Tesi di Dottorato

Niccolò Castronuovo

Combinatorics of pattern avoiding permutations, Dyck paths and Young tableaux

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COMBINATORIA DI PERMUTAZIONI A MOTIVO ESCLUSO, CAMMINI DI DYCK E TABELLE DI YOUNG

Settore Scientifico Disciplinare MAT/02 ALGEBRA

Dottorando
Dott. Niccolò Castronuovo

Relatore
Prof. Arrigo Bonisoli

Correlatore
Prof. Marilena Barnabei

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Introduction

This thesis concerns the study of some combinatorial aspects of pattern avoiding permutations, Dyck paths and Young tableaux.

The theory of permutation patterns goes back to the works of Knuth [40], who, in the 1970's, introduced the definition of pattern avoidance in connection to a stack sorting problem. The first systematic study of these objects appears in the paper of Simion and Schmidt ([54]).

Nowadays, the theory is very rich and widely expanded, with hundreds of papers appeared in the last decades. The most of these works focus their attention on enumerative problems as how to find the cardinality of the set of permutations of length n that avoid one or more fixed patterns. Others study the distribution of particular statistics over sets of pattern avoiding permutations, imitating the classical works of MacMahon, Foata and Schützenberger on the distribution of these statistics over the whole $\mathcal{S}_n$ (see e.g. [13]). In this sense, typical examples of well studied statistics are the number of fixed points, the number of descents, the number of inversions etc...

Some authors prefer to restrict their attention on particular subsets of the set of permutations, e.g. considering the set of pattern avoiding involutions, fixed points free permutations or centrosymmetric permutations.

In particular, the study of centrosymmetric permutations avoiding a fixed pattern is an active area of research. For example, in [27], Egge studies permutation avoiding one or more pattern of length three and fixed under some symmetries; one of these symmetries is in fact the reverse-complement operation. The same author in [28] uses the Robinson-Schensted map and the Schützenberger involution to obtain some enumerative results about centrosymmetric permutations avoiding a descending pattern of arbitrary length. In [9], the authors study the statistic "number of descents" over every set of centrosymmetric permutations avoiding a pattern of length three.

Other papers discuss applications of permutation patterns to others branches of mathematics (analysis, geometry, probability) or to sciences (information theory, biology...).

We remand to the Kitaev's book [38] for an impressive bibliography about pattern avoidance.

In this thesis, we consider the pattern 321 and, more generally, a descending pattern of the form $k\ k-1 ... 3\ 2\ 1$. Our aim is to study in details some properties of 321-avoiding permutations, involutions and centrosymmetric permutations and to obtain new enumerative results about their cardinalities and about the distribution of some well-known statistics over these sets. We want also to study how these permutations are related to lattice paths, in particular Dyck paths, and to standar Young tableaux.

To this end, in Chapter 1, we introduce some standard tools that are used in the thesis.

The first are lattices and posets. In fact, the study of particular order relations over sets of combinatorial objects can be a powerful tool to understand the structure of such sets. Then we recall the definition of Dyck path or, equivalently, of Dyck word. Dyck paths are a central object in combinatorics and formal language theory. As many other objects, they are enumerated by Catalan numbers. For instance, Chapter 6 of [57] points out the role of Dyck words (or Dyck paths) in enumerative combinatorics. From the point of view of formal languages they are generated by a context free grammar, whose important properties were described in [19].

Despite they are easy to define, the combinatorial structure of Dyck paths is very rich. Hence, often, it is possible to study the structure of a combinatorial set mapping bijectively this set over the set of Dyck paths. In the literature, several bijections between the set $\mathcal{D}_n$ of Dyck paths of semilength n and the subset of $\mathcal{S}_n$ of permutations avoiding a pattern of length 3 appear (see e.g. [20]). Most of them are based on the bijections given in [41], [40] and [42]. In this thesis we consider the bijection between $\mathcal{D}_n$ and the set $\mathcal{S}_n(321)$ of 321-avoiding permutation introduced in [31] and essentially due to Knuth [40].

To define this bijection, it is necessary to introduce another set of combinatorial objects, the set of standard Young tableaux.

The combinatorics of Young tableaux has application in representation theory and in the study of symmetric functions. We recall many classical facts about Young tableaux, like the *jeu de taquin*, the Schützenberger involution and the rectification of a tableau.

In the last part of the first chapter we introduce the main facts about the combinatorics of permutations that we will use in the following. In particular we define the concept of pattern avoidance and the main statistics that are usually considered in this field. We also recall the definition and the main properties of the Robinson-Schensted bijection, the Schensted theorem, the Schützenberger theorems and the Knuth equivalences.

Chapter 2 focuses on the set $\mathcal{S}_n$ of permutations of length n and its subset $\mathcal{S}_n(k\ k-1...3\ 2\ 1)$, namely, the set of permutations that avoid a descending pattern of length k $(k \leq n)$.

We introduce an order relation over this set obtained by considering the *dominance order* $\trianglelefteq$ over $Tab(n)$, the set of standard Young tableaux with n boxes, and the product order $\trianglelefteq \times \trianglelefteq$ over the set $SYP(n)$ of pairs of standard Young tableaux of the same shape. This last relation induces, via the Robinson-Schensted bijection, an order relation $\trianglelefteq$ over

$\mathcal{S}_n$, called again dominance order, and hence, by restriction, over $\mathcal{S}_n(k\ k-1...3\ 2\ 1)$.

The choice of this order $\trianglelefteq$ has two reasons: the first one is the importance of the dominance order over the set of standard Young tableaux in representation theory (see e.g. [14]), the second one is that, as we will see, the posets $(\mathcal{S}_n(k\ k-1...3\ 2\ 1), \trianglelefteq)$ have some interesting properties when $k = 3$ or $k = 4$.

Firstly, we study the order $\trianglelefteq$ over $\mathcal{S}_n(k\ k-1...3\ 2\ 1)$. The structure of the poset $(\mathcal{S}_n(k\ k-1...3\ 2\ 1), \trianglelefteq)$ is quite tangled (for $k \geq 4$, it is neither a lattice, nor a graded poset), however, it has some remarkable properties, for example the fact that $\mathcal{S}_n(k\ k-1...3\ 2\ 1)$ is a principal filter in $(\mathcal{S}_n, \trianglelefteq)$.

Afterwards, we restrict our attention to the case of $k = 3$.

We consider the bijection between $\mathcal{S}_n(321)$ and $\mathcal{D}_n$ described in the first Chapter. This bijection is given by the composition of the Robinson-Schensted map RS and a map ψ between $SYP_2(n)$ and $\mathcal{D}_n$, where $SYP_2(n)$ is the set of pairs of standard Young tableaux of the same shape and with at most two rows. In fact, a deep result due to Schensted [49] relates the shape of the tableaux P, Q with the length of the longest increasing and decreasing subsequence of the corresponding permutation σ. In particular, pairs of tableaux with at most two rows correspond to permutations avoiding the pattern 321, namely, not containing a decreasing subsequence of length greater than or equal to three. We will prove that $\psi \circ RS$ is an order isomorphism between $(\mathcal{S}_n(321), \trianglelefteq)$ and the distributive lattice $\mathcal{D}_n$ (endowed with the following order: a path f is smaller than a path g if and only f lies above g). In particular, $(\mathcal{S}_n(321), \trianglelefteq)$ turns out to be a distributive lattice.

We show that the dominance order over $\mathcal{S}_n(321)$ can be described in terms of Knuth equivalences and use the bijection to find some interesting partitions of the sets $\mathcal{S}_n(321)$ and $\mathcal{D}_n$.

Finally, we define a new bijection between $\mathcal{S}_n(4321)$ and a suitable subset of the set of Motzkin paths of length $2n$. This bijection coincides with the previous one, if restricted to the subset $\mathcal{S}_n(321)$ of $\mathcal{S}_n(4321)$.

Chapter 2 is based on [18].

The Schützenberger involution $P \to P^*$ is a shape-preserving bijection of the set of standard Young tableaux into itself, originally stated by M. P. Schützenberger in [50]. It is based upon the jeu de taquin procedure [53] and turns out to be an involution.

On the other hand, the Robinson-Schensted algorithm establishes a bijective correspondence between permutations and pairs of standard Young tableaux of the same shape. If the pair (P, Q) is the image under this correspondence of the permutation σ, then the pair (P^*, Q^*) is the image of σ^{rc}, where r and c are the usual reverse and complement operations over permutations (see [40]). A permutation σ such that $\sigma = \sigma^{rc}$, or, in other terms, such that its corresponding tableaux are fixed by the map $*$, is said to be *centrosymmetric*.

Hence, as a consequence of the Schensted Theorem, if the tableaux P and Q have at most two rows and are fixed by the map $*$, the corresponding permutation avoids 321 and is centrosymmetric.

These facts naturally lead to the problem of finding an analog of the Schützenberger involution over Dyck paths. In other terms, a map $\Gamma : \mathcal{D}_n \to \mathcal{D}_n$ that makes the following diagram commutative.

$$\begin{tikzcd} \mathcal{S}_n(321) \arrow[r, leftrightarrow, "RS"] \arrow[d, leftrightarrow, "rc"] & Tab_2(n) \arrow[r, leftrightarrow, "\psi"] \arrow[d, leftrightarrow, "(*,*)"] & \mathcal{D}_n \arrow[d, leftrightarrow, dotted, "\Gamma"] \\ \mathcal{S}_n(321) \arrow[r, leftrightarrow, "RS"] & Tab_2(n) \arrow[r, leftrightarrow, "\psi"] & \mathcal{D}_n \end{tikzcd} \tag{1}$$

In Chapter 3 we describe such a map and succeed in giving a characterization of the fixed elements of Γ, namely, those Dyck paths which correspond under the map ψ to 321-avoiding centrosymmetric permutations.

We start with the definition of an involution over Dyck prefixes which allows us to define the involution $\Gamma : \mathcal{D}_n \to \mathcal{D}_n$ described above. Then, we give a characterization of those paths which are fixed under the map Γ.

To prove that the diagram (1) commutes, we carry out a deep study of the properties of Young tableaux in $Tab_2(n)$ that are fixed by the Schützenberger involution.

We use these facts to obtain some enumerative results about centrosymmetric involution avoiding 321, Young tableaux fixed by the Schützenberger involution and paths fixed by Γ.

In the last part of the chapter we exploit the previous results to describe a bijection between the set of 321-avoiding centrosymmetric permutations of even length and the set of 321-avoiding involutions of the same length. This allows us to find a relation between the statistics "number of descents" over these two sets.

Chapter 3 is based on [10].

Transformations that do not modify the length of a word was considered by many authors (see [17], [23], [25], [30], [43], [44], [45]). In the last chapter, Chapter 4, we examine three new ones. The transformations that we examine are permutations on the Dyck words of a given length.

We use here a slightly different notion of Dyck word. In fact the Dyck words of this chapter are the same of the previous chapters with a last letter of type "down" added at the end. This modification is necessary because we want to use the so called Cyclic Lemma. This lemma essentially says that, given a word with $2n + 1$ letters of which n are a and $n + 1$ are b, there is only one conjugate of this word that is a Dyck word. This lemma was obtained in [26] long time before the name Dyck path was used to speak of these combinatorial objects.

Using this fact, we can define the three permutations that we want to study.

The first one is an involution which we denote by α. It was recently introduced in [21] in the context of the sandpile (or chip-firing game) model in order to determine the rank of configurations, as defined by Baker and Norine [5], for the case of the complete graph. The definition of α relies on the Cyclic Lemma.

This first involution is added to a very classical one, that is, the symmetry of paths along a vertical line, denoted β here.

The third one, γ, is the composition of these two.

The first two permutations are involutions and it is not difficult to characterize their fixed elements and to compute their number, while the third one has cycles of different lengths.

The main result of the chapter, Theorem 4.24, shows that γ has cycles of odd lengths. In order to prove this result, we remarked that it was sufficient to show that for each word w there exists an odd integer k such that $\gamma^k(w) = w$, since of course this implies that the length of the cycle of γ containing w is also odd. Then the first ingredient for the proof is to consider a subset of the set of Dyck words which we call *smooth* words, namely, Dyck words which do not have UDU or DUD as factors. We then associate to any Dyck word a smooth word that we call its *skeleton*. The last ingredient is to associate to each Dyck word a sequence of integers which allows to rebuild the word from its skeleton. A property of the action of γ on the skeleton and the sequence allows to end the proof.

The fact that the cycles of γ have odd length, allows us to give some information about the interplay between α and β, and a characterization of the fixed points of γ. Despite we are not able to find the number of fixed points of the map γ, we succeed in finding an upper bound for this number.

Chapter 4 is based on [7].

Chapter 1

Preliminary definitions

1.1 Posets and lattices

The theory of ordered structures rose in the work of Dedekind and became an independent theory after the publication of the classcial *Lattice Theory* of Birkhoff [11]. Nowadays, this theory is widely expanded. In this section, we recall only the basic definitions of poset and lattice and we study the principal properties of these algebraic objects, following [11] and [22].

1.1.1 Posets

A *partially ordered set* or *poset* is a set P endowed with an order relation $\mathcal{R}$, that is a relation with the following properties:

a) **Reflexive**

$$a\mathcal{R}a \quad \forall a \in P,$$

b) **Antisymmetric**

$$a\mathcal{R}b \text{ and } b\mathcal{R}a \Rightarrow a = b \quad \forall a, b \in P,$$

and

c) **Transitive**

$$a\mathcal{R}b \text{ and } b\mathcal{R}c \Rightarrow a\mathcal{R}c \quad \forall a, b, c \in P.$$

An *order morphism* between two posets $(P, \leq_p)$ and $(Q, \leq_q)$, is a function $f : P \to Q$ such that

$$x \leq_p y \quad \Rightarrow \quad f(x) \leq_q f(y) \quad \forall x, y \in P.$$

A morphism f between two posets is said to be an *order isomorphism* if it is invertible and the inverse is itself an order morphism. Two posets P and Q are said to be *isomorphic* if there exist an order isomorphism $f : P \to Q$.

An *order anti-morphism* between two posets $(P, \leq_p)$ and $(Q, \leq_q)$, is a function $f : P \to Q$ such that

$$x \leq_p y \quad \Rightarrow \quad f(x) \geq_q f(y) \quad \forall x, y \in P.$$

An *order anti-isomorphism* is an invertible order anti-morphism whose inverse is itself an anti-morphism.

Two elements x and y of P are said to be *confrontable* if $x \leq y$ or $y \leq x$ and *unconfrontable* otherwise. A *linearly ordered set* or *chain* is a poset P in which every two elements are confrontable.

If Q is a subset of the poset $(P, \leq_P)$, the restriction of the order $\leq_P$ to Q is said to be the *induced order* of P over Q; a subset Q of P endowed with the induced order of P is said to be a *suborder* of P.

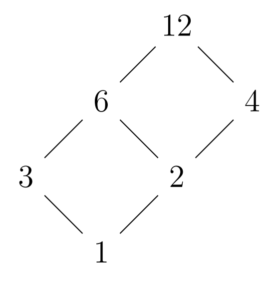
A subset S of a poset P is said to be an *antichain* if the induced order of P over S is the empty relation or, in other terms, if every two elements of S are unconfrontable.

Given two elements x and y of P, the *interval* $[x, y]$ is the set $\{z \in P$ such that $x \leq z \leq y\}$. The element y *covers* the element x, in symbols $y \succ x$ or $x \prec y$, if $[x, y] = \{x, y\}$. Note that, if the poset P is finite, the covering realtion characterizes completely the order relation.

The most natural way to represent the poset $(P, \leq)$ is its *Hasse diagram*; this is the oriented graph of the covering relation of P. The Hasse diagram of P is usually drawed so that, if $x \leq y$, x lies below y in the picture.

Example 1.1. It is possible to endow the set $\mathbb{Z}^+$ of positive integers with the order relation given by the divisibility. The relation $|$ defined by $x|y$ if and only if x divides y is in fact an order relation over this set.

The interval $[1, 12]$ of $(\mathbb{Z}, |)$ has the following Hasse diagram



There exist standar contructions that permit to obtain new posets from one or more given posets. As an example, given two posets $(P, \leq_p)$ e $(Q, \leq_q)$ it is possible to define an order relation $\leq$ over the cartesian product $P \times Q$ setting $(a, b) \leq (c, d)$ if and only if $a \leq_p c$ and $b \leq_q d$. This order over $P \times Q$ is said to be the *product order* of $\leq_P$ and $\leq_Q$.

If, instead, we consider only one poset $(P, \leq)$, it is possible to define another order relation $\leq^*$, called the dual of $\leq$, over the same set P. The dual relation is defined as follows:

$$x \leq^* y \Leftrightarrow y \leq x.$$

It is trivial to verify that $\leq^*$ is in fact an order relation.

The duality is a fundamental tool in order theory. In fact the following proposition holds.

Theorem 1.2 (Duality principle)**.** *Let* ***Prop*** *be a true proposition in the theory of posets. Then the dual proposition* ***Prop*****, obtained from* ***Prop*** *replacing each $\leq$ with a $\geq$ and viceversa, is yet a true proposition.*
The proof of ***Prop**** *is obtained by the proof of* ***Prop*** *replacing each $\leq$ with a $\geq$ and viceversa.*

In a poset $(P, \leq)$ an element x is said to be *minimal* if there exist no $y \in P$ such that $y \leq x$. An element x is said to be the *minimum* if it is the only minimal element or, equivalently, if $x \leq y$ for all $y \in P$. The maximals and the maximum are defined similarly. The maximum and the minimum of a posets, if they exist, are unique.

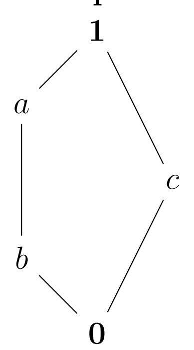
Given two elements a and b in P, a *chain* between a and b is a linearly ordered suborder of P with a as a minimum and b as a maximum. A *maximal* or *saturated* chain is a chain that is not properly contained in other chains.

Now we want to define the rank, a fundamental concept in the study of the structure of posets. Consider a poset $(P, \leq)$ is said to be *ranked* if it satisfies the following properties:

- For all $x, y \in P$, all the chains between x and y are finite (*finite chains* condition)
- For all $x, y \in P$, all the saturated chains between x and y have the same cardinality (*Jordan-Dedekind* condition)
- P has a minimum $\mathbf{0}$

If P is a ranked poset, then the *rank* of the element $x \in P$ is the cardinality of a saturated chain (hence of all the saturated chains) between $\mathbf{0}$ and x minus 1.

Example 1.3. Consider the poset over the set $\{\mathbf{0}, \mathbf{1}, a, b, c\}$ whose Hasse diagram is



Here the Jordan-Dedekind condition is not verified, hence the poset is not ranked.

Example 1.4. Let $\mathcal{B}(U)$ be the Boole's algebra over the finite set U. This poset is ranked and the rank of an element $X \subseteq U$ is $\rho(X) = |X|$.

Also the poset $(\mathbb{Z}, |)$ is ranked and $\rho(p_1^{h_1} p_2^{h_2} ... p_n^{h_n}) = h_1 + h_2 + ... + h_n$.

Let $(P, \leq)$ be a poset. A subset I of P is said to be an *order ideal* if $\forall x \in I$ and $\forall y \in P$,

$$y \leq x \Rightarrow y \in I.$$

If S is a subset of P, the *ideal generated by* S is

$$I(S) := \{x \in P; x \leq s \text{ for some } s \in S\}.$$

If S consists of only on element, $I(S)$ is said to be a *principal ideal*.

Dually, it is possible to define a *filter* $F \subseteq P$ when $\forall x \in F$ and $\forall y \in P$,

$$x \leq y \Rightarrow y \in F.$$

The definitions of *filter generated by a set* S and *principal filter* are obtained similarly.

1.1.2 Lattices

Let $(P, \leq)$ be a poset and let x and y be two elements of P. Consider the set of upper bounds of x and y,

$$\{z \in P | z \geq x \text{ and } z \geq y\}.$$

The minimum of this set, if it exists, is called *supremum* or *join* of x and y and is indicated $x \vee y$. Dually, the maximum of the lower bounds of x and y, if it exists, is called *infimum* or *meet* of x and y, in symbols $x \wedge y$.

Similarly, given a subset S of P, the minimum of the upper bounds of the elements of S, if it exists, is called *supremum* or *join* of S. The definition of *infimum* or *meet* of S is obtained dually.

Now, let $(P, \leq)$ a poset such that, for each pair of elements x and y of P, there exist both $x \vee y$ and $x \wedge y$. Then $(P, \leq)$ is called a *lattice*.

Observe that in a lattice the join and the meet of two elements satisfy the following properties,

- $x \vee x = x$ and $x \wedge x = x$ (Idempotences)
- $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$ (Commutatives)
- $x \vee (y \vee z) = (x \vee y) \vee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ (Associatives)

- $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$ (Absorptions)

In particular, thanks to the associtivity, it is possible to write the join and the meet of a finite number of elements $x_1, x_2, ..., x_n$ of a lattice L as

$$x_1 \vee x_2 \vee ... \vee x_n$$

and

$$x_1 \wedge x_2 \wedge ... \wedge x_n,$$

respectively. As a consequence, the meet and the join of a finite subset Q of a lattice L always exist and are given by the join and the meet of all the elements of Q, respectively.

It is easy to verify that the previous laws characterize the lattice structure. In other terms, if a set L is endowed with two operations $\vee$ and $\wedge$ that satisfy the previous properties, then L is a lattice where

$$x \leq y \Leftrightarrow x \vee y = y$$

(or, equivalently, $x \leq y \Leftrightarrow x \wedge y = x$). In this case and the join and the meet of two elements $x, y \in L$ are exactly $x \vee y$ and $x \wedge y$.

Let $(L, \vee, \wedge)$ and $(L', \vee', \wedge')$ two lattices. A *lattice morphism* is a function $f : L \to L'$ such that

$$f(x \vee y) = f(x) \vee' f(y) \text{ and } f(x \wedge y) = f(x) \wedge' f(y) \quad \forall x, y \in L.$$

A lattice morphism is also an order morphism but, in general, the converse is not true.

A *lattice isomorphism* is a bijective lattice morphism between L and L'. Note that the inverse of a lattice isomorphism is yet a morphism between L' and L. Two lattices L and L' such that a lattice isomorphism between them exists are said to be *isomorphic*.

A *sublattice* of a lattice $(L, \vee, \wedge)$ is a subset S of L closed under the operations $\vee$ and $\wedge$, in symbols,

$$\forall x, y \in S, \qquad x \vee y \in S \text{ and } x \wedge y \in S.$$

In general, the operations $\wedge$ and $\vee$ does not satisfy the distributive property of one with respect to the other. However it is not difficult to find lattices in which these properties are verified. In the following chapters we will describe some of such lattices.

A lattice $(L, \vee, \wedge)$ such that the distributives properties holds, in symbols

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \forall x, y, z \in L$$

and

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \forall x, y, z \in L,$$

is called *distributive*.

Note that the two distributive properties are not independent. In fact it easy to verify that they are completly equivalent,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \Leftrightarrow x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

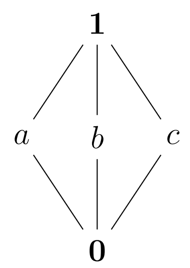
Another interesting type of lattices, are the modular ones. A lattice $(L, \vee, \wedge)$ is said to be *modular* if it satisfies the following *modular law*

$$z \leq x \;\Rightarrow\; x \vee (y \wedge z) = (x \vee y) \wedge z \quad \forall x, y, z \in L.$$

It is immediately seen that if a lattice is distributive then it is modular. The converse is not true as showed by the following example.

Example 1.5. The following Hasse diagram, called M_3, is modular but not distributive in fact

$$a \wedge (b \vee c) = a \neq \mathbf{0} = (a \wedge b) \vee (a \wedge c).$$

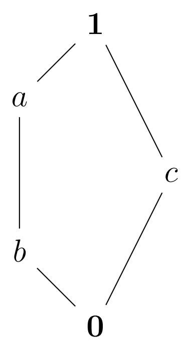


Example 1.6. The following Hasse diagram, called N_5, is not modular nor distributive in fact

$$b \leq a \text{ but } a \vee (c \wedge b) = a \neq b = (a \vee c) \wedge b$$

and

$$a \wedge (b \vee c) = a \neq b = (a \wedge b) \vee (a \wedge c).$$



The lattices of the previous two examples represent in fact the minimal obstructions to modularity and distributivity. This fact is explained in the following theorem.

Theorem 1.7. *A lattice $(L, \wedge, \vee)$ is modular if and only if it does not contains sublattices isomorphic to N_5.*
A lattice $(L, \wedge, \vee)$ is distributive if and only if it does not contains sublattices isomorphic to N_5 and to M_3.

There are many others interesting classes of lattices, as an example the geometric lattices or the semimodular lattices. We do not report their definition here because we do not need them in the following but we refer the interested reader to [2] or [22].

1.2 Dyck paths

Lattice paths are deeply studied in combinatorics. In this section, we introduce the definition of Dyck path or, equivalently, of Dyck word. These paths are very useful tools. In fact, despite they are easy to define, their combinatorial structure is rich. As we will see, fixed a length $2n$ (the length of a Dyck path is always even), the cardinality of the set of Dyck paths of length $2n$ is the n-th Catalan number. This sequence of numbers counts many different sets of combinatorial objects (see [57] and [55]). As a consequence, it is possible to find bijections between the set of Dyck paths of length $2n$ and many of these sets. These bijections allow to study these combinatorial object using the properties of the Dyck paths. Moreover, it is possible to endow the set of Dyck path of length n with an interesting order structure. This order structure will be defined below and will be deeply used in Chapter 2.

1.2.1 Dyck paths

A *Dyck prefix* of length t is a lattice path consisting of t steps $U = (1, 1)$ (*up steps*) and $D = (1, -1)$ (*down steps*), starting at $(0, 0)$ and never going below the x-axis. A vertex of a prefix whose coordinates are (h, k) $(h, k \geq 0)$ is said to be at *position* h and at *height* k.

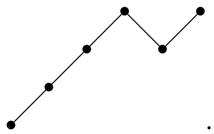
Given a prefix p, a down step whose ending vertex is at height 0 is called *a return* of p. A pair of consecutive steps of the form UD is called a *peak*, which we identify with the vertex of the path in the middle of such a pair. Similarly, a *valley* is a pair of consecutive steps DU and a valley is identified with the vertex in the middle. In this way, the *position* of a peak (valley, respectively) is the abscissa of the corresponding vertex.

The set of Dyck prefixes of length t will be denoted $\mathcal{P}_t$.

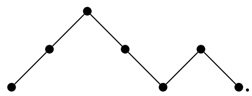
A *Dyck path* of semilength n is a Dyck prefix of length $2n$ whose ending vertex is $(0, 2n)$. A Dyck path d in which the only return is the last (down) step is called *irreducible*. Let $\mathcal{D}_n$ be the set of Dyck path of semilength n.

We recall that each Dyck prefix can be seen as a word in the letters U and D such that, in each position, the number of D's does not exceed the number of U's.

Example 1.8. A Dyck prefix:



A Dyck path:



Note that this path is not irreducible, in fact there is a return at position 4. The path has two peaks, the first at position 2 and the second at position 5, and a valley at position 4.

1.2.2 Cardinalities

The problem of finding the cardinality of the set of Dyck prefixes with a up steps and b down steps ($a \geq b$) is equivalent to the so called "ballot problem": in an election with two candidates, A and B, A receives a votes and B receives b votes. What is the probability that A will be (weakly) ahead of B during the ballot? The answer to this problem is the content of the Bertrand's ballot theorem (see e.g. [3] or [32]):

Theorem 1.9. *The number of Dyck prefixes with a up steps and b down steps ($a \geq b$) is equal to*

$$N_{a,b} := \binom{a+b}{a} - \binom{a+b}{a+1} = \frac{a+1-b}{a+1}\binom{a+b}{a}.$$

In particular the solution to the ballot problem is

$$\frac{N_{a,b}}{\binom{a+b}{a}} = \frac{a+1-b}{a+1}.$$

The numbers $N_{a,b}$ are called the $\{a,b\}$-*ballot numbers.*

The previous result can be employed to find the number of Dyck prefixes of length t and whose ending vertex is at height k. In fact this number is equal to

$$N_{\frac{t+k}{2},\frac{t-k}{2}} = \binom{t}{\frac{t+k}{2}} - \binom{t}{\frac{t+k}{2}+1} = \frac{k+1}{\frac{t+k}{2}+1}\binom{t}{\frac{t+k}{2}} \tag{1.1}$$

because $a = \frac{t+k}{2}$ and $b = \frac{t-k}{2}$.

Hence the total number of Dyck prefixes of length t is

$$|\mathcal{P}_t| = \sum_{k \equiv t \, mod 2, \; 0 \leq k \leq t} \binom{t}{\frac{t+k}{2}} - \binom{t}{\frac{t+k}{2}+1}$$

but this telescoping series reduce to

$$\binom{t}{\frac{t}{2}}$$

if t is even and to

$$\binom{t}{\frac{t+1}{2}}$$

if t is odd. So we obtain the well-known cardinality of $\mathcal{P}_t$:

$$|\mathcal{P}_t| = \binom{t}{\lfloor \frac{t}{2} \rfloor}.$$

As a particular case of formula (1.1) with $k = 0$ and $t = 2n$, the number of Dyck paths of semilength n is given by:

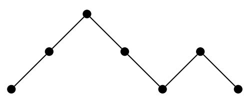
$$|\mathcal{D}_n| = \frac{1}{n+1}\binom{2n}{n}.$$

This last number is called the n-th Catalan number and is denoted by C_n. The importance of the sequence of numbers $\{C_n\}_{n\geq 0}$ in combinatorics is surprisingly deep (see e.g. [57] and [55]).

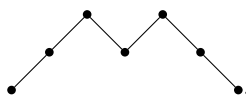
1.2.3 Order over Dyck paths

We endow the set $\mathcal{D}_n$ with the following order: $f \leq g$ if and only if the path f lies above the path g. It is a well-known fact that $(\mathcal{D}_n, \leq)$ is actually a distributive lattice (see e.g. [33]). In the lattice $\mathcal{D}_n$ the covering relation can be expressed in terms of valleys and peaks. In fact, it follows immediately from the definition of the order relation that a path f covers a path g if and only if f is obtained from g by replacing a valley by a peak.

Example 1.10.



is covered by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1.3 Standard Young tableaux

In this section, we recall the fundamental definitions and the basic facts about partitions of an integer, skew tableaux and standar Young tableaux. Afterwards, we will examine some well-known operations over the sets of skew tableaux and standard Young tableaux, the "jeu de taquin", the rectification and the Schützenberger involution, and we will define an order structure over the set of standard Young tableaux, the dominance order. Finally, we wil introduce a bijection between the set of standar Young tableaux with n boxes and with at most two rows and the set of Dyck prefixes of length n. For a detailed introduction to the combinatorics of Young tableaux and their applications in representation theory see e.g. [34].

1.3.1 Partitions

A *partition* of the non-negative integer n is a non-increasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \lambda_3, ...)$ such that $\sum_i \lambda_i = n$, in symbols $\lambda \vdash n$. The infinite tail of zeros is usually suppressed, so we write $\lambda = (\lambda_1, \lambda_2, \lambda_3, ..., \lambda_k)$, where λ_k is the least non-zero integer of λ. If $n = 0$, the only partition is the empty one.

We denote by $\mathfrak{P}_n$ the set of all partitions of n and by $\mathfrak{P}$ their union:

$$\mathfrak{P} = \cup_i \mathfrak{P}_i.$$

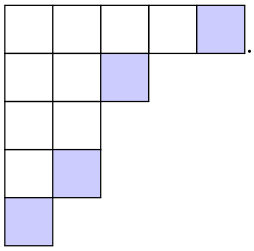
The study of the partition of n is a central theme in combinatorics and in number theory (see e.g. [4]). In particular, the combinatorial properties of the sets $\mathfrak{P}$ and $\mathfrak{P}_n$ are widely studied (see e.g. [2], [56] and [57]). Here we recall only the basic facts that we will need in the following.

A partition of n can be identified with a *Ferrers diagram* with n boxes, namely, a left-justified array of n empty boxes such that each row contains at most as many boxes as the preceding one. The Ferrers diagram corresponding to the partition $(\lambda_1, ..., \lambda_k)$ has k rows of length $\lambda_1, \lambda_2, ..., \lambda_k$, respectively.

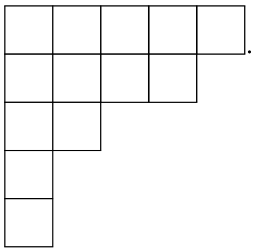
A box of a Ferrers diagram is said to be in *position* (i, j) if it is the box in the i-th row and in the j-th column. A box with no boxes to the right nor below is called *corner box*. The *diagonal* of a Ferrers diagram is the set of the boxes in position (i, i). The *conjugate* of a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, ..., \lambda_k) \vdash n$ is the partition $\lambda^T \vdash n$ whose Ferrers diagram is the symmetric of the Ferrers diagram of λ with respect to the diagonal. In other words

$$\lambda^T = (\underbrace{k, ..., k}_{\lambda_k}, \underbrace{k-1, ..., k-1}_{\lambda_{k-1}-\lambda_k}, ..., \underbrace{1, ..., 1}_{\lambda_1-\lambda_2}).$$

Example 1.11. The partition $\lambda = (5, 3, 2, 2, 1)$ of the integer 13 correspons to following the Ferrers diagram. The shaded boxes are the corner boxes.

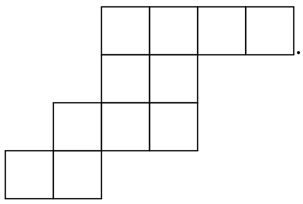


The Ferrers diagram of the partition λ^T is

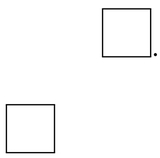


A *skew diagram* is the diagram obtained removing a smaller Ferrers diagram from a larger one that contains it. If the two diagrams correspond to the partitions λ and μ, the resulting skew diagram is denoted by μ/λ. If $\lambda \vdash n$ and $\mu \vdash m$, μ/λ is a skew diagram with $m-n$ boxes.

Example 1.12. The skew diagram $(2,2,1)/(6,4,4,2)$ is



Note that a skew diagram could arise as μ/λ for more than one choice of μ and λ. In this case there can be corner boxes for λ that are also corner box for μ. As an example both $\mu/\lambda = (3,1,1)/(2,1)$ and $\mu/\lambda = (3,2,1)/(2,1)$ represent the skew diagram



In the second case, the box in position $(2,2)$ is a corner box for both μ and λ.

1.3.2 Orders over partitions

Here we recall the definitions of two partial order relations, the first over $\mathfrak{P}$ and the second over $\mathfrak{P}_n$, that will be useful in the following.

- $(\mathfrak{P}, \subseteq)$
  Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, ...)$ and $\tau = (\tau_1, \tau_2, \tau_3, ...)$ be in $\mathfrak{P}$. The *inclusion order* $\subseteq$ over $\mathfrak{P}$ is given by
  $$\lambda \subseteq \tau$$
  if and only if, for all i,
  $$\lambda_i \leq \tau_i.$$
  Equivalently, $\lambda \subseteq \tau$ if and only if the Ferrers diagram of λ is included in the Ferrers diagram of τ. It is a well-known fact (see [57]) that $\mathfrak{P}$ endowed with the partial order relation $\subseteq$ is a distributive lattice called *Young's lattice.*

  In this lattice, the supremum and the infimum of two partitions λ and τ are, respectively, $\lambda \cup \tau$ and $\lambda \cap \tau$ (the intersection and the union of partitions are well defined if we consider the correspondence with Ferrers diagrams). A partition λ covers a partition τ if and only if the Ferrers diagram of λ is obtained by the Ferrers diagram of τ adding a box.

- $(\mathfrak{P}_n, \trianglelefteq)$
  Let λ and τ be two partitions of the integer n. The *dominance order* over $\mathfrak{P}_n$ is given by
  $$\lambda \trianglelefteq \tau$$
  if and only if, for all j,
  $$\lambda_1 + ... + \lambda_j \leq \tau_1 + ... + \tau_j$$
  (see [34]). $\mathfrak{P}_n$ endowed with the order $\trianglelefteq$ results to be a modular (not distributive) lattice.

  In particular, the covering relation is the following: a partition λ covers a partition τ if and only if the Ferrers diagram of τ is obtained by the Ferrers diagram of σ by moving one corner-box in position (i, j) to the first row below the i-th whose length is smaller than j. For a detailed study of this lattice, see [16].

1.3.3 Tableaux

Let A be a subset of $\mathbb{N}$. A *skew tableau* over A is a filling of the boxes of a skew diagram μ/λ with elements of A. The partitions λ and μ will be called the *inner* and the *outer shape* of the skew tableau, respectively. A *tableau* is a skew tableau whose inner shape is empty. Given a tableau P, we denote by *shP* the (outer) shape of P. A skew tableau is said to be *standard* if its elements are different and strictly increasing along rows and columns.

A *standard Young tableau* is a standard tableau P over the set $A = \{1, 2, ..., n\}$, where $shP \vdash n$. The *conjugate tableau* of a standard Young tableau P is the standard Young tableau P^T symmetric of P with respect to the diagonal.

Example 1.13. Three skew tableau. The second is a standard skew tableau and the third is a standard Young tableau.

$$\begin{array}{ccccc} & & 10 & 2 & 5 \\ & 4 & 5 & & \\ 1 & & & & \end{array}, \quad \begin{array}{ccccc} & & 3 & 6 & 7 \\ & 4 & 5 & & \\ 2 & & & & \end{array}, \quad \begin{array}{cccc} 1 & 2 & 3 & 7 \\ 4 & 5 & & \\ 6 & & & \end{array}$$

Observe that a standard Young tableau P with n boxes can be identified with a saturated chain of length $n+1$ of the Young's lattice starting with the empty partition. In fact, we can identify the tableau P with the sequence of Ferrers diagram $(\emptyset, shP_1, shP_2, ..., shP_n)$ where, for every $1 \leq j \leq n$, P_j denotes the tableau induced by the elements $\{1, 2, ..., j\}$ of P. For instance, if

$$P = \begin{array}{cccc} 1 & 2 & 3 & 7 \\ 4 & 5 & & \\ 6 & & & \end{array},$$

then

$$P_4 = \begin{array}{ccc} 1 & 2 & 3 \\ 4 & & \end{array}.$$

Conversly, consider a sequence of partitions $(\emptyset, sh_1, sh_2, ..., sh_n)$ with $sh_i \in \mathfrak{P}_i$ and where $sh_i \subseteq sh_{i+1}$ for all i. Every such sequence gives rise to a standard Young tableau P with $shP_i = sh_i$, since we have an order for the insertion of elements.

Example 1.14. The standard Young tableau

$$\begin{array}{cccc} 1 & 2 & 3 & 7 \\ 4 & 5 & & \\ 6 & & & \end{array}$$

can be identified with the sequence

$$\left(\emptyset, \square, \square\square, \square\square\square, \begin{array}{l}\square\square\square\\ \square\end{array}, \begin{array}{l}\square\square\square\\ \square\square\end{array}, \begin{array}{l}\square\square\square\\ \square\square\\ \square\end{array}, \begin{array}{l}\square\square\square\square\\ \square\square\\ \square\end{array} \right)$$

We denote by $Tab(n)$ the set of standard Young tableaux with n boxes and by $SYP(n)$ the subset of $Tab(n) \times Tab(n)$ given by all pairs of tableaux of the same shape. If $1 \leq k \leq n$, $Tab_k(n)$ denotes the subset of $Tab(n)$ of tableaux with at most k rows. The set $SYP_k(n)$ is defined analogously.

Let T be a standard Young tableau with n boxes. An integer i, $1 \leq i \leq n-1$, is said to be a *descent* of the tableau T if in T the box filled with $i+1$ is in a row strictly below the row of the box filled with i.

Example 1.15. Consider the tableau

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 7 \\ \hline 4 & 5 & 6 & 8 \\ \hline 9 \\ \cline{1-1} \end{array}.$$

The descent of T are $3, 7$ and 8.

1.3.4 Jeu de taquin, rectification and deflation procedure

A fundamental tool in the combinatorics of tableaux is the procedure introduced by M. P. Schützenberger in [50] and called *jeu de taquin* due to its similarity to the famous Loyd's 15-puzzle. The rectification algorithm and the deflation algorithm are based on this procedure.

Given a standard skew tableau T with skew shape μ/λ where $\lambda \vdash n$, $\mu \vdash m$, choose a corner box of λ. Then the *sliding procedure* consists in the following algorithm.

1. Fill the chosen box of λ with the symbol X and call it *the empty box*.

2. Consider the box of μ filled with the smallest integer between the box to the right of X and the box below X (if there is no box to the right or below choose the other one). Swap this integer and the symbol X.

3. Repeat 2. until the empty box becomes a corner box of μ and delete it from μ, hence getting a skew tableau of shape μ'/λ', where $\mu' \vdash m-1$ and $\lambda' \vdash n-1$.

The termination of the algorithm is guaranteed from the fact that the empty square moves to the right or below. Moreover, since at each step we choose the smallest element, the resulting skew tableau is indeed standard.

Example 1.16. Consider the following standard skew tableau with skew shape μ/λ and consider the corner box of λ denoted by the symbol X.

$$\begin{array}{cccc} & & 2 & 6 \\ & X & 4 & 10 \\ & 5 & 8 & \\ 7 & 9 & & \end{array}$$

Then the application of the sliding procedure consists of the following steps:

$$\begin{array}{cccc} & & 2 & 6 \\ & X & 4 & 10 \\ & 5 & 8 & \\ 7 & 9 & & \end{array} \mapsto \begin{array}{cccc} & & 2 & 6 \\ & 4 & X & 10 \\ & 5 & 8 & \\ 7 & 9 & & \end{array} \mapsto \begin{array}{cccc} & & 2 & 6 \\ & 4 & 8 & 10 \\ & 5 & X & \\ 7 & 9 & & \end{array} \mapsto \begin{array}{cccc} & & 2 & 6 \\ & 4 & 8 & 10 \\ & 5 & & \\ 7 & 9 & & \end{array}.$$

Note that the sliding procedure is easily invertible if we know the resulting skew tableau and the corner boxes of the outer shape deleted in the last step of the algorithm.

In fact, if we fix a skew tableau of shape μ'/λ', $\mu' \vdash m-1$ and $\lambda' \vdash n-1$, and a corner box of the outer shape μ', the inverse of the sliding procedure described above is the following algorithm.

1. Fill the chosen box of μ' with the symbol X and call it *the empty box*.

2. Consider the box of μ' filled with the smallest integer between the box to the left of X and the box above X (if there is no box to the left or above choose the other one). Swap this integer and the symbol X.

3. Repeat 2. until the empty box becomes a corner box of λ' and delete it from μ', hence getting a skew tableau of shape μ/λ, where $\mu \vdash m$ and $\lambda \vdash n$.

Now, the the *rectification* of T is the standard Young tableau $Rect(T)$ obtained from T by applying the sliding procedure repeatedly:

1. Choose a corner box of λ.

2. Apply the sliding procedure to T using the box of λ chosen in 1..

3. Return to 1. until the inner shape is empty.

4. Normalize the entries of the tableau from 1 to $m-n$.

Also in this case the termination of the algoritm is trivially verified because the number of boxes of λ is finite. The resulting tableau $Rect(T)$ is a standard Young tableau since, at each step, the skew tableau produced by the step 2. are standard. It is a well-known fact (see e.g. [34]) that the standard Young tableau obtained in this way does not depend on the choice made in step 1.

The map that associates T with its rectification $Rect(T)$ is called *jeu de taquin*.

Example 1.17. Consider the following skew tableau of the previous example. Then the rectification consists of the following steps. At each step, the box filled with X represent the chosen box of the inner shape.

$$\begin{ytableau} \none & \none & 2 & 6 \\ \none & X & 4 & 10 \\ \none & 5 & 8 \\ 7 & 9 \end{ytableau} \mapsto \begin{ytableau} \none & \none & 2 & 6 \\ \none & 4 & 8 & 10 \\ X & 5 \\ 7 & 9 \end{ytableau} \mapsto \begin{ytableau} \none & X & 2 & 6 \\ \none & 4 & 8 & 10 \\ 5 & 9 \\ 7 \end{ytableau} \mapsto \begin{ytableau} \none & 2 & 6 & 10 \\ X & 4 & 8 \\ 5 & 9 \\ 7 \end{ytableau} \mapsto$$

$$\begin{ytableau} X & 2 & 6 & 10 \\ 4 & 8 \\ 5 & 9 \\ 7 \end{ytableau} \mapsto \begin{ytableau} 2 & 6 & 10 \\ 4 & 8 \\ 5 & 9 \\ 7 \end{ytableau} \mapsto \begin{ytableau} 1 & 4 & 8 \\ 2 & 6 \\ 3 & 7 \\ 5 \end{ytableau}$$

Finally, we recall the definition of the *deflation procedure* over a standard Young tableau. If P is a standar Young tableau, the deflation of P is, by definition (see [58]), the tableau $P^{\downarrow} := Rect(P/1)$ where $P/1$ is the standard skew tableau obtained from P by deleting the the top-left box.

Example 1.18. Consider the tableau

$$P = \begin{ytableau} 1 & 2 & 4 & 8 \\ 3 & 5 & 7 & 9 \\ 6 & 10 \end{ytableau}.$$

Then $P^{\downarrow}$ is obtained as follows

$$\begin{ytableau} 1 & 2 & 4 & 8 \\ 3 & 5 & 7 \\ 6 & 8 \end{ytableau} \mapsto \begin{ytableau} X & 2 & 4 & 8 \\ 3 & 5 & 7 & 9 \\ 6 & 10 \end{ytableau} \mapsto \begin{ytableau} 2 & X & 4 & 8 \\ 3 & 5 & 7 & 9 \\ 6 & 10 \end{ytableau} \mapsto \begin{ytableau} 2 & 4 & X & 8 \\ 3 & 5 & 7 & 9 \\ 6 & 10 \end{ytableau}$$

$$\mapsto \begin{ytableau} 2 & 4 & 7 & 8 \\ 3 & 5 & X & 9 \\ 6 & 10 \end{ytableau} \mapsto \begin{ytableau} 2 & 4 & 7 & 8 \\ 3 & 5 & 9 & X \\ 6 & 10 \end{ytableau} \mapsto \begin{ytableau} 2 & 4 & 7 & 8 \\ 3 & 5 & 9 \\ 6 & 10 \end{ytableau} \mapsto \begin{ytableau} 1 & 3 & 6 & 7 \\ 2 & 4 & 8 \\ 5 & 9 \end{ytableau} = P^{\downarrow}.$$

1.3.5 Schützenberger involution

Now we recall the definition of the *Schützenberger involution* over standard Young tableaux (see [50]). It is based on the repeated application to P of the deflation procedure defined above. The image P^* of a tableau $P \in Tab(n)$ under the Schützenberger involution is the standar Young tableau corresponding to the following sequence of shapes

$$(shP^{\overbrace{\downarrow\downarrow\dots\downarrow}^{n}}, shP^{\overbrace{\downarrow\downarrow\dots\downarrow}^{n-1}}, \dots, shP^{\downarrow}, shP).$$

Example 1.19. Consider the tableau

$$P = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}.$$

Then

$$P^{\downarrow} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}, \quad P^{\downarrow\downarrow} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad P^{\downarrow\downarrow\downarrow} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, \quad P^{\downarrow\downarrow\downarrow\downarrow} = \begin{array}{|c|}\hline 1 \\ \hline\end{array}, \quad P^{\downarrow\downarrow\downarrow\downarrow\downarrow} = \emptyset$$

and P^* is

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}.$$

This map is in fact an involution, although this is not obvious. This fact was first proved by Schützenberger [50] but the proof is indirect and based on the Robinson-Schensted map (see Subsection 1.4.3). The result is also contained in [52] and a proof dealing with the more general setting of the sets of order ideals in finite posets is given in [51]. A different direct approach is that of van Leeuwen [58].

Theorem 1.20 (Schützenberger)**.** *The map $S : Tab(n) \longrightarrow Tab(n)$ such that $S(P) := P^*$ is an involution,*

$$S^{-1}(P) = S(P).$$

Equivalently, for all $P \in Tab(n)$,

$$P^{**} = P.$$

A tableau P is said to be *self-dual* if it is fixed by the Schützenberger involution.

1.3.6 Cardinalities

The problem of finding a closed formula for the cardinality $p(n)$ of the set $\mathfrak{P}_n$ of the partitions of the integer n is not trivial. Hardy and Ramanujan found an asymptotic result which show that

$$p(n) \sim \frac{1}{4n\sqrt{3}} exp\left(\pi\sqrt{\frac{2n}{3}}\right) \qquad \text{as } n \to \infty.$$

Rademacher improved the result of Hardy and Ramanujan finding an exact expression for $p(n)$ as a convergent series (see [4]).

The cardinality of the set $Tab(n)$ is

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}(2k-1)!!.$$

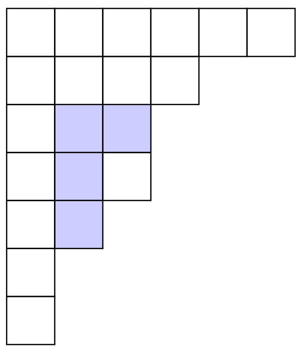
This fact is a simple consequence of the Robinson-Schensted bijection that we will describe in Subsection 1.4.3.

If we fix a partition $\lambda \in \mathfrak{P}_n$ and a box u in position (i, j) in the Ferrers diagram of λ, the *hook length* $h(u)$ of the box u is defined to be the sum of the number of boxes to the right of u plus the number of boxes below u plus one, in symbols,

$$h(u) := \lambda_i - i + \lambda_j^T - j + 1.$$

The following example clarifies the previous definition.

Example 1.21. Consider the following Ferrers diagram and its box u in position $(3, 2)$. Then $h(u) = 4$.



The number of standard Young tableaux P with n boxes and with $shP = \lambda$ is given by the well-known *hook length formula* (see [57]):

Theorem 1.22 (Frame, Robinson, Thrall). *The number f_λ of standar Young tableaux with n boxes and shape λ is*

$$f_\lambda = \frac{n!}{\prod h(u)}$$

where the product in the denominator is taken over all the boxes u of the Ferrers diagram of λ.

Example 1.23. As an example of application of the previous formula we find the number of standar Young tableaux with two rows of length a and b, $(a \geq b)$. In this case, the hook length of a box in position $(2, j)$ is $b - j + 1$, that of a box in position $(1, j)$ with $j \leq b$ is $a - j + 2$ and that of a box in position $(1, j)$ with $j > b$ is $a - j + 1$ hence we have:

$$\frac{(a+b)!}{\prod h(u)} = \frac{(a+b)!}{b!(a-b+2)(a-b+3)...(a+1)(1)(2)...(a-b)} = \frac{a+1-b}{a+1}\binom{a+b}{a}.$$

Note that this number is equal to the number of Dyck prefixes with a up steps and b down steps as seen in Subsection 1.2.2. In fact, in the following Subsection, we will find a bijection between these two sets.

1.3.7 The bijection ψ

We now describe a bijection ψ_n between the set $SYP_2(n)$ and the set $\mathcal{D}_n$ of Dyck paths of semilength n, originally stated in [31], essentially due to Knuth [40] and widely used (see e.g. [1] and [9]).

First of all, we recall the well-known bijection ϕ_n between $Tab_2(n)$ and the set $\mathcal{P}_n$ of Dyck prefixes of length n originally stated in [40]. Given $P \in Tab_2(n)$, $\phi_n(P)$ is the Dyck prefix whose i-th step is an up step if the integer i appears in the first row of P, a down step otherwise.

Note that the restriction of the map ϕ_n furnishes a bijection between the set of Dyck prefixes with a up steps and b down steps $(a + b = n)$ and the set of standard Young tableaux with length of the firt row equal to a and length of the second row equal to b.

The map ϕ_n allows us to define a bijection ψ_n between $SYP_2(n)$ and the set $\mathcal{D}_n$ of Dyck paths of semilength n. Given a pair $(P, Q) \in SYP_2(n)$, consider $p := \phi_n(P)$ and $q := \phi_n(Q)$. The ending vertices of these two prefixes have the same height since P and Q have the same shape.

Define the *inverse* or *symmetric* of a word $p = (p_1, p_2, ..., p_n)$ in the letters U and D as $\overline{p} = (\overline{p}_n, ..., \overline{p}_1)$ where $\overline{D} := U$ and $\overline{U} := D$. Then the Dyck path $\psi_n(P, Q)$ is the juxtaposition of the words q and $\overline{p}$, in symbols,

$$\psi_n(P, Q) = \phi_n(Q)\overline{\phi_n(P)}.$$

In the following we will denote ϕ and ψ the maps defined above, omitting to write the subscript n when it is clear from the context.

The following theorem expresses the main property of the map ϕ and ψ (see [1], [31]).

Theorem 1.24. *Let T be a tableau in $Tab_2(n)$. Then an integer $1 \leq i \leq n-1$ is a descent of T if and only if the Dyck prefix $\phi(T)$ has a peak in position i.*

As a consequence, if $(P, Q) \in SYT_2(n)$, then the Dyck path $\psi(P, Q)$

- *has a peak in position i, $1 \leq i \leq n-1$ if and only if the tableau Q has i as a descent,*
- *has a peak in position i, $n+1 \leq i \leq 2n-1$ if and only if the tableau P has $2n-i$ as a descent,*
- *has a peak in position n if and only if P and Q have the box filled with n in the first row.*

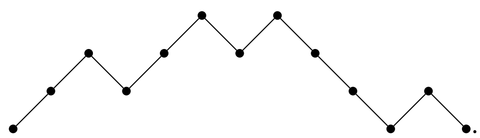
Example 1.25. Consider

$$P = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

and

$$Q = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

so $p = \phi_6(P) = UDUUUD$, $q = \phi_6(Q) = UUDUUD$, $\overline{p} = UDDDUD$ and $\psi_6(P, Q)$ is the path whose corresponding word is $q\overline{p} = UUDUUDUDDDUD$:



In this case the descents of Q and the positions of the peaks in the first half of $q\overline{p}$ are 2 and 5. The descents of P are 1 and 5 and the positions of the peaks in the second half of $q\overline{p}$ are $2n - 5 = 12 - 5 = 7$ and $2n - 1 = 12 - 1 = 11$.

1.4 Permutations

Permutations, i.e. bijective maps over a set, are central objects in mathematics. They have a double interpretation. The first one is the algebraic one. In this interpretation, permutations are thought as elements of a group under the operation of composition. The second one is the combinatorial interpretation. In this case, permutations are thought as words over a finite alphabet in which every letter appears exactly one time. Here we are interested in this second aspect. In this section, we recall some fundamental facts about the combinatorics of permutations. In particular, we introduce the definition of pattern avoiding permutation, the main object of this thesis, and the Robinson-Schensted algorithm that furnishes a link between permutations and Young tableaux. This algorithm will be used extensively in the following chapters.

Combinatorics of permutations is an extensive area of research hence here it is not possible to introduce each branch of this field; we remand the interested reader to [13] for further details.

1.4.1 Permutations

We denote by $\mathcal{S}_n$ the set of permutations of length n, therefore

$$|\mathcal{S}_n| = n!\,,$$

and by $(\mathcal{S}_n, \circ)$ the group of permutations of length n under the operation of composition.

If not otherwise specified, we use the one-line notation for permutations: the permutation $\pi \in \mathcal{S}_n$ that associates the number i with π_i is written $\pi = \pi_1\pi_2...\pi_n$. Alternativley, it is well-known that every permutation in $\mathcal{S}_n$ can be written as a product of cycles in an unique way if we disregard the order of the cycles, e.g. 364152 is equal to the product (26)(134).. When we represent a permutation as a sequence of its cycle, we use the standar convention of writing each cycle starting from its minimum element and ordering the cycles in decreasing order of their first element. As an example, the permutation whose representation in one-line notation is 43617528 has standar cycle representation (8)(23657)(14).

The *cycle type* of a permutation is the sequence of its cycle lengths, written in weakly decreasing order. In particular, the cycle type of a permutation $\pi \in \mathcal{S}_n$ is a partition of n. For example, the permutation whose standar cycle representation is (856)(74)(32)(1) has cycle type $(3, 2, 2, 1) \in \mathfrak{P}_8$. It is well-known that two permutations have the same cycle type if and only if they are conjugate as elements of the group $(\mathcal{S}_n, \circ)$.

A *fixed point* of a permutation $\pi \in \mathcal{S}_n$ is an integer i such that $\pi_i = i$, i.e. a cycle of π with only one element.

An *involution* is a permutation π such that $\pi = \pi^{-1}$. We denote by $\mathcal{I}_n$ the set of involution of length n.

An involution has only cyles of length one or two. Summing over the number of cycles of length two, it is easy to see that

$$|\mathcal{I}_n| = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (2k-1)!!\,.$$

The *reverse* of a permutation $\pi = \pi_1\, \pi_2 \ldots \pi_n \in \mathcal{S}_n$, is the permutation

$$\pi^r = \pi_n\, \pi_{n-1} \ldots \pi_1.$$

Equivalently, $\pi^r = \pi \circ \zeta$ where ζ is the permutation $n\ n-1\ n-2\ \ldots\ 1$. The *complement* of a permutation $\pi = \pi_1\, \pi_2 \ldots \pi_n \in \mathcal{S}_n$, is the permutation

$$\pi^c = n+1-\pi_1\ n+1-\pi_2\ \ldots\ n+1-\pi_n.$$

Equivalently, $\pi^c = \zeta \circ \pi$. Note that $\pi^{rr} = \pi$, and $\pi^{cc} = \pi$, i.e. the reverse and the complement are two involutory maps from $\mathcal{S}_n$ into itself. Moreover we have

$$\pi^{rc} = \pi^{cr}.$$

Since ζ is an involution, $\pi^{rc} = \zeta \circ \pi \circ \zeta = \zeta \circ \pi \circ \zeta^{-1}$, for every permutation π. In particular π and π^{rc} are conjugate and hence have the same cycle type.

A permutation $\pi \in \mathcal{S}_n$ is said to be *centrosymmetric* if it is fixed by the reverse-complement operation, in symbols,

$$\pi^{rc} = \pi,$$

or equivalently

$$\pi = \zeta \circ \pi \circ \zeta.$$

In other terms, π is centrosymmetric if and only if

$$\pi_i = \pi_{n+1-i}$$

for all $i = 1, 2, \ldots, n$.

Note that, if n is odd, $n = 2k+1$, the central element of a centrosymmetric permutation must be fixed:

$$\pi_{k+1} = k+1.$$

We denote by $\mathcal{S}_n^c$ the set of centrosymmetric permutations of length n. Note that $(\mathcal{S}_n^c, \circ)$ is a subgroup of $(\mathcal{S}_n, \circ)$. This subgroup is isomorphic to the Coxeter group of type B (see [12]).

It is easy to find the cardinality of $\mathcal{S}_n^c$, for every n : in each of the first $\left\lfloor \frac{n}{2} \right\rfloor$ positions we have to put an element $i \in \left\{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$ or its complement $n+1-i$. Hence

$$|\mathcal{S}_n^c| = 2^{\left\lfloor \frac{n}{2} \right\rfloor} \left\lfloor \frac{n}{2} \right\rfloor! \,.$$

The set of centrosymmetric involutions of length n is denoted by $\mathcal{I}_n^c$. The number of centrosymmetric involutions of length n is given by the following formula (see [9]):

$$|\mathcal{I}_n^c| = \sum_{k=0}^{\left\lfloor \frac{n}{4} \right\rfloor} \frac{(2\left\lfloor \frac{n}{2} \right\rfloor)!!}{(\left\lfloor \frac{n}{2} \right\rfloor - 2k)!k!2^{2k}} \,.$$

One of the most studied topics in combinatorics of permutations are particular statistics over $\mathcal{S}_n$. A *statistic* over a set A is simply a function F from A to $\mathbb{N}$. As an example, consider the set $A = \mathcal{D}_n$ of Dyck paths of semilength n. A statistic over A is the function $peak : A \to \mathbb{N}$ that associates each path with its number of peak.

Two statistics F and G over A are said to be *equidistributed* if

$$|\{x \in A \text{ such that } F(x) = k\}| = |\{x \in B \text{ such that } F(x) = k\}|,$$

for all $k \in \mathbb{N}$.

In the following we list the definitions of some well-known statistics over $\mathcal{S}_n$ and the most important facts about their equidistribution. The list is not comprehensive, we remand the reader to [13] for further results and references.

- **Number of descents** A *descent* of a permutation π is an index i such that $\pi_i < \pi_{i+1}$. We denote by $Des(\pi)$ the set of descents of π and by $des(\pi)$ its cardinality. The *ascents* are defined analogously.

- **Number of excedences** An *excedence* of a permutation π is an index i such that $\pi_i > i$. The *deficiencies* are defined analogously.

- **Number of left-to-right minimum** A *left-to-right-minimum* of a permutation π is a value π_i such that for all $j < i$, $\pi_j > \pi_i$. *left-to-right maxima*, *right-to-left minima* and *right-to-left maxima* are defined analogously.

- **Major index** The major index of a permutation π is the sum of the descents of π :
$$\text{major index of } \pi := \sum_{i \in Des(\pi)} i.$$

- **Number of inversions** An *inversion* of a permutation π is a pair of values (π_i, π_j) such that $i < j$ and $\pi_i > \pi_j$.

- **Number of fixed points**

- **Number of cycles**

The following are the main equidistribution results about the previous statistics, see e.g. [13].

Theorem 1.26 (Foata)**.** *The statistic "number of descents" and "number of excedences" are equidistributed over $\mathcal{S}_n$.*

The number of permutation of length n and with k descents (equivalently, with k excedences) is called $(n, k) -$ Eulerian number.

Theorem 1.27 (Foata)**.** *The statistics "number of left-to-right minima" and "number of cycles" are equidistibuted over* $\mathcal{S}_n$.

The number of permutation of length n and with k left-to-right minima (equivalently, with k cycles) is called (n, k) – signless Stirling number of the first kind.

Theorem 1.28 (Foata, Schützenberger)**.** *The statistics "number of inversions" and "major index" are equidistibuted over* $\mathcal{S}_n$.

The number of permutation of length n and with k inversions (equivalently, with major index equal to k) is called (n, k) – Mahonian number.

1.4.2 Pattern avoidance

Let τ be a sequence of h integers. The *normalization* $|\tau|$ of τ is the permutation of $\mathcal{S}_h$ whose elements have the same relative order as the elements of τ. For instance, $|5683| = 2341 \in \mathcal{S}_4$. As usual, we say that the permutation $\pi \in \mathcal{S}_n$ *avoids the pattern* $\tau \in \mathcal{S}_h$ if there are no subsequences of π whose normalization is τ. Note that if $h > n$ the previous condition is trivially verified. We denote by $\mathcal{S}_n(\tau)$ the set of permutations of length n that avoid the pattern τ. The sets of involutions, centrosymmetric permutations and centrosymmetric involutions of length n that avoid the pattern τ are denoted by $\mathcal{I}_n(\tau)$, $\mathcal{S}^c_n(\tau)$ and $\mathcal{I}^c_n(\tau)$, respectively.

Since, in the 1970's, Knuth [41] introduced the definition of pattern avoidance in connection to a stack sorting problem, an impressive quantity of works have been done on the subject. The most of these works are concerned with problems of enumeration (exact or asymptotic) of particular classes of pattern avoiding permutations. As examples of these type of results we recall two main thorems about pattern avoidance. The first states the equinumerosity of the permutations of length n avoiding any pattern of length three [54].

Theorem 1.29 (Simion-Schmidt)**.** *For every pattern* $\tau \in \mathcal{S}_3$,

$$|\mathcal{S}_n(\tau)| = C_n, \text{ the } n\text{-th Catalan number.}$$

In general, it is very difficult to find exact formulae for the cardinality of the sets of permutations that avoid one or more patterns. As a matter of fact, a formula for $|\mathcal{S}_n(1324)|$, is not known.

However, in 2004 Marcus and Tardos (see e.g. [13]) proved a conjecture formulated by Stanley and Wilf (independently) in the 1980's about the growth rate of $|\mathcal{S}_n(\tau)|$, $\tau \in \mathcal{S}_h$.

Theorem 1.30 (Marcus-Tardos)**.** *For every pattern* $\tau \in \mathcal{S}_h$, *there exist a constant* C *such that*

$$|\mathcal{S}_n(\tau)| \leq C^n.$$

There exist many other articles in the subject. Some deal with the study of particular statistics, like those listed in the previous subsection, over sets of patterns avoiding permutations (involutions, centrosymmetric permutations, respectively). Others concern applications of permutation patterns to algorithms and complexity theory and connections with other branches of mathematics (order theory, analysis etc). It is not possible to give a comprehensive list of all such works, thus we remand the interested reader to the very complete Kitaev's book [38].

1.4.3 The Robinson-Schensted map

Now we describe the well-known Robinson-Schensted bijection RS.

The bijection RS associates with every permutation $\pi \in \mathcal{S}_n$ a pair of standard Young tableaux $(P, Q) \in SYP(n)$. P is called the *insertion tableau* of π and Q the *recording tableau* of π (see [34]).

This map is based on the *insertion procedure*.

Given a standard tableau P and a number m not occurring as entry of P, the insertion of m in P is the standard tableau obtained with the following algorithm:

1. Consider the first row of P.

2. Look for the smallest entry graeter than m of the considered row. If such an entry exist, replace m with this entry and go to the next step. If such an entry does not exist, put m at the end of the considered row and terminate the procedure.

3. Consider the next row of P and label the entry deleted from P in the last step with the letter m. Insert this number in this row by means of step 2.

It is obvious that the algorithm must terminate, since, at each step, we are moving a row down. Moreover, it is easy to prove that, at each step, the new tableau created is standard (see e.g. [34]). Hence, at the end of the procedure, we obtain a standar tableau with a box more than P. This new box must be a corner box. The number of rows of P is increased by one if and only if, at each application of step 2., the smallest entry greater than the number we are inserting exists.

Example 1.31. Consider the following standard tableau

1	4	6
5	7	
8		

and number 3.

The insertion of 3 in the tableau is given by the following steps:

$$\begin{ytableau} 1 & 4 & 6 \\ 5 & 7 \\ 8 \end{ytableau} \longleftarrow 3 \quad \text{(in the first row } min\{i, i > 3\} = 4\text{)}$$

$$\begin{ytableau} 1 & 3 & 6 \\ 5 & 7 \\ 8 \end{ytableau} \quad \begin{array}{l} \text{(in the second row } min\{i, i > 4\} = 5\text{)} \\ \longleftarrow 4 \end{array}$$

$$\begin{ytableau} 1 & 3 & 6 \\ 4 & 7 \\ 8 \end{ytableau} \quad \begin{array}{l} \text{(in the third row } min\{i, i > 5\} = 8\text{)} \\ \longleftarrow 5 \end{array}$$

$$\begin{ytableau} 1 & 3 & 6 \\ 4 & 7 \\ 5 \end{ytableau}$$

$$\longleftarrow 8$$

$$\begin{ytableau} 1 & 3 & 6 \\ 4 & 7 \\ 5 \\ 8 \end{ytableau}.$$

Observe that the insertion procedure is easily invertible, if we know which is the last created box of the the resulting tableau.

In fact, if we consider a standard tableau T and the entry of a corner box of T, say k, the following procedure, called *extraction procedure*, furnishes a tableau P and a number m and is the inverse of the insertion procedure of m in P.

1. Remove the box filled with k from P.

2. Consider the row of P preceding the one which contained k. Replace k with the greatest entry smaller than k of the considered row of P.

3. Consider the previous row of P and label the entry deleted from P in the last step with the letter k. Apply step 2. to this row and to this element. When the first row of P is reached, the algorithm terminates.

Now, we are in position to describe the *Robinson-Schensted map RS*.

Consider a permutation $\pi = \pi_1\pi_2...\pi_n \in \mathcal{S}_n$. Consider two empty tableaux $P(0) = \emptyset$ and $Q(0) = \emptyset$. Call $P(1)$ the standard tableau resulting from the insertion of π_1 in $P(0)$ and put $Q(1) = \boxed{1}$. For $i = 2, 3, ..., n$, call $P(i)$ the standard tableau resulting from the insertion of π_i in $P(i-1)$. If the new box created in $P(i)$ during this last insertion procedure is in position (j,k), call $Q(i)$ the tableau obtained by $Q(i-1)$ adding a box filled with i in position (j,k).

At the end of the procedure put $P = P(n)$ and $Q = Q(n)$.

Note that, for all $i = 1, 2, ..., n-1$, the $P(i)$ are standard tableaux. $P = P(n)$ is a standar Young tableau because it is standard and it is filled with the integer from 1 to n. The $Q(i)$, $i = 1, 2, ..., n$ are standard Young tableaux. The tableau $Q(i)$ is equal to tableau Q_i, i.e. the tableau induced by the elements $\{1, 2, ..., i\}$ of Q. Moreover the tableaux P and Q correspond to the sequences of shapes

$$P = (\emptyset,\, shP(1),\, shP(2) \dots shP(n))\,,$$

$$Q = (\emptyset,\, shQ(1),\, shQ(2) \dots shQ(n))\,.$$

By construction, $shP(i) = shQ(i)$ for all i. In particular,

$$shP = shQ.$$

Example 1.32. Consider the permutation $\pi = 425361 \in \mathcal{S}_6$. Then $RS(\pi)$ is obtained as follow

$$\mathbf{4}25361 \quad P(1) = \begin{ytableau} 4 \end{ytableau}, \quad Q(1) = \begin{ytableau} 1 \end{ytableau}$$

$$4\mathbf{2}5361 \quad P(2) = \begin{ytableau} 2 \\ 4 \end{ytableau}, \quad Q(2) = \begin{ytableau} 1 \\ 2 \end{ytableau}$$

$$42\mathbf{5}361 \quad P(3) = \begin{ytableau} 2 & 5 \\ 4 \end{ytableau}, \quad Q(3) = \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}$$

$$425\mathbf{3}61 \quad P(4) = \begin{ytableau} 2 & 3 \\ 4 & 5 \end{ytableau}, \quad Q(4) = \begin{ytableau} 1 & 3 \\ 2 & 4 \end{ytableau}$$

$$4253\mathbf{6}1 \quad P(5) = \begin{ytableau} 2 & 3 & 6 \\ 4 & 5 \end{ytableau}, \quad Q(5) = \begin{ytableau} 1 & 3 & 5 \\ 2 & 4 \end{ytableau}$$

$$42536\mathbf{1} \quad P = P(6) = \begin{ytableau} 1 & 3 & 6 \\ 2 & 5 \\ 4 \end{ytableau}, \quad Q = Q(6) = \begin{ytableau} 1 & 3 & 5 \\ 2 & 4 \\ 6 \end{ytableau}.$$

The Robinson-Schensted correspondence is easily invertible. In fact, given two standard Young tableau P and Q put $P(n) = P$ and $Q(n) = Q$. Then, for all $i = n-1, ..., 1$, let (j, k) be the position of the corner box of Q_{i+1} filled with the integer $i+1$. Extract the corner box of $P(i+1)$ in position (j, k) using the extraction procedure, hence obtaining a standard tableau T and an integer k. Put $P(i) := T$ and $\pi_{i+1} = k$.

At the end of the procedure, we obtain a permutation $\pi = \pi_1\pi_2...\pi_n \in \mathcal{S}_n$. This procedure is clearly the inverse of the map RS hence we have proved that the map RS is a bijection.

Theorem 1.33. *The map RS is a bijection between the set $\mathcal{S}_n$ of permutations of length n and the set $SYP(n)$ of pairs of standard Young tableaux with n boxes of the same shape.*

In particular the cardinalities of these two sets are equal,

$$\sum_{\lambda \vdash n} f_\lambda^2 = n!.$$

Knuth [39] widely generalized the bijection RS to biwords and matrices. For this reason, the bijection RS is usually known as the *Robinson-Schensted-Knuth bijection.*

A simlple consequence of the definition of the Robinson-Schensted bijection is the following well-known proposition (see [34]), which describe the link between descents in tableaux and descents in permutations.

Proposition 1.34. *Let π be a permutation and let $RS(\pi) = (P, Q)$. Then the set of the descents of π is equal to the set of the descents of the tableau Q.*

1.4.4 Longest increasing and decreasing subsequences

Given a permutation $\pi = \pi_1\pi_2...\pi_n \in \mathcal{S}_n$, an *increasing subsequence* of π is a subsequence $\pi_{i_1}\pi_{i_2}...\pi_{i_k}$ of π such that

$$\pi_{i_1} < \pi_{i_2} < ... < \pi_{i_k}.$$

Equivalently, an increasing subsequence of length k of π is an occurrence of the pattern $1\ 2\ ...k$ in π.

Similarly it is possible to define a *decreasing subsequence* of π as a subsequence $\pi_{i_1}\pi_{i_2}...\pi_{i_k}$ of π such that

$$\pi_{i_1} > \pi_{i_2} > ... > \pi_{i_k}$$

or, equivalently, as an occurrence of the pattern $k\ k-1\ ...1$ in π.

One of the fundamental properties of the Robinson-Schensted bijection is expressed by the following classical theorem (see [49]).

Theorem 1.35 (Schensted). *Let d and i be the length of the longest decreasing subsequence and of the longest increasing subsequence of a permutation π. Then $RS(\pi)$ is a pair of tableaux with d rows and i columns.*

The Schensted's theorem can be reformulated as follows.

Theorem 1.36. *If a permutation π avoid the pattern $1\ 2\ ...k$ then the tableaux $RS(\pi)$ have at most $k-1$ columns. Similarly, if a permutation π avoid the pattern $k\ k-1\ ...1$ then the tableaux $RS(\pi)$ have at most $k-1$ rows.*

A simple consequence of the Schensted's Theorem and of the definition of the map ψ give in Subsection 2.3.2, is the following corollary.

Corollary 1.37. *The map $\psi \circ RS$ is a bijection between $\mathcal{S}_n(321)$ and $\mathcal{D}_n$.*

In particular the two sets have the same cardinality,

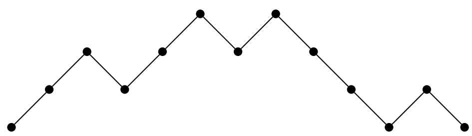
$$|\mathcal{S}_n(321)| = |\mathcal{D}_n| = C_n, \text{ the n-th Catalan number.}$$

This fact is a particular case of the more general result 1.29.

Example 1.38. Consider the permutation $\pi = 231465 \in \mathcal{S}_6(321)$. Then $RS(\pi) = (P, Q)$ with

$$P = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2} \end{array} \quad \text{and} \quad Q = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 \\ \cline{1-2} \end{array}$$

so $\psi \circ RS(\pi)$ is the path



The map $\psi \circ RS$ has been used is several works. As an example, in [1] the authors use this map to prove the equidistribution of some statistics related to the last descent and to the position of the integer n over $\mathcal{S}_n(321)$ and similar statistics over $\mathcal{D}_n$. In [31], instead, it is used to obtain the equidistribution of the statistics "number of fixed points" and "number of excedences" over $\mathcal{S}_n(321)$ and $\mathcal{S}_n(132)$.

The Schensted's theorem has been widely generalized by Green (see [36]) who proved that all the lengths of the rows and of the columns of the tableaux $RS(\pi)$ can be deduced by the increasing and decreasing subsequences in π.

For a permutation $\pi = \pi_1\pi_2...\pi_n \in \mathcal{S}_n$ let $I_j(\pi)$ denote the maximal number of elements of π contained in the union of at most j increasing subsequences. Similarly, let $D_j(\pi)$ be the maximal numer of elements in the union of at most j decreasing subsequences.

Theorem 1.39 (Green). *Let $\lambda = (\lambda_1, \lambda_2, ...\lambda_k)$ be the shape of the tableaux $RS(\pi)$ and let $\lambda^T = (\lambda_1^T, \lambda_2^T, ..., \lambda_{\lambda_1}^T)$ be the conjugate partition. Then, for all $1 \leq j \leq k$,*

$$I_j(\pi) = \lambda_1 + \lambda_2 + ... + \lambda_j$$

and, for all $1 \leq i \leq \lambda_1$,

$$D_i(\pi) = \lambda_1^T + \lambda_2^T + ... + \lambda_i^T.$$

1.4.5 Knuth equivalences

Given $\sigma, \tau \in \mathcal{S}_n$, with $RS(\tau) = (P_\tau, Q_\tau)$ and $RS(\sigma) = (P_\sigma, Q_\sigma)$, the permutations τ and σ are said to be *Knuth equivalent* ($\tau \sim \sigma$) if $P_\sigma = P_\tau$, and *dual-Knuth equivalent* ($\tau \sim_d \sigma$) if $Q_\sigma = Q_\tau$.

In particular, since $RS(\pi) = (P, Q) \Leftrightarrow RS(\pi^{-1}) = (Q, P)$,

$$\sigma \sim \tau \Leftrightarrow \sigma^{-1} \sim_d \tau^{-1}.$$

It is well-known that the relations $\sim$ and $\sim_d$ can be characterized as follows (see e.g. [34]):

- $\tau \sim \sigma$ if and only if τ is obtained from σ by a finite sequence of the following transformations:

$$\begin{cases} yzx \mapsto yxz & \text{where } yzx \text{ are three consecutive letters, } x < y < z \\ yxz \mapsto yzx & \text{where } yxz \text{ are three consecutive letters, } x < y < z \\ xzy \mapsto zxy & \text{where } xzy \text{ are three consecutive letters, } x < y < z \\ zxy \mapsto xzy & \text{where } zxy \text{ are three consecutive letters, } x < y < z \end{cases}$$

- $\tau \sim_d \sigma$ if and only if τ is obtained from σ by a finite sequence of the following transformations:

$$\begin{cases} cab \mapsto bac & \text{where } cab \text{ are three letters, } b = a+1, c = b+1 \\ bac \mapsto cab & \text{where } bac \text{ are three letters, } b = a+1, c = b+1 \\ acb \mapsto bca & \text{where } acb \text{ are three letters, } b = a+1, c = b+1 \\ bca \mapsto acb & \text{where } bca \text{ are three letters, } b = a+1, c = b+1 \end{cases}$$

  Note that in the case of $\sim_d$ the three letters are not necessarily consecutive.

1.4.6 Schützenberger Theorem

The Robinson-Schensted map, when restricted to the subsets $\mathcal{I}_n$ and $\mathcal{S}_n^c$, has interesting properties originally stated in [53, p. 94], [50, §4] and summarized in the following theorem.

Theorem 1.40 (Schützenberger). *If $\pi \in \mathcal{S}_n$ and $RS(\pi) = (P, Q)$, then $RS(\pi^{-1}) = (Q, P)$. In particular π is an involution if and only if $P = Q$.*

If $\pi \in \mathcal{I}_n$ and $RS(\pi) = (P, P)$, then the number of fixed points of π equals the number of columns of odd length of P.

If $\pi \in \mathcal{S}_n$ and $RS(\pi) = (P, Q)$, then $RS(\pi^r) = (P^T, (Q^*)^T)$ and $RS(\pi^c) = ((P^*)^T, Q^T)$. As a consequence, $RS(\pi^{rc}) = (P^*, Q^*)$. In particular, π is centrosymmetric if and only if P and Q are self-dual tableaux.

In particular, the previous results allows us to give explicit formulae for the cardinalities of the sets of standard Young tableaux and of self-dual standard Young tableaux. In fact, recalling the formulae for the number of involutions and centrosymmetric involutions of length n, we have

$$|Tab(n)| = |\mathcal{I}_n| = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (2k-1)!!$$

and

$$|Tab^*(n)| = |\mathcal{I}_n^c| = \sum_{k=0}^{\lfloor \frac{n}{4} \rfloor} \frac{(2\lfloor \frac{n}{2} \rfloor)!!}{(\lfloor \frac{n}{2} \rfloor - 2k)!k!2^{2k}},$$

where $Tab^*(n)$ denotes the set of self-dual standard Young tableaux with n boxes.

Similarly, the following formula holds,

$$\sum_{\lambda \vdash n} (f_\lambda^*)^2 = |\mathcal{S}_n^c| = 2^{\lfloor \frac{n}{2} \rfloor} \left\lfloor \frac{n}{2} \right\rfloor !,$$

where f_λ^* denotes the number of self-dual standard Young tableaux with shape equal to λ.

Moreover, in [27], the author, using a bijection between the set of self-dual standar Young tableaux and the set of *domino tableaux*, found formulae for $\sum (f_\lambda^*)^2$ $(\sum f_\lambda^*)$, where the sum is taken over all the shapes with n boxes and with at most $k-1$ rows, or equivalently (by the Schensted's theorem) for the cardinality of $\mathcal{S}_n^c(k\, k-1 \ldots 1)$ ($\mathcal{I}_n^c(k\, k-1 \ldots 1)$, respectively). Note that these formulae are recursive in k.

To the best of our knowledge, given a shape λ, a general formula for the numbers f_λ^* is not known. In Chapter 3, we will find formulae for f_λ^* where λ is any shape with at most two rows.

1.4.7 Connected permutations

We recall (see [8]) that, given a permutation $\tau \in \mathcal{S}_n$, it is possible to partition the set $\{1, 2, ..., n\}$ into intervals $I_1, I_2, ..., I_t$, with $I_j = \{k_j, k_j + 1, ..., k_j + h_j\}$, $h_j \geq 0$ and $1 = k_1 < k_2 < ... < k_t$, such that $\tau(I_j) = I_j$ for all j.

The restrictions of τ to the intervals I_j in the finest of these decompositions are called the *connected components* of τ.

A permutation τ is said to be *connected* if it has only one connected component.

Note that, in the litterature (see [38]), connected permutations are also called irreducible or indecomposable.

Example 1.41. Consider the permutation $\tau = 3241576 \in \mathcal{S}_7$. The connected components of τ are 3241, 5 and 76.

Note in particular that the fixed points of a permutation are always connected components of this permutation.

In the following, a standard skew tableau is said to be *regular* if it is possible to obtain it from a standar tableau shifting each box to the right a given number of times.

Example 1.42. The skew tableau $\begin{array}{cccc} & & 5 & 6 \\ & 7 & & \\ 8 & & & \end{array}$ is regular because it is possible to obtain it from the standar tableau $\begin{array}{cc} 5 & 6 \\ 7 & \\ 8 & \end{array}$ shifiting to the right the boxes of the first row two times and the boxes of the second row one time.

On the other hand, the standar skew tableau $\begin{array}{ccc} & & 6 \\ 5 & 7 & \\ 8 & & \end{array}$ is not regular.

A simple consequence of the definition of the Robinson-Schensted map is the next lemma that will be useful later.

Lemma 1.43. *Consider a permutation $\tau \in \mathcal{S}_n$. Suppose that $\tau_{k_j}\tau_{k_j+1}...\tau_{k_j+h_j}$ with $h_j \geq 0$ and $1 = k_1 < k_2 < ... < k_t$ are the connected components of τ. Let P and Q be the insertion tableau and the recording tableau of τ. Then P is the juxtaposition of t standard skew tableaux $P_1, P_2, ...P_t$ where P_j is the regular skew tableau obtained by the insertion tableau of $|\tau_{k_j}\tau_{k_j+1}...\tau_{k_j+h_j}|$ normalizing the entries from k_j to $k_j + h_j$ and shifting each box to the right a given number of times. Similarly, Q is the juxtaposition of t standard skew tableaux $Q_1, Q_2, ...Q_t$ where Q_j is the regular skew tableau obtained by the recording tableau of $|\tau_{k_j}\tau_{k_j+1}...\tau_{k_j+h_j}|$ normalizing the entries from k_j to $k_j + h_j$ and shifting each box to the right the same number of times as in P_j. In particular P_j and Q_j are standar skew tableaux over the same set of integers and of the same shape.*

Viceversa, if P and Q are two standard Young tableaux and

- *P is the juxtaposition of t standar skew tableau $P_1, P_2, ..., P_t$*

- *Q is the juxtaposition of t standar skew tableau $Q_1, Q_2, ...Q_t$*
- *for all j, P_j and Q_j are regular skew tableaux over the same set of integers $\{k_j, ..., k_j + h_j\}$, $h_j \geq 0$, and of the same shape*
- *this decomposition of P and Q is the finest with the previous properties*

then the permutation $\tau = RS^{-1}(P, Q)$ has as j-th connected component the sequence of integers obtained by $RS^{-1}(Rect(P_j), Rect(Q_j))$, $1 \leq j \leq t$, normalizing the integers from k_j to $k_j + h_j$.

Example 1.44. Consider the permutation $\tau = 3241576 \in \mathcal{S}_7$, as above. Here the connected components are 3241, 5 and 76 and

$$RS(\tau) = \left(\begin{ytableau} 1 & 4 & 5 & 6 \\ 2 & 7 \\ 3 \end{ytableau}, \begin{ytableau} 1 & 3 & 5 & 6 \\ 2 & 7 \\ 4 \end{ytableau} \right).$$

In this case the tableau P is the juxtaposition of

$$\begin{ytableau} 1 & 4 \\ 2 \\ 3 \end{ytableau}, \quad \begin{ytableau} 5 \end{ytableau} \text{ and } \begin{ytableau} \none & 6 \\ 7 \end{ytableau}.$$

These three tableaux are obtained from the insertion tableaux of $3241 = |3241|$, $1 = |5|$ and $21 = |76|$, i.e.

$$\begin{ytableau} 1 & 4 \\ 2 \\ 3 \end{ytableau}, \quad \begin{ytableau} 1 \end{ytableau} \text{ and } \begin{ytableau} 1 \\ 2 \end{ytableau},$$

normalizing the entries and shifting the boxes of each rows to the right a given number of times.

Similarly the tableau Q is the juxtaposition of

$$\begin{ytableau} 1 & 3 \\ 2 \\ 4 \end{ytableau}, \quad \begin{ytableau} 5 \end{ytableau} \text{ and } \begin{ytableau} \none & 6 \\ 7 \end{ytableau}.$$

These three tableaux are obtained from the recording tableau of $|3241|$, $|5|$ and $|76|$ in the same way.

Viceversa, consider the pairs of standard Young tableaux

$$P = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 6 \\ \hline 2 & 5 & \multicolumn{2}{c}{} \\ \cline{1-2} 7 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}, \quad Q = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & \multicolumn{2}{c}{} \\ \cline{1-2} 6 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}.$$

Here the finest decomposition described above is

$$P_1 = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \quad P_2 = \begin{array}{ccc} \cline{3-3} & & \multicolumn{1}{|c|}{6} \\ \cline{2-3} & \multicolumn{1}{|c|}{5} & \\ \cline{1-2} \multicolumn{1}{|c|}{7} & & \\ \cline{1-1} \end{array} \quad \text{and} \quad Q_1 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \quad Q_2 = \begin{array}{ccc} \cline{3-3} & & \multicolumn{1}{|c|}{7} \\ \cline{2-3} & \multicolumn{1}{|c|}{5} & \\ \cline{1-2} \multicolumn{1}{|c|}{6} & & \\ \cline{1-1} \end{array}$$

and in fact the connected components of $RS^{-1}(P, Q) = 2317546$ are 231 and 7546 which are obtained from $RS^{-1}(Rect(P_1), Rect(Q_1))$ and $RS^{-1}(Rect(P_2), Rect(Q_2))$ normalizing suitably the entries.

Chapter 2

An order over the set of permutations avoiding k k-1...321

2.1 Definition of the dominance order

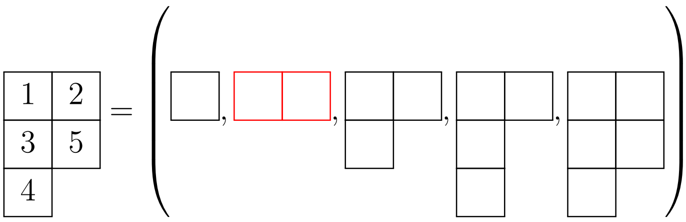
2.1.1 Dominance order for standard Young tableaux

The dominance order over partition of n can be extended to an order over the set $Tab(n)$, called *dominance order over standard tableaux*, by setting

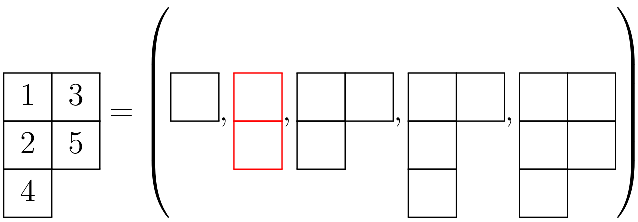
$$P \trianglelefteq Q \tag{2.1}$$

if and only if $shP_j \trianglelefteq shQ_j$ for all $j \leq n$. This order is used in the representation theory of the symmetric group and of the general linear group, see e.g. [14].

Example 2.1. The tableau



is greater than the tableau



Note that, in general, the dominance order defined in Subsection 2.1.1 does not yield a lattice structure over the set of standard Young tableaux. In fact, if we consider the tableaux

$$P = \begin{ytableau} 1 & 2 & 3 & 7 \\ 4 & 5 & 6 \end{ytableau} \quad \text{and} \quad P' = \begin{ytableau} 1 & 2 & 3 & 4 & 5 \\ 6 \\ 7 \end{ytableau}$$

Then two incomparable lower bounds of P and P' are $\begin{ytableau} 1 & 2 & 3 & 7 \\ 4 & 5 \\ 6 \end{ytableau}$ and $\begin{ytableau} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 \end{ytableau}$.

Since P covers both these lower bounds, the infimum of P and P' does not exist.

It will be useful to describe explicitly the covering relation associated with the dominance order.

Proposition 2.2. *If P, P' are two standard tableaux with the same number of boxes, P' covers P in the dominance order (in symbols, $P' \succ P$) if and only if P is obtained from P' by one of the following operations:*

(S) *if $shP = shP'$, choose one box filled with the integer i whose position is right and above the box filled with $i+1$ and swap these two boxes.*

(M) *if $shP \neq shP'$, move one corner-box (i, j) filled with the integer k to the first row below the i-th whose length is smaller than j, with the further condition that between the starting and the ending position of k there are no corner-boxes filled with an integer greater than k.*

Proof. First of all, recall that the covering relation in the lattice of integer partitions of n with the dominance order is the following: the partition σ covers the partition τ if and only if the Ferrers diagram of τ is obtained by the Ferrers diagram of σ by moving one corner-box in position (i, j) to the first row below the i-th whose length is smaller than j (see [16]).

Now we proceed by induction on the number of boxes n. If $n = 1, 2, 3$ the proposition is trivial.

Suppose it true for tableaux with at most $n-1$ boxes and suppose that P and P' are two standard tableaux with n boxes such that $P \prec P'$. Consider P_{n-1} and P'_{n-1}. Since P' covers P, P'_{n-1} covers P_{n-1} and then, by the induction hypothesis, one of the following possibilities is true:

- P_{n-1} is obtained from P'_{n-1} by applying operation **(S)**.

  In this case, the shapes of P and P' must be equal and P and P' are obtained from P_{n-1} and P'_{n-1}, respectively, by adding one box (filled with n) in the same position. So we get P from P' by applying operation **(S)**.

- P_{n-1} is obtained from P'_{n-1} by applying the operation **(M)** to the box filled with the integer j, $j \leq n-1$.

  In this case, P and P' are obtained from P_{n-1} and P'_{n-1}, respectively, by adding the box filled with n. Since P' covers P, the new box must be in a position not between the starting and the ending position of the box filled with j. So we get P from P' by applying the operation **(M)**.

- $P_{n-1} = P'_{n-1}$.

  In this case, the shapes of P and P' are different, so P is obtained from P' by applying operation **(M)** to the corner-box filled with the integer n.

$\square$

Example 2.3. Consider the tableaux

$$P' = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \multicolumn{1}{|c|}{5} \\ \cline{1-1} \end{array}$$

and

$$P = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \multicolumn{1}{|c|}{5} \\ \cline{1-1} \end{array}.$$

Then P' covers P. In fact, P is obtained from P' by applying the operation **(M)** to the box filled with 4 of P'.

Consider now the tableaux

$$R' = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & \multicolumn{1}{c|}{5} \\ \cline{1-2} \end{array}$$

and

$$R = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array}.$$

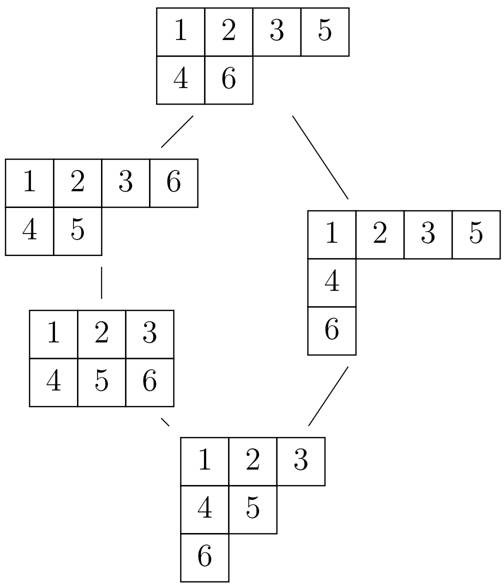
Then R' does not cover the tableau R. In fact, the further condition of operation **(M)** is not verified: there is a box filled with 5 between the starting and the ending position of 4. Actually,

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & \multicolumn{1}{c|}{5} \\ \cline{1-2} \end{array} \succ \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \multicolumn{1}{|c|}{5} \\ \cline{1-1} \end{array} \succ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \multicolumn{1}{|c|}{5} \\ \cline{1-1} \end{array} \succ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array}.$$

The interval

$$\left[\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}\right]$$

is



The previous example shows that the poset $(Tab(n), \trianglelefteq)$ is not ranked.

2.1.2 Dominance order over pairs of tableaux

Consider now the set $SYP(n)$ of all pairs of standard Young tableaux of the same shape in the elements $\{1, 2, ..., n\}$. We can endow the set $SYP(n)$ with the product of the dominance order over standard Young tableaux, namely,

$$(P, Q) \trianglelefteq (P', Q') \text{ if and only if } P \trianglelefteq Q \text{ and } P' \trianglelefteq Q'. \tag{2.2}$$

We call this order relation *dominance order over* $SYP(n)$.

As in the case of $(Tab(n), \trianglelefteq)$, the poset $(SYP(n), \trianglelefteq)$ is neither a lattice nor a ranked poset.

By restriction, we obtain also an order over $SYP_k(n)$, the subset of $SYP(n)$ of pairs of tableaux with at most k rows, $1 \leq k \leq n$. In this case the poset $SYP_k(n)$ has a minimum, namely the pair $(T_k(n), T_k(n))$, where:

$$T_k(n) = \begin{array}{|c|c|c|l} \hline 1 & k{+}1 & 2k{+}1 & \dots \\ \hline 2 & k{+}2 & \vdots & \\ \cline{1-2} 3 & k{+}3 & & \\ \cline{1-2} \vdots & \vdots & & \\ \vdots & \vdots & & \\ \cline{1-2} k & 2k & & \\ \cline{1-2} \end{array}$$

For example, if $n = 9$

$$T_4(9) = \begin{array}{|c|c|c|} \hline 1 & 5 & 9 \\ \hline 2 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & 7 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & 8 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}.$$

Proposition 2.2 implies that $(T_k(n), T_k(n))$ is in fact the minimum of $SYP_k(n)$.

2.1.3 Dominance order over $\mathcal{S}_n$

Schensted's theorem implies that the Robinson-Schensted bijection maps the set $\mathcal{S}_n(k+1\ k\ ...3\ 2\ 1)$ of permutations of length n that avoid the pattern $k+1\ k\ ...3\ 2\ 1$ to the set $SYP_k(n)$. This fact allows us to define an order relation over the set $\mathcal{S}_n(k+1\ k\ ...3\ 2\ 1)$, as follows:

$$\pi \trianglelefteq \sigma \Longleftrightarrow RS(\pi) \trianglelefteq RS(\sigma) \tag{2.3}$$

where the $\trianglelefteq$ on the right-hand side denotes the dominance order over SYP(n). In particular, we have an order over $\mathcal{S}_n = \mathcal{S}_n(n+1\ n\ ...2\ 1)$ and the order over $\mathcal{S}_n(k+1\ k\ ...3\ 2\ 1)$, $1 \leq k \leq n-1$, is simply the restriction of this order over $\mathcal{S}_n$. We call this order *dominance order over permutations.*

2.1.4 Structure of the poset $(\mathcal{S}_n, \trianglelefteq)$

Though the structure of the posets $(\mathcal{S}_n, \trianglelefteq)$ and $(SYP(n), \trianglelefteq)$ is quite difficult to describe, these posets have some remarkable symmetries given in the next proposition. Recall that, if τ is a partition of the integer n, τ^T is the conjugate partition and, similarly, if P is a standard tableau, P^T is the conjugate tableau.

Proposition 2.4. *In the poset $SYP_k(n)$, for every k with $1 \leq k \leq n$, the map $(P, Q) \mapsto (Q, P)$ is an order isomorphism. Equivalently, in the poset $\mathcal{S}_n(k+1\ k\ k-1 ... 1)$, the map $\pi \mapsto \pi^{-1}$ is an order isomorphism.*

In the poset $SYP(n)$, the map $(P, Q) \mapsto (P^T, Q^T)$ is an order anti-isomorphism.

Proof. It follows directly from the definitions of the order relations given in Subsections 2.1.2 and 2.1.3. ☐

Now consider the permutation

$$\Theta_k(n) := \left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & k & k+1 & \cdots & & 2k & \cdots & & & & & & & \cdots & n \\ k & k-1 & k-2 & \cdots & 1 & 2k & 2k-1 & \cdots & k+1 & \cdots & tk & tk-1 & \cdots & (t-1)k+1 & n & n-1 & \cdots & tk+1 \end{smallmatrix}\right),$$

where $t = \left\lfloor \frac{n}{k} \right\rfloor$. For example, if $n = 11$,

$$\Theta_4(11) := \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 & 11 & 10 & 9 \end{smallmatrix}\right).$$

We can now state the main result of this section.

Theorem 2.5. *For every $1 \leq k \leq n$, the set $\mathcal{S}_n(k+1\ k ... 2\ 1)$ is a principal filter in $\mathcal{S}_n$ generated by the permutation $\Theta_k(n)$, namely,*

$$\mathcal{S}_n(k+1\ k ... 2\ 1) = \{\sigma \in \mathcal{S}_n; \sigma \trianglerighteq \Theta_k(n)\}.$$

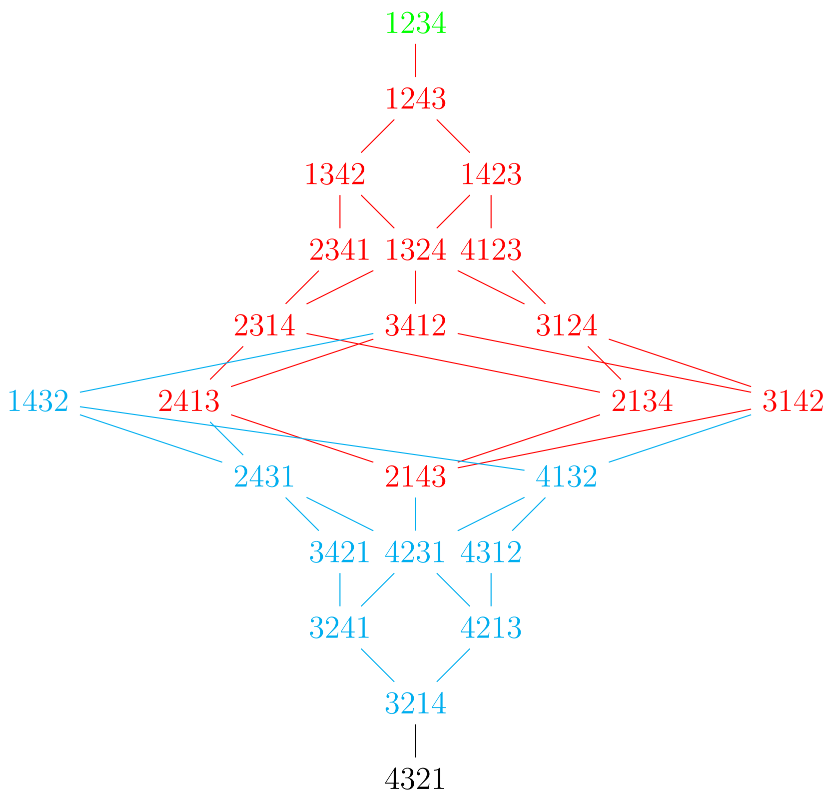
Proof. Recall that the poset $SYP_k(n)$ has a minimum, namely, the pair $(T_k(n), T_k(n))$. The permutation corresponding to this pair via the inverse of the Robinson-Schensted correspondence is $\Theta_k(n)$ and hence it is the minimum of $\mathcal{S}_n(k+1\ k ... 3\ 2\ 1)$ in the dominance order defined in Subsection 2.1.3. ☐

As an immediate consequence we get the following.

Corollary 2.6. *Consider the set $\mathcal{S}_n$ with the order relation defined above. In this poset there is a chain of principal filters, given by*

$$\mathcal{S}_n(21) \subset \mathcal{S}_n(321) \subset ... \subset \mathcal{S}_n.$$

Example 2.7. The Hasse diagram of the chain $\mathcal{S}_4(21) \subset \mathcal{S}_4(321) \subset \mathcal{S}_4(4321) \subset \mathcal{S}_4$ is



2.2 The case of at most two rows

2.2.1 Structure of the poset $(\mathcal{S}_n(321), \trianglelefteq)$

The dominance order defined above reveals to have some remarkable properties when restricted to the set $SYP_2(n)$ of pairs of standard Young tableaux with at most two rows, namely, to permutations avoiding the pattern 321.

In fact, the order relation over $\mathcal{S}_n(321)$ induced by the order over $\mathcal{D}_n$ via the bijection $\psi \circ RS$ (see Corollary 1.37) is precisely the dominance relation defined in Subsection 2.1.3. This fact is an immediate consequence of the following result.

Theorem 2.8. *Consider the set $SYP_2(n)$ endowed with the dominance order defined in Subsection 2.1.2. Then the map ψ is an order isomorphism between $SYP_2(n)$ and $\mathcal{D}_n$.*

Proof. Given two Dyck paths f, g of semilength n, suppose that f covers g (in symbols, $f \succ g$). Applying the map ψ_n^{-1} to f and g, we obtain two pairs of tableaux (P_f, Q_f) and (P_g, Q_g) where $shP_f = shQ_f$ and $shP_g = shQ_g$. Since f covers g, f is obtained from g

by the replacement of a valley with a peak. If this valley (and this peak) is at position $i, i+1$, then the following cases are possible:

1. If $i < n$, then $P_f = P_g$, $shQ_g = shQ_g$ and Q_f is obtained from Q_g by swapping the boxes filled with $i+1$ and i. Note that, in Q_g, the box filled with $i+1$ must be right and above the box filled with i.

2. If $i > n$, then $Q_f = Q_g$, $shP_g = shP_g$ and P_f is obtained from P_g by swapping the boxes filled with $i+1$ and i. Note that, in P_g, the box filled with $i+1$ must be right and above the box filled with i.

3. If $i = n$, then $shP_g = shQ_g \neq shP_f = shQ_f$ and P_f and P_g are obtained from Q_f and Q_g by moving the corner box filled with the integer n from the last position of the second row to the end of the first row, respectively.

The previous cases, by proposition 2.2, correspond exactly to the covering relation over the set $SYP_2(n)$. So we have proved:

$$g \prec f \Longleftrightarrow (P_g, Q_g) \prec (P_f, Q_f).$$

$\square$

In this way $(SYP_2(n), \trianglelefteq)$ result to be a distributive lattices isomorphic to $\mathcal{D}_n$.

Corollary 2.9. *Consider the set $\mathcal{S}_n(321)$ endowed with the dominance order defined in Subsection 2.1.3. Then the map $\psi \circ RS$ is an order isomorphism between $\mathcal{S}_n(321)$ and $\mathcal{D}_n$.*

The maximum and the minimum elements of the lattice $(SYP_2(n), \trianglelefteq)$ are:

$$\left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & n \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & n \\ \hline \end{array} \right)$$

and

$$\begin{cases} \left(\begin{array}{|c|c|c|c|c|} \hline 1 & 3 & \cdots & n-2 & n \\ \hline 2 & 4 & \cdots & n-1 \\ \cline{1-4} \end{array} \, , \, \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & \cdots & n-2 & n \\ \hline 2 & 4 & \cdots & n-1 \\ \cline{1-4} \end{array} \right) & \text{if } n \text{ is odd} \\ \left(\begin{array}{|c|c|c|c|} \hline 1 & 3 & \cdots & n-1 \\ \hline 2 & 4 & \cdots & n \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|c|} \hline 1 & 3 & \cdots & n-1 \\ \hline 2 & 4 & \cdots & n \\ \hline \end{array} \right) & \text{if } n \text{ is even} \end{cases}$$

that correspond to the paths $U^n D^n$ and $(UD)^n$, respectively, and to the 321-avoiding permutations

$$123...n$$

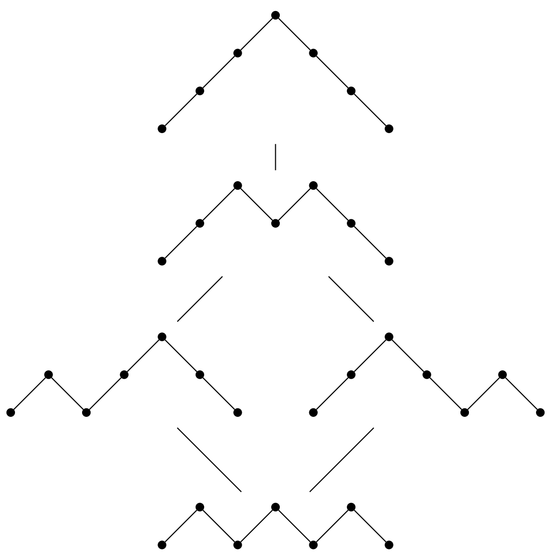
and

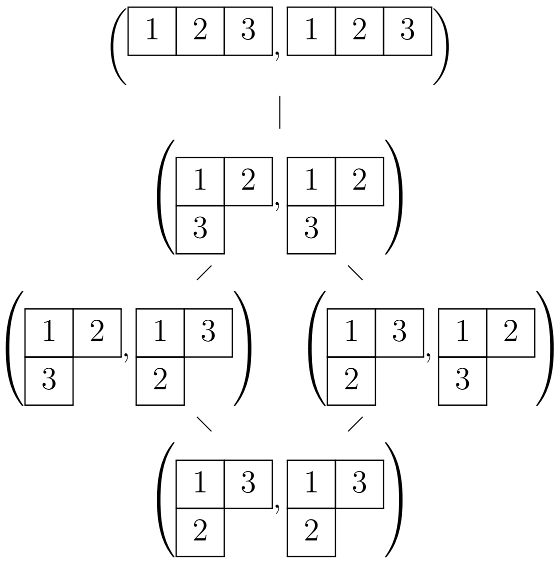
$$\begin{cases} 2\;1\;4\;3\;...\;n-1\;n-2\;n & \text{if } n \text{ is odd} \\ 2\;1\;4\;3\;...\;n\;n-1 & \text{if } n \text{ is even} \end{cases},$$

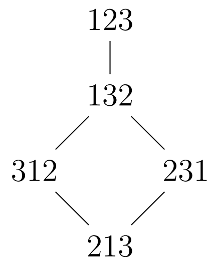
respectively.

Note, in particular, that the elements of $SYP_2(n)$ of the form (P, P), i.e., the involutions in $\mathcal{S}_n(321)$, correspond to symmetric paths. In particular, the set $Tab_2(n)$ of standard Young tableaux with at most two rows endowed with the dominance order defined in Subsection 2.1.1 is a distributive lattice isomorphic to the sublattice of $\mathcal{D}_n$ of symmetric paths.

Example 2.10. The following are the Hasse diagrams of the posets $(\mathcal{D}_3, \leq)$, $SYP_2(3)$ and $(\mathcal{S}_3(321), \trianglelefteq)$, respectively. The three diagrams are in fact identical.







In the following, we will need the next lemma that describes the operations of sup and inf in $SYP_2(n)$ and $Tab_2(n)$.

Lemma 2.11. *In the lattice $Tab_2(n)$ the infimum and the supremum are given by*

$$inf(P,P') = (inf(shP_1, shP'_1), inf(shP_2, shP'_2), ..., inf(shP_n, shP'_n))$$

and

$$sup(P,P') = (sup(shP_1, shP'_1), sup(shP_2, shP'_2), ..., sup(shP_n, shP'_n)).$$

In the lattice $SYP_2(n)$ the infimum and the supremum are given by

$$inf((P,Q),(P',Q')) = (inf(P,P'), inf(Q,Q'))$$

and

$$sup((P,Q),(P',Q')) = (sup(P,P'), sup(Q,Q')).$$

Proof. First of all we study the supremum and the infimum of two tableaux P and P' with at most two rows in the dominance order defined in Subsection 2.1.1. We identify P and P' with the sequences of shapes $(shP_1, shP_2, ..., shP_n)$ and $(shP'_1, shP'_2, ..., shP'_n)$. Note that the sequence $(inf(shP_1, shP'_1), inf(shP_2, shP'_2), ..., inf(shP_n, shP'_n))$ satisfies the condition $inf(shP_j, shP'_j) \subseteq inf(shP_{j+1}, shP'_{j+1})$ for all $1 \leq j \leq n-1$ (here the infimum are taken in the lattice of partitions of the integer $1, 2, 3, ..., n$). In fact, if $shP_j = (a, j-a)$ and $shP'_j = (b, j-b)$ (with $a \geq j-a$ and $b \geq j-b$) then $inf(shP_j, shP'_j) = (min(a, b), j - min(a, b))$. Since $shP_{j+1} = (a, j+1-a)$ or $(a+1, j-a-1)$ and $shP_{j+1} = (b, j+1-b)$ or $(b+1, j-b-1)$, $inf(shP_j, shP'_j) \subseteq inf(shP_{j+1}, shP'_{j+1})$. So such sequence of shapes corresponds to a tableau and this tableau is the greates lower bound of P and P':

$$inf(P, P') = (inf(shP_1, shP'_1), inf(shP_2, shP'_2), ..., inf(shP_n, shP'_n)).$$

Similarly

$$sup(P, P') = (sup(shP_1, shP'_1), sup(shP_2, shP'_2), ..., sup(shP_n, shP'_n)).$$

Consider now the pairs (P, Q) and (P', Q'). Since $shP = shQ$ and $shP' = shQ'$, $sh(inf(P, P')) = sh(inf(Q, Q'))$ so the infimum operation considered in the lemma is well defined:

$$inf((P, Q), (P', Q')) = (inf(P, P'), inf(Q, Q')).$$

Similarly, the supremum operation is well defined:

$$sup((P, Q), (P', Q')) = (sup(P, P'), sup(Q, Q')).$$

□

2.2.2 The Bruhat order

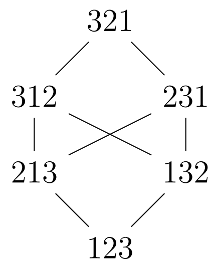
The set $\mathcal{S}_n$ of all the permutations of length n can be endowed with several order relations. One of the most important, thanks to its connections with Coxeter systems and inversions, is the Bruhat order. Here we consider only the so called strong Bruhat order. For the definitions and the properties of the weak Bruhat orders, see [12].

Recall that every permutation in $\mathcal{S}_n$ can be written as a product of cycles in an unique way if we disregard the order of the cycles, e.g. 364152 is equal to the product (26)(134). Moreover, we denote by $inv(\sigma) := |\{(i, j);\ i < j \text{ and } \sigma(i) > \sigma(j)\}|$ the *inversions number* of a permutation σ. Now, given a permutation $\pi \in \mathcal{S}_n$ written as a product of cycles, we consider the following relation $\mathcal{R}$ on $\mathcal{S}_n$

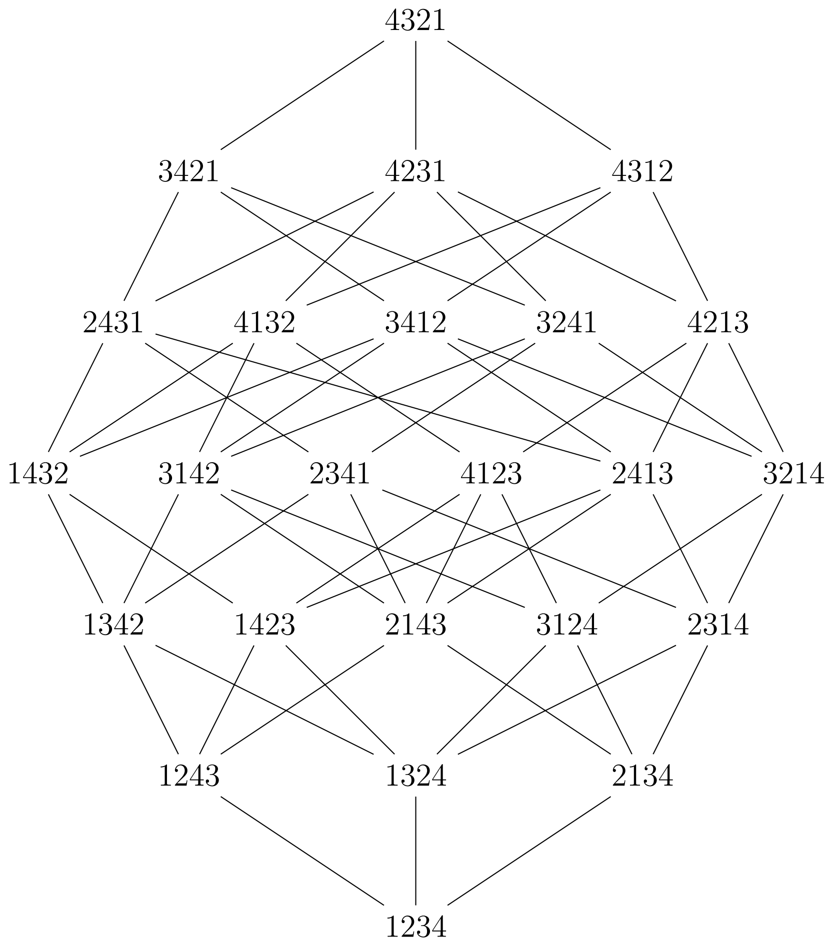
$$\pi \mathcal{R} \tau \text{ if and only if } \tau = (i, j)\pi \text{ and } inv(\pi) + 1 = inv(\tau).$$

The *strong Bruhat order* $\leq_B$ is the transitive and reflexive closure of this relation. The relation obtained in this way is also antisymmetric and hence an order relation. In the following the strong Bruhat order will be called simply *Bruhat order*.

Example 2.12. The following diagram shows the Hasse diagram of $(\mathcal{S}_3, \leq_B)$

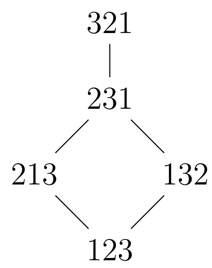


and of $(\mathcal{S}_4, \leq_B)$



Given a subset $S \subseteq \mathcal{S}_n$, we can consider the induced order $\leq_B$ over S. In the following we consider the Bruhat order induced over the subset $\mathcal{S}_n(312)$.

Example 2.13. The following is the Hasse diagram of $(\mathcal{S}_3(312), \leq_B)$



In the previous Subsections, we described a bijective map ψ between $\mathcal{D}_n$ and $\mathcal{S}_n(321)$. Theorem 2.8 affirms that ψ is in fact an order isomorphism between these two sets. We point out that the set $\mathcal{D}_n$ corresponds also to the set $\mathcal{S}_n(312)$, via the bijection KR originally stated in [41] and studied in [42]. The following result states that, if the set of Dyck paths is endowed with the order $\leq$ defined above, the corresponding order over $\mathcal{S}_n(312)$ is the Bruhat order (see [9]).

Theorem 2.14. *The map KR is an order isomorphism between $(\mathcal{D}_n, \leq)$ and $(\mathcal{S}_n(312), \leq_B)$.*

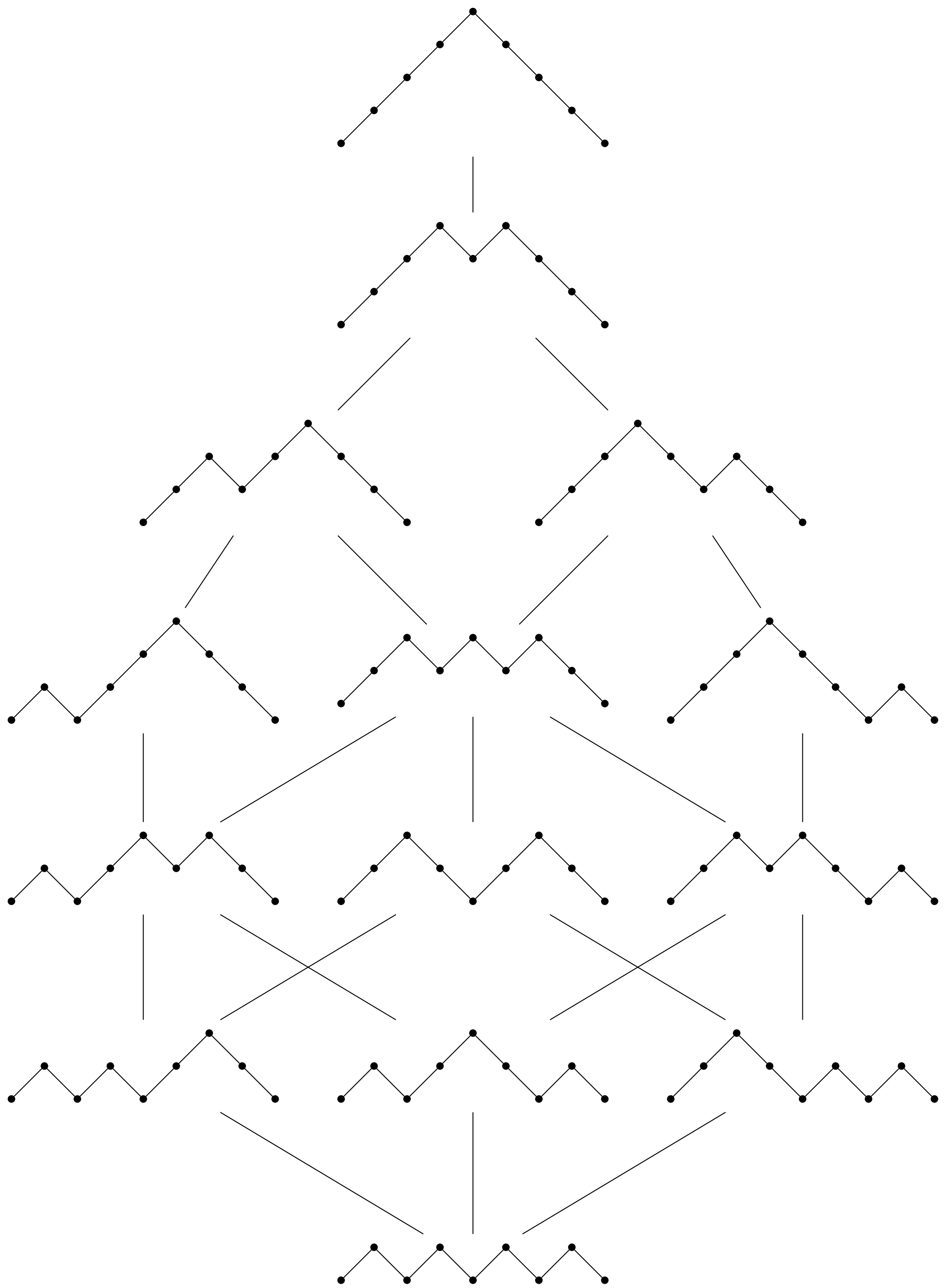
As a consequence, $(\mathcal{S}_n(312), \leq_B)$ results to be a lattice. Note that this is not the case for $(\mathcal{S}_n(321), \leq_B)$, since this set endowed with the Bruhat order has no maximum hence it is not a lattice.

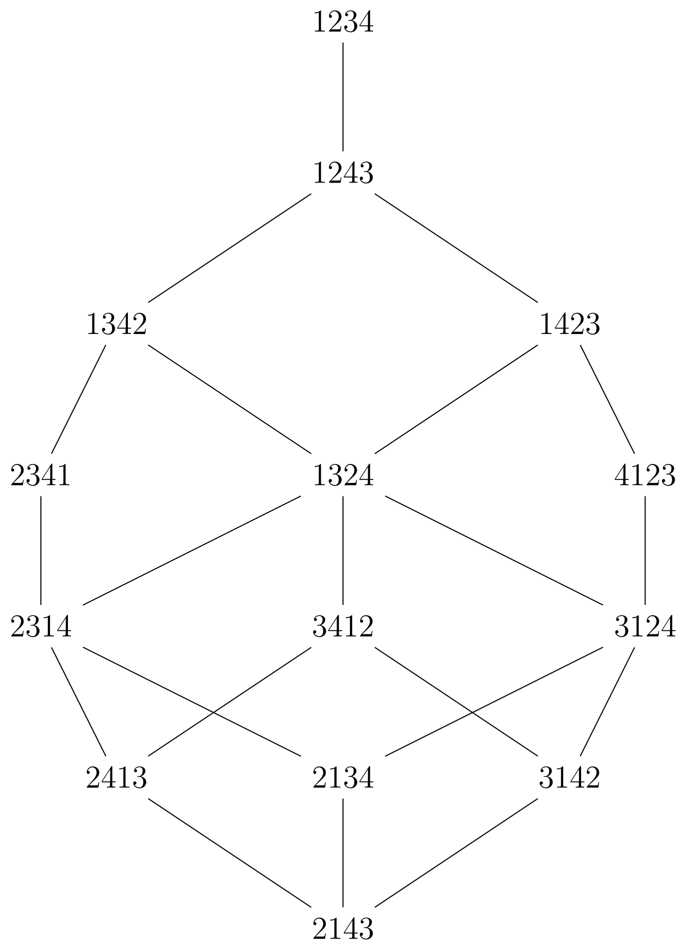
In particular, by means of the previous theorem and of Theorem 2.8 we obtain the following corollary.

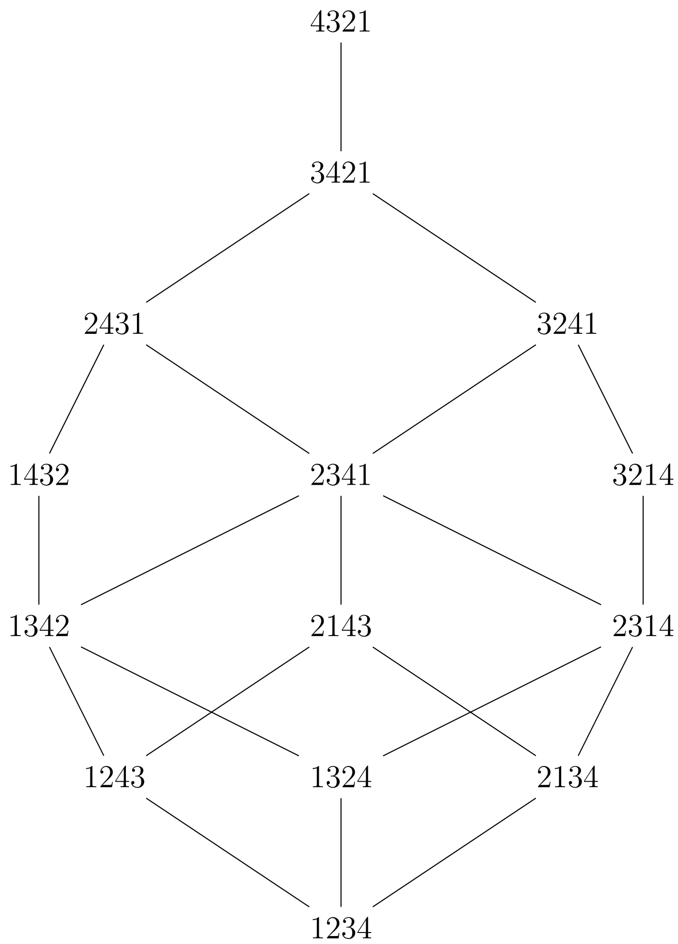
Corollary 2.15. *The map $KR \circ \psi \circ RS$ is a lattice isomorphism between $(\mathcal{S}_n(321), \trianglelefteq)$ and $(\mathcal{S}_n(312), \leq_B)$.*

The map $KR \circ \psi \circ RS$ appears also in the classifcation [20] of the bijections between these two sets.

Example 2.16. The following are the Hasse diagrams of the posets $(\mathcal{D}_4, \leq)$, $(\mathcal{S}_4(321), \trianglelefteq)$ and $(\mathcal{S}_4(312), \leq_B)$, respectively. The three diagrams are in fact identical.




1234
1243
1342
1423
2341
1324
4123
2314
3412
3124
2413
2134
3142
2143



2.2.3 Description of the order $(\mathcal{S}_n(321), \trianglelefteq)$

We now show that the order relation over $\mathcal{S}_n(321)$ defined in Subsection 2.1.3 can be described in terms of Knuth equivalences.

Given $\pi = \pi_1\pi_2...\pi_n \in \mathcal{S}_n$, define ${}^j\pi$ to be the permutation of $\mathcal{S}_j$ obtained by considering the integers $1, 2, ..., j$ in the same order as they appear in π. For example if $\pi = 5143726$ then ${}^4\pi = 1432$. Note that, if P is the insertion tableau of π, then P_j is the insertion tableau of ${}^j\pi$. Moreover, if Q is the recording tableau of π, then Q_j is the recording tableau of $|\pi_1\pi_2...\pi_j|$, the normalization of $\pi_1\pi_2...\pi_j$.

If $\pi \in \mathcal{S}_n(321)$, consider the set $A(\pi)$ of the last elements of the maximal increasing subsequences of π. As an example, the maximal increasing subsequences of $\pi = 13254$ are 135, 125, 134, 124 and hence $A(\pi) = \{4, 5\}$. Note that $A(\pi)$ has at most two elements. In fact, the greatest elements of two increasing subsequences (if different) are in decreasing order. Denote with $l(\pi)$ the length of a maximal increasing subsequence of π and by

$M(\pi)$ the maximum of $A(\pi)$.

Now we can state the characterization of the covering relation over $\mathcal{S}_n(321)$.

Theorem 2.17. *Consider the dominance order over $\mathcal{S}_n(321)$. In this order, a permutation $\sigma = \sigma_1...\sigma_n$ covers a permutation $\pi = \pi_1...\pi_n$ (in symbols, $\sigma \succ \pi$) if and only if one of the following conditions is satisfied:*

P(1) *π and σ are dual-Knuth equivalent and there exists an integer j with $1 \leq j \leq n-2$ such that*

- *${}^{j}\pi$ and ${}^{j}\sigma$ are Knuth equivalent*
- *${}^{j+1}\pi$ is obtained from ${}^{j}\pi$ by inserting $j+1$ in a position to the left of $M({}^{j}\pi)$ and ${}^{j+1}\sigma$ is obtained from ${}^{j}\sigma$ by inserting $j+1$ in a position to the right of $M({}^{j}\sigma)$*
- *${}^{j+2}\pi$ is obtained from ${}^{j+1}\pi$ by inserting $j+2$ in a position to the right of $M({}^{j+1}\pi)$ and ${}^{j+2}\sigma$ is obtained from ${}^{j+1}\sigma$ by inserting $j+2$ in a position to the left of $M({}^{j+1}\sigma)$*
- *${}^{k}\pi$ and ${}^{k}\sigma$ with $k \geq j+3$, are obtained from ${}^{j+2}\pi$ and ${}^{j+2}\sigma$, respectively, by inserting $j+3, j+4, ..., k$ in the same positions*

P(2) *π and σ are Knuth equivalent and there exists an integer $1 \leq j \leq n$ such that*

- *$|\pi_1...\pi_{j+1}|$ and $|\sigma_1...\sigma_{j+1}|$ are dual-Knuth equivalent*
- *$l(|\pi_1...\pi_j\pi_{j+1}|) = l(|\pi_1...\pi_j|)$ and $l(|\sigma_1...\sigma_j\sigma_{j+1}|) = l(|\sigma_1...\sigma_j|) + 1$*
- *$l(|\pi_1...\pi_j\pi_{j+1}\pi_{j+2}|) = l(|\pi_1...\pi_j\pi_{j+1}|)+1$ and $l(|\sigma_1...\sigma_j\sigma_{j+1}\sigma_{j+2}|) = l(|\sigma_1...\sigma_j\sigma_{j+1}|)$*
- *$\forall k \geq j+3$, $l(|\pi_1...\pi_k|) = l(|\sigma_1...\sigma_k|)$*

P(3) *${}^{n-1}\pi$ and ${}^{n-1}\sigma$ are Knuth equivalent, $|\pi_1...\pi_{n-1}|$ and $|\sigma_1...\sigma_{n-1}|$ are dual-Knuth equivalent, $\sigma_n = n$, $l(\pi) = l({}^{n-1}\pi)$ and $l(\sigma) = l({}^{n-1}\sigma) + 1$.*

In particular, the bijection ψ defined above is a poset isomorphism between the posets $\mathcal{S}_n(321)$ and $\mathcal{D}_n$, hence, $\mathcal{S}_n(321)$ results to be a distributive lattice.

Proof. We want to prove that the dominance order over $\mathcal{S}_n(321)$ has the stated covering relation. Suppose that, in this order, $\pi \prec \sigma$ and $RS(\pi) = (P, Q)$, $RS(\sigma) = (P', Q')$.

Recall that in the dominance order defined in Subsection 2.1.2, $(P, Q) \prec (P', Q')$ if and only if one of the following condition is satisfied

(C1) $shP = shQ = shP' = shQ'$, $Q = Q'$ and $P \prec P'$

(C2) $shP = shQ = shP' = shQ'$, $P = P'$ and $Q \prec Q'$

(C3) $shP = shQ \neq shP' = shQ'$, $P \prec P'$ and $Q \prec Q'$

The three previous cases correspond respectively to the three cases **P(1)**, **P(2)**, **P(3)**.

In fact, in case **(C1)**, since $Q = Q'$, π and σ are dual-Knuth equivalent. By proposition 2.2, P is obtained from P' by choosing one box filled with the integer $j+1$ in the first row, one box filled with the integer $j+2$ in the second row and swapping these boxes. So the tableaux P_j and P'_j are equal, hence ${}^j\pi$ and ${}^j\sigma$ are Knuth equivalent, and the tableaux P_{j+1} and P'_{j+1} have the boxes filled with $j+1$ in the second and first row, respectively.

Considering the definition of the insertion procedure, this implies that ${}^{j+1}\pi$ is obtained from ${}^j\pi$ by adding the element $j+1$ to the left of $M({}^j\pi)$ and that ${}^{j+1}\sigma$ is obtained from ${}^j\sigma$ by adding $j+1$ to the right of $M({}^j\sigma)$. Similarly, ${}^{j+2}\pi$ and ${}^{j+2}\sigma$ are obtained as described. All the ${}^k\pi$ and ${}^k\sigma$ with $k \geq j+3$ have the elements $j+3, j+4, ..., k$ in the same position because $Q = Q'$ and P and P' differ only in the boxes filled with $j+1$ and $j+2$.

In case **(C2)**, note that, during the insertion procedure, the insertion of the $(j+1)$th element does not modify the tableau Q_j but it only adds the box $j+1$ at the end of the first or of the second row of Q_j.
Since, in this case, $P = P'$ and Q is obtained from Q' by swapping the boxes $j+1$ and $j+2$, $|\pi_1...\pi_j|$ and $|\sigma_1...\sigma_j|$ are dual-Knuth equivalent and $l(|\pi_1...\pi_j\pi_{j+1}|) = l(|\pi_1...\pi_j|)$, $l(|\sigma_1...\sigma_j\sigma_{j+1}|) = l(|\sigma_1...\sigma_j|) + 1$. Similarly, considering Q_{j+2} and Q'_{j+2} we have $l(|\pi_1...\pi_j\pi_{j+1}\pi_{j+2}|) = l(|\pi_1...\pi_j\pi_{j+1}|) + 1$ and $l(|\sigma_1...\sigma_j\sigma_{j+1}\sigma_{j+2}|) = l(|\sigma_1...\sigma_j\sigma_{j+1}|)$ and , considering Q_k and Q'_k, $k \geq j+3$, we have $l(|\pi_1...\pi_k|) = l(|\sigma_1...\sigma_k|)$.

In the same way, in case **(C3)**, the tableaux have the following form:

$$P = \begin{array}{|c|c|c|c|c|}\hline a_1 & a_2 & \cdots & \cdots & a_l \\ \hline a_{l+1} & \cdots & a_{n-1} & n & \\ \cline{1-4}\end{array}, \quad P' = \begin{array}{|c|c|c|c|c|c|}\hline a_1 & a_2 & \cdots & \cdots & a_l & n \\ \hline a_{l+1} & \cdots & a_{n-1} & \multicolumn{3}{c}{} \\ \cline{1-3}\end{array}$$

and

$$Q = \begin{array}{|c|c|c|c|c|}\hline b_1 & b_2 & \cdots & \cdots & b_l \\ \hline b_{l+1} & \cdots & b_{n-1} & n & \\ \cline{1-4}\end{array}, \quad Q' = \begin{array}{|c|c|c|c|c|c|}\hline b_1 & b_2 & \cdots & \cdots & b_l & n \\ \hline b_{l+1} & \cdots & b_{n-1} & \multicolumn{3}{c}{} \\ \cline{1-3}\end{array}.$$

In particular, $P_{n-1} = P'_{n-1}$, $Q_{n-1} = Q'_{n-1}$ and $\sigma_n = n$. So ${}^{n-1}\sigma = \sigma_1\sigma_2...\sigma_{n-1} = |\sigma_1\sigma_2...\sigma_{n-1}|$. This implies that $l({}^{n-1}\sigma) = l(|\sigma_1...\sigma_{n-1}|)$ (so also $l({}^{n-1}\pi) = l(|\pi_1...\pi_{n-1}|)$), that ${}^{n-1}\pi$ and ${}^{n-1}\sigma$ are Knuth equivalent and that $|\pi_1...\pi_{n-1}|$ and $|\sigma_1...\sigma_{n-1}|$ are dual-Knuth equivalent. Since P and Q are obtained from P_{n-1} and Q_{n-1} adding n in the second row and P' and Q' are obtained from P'_{n-1} and Q'_{n-1} adding n in the first row, $l(\pi) = l({}^{n-1}\pi)$ and $l(\sigma) = l({}^{n-1}\sigma) + 1$.

Viceversa, it is easy to see with similar arguments that the conditions given in the theorem imply respectively the conditions **(C1)**, **(C2)** and **(C3)** that characterize the covering relation in $SYP_2(n)$. $\square$

The previous theorem allows us to describe the order relation over $\mathcal{S}_n(321)$ without using the Robinson-Schensted map.

Example 2.18. Consider the permutations $\sigma = 4571236$ and $\pi = 3571246$. Then

$$RS(\sigma) = \left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 & 7 \\ \cline{1-3} \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 7 \\ \hline 4 & 5 & 6 \\ \cline{1-3} \end{array} \right)$$

and

$$RS(\pi) = \left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 6 \\ \hline 3 & 5 & 7 \\ \cline{1-3} \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 7 \\ \hline 4 & 5 & 6 \\ \cline{1-3} \end{array} \right)$$

so $\pi \prec \sigma$, π and σ are dual-Knuth equivalent and in fact we are in case $\mathbf{P(1)}$ of the theorem:
$j = 2$, $M(^2\sigma) = 2$ and $M(^2\pi) = 2$.

j	$^j\pi$	$^j\sigma$
2	12	12
3	312	123
4	3124	4123
5	35124	45123
6	351246	451236
7	3571246	4571236

Consider the permutations $\sigma = 213784596$ and $\pi = 217893456$. Then

$$RS(\sigma) = \left(\begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 5 & 6 \\ \hline 2 & 7 & 8 & 9 \\ \cline{1-4} \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 5 & 8 \\ \hline 2 & 6 & 7 & 9 \\ \cline{1-4} \end{array} \right)$$

and

$$RS(\pi) = \left(\begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 5 & 6 \\ \hline 2 & 7 & 8 & 9 \\ \cline{1-4} \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 5 & 9 \\ \hline 2 & 6 & 7 & 8 \\ \cline{1-4} \end{array} \right)$$

so $\pi \prec \sigma$, π and σ are Knuth equivalent and we are in case $\mathbf{P(2)}$ of the theorem:
$j = 7$,

j	$\lvert\pi_1 ... \pi_j\rvert$	$\lvert\sigma_1 ... \sigma_j\rvert$
7	2156734	2136745
8	21678345	21367458
9	217893456	213784596

As an example of case $\mathbf{P(3)}$, consider $\sigma = 1324$ and $\pi = 3412$. Then

$$RS(\sigma) = \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} , \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \right)$$

and

$$RS(\pi) = \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} , \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \right)$$

so $\pi \prec \sigma$ and we have: $\sigma_4 = 4$, ${}^3\pi = 312$, ${}^3\sigma = 132$, $|\pi_1\pi_2\pi_3| = 231$ and $|\sigma_1\sigma_2\sigma_3| = 132$.

2.2.4 Partitions of $\mathcal{D}_n$

We conclude this chapter with an application of the bijection ϕ to the study of the lattice $\mathcal{D}_n$.

In the literature, several interesting partitions of the lattice $\mathcal{D}_n$ into sublattices has been considered, in particular in connection with the ECO method (see [33] and [6]). Now we describe other partitions of $\mathcal{D}_n$ obtained by using the map ϕ. Consider the following sets:

$$\mathcal{S}_n(321)_k : \text{ the subset of } \mathcal{S}_n(321) \text{ of permutations } \pi \text{ with } l(\pi) = k$$

where $l(\pi)$ is the length of a maximal increasing subsequence of π,

$$\mathcal{S}_n(321)_{(P,)} : \text{ the subset of } \mathcal{S}_n(321) \text{ of permutations } \pi \text{ whose insertion tableau is } P$$

and

$$\mathcal{S}_n(321)_{(,Q)} : \text{ the subset of } \mathcal{S}_n(321) \text{ of permutations } \pi \text{ whose recording tableau is } Q.$$

Note that the set $\mathcal{S}_n(321)_k$ corresponds, via the map RS, to the subset of $SYP_2(n)$ of pairs of tableaux with shape $(k, n-k)$.

Theorem 2.19. *The family of sets $\mathcal{S}_n(321)_k$ with $\lceil \frac{n}{2} \rceil \leq k \leq n$, is a partition of $\mathcal{S}_n(321)$ into sublattices. This partition corresponds to the partition of $\mathcal{D}_n$ given by the sublattices $\mathcal{D}_{n,k}$ of paths with exactly k up steps in the first half and k down steps in the second half.*

The family of sets $\mathcal{S}_n(321)_{(P,)}$ with $P \in Tab_2(n)$ and $shP = (k, n-k)$ is a partition of $\mathcal{S}_n(312)_k$ into sublattices. This partition corresponds to the partition of $\mathcal{D}_{n,k}$ given by the sublattices of paths with a fixed second half.

Similarly, the family of sets $\mathcal{S}_n(321)_{(,Q)}$ with $Q \in Tab_2(n)$ and $shQ = (k, n-k)$ is a partition of $\mathcal{S}_n(312)_k$ into sublattices and correspond to Dyck paths with a fixed first half.

Proof. Follows directly by a careful analysis of the properties of the bijection ϕ and by Lemma 2.11. $\square$

2.3 The case of at most three rows

2.3.1 Motzkin paths

A *Motzkin path* of length t is a lattice path consisting of steps $U = (1, 1)$, $D = (1, -1)$ and $H = (1, 0)$, starting at $(0, 0)$, ending at $(t, 0)$, and never going below the x-axis. Similarly, a *Motzkin prefix* of length n is a lattice path consisting of n steps $U = (1, 1)$, $D = (1, -1)$ and $H = (1, 0)$, starting at $(0, 0)$, and never going below the x-axis. We recall that a Dyck prefix can be seen as a Motzkin prefix without H steps. We denote by $\mathcal{M}_t$ the set of Motzkin paths of length t.

As in the case of Dyck paths, it is possible to endow the set of Motzkin paths of length t with the order: $f \leq g$ if and only if the path f lies above the path g. Considering this order relation, $\mathcal{M}_t$ becomes a distributive lattice (see e.g. [33]).

In this section we consider a subset of $\mathcal{M}_t$ and an order over this subset different from the previous one. The subset we are interested in is defined as follows.

Given $p \in \mathcal{M}_{2n}$ and $i \leq n$, we denote by $(U, p, \leq i)$ the number of up steps U whose position is smaller than or equal to i. Analogously we denote by $(D, p, \leq i)$, $(H, p, \leq i)$ the number of steps D and H whose position is smaller than or equal to i and by $(U, p, \geq i)$, $(D, p, \leq i)$, $(H, p, \leq i)$ the number of steps U, D and H respectively whose position is greater than or equal to i, respectively.

Let $\widehat{M_{2n}}$ be the subset of $\mathcal{M}_{2n}$ of paths p with the following properties:

- $\forall i$ with $1 \leq i \leq n$, $(U, p, \leq i) \geq (D, p, \leq i) \geq (H, p, \leq i)$
- $\forall i$ with $n + 1 \leq i \leq 2n$, $(D, p, \geq i) \geq (U, p, \geq i) \geq (H, p, \geq i)$
- $(H, p, \leq n) = (H, p, \geq n + 1)$

Note that the set $\mathcal{D}_n$ of Dyck paths of semilength n is a subset of $\widehat{M_{2n}}$.

2.3.2 The bijection Ψ

We associate with a given pair $(P, Q) \in SYP_3(n)$ two Motzkin prefixes p, q in the following way: the prefix p has steps U, D, H in all the positions given by the elements of the first, second and third row of P, respectively, q is built in the same way using the tableau Q. Now we have a Motzkin path $q\overline{p}$ where $\overline{p}$ is the inverse or symmetric of the word $p = (p_1, ..., p_n)$ defined as $\overline{p} := (\overline{p}_n, ..., \overline{p}_1)$ where $\overline{U} := D$, $\overline{D} := U$ and $\overline{H} := H$.

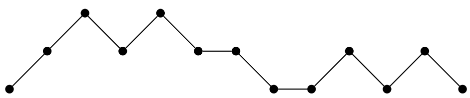
Similarly to the case of two rows, define Ψ to be the map $(P, Q) \mapsto q\overline{p}$.

It is immediately seen that Ψ is a bijection between $SYP_3(n)$ and $\widehat{M_{2n}}$, hence $\Psi \circ RS$ is a bijection between $\mathcal{S}_n(4321)$ and $\widehat{M_{2n}}$.

Example 2.20. Consider the permutation $\pi \in \mathcal{S}_6(321)$, $\pi = 251643$. Then $RS(\pi) = (P, Q)$ with

$$P = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \quad \text{and} \quad Q = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$$

so $p = UDUDHU$, $q = UUDUDH$, $\overline{p} = DHUDUD$ and we obtain the path $\Psi(P, Q)$ whose word is $q\overline{p} = UUDUDHDHUDUD$:



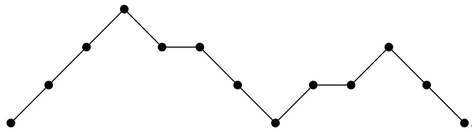
2.3.3 Order relation over $\widehat{M}_{2n}$

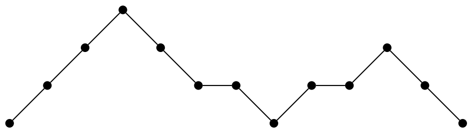
Now, consider the order over $\widehat{M}_{2n}$ whose covering relation is given by: $f \prec g$ if and only if the word $g = (g_1, ..., g_n | g_{n+1}, ..., g_{2n})$ is obtained from the word $f = (f_1, ..., f_n | f_{n+1}, ..., f_{2n})$ by one of the following substitutions

- $...DU... \mapsto ...UD...$ where DU is a pair of consecutive steps of f
- $...HD... \mapsto ...DH...$ where HD is a pair of consecutive steps in the first half of f
- $...HU... \mapsto ...UH...$ where HU is a pair of consecutive steps in the first half of f
- $...DH... \mapsto ...HD...$ where DH is a pair of consecutive steps in the second half of f
- $...UH... \mapsto ...HU...$ where UH is a pair of consecutive steps in the second half of f
- $...H|H... \mapsto ...U|D...$ where $H|H$ is a pair of consecutive steps symmetric with respect to the half of f
- $...D\underbrace{H...H}_{k_1}|\underbrace{H...H}_{k_2}U... \mapsto ...U\underbrace{H...H}_{k_1}|\underbrace{H...H}_{k_2}D...$ where D, U are, respectively, in the first half and in the second half of f and $k_1, k_2 \geq 0$
- $...H\underbrace{U...U}_{k_1}|\underbrace{D...D}_{k_2}H... \mapsto ...D\underbrace{U...U}_{k_1}|\underbrace{D...D}_{k_2}U...$ where H, H are, respectively, in the first half and in the second half of f and $k_1, k_2 \geq 0$

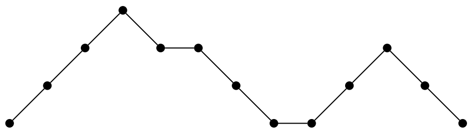
where the bar indicates the half of the path.

Example 2.21. Consider the following paths f, g_1 and g_2 in $\widehat{M}_{12}$ given by:

$f :=$ ,

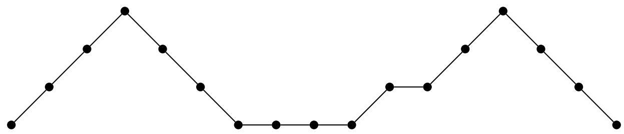
$g_1 :=$ 

and

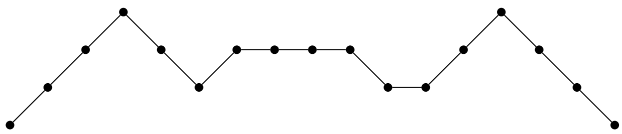
$g_2 :=$ 

then $f \prec g_1$ and $f \prec g_2$. In fact, g_1 is obtained from f by the substitution $...HD... \mapsto ...DH...$ where HD is a pair of consecutive steps in the first half of f and g_2 is obtained from f by the substitution $...UH... \mapsto ...HU...$ where UH is a pair of consecutive steps in the second half of f.

Consider the paths f' and g' in $\widehat{M}_{16}$ given by:

$f' :=$ 

and

$g' :=$ 

then $f' \prec g'$. In fact, g' is obtained from f' by the following substitution $...DHH|HU... \mapsto ...UHH|HD...$ where the bar $|$ indicates the half of the path.

Note that this covering relation extends the covering relation over the set $\mathcal{D}_n$ of Dyck paths, where $f \prec g$ if and only if g is obtained from f by the substitution of consecutive steps: $...DU... \mapsto ...UD...$

2.3.4 Isomorphism theorem

We have the following theorem.

Theorem 2.22. *The map Ψ is an order isomorphism between $SYP_3(n)$ and $\widehat{M_{2n}}$ which extends the bijection ψ between $SYP_2(n)$ and $\mathcal{D}_n$ defined in Subsection 1.3.7. In particular, $\mathcal{D}_n$ is a principal filter in $\widehat{M_{2n}}$ generated by the element $(UD)^n$.*

Proof. The proof is similar to the proof of Theorem 2.8. □

As a consequence, by Gessel's formula (see [13]),

$$|\widehat{M_{2n}}| = |\mathcal{S}_n(4321)| = \frac{1}{(n+1)^2(n+2)} \sum_{k=0}^{n} \binom{2k}{k} \binom{n+1}{k+1} \binom{n+2}{k+1}.$$

Note in particular that, if $f \prec g$ and the paths f and g differs in two consecutive positions in the first half or in two consecutive positions in the second half, then the corresponding tableaux P_f, Q_f, P_g and Q_g have the same shape. Otherwise, if $f \prec g$ and f and g differs in two positions, one in the first half and the other in the second half of the path, then $shP_f = shQ_f \neq shP_g = shQ_g$.

Observe that, both in the case of $\mathcal{D}_n$ and of $\widehat{M_{2n}}$, in the construction of the maps ψ and Ψ, we consider a path as a pair of words (p, q) where p and q have the same number of letters U, D and H and the number of letters H is smaller than the number of letters D that is smaller than the number of letters U. In the same way, it is possible to define a bijection between $\mathcal{S}_n(k+1\ k\ k-1...321)$ and the set of pairs (p, q) of words of length n over the alphabet $U_1, U_2, ..., U_k$ such that p and q have the same number of letters U_i, for all $1 \leq i \leq k$, and the number of letters U_i is smaller than the number of letters U_{i+1}, for all $1 \leq i \leq k-1$. So, if we call $\mathcal{L}_{n,k}$ this set of pairs of words, we have

$$|\mathcal{S}_n(k+1\ k\ k-1...321)| = |\mathcal{L}_{n,k}|$$

for all k. This words correspond obviously to standard Young tableaux with at most k rows.

Chapter 3

Schützenberger involution over Dyck paths

3.1 Self-dual tableaux with at most two rows

3.1.1 Rectification of tableaux with at most two rows

In Subsection 1.3.4 we introduced the sliding procedure and the inverse of the sliding procedure. It is immediately seen that if we start from a standard Young tableau P, apply the inverse slide move a given number of times (choosing the starting corner boxes of the outer shape arbitrarily) getting a standard skew tableau T, then $Rect(T) = P$. On the other hand, there are different standard skew tableaux of the same shape with the same rectification. For example, consider the standard skew tableaux

$$T_1 = \begin{array}{ccc} & & \boxed{5} \\ & \boxed{4} & \\ \boxed{6} & & \end{array} \qquad \text{and} \qquad T_2 = \begin{array}{ccc} & & \boxed{5} \\ & \boxed{6} & \\ \boxed{4} & & \end{array}.$$

We have

$$Rect(T_1) = Rect(T_2) = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

However, if we restrict our attention to standard skew tableaux with at most two rows, this situation can not occur. In fact, we have

Lemma 3.1. *Let T_1, T_2 be two standard skew tableaux in the same symbols, of the same skew shape μ/λ with at most two rows. Then,*

$$Rect(T_1) = Rect(T_2) \Rightarrow T_1 = T_2.$$

Proof. Set $Rect(T_1) = Rect(T_2) = P$. Note that both T_1 and T_2 can be obtained from P (up to a renaming of symbols) by applying a certain number of reverse sliding operations. In the present case we have only two kinds of reverse sliding, namely:

R_1 apply the reverse slide with respect to an outside corner on the right of the first row of μ.

R_2 apply the reverse slide with respect to an outside corner on the right of the second row of μ.

The key fact is that, if T is a standard skew tableau with at most two rows, then $R_2(R_1(T)) = T$. In fact, let

$$T = \begin{array}{ccccccc} & & a_1 & \cdots & \cdots & \cdots & a_h \\ b_1 & \cdots & b_j & \cdots & b_k & & \end{array}.$$

The standardness of S implies

$$a_1 < b_j < b_{j+1} < ... < b_k, \quad a_2 < b_{j+1} < ... < b_k, \quad \cdots, \quad a_{k-j+1} < b_k.$$

This yields

$$\begin{array}{cccccccc} & & a_1 & \cdots & \cdots & \cdots & a_h & X \\ b_1 & \cdots & b_j & \cdots & b_k & & & \end{array} \xrightarrow{R_1} \begin{array}{ccccccccc} & & & a_1 & \cdots & \cdots & \cdots & \cdots & a_h \\ b_1 & \cdots & b_j & b_{j+1} & \cdots & b_k & X & & \end{array}$$

$$\xrightarrow{R_2} \begin{array}{ccccccc} & & a_1 & \cdots & \cdots & \cdots & a_h \\ b_1 & \cdots & b_j & \cdots & b_k & & \end{array} = T.$$

Hence, both T_1 and T_2 are obtained from P by a certain number (n_1, n_2, say) of operations R_2 followed by a number m_1, m_2 of operations R_1. Let (ℓ_1, ℓ_2) be the shape of the tableau P. We notice that, since P is a standard tableau, if we apply s times the operation R_2 to P we get a skew tableau with outer shape $(\ell_1, \ell_2 + s)$ and inner shape (s). A subsequent application of t operations R_1 yields a skew tableau of outer shape $(\ell_1 + t, \ell_2 + s)$ and inner shape $(s + t)$. Since T_1 and T_2 have the same skew shape, we must have $n_1 = n_2$ and $m_1 = m_2$, which implies $T_1 = T_2$. $\square$

3.1.2 Even number of boxes

Our next goal is to give a characterization of self-dual tableaux with at most two rows and an even number of boxes.

Let P be a standard Young tableau with $n = 2\ell$ boxes and with at most two rows. Then P can be decomposed as $P = P_1P_2$, where $P_1 \in Tab(\ell)$ is the standard Young tableau containing the first ℓ symbols of P, while P_2 is the standard skew tableau containing the remaining ℓ symbols. The pair (P_1, P_2) will be called the *central decomposition* of P, and the tableau P_1 and P_2 will be called the *first half tableau* and the *second half tableau* of P, respectively.

Proposition 3.2. *Let $P \in Tab_2(2\ell)$, with central decomposition (P_1, P_2). Then, the tableau P^* is the unique standard tableau with the same shape of P whose central decomposition is (Q_1, Q_2), where*

$$Q_1 = Rect(P_2)^* \quad \textit{and} \quad P_1 = Rect(Q_2)^*. \tag{3.1}$$

In particular $P = P^*$ whenever

$$P_1 = Rect(P_2)^*.$$

Proof. Identities (3.1) are a straightforward consequence of the definition of the Schützenberger involution. The uniqueness of the standard tableau P^* follows from Lemma 3.1. □

Example 3.3. Consider the self-dual tableau

$$P = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 7 & 9 & 12 \\ \hline 4 & 6 & 8 & 10 & 11 & \multicolumn{2}{c}{} \\ \cline{1-5} \end{array}.$$

Here we have

$$P_1 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}, \quad P_2 = \begin{array}{ccccc} \cline{3-5} & & \multicolumn{1}{|c|}{7} & 9 & \multicolumn{1}{c|}{12} \\ \hline \multicolumn{1}{|c|}{8} & 10 & \multicolumn{1}{c|}{11} & & \\ \cline{1-3} \end{array}$$

and

$$Rect(P_2) = P_1^* = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}.$$

In the following theorem we succeed in enumerating self-dual standard Young tableaux with a fixed first half tableau.

Theorem 3.4. *The number of self-dual tableaux $P \in Tab_2(2\ell)$ with a fixed first half tableau P_1 such that $shP_1 = (\ell - t, t)$ is*

$$\ell - 2t + 1.$$

As a consequence, the number of self-dual tableaux $P \in Tab_2(2\ell)$ such that the first half tableau of P has shape $(\ell - t, t)$ is

$$\frac{(\ell - 2t + 1)^2}{\ell - t + 1}\binom{\ell}{t}.$$

Proof. Let $P_1 \in Tab_2(\ell)$ with $shP_1 = (\ell - t, t)$, $t \leq \lfloor \frac{\ell}{2} \rfloor$, and choose a shape $\lambda = (2\ell - j, j)$, with $j \leq \ell$.

Now, if $j < 2t$, it is impossible that the rectification of the second half of a tableau of shape λ is P_1. Hence there are no self-dual tableaux P with first half equal to P_1 and shape λ.

If $2t \leq j$, Proposition 3.2 and Lemma 3.1 implies that there exists a unique self-dual tableau $P \in Tab_2(2\ell)$ of shape λ whose first half tableau is P_1. Since the number of integers j such that $2t \leq j \leq \ell$ is $\ell - 2t + 1$, we have the first assertion.

Hence, the number of self-dual tableaux $P \in Tab_2(2\ell)$ whose first half tableau has shape $(\ell - t, t)$ is $(\ell - 2t + 1) f_{\ell - t, t}$, where $f_{\ell - t, t}$ is the total number of standard Young tableaux of shape $(\ell - t, t)$. It is an immediate consequence of the hook length formula (see example 1.23) that $f_{(\ell - t, t)}$ equals the ballot number

$$\frac{\ell - 2t + 1}{\ell - t + 1} \binom{\ell}{t}.$$

This gives the second assertion. $\square$

Now we enumerate self-dual tableaux $P \in Tab_2(n)$ with a given shape $(n - j, j)$.

Theorem 3.5. *The number f^*_λ of self-dual tableaux $P \in Tab_2(2\ell)$ of shape $\lambda = (2\ell - j, j)$ is*

$$f^*_\lambda = \binom{\ell}{\lfloor \frac{j}{2} \rfloor}.$$

Proof. Proposition 3.2 implies that every self-dual tableau of shape λ is uniquely determined by its first half tableau P_1. The only restriction over P_1 is that, if $shP_1 = (\ell - t, t)$, $j \geq 2t$. Hence:

$$f^*_\lambda = \sum_{0 \leq t \leq \lfloor \frac{j}{2} \rfloor} f_{(\ell - t, t)},$$

where $f_{(l-t,t)}$ is the number of standard Young tableaux of shape $(l - t, t)$.

Recalling that, as we mentioned above, $f_{(\ell - t, t)}$ equals the ballot number

$$\frac{\ell - 2t + 1}{\ell - t + 1} \binom{\ell}{t} = \binom{\ell}{\ell - t} - \binom{\ell}{\ell - t + 1},$$

we get:

$$f^*_\lambda = \sum_{0 \leq t \leq \lfloor \frac{j}{2} \rfloor} \left(\binom{\ell}{\ell - t} - \binom{\ell}{\ell - t + 1} \right) =$$

$$\sum_{0 \leq t \leq \lfloor \frac{j}{2} \rfloor} \left(\binom{\ell}{t} - \binom{\ell}{t - 1} \right) = \binom{\ell}{\lfloor \frac{j}{2} \rfloor},$$

where the last equality is due to the fact that the preceding sum is telescopic. $\square$

Note that, surprisingly, the sequence $f^*_{(n-j,j)}$, $0 \le j \le \frac{n}{2}$ is exactly the $\frac{n}{2} - th$ row of the Tartaglia triangle rearranged in increasing order.

3.1.3 Odd number of boxes

Now we examine the easier case of self-dual tableaux with at most two rows and an odd number n of boxes.

Let P be a standard Young tableau with $n = 2\ell+1$ boxes and with at most two rows. Then P can be decomposed as $P = P_1[\ell+1]P_2$, where $P_1 \in Tab(\ell)$ is the standard Young tableau containing the first ℓ symbols of P, P_2 is the standard skew tableau containing the last ℓ symbols of P and $[\ell+1]$ denotes the box of P containing the symbol $\ell+1$. The triple $(P_1, [\ell+1], P_2)$ will be called the *central decomposition* of P, and the tableau P_1 and P_2 will be called the *first half tableau* and the *second half tableau* of P, respectively.

As in the even case, we have a characterization of the self-dual tableaux.

Proposition 3.6. *Let $P \in Tab_2(2\ell+1)$ and let $(P_1, [\ell+1], P_2)$ be the central decomposition of P. Then P is self-dual if and only if the box $[\ell+1]$ is in the first row of P and P_2 is obtained from P_1 shifting the boxes of the first row to the right a given number of times and then normalizing the entries from $\ell+2$ to $2\ell+1$.*

Proof. Schützenberger Theorem 1.40 implies that the tableau P is self-dual if and only if it is the insertion tableau of a centrosymmetric permutation π.

Since P is a tableau with at most two rows and with $2\ell+1$ boxes, by the Schensted Theorem 1.35, π is a 321-avoiding permutation of length $2\ell+1$.

As we noted in subsection 1.4.1, a centrosymmetric permutation of odd length $2\ell+1$ fixes its central element $\ell+1$.

As a consequence, since π avoids the pattern 321, $\pi_1\pi_2...\pi_\ell$ is a permutation of the first ℓ positive integers and $\pi_{\ell+2}\pi_{\ell+3}...\pi_{2\ell+1}$ is a permutation of the integers from $\ell+2$ to $2\ell+1$. Moreover, by the definition of centrosymmetric permutation, we have $\pi_i = n+1-\pi_{n+1-i}$ for all $1 \le i \le \ell$ and hence

$$\pi_1\pi_2...\pi_\ell = |\pi_{\ell+2}\pi_{\ell+3}...\pi_{2\ell+1}|.$$

Now the result follows immediately from Lemma 1.43. $\square$

Example 3.7. Consider the self-dual tableau

$$P = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 2 & 9 \\ \cline{1-2} \end{array}.$$

Here

$$P_1 = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array}$$

and

$$P_2 = \begin{array}{ccc} & & \boxed{6}\boxed{7}\boxed{8} \\ \boxed{9} & & \end{array}.$$

P_2 is obtained from P_1 by moving the first row to the right and normalizing the entries.

The following theorem is a simple consequence of Proposition 3.6.

Theorem 3.8. *A self-dual tableaux $P \in Tab_2(2\ell+1)$ is uniquely determined by its first half tableau P_1. In particular, if $shP_1 = (\ell - t, t)$ then $shP = (2\ell + 1 - 2t, 2t)$. As a consequence, the number of self-dual tableaux $P \in Tab_2(2\ell+1)$ such that the first half tableau of P has shape $\lambda = (\ell - t, t)$ is*

$$f_\lambda = \frac{\ell - 2t + 1}{\ell - t + 1}\binom{\ell}{t}.$$

Now we enumerate self-dual tableaux $P \in Tab_2(2\ell+1)$ with a given shape $(2\ell + 1 - j, j)$.

Theorem 3.9. *The number f^*_λ of self-dual tableaux $P \in Tab_2(2\ell+1)$ of shape $\lambda = (2\ell+1-j, j)$ is*

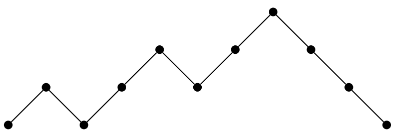
$$f^*_\lambda = \begin{cases} 0 & \textit{if } j \textit{ is odd} \\ f_{(\ell-\frac{j}{2},\frac{j}{2})} = \frac{\ell-j+1}{\ell-\frac{j}{2}+1}\binom{\ell}{\frac{j}{2}} & \textit{otherwise} \end{cases}.$$

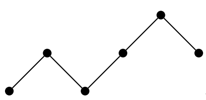
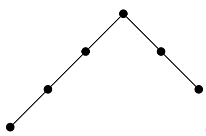
Proof. It is an immediate consequence of the previous theorem. □

3.2 An involution over Dyck paths

3.2.1 Decompositions of paths and prefixes

A Dyck path $d \in \mathcal{D}_n$ can be seen as the juxtaposition of two words q and $\overline{p}$, the first and the second half of d, where p, q are two Dyck prefixes in $\mathcal{P}_n$ whose ending vertices have the same height. $\overline{p}$ denotes as usual the symmetric of the word p. The pair (p, q) will be called the *middle vertex decomposition* of the Dyck path d.

Example 3.10. If $d =$ 
then

$p =$  , $q =$  .

Consider a Dyck prefix $p \in \mathcal{P}_n$; we say that an up step U in p is *closed* by a down step D if the prefix p can be decomposed as $p = p_1 U d D f$, where p_1 is a Dyck prefix, d is a Dyck path and f is an arbitrary word in the letters U and D. In other terms, if in the word corresponding to p we replace each up step by an open parenthesis and each down step by a closed parenthesis, then an up step U is closed by a down step D if the parenthesis corresponding to D closes the parenthesis corresponding to U.

It is immediately seen that every Dyck prefix p can be decomposed into a sequence of non-closed up steps and Dyck paths,

$$p = U^{k_1}\ d_1\ U^{k_2}\ d_2\ ...\ U^{k_r}\ d_r\ U^{k_{r+1}}$$

where $k_1 \geq 0$, $k_{r+1} \geq 0$, and $k_i \geq 1$ for $2 \leq i \leq r$ and d_i is a (non-empty) Dyck path for $1 \leq i \leq r$. We call this decomposition the *canonical decomposition* of p.

3.2.2 Definition of the involution Γ

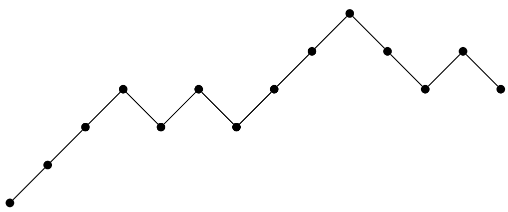
We define a map $* : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ as follows. Let $p \in \mathcal{P}_n$ with canonical decomposition

$$p = U^{k_1}\ d_1\ U^{k_2}\ d_2\ ...\ U^{k_r}\ d_r\ U^{k_{r+1}}.$$

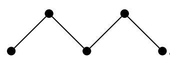
Then

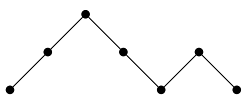
$$p^* := U^{k_{r+1}}\ \overline{d}_r\ U^{k_r}\ \overline{d}_{r-1}\ ...\ \overline{d}_1\ U^{k_1}.$$

Example 3.11. If

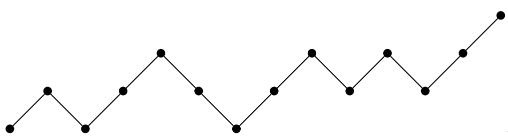
$p =$ 

then $p = U^2\ d_1\ U\ d_2$, where

$d_1 =$ ,

$d_2 =$ 

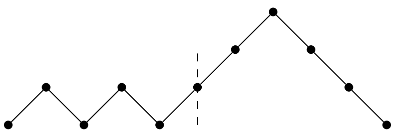
so we have

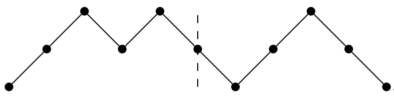
$p^* =$ .

The map $*$ defined above allows us to define a map $\Gamma : \mathcal{D}_n \longrightarrow \mathcal{D}_n$ which will be a basic tool in the following. Let d be a Dyck path of semilength n with middle vertex decomposition (p, q). Then

$$\Gamma(d) := q^* \overline{p}^*.$$

Note that the map Γ is well defined since the ending vertices of p^* and q^* have the same height.

Example 3.12. If $d =$ 

then $\Gamma(d) =$ .

A Dyck prefix fixed by $*$ and a Dyck path fixed by the map Γ will be said to be *self-dual*.

3.2.3 Properties of the involution Γ

The following results, which describe the main properties of the maps $*$ and Γ, are trivial consequences of the definitions.

Proposition 3.13.

- *The map* $* : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ *is an involution.*
- *A Dyck prefix* p *is self-dual whenever its canonical decomposition is*

$$p = U^{k_1}\ d_1\ U^{k_2}\ d_2\ \dots\ \overline{d}_2\ U^{k_2}\ \overline{d}_1\ U^{k_1}.$$

- *If the prefix* p *has a peak at position* i*, then* p^* *has a peak at position* $n - i$*.*

Corollary 3.14. *The statistics "length of the first sequence of non-closed up steps" and "length of the last sequence of non-closed up steps" are equidistributed over* $\mathcal{P}_n$*.*

Proposition 3.15.

- *The map* $\Gamma : \mathcal{D}_n \longrightarrow \mathcal{D}_n$ *is an involution.*
- *A Dyck path* d *is self-dual if and only if* $p = p^*$ *and* $q = q^*$*, where* (p, q) *is the middle vertex decomposition of* d*.*

3.3 The correspondence with the Schützenberger involution

3.3.1 Deflation procedure over Dyck paths

The map ψ described in Subsection 1.3.7 allows us to describe a link between the map $* : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ defined in Subsection 3.2.2 and the Schützenberger involution. More precisely, we will show that, for every $P \in Tab_2(n)$,

$$\phi(P^*) = \phi(P)^*.$$

As a consequence, if $(P, Q) \in SYP_2(n)$, then

$$\psi(P^*, Q^*) = \Gamma(\psi(P, Q)).$$

In order to prove this result, we need to translate the deflation procedure into a procedure over the set of Dyck prefixes.

Given $P \in Tab_2(n)$, we set $p := \phi_n(P)$ and $p^{\downarrow} := \phi_{n-1}(P^{\downarrow})$. Note that $p^{\downarrow} \in \mathcal{P}_{n-1}$.

Lemma 3.16. *Let $P \in Tab_2(n)$ and let $p = \phi(P)$. Then the prefix $p^{\downarrow}$ is obtained from p in the following way. Consider the first (up) step of p, U_1.*

(C1) *If U_1 is non-closed, delete U_1 from p.*

(C2) *If U_1 is closed by the down step $\widetilde{D}$, delete U_1 from p and replace $\widetilde{D}$ with an up step.*

Proof. Suppose that $P \in Tab_2(n)$ is the following tableau of shape $(l, n-l)$

$$P = \begin{array}{|c|c|c|c|c|c|c|} \hline a_1 & a_2 & \dots & \dots & \dots & \dots & a_l \\ \hline a_{l+1} & \dots & a_{n-1} & a_n \\ \cline{1-4} \end{array}.$$

Since P is standard, $a_1 < a_2 < \dots a_l$, $a_{l+1} < a_{l+2} < \dots a_n$ and $a_j < a_{l+j}$ for all $1 \leq j \leq n-l$. These inequalities imply that $a_{l+j} \geq 2j$ for all $1 \leq j \leq n-l$ and $a_{j+1} \leq 2j+1$.

Now we apply the deflation procedure to P. Suppose that, at the j-th step of this procedure, $1 \leq j \leq l-1$, the empty box is in the following position:

$$\begin{array}{c|c|c|c} \dots & X & a_{j+1} & \dots \\ \cline{2-3} \dots & a_{l+j} & \multicolumn{1}{c}{\dots} & \\ \cline{2-2} \end{array}.$$

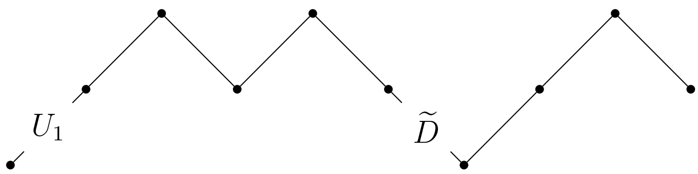
Then we must choose the smaller element between a_{j+1} and a_{l+j}. Since $2j \leq a_{l+j}$ and $a_{j+1} \leq 2j+1$, we have $a_{l+j} < a_{j+1}$ if and only if $a_{l+j} = 2j$ (and hence $a_{j+1} = 2j+1$).

Therefore, if there exists an index j such that $a_{l+j} = 2j$, during the deflation procedure the empty box moves to the second row, while it remains in the first row otherwise.

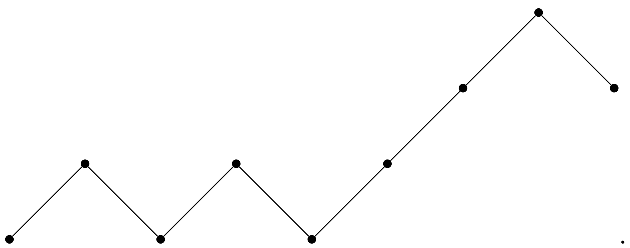
Now, let $\widehat{j}$ be the least integer (if any) such that $a_{l+\widehat{j}} = 2\widehat{j}$. By the definition of the map ϕ, it follows that the first up step U_1 of p is closed by a down step $\widetilde{D}$ if and only if $\widetilde{D}$ is at position $2\widehat{j}$ as a letter in the word p.

So, if such a $\widehat{j}$ exists, i.e., if U_1 is closed by some $\widetilde{D}$, we obtain $p^{\downarrow}$ from p by deleting U_1 and replacing $\widetilde{D}$ by an up step; if such a $\widehat{j}$ does not exist, i.e., if U_1 is not closed, we obtain $p^{\downarrow}$ from p by simply deleting U_1. $\square$

Example 3.17. Consider the tableau $P = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 7 & 8 \\ \hline 3 & 5 & 6 & 9 \\ \cline{1-4}\end{array}$. It can be easily checked that $P^{\downarrow} = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 5 & 6 & 7 \\ \hline 2 & 4 & 8 \\ \cline{1-3}\end{array}$. The prefixes p and $p^{\downarrow}$ are:

$p =$ 

and

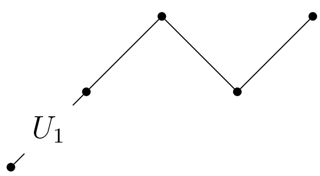
$p^{\downarrow} =$ 

Consider now the tableau $P = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$.

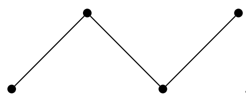
We have

$$P^{\downarrow} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

and the corresponding prefixes are

$p =$ 

and

$p^{\downarrow} =$  .

3.3.2 Correspondence theorem

We are now in position to prove the main result of this section.

Theorem 3.18. *Let $P \in Tab_2(n)$. Then*

$$\phi(P^*) = \phi(P)^*.$$

As a consequence, if $(P, Q) \in SYP_2(n)$, then

$$\psi(P^*, Q^*) = \Gamma(\psi(P, Q)).$$

Proof. Let $P \in Tab_2(n)$ and $p = \phi(P) \in \mathcal{P}_n$. Write p as

$$p = U^{k_1}\ d_1\ U^{k_2}\ d_2\ ...\ U^{k_r}\ d_r\ U^{k_{r+1}},$$

where $d_1, d_2, ..., d_r$ are Dyck paths.

We want to prove that

$$\phi(P^*) = p^* = U^{k_{r+1}}\ \overline{d}_r\ U^{k_r}\ \overline{d}_{r-1}\ ...\ \overline{d}_1\ U^{k_1}. \tag{3.2}$$

First of all we note that $\phi(P^*)$ can be obtained from p by the following procedure:

a) Set $p^{0\downarrow} := p$ and let $e_0 \in \mathcal{P}_0$ be the empty path.

b) Apply to p the deflation procedure described above, and set $e_1 := Ue_0 = U$ if the first up step of p is non-closed, $e_1 := De_0 = D$ otherwise.

c) For every $j = 1, ..., n-1$, set $p^{j\downarrow} = p^{\underbrace{\downarrow\downarrow \cdots \downarrow}_{j}}$, and set $e_{j+1} := Ue_j$ if the first up step of $p^{j\downarrow}$ is non-closed, $e_{j+1} := De_j$ otherwise.

d) Set $\phi(P^*) := e_n$.

In fact, it is sufficient to notice that the $(n-j+1)$-th box of P^* is created at the j-th application of the deflation procedure to P. This box will be in the first row whenever the $(n-j+1)$-th step of $\phi(P^*)$ is an up step and this happens if we are in the case **(C1)** of the previous lemma.

Now we prove that the identity (3.2) is true when P has a rectangular shape. In this case n must be even, $n = 2\ell$, say, and $p = \phi(P)$ is a Dyck path of semilength ℓ, whence $p^* = \overline{p}$.

We prove that in this case

$$\phi(P^*) = \overline{p}$$

by complete induction on ℓ.

If $\ell = 1$ the assertion is trivially true.

Suppose the assertion true for every Dyck path of semilength j, $1 \leq j \leq \ell - 1$, and consider a Dyck path d of semilength ℓ. Decompose d as $f_1 f_2 ... f_k$, where the f_i's are the irreducible components of d. Let ℓ_i be the semilength of the Dyck path f_i. Write $f_1 = U_1 g \widetilde{D}$, where g is a Dyck path. We now apply to d the above procedure. Since $\widetilde{D}$ closes U_1, at the first step, we get

$$d^{\downarrow} = gUf_2f_3...f_k, \qquad e_1 = D.$$

Since g is a Dyck path of semilength $< \ell$, by the induction hypothesis we obtain

$$d^{\ell_1 \downarrow} = f_2 f_3 ... f_k, \qquad e_{\ell_1} = \overline{f}_1.$$

Similarly,

$$d^{\ell_1 + l_2 \downarrow} = f_3 ... f_k, \qquad e_{\ell_1 + \ell_2} = \overline{f}_2 \overline{f}_1 = \overline{f_1 f_2}$$

$$...$$

$$d^{\ell \downarrow} = \emptyset, \qquad e_\ell = \overline{p}.$$

Now we turn to the general case. Let $P \in Tab_2(n)$ be a Young tableau and let $p = \phi(P) \in \mathcal{P}_n$ be the corresponding prefix; write p as

$$p = U^{k_1}\ d_1\ U^{k_2}\ d_2\ ...\ U^{k_r}\ d_r\ U^{k_{r+1}},$$

where d_i is a Dyck path of semilength ℓ_i.

The first $k_1 + 1$ steps of the procedure produce the following pairs of prefixes:

$$p^{0\downarrow} = p, \qquad e_0 = \emptyset,$$

$$p^{\downarrow} = U^{k_1 - 1}\ d_1\ U^{k_2}\ d_2\ ...\ U^{k_r}\ d_r\ U^{k_{r+1}}, \qquad e_{k_1} = U$$

$$...$$

$$p^{k_1 \downarrow} = d_1\ U^{k_2}\ d_2\ ...\ U^{k_r}\ d_r\ U^{k_{r+1}}, \qquad e_{k_1} = U^{k_1}.$$

At this point, the initial segment of $p^{k_1 \downarrow}$ is the Dyck path d_1, hence, by previous considerations, we get:

$$p^{k_1 + \ell_1 \downarrow} = U^{k_2}\ d_2\ ...\ U^{k_r}\ d_r\ U^{k_{r+1}}, \qquad e_{k_1} = \overline{d}_1\ U^{k_1}$$

$$\dots$$

$$p^{n\downarrow} = \emptyset, \qquad e_n = \phi(P^*) = U^{k_{r+1}}\ \overline{d}_r\ U^{k_r}\ \overline{d}_{r-1}\ \dots\ \overline{d}_1\ U^{k_1} = p^*.$$

□

Note that Theorem 3.18 gives a simple method to compute the image under the Schützenberger involution of a tableau $P \in Tab_2(n)$ when P has a rectangular shape.

Corollary 3.19. *Let $P \in Tab_2(2\ell)$ be a rectangular tableau:*

$$P = \begin{array}{|c|c|c|c|c|} \hline a_1 & a_2 & \dots & \dots & a_\ell \\ \hline a_{\ell+1} & \dots & \dots & \dots & a_{2\ell} \\ \hline \end{array}.$$

Then

$$P^* = \begin{array}{|c|c|c|c|c|} \hline b_1 & b_2 & \dots & \dots & b_\ell \\ \hline b_{\ell+1} & \dots & \dots & \dots & b_{2\ell} \\ \hline \end{array}$$

if and only if $b_j = 2\ell + 1 - a_{2\ell+1-j}$ for all $1 \le j \le 2\ell$. In particular $P = P^*$ if and only if $a_{2\ell-j+1} = 2\ell + 1 - a_j$ for all $1 \le j \le \ell$.

Proof. Consider $p = \phi(P)$. Since P has a rectangular shape, the Dyck prefix p is indeed a Dyck path, hence $p^* = \overline{p}$. This gives the assertion. □

3.4 Centrosymmetric permutations avoiding 321

3.4.1 Some bijections

Now we consider the set $\mathcal{S}_n^c(321)$ of centrosymmetric permutations avoiding 321.

First of all, we recall from the previous chapters that the composition $\psi \circ RS$

$$\mathcal{S}_n(321) \xrightarrow{RS} \mathrm{SYP}_2(n) \xrightarrow{\psi} \mathcal{D}_n$$

is a bijection between $\mathcal{S}_n(321)$ and $\mathcal{D}_n$.

The following corollary of the Schützenberger Theorem 1.40 describe what happens if we restrict the map $\psi \circ RS$ to the set of involutions that avoid 321 or to the set of centrosymmetric permutations that avoid 321.

Corollary 3.20. *We have the following bijections*

- $\mathcal{I}_n(321) \xleftrightarrow{RS} \{(P,P) \in SYP_2(n)\} \xleftrightarrow{\psi} \{d \in \mathcal{D}_n,\ d \text{ symmetric}\}$,

- $\mathcal{S}_n^c(321) \xleftrightarrow{RS} \{(P,Q) \in SYP_2(n) \text{ with } P = P^*,\ Q = Q^*\}$
$\xleftrightarrow{\psi} \{d \in \mathcal{D}_n,\ d \text{ self-dual}\}$,

- $\mathcal{I}_n^c(321) \xleftrightarrow{RS} \{(P,P) \in SYP_2(n) \text{ with } P = P^*\}$
$\xleftrightarrow{\psi} \{d \in \mathcal{D}_n,\ d \text{ symmetric and self-dual}\}$.

Furthermore, every involution $\pi \in \mathcal{I}_n(321)$ with k fixed points corresponds, via the map RS, to a pair of equal tableaux $(P,P) \in SYP_2(n)$ with $shP = (\lambda_1, \lambda_2)$ and $\lambda_1 - \lambda_2 = k$. Such a pair corresponds, via the map ψ, to a symmetric Dyck path d whose central vertex has height k.

Proof. It follows immediately from the Schützenberger Theorem 1.40 and from the Theorem 3.18. □

As a consequence of the previous result and of Theorems 3.5 and 3.9 we have the following Proposition.

Proposition 3.21. *The number of permutations $\pi \in \mathcal{I}_{2\ell}^c(321)$ with $2\ell - 2j$ fixed points is*

$$\binom{\ell}{\lfloor \frac{j}{2} \rfloor}.$$

It is also the number of self-dual symmetric Dyck paths in $\mathcal{D}_{2\ell}$ whose central vertex has height $2\ell - 2j$.

The number of permutations $\pi \in \mathcal{I}_{2\ell+1}^c(321)$ with $2\ell + 1 - 2j$ fixed points is

$$\begin{cases} 0 & \text{if } j \text{ is odd} \\ f_{(\ell - \frac{j}{2}, \frac{j}{2})} = \frac{\ell - j + 1}{\ell - \frac{j}{2} + 1}\binom{\ell}{\frac{j}{2}} & \text{otherwise} \end{cases}.$$

It is also the number of self-dual symmetric Dyck paths in $\mathcal{D}_{2\ell+1}$ whose central vertex has height $2\ell + 1 - 2j$.

As a corollary of Proposition 3.21 we reobtain the following well-known enumerative results (see [27]).

Corollary 3.22.

$$|\mathcal{I}_{2\ell}^c(321)| = 2^\ell,$$

$$|\mathcal{S}_{2\ell}^c(321)| = \binom{2\ell}{\ell},$$

$$|\mathcal{S}_{2\ell+1}^c(321)| = C_\ell, \text{ the } \ell\text{-th Catalan number,}$$

and

$$|\mathcal{I}_{2\ell+1}^c(321)| = \binom{\ell}{\lfloor \frac{\ell}{2} \rfloor}.$$

Proof. Proposition 3.21 yields the following equations.

- $$|\mathcal{I}^c_{2\ell}(321)| = \sum_{j=0}^{\ell} \binom{\ell}{\lfloor \frac{j}{2} \rfloor} = \sum_{j=0}^{\ell} \binom{\ell}{j} = 2^{\ell},$$

  where the last equality is an obvious consequence of the Newton's binomial formula.

- $$|\mathcal{S}^c_{2\ell}(321)| = \sum_{j=0}^{\ell} \binom{\ell}{\lfloor \frac{j}{2} \rfloor}^2 = \sum_{j=0}^{\ell} \binom{\ell}{j}^2 = \binom{2\ell}{\ell},$$

  where the last equality is a particular case of the *Vandermonde identity* (see e.g. [2]):

  $$\sum_j \binom{r}{m+j}\binom{s}{n-j} = \binom{r+s}{m+n}, \quad r, s, m, n \in \mathbb{Z} \tag{3.3}$$

  with $r = s = n = l$ and $m = 0$.

- $$|\mathcal{S}^c_{2\ell+1}(321)| = \sum_{j=\leq l,\ j \text{ even}} \frac{\ell - j + 1}{\ell - \frac{j}{2} + 1}\binom{\ell}{\frac{j}{2}} = \sum_{j=\leq l,\ j \text{ even}} \left(\binom{l}{\frac{j}{2}} - \binom{l}{\frac{j}{2}-1}\right)^2 =$$

  $$\sum_{t=0}^{\lfloor \frac{l}{2} \rfloor} \left(\binom{l}{t} - \binom{l}{t-1}\right)^2 = \sum_{t=0}^{\lfloor \frac{l}{2} \rfloor} \binom{l}{t}^2 + \binom{l}{t-1}^2 - 2\binom{l}{t}\binom{l}{t-1} =$$

  $$\sum_{t=0}^{\lfloor \frac{l}{2} \rfloor} \binom{l}{t}^2 + \sum_{t=0}^{\lfloor \frac{l}{2} \rfloor} \binom{l}{l-t+1}^2 - \sum_{t=0}^{\lfloor \frac{l}{2} \rfloor} \binom{l}{t}\binom{l}{t-1} - \sum_{t=0}^{\lfloor \frac{l}{2} \rfloor} \binom{l}{l-t}\binom{l}{l-t+1}. \tag{3.4}$$

  If l is even, we obtain:

  $$\sum_{t=0}^{l} \binom{l}{t}^2 - \sum_{t=0}^{l} \binom{l}{t}\binom{l}{t-1} = \binom{2l}{l} - \sum_{t=0}^{l} \binom{l}{t}\binom{l}{t-1} = \binom{2l}{l} - \binom{2l}{l-1},$$

  i.e., the l-th Catalan number C_l. The last equality is due to the Vandermonde identity (3.3) with $r = s = l$, $m = 0$ and $n = l + 1$.

  If l is odd, from (3.4), we obtain:

  $$\sum_{t=0}^{l} \binom{l}{t}^2 - \binom{l}{\frac{l+1}{2}}^2 - \sum_{t=0}^{\frac{l-1}{2}} \binom{l}{t}\binom{l}{t-1} - \sum_{t=0}^{\frac{l-1}{2}} \binom{l}{l-t}\binom{l}{l-t+1} =$$

$$\binom{2l}{l} - \binom{l}{\frac{l+1}{2}}\binom{l}{\frac{l-1}{2}} - \sum_{t=0}^{\frac{l-1}{2}} \binom{l}{t}\binom{l}{t-1} - \sum_{t=0}^{\frac{l-1}{2}} \binom{l}{l-t}\binom{l}{l-t+1} =$$

$$\binom{2l}{l} - \sum_{t=0}^{l} \binom{l}{t}\binom{l}{t-1} = \binom{2l}{l} - \binom{2l}{l-1} = C_l$$

as above. Hence:

$$|\mathcal{S}^c_{2\ell+1}(321)| = C_l.$$

Note that this result is a trivial consequence of the fact that, if $\pi = \pi_1\pi_2...\pi_{2\ell+1}$ is a centrosymmetric permutation of odd length, then $\pi_{\ell+1} = \ell+1$ and $\pi_1...\pi_\ell = |\pi_{\ell+2}...\pi_{2\ell+1}|$.

- $$|\mathcal{I}^c_{2\ell+1}(321)| = \sum_{j\leq l,\ j \text{ even}} \frac{\ell-j+1}{\ell-\frac{j}{2}+1}\binom{\ell}{\frac{j}{2}} = \sum_{j\leq l,\ j \text{ even}} \binom{l}{\frac{j}{2}} - \binom{l}{\frac{j}{2}-1}$$

  but this telescoping sum reduces to

  $$-\binom{l}{-1} + \binom{l}{\frac{\hat{j}}{2}} \text{ where } \hat{j} = \begin{cases} l & \text{if } l \text{ is even} \\ l-1 & \text{else} \end{cases}$$

  so the sum is equal to $\binom{l}{\lfloor\frac{l}{2}\rfloor}$.

□

3.4.2 Involutions avoiding 321

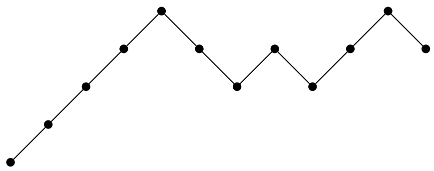
Now we restrict our attention to the set $\mathcal{I}_n(321)$. We want to construct the prefix corresponding to $\tau \in \mathcal{I}_n(321)$ via the map $\phi_n \circ RS$ in an easier way and, conversely, given a prefix $p \in \mathcal{P}_n$ we want to recover the corresponding τ.

It follows essentially from Proposition 3 of [31] that τ has i as a fixed point if and only if at the i-th step of p there is a non-closed up step U. In fact, the involution τ corresponds, via the map $\psi_n \circ RS$, to the symmetric Dyck path $d = p\bar{p}$ and the non-closed up steps of the first half of this path correspond to the *centered tunnels* considered in [31]. Moreover, if τ is an involution avoiding 321, the normalizations of its connected components of length greater than two are involutions without fixed points.

Hence, as a consequence of Lemma 1.43 and of Proposition 3 of [31], we have the following theorem.

Theorem 3.23. *Consider an involution $\tau \in \mathcal{I}_n(321)$ and let p be the corresponding prefix. Suppose that $\tau_{k_j}\tau_{k_j+1}...\tau_{k_j+h_j}$ with $h_j \geq 0$ and $1 = k_1 < k_2 < ... < k_t$, are the connected components of τ. If $h_j = 0$, the fixed point τ_{k_j} corresponds to a non-closed* *up steps at position k_j in p. If $h_j > 0$, the connected component $\tau_{k_j}\tau_{k_j+1}...\tau_{k_j+h_j}$ of τ corresponds to an irreducible component in positions $k_j, ..., k_j + h_j$ of a Dyck path d in the canonical decomposition of p. Moreover, this irreducible component is given by the path $\phi_{h_j} \circ RS(|\tau_{k_j}\tau_{k_j+1}...\tau_{k_j+h_j}|)$.*

Example 3.24. The prefix

$p =$  $= U^2 d_1 U d_2$

corresponds to the involution $\pi = 1\ 2\ 5\ 6\ 3\ 4\ 8\ 7\ 9\ 11\ 10$. Here the fixed points are 1,2 and 9 that correspond to the non-closed up steps of p. The others connected components of π are 5 6 3 4, 8 7 and 11 10. The normalizations of this connected components are 3412, 21 and 21. The first two of these involutions correspond to the irreducible components of d_1, the last corresponds to the (irreducible) path d_2.

3.4.3 A map between $\mathcal{I}_{2\ell}(321)$ and $\mathcal{S}^c_{2\ell}(321)$

We recall that the central binomial coefficient $\binom{2\ell}{\ell}$ is the cardinality of the set $\mathcal{I}_{2\ell}(321)$ (see [35], [54]).

In fact, Corollary 3.20 implies that, in general, there exists a bijection between the set of 321-avoiding involutions of length t and the set of symmetric Dyck paths of semilength t (or, equivalently, Dyck prefixes of length t). As noted in subsection 1.2.2, the cardinality of this last set is the central binomial coefficient:

$$|\mathcal{I}_n(321)| = \binom{t}{\lfloor \frac{t}{2} \rfloor}.$$

As seen in Subsection 3.4.1, $\binom{2\ell}{\ell}$ is also the cardinality of the set $\mathcal{S}^c_{2\ell}(321)$.

Our next goal is to describe a bijection between the sets $\mathcal{S}^c_{2\ell}(321)$ and $\mathcal{I}_{2\ell}(321)$.

Given a permutation $\pi \in \mathcal{S}^c_{2\ell}(321)$, set $RS(\pi) = (P, Q)$. By Theorem 1.40 we have $P = P^*$ and $Q = Q^*$. Consider the decompositions $P = P_1P_2$ and $Q = Q_1Q_2$ with $shP_1 = (\ell - t, t)$, $(t \leq \lfloor \frac{\ell}{2} \rfloor)$, and $shQ_1 = (\ell - s, s)$, $(s \leq \lfloor \frac{\ell}{2} \rfloor)$.

- If $t \leq s$, let $\widehat{P} = P_1\widehat{Q_2}$ be the tableau whose first (second) row is obtained by juxtaposing the first (second) row of P_1 and Q_2.
- If $t > s$, let $\widehat{Q} = Q_1\widehat{P_2}$ the tableau obtained from Q_1 and P_2 in the same way.

We define a map

$$\Upsilon : \mathcal{S}^c_{2\ell}(321) \longrightarrow \mathcal{I}_{2\ell}(321)$$

as follows:

$$\Upsilon(\pi) := \begin{cases} RS^{-1}(\widehat{P}, \widehat{P}) & \text{if } t \leq s \\ RS^{-1}((\widehat{Q})^*, (\widehat{Q})^*) & \text{if } t > s \end{cases}.$$

Note that the first half tableau of both $\widehat{P}$ and $\widehat{Q}^*$ is P_1. In fact, if $t > s$, $\widehat{Q}^* = (Q_1\widehat{P_2})^* = Rect(\widehat{P_2})^* S = Rect(P_2)^* S = P_1 S$, where S is a skew tableau such that $shQ_1\widehat{P_2} = shP_1S$ and, since $*$ is an involution, $Rect(S)^* = Q_1$.

Example 3.25. Consider the permutation $\pi = 142536 \in \mathcal{S}^c_6(321)$. Then

$$RS(\pi) = \left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 6 \\ \hline 3 & 5 \\ \cline{1-2} \end{array} \right)$$

so we have

$$\Upsilon(\pi) = RS^{-1} \left(\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 6 \\ \hline 5 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 6 \\ \hline 5 \\ \cline{1-1} \end{array} \right) = 123546 \in \mathcal{I}_6(321).$$

Consider now the permutation $\pi = 246135 \in \mathcal{S}^c_6(321)$.

$$RS(\pi) = \left(\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array} \right)$$

so we have

$$\Upsilon(\pi) = RS^{-1} \left(\left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 \\ \cline{1-2} \end{array} \right)^*, \left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 \\ \cline{1-2} \end{array} \right)^* \right) = 214356 \in \mathcal{I}_6(321).$$

Theorem 3.26. *The map $\Upsilon : \mathcal{S}^c_{2\ell}(321) \longrightarrow \mathcal{I}_{2\ell}(321)$ is a bijection and*

$$\Upsilon(\pi)^{rc} = \Upsilon(\pi^{-1}).$$

Moreover, if $\pi \in \mathcal{I}^c_{2\ell}(321)$, then $\Upsilon(\pi) = \pi$.

Proof. As we have already proved, the sets $\mathcal{S}^c_{2\ell}(321)$ and $\mathcal{I}_{2\ell}(321)$ have the same cardinality, hence, it is sufficient to prove that Υ is surjective. Let σ be an involution in $\mathcal{I}_{2\ell}(321)$ and T be the corresponding tableau. Write T as $T = T_1T_2$. We want to find a pair of self-dual tableaux $(P, Q) \in \mathrm{SYP}_2(n)$ such that $\Upsilon(RS^{-1}(P, Q)) = \sigma$. By previous considerations we deduce that the first half tableaux P_1, Q_1 of P, Q, respectively, must be

$$P_1 = T_1 \quad \text{and} \quad Q_1 = Rect(T_2)^*.$$

Set $shP_1 = (\ell - t, t)$, $shQ_1 = (\ell - s, s)$ and suppose that the second row of T_2 contains j boxes. Since $Q_1 := Rect(T_2)^*$, $j \geq s$.

- If $t \leq s$, by Proposition 3.2, there exists a self-dual tableau Q such that $Q = Q_1Q_2$, with $\widehat{Q_2} = T_2$ (i.e., T_2 is obtained from Q_2 by sliding its first row to the right).

  Then, we can find a self-dual tableau P with the same shape of Q and whose first half tableau equals P_1. So, in this case, $T = P_1\widehat{Q_2}$.

- If $t > s$, $T^* = Q_1S'$ where $Rect(S')^* = T_1 = P_1$. As in the previous case, we can find a self-dual tableau $P = P_1P_2$ with $\widehat{P_2} = S'$. So, in this case, $T^* = Q_1\widehat{P_2}$, hence $T = (Q_1\widehat{P_2})^*$.

This proves the surjectivity of Υ.

Now we prove that $\Upsilon(\pi)^{rc} = \Upsilon(\pi^{-1})$. As above, suppose that $RS(\pi) = (P, Q)$, with $P = P_1P_2$, $Q = Q_1Q_2$ and $shP_1 = (\ell - t, t)$, $shQ_1 = (\ell - s, s)$. Note that, if $t = s$, $P_1\widehat{Q_2} = (Q_1\widehat{P_2})^*$. Then, by Theorem 1.40, we have

$$\Upsilon(\pi^{-1}) = \begin{cases} RS^{-1}(Q_1\widehat{P_2}, Q_1\widehat{P_2}) & \text{if } s \leq t \\ RS^{-1}((P_1\widehat{Q_2})^*, (P_1\widehat{Q_2})^*) & \text{if } s > t \end{cases} = \Upsilon(\pi)^{rc}.$$

The fact that $\Upsilon(\pi) = \pi$ if $\pi \in \mathcal{I}^c_{2\ell}$ follows from the construction of the map Υ. □

Finally we show how the bijection Υ allows us to give a link between the statistic "number of descents" over the sets $\mathcal{S}^c_n(321)$ and $\mathcal{I}_n(321)$.

Theorem 3.27. *For every* $\pi \in \mathcal{S}^c_{2\ell}(321)$ *we have*

$$des(\Upsilon(\pi)) = \left\lfloor \frac{des(\pi) + des(\pi^{-1})}{2} \right\rfloor.$$

Proof. First of all, denote by $Des(T)$ the set of descents of a standard Young tableau T and by $des(T)$ its cardinality. Similarly, denote by $Peak(p)$ the set of positions of the peak of the prefix (or of the path) p and by $peak(p)$ its cardinality.

Now, consider a permutation $\pi \in \mathcal{S}_n(321)$. By Proposition 1.34, a descent of π corresponds, via the map RS, to a descent of its recording tableau Q.

Let (p, q) be the middle vertex decomposition of the Dyck path $(\psi \circ RS)(\pi)$. Then, by Theorem 1.24, $i \in Des(\pi)$ whenever the prefix q has a peak at position i.

Since $RS(\pi) = (P, Q)$ if and only if $RS(\pi^{-1}) = (Q, P)$, a descent i of π^{-1} corresponds to a descent i of the tableau P and to a peak at position i of the prefix p.

From the above considerations, it follows that

$$des(\pi) = des(Q) = peak(q)$$

and

$$des(\pi^{-1}) = des(P) = peak(p).$$

Now, suppose that $\pi \in \mathcal{S}^c_{2\ell}(321)$. Using the same notation as above, suppose that $P = P_1P_2$ and $Q = Q_1Q_2$ and set $p = p_1p_2$ and $q = q_1q_2$, where p_1 and p_2 (q_1 and q_2) are the first and the second half of the Dyck prefix p (q, respectively).

Proposition 3.13 implies that

$$peak(p_1) = peak(p_2)$$

and

$$peak(q_1) = peak(q_2).$$

Suppose that p and q have j down steps and that p_1 and q_1 have t and s down steps, respectively. This implies that

$$shP = shQ = (2\ell - j, j), \quad shP_1 = (\ell - t, t), \quad shQ_1 = (\ell - s, s).$$

- If $t \leq s$, the juxtaposition p_1q_2 is a Dyck prefix, and

  $$des(\Upsilon(\pi)) = des(P_1\widehat{Q_2}) = peak(p_1q_2) = peak(p_1) + peak(q_2) + \epsilon$$

  where

  $$\epsilon = \begin{cases} 1 & \text{if } l \in Peak(p_1q_2) \\ 0 & \text{otherwise} \end{cases}.$$

  Consider the canonical decompositions of the self-dual prefixes p and q:

  $$p = U^{a_1}\ d_1\ U^{a_2}\ ...\ U^{a_2}\ \overline{d}_1\ U^{a_1},$$

  $$q = U^{b_1}\ d_1\ U^{b_2}\ ...\ U^{b_2}\ \overline{d}_1\ U^{b_1}.$$

  If $\epsilon = 1$, p_1 ends with an up step and q_2 starts with a down step. Since q is self-dual, q_1 must end with an up step. Hence, the path q has a central peak. Suppose that p_2 starts with an up step:

  $$p = \underbrace{U^{a_1}\ d_1\ U^{a_2}\ ...\ d_{k-1}\ U^{a_k}}_{p_1}\underbrace{U^{a_k}\ \overline{d}_{k-1}\ ...\ \overline{d}_1\ U^{a_1}}_{p_2},$$

$$q = \underbrace{U^{b_1}\ d_1\ U^{b_2}\ ...\ U^{b_r}\ d_r\ U^m}_{q_1}\underbrace{D^m\ \overline{d}_r\ U^{b_r}\ ...\ \overline{d}_1\ U^{b_1}}_{q_2}$$

where $m > 0$ is the length of the last sequence of consecutive closed up steps of q_1 (since q has a central peak, q_1 ends with a sequence of closed up steps).

Now, $t \leq s$ if and only if

$$\ell - 2t \geq \ell - 2s$$

where $\ell - 2t$ is the height of the ending vertex of p_1 and $\ell - 2s$ is the height of the ending vertex of q_1:

$$\ell - 2t = \sum_{i=1}^{k} a_i \quad \text{and} \quad \ell - 2s = \sum_{i=1}^{r} b_i + m$$

hence

$$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{r} b_i + m.$$

Moreover, the ending vertices of p and q must have the same height:

$$2\sum_{i=1}^{k} a_i = 2\sum_{i=1}^{r} b_i.$$

These equations imply $0 \geq m$, hence we get a contradiction. So we can conclude that also p_2 starts with a down step and, hence, p also has a central peak.

It follows that

$$\epsilon = \begin{cases} 1 & \text{if } \ell \in Peak(p) \cap Peak(q) \\ 0 & \text{otherwise} \end{cases}.$$

By the previous considerations we obtain

$$des(\Upsilon(\pi)) = des(P_1\widehat{Q_2}) = peak(p_1 q_2) = peak(p_1) + peak(q_2) + \epsilon =$$

$$\left\lfloor \frac{peak(p) + peak(q)}{2} \right\rfloor = \left\lfloor \frac{des(\pi) + des(\pi^{-1})}{2} \right\rfloor.$$

- If $t > s$, the path $q_1 p_2$ is a Dyck prefix, and

$$des(\Upsilon(\pi)) = des((Q_1 P_2)^*) = peak((q_1 p_2)^*)$$

but, by Proposition 3.13, this cardinality equals $peak(q_1 p_2)$. Reasoning as above, we get the assertion.

□

3.4.4 The statistic "number of fixed points" over $\mathcal{S}_n^c(321)$

In Proposition 3.21 we found the number of centrosymmetric involutions of length N and with K fixed points, for every pair of integers N and K. Now, it is natural to ask what is the number $m_{N,K}$ of centrosymmetric permutations of length N and with K fixed points. This number does not follow from the above considerations.

First of all, we note that the previous problem is trivial if N is odd, $N = 2n+1$. In fact, as already mentioned, in this case every permutation $\pi \in \mathcal{S}_N^c(321)$ can be written in the form $\pi = \pi_1 \ldots \pi_n\, n+1\, \pi_{n+2} \ldots \pi_{2n+1}$ where $\pi_1 \ldots \pi_n$ is a permutation in $\mathcal{S}_n(321)$ and $|\pi_{n+2} \ldots \pi_{2n+1}| = \pi_1 \ldots \pi_n$. So the number $m_{2n+1,K}$ is equal to zero if K is even and is equal to the number of permutation in $\mathcal{S}_n(321)$ with k fixed points if $K = 2k+1$. The statistic "number of fixed points" over sets of permutations avoiding a pattern of length 3 is well studied (see e.g. [48] and [29]). In particular, it is known (see [48]) that the number of 321-avoiding permutations with k fixed points and of length n, and hence $m_{2n+1,2k+1}$, is equal to

$$\sum_{j=0}^{n-k}(-1)^j \frac{j+k+1}{n+1}\binom{2n-k-j}{n}\binom{j+k}{k}.$$

Similarly, by the definition of centrosymmetric permutation it follows that if N is even and K is odd then $m_{N,K} = 0$. In fact every permutation $\pi \in \mathcal{S}_N^c(321)$ is of the form $\pi = \pi_1 \ldots \pi_{N/2}\, N+1-\pi_{N/2} \ldots N+1-\pi_1$.

To calculate the number $m_{N,K}$ when both N and K are even, say $N = 2n$ and $K = 2k$, we use a bijection Ω between the set $\mathcal{S}_{2n}^c(321)$ and $\mathcal{P}_{2n}$, the set of Dyck prefixes of length $2n$. This bijection is based on a bijection $\Phi : \mathcal{S}_{2n}^c(123) \to \mathcal{P}_{2n}$ introduced in [8] with the aim of studing the statistic "number of descents" over $\mathcal{S}_{2n}^c(123)$. We define $\Omega : \mathcal{S}_{2n}^c(123) \to \mathcal{P}_{2n}$ as

$$\Omega(\pi) := \Phi(\pi^r),$$

where π^r denotes the reverse of the permutation π.

Now we recall the definition of the map Φ. Every permutation $\pi \in \mathcal{S}_{2n}^c$ is completely determined by its first n values, namely by the word

$$w(\pi) := \pi_1\pi_2 \ldots \pi_n.$$

This word can be written as $w(\pi) = x_1w_1x_2w_2 \ldots x_tw_t$ where the integers x_i are the left-to-right minima of π appearing within the first n positions and the $w_j{}'s$ are (possbily empty) words. Denote by l_i the length of w_i. Define a family of alphabets as follows.

- Let A_0 be the alphabet $\{1, 2, \ldots, 2n\}$.
- For $i > 0$, let A_i be obtained from A_{i-1} by removing the letters x_i, the complement of x_i to $2n+1$, namely $2n+1-x_i$, the letters of the word w_i and all the complements to $2n+1$ of these letters.

In this way, the permutations $\pi \in \mathcal{S}_{2n}^c(123)$ can be recursively characterized as the permutations π of $\mathcal{S}_{2n}^c$ such that

$$w(\pi) = x_1 w_1 w(\pi'),$$

where

- $x_1 \geq n$,
- w_1 is empty or $w_1 = 2n\,2n-1\,2n-2\ldots 2n-l_1+1$, with $2n-l_1+1 > x_1$,
- π' is a permutation over the set A_1 and $|\pi'| \in \mathcal{S}_{2n-2(l_1+1)}^c(123)$,
- the first entry in π' is less than x_1.

Example 3.28. The permutation $\pi = 11\,16\,15\,9\,7\,14\,13\,12\,5\,4\,3\,10\,8\,2\,1\,6$ is in $\mathcal{S}_{16}^c(123)$. In fact $w(\pi) = 11\,16\,15\,9\,7\,14\,13\,12$ where $x_1 = 11$, $w_1 = 16\,15$, $x_2 = 9$, $w_2 = \emptyset$, $x_3 = 7$, $w_3 = 14\,13\,12$. Here we have that π' is the unique permutation over $A_1 = \{3,4,5,7,8,9,10,12,13,14\}$ whose normalization is $6\,4\,10\,9\,8\,3\,2\,1\,7\,5 \in \mathcal{S}_{10}^c(123)$.

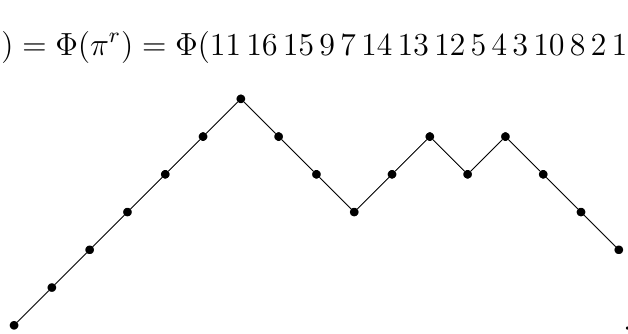
Now, it is possible to define recursively the map $\Phi : \mathcal{S}_{2n}^c(123) \to \mathcal{P}_{2n}$ as follows. Let $\pi \in \mathcal{S}_{2n}^c(123)$ and let $w(\pi) = x_1 w_1 w(\pi')$ be its associated word. Let, as above, l_1 be the the length of w_1. Define the image of π under the map Φ as

$$\Phi(\pi) = \begin{cases} U^{2n+1-x_1} D^{l_1+1} \overline{\Phi(\pi')} \text{ where } \overline{\Phi(\pi')} \text{ is obtained from } \Phi(\pi') \text{ by deleting} \\ \text{the leftmost } 2n - x_1 - l_1 \text{ steps, if } x_1 > n, \\ \\ U^{n+1} D l_1 \widehat{\Phi(\pi')} \text{ where } \widehat{\Phi(\pi')} \text{ is obtained from } \Phi(\pi') \text{ by deleting} \\ \text{the leftmost } n - l_1 - 1 \text{ steps, if } x_1 = n. \end{cases}$$

It easy to check that $\Phi(\pi) \in \mathcal{P}_{2n}$ and, in [8], the authors prove that Φ is invertible, hence a bijection. As a consequence, $\Omega := \Phi \circ (\cdot)^r$ is a bijection between $\mathcal{S}_{2n}^c(321)$ and $\mathcal{P}_{2n}$.

Example 3.29. Consider the permutation $\pi = 6\,1\,2\,8\,10\,3\,4\,5\,12\,13\,14\,7\,9\,15\,16\,11$. Then

$$\Omega(\pi) = \Phi(\pi^r) = \Phi(11\,16\,15\,9\,7\,14\,13\,12\,5\,4\,3\,10\,8\,2\,1\,6) =$$



.

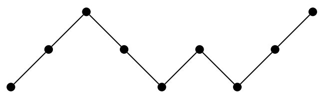
One of the main properties of the bijection Φ (see [8]) is the following. If we consider the *right connected components* of the permutation $\pi \in \mathcal{S}_{2n}(123)$ (i.e., by definition, the reverses of the connected components of π^r) then such a component in the first half of π correspond to an irreducible component of the path $\Phi(\pi)$. In particular the number of right connected components of $\pi \in \mathcal{S}^c_{2n}(123)$ is equal to

$$\begin{cases} 2ret(\Phi(\pi)) & \text{if } \Phi(\pi) \text{ is a Dyck path} \\ 2ret(\Phi(\pi)) + 1 & \text{otherwise} \end{cases}$$

where $ret(\Phi(\pi))$ is the number of returns of the prefix $\Phi(\pi)$.

A fact that is implicit in the prove of this result, is that the right connected components of $\pi \in \mathcal{S}_{2n}(123)$ composed by only one element and located in the first half of π correspond to irreducible paths of height one, i.e. peaks at height one, of $\Phi(\pi)$. Equivalently, the fixed points of the first half of the permutation $\pi \in \mathcal{S}_{2n}(321)$ correspond to peaks of height one in $\Omega(\pi)$. As a consequence we have that the number $m_{2n,2k}$ is equal to the number of Dyck prefixes of length $2n$ with k peak at height one.

Example 3.30. Consider the permutation $\pi = 21354687 \in \mathcal{S}^c_8(321)$. Then $\pi^r = 78645312 \in \mathcal{S}^c_8(123)$ and the connected components of π are 21, 3, 54, 6, and 87. In particular the fixed points are 3 and 6.

$\Omega(\pi) =$ 

has a peak of height one that corresponds to the fixed point 3.

Using this description, it easy to calculate the number $m_{2n,2k}$. In fact, a Dyck prefix p has k peak at height one if and only if the prefix obtained from p by removing each of these peaks is formed by a given number j of irreducible Dyck paths of length greater than two eventually followed by a Dyck prefix with no returns. Hence we have,

$$m_{2n,2k} = \sum_{j \geq 0} \left(\sum_{\substack{2r_1+2r_2+\ldots+2r_j+t=2n-2k,\\ r_i \geq 2\, \forall i,\ t \geq 0}} \binom{(j+1)+k-1}{k} C_{r_1-1} C_{r_2-1} \cdots C_{r_j-1} \binom{t-1}{\lfloor \frac{t-1}{2} \rfloor} \right),$$

where C_n is the n-th Catalan number and where $\binom{t-1}{\lfloor \frac{t-1}{2} \rfloor}$ is intended to be equal to 1 if $t = 0$.

The previous expression can be rewritten in another form noting that Deutsch ([24]) found a formula for the number of Dyck paths of semilength r and with k peaks at height one,

$$a_{r,k} := \begin{cases} \sum_{h=0}^{\lfloor \frac{r-k}{2} \rfloor} \frac{h}{r-k-h} \binom{k+h}{h} \binom{2r-2k-2h}{r-k} & \text{if } k < r \\ 1 & \text{if } k = r \\ 0 & \text{otherwise} \end{cases}$$

(for the generating function of Dyck paths counted by number of peaks at height one see also [47]).

Using the previous formula we have

$$m_{2n,2k} = \sum_{\substack{2r+t=2n,\\ r\geq 0, t\geq 0}} a_{r,k} \binom{t-1}{\lfloor \frac{t-1}{2} \rfloor}$$

where, as above, $\binom{t-1}{\lfloor \frac{t-1}{2} \rfloor}$ is intended to be equal to 1 if $t = 0$

We summarize these results in the following proposition.

Proposition 3.31. *The number $m_{N,K}$ of permutations $\pi \in \mathcal{S}_N^c(321)$ with K fixed points is given by*

$$\begin{cases} 0 & \text{if } N \not\equiv K mod 2 \\ \displaystyle\sum_{j=0}^{n-k} (-1)^j \frac{j+k+1}{n+1} \binom{2n-k-j}{n} \binom{j+k}{k} & \text{if } N = 2n+1,\ K = 2k+1, \\ \displaystyle\sum_{\substack{2r+t=2n,\\ r\geq 0, t\geq 0}} a_{r,k} \binom{t-1}{\lfloor \frac{t-1}{2} \rfloor} & \text{if } N = 2n,\ K = 2k. \end{cases}$$

Observation 3.32. In this chapter we have studied the distribution of the statistic "number of fixed points" over sets of involutions and centrosymmetric permutations. If π is an involution or a centrosymmetric permutation, the number of excedences of π is equal to the number of deficiencies: if π is an involution, π_i is an excedence if and only if i is a deficiency; if π is centrosymmetric, i is and excedence if and only if $n + 1 - i$ is a deficiency. Therefore, in these cases, the joint distribution of "number of excedances," "number of deficiencies" and "number of fixed points" is trivially recovered once we know the second: if π is an involution or a centrosymmetric permutation of length n, π has k excedences (or deficiencies) if and only if it has $n - 2k$ fixed points.

Chapter 4

The reverse and the complement over the set of Dyck words

4.1 Dyck words

4.1.1 Dyck words and their parameters

In this Chapter, we will consider words on the alphabet with two letters $A = \{U, D\}$. For a word w, we denote $|w|_x$ (where $x \in \{U, D\}$) the number of occurrences of the letter x in w. Hence the *length* of w, denoted $|w|$, is equal to $|w|_U + |w|_D$. We use also the mapping δ associating to any word w the integer $\delta(w) = |w|_U - |w|_D$.

The word u is a *prefix* of w if $w = uv$. This prefix is *strict* if $u \neq w$.

We will use the following notation. For any word $w = w_1 w_2 \cdots w_m$ where $w_i \in A$ we denote by $\widetilde{w}$ the mirror image of w, that is, the word obtained by reading w from right to left, hence $\widetilde{w} = w_m w_{m-1} \cdots w_1$. Moreover, we denote by $\widehat{w}$ the word obtained from w by replacing any occurrence of b by an occurrence of U and vice versa, giving $\widehat{w} = \overline{w}_1 \overline{w}_2 \cdots \overline{w}_m$, where, as above, $\overline{D} = U$ and $\overline{U} = D$. In the notation of the previous chapters, we have $\widehat{w} = \overline{\widetilde{w}} = \widetilde{\overline{w}}$.

As we recalled in Chapter 1, it is possible to identify a Dyck path with a word in the letters U and D. The up steps of the path are represented by the letter U, and the down steps, are represented by the letter D. Paths are better for the visual intuition and words are better for writing proofs, but of course they represent in different ways the same combinatorial object. For this reason, almost all the examples of the chapter will be given in the language of Dyck paths, while proofs are mostly written with Dyck words.

A word that correspond to a Dyck path is said to be a *Dyck word*. In fact a Dyck word is often defined as a word w such that $\delta(w) = 0$ and $\delta(u) \geq 0$ for any prefix u of w.

In this chapter we consider a slight modification of the definition of a Dyck word adding a letter D at the end of it. We define for each non negative integer n the set $\mathcal{DW}_n$ of Dyck words of length $2n+1$ first by considering A_n to be the set of words on the alphabet A having n occurrences of the letter U and $n+1$ occurrences of the letter D. Then the set $\mathcal{DW}_n$ is the subset of words w in A_n satisfying

$$\delta(w') \geq 0 \text{ for any strict prefix } w' \text{ of } w. \tag{4.1}$$

Notice that $\mathcal{DW}_0 = \{D\}$ and $\mathcal{DW}_1 = \{UDD\}$.

Many parameters are defined for Dyck words. We recall here the definition of those we use in this chapter. As we said in the first chapter, a *peak* of a Dyck word is an occurrence of the letter U followed by an occurrence of D – or, in the Dyck path language, the vertex between an up step and a down step. Any word in $\mathcal{DW}_n$ for $n \neq 0$ has at least one peak and at most n peaks.

The *height* of a Dyck word w is the maximum value of $\delta(u)$, as u ranges over all prefixes of w.

4.1.2 The Cyclic Lemma

In the first chapter we found that number of Dyck path of semilength n is given by the $n-$th Catalan number using the ballot theorem. As a trivial consequence, we have that also the cardinality of $\mathcal{DW}_n$ is given by this number:

$$|\mathcal{DW}_n| = C_n = \frac{(2n)!}{n!(n+1)!}. \tag{4.2}$$

Another elegant way to obtain the above formula is to use the so-called *Cyclic Lemma* (see. e.g. [26]) which will be used very often in this chapter. This Lemma considers the set A_n and uses the operation of conjugation. Two words w and w' on an alphabet A are *conjugate* if there exist u and v such that $w = uv$ and $w' = vu$. In other terms, if you write a word w in counterclockwise order, wrapped around a circle, the conjugates of w are then all the words obtained by starting reading at any letter in counterclockwise order and making a full circle.

Lemma 4.1. ***(Cyclic Lemma)*** *Any word w of A_n has exactly one decomposition into two factors $w = uv$ such that vu is an element of $\mathcal{DW}_n$. Moreover the decomposition into two factors $w = uv$ is such that u is the smallest prefix of w attaining the minimal value for $\delta(u)$.*

It is well-known that the number of different conjugates of a word w divides its length and that if a word uv is equal to one of its conjugates vu, where $u, v \neq \emptyset$, then there exist a word q and an integer $k > 1$ such that $uv = q^k$ (see [46, Paragraph 1.3]). Since the words in A_n contain n occurrences of the letter U and $n+1$ occurrences of D, it follows that an element in A_n has $2n+1$ different conjugates. Clearly the number of elements of A_n is $\binom{2n+1}{n}$. This proves Formula (4.2).

4.2 Involutions for words of $\mathcal{DW}_n$

4.2.1 The involution related to the Baker Norine theorem on graphs

We define an involution α on Dyck words in $\mathcal{DW}_n$ by setting $\alpha(w)$ to be the unique conjugate of $\widetilde{w}$ which belongs to $\mathcal{DW}_n$. Recall that $\widetilde{w}$ denotes the mirror image of w, that is, if $w = w_1 w_2 \cdots w_{2n+1}$, then $\widetilde{w} = w_{2n+1} w_{2n} \cdots w_2 w_1$.

Observe that this map was introduced in [21] in order to compute the rank of configurations in complete graphs where it was proved that it keeps invariant the *dinv* parameter (extensively studied in [37]).

Example 4.2. Let $w = UUDUUDUDDDUDD$ then $\widetilde{w} = DDUDDDUDUUDUU$. This decomposes as

$$\widetilde{w} = (DDUDDD)(UDUUDUU)$$

and the conjugate

$$(UDUUDUU)(DDUDDD)$$

is in $\mathcal{DW}_n$, hence

$$\alpha(UUDUUDUDDDUDD) = UDUUDUUDDUDDD.$$

Proposition 4.3. *The mapping α is an involution.*

Proof. Let w be a word in $\mathcal{DW}_n$ and set $v = \alpha(w)$. Consider the word $\widetilde{w}$ and decompose it as $\widetilde{w} = w'w''$, where w' is the shortest prefix of w such that $\delta(w')$ is minimal. Then we have $v = w''w'$. To compute $\alpha(v)$ we consider $\widetilde{v}$ which is equal to $\widetilde{w'}\widetilde{w''}$. But since $\widetilde{w} = w'w''$ we have $w = \widetilde{w''}\widetilde{w'}$ showing that w is a conjugate of $\widetilde{v}$ which is in $\mathcal{DW}_n$. The unicity of such a conjugate given by the cyclic lemma implies $\alpha(v) = w$. $\square$

In order to determine the number of fixed points of α we need the following characterization.

Proposition 4.4. *The Dyck word w satisfies $\alpha(w) = w$ if and only if w is the concatenation of two palindromes, namely,*

$$w = v\,u, \quad \textit{with} \quad v = \widetilde{v}, \quad u = \widetilde{u}.$$

Moreover each word in $\mathcal{DW}_n$ has at most one decomposition as a concatenation of two palindromes.

Proof. Let $w \in \mathcal{DW}_n$ such that $w = \alpha(w)$. Consider its mirror image $\widetilde{w}$ and write it as $\widetilde{w} = u\,v$ such that $vu \in \mathcal{DW}_n$; then $\alpha(w) = v\,u$. On the other hand, using the relation $\widetilde{u\,v} = \widetilde{v}\,\widetilde{u}$, which is true for any pair of words u, v, we have: $w = \widetilde{v}\,\widetilde{u}$. Since $w = \alpha(w)$ and $|v| = |\widetilde{v}|$, we get

$$v = \widetilde{v}, \qquad\qquad u = \widetilde{u}.$$

Conversely, assume that $w = u\,v$ is such that $v = \widetilde{v}, u = \widetilde{u}$. Then $\widetilde{w} = \widetilde{v}\,\widetilde{u} = v\,u$. This shows that w is the unique conjugate of $\widetilde{w}$ which lies in $\mathcal{DW}_n$, proving that $\alpha(w) = w$.

The unicity of the decomposition of w as a concatenation of two palindromes comes from the fact that if $w = u'v'$, where u' and v' are two palindromes different from u, v, then $\widetilde{w} = \widetilde{v'}\widetilde{u'} = v'u'$, thus $\widetilde{w}$ would have two different decompositions vu and $v'u'$ such that $uv = u'v' \in \mathcal{DW}_n$, contradicting the Cyclic Lemma. □

Then we have, as done in [15, Theorem 4]:

Corollary 4.5. *The Dyck word w satisfies $\alpha(w) = w$ if and only if w is the conjugate of a palindrome.*

Proof. If w has a conjugate which is a palindrome then it can be written as $w = w'w''$, and $w''w'$ is a palindrome. Since the length of w is odd we can write $w''w' = ux\widetilde{u}$ where $x \in \{U, D\}$. If $|w''| \le |u|$ we have $u = w''v$ for some v and $w' = vx\widetilde{u}$. Thus

$$w'w'' = vx\widetilde{u}w''. \tag{4.3}$$

But since $u = w''v$ we have $\widetilde{u} = \widetilde{v}\widetilde{w}''$, whence

$$w = w'w'' = vx\widetilde{v}\widetilde{w}''w'', \tag{4.4}$$

that is, w is the concatenation of the palindromes $vx\widetilde{v}$ and $\widetilde{w}''w''$.

If $|w''| > |u|$ we have $w'' = uxv$ and $\widetilde{u} = vw'$, hence

$$w = w'w'' = w'uxv = w'\widetilde{w}'\widetilde{v}xv,$$

which is the concatenation of two palindromes.

Conversely, suppose that w is the concatenation of the two palindromes u and v. Since w is of odd length, then one of u, v is of odd length and the other one of even length. Suppose u is of odd length. Then $u = u'x\widetilde{u}'$ where $x \in \{U, D\}$ and $v = v'\widetilde{v}'$. We have

$$w = uv = u'x\widetilde{u}'v'\widetilde{v}'. \tag{4.5}$$

This shows that the conjugate of w equal to $\widetilde{v}'u'x\widetilde{u}'v'$ is a palindrome. A similar construction holds if v is of odd length, and u of even length. □

Theorem 4.6. *The number of fixed points of α in $\mathcal{DW}_n$ is equal to the central binomial coefficient $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

Proof. We give a bijection between words of length n with $\lfloor \frac{n}{2} \rfloor$ occurrences of U and words in $\mathcal{DW}_n$ which are fixed by α. Let w be any word of length n with with $\lfloor \frac{n}{2} \rfloor$ occurrences of U and consider the word $wx\widetilde{w}$ where $x = U$ if n is odd and $x = D$ if n is even. This word has n occurrences of U and $n+1$ occurrences of D in both cases. By the cyclic Lemma, it has a unique conjugate in $\mathcal{DW}_n$. All these words are obviously distinct and by Proposition 4.4 they are fixed points of α. $\square$

Example 4.7. Consider the 10 words of length 5 with 2 occurrences of U:

$$UUDDD, UDUDD, UDDUD, UDDDU, DUUDD,$$

$$DUDUD, DUDDU, DDUUD, DDUDU, DDDUU.$$

Building $wU\widetilde{w}$ for each of these words, we obtain:

$$UUDDDUDDDUU, UDUDDUDDUDU, UDDUDUDUDDU, UDDDUUUDDDU,$$

$$DUUDDUDDUUD, DUDUDUDUDUD, DUDDUUUDDUD, DDUUDUDUUDD,$$
$$DDUDUUUDUDD, DDDUUUUUDDD.$$

Now computing the conjugate of each of these words which belongs to $\mathcal{DW}_5$ we get:

$$UUUUDDDUDDD, UDUUDUDDUDD, UUDDUDUDUDD, UUUDDDUUDDD,$$

$$UUDDUUDDUDD, UDUDUDUDUDD, UUUDDUDDUDD, UUDUDUUDDDD,$$
$$UDUUUDUDDDD, UUUUUDDDDDD.$$

The words above can be decomposed as the concatenation of two palindromes as follows:

$$(UUUU)(DDDUDDD),\ (UDUUDU)(DDUDD),\ (UU)(DDUDUDUDD),$$

$$(UUU)(DDDUUDDD),\ (UUDDUU)(DDUDD),\ (UDUDUDU)(DD)$$
$$(UUU)(DDUDDUDD),\ (UUDUDUU)(DDDD), (UDUUUDU)(DDDD),$$
$$(UUUUU)(DDDDDD).$$

4.2.2 Symmetry

The easiest involution on Dyck paths is the symmetry, denoted β here. We considered this map also in the previous chapters. It consists in seeing the path from right to left and replacing each up step by a down step, and vice versa. In the actual setting we have to take in account that now we are adding a letter D at the end of each word (or path). Hence we translate this definition to the words of $\mathcal{DW}_n$ by setting, for $w = w_1w_2, \cdots w_m$, where $m = 2n+1$,

$$\beta(w) = \overline{w}_{m-1}\overline{w}_{m-2}\ldots\overline{w}_2\overline{w}_1w_m.$$

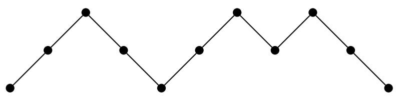
Recall that $\overline{w_i}$ means changing U into D and D into U.

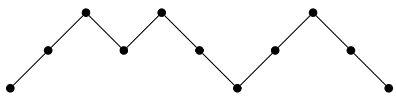
Using the notation of the previous chapters we have that, if $w = w'D$, then $b(w) = \overline{w'}D$.

Example 4.8. If $w = UUDDUUDUDDD$, then

$$\beta(w) = UUDUDDUUDDD.$$

The corresponding Dyck paths are given below.





Notice that this map β is such that for $w \in \mathcal{DW}_n$ we have $\beta(w) \in \mathcal{DW}_n$, and that w and $\beta(w)$ have the same number of peaks and the same height.

Proposition 4.9. *The number of elements w of $\mathcal{DW}_n$ such that $w = \beta(w)$ is the central binomial coefficient $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

Proof. A word in $\mathcal{DW}_n$ is fixed by β if and only if the corrispondig Dyck path is symmetric. As we noted in Chapter 1, the number of such paths i.e. the number of Dyck prefixes of length n, is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. $\square$

4.3 The permutation γ

4.3.1 The definition of the map γ

The permutation γ is defined on $\mathcal{DW}_n$ as the composition of α and β:

$$\gamma(w) = \alpha(\beta(w)).$$

In order to give a direct determination of $\gamma(w)$ we need to introduce the following definition.

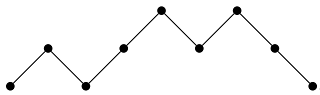
The *principal prefix* of a word $w \in \mathcal{DW}_n$ is the shortest prefix u of w such that $\delta(u)$ is maximum.

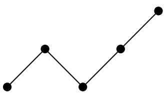
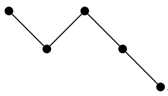
Proposition 4.10. *For $w = w'D \in \mathcal{DW}_n$, the word $\gamma(w)$ is the unique conjugate of the word $\widehat{w'}D$ belonging to $\mathcal{DW}_n$. Moreover, if $w = uvD$, where u is the principal prefix of w, we have:*

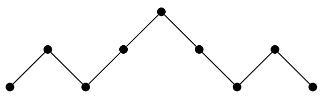
$$\gamma(w) = \widehat{v}D\widehat{u}.$$

Proof. By definition of α, $\alpha(\beta(w))$ is the unique conjugate of the mirror image of $\beta(w)$ which is in $\mathcal{DW}_n$. Denoting $w = w'D$ we have that the mirror image of $\beta(w'D)$ is equal to $D\widehat{w'}$ which has the same conjugates as $\widehat{w'}D$. The shortest prefix v' of $\widehat{w'}D$ such that $\delta(v')$ is minimal is equal to $\widehat{u}$, where u is the principal prefix of w. □

Example 4.11. Consider the Dyck path

$w =$ .

Here $u =$  and $v =$ . Hence,

$\gamma(w) =$ .

The following result is an immediate consequence of Proposition 4.10.

Corollary 4.12. *Let $w \in \mathcal{DW}_n$. The word $\gamma(w)$ has the same height and the same number of peaks as w.*

We point out that the definition of the map γ and the fact that α and β are involutions imply immediately the following commutation rules:

$$\beta \circ \gamma = \gamma^{-1} \circ \beta, \qquad \alpha \circ \gamma = \gamma^{-1} \circ \alpha.$$

Hence, α and γ generate a dihedral group.

4.3.2 Smooth words and skeleton for words in $\mathcal{DW}_n$

Consider a Dyck word $w \in \mathcal{DW}_n$, $w = w_1w_2...w_m$, where $w_i \in \{U, D\}$ and $m = 2n+1$.

An integer i, $i = 1, 2, \ldots, 2n+1$, is said to be an *active site* of w if both $w_{i-1}w_i$ and w_iw_{i+1} are different from UD.

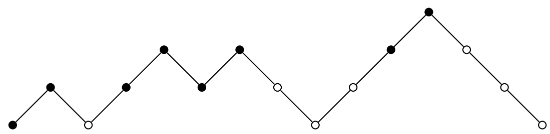
In other words, if we see w as a lattice path, the integer i is an active site for w if the i-th vertex of w is neither a peak nor the vertex preceding a peak.

The number of active sites of a Dyck word w depends only on the length and the number of peaks of w. In fact, we have the following result, whose proof is straightforward.

Proposition 4.13. *If w is a Dyck word of length $2n+1$, then the number of active sites of w is $2n+1-2\,\mathrm{peak}(w)$, where $\mathrm{peak}(w)$ denotes the number of peaks of w. In particular the number of active sites of w is always odd.*

Recalling that the map γ preserves the number of peaks of a word (Corollary 4.12), the previous Proposition implies that γ preserves also the number of active sites.

Example 4.14. In the following Dyck path the white vertices are the active sites.



A Dyck word w is said to be *smooth* if w may be written as

$$w = U^{i_1}D^{j_1}U^{i_2}D^{j_2}\cdots U^{i_k}D^{j_k},$$

where k is the number of peaks of w and $i_s > 1$, $j_t > 1$ for all s,t.
Roughly speaking, w is smooth whenever it can not be obtained from a shorter word by adding one peak in any active site.

The definition of a smooth word implies immediately that a Dyck word w is smooth if and only if it does not contain any sequence DUD or UDU. As a consequence, if w is smooth so is $\gamma(w)$.

Given a smooth word

$$w = U^{i_1}D^{j_1}U^{i_2}D^{j_2}\cdots U^{i_k}D^{j_k},$$

consider the operation θ which decreases by 1 all the exponents, giving

$$\theta(w) = U^{i_1-1}D^{j_1-1}U^{i_2-1}D^{j_2-1}\cdots U^{i_k-1}D^{j_k-1}.$$

As a consequence, if at least one of the exponents is equal to 2, then $\theta(w)$ is not smooth. However we have:

Lemma 4.15. *Let w be a smooth word in $\mathcal{DW}_n$. Then,*

i) $\gamma(\theta(w)) = \theta(\gamma(w))$.

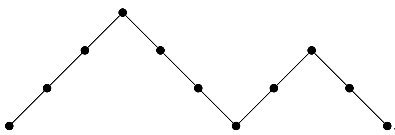
ii) $\theta(w) = \theta(w')$ *implies* $w = w'$.

iii) The words w *and* $\theta(w)$ *are in cycles of* γ *of the same length.*

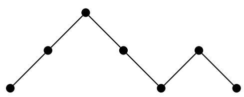
Proof. The first assertion follows from the fact that if $U^{i_1}D^{j_1}U^{i_2}D^{j_2}\cdots U^{i_h}$ is the principal prefix of w then $U^{i_1-1}D^{j_1-1}U^{i_2-1}D^{j_2-1}\cdots U^{i_h-1}$ is the principal prefix of $\theta(w)$. The second assertion follows immediately from the definition of θ, and the third one is a consequence of the first two. $\square$

Notice that here we are considering γ as an operator acting on all Dyck words of any length at the same time. Observe that γ preserves the length of the Dyck word, whereas θ changes the length of a smooth Dyck word.

Example 4.16. Consider the Dyck path

$w =$ .

Then

$\theta(w) =$ 

We now associate to any element $w \in \mathcal{DW}_n$ a smooth word $\mathrm{Sk}(w)$, given by:

The skeleton $\mathrm{Sk}(w)$ of a word w obtained by deleting iteratively all the factors UD of w which are followed by U or preceded by D.

Note that the above definition is well posed, since the end result is obviously independent of the order in which the factors UD are removed.

The following result, whose proof is straightforward, will be useful in the sequel.

Proposition 4.17. *The words* w *and* $\mathrm{Sk}(w)$ *have the same height and the same number of active sites.*

Example 4.18. Let

$$w = U\ D\ U\ U\ D\ U\ D\ D\ U\ U\ D\ D\ U\ D\ U\ D\ U\ U\ U\ D\ D\ D\ D.$$

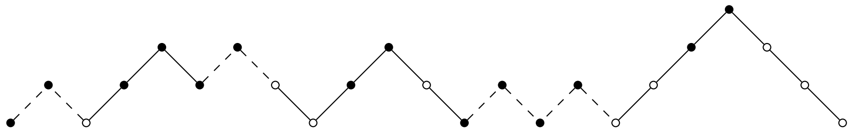
We get:

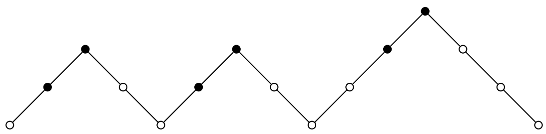
$$\mathrm{Sk}(w) = U\ U\ D\ D\ U\ U\ D\ D\ U\ U\ U\ D\ D\ D\ D,$$

since we delete the factors UD which are delimited in the equation below by parenthesis:

$$w = (U\ D)\ U\ U\ D\ (U\ D)\ D\ U\ U\ D\ D\ (U\ D)\ (U\ D)\ U\ U\ U\ D\ D\ D\ D.$$

The Dyck words w and $\mathrm{Sk}(w)$ correspond to the following Dyck paths.

$w =$ 

$\mathrm{Sk}(w) =$ 

The nine white vertices in each path denote the active sites and the dashed peaks are removed from w to get $\mathrm{Sk}(w)$.

The word w can be obtained from $\mathrm{Sk}(w)$ as soon as the positions of the active sites where the factors UD were deleted are known. This can be done by a sequence $(p_1, p_2, \cdots, p_\ell)$ that gives for each active site the number of factors UD that were deleted. For our example the number of active sites in this sequence is 9, and the values are given by:

$$(1, 1, 0, 0, 2, 0, 0, 0).$$

In the sequel we will denote this sequence by $\mathrm{ins}(w)$. Conversely, a smooth word $\breve{w}$ and a sequence $I = (i_1, i_2, \dots i_\ell)$ of integers, where ℓ is the number of active sites in $\breve{w}$, determine a unique word w such that $\mathrm{Sk}(w) = \breve{w}$ and $\mathrm{ins}(w) = I$.

We have a first trivial lemma showing the effect of deleting a factor UD in a word when this factor is followed by an occurrence of U or preceded by an occurrence of D.

Lemma 4.19. *Let $w = uUDv \in \mathcal{DW}_n$ be such that u ends with D or v begins with U. Let $\mathrm{ins}(w) = (i_1, i_2, \dots, i_\ell)$ and set $w' = uv$. Then we have*

$$Sk(w) = Sk(w') \quad \textit{and} \quad \mathrm{ins}(w') = (i_1, i_2, \dots, i_{k+1} - 1, \dots, i_\ell), \tag{4.6}$$

where k is the number of active sites of w appearing in u.

The action of γ in relation with the deletion of a factor UD obeys to the rules expressed in the following Lemma.

Lemma 4.20. *If w' is obtained from w by deleting a factor UD followed by an occurrence of U or preceded by an occurrence of D, then $\gamma(w')$ is obtained from $\gamma(w)$ in the same way. Moreover, let ℓ be the number of active sites of w and ℓ_0 the number of those situated in its principal prefix. Assume that the factor UD deleted from w is situated in the i-th active site of w. Then:*

- *If $i \leq \ell_0$ then $\gamma(w')$ is obtained from $\gamma(w)$ by deleting UD in its $(\ell - \ell_0 + i)$-th active site.*

- *If $i > \ell_0$ then $\gamma(w')$ is obtained from $\gamma(w)$ by deleting UD in its $(i - \ell_0)$-th active site.*

Proof. Assume $w = w^{(1)}UDUw^{(2)}D$ and $w' = w^{(1)}Uw^{(2)}D$. The case $w = w^{(1)}DUDw^{(2)}D$ and $w' = w^{(1)}Dw^{(2)}D$ can be treated similarly.

Let u be the principal prefix of w. Then we have $w = uvD$ and $\gamma(w) = \widehat{v}D\widehat{u}$.

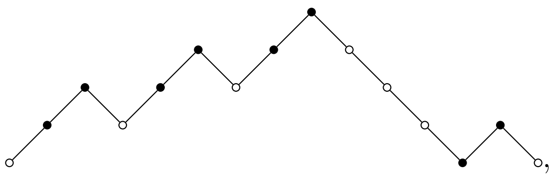
- If $|w^{(1)}U| < |u|$ then we have $u = w^{(1)}UDUw^{(3)}$, with $w^{(3)} \neq \emptyset$ since the principal prefix cannot end with UDU, and
$$\gamma(w) = \widehat{v}D\widehat{w}^{(1)}DUD\widehat{w}^{(3)}.$$
We notice that the number of active sites in $\widehat{v}$ is $\ell - \ell_0$ and their number in $\widehat{w}^{(1)}$ is i, giving the first part of the Lemma, since $\gamma(w') = \widehat{v}D\widehat{w}^{(1)}b\widehat{w}^{(3)}$.

- If $w^{(1)}U = u$, then we have $i = \ell_0 + 1$ and
$$\gamma(w) = UD\widehat{w}^{(2)}D\widehat{w}^{(1)}D, \qquad \gamma(w') = \widehat{w}^{(2)}D\widehat{w}^{(1)}D,$$
thus $\gamma(w')$ is obtained from $\gamma(w)$ by deleting UD in the first position in accordance with $i - \ell_0 = 1$.

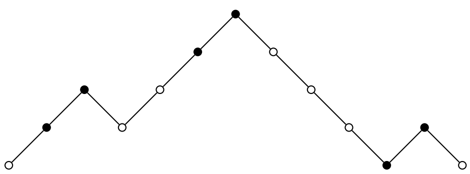
- If $|w^{(1)}U| > |u|$ then we have $w^{(1)}U = uw^{(3)}$ where $w^{(3)}$ ends with an occurrence of U, thus $w = uw^{(3)}DUw^{(2)}D$ and
$$\gamma(w) = \widehat{w}^{(3)}UD\widehat{w}^{(2)}D\widehat{u}$$
The number of active sites in $\widehat{w}^{(3)}$ is $i - \ell_0$, proving the second part of the Lemma.

□

Example 4.21. Consider the following path

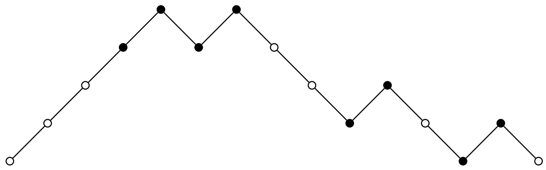
$w =$ ,

where the white vertices are the active sites. Here $\ell = 7$ and $\ell_0 = 3$. w is obtained from the path

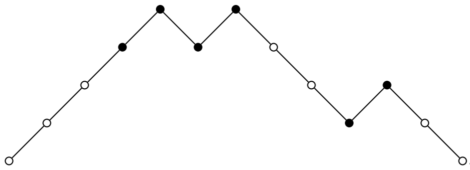
$w' =$ 

by adding a peak in the third active site. Hence $i = 3 \leq \ell_0$.

We have

$\gamma(w) =$ 

and

$\gamma(w') =$ .

$\gamma(w)$ is obtained from $\gamma(w')$ by adding a peak in the active site number $\ell - \ell_0 + i = 7 - 3 + 3 = 7$.

We now compare $(\mathrm{Sk}(w), \mathrm{ins}(w))$ and $(\mathrm{Sk}(\gamma(w)), \mathrm{ins}(\gamma(w))$.

We denote by cyc the cyclic shift of a finite sequence. It is given by:

$$\mathrm{cyc}((i_1, i_2, \ldots, i_n)) = (i_2, i_3, \ldots, i_n, i_1).$$

Lemma 4.22. *Let $w \in \mathcal{DW}_n$. Then we have*

$$\mathrm{Sk}(\gamma(w)) = \gamma(\mathrm{Sk}(w)),$$

and

$$\mathrm{ins}(\gamma(w)) = \mathrm{cyc}^{\ell_0}(\mathrm{ins}(w)),$$

where ℓ_0 is the number of active sites contained in the principal prefix of w.

Proof. This Lemma is obtained by applying the two Lemmas 4.19 and 4.20 sufficiently many times. □

4.3.3 Orbits of γ

We are now in the position to prove our main result.

Theorem 4.23. *Let w be a word in $\mathcal{DW}_n$. Then there exists an odd integer k such that $\gamma^k(w) = w$.*

Proof. The proof proceeds by induction on n. The assertion is trivially true for $n = 1$. Assume that for any $n' < n$ and for each word $w' \in \mathcal{DW}_{n'}$ there exists an odd integer k' such that $\gamma^{k'}(w') = w'$. Let $w \in \mathcal{DW}_n$. We distinguish two cases:

- If w is smooth then $w' = \theta(w)$ is an element of $\mathcal{DW}_{n'}$ with $n' < n$. By the inductive hypothesis there exists an odd integer k' such that $\gamma^{k'}(w') = w'$. But by Lemma 4.15 $\gamma(\theta(w)) = \theta(\gamma(w))$, hence $\gamma^{k'}(w) = w$.

- If w is not smooth then its skeleton $\breve{w}$ belongs to $\mathcal{DW}_{n'}$ with $n' < n$. By the inductive hypothesis we have that there exists an odd integer k' such that $\gamma^{k'}(\breve{w}) = \breve{w}$.

  Set $\text{ins}(w) = (i_1, i_2, \ldots, i_\ell)$. Denote by ℓ_i, $0 \leq i \leq k'-1$, the number of active sites in the principal prefixes of w, $\gamma(w)$, ... $\gamma^{k'-1}(w)$, respectively, and set $L = \sum_{i=0}^{k'-1} \ell_i$. By Lemma 4.22, we have

$$\begin{array}{ll}
\text{Sk}(\gamma(w)) = \gamma(\breve{w}), & \text{ins}(\gamma(w)) = \text{cyc}^{\ell_0}(\text{ins}(w)),\\
\text{Sk}(\gamma^2(w)) = \gamma^2(\breve{w}), & \text{ins}(\gamma^2(w)) = \text{cyc}^{\ell_0+\ell_1}(\text{ins}(w)),\\
\ldots & \\
\text{Sk}(\gamma^{k'}(w)) = \gamma^{k'}(\breve{w}) = \breve{w}, & \text{ins}(\gamma^{k'}(w)) = \text{cyc}^{L}(\text{ins}(w)),\\
\ldots & \\
\text{Sk}(\gamma^{k'\ell}(w)) = \gamma^{k'\ell}(\breve{w}) = \breve{w}, & \text{ins}(\gamma^{k'\ell}(w)) = \text{cyc}^{\ell L}(\text{ins}(w)).
\end{array}$$

  Now, ℓL is a multiple of the length of the sequence $\text{ins}(w)$, so we have that $\text{cyc}^{\ell L}(\text{ins}(w)) = \text{ins}(w)$, giving $w = \gamma^{k'\ell}(w)$. Since the two integers ℓ and k' are odd we get the assertion.

□

In the following we denote by $\text{Orb}(w)$ the orbit of the word w under the action of the map γ, namely, the set $\{\gamma(w), \gamma^2(w), \ldots, \gamma^{k-1}(w)\}$, where k is the minimum positive integer such that $\gamma^k(w) = w$.

Theorem 4.24. *For every $w \in \mathcal{DW}_n$ the orbit of w has odd cardinality.*

Proof. The assertion is an immediate consequence of the preceding result, since the cardinality of $\text{Orb}(w)$ must divide any k such that $\gamma^k(w) = w$. □

In the table below we give for each n the number of cycles of lengths $1, 3, \ldots, 2n-3$ of the permutation γ on $\mathcal{DW}_n$, and the list of lengths of the cycles which are greater than $2n-3$. This table was obtained by a computer program that generates all Dyck words in $\mathcal{DW}_n$ with $2 \leq n \leq 10$ and determines the cycles of the permutation γ.

	1	3	5	7	9	11	13	15	17	other values
n=2	2									
n=3	2	1								
n=4	3	2	1							
n=5	3	4	4	1						
n=6	4	5	11	4	1					21
n=7	2	10	21	14	6	1				21,45
n=8	6	10	44	34	24	6	1			21,27,33,45,77
n=9	5	17	67	83	74	27	8	2		21,27,33,45,55 65,77,117
n=10	4	17	119	162	212	92	44	11	1	21,27,33,45,55 65,77,91,105 117,165,273

Proposition 4.25. *For every odd number $2n+1$ there exists a word $w \in \mathcal{DW}_{n+2}$ such that* $\mathrm{Orb}(w)$ *has length* $2n+1$.

Proof. Consider the word $w = UDU^{n+1}D^{n+2}$. Notice that the principal prefix of w has n active sites. We have

$$\mathrm{Sk}(w) = U^{n+1}D^{n+2}, \qquad \mathrm{ins}(w) = \underbrace{(1,0,\dots,0)}_{2n+1}.$$

The word $\mathrm{Sk}(w)$ is obviously fixed under γ, hence, by Lemma 4.22,

$$\mathrm{Sk}(\gamma(w)) = \mathrm{Sk}(w), \qquad \mathrm{ins}(\gamma(w)) = \mathrm{cyc}^n(\mathrm{ins}(w)).$$

Since $gcd(n, 2n+1) = 1$, we deduce that $|\mathrm{Orb}(w)| = 2n+1$. $\square$

In order to give some additional properties of the orbits of γ we build a graph G_n whose vertices are the words in $\mathcal{DW}_n$. The edges of G_n are all the pairs $(w, \beta(w))$ and $(w, \alpha(w))$, thus the fixed points of β and α correspond to loops in G_n. Taking the convention that a loop contributes for 1 in the degree of the vertex adjacent to it, all vertices in G_n have degree 2.

The graph G_n has some elementary properties:

Lemma 4.26. *Each connected component C_i of G_n corresponds to an orbit of the group generated by α and β and satisfies one of the conditions below:*

- *C_i has a unique vertex incident with two loops.*
- *C_i is a path of odd length connecting two vertices each one being incident with a loop.*

- *C_i is a cycle of length $4p+2$, where $p \geq 1$.*

Proof. One of the first trivial results in graph theory states that a graph with no loops whose vertices have degree $0, 1$ or 2 is the union of isolated vertices, paths and cycles. Let G'_n be the graph obtained from G_n by deleting every loop. Notice that the connected components of the graph G'_n have the same blocks of vertices as the connected components of G_n. Thus the graph G'_n satisfies the condition stated above. Let C_i be a connected component of G'_n. Then:

- if C_i is an isolated vertex, then it corresponds in G_n to a vertex incident with two loops.

- If C_i is a path, it corresponds in G_n to a path connecting two vertices each one being incident with a loop. By definition of G_n this path may be written $w^{(1)}, w^{(2)}, \cdots, w^{(p)}$. Without loss of generality (by exchanging β and α) , we may suppose that $\beta(w^{(1)}) = w^{(1)}$, $\beta(w^{(i)}) = w^{(i+1)}$ for i even and $\alpha(w^{(i)}) = w^{(i+1)}$ for $i \neq 1$ odd. If p is even, $p = 2q$, then the cycle of γ containing $w^{(1)}$ is equal to:

$$(w^{(1)}, w^{(2)}, w^{(4)}, \cdots, w^{(2q)}, w^{(2q-1)}, \cdots, w^{(3)}),$$

  hence it has even length contradicting Theorem 4.24.

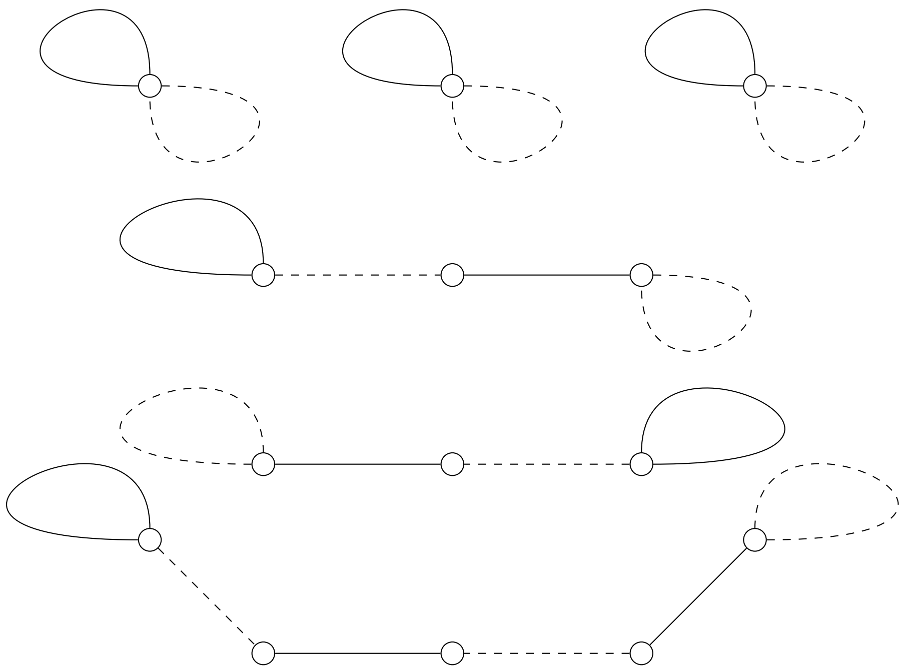
- If C_i is a cycle, it is also a cycle in G_n, hence this cycle may be written $C_i = w^{(1)}, w^{(2)}, \cdots w^{(p)}$. The images under β and α alternate in this cycle, hence, p is even. Let $p = 2q$. Computing the cycle $\mathcal{C}$ of γ containing $w^{(1)}$ we get

$$\mathcal{C} = (w^{(1)}, w^{(3)}, \cdots w^{(2q-1)}).$$

  As a consequence of Theorem 4.24 we have that q is odd, completing the proof.

□

Example 4.27. The graph G_4



As an immediate consequence we have:

Proposition 4.28. *Any orbit $\mathcal{O}$ of γ satisfies one and only one of the two assertions below:*

1. *Any w in $\mathcal{O}$ satifies $\beta(w) \notin \mathcal{O}$, and $\alpha(w) \notin \mathcal{O}$, moreover there exists an orbit $\mathcal{O}'$ different from $\mathcal{O}$ such that $w \in \mathcal{O} \Leftrightarrow \beta(w) \in \mathcal{O}'$ and $w \in \mathcal{O} \Leftrightarrow \alpha(w) \in \mathcal{O}'$.*
2. *There exists w' and w'' in $\mathcal{O}$ such that $\beta(w') = w'$ and $\alpha(w'') = w''$. Moreover for any $w \in \mathcal{O}$ we have $\alpha(w) \in \mathcal{O}$ and $\beta(w) \in \mathcal{O}$.*

Propositions 4.28 and 4.4 imply immediately the following.

Corollary 4.29. *The pair of statistics (number of peaks, height) is equidistributed over the set of symmetric words in $\mathcal{DW}_n$ and the set of words in $\mathcal{DW}_n$ that are the concatenation of two palindromes.*

4.3.4 The fixed points of γ

We are now interested in the fixed points of γ.

Proposition 4.30. *Let $w \in \mathcal{DW}_n$ be such that $\gamma(w) = w$, and let p be the length of the principal prefix u of w. Then we have*

$$w_{m-p} = w_m = D,$$

and for all $i \neq m-p$

$$w_{p+i} = \overline{w_i}$$

where $p+i$ is taken $\mod m$, where $m = 2n+1$ is the common length of the words in $\mathcal{DW}_n$.

Proof. Let u be the principal prefix of w. Then we have $w = uvD$ and $\gamma(w) = \widehat{v}D\widehat{u}$, giving:

$$uvD = \widehat{v}D\widehat{u},$$

thus

$$w_1 w_2 \cdots w_p w_{p+1} \cdots w_{m-1} D = \overline{w}_{p+1} \cdots \overline{w}_{m-1} D \overline{w}_1 \cdots \overline{w}_p.$$

Hence

$$w_i = \overline{w}_{p+i} \text{ for } i = 1, 2, \ldots m-p-1,$$

$$w_{m-p} = w_m = D \quad \text{and} \quad w_i = \overline{w}_{p+i-m} \text{ for } m-p < i \leq m.$$

The result now follows from the fact that $p+i-m \equiv p+i \mod m$.

□

Corollary 4.31. *Let $w \in \mathcal{DW}_n$ be such that $\gamma(w) = w$ and $m = 2n+1$. Let p be the length of the principal prefix of w. Then the permutation $\tau \in S_m$ defined by $\tau(i) \equiv i+p \mod m$ has only one cycle. Hence*

$$gcd(m,p) = 1.$$

Proof. It is well-known that the permutation τ has cycles of the same length, which is equal to $\frac{m}{gcd(m,p)}$. Since m is odd this common length cannot be even. By Proposition 4.30 we have that the letters w_i and $w_{\tau(i)}$ of the word w are different except when $\tau(i) = m$. Hence, if a cycle of τ does not contain m, its elements correspond to occurrences of letters in w alternating between U and D. This implies that such a cycle must be of even length. We conclude that such a cycle does not exist and that τ has only one cycle.

□

Theorem 4.32. *For any $p < m$ such that $gcd(m,p) = 1$ there is at most one word w in $\mathcal{DW}_n$ satisfying the two conditions:*

- $\gamma(w) = w$.
- *The principal prefix of w has length p.*

  Moreover each such word satisfies

$$\alpha(w) = \beta(w) = w.$$

Proof. We have seen that if w satisfies the condition $w = \gamma(w)$, the permutation τ has only one cycle which we can write $(i_1 = m, i_2, \ldots, i_m)$. With this notation w_{i_j} is equal to U if j is even and equal to D if j is odd. We also have $i_{j+1} \equiv i_j + p \mod m$. This gives a unique word.

In order to prove that $\beta(w) = w$ we have to show that $w_k = \overline{w}_{m-k}$ for any $k < m$, or equivalently $w_{i_j} = \overline{w}_{m-i_j}$ for any j such that $i_j < m$. Note that $i_j \equiv jp \mod m$. Hence we have to prove that if k is such that $m - i_j = kp$, then one of j, k is even and the other one is odd. This follows from the fact that m is odd and $gcd(m,p) = 1$, since we have:

$$kp \equiv m - i_j \equiv m - pj \mod m,$$

giving

$$p(j+k) \equiv 0 \mod m.$$

Since $\gamma = \alpha \circ \beta$, we get immediately also $\alpha(w) = w$.

□

This result implies the following characterization of the fixed points of γ.

Proposition 4.33. *A word w in $\mathcal{DW}_n$ is fixed under γ whenever the following two conditions are satisfied:*

- $\gamma(\mathrm{Sk}(w)) = \mathrm{Sk}(w)$*, and*
- *$ins(w) = (i, i, \ldots, i)$ for some $i \geq 0$.*

Proof. Let w be a Dyck word satisfying the two conditions above. Then by Lemma 4.22 we get $\gamma(w) = w$.

Conversely, take $w \in \mathcal{DW}_n$ such that $w = \gamma(w)$. Lemma 4.22 implies immediately that also $\mathrm{Sk}(w)$ is fixed under γ and $\mathrm{ins}(w) = cyc^{\ell_0}(\mathrm{ins}(w))$, where ℓ_0 is number of active sites in the principal prefix of w. Set

$$\mathrm{Sk}(w) = U^{i_1} D^{j_1} U^{i_2} D^{j_2} \cdots U^{i_k} D^{j_k},$$

where k is the number of peaks of w and $i_s > 1$, $j_t > 1$ for all s, t. Hence, by Propositions 4.13 and 4.17, the words w and $\mathrm{Sk}(w)$ have

$$\ell = \sum_{s=1}^{k} (i_s + j_s) \; - 2k$$

active sites, and both their principal prefixes have

$$\ell_0 = \sum_{s=1}^{h-1} (i_s + j_s) \; - 2h + 1$$

active sites, for some $h \leq k - 1$.

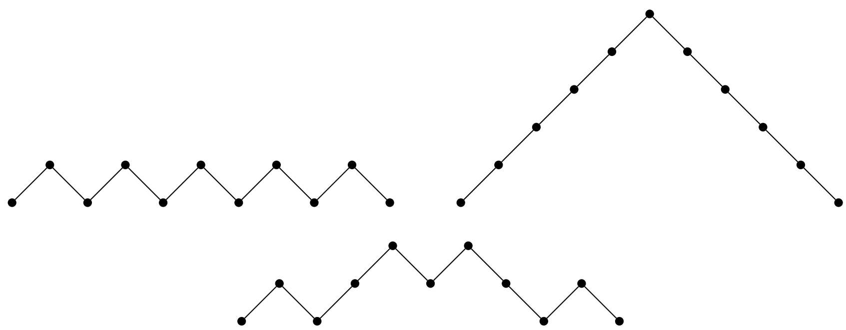
Consider now the word

$$\breve{w} = \theta(\mathrm{Sk}(w)) = U^{i_1 - 1} D^{j_1 - 1} U^{i_2 - 1} D^{j_2 - 1} \cdots U^{i_k - 1} D^{j_k - 1}.$$

Observe that by Proposition 4.15 $\breve{w}$ is fixed under γ.

It is immediately checked that $\breve{w}$ has length ℓ and its principal prefix has length ℓ_0. Corollary 4.31 implies now that $gcd(\ell, \ell_0) = 1$. Since the sequence $\mathrm{ins}(w)$ has length ℓ and $\mathrm{ins}(w) = cyc^{\ell_0}(\mathrm{ins}(w))$, we deduce that $\mathrm{ins}(w)$ must be a constant sequence. $\square$

Example 4.34. The three Dyck paths of semilength 5 fixed by the action of γ.



Theorem 4.32 gives a way to build all words $w \in \mathcal{DW}_n$ such that $\gamma(w) = w$. We use the following algorithm:

Let $m = 2n+1$. Take all integers p such that $p \leq n$ and $gcd(2n+1, m) = 1$. For each such p consider the permutation τ on $\{1, 2, \cdots, m\}$ such that $\tau(i) \equiv i + p \mod m$. This permutation has only one cycle, which may be written $(i_1, i_2, \cdots i_m)$ with $i_1 = p, i_m = m$. Let w be the word given by $w_i = U$ if $i = i_j$ with j odd and $i \neq m$, and $w_i = D$ if $i = m$ or $i = i_j$ with j even. If w is in $\mathcal{DW}_n$, then $\gamma(w) = w$.

Example 4.35. Consider $n = 5$ and $p = 4$, then $m = 11$ and the permutation τ is the cycle

$$(4, 8, 1, 5, 9, 2, 6, 10, 3, 7, 11),$$

thus $w_i = U$ for $i = 4, 1, 9, 6, 3$ and $w_i = D$ for $i = 8, 5, 2, 10, 7, 11$ giving the word $w = UDUUDUDDUDD$ which is in $\mathcal{DW}_n$ and is such that $\gamma(w) = w$.

Taking $n = 5, p = 3$ gives

$$\tau = (3, 6, 9, 1, 4, 7, 10, 2, 5, 8, 11).$$

Since this implies $w_1 = b$, we have $w \notin \mathcal{DW}_n$.

Combining Theorem 4.32 and Proposition 4.33 we have:

Corollary 4.36. *The number of words w fixed by γ in $\mathcal{DW}_n$ is upper bounded by:*

$$\min(n, \phi(2n + 1)),$$

where ϕ is the Euler totient function.

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