (X, π) over $A \in \mathbf{Art}_{\mathbb{C}}$ is the data of

a deformation $X \xrightarrow{i} \mathcal{X} \xrightarrow{p} \text{Spec } A$ of X over A as in Definition 6.2.1, and

a global section $\tilde{\pi} \in H^0(X; \bigwedge^2 \Theta_{\mathcal{X}/A})$ such that $[\tilde{\pi}, \tilde{\pi}]$ and such that $\tilde{\pi}$ restricts to π on the closed fiber, that is, $\bigwedge \Theta_{\mathcal{X}/A} \rightarrow \bigwedge \Theta_X : \tilde{\pi} \rightarrow \pi$.

Given two deformations $(\mathcal{X}_0, \tilde{\pi}_0)$, $(\mathcal{X}_1, \tilde{\pi}_1)$ an equivalence between them is an equivalence between \mathcal{X}_0 and \mathcal{X}_1 such that the induced isomorphism $\bigwedge \Theta_{\mathcal{X}_0/A} \rightarrow \bigwedge \Theta_{\mathcal{X}_1/A}$ sends $\tilde{\pi}_0$ to $\tilde{\pi}_1$. To every holomorphic Poisson manifolds (X, π) we associate

a formal pointed groupoid $\text{Del}_{(X, \pi)} : \mathbf{Art}_{\mathbb{C}} \rightarrow \mathbf{Grpd}_*$, sending A to the groupoid whose objects are deformations of (X, π) over A and whose arrows are equivalences between them;

a formal pointed set $\text{Def}_{(X, \pi)} : \mathbf{Art}_{\mathbb{C}} \rightarrow \mathbf{Set}_* : A \rightarrow \pi_0(\text{Del}_{(X, \pi)}(A))$, sending A to the set of equivalence classes of deformations of (X, π) over A .

Remark 6.2.3. Equivalently, a deformation of (X, π) over A is a sheaf $\mathcal{O}_{\mathcal{X}}$ of flat Poisson A -algebras on X (that is, flat A -algebras equipped with an A -bilinear Poisson bracket) and a sheaves of Poisson A -algebras morphism $\mathcal{O}_{\mathcal{X}} \rightarrow \mathcal{O}_X$ wich trivializes locally, that is, for all $x \in X$ there is an open $U \subset X$ containing x and an isomorphism $\mathcal{O}_{\mathcal{X}}(U) \rightarrow \mathcal{O}_X(U) \otimes A$ of sheaves of Poisson A -algebras over \mathcal{O}_X , where we consider the Poisson structure on $\mathcal{O}_X(U) \otimes A$ given via scalar extension by A .

Given a covering \mathcal{U} of a Poisson manifold (X, π) by Stein open sets, via the Čech construction it is defined a semicosimplicial differential Gerstenhaber algebra (with the Lichnerowicz-Poisson differential, cf. the previous section)

$$\bigwedge \Theta_X(\mathcal{U})_{\bullet} : \prod_i \bigwedge \Theta_X(U_i) \rightrightarrows \prod_{i,j} \bigwedge \Theta_X(U_{ij}) \rightrightarrows \prod_{i,j,k} \bigwedge \Theta_X(U_{ijk}) \cdots$$

with the usual face operators given by restriction. The part in degrees ≥ 1 is a semicosimplicial differential Gerstenhaber subalgebra, since the associated semicosimplicial dg Lie algebra

$$\bigwedge \Theta_X^{\geq 1}[1](\mathcal{U})_{\bullet} : \prod_i \bigwedge^{\geq 1} \Theta_X[1](U_i) \rightrightarrows \prod_{i,j} \bigwedge^{\geq 1} \Theta_X[1](U_{ij}) \rightrightarrows \prod_{i,j,k} \bigwedge^{\geq 1} \Theta_X[1](U_{ijk}) \cdots$$

is concentrated in degrees ≥ 0 , it satisfies the hypotheses of Theorem 5.3.12 and Corollary 5.3.15.

Theorem 6.2.4. *The totalization $\text{Tot}(\bigwedge^{\geq 1} \Theta_X[1](\mathcal{U})_{\bullet})$ (or, if we want a dg Lie algebra, the Thom-Whitney totalization $\text{Tot}_{TW}(\bigwedge^{\geq 1} \Theta_X[1](\mathcal{U})_{\bullet})$) governs the deformations of (X, π) . More precisely, there are equivalences of formal pointed groupoids*

$$\text{Del}_{\text{Tot}_{TW}(\bigwedge^{\geq 1} \Theta_X[1](\mathcal{U})_{\bullet})} \simeq \text{Del}_{\text{Tot}(\bigwedge^{\geq 1} \Theta_X[1](\mathcal{U})_{\bullet})} \simeq \text{Del}_{(X, \pi)}.$$

Proof. Let $A \in \mathbf{Art}_{\mathbb{C}}$ and $(\mathcal{X}, \tilde{\pi})$ be a deformation of (X, π) over A , this trivializes over each U_i and thus we have isomorphisms $\bigwedge \Theta_{\mathcal{X}/A}(U_i) \cong \bigwedge \Theta_X(U_i) \otimes A$ of Gerstenhaber algebras over $\bigwedge \Theta_X(U_i)$. As we said, the sheaf $\bigwedge \Theta_{\mathcal{X}/A}$ can be reconstructed by the local trivial pieces via a family of transition automorphisms $e^{\text{ad}_{\eta_{ij}}} : \bigwedge \Theta_{\mathcal{X}/A}(U_{ij}) \rightarrow \bigwedge \Theta_{\mathcal{X}/A}(U_{ij})$, where $\eta_{ij} \in \Theta_X(U_{ij}) \otimes \mathfrak{m}_A$, satisfying the cocycle condition $\eta_{ij} \circ \eta_{jk} = \eta_{ik}$ in the Lie algebra $\Theta_X(U_{ijk}) \otimes \mathfrak{m}_A$. Since $\tilde{\pi}$ restricts to π , over each U_i we have $\tilde{\pi}|_{U_i} = \pi|_{U_i} + \sigma_i$, where $\sigma_i \in \bigwedge^2 \Theta_X(U_i) \otimes \mathfrak{m}_A$: then $[\tilde{\pi}, \tilde{\pi}] = 0$ is equivalent to $[\pi|_{U_i}, \sigma_i] + \frac{1}{2}[\sigma_i, \sigma_i] = 0$, $\forall i$, that is, σ_i is a Maurer-Cartan element of the dg Lie algebra $(\bigwedge^{\geq 1} \Theta_X[1](U_i) \otimes \mathfrak{m}_A, d_{\pi|_{U_i}}, [\cdot, \cdot])$ for all $U_i \in \mathcal{U}$. Finally, since the local sections $\tilde{\pi}|_{U_i}$ glue to the global $\tilde{\pi}$, on double intersections we have that $e^{\text{ad}_{\eta_{ij}}}(\pi|_{U_{ij}} + \sigma_j|_{U_{ij}}) = \pi|_{U_{ij}} + \sigma_i|_{U_{ij}}$, which becomes the equation $e^{\eta_{ij}} * \sigma_j|_{U_{ij}} = \sigma_i|_{U_{ij}}$ in the dg Lie algebra $(\bigwedge^{\geq 1} \Theta_X[1](U_{ij}) \otimes \mathfrak{m}_A, d_{\pi|_{U_{ij}}}, [\cdot, \cdot])$, where $*$ denotes the Gauge action. Conversely, the previous discussion shows that given the open covering \mathcal{U} of X by Stein open subsets a deformation $(\mathcal{X}, \tilde{\pi})$ of (X, π) over A is determined up to equivalence by the following data:

for all $U_i \in \mathcal{U}$ a solution $\sigma_i \in \bigwedge^2 \Theta_X(U_i) \otimes \mathfrak{m}_A$ to the Maurer-Cartan equation in the dg Lie algebra $(\bigwedge^{\geq 1} \Theta_X[1](U_i) \otimes \mathfrak{m}_A, d_{\pi|_{U_i}}, [\cdot, \cdot])$ and

for all double intersections U_{ij} a section $\eta_{ij} \in \Theta_X(U_{ij}) \otimes \mathfrak{m}_A$ such that $e^{\eta_{ij}} * \sigma_j|_{U_{ij}} = \sigma_i|_{U_{ij}}$ in the dg Lie algebra $(\bigwedge^{\geq 1} \Theta_X[1](U_{ij}) \otimes \mathfrak{m}_A, d_{\pi|_{U_{ij}}}, [\cdot, \cdot])$, and such that moreover

.the cocycle conditions $\eta_{ij} \circ \eta_{jk} = \eta_{ik}$ are satisfied in the Lie algebras $\Theta_X(U_{ijk}) \otimes \mathfrak{m}_A$.

If we compare with Definition 5.3.10 we see that the above data is exactly the datum of an object in the groupoid of descent data $\text{Tot}(\text{Del}_{\bigwedge_{\Theta_X^{\geq 1}[1](\mathcal{U})} \bullet}(A))$ of the semicosimplicial groupoid

$$\prod_i \text{Del}_{\bigwedge^{\geq 1} \Theta_X[1](U_i)}(A) \rightrightarrows \prod_{i,j} \text{Del}_{\bigwedge^{\geq 1} \Theta_X[1](U_{ij})}(A) \rightrightarrows \prod_{i,j,k} \text{Del}_{\bigwedge^{\geq 1} \Theta_X[1](U_{ijk})}(A) \cdots$$

Equivalences of deformations can be treated similarly: given a pair of deformation $(\mathcal{X}_1, \tilde{\pi}_1)$ and $(\mathcal{X}_2, \tilde{\pi}_2)$, and once we have fixed trivializations $\bigwedge \Theta_{\mathcal{X}_p/A}(U_i) \xrightarrow{\cong} \bigwedge \Theta_X(U_i) \otimes A$, $p = 1, 2$, of both over the various $U_i \in \mathcal{U}$, we denote by (σ_i, η_{ij}) and (τ_i, θ_{ij}) the corresponding descent data as in the previous discussion: then an equivalence between $(\mathcal{X}_1, \tilde{\pi}_1)$ and $(\mathcal{X}_2, \tilde{\pi}_2)$ is exactly the data

for all $U_i \in \mathcal{U}$ of a section $\xi_i \in \Theta_X(U_i) \otimes \mathfrak{m}_A$ such that $e^{\xi_i} * \sigma_i = \tau_i$ in the dg Lie algebra $(\bigwedge^{\geq 1} \Theta_X[1](U_i) \otimes \mathfrak{m}_A, d_{\pi|_{U_i}}, [\cdot, \cdot])$ and such that moreover

the various e^{ξ_i} glue to a global $\mathcal{O}_{\mathcal{X}_1} \rightarrow \mathcal{O}_{\mathcal{X}_2}$, that is, on double intersections the identities $e^{\xi_i|_{U_{ij}}} e^{\eta_{ij}} = e^{\theta_{ij}} e^{\xi_j|_{U_{ij}}}$ are satisfied, or equivalently we have $\xi_i|_{U_{ij}} \circ \eta_{ij} = \theta_{ij} \circ \xi_j|_{U_{ij}}$ in the Lie algebras $\Theta_X(U_{ij}) \otimes \mathfrak{m}_A$.

Again, comparing with Definition 5.3.10 these are exactly the morphisms between the objects (σ_i, η_{ij}) and (τ_i, θ_{ij}) in the groupoid of descent data $\text{Tot}(\text{Del}_{\bigwedge_{\Theta_X^{\geq 1}[1](\mathcal{U})} \bullet}(A))$. To sum up: there is an equivalence of pointed groupoids $\text{Del}_{(X, \pi)}(A) \xrightarrow{\sim} \text{Tot}(\text{Del}_{\bigwedge_{\Theta_X^{\geq 1}[1](\mathcal{U})} \bullet}(A))$, and since it clear that this is natural in A by Theorem 5.3.12 and Corollary 5.3.15 we get equivalences of formal pointed groupoids

$$\text{Del}_{(X, \pi)} \simeq \text{Tot}(\text{Del}_{\bigwedge_{\Theta_X^{\geq 1}[1](\mathcal{U})} \bullet}) \cong \text{Del}_{\text{Tot}(\bigwedge_{\Theta_X^{\geq 1}[1](\mathcal{U})} \bullet)} \simeq \text{Del}_{\text{Tot}_{TW}(\bigwedge_{\Theta_X^{\geq 1}[1](\mathcal{U})} \bullet)} .$$

□

Remark 6.2.5. We notice that the previous argument is algebraic in nature, in particular, it can be easily extended to every algebraic Poisson manifold defined over a field of characteristic 0: roughly, it is sufficient to replace holomorphic with algebraic and Stein with affine and everything still works.

Comparing Theorem 5.3.12 and Corollary 5.3.15 we see that the more natural model to consider would be $\text{Tot}(\bigwedge_{\Theta_X^{\geq 1}[1](\mathcal{U})} \bullet)$, and in fact the underlying dg space is the usual Čech complex of cochains on the (nerve of the) covering \mathcal{U} with coefficients in the sheaf of dg Lie algebras $\bigwedge_{\Theta_X^{\geq 1}[1]}$ over X . On the other hand, in some situations we may want to work only with dg Lie algebras, and in this case, although a natural homotopical construction, the Thom-Whitney totalization is not really a familiar object to work with: we shall use Dolbeault's resolutions in order to describe

another dg Lie algebra governing Poisson deformations (of course this only works in the complex analytic setting).

Given a locally free sheaf \mathcal{E} on a complex manifold X we shall denote by $\mathcal{A}_X^{0,j}(\mathcal{E})$ the sheaf of differentiable forms of type $(0, j)$ with values in \mathcal{E} . The Dolbeault resolution of a bounded below complex

$$(\mathcal{E}^*, \delta): \quad 0 \rightarrow \mathcal{E}^i \xrightarrow{\delta} \mathcal{E}^{i+1} \xrightarrow{\delta} \dots$$

of locally free sheaves on a complex manifold is the sheaf of dg vector spaces $\mathcal{A}_X^{0,*}(\mathcal{E}^*)$, where

$$\mathcal{A}_X^{0,*}(\mathcal{E})^i = \bigoplus_{j+h=i} \mathcal{A}_X^{0,j}(\mathcal{E}^h),$$

and the differential $\bar{\partial}_{\mathcal{E}^*}$ is defined by the formula

$$\bar{\partial}_{\mathcal{E}^*}: \mathcal{A}_X^{0,j}(\mathcal{E}^h) \rightarrow \mathcal{A}_X^{0,j+1}(\mathcal{E}^h) \oplus \mathcal{A}_X^{0,j}(\mathcal{E}^{h+1}), \quad \bar{\partial}_{\mathcal{E}^*}(\phi \otimes e) = \bar{\partial}\phi \otimes e + (-1)^j \phi \otimes \delta e.$$

According to Dolbeault's lemma, the natural inclusion $\mathcal{E}^* \rightarrow \mathcal{A}_X^{0,*}(\mathcal{E}^*)$ is a quasi-isomorphism.

Similarly we denote by $A_X^{0,*}(\mathcal{E}^*)$ the dg space of global sections of the Dolbeault resolution; more generally, for every open subset $U \subset X$ we shall denote by $A_U^{0,*}(\mathcal{E}^*)$ the dg space of sections of $\mathcal{A}_X^{0,*}(\mathcal{E}^*)$ over U . Notice that, by Dolbeault theorem, the cohomology $A_X^{0,*}(\mathcal{E}^*)$ is isomorphic to the hypercohomology of \mathcal{E}^* .

Let (\mathcal{E}^*, δ) be a bounded below complex of locally free sheaves on a complex manifold X and let $\mathcal{U} = \{U_i\}$ be an open Stein covering of X . Thus we have a natural morphism of semicosimplicial dg vector spaces:

$$\begin{array}{ccccccc} \mathcal{E}^*(\mathcal{U})_\bullet : & \prod_i \mathcal{E}^*(U_i) & \rightrightarrows & \prod_{i,j} \mathcal{E}^*(U_{ij}) & \rightrightarrows & \prod_{i,j,k} \mathcal{E}^*(U_{ijk}) & \cdots \\ \downarrow & \downarrow & & \downarrow & & \downarrow & \\ A_{\mathcal{U}}^{0,*}(\mathcal{E}^*)_\bullet : & \prod_i A_{U_i}^{0,*}(\mathcal{E}^*) & \rightrightarrows & \prod_{i,j} A_{U_{ij}}^{0,*}(\mathcal{E}^*) & \rightrightarrows & \prod_{i,j,k} A_{U_{ijk}}^{0,*}(\mathcal{E}^*) & \cdots \end{array}$$

Since $A_X^{0,*}(\mathcal{E}^*)$ is the equalizer of $\partial_0, \partial_1: A_{\mathcal{U}}^{0,*}(\mathcal{E}^*)_0 \rightarrow A_{\mathcal{U}}^{0,*}(\mathcal{E}^*)_1$ and every map

$$\mathcal{E}^*(U_{i_1 \dots i_k}) \rightarrow A_{U_{i_1 \dots i_k}}^{0,*}(\mathcal{E}^*)$$

is a quasi-isomorphism, according to the following Remark 6.2.6 we get diagrams of quasi-isomorphisms

$$\begin{array}{ccc} \text{Tot}_{TW}(\mathcal{E}^*(\mathcal{U})_\bullet) & & \text{Tot}(\mathcal{E}^*(\mathcal{U})_\bullet) \\ \downarrow & & \downarrow \\ A_X^{0,*}(\mathcal{E}^*) \xrightarrow{e'} \text{Tot}_{TW}(A_{\mathcal{U}}^{0,*}(\mathcal{E}^*)_\bullet) & & A_X^{0,*}(\mathcal{E}^*) \xrightarrow{e} \text{Tot}(A_{\mathcal{U}}^{0,*}(\mathcal{E}^*)_\bullet) \end{array}$$

Remark 6.2.6. Let L_\bullet be a semicosimplicial dg Lie algebra and denote by $H = \{x \in L_0 \mid \partial_0 x = \partial_1 x\}$ the equalizer of $\partial_0, \partial_1: L_0 \rightarrow L_1$: we remark that if $x \in H$ then x is also in the equalizer of the $(n+1)$ iterated faces $L_0 \rightarrow L_n$ for all $n \geq 1$. This clearly implies, by the universal property of $\text{Tot}(-)$ (as explained in Remark 5.3.1), that the family of maps $H \rightarrow C(\Delta_n; L_n): x \rightarrow \pi^*(\partial_0^n x)$, where $\partial_0^n: L_0 \rightarrow L_n$ is the iterated face and $\pi^*: L_n \rightarrow C(\Delta_n; L_n)$ is pullback by the terminal morphism $\pi: \Delta_n \rightarrow \Delta_0$, induces a strict morphism $e: H \rightarrow \text{Tot}(L_\bullet)$ of L_∞ algebras. In the same way it is defined a morphism of dg Lie algebras $e': H \rightarrow \text{Tot}_{TW}(L_\bullet)$, and moreover the two are compatible with the natural quasi-isomorphism $\int: \text{Tot}_{TW}(L_\bullet) \rightarrow \text{Tot}(L_\bullet)$. In the above diagram e is a quasi-isomorphism, and thus so is e' , since the whole diagram induces the usual isomorphism between Dolbeault and Čech (hyper)cohomology.

We apply the previous considerations to the complex of locally free sheaves on X

$$\bigwedge^{\geq 1} \Theta_X[1] : \quad 0 \rightarrow \Theta_X \xrightarrow{d_\pi} \bigwedge^2 \Theta_X \xrightarrow{d_\pi} \bigwedge^3 \Theta_X \cdots ,$$

where $\bigwedge^i \Theta_X$ is in degree $i-1$. In this case the complex $A_X^{0,*}(\bigwedge^{\geq 1} \Theta_X[1])$ admits a natural structure of dg Lie algebra, where the bracket is the antiholomorphic extension of the Schouten-Nijenhuis bracket on $A_X^{0,0}(\bigwedge^{\geq 1} \Theta_X[1])$.

Theorem 6.2.7. *The dg Lie algebra $A_X^{0,*}(\bigwedge^{\geq 1} \Theta_X[1])$ controls the deformation of (X, π) . More precisely, there is an equivalence of formal pointed groupoids*

$$\mathrm{Del}_{A_X^{0,*}(\bigwedge^{\geq 1} \Theta_X[1])} \simeq \mathrm{Del}_{(X, \pi)}$$

Proof. The previous diagram of quasi-isomorphisms (the one on the left) is clearly a diagram of dg Lie algebras: we know by the classical result of Goldman and Milson [42] that a quasi-isomorphism $f : L \rightarrow M$ of dg Lie algebras concentrated in degrees ≥ 0 induces an equivalence of formal pointed groupoids $\mathrm{Del}_L \rightarrow \mathrm{Del}_M$, thus the result follows from Theorem 6.2.4. \square

6.3 Coisotropic deformations

The aim of this section is to study infinitesimal embedded coisotropic deformations of a coisotropic submanifold $Z \subset X$ of a holomorphic Poisson manifold (X, π) : we shall begin by studying coisotropic deformations of the triad (X, Z, π) , and as a first step deformations of the (complex manifold, complex submanifold) structure on the pair (X, Z) .

Definition 6.3.1. A deformation of the (complex manifold, complex submanifold) structure on the pair (X, Z) over A is a deformation \mathcal{X} of X over A , as in Definition 6.2.1, together with a sheaf $\mathcal{I}_Z \subset \mathcal{O}_{\mathcal{X}}$ of A -flat ideals and a morphism

$$\begin{array}{ccc} \mathcal{I}_Z \hookrightarrow & \mathcal{O}_{\mathcal{X}} \\ \downarrow & \downarrow \\ \mathcal{I}_Z \hookrightarrow & \mathcal{O}_X \end{array}$$

of pairs of sheaves of A -algebras which locally trivializes, that is, for all $x \in X$ there is a neighborhood $x \in U \subset X$ and an isomorphism

$$\begin{array}{ccc} \mathcal{I}_Z(U) \hookrightarrow & \mathcal{O}_{\mathcal{X}}(U) \\ \downarrow & \downarrow \\ \mathcal{I}_Z(U) \otimes A \hookrightarrow & \mathcal{O}_X(U) \otimes A \end{array}$$

of pairs of A -algebras over $\mathcal{I}_Z(U) \hookrightarrow \mathcal{O}_X(U)$. Equivalences between deformations of (Z, X) over A are isomorphisms of pairs of sheaves over $\mathcal{I}_Z \hookrightarrow \mathcal{O}_X$.

Recall that we denote by $\Theta_X(-\log Z) \subset \Theta_X$ the sheaf of vector fields tangent everywhere to Z . Given a covering \mathcal{U} of X by Stein open sets, a deformation of the pair (X, Z) over A trivializes globally over each $U_i \in \mathcal{U}$, and as in the previous section we can recover the whole deformation

by gluing together trivial deformations via a family of transition automorphisms on the double intersections, satisfying moreover the cocycle condition on triple intersections. It is easy to see that $e^{\eta_{ij}} : \mathcal{O}_X(U_{ij}) \otimes A \rightarrow \mathcal{O}_X(U_{ij}) \otimes A$ sends $\mathcal{I}_Z(U_{ij}) \otimes A$ into itself, where $\eta_{ij} \in \Theta_X(U_{ij}) \otimes \mathfrak{m}_A$, if and only if $\eta_{ij} \in \Theta_X(-\log Z)(U_{ij}) \otimes \mathfrak{m}_A$: in other words, the group of self-equivalences of the trivial deformation $\mathcal{I}_Z(U_{ij}) \otimes A \hookrightarrow \mathcal{O}_X(U_{ij}) \otimes A$ is the exponential group $\exp(\Theta_X(-\log Z)(U_i) \otimes \mathfrak{m}_A)$.

Definition 6.3.2. Given a holomorphic Poisson manifold (X, π) and a coisotropic submanifold $Z \subset X$, a coisotropic deformation of (X, Z, π) over $A \in \mathbf{Art}_{\mathbb{C}}$ is a deformation $(\mathcal{X}, \tilde{\pi})$ of (X, π) together with a sheaf of coisotropic ideals $\mathcal{I}_{\mathcal{Z}} \subset \mathcal{O}_{\mathcal{X}}$ such that $\mathcal{I}_{\mathcal{Z}} \hookrightarrow \mathcal{O}_{\mathcal{X}}$ is a deformation of the pair (X, Z) over A . Together with the obvious notion of equivalence, we associate to (X, Z, π) a formal pointed groupoid and a formal pointed set

$$\mathrm{Del}_{(X,Z,\pi)}^{\mathrm{co}} : \mathbf{Art}_{\mathbb{C}} \rightarrow \mathbf{Grpd}_*, \quad \mathrm{Def}_{(X,Z,\pi)}^{\mathrm{co}} : \mathbf{Art}_{\mathbb{C}} \rightarrow \mathbf{Set}_*.$$

We take \mathcal{U} as usual. Recall from Proposition 6.1.9 the sheaf of differential Gerstenhaber subalgebras $\mathcal{L}_Z \subset \bigwedge \Theta_X$: the semicosimplicial dg Lie algebra $\mathcal{L}_Z^{\geq 1}[1](\mathcal{U})_{\bullet}$ is defined as in the previous section via the usual Čech construction, this is concentrated in degrees ≥ 0 and in particular it satisfies the assumptions of Theorem 5.3.12 and Corollary 5.3.15.

Theorem 6.3.3. *There are equivalences of formal pointed groupoids*

$$\mathrm{Del}_{\mathrm{Tot}_{\mathcal{T}w}(\mathcal{L}_Z^{\geq 1}[1](\mathcal{U})_{\bullet})} \simeq \mathrm{Del}_{\mathrm{Tot}(\mathcal{L}_Z^{\geq 1}[1](\mathcal{U})_{\bullet})} \simeq \mathrm{Del}_{(X,Z,\pi)}^{\mathrm{co}}.$$

Proof. Given a deformation $(\mathcal{X}, \mathcal{Z}, \tilde{\pi})$ of (X, Z, π) over A , the deformation $(\mathcal{X}, \mathcal{Z})$ of the pair (X, Z) is determined up to equivalence by the family of transition automorphisms $e^{\eta_{ij}}$, where $\eta_{ij} \in \Theta_X(-\log Z)(U_{ij}) \otimes \mathfrak{m}_A = \mathcal{L}_Z^1(U_{ij}) \otimes \mathfrak{m}_A$: moreover, as in the proof of (1) \Leftrightarrow (2) in Proposition 6.1.9 we see that the ideal $\mathcal{I}_Z(U_i) \otimes A \subset \mathcal{O}_X(U_i) \otimes A$ is coisotropic with respect to the Poisson bivector $\tilde{\pi}|_{U_i} = \pi|_{U_i} + \sigma_i$ if and only if $\sigma_i \in \mathcal{L}_Z^2(U_i) \otimes \mathfrak{m}_A \subset \bigwedge^2 \Theta_X(U_i) \otimes \mathfrak{m}_A$. As in the proof of 6.2.4, the deformation $(\mathcal{X}, \mathcal{Z}, \tilde{\pi})$ is determined up to equivalence by the corresponding object (σ_i, η_{ij}) in the groupoid of descent data $\mathrm{Tot}(\mathrm{Del}_{\mathcal{L}_Z^{\geq 1}[1](\mathcal{U})_{\bullet}}(A))$ of the semicosimplicial groupoid

$$\prod_i \mathrm{Del}_{\mathcal{L}_Z^{\geq 1}[1](U_i)}(A) \rightrightarrows \prod_{i,j} \mathrm{Del}_{\mathcal{L}_Z^{\geq 1}[1](U_{ij})}(A) \rightrightarrows \prod_{i,j,k} \mathrm{Del}_{\mathcal{L}_Z^{\geq 1}[1](U_{ijk})}(A) \cdots$$

Equivalences between deformations can be treated in the exact same way as in 6.2.4, and since this is natural in A there is an equivalence of formal pointed groupoids $\mathrm{Del}_{(X,Z,\pi)}^{\mathrm{co}} \simeq \mathrm{Tot}(\mathrm{Del}_{\mathcal{L}_Z^{\geq 1}[1](\mathcal{U})_{\bullet}})$, so the thesis follows from Theorem 5.3.12 and Corollary 5.3.15. \square

As an application of the above result we are able to give the analog of Kodaira's stability theorem for coisotropic submanifolds. Let $\bigwedge^{\geq 1} \mathcal{N}_{Z|X}[1]$ be the complex of sheaves on X

$$\bigwedge^{\geq 1} \mathcal{N}_{Z|X}[1] : \quad 0 \rightarrow \mathcal{N}_{Z|X} \xrightarrow{d_{\pi}} \bigwedge^2 \mathcal{N}_{Z|X} \xrightarrow{d_{\pi}} \bigwedge^3 \mathcal{N}_{Z|X} \cdots, \quad (6.3.1)$$

where $\bigwedge^i \mathcal{N}_{Z|X}$ is in degree $i - 1$. We notice that the Lichnerowicz-Poisson differential induces $\bigwedge^i \mathcal{N}_{Z|X} \xrightarrow{d_{\pi}} \bigwedge^{i+1} \mathcal{N}_{Z|X}$ thanks to Proposition 6.1.9.

Corollary 6.3.4 (Stability of coisotropic submanifolds). *Let (X, π) be a compact holomorphic Poisson manifold and let Z be a coisotropic submanifold. Let $\mathcal{X} \rightarrow (B, 0)$ be a Poisson deformation of (X, π) over a germ of complex space $(B, 0)$. If $\mathbb{H}^1(Z, \bigwedge^{\geq 1} \mathcal{N}_{Z|X}[1]) = 0$ then, after a possible shrinking of B , there exists a family of coisotropic submanifolds $\mathcal{Z} \subset \mathcal{X}$ which is smooth over B and such that $\mathcal{Z}_0 = Z$.*

Proof. Following the same standard argument used in the proof of Theorem 8.1 of [49], involving relative Douady space and Artin's theorem on the solution of analytic equations, it is not restrictive to assume B a fat point, that is, $B = \text{Spec } A$ for some $A \in \mathbf{Art}_{\mathbb{C}}$. Thus the stability theorem is proved whenever we show that the natural transformation of formal pointed sets

$$\text{Def}_{(X,Z,\pi)}^{\text{co}} \longrightarrow \text{Def}_{(X,\pi)}$$

is smooth (cf. Remark 6.4.4). Fixing an open Stein covering $\mathcal{U} = \{U_i\}$ of X , the above natural transformation is induced by the inclusion of differential graded Lie algebras

$$\text{Tot}(\mathcal{L}_{\mathbb{Z}}^{\geq 1}[1](\mathcal{U})_{\bullet}) \xrightarrow{i} \text{Tot}\left(\bigwedge^{\geq 1} \Theta_X[1](\mathcal{U})_{\bullet}\right).$$

According to standard smoothness criterion, see e.g. [76], the morphism $\text{Def}_{(X,Z,\pi)}^{\text{co}} \longrightarrow \text{Def}_{(X,\pi)}$ is smooth whenever i is surjective on H^1 and injective on H^2 . By the definition of $\mathcal{L}_{\mathbb{Z}}^{\geq 1}$ we have an exact sequence of complexes of coherent sheaves

$$0 \rightarrow \mathcal{L}_{\mathbb{Z}}^{\geq 1}[1] \rightarrow \bigwedge^{\geq 1} \Theta_X[1] \rightarrow \bigwedge^{\geq 1} \mathcal{N}_{Z|X}[1] \rightarrow 0,$$

and the thesis follows from the hypercohomology long exact sequence. \square

Next we study embedded coisotropic deformations. Recall that given a pair (X, Z) consisting of a complex manifold X and a locally closed complex submanifold $Z \subset X$ the local Hilbert functor is the formal pointed set $\text{Hilb}_{Z|X} : \mathbf{Art}_{\mathbb{C}} \rightarrow \mathbf{Set}$ sending $A \in \mathbf{Art}_{\mathbb{C}}$ to the set of sheaves of A -flat ideals $\mathcal{I}_{\mathbb{Z}} \subset \mathcal{O}_X \otimes A$ such that $\mathcal{I}_{\mathbb{Z}} \otimes_A \mathbb{C} = \mathcal{I}_{\mathbb{Z}}$. In other words, $\text{Hilb}_{Z|X}$ is the functor of formal embedded deformations of Z in X . If $U \subset X$ is open Stein then $\mathcal{I}_{\mathbb{Z}}(U) \hookrightarrow \mathcal{O}_X(U) \otimes A$ is isomorphic as a pair to $\mathcal{I}_{\mathbb{Z}}(U) \otimes A \hookrightarrow \mathcal{O}_X(U) \otimes A$, and thus there is $\eta \in \Theta_X(U) \otimes \mathfrak{m}_A$ such that $\mathcal{I}_{\mathbb{Z}}(U) = e^{\eta}(\mathcal{I}_{\mathbb{Z}}(U) \otimes A)$.

Definition 6.3.5. Given a holomorphic Poisson manifold (X, π) and a coisotropic submanifold $Z \subset X$, the local coisotropic Hilbert functor is the formal pointed set $\text{Hilb}_{Z|X}^{\text{co}} : \mathbf{Art}_{\mathbb{C}} \rightarrow \mathbf{Set}_{*}$ sending A to the set of sheaves of A -flat *coisotropic* ideals $\mathcal{I}_{\mathbb{Z}} \subset \mathcal{O}_X \otimes A$ such that $\mathcal{I}_{\mathbb{Z}} \otimes_A \mathbb{C} = \mathcal{I}_{\mathbb{Z}}$.

Let $\mathcal{K}_{\mathbb{Z}}^{\geq 1}$ be the homotopy fiber of the inclusion of sheaves of (non negatively graded) dg Lie algebras $\mathcal{L}_{\mathbb{Z}}^{\geq 1}[1] \hookrightarrow \bigwedge^{\geq 1} \Theta_X[1]$ (the notation is a little ambiguous, this is not the non negatively graded part of the homotopy fiber \mathcal{K}_Z of the inclusion $\mathcal{L}_Z[1] \hookrightarrow \bigwedge \Theta_X[1]$). We take \mathcal{U} as usual and denote by $\mathcal{K}_{\mathbb{Z}}^{\geq 1}(\mathcal{U})_{\bullet}$ the associated semicosimplicial dg Lie algebra (concentrated in degrees ≥ 0).

Theorem 6.3.6. *There are equivalences of formal pointed groupoids*

$$\text{Del}_{\text{Tot}_{TW}}(\mathcal{K}_{\mathbb{Z}}^{\geq 1}(\mathcal{U})_{\bullet}) \simeq \text{Del}_{\text{Tot}}(\mathcal{K}_{\mathbb{Z}}^{\geq 1}(\mathcal{U})_{\bullet}) \simeq \text{Hilb}_{Z|X}^{\text{co}},$$

where $\text{Hilb}_{Z|X}^{\text{co}}$ is regarded as a formal pointed groupoid via the inclusion $\mathbf{Set}_{*} \rightarrow \mathbf{Grpd}_{*}$.

Proof. We shall first show that $\text{Del}_{\mathcal{K}_{\mathbb{Z}}^{\geq 1}(\mathcal{U})} \simeq \text{Hilb}_{U \cap Z|U}^{\text{co}}$ for a Stein open $U \subset X$, then the theorem will follow from descent of Deligne groupoids as in the previous cases. We notice that the mapping cocone $\text{coC}(f)$ and the homotopy fiber $K(f)$ of a morphism $f : L \rightarrow M$ of dg Lie algebras are particular cases of totalization and Thom-Whitney totalization, respectively applied to the semicosimplicial dg Lie algebra

$$L \begin{array}{c} \xrightarrow{0} \\ \rightrightarrows \\ \xrightarrow{f} \end{array} M \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} 0 \cdots$$

If L and M are concentrated in degrees ≥ 0 Theorem 5.3.12 and Corollary 5.3.15 apply, and in this case they tell us that $\text{Del}_{\text{coC}(f)}$ and $\text{Del}_{K(f)}$ are respectively isomorphic and equivalent to the formal groupoid (sending $A \in \mathbf{Art}_{\mathbb{C}}$ to the groupoid) whose objects are pairs (l, a) , where $l \in \text{MC}(L \otimes \mathfrak{m}_A)$ and $a \in M^0 \otimes \mathfrak{m}_A$ is such that $e^a * 0 = f(l)$ (notice that the cocycle condition is automatically satisfied) and whose morphism $(l, a) \rightarrow (l', a')$ are the $b \in L^0 \otimes \mathfrak{m}_A$ such that $e^b * l = l'$ and $e^{f(b)} e^a = e^{a'}$, that is, $f(b) \circ a = a'$ in the Lie algebra $M^0 \otimes \mathfrak{m}_A$. When f is moreover the inclusion $L \subset M$ of a dg Lie subalgebra, objects in this groupoid are just the $\eta \in M^0 \otimes \mathfrak{m}_A$ such that $e^\eta * 0 \in L^1 \otimes \mathfrak{m}_A$: in this case there is at most one morphism between two objects and thus $\mathbf{Grpd}_* \xrightarrow{\pi_0(-)} \mathbf{Set}_* \rightarrow \mathbf{Grpd}_*$, where the second map is the inclusion, induces equivalences of formal pointed groupoids $\text{Del}_{\text{coC}(f)} \simeq \text{Del}_{K(f)} \simeq \text{Def}_{\text{coC}(f)} \cong \text{Def}_{K(f)}$.

In the case of the inclusion of non negatively graded dg Lie algebras $\mathcal{L}_{\mathbb{Z}}^{\geq 1}[1](U) \hookrightarrow \bigwedge^{\geq 1} \Theta_X[1](U)$ we see that objects in $\text{Del}_{\mathcal{K}_{\mathbb{Z}}^{\geq 1}(U)}(A)$ are the $\eta \in \Theta_X(U) \otimes \mathfrak{m}_A$ such that $e^\eta * 0 \in \mathcal{L}_{\mathbb{Z}}^2(U) \otimes \mathfrak{m}_A$, and morphisms $\eta \rightarrow \theta$ are the $\alpha \in \Theta(-\log Z)(U) \otimes \mathfrak{m}_A$ such that $\alpha \circ \eta = \theta$ in the Lie algebra $\Theta_X(U) \otimes \mathfrak{m}_A$. We define an isomorphism of sets $\text{Def}_{\mathcal{K}_{\mathbb{Z}}^{\geq 1}(U)}(A) \xrightarrow{\sim} \text{Hilb}_{U^{\text{co}} \cap Z|U}(A)$ by sending the Gauge equivalence class of η to the ideal $e^{-\eta}(\mathcal{I}_{Z \cap U} \otimes A) \subset \mathcal{O}_U \otimes A$, we have to show that this is well defined: given a morphism $\alpha \circ \eta = \theta$ between η and θ , since $\alpha \in \Theta(-\log Z)(U) \otimes \mathfrak{m}_A$ we have $e^{-\alpha}(\mathcal{I}_{Z \cap U} \otimes A) = \mathcal{I}_{Z \cap U} \otimes A$ and thus also $e^{-\eta}(\mathcal{I}_{Z \cap U} \otimes A) = e^{-\theta}(\mathcal{I}_{Z \cap U} \otimes A)$, so it remains to show that $e^{-\eta}(\mathcal{I}_{Z \cap U} \otimes A)$ is a coisotropic ideal, but since $e^{\text{ad}_\eta} : \bigwedge \Theta_X(U) \otimes \mathfrak{m}_A \rightarrow \bigwedge \Theta_X(U) \otimes \mathfrak{m}_A$ is a Gerstenhaber automorphism, this is equivalent to say that $\mathcal{I}_{Z \cap U} \otimes A$ is coisotropic with respect to the Poisson bracket induced by $e^{\text{ad}_\eta}(\pi|_U) = \pi|_U + e^\eta * 0$, and as in the proof of Proposition 6.1.9 this is equivalent to $e^\eta * 0 \in \mathcal{L}_{\mathbb{Z}}^2(U) \otimes \mathfrak{m}_A$. We have to show that this is an isomorphism: this is injective since $e^{-\eta}(\mathcal{I}_{Z \cap U} \otimes A) = e^{-\theta}(\mathcal{I}_{Z \cap U} \otimes A)$ if and only if $\theta \circ (-\eta) =: \alpha \in \Theta(-\log Z)(U) \otimes \mathfrak{m}_A$, and thus α is a morphism from η to θ in $\text{Del}_{\mathcal{K}_{\mathbb{Z}}^{\geq 1}(U)}(A)$, this is surjective since as we said if U is open Stein then every $\mathcal{I}_{Z|U}$ in $\text{Hilb}_{Z \cap U|U}(A)$ (and in particular in $\text{Hilb}_{Z \cap U|U}^{\text{co}}(A)$) is of the form $e^{-\eta}(\mathcal{I}_{Z \cap U} \otimes A)$ for some $\eta \in \Theta_X(U) \otimes \mathfrak{m}_A$. Finally, it is clear that this is natural in A , so we have constructed the promised equivalence $\text{Del}_{\mathcal{K}_{\mathbb{Z}}^{\geq 1}(U)} \simeq \text{Hilb}_{U^{\text{co}} \cap Z|U}^{\text{co}}$ of formal pointed groupoids.

It is clear by definition that $U \rightarrow \text{Hilb}_{U^{\text{co}} \cap Z|U}^{\text{co}}(A)$ is a sheaf of pointed sets on X , in particular this means that the formal pointed set $\text{Hilb}_{Z|X}^{\text{co}}$, regarded as a formal pointed groupoid, is canonically isomorphic to its groupoid of descent data, that is, the totalization of the semicosimplicial formal pointed groupoid

$$\prod_i \text{Hilb}_{U_i^{\text{co}} \cap Z|U_i}^{\text{co}} \rightrightarrows \prod_{i,j} \text{Hilb}_{U_{ij}^{\text{co}} \cap Z|U_{ij}}^{\text{co}} \rightrightarrows \prod_{i,j,k} \text{Hilb}_{U_{ijk}^{\text{co}} \cap Z|U_{ijk}}^{\text{co}} \cdots$$

and by the first part of the proof this is equivalent to the semicosimplicial formal pointed groupoid

$$\prod_i \text{Del}_{\mathcal{K}_{\mathbb{Z}}^{\geq 1}(U_i)} \rightrightarrows \prod_{i,j} \text{Del}_{\mathcal{K}_{\mathbb{Z}}^{\geq 1}(U_{ij})} \rightrightarrows \prod_{i,j,k} \text{Del}_{\mathcal{K}_{\mathbb{Z}}^{\geq 1}(U_{ijk})} \cdots$$

Since totalization commutes with equivalences (Remark 5.3.11) we finally get an equivalence of formal pointed groupoids $\text{Hilb}_{Z|X}^{\text{co}} \simeq \text{Tot}(\text{Del}_{\mathcal{K}_{\mathbb{Z}}^{\geq 1}(U)_\bullet})$. Now the thesis follows as usual from Theorem 5.3.12 and Corollary 5.3.15. \square

In the final part of the section we compare Theorem 6.3.6 with the more usual approach to coisotropic deformations via the homotopy Lie algebroid of Oh-Park [87] and Cattaneo-Felder [19]. As a first step we shall use Dolbeault resolutions to determine a more amenable dg Lie algebra governing the embedded coisotropic deformations of Z in X . Let $\bigwedge^{\geq 1} \mathcal{N}_{Z|X}[1]$ be the complex of sheaves on X as in equation (6.3.1), we define $L_{Z|X} \subset A_X^{0,*}(\bigwedge^{\geq 1} \Theta_X[1])$ by the short exact sequence

$$0 \rightarrow L_{Z|X} \xrightarrow{x} A_X^{0,*}(\bigwedge^{\geq 1} \Theta_X[1]) \xrightarrow{P} A_Z^{0,*}(\bigwedge^{\geq 1} \mathcal{N}_{Z|X}[1]) \rightarrow 0, \quad (6.3.2)$$

where P is the natural projection map and χ is the inclusion: then we see as in the proof of Proposition 6.1.9 that $L_{Z|X} \subset A_X^{0,*}(\bigwedge^{\geq 1}\Theta_X[1])$ is a dg Lie subalgebra.

Theorem 6.3.7. *The homotopy fiber $K(\chi)$ of the inclusion $L_{Z|X} \xrightarrow{\chi} A_X^{0,*}(\bigwedge^{\geq 1}\Theta_X[1])$ governs the infinitesimal embedded coisotropic deformations of Z in X .*

Proof. Since the functor $\text{Tot}_{TW}(-) : \mathbf{DG}^{\Delta} \rightarrow \mathbf{DG}$ is exact there is a short exact sequence

$$0 \rightarrow \text{Tot}_{TW}(\mathcal{L}_{\bar{Z}}^{\geq 1}[1](\mathcal{U})_{\bullet}) \xrightarrow{\alpha} \text{Tot}_{TW}(\bigwedge^{\geq 1}\Theta_X[1](\mathcal{U})_{\bullet}) \rightarrow \text{Tot}_{TW}(\bigwedge^{\geq 1}\mathcal{N}_{Z|X}[1](\mathcal{U})_{\bullet}) \rightarrow 0$$

and $\text{Tot}_{TW}(\mathcal{K}_{\bar{Z}}^{\geq 1}(\mathcal{U})_{\bullet})$ is isomorphic to the homotopy fiber of α . The above sequence is part of a 3×3 diagram with exact rows, with the first two columns made by morphisms of differential graded Lie algebras, the second two columns as in Remark 6.2.6 and where every vertical arrow is a quasi-isomorphism

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Tot}_{TW}(\mathcal{L}_{\bar{Z}}^{\geq 1}[1](\mathcal{U})_{\bullet}) & \xrightarrow{\alpha} & \text{Tot}_{TW}(\bigwedge^{\geq 1}\Theta_X[1](\mathcal{U})_{\bullet}) & \longrightarrow & \text{Tot}_{TW}(\bigwedge^{\geq 1}\mathcal{N}_{Z|X}[1](\mathcal{U})_{\bullet}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K & \longrightarrow & \text{Tot}_{TW}(A_U^{0,*}(\bigwedge^{\geq 1}\Theta_X[1])_{\bullet}) & \longrightarrow & \text{Tot}_{TW}(A_U^{0,*}(\bigwedge^{\geq 1}\mathcal{N}_{Z|X}[1])_{\bullet}) \longrightarrow 0 \\ & & \uparrow & & \uparrow e & & \uparrow e \\ 0 & \longrightarrow & L_{Z|X} & \xrightarrow{\chi} & A_X^{0,*}(\bigwedge^{\geq 1}\Theta_X[1]) & \longrightarrow & A_Z^{0,*}(\bigwedge^{\geq 1}\mathcal{N}_{Z|X}[1]) \longrightarrow 0 \end{array}$$

This diagram induces a quasi-isomorphism between the homotopy fibers of α and χ . \square

Now we review the construction of the homotopy Lie algebroid: the main ingredient was the subject of study of Chapter 4, namely, higher derived brackets. In order to apply the higher derived brackets construction we look for a splitting $A_Z^{0,*}(\bigwedge^{\geq 1}\mathcal{N}_{Z|X}[1]) \rightarrow A_X^{0,*}(\bigwedge^{\geq 1}\Theta_X[1])$ of the exact sequence (6.3.2): in the analog situation in the differentiable setting one performs such a choice via the identification of the normal bundle $N_{Z|X}$ with a tubular neighborhood of Z in X^4 , in the complex analytic setting this is not possible anymore and we have to work from the outset in the rather restrictive hypothesis that $X = E$ is the total space of a holomorphic vector bundle $p: E \rightarrow Z$ over Z , which is embedded in E as the zero section. In this case there is a natural identification $E \cong N_{Z|E}$, and thus a natural identification $p^*N_{Z|E} \cong p^*E$ between the pullback bundle $p^*N_{Z|E} \rightarrow E$ and the sub-bundle $p^*E \subset TE$ of vertical tangent vectors: this induces a morphism $\mathcal{N}_{Z|E} \rightarrow p_*\Theta_E$ of sheaves on Z , sending a section ξ of $N_{Z|E}$ to the vector field constantly ξ_p along the fiber E_p , by multiplicative extension we also get $\bigwedge^{\geq 1}\mathcal{N}_{Z|E} \rightarrow p_*\bigwedge^{\geq 1}\Theta_E$. Thus, for every open subset $U \subset Z$ we have a morphism

$$\bigwedge^{\geq 1}\mathcal{N}_{Z|E}[1](U) \rightarrow \bigwedge^{\geq 1}\Theta_E[1](p^{-1}(U))$$

whose image is an abelian graded Lie subalgebra. Acting via pull-back on differential forms, we obtain a splitting

$$\sigma: A_Z^{0,*}(\bigwedge^{\geq 1}\mathcal{N}_{Z|E}[1]) \longrightarrow A_E^{0,*}(\bigwedge^{\geq 1}\Theta_E[1]) \quad (6.3.3)$$

of the exact sequence (6.3.2) whose image is an abelian graded Lie subalgebra.

⁴We could recover by Lemma 4.1.17 the fact already proved in [20] that the resulting L_{∞} algebra does not depend on this choice up to L_{∞} isomorphism.

Given a Poisson bivector π on E seen as an element of $A_E^{0,0}(\Lambda^2 \Theta_E)$, we see as in the proof of Proposition 6.1.9 that if $Z \subset E$ is a coisotropic submanifold then $\pi \in L_{Z|E}^1$. In this case the derivation $D = \bar{\partial} + d_\pi \in \text{Der}(A_E^{0,*}(\Lambda^{\geq 1} \Theta_E[1]))$ sends the graded Lie subalgebra $L_{Z|E}$ into itself, thus we are in the algebraic setup of Voronov's construction of higher derived brackets and there is an induced $L_\infty[1]$ algebra structure on $A_Z^{0,*}(\Lambda^{\geq 1} \mathcal{N}_{Z|E}[1])$: explicitly

$$\Phi(D)_1(\xi) = (\bar{\partial} + d_\pi)(\xi),$$

$$\Phi(D)_n(\xi_1 \odot \cdots \odot \xi_n) = P([\cdots [d_\pi(\sigma(\xi_1)), \sigma(\xi_2)], \cdots], \sigma(\xi_n)), \quad n \geq 2,$$

where $P: A_E^{0,*}(\Lambda^{\geq 1} \Theta_E[1]) \rightarrow A_Z^{0,*}(\Lambda^{\geq 1} \mathcal{N}_{Z|E}[1])$ is the projection (notice that $\Phi(\bar{\partial})_n = 0$ for $n \geq 2$ since the image of σ is $\bar{\partial}$ -closed).

According to Theorem 4.1.7 the L_∞ algebra structure on the space $A_Z^{0,*}(\Lambda^{\geq 1} \mathcal{N}_{Z|E})$ (induced via décalage) is weakly equivalent to the homotopy fiber of $L_{Z|X} \xrightarrow{\chi} A_E^{0,*}(\Lambda^{\geq 1} \Theta_E[1])$, thus by homotopy invariance of Def_- (cf. [75]) we obtain the following corollary of Theorem 6.3.7.

Corollary 6.3.8. *In the previous hypotheses there is an isomorphism of formal pointed sets*

$$\text{Def}_{A_Z^{0,*}(\Lambda^{\geq 1} \mathcal{N}_{Z|E})} \cong \text{Hilb}_{Z|E}^{\text{co}}.$$

Remark 6.3.9. Since the L_∞ algebra $A_Z^{0,*}(\Lambda^{\geq 1} \mathcal{N}_{Z|E})$ is concentrated in degrees ≥ 1 there is an isomorphism of formal sets $\text{Def}_{A_Z^{0,*}(\Lambda^{\geq 1} \mathcal{N}_{Z|E})} \cong \text{MC}_{A_Z^{0,*}(\Lambda^{\geq 1} \mathcal{N}_{Z|E})}$, thus the corollary shows that for all $A \in \mathbf{Art}_{\mathbb{C}}$ there is a bijective correspondence between the set $\text{Hilb}_{Z|E}^{\text{co}}(A)$ and the set of solutions $\xi \in A_Z^{0,0}(\mathcal{N}_{Z|E}) \otimes \mathfrak{m}_A$ of the Maurer-Cartan equation $\sum_{n \geq 1} \frac{1}{n!} \Phi(D)_n(\xi^{\odot n}) = 0$ in the nilpotent $L_\infty[1]$ algebra $A_Z^{0,*}(\Lambda^{\geq 1} \mathcal{N}_{Z|E}[1]) \otimes \mathfrak{m}_A$.

6.4 Deformations induced by the anchor map

In this section we show that deformations induced by the anchor map are unobstructed, to this end we need the following easy criterion for a dg Lie algebra to be homotopy abelian (this was proved in [33], cf. also [2]).

Theorem 6.4.1. *Let $L = (L, d, [\cdot, \cdot])$ be a dg Lie algebra and $(L[1], q_1, q_2, 0, \dots, 0, \dots)$ the corresponding $L_\infty[1]$ algebra: if there is $h \in \text{Hom}^{-1}(L^{\wedge 2}, L)$ such that its image under décalage $\text{déc}(h) =: r \in \text{Hom}^0(L[1]^{\odot 2}, L[1])$ satisfies $[r, q_1] = q_2$ and $[r, q_2] = 0$ in the graded Lie algebra $\overline{\text{CE}}(L[1])$, then L is homotopy abelian.*

Proof. We consider r as a coderivation of $\overline{\mathcal{S}}(L[1])$: since $r: L[1]^{\odot n} \rightarrow L[1]^{\odot n-1}$ it is well defined $e^r: \overline{\mathcal{S}}(L[1]) \rightarrow \overline{\mathcal{S}}(L[1])$ and since r is a degree zero coderivation this is an automorphisms of coalgebras. This also induces an automorphisms of graded Lie algebras $e^{[r, -]}: \overline{\text{CE}}(L[1]) \rightarrow \overline{\text{CE}}(L[1])$, and by hypothesis

$$e^{[r, -]}(q_1) = q_1 + [r, q_1] + \frac{1}{2}[r, [r, q_1]] + \cdots = q_1 + q_2 + \frac{1}{2}[r, q_2] + \cdots = q_1 + q_2.$$

Since $e^{[r, -]}(q_1) = e^r \circ q_1 \circ e^{-r}$ this says that $e^r: (L[1], q_1, 0, \dots, 0, \dots) \rightarrow (L[1], q_1, q_2, 0, \dots, 0, \dots)$ is an isomorphism of $L_\infty[1]$ algebras, and then $(L, d, [\cdot, \cdot])$ is homotopy abelian. \square

Corollary 6.4.2. *Let*

$$L_\bullet : \quad L_0 \rightrightarrows L_1 \rightrightarrows L_2 \rightrightarrows \cdots$$

be a semicosimplicial differential graded Lie algebra. If there is a family of degree minus one $h_n : L_n \wedge L_n \rightarrow L_n$ satisfying the assumptions of the previous theorem and assembling to a morphism of semicosimplicial sets $h_\bullet : L_\bullet \times L_\bullet \rightarrow L_\bullet$, then $\text{Tot}_{TW}(L_\bullet)$ is homotopy abelian.

Proof. The Thom-Whitney totalization identifies naturally with a dg Lie subalgebra of the product $\prod_{n \geq 0} \Omega(\Delta_n; L_n)$, cf. [34, 2] for an explicit description. By scalar extension, every map h_n extends to a bilinear map on $\Omega(\Delta_n; L_n)$ and it is clear that the induced

$$h = \prod_{n \geq 0} h_n : \left(\prod_{n \geq 0} \Omega(\Delta_n; L_n) \right) \wedge \left(\prod_{n \geq 0} \Omega(\Delta_n; L_n) \right) \rightarrow \prod_{n \geq 0} \Omega(\Delta_n; L_n)$$

satisfies the assumptions of Theorem 6.4.1. The fact that the h_n assemble to a semicosimplicial h_\bullet says moreover that the above h restricts to $h : \text{Tot}_{TW}(L_\bullet) \wedge \text{Tot}_{TW}(L_\bullet) \rightarrow \text{Tot}_{TW}(L_\bullet)$ satisfying the assumptions of Theorem 6.4.1. \square

Corollary 6.4.3. *In the hypotheses of Theorem 6.4.1, if $M \subset L$ is dg Lie subalgebra such that $h(M \wedge M) \subset M$, then M and the homotopy fiber of the inclusion $M \rightarrow L$ are homotopy abelian.*

Proof. The first claim is clear, the second is the previous corollary applied to the semicosimplicial dg Lie algebra

$$L \begin{array}{c} \xrightarrow{0} \\ \rightrightarrows \\ \xrightarrow{f} \end{array} M \rightrightarrows 0 \cdots$$

\square

Remark 6.4.4. The importance of homotopy abelian dg Lie algebras in deformation theory lies in the fact that their deformation functors are smooth [76]: this means that if L is homotopy abelian and $B \twoheadrightarrow A$ is an epimorphism in $\mathbf{Art}_{\mathbb{K}}$, then $\text{Def}_L(B) \rightarrow \text{Def}_L(A)$ is surjective. If we have a formal moduli problem $M : \mathbf{Art}_{\mathbb{K}} \rightarrow \mathbf{Set}_*$ governed by L , this tells us that every first order infinitesimal deformation $\alpha \in H^1(L) \cong T^1 M := M(\mathbb{K}[t]/(t^2))$ is tangent to a deformation over the ring $\mathbb{K}[[t]]$ of formal power series. We are going to use a relative version of this latter observation: if $M : \mathbf{Art}_{\mathbb{K}} \rightarrow \mathbf{Set}_*$ is a formal moduli problem governed by a dg Lie algebra L and we are given a morphism of dg Lie algebras $f : L' \rightarrow L$ with L' homotopy abelian, then every first order infinitesimal deformation $\alpha \in H^1(L) \cong T^1 M$ in the image of $H^1(f) : H^1(L') \rightarrow H^1(L)$ is tangent to a deformation over the ring $\mathbb{K}[[t]]$.

Recall from Section 6.1 that a Poisson bivector π on X induces the Koszul bracket $[\cdot, \cdot]_\pi$ on $\Omega_X[1]$.

Lemma 6.4.5. *If (X, π) is a holomorphic Poisson manifold, for all open $U \subset X$ the dg Lie algebra $(\Omega_X[1](U), \partial, [\cdot, \cdot]_\pi)$ satisfies the assumptions of Theorem 6.4.1, with h given by*

$$h : \Omega_X^i(U) \otimes \Omega_X^j(U) \rightarrow \Omega_X^{i+j-2}(U), \quad h(\alpha, \beta) = (-1)^i (\mathbf{i}_\pi(\alpha \wedge \beta) - \mathbf{i}_\pi(\alpha) \wedge \beta - \alpha \wedge \mathbf{i}_\pi(\beta)).$$

Given an open covering \mathcal{U} of X , there is induced a sequence of linear maps on the semicosimplicial dg Lie algebra $\Omega_X^[1](\mathcal{U})_\bullet$ as in the hypotheses of Corollary 6.4.2.*

Proof. Compare with the discussion of Koszul brackets in Section 6.1. We see that r in the claim of Theorem 6.4.1 is given by $r = -\mathcal{K}(\mathbf{i}_\pi)_2$, moreover $q_1 = -\partial$ and $q_2 = \mathcal{K}(\mathbf{l}_\pi)_2$: these satisfy $[-\mathcal{K}(\mathbf{i}_\pi)_2, -\partial] = \mathcal{K}([\mathbf{i}_\pi, \partial])_2 = \mathcal{K}(\mathbf{l}_\pi)_2$ and also $[\mathcal{K}(\mathbf{i}_\pi)_2, \mathcal{K}(\mathbf{l}_\pi)_2] = \mathcal{K}([\mathbf{i}_\pi, \mathbf{l}_\pi])_3 = \mathcal{K}(\mathbf{i}_{[\pi, \pi]})_3 = 0$, since both \mathbf{i}_π and \mathbf{l}_π are differential operators of order ≤ 2 . The last claim is clear. \square

For all $k \geq 0$ we denote by $\Omega_X^{\geq k}$ the part of Ω_X concentrated in degrees $\geq k$, in particular $\Omega_X^{\geq 0} = \Omega_X$: this is a sheaf of differential Gerstenhaber subalgebras for all k . We notice that for $k \neq 1$ the sheaf of dg Lie subalgebras $\Omega_X^{\geq k}[1]$ is stable with respect to the operator h defined in the previous Lemma 6.4.5: this and Theorem 6.4.1 immediately imply the following proposition.

Proposition 6.4.6. *Let (X, π) be a holomorphic Poisson manifold. For every nonnegative integer $k \neq 1$, $(\Omega_X^{\geq k}[1], \partial, [\cdot, \cdot]_\pi)$ is a sheaf of homotopy abelian dg Lie algebras on X , and given an open covering \mathcal{U} of X the Thom-Whitney totalization $\text{Tot}_{TW}(\Omega_X^{\geq k}[1](\mathcal{U})_\bullet)$ is homotopy abelian.*

Now we consider a closed submanifold $Z \subset X$: we denote by $\mathcal{J}_Z \subset \Omega_X$ the sheaf of dg ideals of forms vanishing along Z , that is, the kernel of the restriction $\Omega_X \rightarrow \Omega_Z$, in particular its degree zero part is $\mathcal{J}_Z^0 = \mathcal{I}_Z$; also recall the sheaf \mathcal{L}_Z from Proposition 6.1.9.

Proposition 6.4.7. *Let (X, π) be a holomorphic Poisson manifold and $Z \subset X$ a coisotropic submanifold. The sheaf $\mathcal{J}_Z \subset \Omega_X$ is a sheaf of differential Gerstenhaber subalgebras, moreover, it is closed with respect to the operator h introduced in Lemma 6.4.5, finally, it is sent into \mathcal{L}_Z by the anchor map $\pi^\# : \Omega_X \rightarrow \bigwedge \Theta_X$. Conversely, each one of these conditions is equivalent to Z being coisotropic.*

Proof. We choose local holomorphic coordinates z_1, \dots, z_n such that $Z = \{z_1 = \dots = z_p = 0\}$. As in the proof of Proposition 6.1.9, Z is coisotropic if and only if $\pi_{ij} = -\mathbf{i}_\pi(\partial z_i \wedge \partial z_j) \in \mathcal{I}_Z$ for all $1 \leq i, j \leq p$, since

$$\mathbf{i}_\pi(\partial z_i \wedge \partial z_j) = -h(\partial z_i, \partial z_j) = \pi^\#(\partial z_i)(z_j) = [\partial z_i, z_j]_\pi$$

each one of the conditions in the claim of the proposition implies that Z is coisotropic.

Since $\mathcal{J}_Z \subset \Omega_X$ is the multiplicative ideal generated by $S = \{z_j, \partial z_i\}_{1 \leq i, j \leq p}$, since $\pi^\#$ is a morphism of graded algebras and recalling Remark 6.1.2 this also shows the converse except for h -closeness. The latter is equivalent to $\mathbf{i}_\pi(\alpha \wedge \beta) \in \mathcal{J}_Z$ for all $\alpha, \beta \in \mathcal{J}_Z$ and it is not restrictive to take as α an element of S : if $\alpha = z_j$, $1 \leq j \leq p$, then $\mathbf{i}_\pi(z_j \wedge -) = z_j \wedge \mathbf{i}_\pi(-)$ and since $z_j \wedge - : \Omega_X \rightarrow \mathcal{J}_Z$ we are done, if $\alpha = \partial z_i$, $1 \leq i \leq p$, then \mathcal{J}_Z is $\mathbf{i}_\pi(\partial z_i \wedge -)$ -closed if and only if it is $[\mathbf{i}_\pi, \partial z_i \wedge -]$ -closed, but now formula (6.1.4) shows

$$[\mathbf{i}_\pi, \partial z_i \wedge -] = [\mathbf{i}_\pi, [\partial, z_i \wedge -]] = [\mathbf{l}_\pi, z_i \wedge -] = [z_i, -]_\pi,$$

and we already observed that $[z_i, -]_\pi$ -closeness follows from Remark 6.1.2.

We notice that for a coisotropic submanifold Z the subspace \mathcal{J}_Z is not \mathbf{i}_π -closed in general. □

We denote by $\mathcal{J}_Z^{\geq k}$ the part of \mathcal{J}_Z in degrees $\geq k$ and by $\mathcal{H}_Z^{\geq k}$ the homotopy fiber of the inclusion of sheaves of dg Lie algebras $\mathcal{I}_Z^{\geq k}[1] \hookrightarrow \Omega_X^{\geq k}[1]$: in particular $\mathcal{H}_Z^{\geq 0} = \mathcal{H}_Z$ is the homotopy fiber of $\mathcal{J}_Z[1] \rightarrow \Omega_X[1]$.

Proposition 6.4.8. *Let (X, π) be holomorphic Poisson manifold, $Z \subset X$ a coisotropic submanifold and k a nonnegative integer. If $k \neq 1$, then $\mathcal{J}_Z^{\geq k}[1]$, $\mathcal{H}_Z^{\geq k}$ are sheaves of homotopy abelian dg Lie algebras on X . Moreover, for an open covering \mathcal{U} of X the Thom-Whitney totalizations $\text{Tot}_{TW}(\mathcal{J}_Z^{\geq k}[1](\mathcal{U})_\bullet)$ and $\text{Tot}_{TW}(\mathcal{H}_Z^{\geq k}(\mathcal{U})_\bullet)$ are homotopy abelian.*

Proof. Immediate from Proposition 6.4.7 and Theorem 6.4.1. □

On the other hand, we are specifically interested in the only case not covered by the previous proposition, namely, the $k = 1$ case. In the same assumption of Proposition 6.4.8, and fixed once and for all a covering \mathcal{U} of X by Stein open sets, we have a commutative diagram of dg Lie algebras

$$\begin{array}{ccccc} \mathrm{Tot}_{TW}(\mathcal{H}_Z(\mathcal{U})_\bullet) & \longrightarrow & \mathrm{Tot}_{TW}(\mathcal{J}_Z[1](\mathcal{U})_\bullet) & \longrightarrow & \mathrm{Tot}_{TW}(\Omega_X[1](\mathcal{U})_\bullet) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Tot}_{TW}(\mathcal{H}_Z^{\geq 1}(\mathcal{U})_\bullet) & \longrightarrow & \mathrm{Tot}_{TW}(\mathcal{J}_Z^{\geq 1}[1](\mathcal{U})_\bullet) & \longrightarrow & \mathrm{Tot}_{TW}(\Omega_X^{\geq 1}[1](\mathcal{U})_\bullet) \end{array}$$

where the rows are the associated homotopy fiber sequences, the dg Lie algebras in the top row are homotopy abelian and the vertical arrows are the inclusions.

Lemma 6.4.9. *If the Hodge to de Rham spectral sequence of X degenerate at E_1 , then the dg Lie algebra $\mathrm{Tot}_{TW}(\Omega_X^{\geq 1}[1](\mathcal{U})_\bullet)$ is homotopy abelian. If the Hodge to de Rham spectral sequence of Z degenerate at E_1 , then the dg Lie algebra $\mathrm{Tot}_{TW}(\mathcal{H}_Z^{\geq 1}(\mathcal{U})_\bullet)$ is homotopy abelian.*

Proof. If we show that the hypotheses imply that the vertical arrows in the previous diagram are injective in cohomology, then the thesis follows from Theorem 3.3.5: to this end we may replace $\mathrm{Tot}_{TW}(-)$ with $\mathrm{Tot}(-)$. The first item follows once we recall that the Hodge to de Rham spectral sequence of a smooth complex manifold X may be defined as the spectral sequence associated to the filtration of Čech (double) complexes $F^k = C(\mathcal{U}, \Omega_X^{\geq k})$ (see e.g. [28]), the second item follows from the same reason, once we point out that for every Stein open subset $U \subset X$ and every $k \geq 0$ the complexes $\mathcal{J}_Z^{\geq k}(U)$ and $\Omega_Z^{\geq k}(U)$ are quasi-isomorphic. \square

Next we consider the following commutative diagram of dg Lie algebras, where the vertical arrows are the anchor maps.

$$\begin{array}{ccccc} \mathrm{Tot}_{TW}(\mathcal{H}_Z^{\geq 1}(\mathcal{U})_\bullet) & \longrightarrow & \mathrm{Tot}_{TW}(\mathcal{J}_Z^{\geq 1}[1](\mathcal{U})_\bullet) & \longrightarrow & \mathrm{Tot}_{TW}(\Omega_X^{\geq 1}[1](\mathcal{U})_\bullet) \\ \downarrow \pi^\# & & \downarrow \pi^\# & & \downarrow \pi^\# \\ \mathrm{Tot}_{TW}(\mathcal{K}_Z^{\geq 1}(\mathcal{U})_\bullet) & \longrightarrow & \mathrm{Tot}_{TW}(\mathcal{L}_Z^{\geq 1}[1](\mathcal{U})_\bullet) & \longrightarrow & \mathrm{Tot}_{TW}(\Lambda^{\geq 1}\Theta_X[1](\mathcal{U})_\bullet) \end{array} \quad (6.4.1)$$

In particular there are morphisms of deformation functors induced by the anchor map

$$\pi^\# : \mathrm{Def}_{\mathrm{Tot}_{TW}(\mathcal{J}_Z^{\geq 1}(\mathcal{U})_\bullet)} \rightarrow \mathrm{Hilb}_{Z|X}^{\mathrm{co}}, \quad \pi^\# : \mathrm{Def}_{\mathrm{Tot}_{TW}(\Omega_X^{\geq 1}[1](\mathcal{U})_\bullet)} \rightarrow \mathrm{Def}_{(X,\pi)},$$

which at first order reduce to the anchor map in cohomology

$$\pi^\# : H^1(\mathrm{Tot}_{TW}(\mathcal{J}_Z^{\geq 1}(\mathcal{U})_\bullet)) = \mathbb{H}^1(Z; \Omega_Z^{\geq 1}) \rightarrow T^1 \mathrm{Hilb}_{Z|X}^{\mathrm{co}},$$

$$\pi^\# : H^1(\mathrm{Tot}_{TW}(\Omega_X^{\geq 1}[1](\mathcal{U})_\bullet)) = \mathbb{H}^2(X; \Omega_X^{\geq 1}) \rightarrow T^1 \mathrm{Def}_{(X,\pi)}.$$

Whenever the Hodge to de Rham spectral sequence of Z (resp.: X) degenerates at E_1 we have an isomorphism $\mathbb{H}^1(Z, \Omega_Z^{\geq 1}) \simeq H^0(Z, \Omega_Z^1)$ (resp.: $\mathbb{H}^2(X, \Omega_X^{\geq 1}) \simeq H^0(X, \Omega_X^2) \oplus H^1(X, \Omega_X^1)$).

Theorem 6.4.10. *In the notation above, if the Hodge to de Rham spectral sequence of Z degenerates at E_1 , then for every $\omega \in H^0(Z, \Omega_Z^1)$ the first order embedded coisotropic deformation $\pi^\#(\omega)$ extends to an embedded coisotropic deformation of Z over $\mathrm{Spec} \mathbb{C}[[t]]$.*

Proof. Clear by Remark 6.4.4 and Lemma 6.4.9. \square

Theorem 6.4.11. *In the notation above, if the Hodge to de Rham spectral sequence of X degenerates at E_1 then for every $\omega \in H^0(X, \Omega_X^2) \oplus H^1(X, \Omega_X^1)$ the first order deformation $\pi^\#(\omega)$ extends to a deformation of (X, π) over $\text{Spec } \mathbb{C}[[t]]$.*

Proof. As above. \square

Theorem 6.4.11 has been proved in a different way by Hitchin [47] under the additional assumption that X is compact Kähler. As a further application we can generalize to coisotropic submanifolds part of classical results by McLean and Voisin about deformation of Lagrangian submanifolds [82, 104].

Corollary 6.4.12. *Let Z be a compact coisotropic submanifold of a holomorphic Poisson manifold (X, π) . If the Hodge to de Rham spectral sequence of Z degenerates at E_1 and the anchor map*

$$\pi^\# : H^0(Z, \Omega_Z^1) \rightarrow H^0(Z, \mathcal{N}_{Z|X})$$

is surjective, then every small embedded deformation of Z is coisotropic and the Hilbert functor $\text{Hilb}_{Z|X} = \text{Hilb}_{Z|X}^{\text{co}}$ is unobstructed.

Proof. Since Z is compact, by the argument used in Corollary 6.3.4 it is sufficient to consider infinitesimal deformations. It is now sufficient to apply Theorem 6.4.10. \square

Obviously the above corollary fails without the assumption about the anchor map. For instance, if $Z = p$ is a point, then Z is coisotropic if and only if π vanishes at p ; this shows that in general $\text{Hilb}_{Z|X}^{\text{co}}$ is obstructed and strictly contained in $\text{Hilb}_{Z|X}$. Corollary 6.4.12 holds in particular for Lagrangian submanifolds of a holomorphic symplectic manifold; a different proof of this case, based on Ran-Kawamata's T^1 -lifting theorem, is given in [61].

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