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Algorithms and quantifications in amenable and sofic groups

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Algorithms and quantifications in amenable and sofic groups

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Introduction

Sofic groups were introduced by Gromov [22] as a common generalization of amenable and residually finite groups. For both the classes *algorithmic problems* have been studied, for example the solvability of the *word problem* for residually finite, finitely presented groups and the subrecursivity of the Følner function for amenable groups.

The topic of this dissertation is the investigation of computability and decidability of some problems related to soficity.

And when the problem is to understand the *effectiveness* of a property the natural tool is the *quantification* of this property.

A group Γ is sofic if for all $\epsilon > 0$, for any finite subset K of Γ there exist a $k \in \mathbb{N}$ and a map $\phi : \Gamma \to Sym(k)$, where $Sym(k)$ denotes the symmetric group of degree k, such that:

$$d_H(\phi(gh), \phi(g)\phi(h)) \leq \epsilon, \quad g, h \in K;$$

$$d_H(\phi(g), \phi(h)) \geq 1 - \epsilon, \quad g, h \in K : \; g \neq h.$$

Here, the function d_H is the normalized Hamming distance on the symmetric group: $d_H(\sigma, \tau) = \frac{|\{i \in \{1,2,\ldots k\} : \sigma i \neq \tau i\}|}{k}$. We call ϕ a (K, ϵ)-approximation of Γ. If we also ask for the existence of (K,0)-approximations for all finite subset

K of Γ we characterize the *local embeddability into finite groups (LEF)*, moreover if we want that these approximations are actually homomorphisms we have all and only groups that are residually finite. From this point of view it is clear that residually finite groups, and, more generally, LEF groups, are sofic. LEF groups were introduced in [37] and for these groups we have that the multiplicative table of any finite subset is the same as the multiplicative table of a subset of a finite group. In a sofic group the multiplicative table of any finite subset is arbitrarily close, in the sense of Hamming distance, to a multiplicative table of a subset of a symmetric group.

In the last fifteen years many conjectures on groups were proved for sofic groups: Gottschalk's surjunctivity conjecture [22,38], Kaplansky stable finiteness [10,14], algebraic eigenvalues conjecture [36], determinant conjecture [15], Connes' embedding conjecture (since sofic groups are hyperlinear)[15,34]. Now we know many stability properties for these groups: subgroups, direct products, direct limits, inverse limits, amenable extensions, free product over amenable amalgamation [16,17,32] and others, but we don't know if there exists a non sofic group and this is the main problem of the subject.

There are also a lot of equivalent ways to define soficity. For some of these definitions and for some results what we really use is the *bi-invariance* of normalized Hamming distance and some finiteness properties of symmetric groups. Simply a metric d on a group G is bi-invariant if $d(gx, gy) = d(x, y) = d(xg, yg)$ for every $g, x, y \in G$. The Hamming distance is bi-invariant on symmetric groups, but also the discrete distance, the Hilbert-Schmidt distance on unitary groups of matrices... So we can define other properties of this kind of metric approximations, there is a short presentation in [1]. And in general for a more complete survey on sofic groups it is possible to see [33] and [8].

We have explained why the class of residually finite groups is sofic, now we see the other class, the amenable groups. Amenability property has a lot of equivalent definitions, varying from representation theoretic, topological, algebraic, theory of dynamical systems, theoretical computer science. Here we present the *Følner condition* which is the closest to the given definition of soficity. A group Γ is amenable if for every finite subset K, for any $\epsilon > 0$ there exists a finite non empty subset F of Γ which is ϵ-invariant by left multiplication of all elements in K, that is:

$$\frac{|F \cap kF|}{|F|} \geq 1 - \epsilon \quad \forall k \in K.$$

If we ask the condition for $\epsilon = 0$ we obtain exactly the finite groups (take $F = \Gamma$). Moreover notice that when Γ is finite for each $k \in K$ we can associate a bijection on F, the left multiplication by k and it is easy to check that this association is a (K,0)-approximation (and actually a real injective homomorphism in the symmetric group of Γ, as Cayley's Theorem states).

So the idea for a general amenable group is, given an element $k \in K$, to map into a bijection of F that on $F \cap kF$ is the left multiplication by k, but $F \cap kF$ is a large part of F and it is possible to see that this map is a (K', ϵ')-approximation for some finite subset K' and some $\epsilon' > 0$. At this qualitative (and not quantitative) level, if we just want to prove that Γ is sofic, it's enough to show that for each possible (K', ϵ') of soficity we can find the corresponding (K, ϵ) for amenability. We might be tempted to understand how they are related and quantify this relation... but first we specialize our subject to finitely generated groups.

The LEF, sofic and Følner conditions are local, defined and checked on finite subsets, so a group Γ is LEF, amenable or sofic if (and only if) every finitely generated subgroups of Γ are resp LEF, amenable or sofic. So it is very natural to consider just finitely generated groups.

If Γ is a group generated by a finite subset $X = \{x_1, \dots x_d\}$, denote by $\mathbb{F}_X \cong \mathbb{F}_d$ the free group generated by X, we have a natural epimorphism

$$\pi : \mathbb{F}_X \to \Gamma$$

that maps a word $\omega \in \mathbb{F}_X$ to the corresponding element $\omega(x_1, \dots x_d)$ in the group Γ. In the free group we have the word length and so for $g \in \Gamma$ we define the word length $|g|$ as the length of shortest word projecting into g. We can also give a presentation, Klein did it first, in this way:

$$\Gamma = \langle X | R \rangle$$

then R is a subset of $\mathbb{F}_X$ such that the normal closure $R^{\mathbb{F}_X} = \ker \pi$. We say that Γ is finitely presented if there exists a finite R with this property.

For an amenable, finitely generated group Γ with finite set of generators X we say that $F \subset \Gamma$ is *n-Følner* if F is finite, non empty and:

$$\frac{|F \setminus xF|}{|F|} \leq \frac{1}{n}, \ \forall x \in X \cup X^{-1}.$$

We present the Følner function, first defined by Vershik:

$$F_{\Gamma,X}(n) = \min\{|F|, \ F \subset \Gamma, \ F \text{ is } n\text{-Følner } \}.$$

Suppose that F is n-Følner, it is easy to see that, if we consider for example the element $x_1 x_2 \in \Gamma$, we have $\frac{|F \setminus x_1 x_2 F|}{|F|} \leq \frac{2}{n}$ and in general

$$\frac{|F \setminus gF|}{|F|} \leq \frac{|g|}{n}. \tag{1}$$

But for a finite subset of Γ we have a bound for the length of its elements. This might convince that given a finite subset K and an $\epsilon > 0$ there exists an n large enough to obtain a Følner set for K and ϵ. And the existence of that minimum for each n is equivalent to amenability. Actually $F_{\Gamma,X}$ is a sort of quantification of amenability.

The asymptotic behaviour of the Følner function does not depend on the choice of generators: this means that if X and X' are two finite sets of generators of Γ, there exists $C > 0$ such that $C^{-1}F_{\Gamma,X'}(C^{-1}n) \leq F_{\Gamma,X}(n) \leq CF_{\Gamma,X}(Cn)$. This function is very well studied (see for example [18, 19, 23, 30, 35]), it is related to other famous functions as the isoperimetric profile and the return probability in a random walk on a group, but actually it is still unknown its precise asymptotic behaviour for many amenable groups.

At this point it is natural to try something similar for soficity of finitely generated group. So we start with another definition of soficity:

Definition. *A finitely generated group Γ is sofic if for $n \in \mathbb{N}$ there exist $k \in \mathbb{N}$ and $(\sigma_1, \sigma_2, \dots \sigma_d) \in Sym(k)^d$ with the following property:*

$$\ell_H(\omega(\sigma_1, \dots, \sigma_d)) \begin{cases} \leq \frac{1}{n}, & \textit{if } \omega \in B_n \cap \ker \pi \\ \geq 1 - \frac{1}{n}, & \textit{if } \omega \in B_n \setminus \ker \pi \end{cases} \tag{2}$$

*where B_n is the ball of radius n in the free group $\mathbb{F}_d$ and ℓ_H is the normalized Hamming length: $\ell_H(\sigma) = d_H(\sigma, 1)$. We call the d-tuple $(\sigma_1, \dots, \sigma_d)$ an n-*approximation.

All information about the multiplicative table of a finitely generated group is in the kernel of π. For every $n \in \mathbb{N}$ we define $K_{\Gamma,X}(n) \in \mathbb{N}$ to be the minimum rank k of $Sym(k)$ containing n-approximations. We call the map $n \mapsto K_{\Gamma,X}(n)$ *sofic dimension growth.* Arzhantseva and Cherix first introduced and studied this function in the more general setting of metric approximations in an unpublished paper [2]. Some of the results about $K_{\Gamma,X}$ in this text (there will be a list) were already proved in [2] but our proofs are independent. For example the independence on generators up to asymptotic equivalence. Another related work with the same idea to quantify the soficity by the rank of approximations is the one about sofic profile in [12].

Coming back to our proof of soficity for amenable groups, in the case of finite generation, we can provide the following quantification, simply using inequality (1):

$$K_{\Gamma,X}(n) \leq F_{\Gamma,X}(n^2).$$

In the last part of this dissertation many of the results are exactly a quantitative version of some known properties of sofic groups or related with them. But in some case the procedures of classical proofs loose all quantitative information, forcing us to find other proofs.

In [18] it is proved that there exist finitely generated groups with Følner function growing faster than any *recursive* function, in [23] it is asked if that it's possible also for some finitely presented groups. So we need the definition of recursive function and this brings us closer to the heart of the thesis.

Given a function $f : \mathbb{N} \to \mathbb{N}$, informally, the Church-Turing thesis states that every way to define the computability of f is equivalent to the computability definition given by a Turing machine. More formally, Godel, Church and Turing defined in three different ways the class of recursive, or computable functions. In 1936 Turing and Church proved that the class of computable functions is the same in the three definitions. We give, later, another equivalent definition for computability, by using the Minsky machines.

A 2-glasses Minsky machine is a model of computation. Imagine we are given two glasses, A and B, and an infinite number of coins. The glasses can contain any finite number of coin each. The machine can put a coin into a glass, can check if a glass is empty, and if not can take a coin out from it. The program of the machine consists in a finite sequence of numbered instructions, $0, 1, \dots N$. The instant configuration of the machine is given by $(i, \epsilon_A, \epsilon_B)$ where $i \in \{0, 1, \dots N\}$ is the instruction that the machine is reading, $\epsilon_A \in \mathbb{N}$ and $\epsilon_B \in \mathbb{N}$ are, respectively, the number of coins in A and in B. The instruction has one of these forms:

$$i \to Add(A), j$$

$$i, \epsilon_A > 0 \to Sub(A), j$$

$$i, \epsilon_A = 0 \to j$$

$$0 \to STOP.$$

In the first instruction the machine is in i, adds a coin to A and goes to j; in the second it checks if A is empty, if not it takes a coin and goes in j; in the third it checks if A is empty and goes to j, if it is not empty goes to $i+1$. The same instructions hold for B. At the beginning the machine has input $(1, n, m)$, it starts to read and execute the instructions, it stops when it goes to the instruction 0.

We are now in position to define a recursive function.

Definition. *A function $f : \mathbb{N} \to \mathbb{N}$ is computable (recursive) if there exists a Minsky machine with INPUT $(1, 2^n, 0)$ and OUTPUT $(0, 0, 2^{f(n)})$.*

Example. *The program:*

- $1, \epsilon_A = 0 \to 0$
- $2, \epsilon_A > 0 \to Sub(A), 3$
- $3, \to Add(B), 4$
- $4, \to Add(B), 1$

with INPUT $(1, n, 0)$ *has OUTPUT* $(0, 0, 2n)$.
Therefore the function $n \mapsto n + 1$ *is computable.*

By a theorem of Minsky all Turing-computable functions are Minsky-computable functions. Now it's simple to extend the concept of computability to other things, at first we can say algorithm, effective procedure, Turing machine and Minsky machine and obtain the same concept of computability. Moreover we say that a subset $A \subset \mathbb{N}$ is computable (recursive) if χ_A, the characteristic function of A, is computable. More generally we may define computability of subsets of a countable set with a given enumeration.

Given a finitely presented group

$$\Gamma = \langle X | R \rangle,$$

in $\mathbb{F}_X$ we can fix an enumeration of the reduced words and ask if $\ker \pi$, the normal closure of the finite set R, is recursive or not. Actually this is equivalent to asking for the existence of an algorithm with:
INPUT $\omega \in \mathbb{F}_X$
OUTPUT 0 if $\omega \in \ker \pi$, 1 if $\omega \notin \ker \pi$.

This problem is called the word problem and it was formulated by Dehn in 1911, many years before the study about computability started. Then he

gave an explicit algorithm to solve the word problem for the fundamental group of closed orientable two-dimensional manifolds of genus greater than or equal to 2 (but now we can use the same algorithm for hyperbolic groups). Magnus in 1935 solved the word problem for one-relator groups (finitely presented group with $|R| = 1$). But to see a negative answer we should wait till the studies of Turing, in particular he proves the non computability of the halting problem: there's no algorithm with:
INPUT (M, x) where M is an algorithm and x is an input for M,
OUTPUT 1 if M stops with input x, 0 if M doesn't stop with x.
After this Markov and Post in 1947 constructed finitely presented semigroups with unsolvable word problem: the idea is to simulate a Turing machine with a semigroup and obtain that the computability of word problem implies the computability of halting problem. In the fifties we finally have a finitely presented group with unsolvable word problem, by Novikov and Boone. But maybe the theorem that crowned the theory of computability inside the theory of groups is the Higmann embedding theorem 1960s: a finitely generated group is a subgroup of a finitely presented group if and only if it is recursively presented (Γ is recursively presented if there is an algorithm that lists all elements in R). So the answer to a purely algebraic question is provided by the theory of recursive functions. In particular, since it is easy to find recursively presented groups with unsolvable word problem, the theorem easily provides the existence of finitely presented groups with unsolvable word problem.

We can now come back to our classes of groups. If Γ is finitely presented, residually finite we know that for each $g \in \Gamma \setminus \{1\}$ there exists an homomor-

phism ϕ onto a finite group such that $\phi(g) \neq 1$. Since Γ is finitely presented it's easy to see that we can list all the words in $\ker \pi$. Consider the following algorithm:
Given $\omega \in \mathbb{F}_X$ as input we list all elements in $\ker \pi$ and stop if we find ω. Simultaneously we can list all homomorphisms onto finite groups (again thanks to the finite presentation of Γ and the possibility to list all finite groups) and stop if we find ϕ such that $\phi(\pi(\omega)) \neq 1$ (we can check it because finite groups have solvable word problem).
Then if $\omega \in \ker \pi$ we find it in the first list and the algorithm stops, if $\omega \notin \ker \pi$ we know that $\pi(\omega) \in \Gamma \setminus \{1\}$ and then the second algorithm stops. This is the Malcev algorithm to solve the word problem in finitely presented residually finite groups.

On the other hand Kharlampovich constructed in [24] finitely presented solvable of step 3 (and therefore amenable) groups, with unsolvable word problem. Again the idea is to simulate Minsky machines with unsolvable halting problem.

In some sense for finitely presented groups the residual finiteness is *effective*: it is possible to compute the homomorphisms onto finite groups. This gives the solvability of the word problem for this kind of groups. What happens in the class of amenable and sofic groups? It is natural to try to understand which are the *effectively amenable groups* and *effectively sofic groups*. We thank A. Thom who gave us the idea to study effective soficity.

The definition of effective soficity is very natural, this means that there exists an algorithm with:
INPUT $n \in \mathbb{N}$
OUTPUT $k \in \mathbb{N}$,$(\sigma_1, \dots \sigma_d) \in Sym(k)$ the n-approximation.
For effective amenability we want an algorithm with:

INPUT $n \in \mathbb{N}$

OUTPUT $F \subset \mathbb{F}_X$ such that $\pi(F) \subset \Gamma$ is n-Følner.

It has no sense in fact asking as output a subset for a group that in general has unsolvable word problem, but we can search for the preimages of Følner sets. In particular it means that if this algorithm exists we can list the elements of a Følner set, but maybe with repetitions.

The most important facts of the thesis are:

$$\Gamma \text{ effectively sofic } \Longleftrightarrow \Gamma \text{ sofic with solvable WP },$$

$$\Gamma \text{ effectively amenable } \;\underset{\Longleftarrow}{\not\Longrightarrow}\; \Gamma \text{ amenable with solvable WP.}$$

It is natural to ask for the relations with the Følner function, also to show examples of amenable groups which are not effectively amenable. A function $f : \mathbb{N} \to \mathbb{N}$ is subrecursive if there exists a recursive upper bound. It is known that there exist functions which are not subrecursive. In [18] for each function f finitely generated groups with Følner function greater than f are constructed. If we start with a non subrecursive function f we obtain a non subrecursive Følner function. Noting that the effective amenability implies the subrecursivity of F, then these groups are not effectively amenable (and therefore have unsolvable word problem).

We don't know if subrecursivity of F_Γ is equivalent to computability of Følner sets in Γ. For the analogue question about soficity we have the answer, again thanks to Kharlampovich groups:

$$K_\Gamma \text{ subrecursive } \not\Longrightarrow \Gamma \text{ effectively sofic (and then solvable WP)}$$

because the sofic dimension growth is bounded from above by the Følner function which is, in the case of Kharlampovich groups, subrecursive. And we know that the word problem is unsolvable.

Finally we give some stability properties for effectively amenable groups and for groups with subrecursive sofic dimension growth. For the first ones we construct explicit preimages of Følner sets for some amenable extensions and for the second ones we find several upper bounds for K_Γ, in particular for direct products, semidirect products with amenable groups and free products.

Many questions, that we will present throughout the text, remain open. A natural ambitious program would be to find some good algorithmic property for sofic groups, maybe so good that it wouldn't be true for all finitely generated groups.

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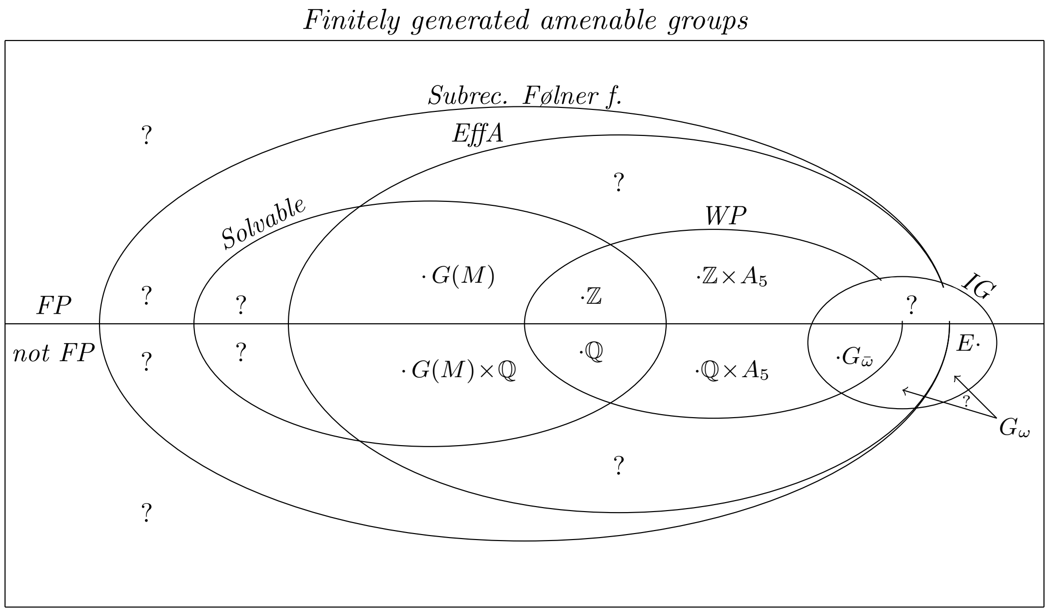
List of definitions and results



LEGENDA

FP	*finitely presented groups;*
WP	*group with solvable word problem;*
EffA	*effectively amenable groups (groups with computable Følner sets);*
IG	*intermediate growth groups;*
$G(M)$	*Kharlampovich group (see Section 1.7);*
E	*Erschler groups with non subrecursive Følner function in* [18]*;*
$G_{\bar\omega}$	*Grigorchuk groups with $\bar\omega$ computable (see* [21]*);*
G_ω	*Grigorchuk groups with ω non computable (see* [21]*).*

Solvable groups are disjoint from IG groups by Milnor's Theorem. For intermediate growth groups the subrecursivity of Følner function is equivalent to the computability of Følner sets (we know that a subsequence of the balls is a Følner sequence). Grigorchuk asked for the growth rate of the Følner function of G_ω, generalization of the Grigorchuk group described in [21]. Gromov asked for amenable groups with non subrecursive Følner function; there are only not finitely presented examples by Erschler [18]. The remaining non trivial inclusions and questions will be discussed in the sequel.

Chapter 2

- Definition of computable Følner sets or effective amenability (EffA),
- WP+amenability implies EffA,
- Kharlampovich groups are EffA and then EffA does not imply WP,
- f.g. extensions of an amenable solvable WP groups by abelian groups are EffA,
- semidirect products between f.g. groups EffA are EffA,
- N, G, K f. g. groups such that:$1 \to N \to G \to K \to 1$, N EffA, distortion function Δ_N^G subrecursive, K amenable solvable WP, then G is EffA.

Chapter 3

- amenable+WP implies subrecursive Følner function F
- upper bound Følner function Kharlampovich groups and solvable groups [1]
- upper bound Følner function semidirect product
- upper bound Følner function amenable extension in term of distortion function
- definition of function LEF, quantitative version of LEF property and comparison with function R quantifying residual finiteness [2]
- definition of sofic dimension growth K, asymptotic invariance, comparison with other version of sofic growth [2]
- comparison between LEF, F and K functions [2]
- definition of function Stb, quantitative version of weak stability of a finite set of relations
- remark about the arbitrary large subrecursive growth of Stb
- definition of computability of sofic approximation or effective soficity (EffS)
- EffS if and only if WP+soficity
- subrecursivity of K in Kharlampovich group (K subrecursive $\not\Rightarrow$ EffS)
- upper bound for K of direct products[2]
- upper bound for K of semidirect products with amenable groups
- upper bound for K of free product.

[1] different proof of something known

[2] independent proof of something already proved in the unpublished work [2]

Chapter 1

Preliminaries and notation

In this chapter we fix for good some notation, essentially about finitely generated groups. For example the symbols and the description of the presentation of the finitely generated group Γ will be the same, unless explicit mention, troughout the whole thesis.

Moreover we report some classic definitions and some results without proofs. In the thesis we will quantify some of these results (then we will prove stronger versions). Actually in some cases we present alternative version of known results or definitions, for example we prove the general equivalence between metric approximation on subsets or on generators, that we will quantify in the last chapter in the case of soficity.

1.1 Finitely generated groups

Let Γ be a group, a finite subset $X = \{x_1, \ldots, x_d\}$ *generates* Γ if $\langle X \rangle := \bigcup_{n\in\mathbb{N}}(X \cup X^{-1})^n = \Gamma$. Consider $\mathbb{F}_X$, the free group on X: the group of all free reduced words in $x_1, \ldots, x_d, x_1^{-1}, \ldots, x_d^{-1}$. When it is possible we will use the same symbol for the generator x_i in the group Γ and the generator in $\mathbb{F}_X$.

We have a canonical epimorphism $\pi : \mathbb{F}_X \longrightarrow \Gamma$, the homomorphism extension of the map $x_i \mapsto x_i$. In general given a word $\omega \in \mathbb{F}_d$, given d elements $g_1, \ldots, g_d$ of a group G, we denote by $\omega(g_1, \ldots, g_d)$ the obvious element of G. Then $\pi(\omega) = \omega(x_1, \ldots, x_d)$.

The kernel $\ker \pi$ is the normal subgroup of the words of $\mathbb{F}_X$ trivial in Γ. If R is a subset of $\mathbb{F}_X$ such that $\ker \pi = R^{\mathbb{F}_X}$, where $R^{\mathbb{F}_X}$ is the *normal closure* of R in $\mathbb{F}_X$, then we shortly write:
$\Gamma = \langle X|R\rangle$, where $\langle X|R\rangle$ is a *presentation* of Γ.

A classical result about presentations:

Theorem 1.1.1. *If $\Gamma = \langle X|R\rangle$, $\Gamma' = \langle Y|S\rangle$ and $\Gamma \cong \Gamma'$ then by using a sequence of Tietze Transformations R^+, R^-, X^+, X^- we can transform $\langle X|R\rangle$ to $\langle Y|S\rangle$.*

In particular if X and Y are finite then X^+ and X^- appear just a finite number of times in the sequence. We'll recall Tietze Transormations in Subsection 3.3.1, for a general reference see [28].

For each element $g \in \Gamma$ we define $|\cdot|$ as the word length with respect to X, $|g| := \min\{|\omega|,\ \omega \in \mathbb{F}_X, \pi(\omega) = g\}$, where $|\omega|$ is the usual length of a reduced word in $\mathbb{F}_X$. Sometimes we write $|g|_X$ to underline the dependence of the length on the set of generators. We denote with B_n the ball of radius n in $\mathbb{F}_X$ and with $B_n(\Gamma) := \pi(B_n)$ in Γ. We denote by e the identity element of Γ and by 1_G the identity element of a generic group G. For a subset $\mathcal{A} \subset \mathbb{F}_X$ we denote $|\mathcal{A}|_w := \max\{|\omega|,\ \ \omega \in \mathcal{A}\}$ and for a subset $A \subset \Gamma$ we denote $|A|_X := \max\{|a|_X,\ \ a \in A\}$.

1.2 Amenable groups

The class of amenable groups appear in many areas of mathematics, in particular there are a lot of different equivalent characterizations.

Theorem 1.2.1. *For a discrete group G, the following are equivalent:*

- *G has no paradoxical decomposition;*
- *there exists a state in $\ell^\infty(G)$ which is invariant under the left translation action;*
- *G has an approximate invariant mean;*
- *G satisfies the Følner condition;*
- *the trivial representation of G is weakly contained in the regular representation of G;*
- *the C^*-algebra and the reduced C^*-algebra of G coincide.*
- *any continuous affine action of G on a nonempty compact convex subset of a locally convex space has a fixed point.*

A discrete group G is amenable if one of the conditions in Theorem 1.2.1 holds. Actually the list is not complete: if we restrict to finitely generated groups, for example, Γ is amenable if and only if the spectral radius of the Markov operator associated to (Γ, X) is 1 (Kesten-Day Theorem). But also Γ is amenable if and only if $\lim_{n\to\infty} |B_n \cap \ker \pi|^{\frac{1}{n}} = 2|X| - 1$ (Grigorchuk-Cohen cogrowth theorem). We also have a specialized Følner property for Γ and this is the unique definition that we will use:

Definition 1.2.1. Γ *is amenable if for every* $\epsilon > 0$ *there exists a finite nonempty subset* $F \subset \Gamma$ *such that*

$$\frac{|F \setminus xF|}{|F|} \leq \epsilon, \quad \forall x \in X \cup X^{-1}.$$

The class of amenable groups contains finite groups, abelian groups and more generally groups with subexponential growth. It is closed under the operations of taking subgroups, taking quotients, taking extensions, and taking inductive limits. Non abelian free groups and groups containing non abelian free subgroups are not amenable.

For proofs, references and systematic treatment see [9, 11].

1.3 Residually finite groups

The class of residually finite groups is a generalization of the class of finite groups, very different from amenability.

Definition 1.3.1. *A group* G *is* residually finite *if for each element* $g \in G$ *with* $g \neq 1_G$*, there exist a finite group* F *and a homomorphism* $\Phi : G \to F$ *such that* $\Phi(g) \neq 1_F$*.*

It is easy to see that it is equivalent to the fact that finite index normal subgroups of G separate the elements of G or that G is embeddable into a direct product of finite groups. Finite groups are obviously residually finite, finitely generated abelian, and more generally metabelian, groups are residually finite and the class of residually finite groups is closed under taking subgroup, taking projective limits and finite extensions.

For a given group F a map $\phi : \Gamma \to F$ is a homomorphism if and only if $\phi(\pi(r)) = 1_F$ for all $r \in R$ and we can completely describe a homomorphism fixing $\phi(x_1), \dots, \phi(x_d) \in F$.

Then, if Γ is residually finite, for each $\omega \in \mathbb{F}_X \setminus \ker \pi$, we can find a finite group F and $f_1, \dots f_d \in F$ such that $\omega(f_1, \dots f_d) \neq 1_F$ and $r(f_1, \dots f_d) = 1_F$ for all $r \in R$.

Finally with a simple direct-product we can obtain an apparently stronger form of residual finiteness in the case of finitely generated groups:

Proposition 1.3.1. *Γ is residually finite if and only if for every $n \in \mathbb{N}$ we have a finite group F and d elements $f_1, \dots, f_d \in F$ such that:*

$$\omega(f_1, \dots, f_d) \begin{cases} = 1_F & \text{if } \omega \in \ker \pi \\ \neq 1_F, & \text{if } \omega \in B_n \setminus \ker \pi. \end{cases} \tag{1.1}$$

In this way the residual finiteness appears as an approximation property of Γ by finite groups. We see now a large generalization of this.

1.4 Metric approximations

An *invariant length group* is a group G equipped with a function $\ell : G \to \mathbb{R}^+$ such that, $\forall g, h \in G$ we have:

$$\ell(g) = 0 \Leftrightarrow g = 1_G, \qquad \ell(g^{-1}) = \ell(g),$$
$$\ell(gh) \leq \ell(g) + \ell(h), \qquad \ell(g^h) = \ell(g).$$

For each group G the trivial length, 0 on 1_G and 1 elsewhere, is invariant. Another example is the symmetric group $Sym(k)$ with the *normalized Hamming length*, the normalized number of points not fixed by a permutation. That is, for $\sigma \in Sym(k)$:

$$\ell_H(\sigma) := \frac{|\{i \in \{1, 2, \dots, k\} : \sigma i \neq i\}|}{|k|}.$$

Given such a length, we can define the associated distance on G, simply by $d(g,h) := \ell(gh^{-1})$, that is bi-invariant: $d(g,h) = d(xg,xh) = d(gy,hy)$. Viceversa from a bi-invariant distance d we obtain an invariant length $\ell(g) := d(g, 1_G)$.

We can speak about continuous or isometric maps or homomorphisms between length invariant groups. In particular every injective homomorphism is isometric with the trivial length. Another example of isometric homomorphism is the inclusion of the symmetric group on a finite set A into the symmetric group on $A \times B$, where B is a finite set:

$$i : Sym(A) \hookrightarrow Sym(A \times B)$$

$$i(\sigma)(a,b) = (\sigma a, b), \ \ \forall a \in A, \ \forall b \in B, \ \forall \sigma \in Sym(A).$$

Definition 1.4.1. *Given a group G and a group F equipped with a bi-invariant distance d, for $K \subset G$ and $\epsilon > 0$, a map $\phi : G \to F$ is (K,ϵ)-approximation of G by F if:*

$$\begin{aligned} &d(\phi(gh), \phi(g)\phi(h)) \leq \epsilon, && g,h \in K; \\ &d(\phi(g), \phi(h)) \geq 1 - \epsilon, && g,h \in K, g \neq h. \end{aligned}$$

For each class of invariant length groups $\mathcal{C}$ we can consider the following property. *G is approximable by the class $\mathcal{C}$* if for each finite subset $K \subset G$ and for each $\epsilon > 0$, there exist $C \in \mathcal{C}$ and a (K,ϵ)-approximation of G by C.

If the groups of the class $\mathcal{C}$ are equipped with the trivial length we just say that G is locally embeddable into $\mathcal{C}$. The two main examples are *local embeddability into finite* and *into amenable groups* (resp. LEF and LEA). In these cases we can choose $\epsilon = 0$ and obtain $(K,0)$-approximations that simply are locally multiplicative and locally injective maps, so the approximation is purely algebraic. The first introduction of LEF property is in [37]. In Subsection 3.2 we give a direct proof, in quantitative version, of the fact that residually finite groups are LEF and that finitely presented LEF groups are residually finite.

For finitely generated groups it is possible express the the approximation fixing the images of the generators:

Proposition 1.4.1. Γ *is approximable by a class of invariant length groups* $\mathcal{C}$ *if and only if for every* $n \in \mathbb{N}$ *there exist* $C \in \mathcal{C}$ *and* $c_1, \dots c_d \in C$ *such that:*

$$\ell_C(\omega(c_1, \dots, c_d)) \begin{cases} \leq \frac{1}{n}, & \text{if } \omega \in B_n \cap \ker \pi \\ \geq 1 - \frac{1}{n}, & \text{if } \omega \in B_n \setminus \ker \pi. \end{cases} \tag{1.2}$$

Proof. *If* part.
For each finite subset $K \subset \Gamma$, for each $\epsilon > 0$ we fix $n \in \mathbb{N}$ such that $4|K|_X \leq n$ and $n^{-1} \leq \epsilon$. We find $C \in \mathcal{C}$ and $c_1, \dots c_d \in C$ such that conditions (1.2) hold for this n.

For all $g \in \Gamma$ we choose $w_g \in \mathbb{F}_d$ such that $g = \pi(w_g)$ and $|w_g| = |g|$. In this way we can define a map:

$$\phi \ : \Gamma \to C$$
$$g \longmapsto w_g(c_1, \dots, c_d).$$

For $g, h \in K$

$$\begin{aligned} d_C(\phi(gh), \phi(g)\phi(h)) &= \ell_C(w_{gh}(c_1, \dots, c_d)^{-1} w_g(c_1, \dots c_d) w_h(c_1, \dots, c_d)) \\ &= \ell_C(w(c_1, \dots, c_d)) \end{aligned}$$

where $w := w_{gh}^{-1} w_g w_h$.
But $|w| \leq |w_{gh}| + |w_g| + |w_h| = |gh| + |g| + |h| \leq 4|K|_X \leq n$, moreover

$w \in \ker \pi$ and then $d_C(\phi(gh), \phi(g)\phi(h)) \leq \frac{1}{n}$.

For $g, h \in K$, $g \neq h$,

$$d_C(\phi(g), \phi(h)) = \ell_C(w_g^{-1} w_h(c_1, \ldots, c_d)) \geq 1 - \frac{1}{n}$$

because $w_g^{-1} w_h \in B_n \setminus \ker \pi$. And so the map ϕ is a (K, ϵ)-approximation.

Only if part.

For each $N \in \mathbb{N}$ consider $\phi : \Gamma \to C$, a $(B_N(\Gamma), \frac{1}{N})$-approximation. We want to find N such that the elements $c_1 := \phi(x_1)$, $c_2 := \phi(x_2), \ldots,$ $c_d := \phi(x_d)$ respect the conditions in (1.2).

We have:

$$\frac{1}{N} \geq d_C(\phi(e), \phi(e)\phi(e)) = \ell_H(\phi(e)),$$

For all $w \in B_n$, $w(x_1, x_2, \ldots, x_d) \in B_N(\Gamma)$ if $N \geq n$, then by the triangle inequality:

$$d_C(w(\phi(x_1), \phi(x_2), \ldots, \phi(x_d)), \phi(w(x_1, x_2, \ldots, x_d))) \leq n\frac{1}{N}.$$

If $w \in \ker \pi$ then $w(x_1, x_2, \ldots, x_d) = \pi(w) = 1_\Gamma$ and

$$\ell_C(w(\phi(x_1), \phi(x_2), \ldots, \phi(x_d))) \leq (n+1)\frac{1}{N}.$$

If $w \notin \ker \pi$:

$$\ell_C\big(w(\phi(x_1), \ldots, \phi(x_d))\big) \geq d_C\big(w(\phi(x_1), \ldots, \phi(x_d)), \phi(1_\Gamma)\big) - \frac{1}{N} \geq$$

$$d_C\big(\phi(w(x_1, \ldots, x_d)), \phi(1_\Gamma)\big) - d_C\big(w(\phi(x_1), \ldots, \phi(x_d)), \phi(w(x_1, \ldots, x_d))\big) - \frac{1}{N}$$

$$\geq 1 - (n+2)\frac{1}{N}.$$

So if we choose $N = n(n+2)$ we have that for $c_1, c_2, \ldots, c_d$ the conditions (1.2) hold. $\square$

1.5 Sofic groups

In the language of the previous section, we could define *soficity* as the approximability by the class of finite symmetric groups equipped with normalized Hamming distance. Explicitly:

Definition 1.5.1. *A group G is sofic if for all $\epsilon > 0$, for any finite subset $K \subset G$ there exist a $k \in \mathbb{N}$ and a map $\phi : \Gamma \to Sym(k)$ such that:*

$$d_H(\phi(gh), \phi(g)\phi(h)) \leq \epsilon, \qquad g, h \in K;$$

$$d_H(\phi(g), \phi(h)) \geq 1 - \epsilon, \qquad g, h \in K, \ g \neq h.$$

It easy to see that finite groups are sofic, but more generally:

Proposition 1.5.1. *Amenable groups are sofic.*

Proposition 1.5.2. *If G is locally embeddable into the class of sofic groups the G is sofic.*

Then LEF and LEA groups are sofic.

We report a short summary of the stability property of sofic groups from [8]:

Theorem 1.5.1. *The class of sofic groups are closed with respect to the following operations:*

- *Subgroups;*
- *Direct limits;*
- *Direct products;*
- *Inverse limits;*

- *Extension by amenable groups*
- *Free product;*
- *Free product amalgamated over amenable groups;*
- *HNN extension over amenable groups;*
- *Graph product.*

Question. *Does there exist a non sofic: group, finitely presented group, group with solvable word problem, hyperbolic group, one-relator group, group with Haagerup property, group approximable by finite groups with any invarian length (weakly sofic), group approximable by general linear groups equipped with normalized rank distance (linear sofic), hyperlinear group?*

Here some references for introductions and works on sofic groups: [1, 8, 9, 14–17, 22, 32, 33, 36, 38].

1.6 Algorithmic problems in finitely generated groups

The informal presentations of Minsky machines and recursive functions in the Introduction are enough for our treatise, so we consider them as given.

The fundamental algorithmic problems for finitely generated groups are:

word problem:

IN $\omega \in \mathbb{F}_X$,

OUT 0 if $\omega \in \ker \pi$ and 1 otherwise;

conjugacy problem:

IN $\omega_1, \omega_2 \in \mathbb{F}_X$,

OUT 0 if $\pi(\omega_1)$ and $\pi(\omega_2)$ are conjugates in Γ and 1 otherwise;

isomorphism problem:

IN (X, R), $R \subset \mathbb{F}_X$,

OUT 0 if $\Gamma \cong \{e\}$, 1 otherwise;

and more generally

IN (X, R), (X', R'),

OUT 0 if the two presentations generate isomorphic groups, 1 otherwise.

And we have that the isomorphism problem implies the conjugacy problem that implies the word problem. But the three are undecidable in general. Actually we often restrict to recursively presented groups that means that R is recursively enumerable (there exists a Minsky machine that lists all elements in R). Under the hypothesis of recursive presentation $\ker \pi$ is recursively enumerable.

Remark 1.6.1. *If Γ is finitely presented the set of isomorphic finite presentations of Γ are recursively enumerable. It implies that given two presentations of the same group the sequence of Tietze transformations from the first presentation to the second one is computable.*

Theorem 1.6.1. *The following conditions imply the solvability of the word problem for a finitely generated group Γ:*

- Γ *is finite;*
- Γ *is abelian;*
- Γ *is recursively presented and simple;*
- Γ *is finitely presented and residually finite;*
- Γ *is hyperbolic;*
- Γ *is embeddable into a simple subgroup of a finitely presented group.*

In particular the last condition is equivalent to the word problem (Boone-Higman theorem).

The first examples of unsolvable word problem in finitely presented groups were given by Boone and Novikov. But for our purpose we introduce the first examples of finitely presented solvable groups with unsolvable word problem, by Kharlampovich.

1.7 Kharlampovich groups

We refer directly to [26] for a nice description of the Kharlampovich group, first constructed in [24]. In this section we shortly present the generator sets and recall the properties needed in the sequel.

We start with a prime p and a Minsky machine M with K glasses and $0, 1, \dots N$ instructions for which the set of inputs for which M will stop is not recursive. We denote the group $G(M)$.

Consider the letters $q_0, \dots q_N$ associated to the instructions and $A_0, \dots A_K$ associated to the glasses (plus 0). Consider the free abelian monoid generated by $A_0, \dots A_K$, let U_0 be the divisors of $A_0 A_1 \dots A_K$ in this monoid (in other words the parts of $\{0, 1, \dots K\}$). Finally U is the set of the words $q_j w$, $j = 0, \dots K$, $w \in U_0$.

Now we can describe the generators, divided into three sets:

$$L_0 := \{x_u : \ u \in U\};$$

$$L_1 := \{A_i : \ i = 0, \dots K\};$$

$$L_2 := \{a_i, \, a_i', \, \widetilde{a}, \widetilde{a}' : \ i = 1, \dots K\}.$$

There are two kinds of relations, dependent or independent on M. The first simulate the instructions of M and finally give the unsolvability of the word problem. The independent relations are related to the algebraic structure of the group, we simply present some consequences.

Denoting:

$$H_j := \langle L_j \rangle,\ j = 0, 1, 2,\ \ H := \langle L_1 \cup L_2 \rangle,$$

- H_j is abelian,
- H_0, H_1 are of exponent p,
- $H_1^H =$ is abelian of exponent p,
- $H = H_1^{H_2} \rtimes H_2$,
- $H_0^G(M)$ is abelian of exponent p,
- $G(M) = {H_0}^G(M) \rtimes H$

Then $G(M)$ is semidirect product of a metabelian group with an abelian group of exponent p.

Chapter 2

Shape of Følner sets

In this chapter we introduce the Følner sets and we study their shape for suitable classes of groups. But actually we are interested in an effective construction so we give the definition of computable Følner sets by the preimages of Følner sets in the free groups, where the word problem is solvable and it's well defined what it means to give a subset like an output of an algorithm.

In fact for groups with solvable word problem it is easy to check the computability of the Følner sets but actually we are more interested in the setting of groups with unsolvable word problem. We prove in a very elementary way that the Kharlampovich group, finitely presented, solvable and then amenable group, with unsolvable word problem, has computable Følner sets. We also obtain some information about the Følner sets of the extensions, therefore we find some stability properties for the class of groups with computable Følner sets.

In particular we prove that semidirect products between finitely presented groups with computable Følner sets have computable Følner sets and the computability of Følner sets for a group that is extension of an amenable group with solvable word problem by a group with computable Følner sets with subrecursive distortion function. Some of these results will be used in the next Chapter to obtain bounds for the cardinality of these Følner sets (Følner function).

But now we start with one of the classical definitions of Følner sets:

Definition 2.0.1. *Let* Γ *be an amenable finitely generated group,* X *a finite set of generators. For all* $n \in \mathbb{N}$ *we define the family of* n*-Følner sets, the Følner sets with at least* $\frac{1}{n}$ *approximation. We write:*

$$\mathfrak{F}\o l_{\Gamma,X}(n) := \{F \in \Gamma,\, F \neq \emptyset,\, F \text{ finite },\, \frac{|F \setminus xF|}{|F|} \leq n^{-1},\, \forall x \in X \cup X^{-1}\}.$$

The simplest class of amenable groups for which we know the Følner sets is the class of finite groups. If $|\Gamma| < \infty$ we clearly have $\Gamma \in \mathfrak{F}\o l_{\Gamma,X}(n)$ for all $n \in \mathbb{N}$.

There are a lot of (asymptotically) equivalent ways to define Følner sets, also inside the finitely generated groups. This one in particular is such that in the lattice of $\mathbb{Z}^2$ the $n \times n$ squares are n-Følner, and it will be useful.

2.1 Computable Følner sets

We are interested in computing these Følner sets. A priori we don't know if it is possible just for groups with solvable word problem or not, for this reason we give the definition on the preimages of Følner set inside the free group: recall that $\pi : \mathbb{F}_X \to \Gamma$ is the canonical projection into the finitely generated group Γ

Definition 2.1.1. Γ *has* computable Følner sets *if there exists an algorithm with:*

INPUT: $n \in \mathbb{N}$

OUTPUT: $F \subset \mathbb{F}_X$ *finite, such that* $\pi(F) \in \mathfrak{F}\o l_{\Gamma,X}(n)$.

At first we can see that this definition does not depend on the choice of presentation for finitely presented groups:

Proposition 2.1.1. *Let* $\langle X, R\rangle$ *and* $\langle X', R'\rangle$ *be two finite presentations of the same group* Γ*, inducing respectively* $\pi : \mathbb{F}_X \to \Gamma$ *and* $\pi' : \mathbb{F}_{X'} \to \Gamma$*. The Følner sets of* Γ *are computable by* π *if and only if they are computable by* π'*.*

Proof. By Remark 1.6.1 we can algorithmically find the Tietze transformations and then the homomorphism $\phi : \mathbb{F}_X \to \mathbb{F}_{X'}$ such that $\pi' \circ \phi = \pi$ is computable. □

A very obvious fact is that:

Theorem 2.1.1. *A finitely generated amenable group with solvable word problem has computable Følner sets.*

Proof. By the solvability of the word problem, given a finite set $F \subset \mathbb{F}_X$ it is possible to compute $|\pi(F)|$ and $|\pi(F) \setminus \pi(xF)|$, $\forall x \in X$. So we can construct an algorithm with:
INPUT: $F \subset \mathbb{F}_X$ finite,
OUTPUT: answer to '$\pi(F) \in \mathfrak{F}\o l_{\Gamma,X}(n)$?'
Finally we start a (fixed) enumeration of the finite subsets of F_X and apply this algorithm for each. We stop when we obtain -yes-. □

A natural question is if the converse is true, that is, if the class of groups with computable Følner sets is the class of amenable groups with solvable word problem. The answer is NO, this chapter is dedicated to prove this and some stability properties of this class of groups.

2.2 Geometry of Følner sets

The family of Følner sets has some good properties:

Proposition 2.2.1.

(i) $A, B \in \mathfrak{F}\text{ø}l_{\Gamma,X}(n)$ *and* $A \cap B = \emptyset$ *implies* $A \cup B \in \mathfrak{F}\text{ø}l_{\Gamma,X}(n)$*;*

(ii) $A \in \mathfrak{F}\text{ø}l_{\Gamma,X}(n)$ *implies* $Ag \in \mathfrak{F}\text{ø}l_{\Gamma,X}(n)$ $\forall g \in \Gamma$*;*

(iii) $A \cup B \in \mathfrak{F}\text{ø}l_{\Gamma,X}(n)$, $A \cap B = A \cap XB = \emptyset$
implies $A \cup Bg \in \mathfrak{F}\text{ø}l_{\Gamma,X}(n)$ $\forall g \in \Gamma$ *such that* $A \cap Bg = \emptyset$.

Proof. (i) First it is clear that if $A \cap B = \emptyset$ we have $|A \cup B| = |A| + |B|$. In general $(A \cup B) \setminus x(A \cup B) \subset (A \setminus xA) \cup (B \setminus xB)$. Finally the sum of the two inequalities $|A \setminus xA| \leq n^{-1}|A|$ and $|B \setminus xB| \leq n^{-1}|B|$ gives us the thesis.

(ii) The translation from the right doesn't change the cardinality of the sets:
$|A| = |Ag|$ and $|A \setminus xA| = |(A \setminus xA)g| = |Ag \setminus xAg|$.

(iii) In this case we have $(A \cup B) \setminus x(A \cup B) = (A \setminus xA) \cup (B \setminus xB)$ so:

$$\frac{|(A \cup Bg) \setminus x(A \cup Bg)|}{|A \cup Bg|} \leq \frac{|(A \setminus xA)| + |(Bg \setminus xBg)|}{|A| + |Bg|} = \frac{|(A \setminus xA) \cup (B \setminus xB)|}{|A| + |B|} =$$

$$= \frac{|(A \cup B) \setminus x(A \cup B)|}{|A \cup B|} \leq n^{-1}.$$

□

These properties give us some information about the shape of Følner sets or at least the shape that we can choose. In fact given a Følner set $F \in \mathfrak{F}\text{ø}l_{\Gamma,X}(n)$, consider $f \in F$, we have $Ff^{-1} \in \mathfrak{F}\text{ø}l_{\Gamma,X}(n)$ and $|F| = |Ff^{-1}|$, and clearly $e \in Ff^{-1}$. The other important property that it is possible to have is the connection.

Definition 2.2.1. *We say that $K \subset \Gamma$ is* left-connected *(resp.* right-connected*) if for any $A, B \subset K$ such that $K = A \bigsqcup B$ we have $A \cap BX \neq \emptyset$, (resp. $A \cap XB \neq \emptyset$).*

It's simple to notice that a left[right]-connected subset K describes a connected subgraph of the left[right] Cayley graph.

To have connection we need a stronger property of invariance for Følner sets:

Definition 2.2.2.

$$\mathfrak{F}\o l'_{\Gamma,X}(n) := \{F, \text{ non empty finite subset of } \Gamma : \ \frac{|\partial_X F|}{|F|} \leq \frac{1}{n}\},$$

where $\partial_X F := \{f \in F : \exists\, x \in X \cup X^{-1} : xf \notin F\}$.

But X is finite and $\partial_X F = \bigcup_{X \cup X^{-1}} (F \setminus xF)$, and then

$$\frac{|F \setminus xF|}{|F|} \leq \frac{|\partial_X F|}{|F|} \leq \sum_{X \cup X^{-1}} \frac{|F \setminus x'F|}{|F|}.$$

It implies:

$$\mathfrak{F}\o l_{\Gamma,X}(|X \cup X^{-1}|n) \subset \mathfrak{F}\o l'_{\Gamma,X}(n) \subset \mathfrak{F}\o l_{\Gamma,X}(n).$$

So in particular the definition of amenability is unaffected by the choice of this different type of Følner sets. Often we are interested in the *optimal Følner sets*, which are the smallest sets in $\mathfrak{F}\o l_{\Gamma,X}(n)$ and $\mathfrak{F}\o l'_{\Gamma,X}(n)$. They will be important also to quantify amenability.

For the optimal Følner sets, at least for those in $\mathfrak{F}\o l'_{\Gamma,X}(n)$, we have the connection. This is a well known fact:

Lemma 2.2.1. *For $F \in \mathfrak{F}\o l'_{\Gamma,X}(n)$, if for every $F' \in \mathfrak{F}\o l'_{\Gamma,X}(n)$ we have $|F| \leq |F'|$, then F is right-connected.*

Proof. By contradiction. Suppose that F can be written non-trivially as $F = A_1 \sqcup A_2$ and that for any $x \in X$ we have $xA_1 \cap A_2 = A_1 \cap xA_2 = \emptyset$. In particular $|A_1| < |F|$ and $|A_2| < |F|$. Then by minimality

$$|\partial_X A_1| > n^{-1}|A_1|$$

$$|\partial_X A_2| > n^{-1}|A_2|$$

and then the sum:

$$|\partial_X A_1| + |\partial_X A_2| > n^{-1}(|A_1| + |A_2|).$$

But $\partial_X F = \partial_X A_1 \sqcup \partial_X A_2$ because:
if $g \in \partial_X F$ then $g \in F$ and there exists $i \in \{1, 2\}$ such that $g \in A_i$, but also there is $x \in X$ such that $xg \notin F$ that implies $xg \notin A_i$ and then the inclusion;
if $g \in \partial_X A_i$ then $g \in A_i \subset F$ and there exists $x \in X$ sucht that $xg \notin A_i$, but $xA_1 \cap A_2 = A_1 \cap xA_2 = \emptyset$ and then $xg \notin A$.
Finally $\frac{|\partial_X F|}{|F|} > n^{-1}$, contradicting the hypothesis. $\square$

So if Γ is amenable we have $F \in \mathfrak{F}\o l_{\Gamma,X}(n)$ that is right-ed, in particular we can assume $e \in F$ and then $|F| \subset B_{|F|}(\Gamma)$: we have a bound for the word length of elements in F.

Finally the result that allowed us to check the invariance property only under the action of generators for amenability:

Lemma 2.2.2. *For any $F \in \mathfrak{F}\o l_{\Gamma,X}(n)$ and for all $g \in \Gamma$ we have:*

$$\frac{|F \setminus gF|}{|F|} \leq |g|n^{-1}. \tag{2.1}$$

Proof. By induction on $|g|$: if $|g| = 1$ it is trivial. Suppose $g = xg'$ with $x \in X \cup X^{-1}$ and $|g'| = |g| - 1$, by induction hypothesis for g' the (2.1) holds. Then:

$$\frac{|F \setminus xg'F|}{|F|} = \frac{|x^{-1}F \setminus g'F|}{|F|}$$

and recalling that $A \setminus B \subset A \setminus C \cup C \setminus B$, we have:

$$\frac{|F \setminus xg'F|}{|F|} \leq \frac{|x^{-1}F \setminus F|}{|F|} + \frac{|F \setminus g'F|}{|F|} \leq \frac{|F \setminus xF|}{|F|} + |g'|n^{-1} \leq |g|n^{-1}.$$

□

2.3 Abelian groups

We first consider the simple case of the free abelian group:

Example 2.3.1. *Consider $\mathbb{Z}^d$ with generators $x_1, x_2, \dots x_d$.*

Define $C_n = \{x_1^{i_1} x_2^{i_2} \dots x_d^{i_d} : \quad i_1, i_2, \dots i_d \in \{0, 1, \dots n-1\}\}$. Then

$$C_n \setminus x_j C_n = \{x_1^{i_1} \dots x_{j-1}^{i_{j-1}} x_{j+1}^{i_{j+1}} \dots x_d^{i_d} : \quad i_1, \dots i_{j-1}, i_{j+1} \dots i_d \in \{0, 1, \dots n-1\}\},$$
$$C_n \setminus x_j^{-1} C_n = \{x_1^{i_1} \dots x_{j-1}^{i_{j-1}} x_j^{n} x_{j+1}^{i_{j+1}} \dots x_d^{i_d} : \quad i_1, \dots i_{j-1}, i_{j+1} \dots i_d \in \{0, 1, \dots n-1\}, \}$$

in the group the only relations are those given by the commutativity, so we have $|C_n| = n^d$ and $|C_n \setminus x_j^{\pm 1} C_n| = n^{d-1}$. This implies that $C_n \in \mathfrak{F}\o l_{\mathbb{Z}^d, X}(n)$.

Following the construction for $\mathbb{Z}^d$ we build the Følner set for a general abelian finitely generated group. But we want an algorithmic description of the preimages in the free group.

Definition 2.3.1.

$$\mathfrak{C}_n(x_1, \dots x_d) := \{x_1^{i_1} x_2^{i_2} \dots x_d^{i_d} : \quad i_1, i_2, \dots i_d \in \{0, 1, \dots n-1\}\} \subset \mathbb{F}_X.$$

Now we want to prove that $\pi(\mathfrak{C}_n(x_1, \ldots x_d)) \subset \Gamma$ is n-Følner. More generally we define C_n for arbitrary elements and we have:

Proposition 2.3.1. *Let $y_1, y_2, \ldots y_s$ be commuting elements of Γ not necessarily distinct. Set*

$$C_n(y_1, y_2, \ldots y_s) := \{y_1^{i_1} y_2^{i_2} \ldots y_s^{i_s} : \quad i_1, i_2, \ldots i_s \in \{0, 1, \ldots n-1\}\}.$$

We have:

$$\frac{|C_n(y_1, y_2, \ldots y_s) \setminus y_j^{\pm 1} C_n(y_1, y_2, \ldots y_s)|}{|C_n(y_1, y_2, \ldots y_s)|} \leq n^{-1}, \quad \forall j \in \{0, 1, \ldots d\}.$$

Proof. We write C_n instead of $C_n(y_1, y_2, \ldots y_s)$:

- $C_n(y_1, y_2, \ldots y_s) = C_n(y_1) C_n(y_2, y_3 \ldots y_s)$

  and

  $$C_n(y_1) \setminus y_1 C_n(y_1) = \begin{cases} \emptyset & \text{if } y_1 \text{ has order less or equal to } n \\ \{e\} & \text{otherwise.} \end{cases}$$

- $C_n \setminus y_1 C_n \subset C_n(y_2, y_3, \ldots, y_s)$
  because in $C_n \setminus y_1 C_n \subset C_n(y_1) \setminus y_1 C_n(y_1) C_n(y_2, y_3, \ldots, y_s)$

- $y_1^k [C_n \setminus y_1 C_n] \subset C_n$ for $k \in \{0, 1, \ldots n-1\}$
  if $g \in y_1^k [C_n \setminus y_1 C_n]$ so there exist $\widehat{i}_2, \ldots \widehat{i}_s \in \{0, 1, \ldots n-1\}$ such that $g = y_1^k y_2^{\widehat{i}_2} \ldots y_s^{\widehat{i}_s}$ and so $g \in C_n$.

- $y_1^k [C_n \setminus y_1 C_n] \cap [C_n \setminus y_1 C_n] = \emptyset \;\; \forall k \in \{1, \ldots n-1\}$,
  if $g \in y_1^k [C_n \setminus y_1 C_n]$ so $g = y_1^k y_2^{\widehat{i}_2} \ldots y_s^{\widehat{i}_s}$, but $k \neq 0$ and then $g \notin C_n \setminus y_1 C_n$.

So C_n contains n disjoint translations of $C_n \setminus y_1 C_n$, that is:

$$C_n \supset \bigsqcup_{k=0}^{n-1} y^k [C_n \setminus y_1 C_n].$$

It is easy do the same with y_1^{-1} and we have

$$C_n \supset \bigsqcup_{k=0}^{n-1} y^{-k} [C_n \setminus y_1^{-1} C_n].$$

So $\frac{|C_n \setminus y_1^{\pm 1} C_n|}{|C_n|} \leq n^{-1}$. □

In particular if Γ is abelian $C_n(x_1, x_2, \dots x_d) \in \mathfrak{F}\o l_{\Gamma,X}(n)$.

Remark 2.3.1. *Actually we can prove it also noting that the product of two n-Følner sets is an n-Følner set for the direct product of the groups, the products of two "cubes" C_n is still a cube. But the cubes are n-Følner sets for $\mathbb{Z}^d$ and clearly for finite groups (for n big enough) so the n-cubes are n-Følner for finite direct products of finite or free abelian groups, so for finitely generated abelian groups.*

Finally $\pi(\mathfrak{C}_n(x_1, \dots x_d)) = C_n(x_1, \dots x_d)$.

Corollary 2.3.1. *If Γ is abelian then it has computable Følner sets.*

Actually the ordering in the set doesn't matter if we project into an abelian group, in this case, for a finite subset $Y \subset \mathbb{F}_X$, we define $\mathfrak{C}_n(Y)$ and $C_n(Y) = \pi(\mathfrak{C}_n(Y))$.

2.4 Kharlampovich groups: The Revenge

We give a very simple and effective construction of n-Følner sets for a particular semidirect product that is the one that was used twice to construct the Kharlampovich groups:

Theorem 2.4.1. *Let $\Gamma = \langle L_1 \cup L_2 \rangle$ be a finitely generated group, L_1 and L_2 two finite disjoint subsets and respectively H_1 and H_2 the subgroups that they generate. Suppose that H_2 is amenable, H_1^{Γ} is abelian and $\Gamma = H_1^{H_2} \rtimes H_2$, then:*

$$AC_n(L_1^A) \in \mathfrak{F}\o l_\Gamma(n), \quad \forall A \in \mathfrak{F}\o l_{H_2}(n).$$

where $L_1^A = \{a^{-1}xa : \ a \in A, x \in L_1\}$.

Proof. $B := C_n(L_1^A)$, $|AB| = |A||B|$ because $A \subset H_2$ and $B \subset H_1^H$ and $H_2 \cap H_1^{H_2} = \{e\}$.
For $x \in L_2 \cup L_2^{-1}$ we have:

$$\frac{|AB \setminus xAB|}{|AB|} \leq \frac{|A \setminus xA||B|}{|A||B|} \leq n^{-1}.$$

For $x \in L_1 \cup L_1^{-1}$, using Proposition 2.3.1, we have:

$$\frac{|AB \setminus xAB|}{|AB|} = \frac{|\{ab : \ a \in A, \ b \in B : ab \notin xAB\}|}{|A||B|} =$$

$$= \frac{|\{ab : \ a \in A, \ b \in B : b \notin a^{-1}xAB\}|}{|A||B|} \leq \frac{|\{ab : \ \ a \in A, \ b \in B : b \notin a^{-1}xaB\}|}{|A||B|} \leq$$

$$\leq \frac{|\bigcup_{a \in A} a(B \setminus a^{-1}xaB)|}{|A||B|} \leq n^{-1}.$$

□

In the case of Kharlampovich group $G(M)$ we have (see Subsection 1.7):

$$C_n(L_2) \in \mathfrak{F}\o l_{H_2}(n),$$

$$C_n(L_2)C_n(L_1^{C_n(L_2)}) \in \mathfrak{F}\o l_H(n),$$

but H_1^H is of exponent p, so for $n \geq p$ we have $C_n = C_p$ in H_1^H and the same holds in $H_0^{G(M)}$, so finally:

$$C_n(L_2)C_p(L_1^{C_n(L_2)})C_p(L_0^{C_n(L_2)C_p(L_1^{C_n(L_2)})}) \in \mathfrak{F}\o l_{G(M)}(n).$$

So we have a finitely presented group $G(M)$ with unsolvable word problem with computable Følner sets: we have an algorithm with input n and output a finite subset of the free group projecting onto an n-Følner set in $G(M)$. Of course we don't know if some different words of the set represent the same element in the group. And we have also a bound from above for the cardinality of these sets.

Corollary 2.4.1. *The class of finitely presented groups with computable Følner sets is larger than the class of finitely presented amenable groups with solvable word problem.*

2.5 Amenable extensions

Amenability is stable under semidirect products and more generally under amenable extensions, the most common proofs of this do not use the characterization of amenability by Følner sets. The book [11] is one of the exceptions and in [25] it was shown explicitly that a Følner net for the semidirect product is given by the product of the Følner nets of the factor groups. But it's not an effective procedure to have, fixing $n \in \mathbb{N}$, an n-Følner set.

We first consider general abelian extensions, but apparently the procedure doesn't ensure the computability of the Følner sets:

Proposition 2.5.1. *If Γ is finitely generated by X and $N \triangleleft \Gamma$ is an abelian normal subgroup, denoting with $\rho : \Gamma \to \Gamma/N$ the projection:*

$$AC_{4n|A^{-1}XA\cap N|}(A^{-1}XA \cap N) \in \mathfrak{F}\o l_{\Gamma,X}(n),$$

for each $A \subset \Gamma$ such that $|A| = |\rho(A)|$ and $\rho(A) \in \mathfrak{F}\o l_{\Gamma/N,\rho(X)}(2n)$.

Proof. Consider $S := A^{-1}XA \cap N$, it is finite and $|S| \leq |A|^2|X|$.
Denote $B := C_{4n|S|}(S)$, $B \subset N$ and then by Lemma 2.3.1 we have

$\frac{|B \setminus sB|}{|B|} \leq (4n|S|)^{-1}$ for all $s \in S \cup S^{-1}$.

Consider the set $F := AB$, notice that $|F| = |A||B|$ because the intersection $A \cap B$ has at most one element since $\rho_{|A}$ is injective and ρ sends B to the identity of Γ/N. So for $g \in F$ we write $g = ab$, $a \in A$, $b \in B$ in a unique way (again because $\rho_{|A}$ is injective and $\rho(g) = \rho(a)$) and we write $A' := \rho(A)$, that is $2n$-Følner.

For each $x \in X \cup X^{-1}$, the set $F \setminus xF$ is disjoint union of E_1 and E_2:

$$E_1 = \{g \in F \setminus xF : \rho(g) \notin \rho(x)A'\}$$

$$E_2 = \{g \in F \setminus xF : \rho(g) \in \rho(x)A'\}.$$

If $g \in E_1$, since $\rho(g) = \rho(a) \notin \rho(x)A'$ so $\rho(a) \in A' \setminus \rho(x)A'$. But ρ is injective on A then:

$$\frac{|E_1|}{|F|} = \frac{|A' \setminus \rho(x)A'||B|}{|A||B|} \leq (2n)^{-1}.$$

If $g \in E_2$ then $\rho(a) \in \rho(x)A' = \rho(xA)$, hence there exist $a' \in A, s \in N$ such that $as = xa'$. So $s \in S \cup S^{-1}$ and $g = xa's^{-1}b$, but since $g \notin xAB$ then $b \notin sB$:

$$\frac{|E_2|}{|F|} \leq \frac{|\{xa's^{-1}b,\ a' \in A,\ s \in S \cup S^{-1},\ b \in B \setminus sB\}|}{|A||B|} \leq \sum_{s \in S \cup S^{-1}} \frac{|B \setminus sB|}{|B|} \leq (2n)^{-1}.$$

$\square$

Consider the case of Γ/N amenable with solvable word problem and with the set $\rho(X)$ as generators. If $\pi_{\Gamma/N} : \mathbb{F}_X \to \Gamma/N$ is the canonical epimorphism, for every n we can compute $\mathcal{A} \in \mathbb{F}_X$ such that $\pi_{\Gamma/N}(\mathcal{A}) \in \mathfrak{F}\text{ø}l_{\Gamma/N,\rho(X)}(2n)$ (by Theorem 2.1.1), but also with $|\mathcal{A}| = |\pi_{\Gamma/N}(\mathcal{A})|$, by the solvability of the word problem.

But then $A := \pi_\Gamma(\mathcal{A})$ is such that $\rho(A) = \pi_{\Gamma/N}(\mathcal{A}) \in \mathfrak{F}\o l_{\Gamma/N,\rho(X)}(2n)$ and $|A| = |\rho(A)|$, because:

$$|\rho(A)| \leq |A| \leq |\mathcal{A}| = |\pi_{\Gamma/N}(\mathcal{A})|.$$

Moreover, given an element $\omega \in \mathcal{A}^{-1}X\mathcal{A}$ we can compute if $\pi_{\Gamma/N}(\omega) = 1$ or not, and then we can compute the preimage of $A^{-1}XA \cap N$ in $\mathbb{F}_X$ and finally we can compute a preimage of the n-Følner sets for Γ.

Corollary 2.5.1. *A finitely presented group that is an extension of an amenable group with solvable word problem by an abelian group has computable Følner sets.*

This implies again that Kharlampovich group has computable Følner sets, because it is an abelian extension of a finitely presented metabelian, and therefore residually finite with solvable WP, group.

Notice that the abelian group could be not finitely generated.

Remark 2.5.1. *In general if we observe that* $AC_{4n|S|}(S) \subset AC_{4n|A|^2|X|}(A^{-1}XA)$ *so even if* Γ/N *has non-computable Følner sets, we can compute sets containing them.*

The situation is clearer if the extension splits. In this case we can also consider more generally the extension by an amenable group.

Theorem 2.5.1. *Let* $N = \langle X|R_1\rangle$ *and* $H = \langle Y|R_2\rangle$ *be finitely generated groups and let* $\phi : H \to Aut(N)$ *be homomorphism. Let* $c := \max\{|\phi_y(x)|_X : x \in X,\ y \in Y\}$ *and consider* $N \rtimes H = \langle X, Y|R_1,\, R_2,\, x^y = \phi_y(x)\ \forall x \in X,\, \forall y \in Y\rangle$ *then:*

$$AB \in \mathfrak{F}\o l_{N\rtimes H}(n)$$

for every $A \in \mathfrak{F}\o l_H(n)$*, and every* $B \in \mathfrak{F}\o l_N(nc^{|A|_Y})$*.*
(Remember that $|A|_Y = \max\{|a|_Y, a \in A\}$*).*

Proof. $|AB| = |A||B|$ because $A \subset H$ and $B \subset N$,
for $y \in Y \cup Y^{-1}$:

$$\frac{|AB \setminus yAB|}{|AB|} \leq \frac{|A \setminus yA||B|}{|A||B|} \leq n^{-1}.$$

For $x \in X \cup X^{-1}$:

$xab = aa^{-1}xab = a\phi_a(x)b$, so

$\{ab \in AB : xab \notin AB\} \subset \{ab \in AB : \phi_a(x)b \notin B\}$.

But $|\phi_a(x)|_X \leq c^{|a|_Y} \leq c^{|A|_Y}$. Then, using Lemma 2.2.2:

$$\frac{|AB \setminus xAB|}{|AB|} = \frac{|\bigcup_{a\in A} a[B \setminus \phi_a(x)B]|}{|A||B|} \leq \frac{\sum_{a\in A} |B \setminus \phi_a(x)B|}{|A||B|} \leq \frac{|\phi_a(x)|_X}{c^{|A|_Y} n} \leq n^{-1}.$$

□

We can observe that the thesis is true also if $B \in \mathfrak{F}\o l_H(n')$ with $n' \geq ns^{|A|_Y}$. In particular this implies that the semidirect product between two finitely generated groups with computable Følner sets has computable Følner sets because if we know a preimage for the Følner set A we also have a bound for $|A|_Y$ and so we know a right input for algorithm computing B, Følner set of N.

Corollary 2.5.2. *The semidirect product between two finitely generated groups with computable Følner sets has computable Følner sets.*

Finally we give the computation for the general extension by amenable finitely generated groups. Before this, we recall the definition of the distortion function.

Definition 2.5.1. *We call the map* $\Delta_N^G : \mathbb{N} \to \mathbb{N}$ *with:*

$$\Delta_N^G(n) := \max\{|\omega|_Y : \omega \in N, |\omega|_X \leq n\}$$

the distortion function.

Theorem 2.5.2. *Let Γ be generated by the finite set X and N be a normal subgroup of Γ generated by the finite set Y. Let $\rho : \Gamma \to K := \Gamma/N$ be the projection to the quotient. Then:*

$$AB \in \mathfrak{F}\o l_{\Gamma,X}(n),$$

$$\textit{if } A' := \rho(A) \in \mathfrak{F}\o l_{K,\rho(X)}(2n),\ |A| = |A'|,\ |A|_X \leq |A'|_{\rho(X)}$$

$$\textit{and } B \in \mathfrak{F}\o l_{N,Y}(2n|A'|^2|X \cup X^{-1}|\Delta_N^G(2|A'|_{\rho(X)} + 1)).$$

Proof. Denoting with $F := AB$ it is easy to see that $|F| = |A'||B|$ because ρ is injective on A.

For each $x \in X \cup X^{-1}$, the set $F \setminus xF$ is disjoint union of E_1 and E_2:

$$E_1 = \{g \in F \setminus xF : \rho(g) \notin \rho(x)A'\}$$

$$E_2 = \{g \in F \setminus xF : \rho(g) \in \rho(x)A'\}.$$

We can write $g = ab$, with $b \in B$, in a unique way.

If $g \in E_1$, since $\rho(g) = \rho(a) \notin \rho(x)A'$ so $\rho(a) \in A' \setminus \rho(x)A'$, ρ is injective on A so:

$$\frac{|E_1|}{|F|} = \frac{|A' \setminus \rho(x)A'||B|}{|A'||B|} \leq (2n)^{-1}.$$

If $g \in E_2$ then $\rho(g) = \rho(a) \in \rho(x)A'$ hence there exists $a' \in A$ such that $\rho(a) = \rho(x)\rho(a')$. The images by ρ of a and xa' are the same so there is $s \in N$ such that $as = xa'$.

If we call $S := A^{-1}XA$ we see that $s \in S \cup S^{-1}$ and $|S \cup S^{-1}| \leq |A|^2|X \cup X^{-1}|$. Then $g = xa's^{-1}b$, but since $g \notin xAB$ then $b \notin sB$:

$$\frac{|E_2|}{|F|} \leq \frac{|\{xa's^{-1}b,\ a' \in A,\ s \in R,\ b \in B \setminus sB\}|}{|A'||B|} \leq \sum_{s \in S \cup S^{-1}} \frac{|B \setminus sB|}{|B|}.$$

We have a bound for $|S|$, we need a bound for the length of the elements in $S \cup S^{-1}$:

$|s|_Y \leq \Delta_N^\Gamma(|s|_X)$. But $|s|_X = |a^{-1}xa'|_X \leq 2|A|_X + 1 \leq 2|A'|_{\rho(X)} + 1$

And so by Lemma 2.2.2:

$$\frac{|B \setminus sB|}{|B|} \leq (2n|A'|^2|X \cup X^{-1}|)^{-1} \leq \frac{|S \cup S^{-1}|}{2n}.$$

Finally $\dfrac{|F \setminus xF|}{|F|} = \dfrac{|E_1|}{|F|} + \dfrac{|E_2|}{|F|} \leq n^{-1}$.

□

Suppose that N and K have computable Følner sets. For each k we can construct $\mathcal{A} \subset \mathbb{F}_X$ such that $\pi_K(\mathcal{A}) \in \mathfrak{F}\o l_K(k)$. We denote $A' := \pi_K(\mathcal{A})$. If we consider $A := \pi_\Gamma(\mathcal{A})$, it is clear that $\rho(A) = A' \in \mathfrak{F}\o l_K(k)$ as in hypothesis of the theorem. But we want the bound also on the cardinality and on the length: if $|\mathcal{A}| > |A'|$ we could have $|A| > |A'|$. But if we restrict to the case in which K has solvable word problem we can detect $\mathcal{A}$ such that $|\pi_K(\mathcal{A})| = |\mathcal{A}|$ and $|\pi_K(\mathcal{A})|_{\rho(X)} = |\mathcal{A}|_w$. So we can compute a preimage for a set A respecting the hypothesis of the theorem. For the set B we just need the computability (of a bound) of the number $2n|A'|^2|X \cup X^{-1}|\Delta_N^G(2|A'|_{\rho(X)} + 1)$. While we have automatically (by to the computability of Følner sets in K) a computable bound for $|A'|$ and $|A'|_{\rho(X)}$, it is possible that Δ_N^G is not subrecursive, see for example [3], also in solvable groups, see [13].

Corollary 2.5.3. *Let N, G, K finitely generated groups such that:*

$$1 \to N \to G \to K \to 1.$$

If N has computable Følner sets, Δ_N^G is subrecursive, K is amenable with solvable word problem, then G has computable Følner sets.

Remark 2.5.2. *By the Remark 2.5.1 we know that a finitely generated solvable group* Γ *has the following property that we can call* subcomputability of Følner sets

There exist an algorithm with:

INPUT $n \in \mathbb{N}$

OUTPUT $F \subset \mathbb{F}_X : \exists A \in F\o l_{\Gamma,X} : A \subset \pi(F)$.

We can conclude this statement also observing that the free solvable groups have solvable word problem and then they have computable Følner sets. If we look for example at the proof of [18, Lemma 2.2], *we can see that even if the projection of a n-Følner set is not n-Følner, it contains an n-Følner set. We know something more:* $\rho : G_1 \to G_2$ *a projection,* G_1 *generated by* X_1 *and* G_2 *by* $X_2 := \rho(X_1)$*: if* $A \in \mathfrak{F}\o l_{G_1,X_1}(n)$ *then there exists* $i \in \mathbb{N}$, $1 \le i \le |A|$ *such that the set* $B_i := \{b \in G_2 : |\rho^{-1}(b) \cap A| \ge i\}$ *belongs to* $F\o l_{G_2,X_2}(n)$.

But if we haven't solvable word problems apparently we cannot compute B_i *and we cannot say which i is right.*

- *Have solvable groups computable Følner sets?*
- *Is computability of Følner sets stable under quotients?*
- *Does subcomputability imply computability of Følner sets?*

The questions are in order of generality, apparently are distinct questions: while for finitely generated amenable groups and solvable groups we have a "universal group" for which in its quotients all groups of the class can be embedded, it is not the case of finitely generated amenable group. The examples to see this are the Erschler's groups in [18] *and they also are examples of amenable groups with non computable Følner sets. Infact another neces-*

sary condition for the computability of Følner sets is the subrecursivity of the Følner function, we will see it in the next chapter.

Chapter 3

Quantifications

One of the obstructions to compute approximations with arbitrary precision is the possibility that the growth of the dimension of these approximations is faster than any recursive function. This is what happens for example for the non computability of Følner sets: actually we have found examples just in this way. So on purpose to study effectiveness of some properties, especially of approximation properties, it is natural to try to quantify them.

In this Chapter we start with the definition of a famous function, the *Følner function*, introduced by Vershik. In particular we use an asymptotically equivalent version compatible with our definition of Følner sets. The Følner function is the cardinality of the optimal Følner sets and then in some sense it quantifies amenability. Vershik conjectured for it the possibility of a growth faster than any iterated exponentials, Erschler in [18] shows finitely generated groups with Følner function faster then any recursive function (while it is open for finitely presented groups). With the theory of the previous chapter we know that these groups have unsolvable word problem and non computable Følner sets. It's just a short step from effective constructions to the quantifications, so we give as corollaries some upper bounds for Følner functions even if at least those about solvable groups are already known. The final goal of the section is that the class of finitely generated groups with subrecursive Følner function is closed under extension with subrecursive distortion.

In the following section we present the depth function introduced by K. Bou-Rabee to quantify the residual finiteness and an analogous function for the local embeddability into finite groups and the comparisons of these two functions.

In the last section we study a quantificative function for soficity. The first introduction of the *sofic dimension growth* is in an unpublished work of Arzhantseva and Cherix [2]. The idea to study algorithmic questions in this setting was suggested to me by Thom and the paper [12]. Cornulier in fact introduced his own version of quantification of soficity, the sofic profile.

We prove independently some results that will be also in [2]. Then we introduce a quantifying version of the weak stability (see [4]) and observe that there's no uniform recursive bound for all weak stable groups. We prove that under sofic condition the computability of sofic approximations is equivalent to the word problem and is not equivalent to subrecursivity of sofic dimension growth. Finally we give an upper bound for sofic dimension growth of direct products, free products and splitting amenable extensions of sofic groups.

3.1 Følner function

Definition 3.1.1. *For an amenable group* Γ *generated by a finite set* X*, remember that for all* $n \in \mathbb{N}$*:*

$$\mathfrak{F}\o l_{\Gamma,X}(n) = \{F \subset \Gamma,\ F \neq \emptyset,\ F \text{ finite },\ \frac{|F \setminus xF|}{|F|} \leq n^{-1},\ \forall x \in X \cup X^{-1}\}.$$

So we define the Følner function $F : \mathbb{N} \to \mathbb{N}$ *as*

$$F_{\Gamma,X}(n) := \min\{|A|, A \in \mathfrak{F}\o l_{\Gamma,X}(n)\}.$$

This is a well studied function. It is shown that the asymptotic behaviour does not depend on presentation. The first simple remark is that if Γ is a group with computable Følner sets then the function F is subrecursive. So in particular, as a consequence of Theorem 2.1.1:

Proposition 3.1.1. *If* Γ *is finitely generated amenable group with solvable word problem than* F_Γ *is subrecursive.*

We know that there exist finitely generated groups with F_Γ non subrecursive [18], so these groups have unsolvable word problem, and in general there exist finitely generated groups with non computable Følner sets, but we don't know examples of groups with subrecursive Følner function and non computable Følner sets.

Remark 3.1.1. Γ *has subcomputable Følner sets if and only if* F_Γ *is subrecursive.*

If the preimage of a set containing an n-Følner is computable then we can compute a bound for the cardinality of an n-Følner and thus F_Γ *is subrecursive. Viceversa by Lemma 2.2.1 we have n-Følner sets contained in* $B_{F_\Gamma(|X \cup X^{-1}|n)}(\Gamma)$*, so if* F_Γ *is subrecursive we can compute a ball large enough* *to contain an n-Følner.*

So we can reformulate the last question in Remark 2.5.2 in the following way:

Question 1. *Does the subrecursivity of F_Γ imply the computability of Følner sets of Γ?*

We will see that the analogous question for soficity has negative answer.

In this section we can translate the results from the previous chapter to obtain upper bounds for the Følner function for some amenable extensions. Asymptotically equivalent bounds for solvable groups could be also found as corollaries of the works [19] and [18], or using the comparison with the Følner function in free solvable groups in [35].

The simplest case is that of finite groups, where the Følner function is eventually constant, and of abelian groups: if $\Gamma = \langle X \rangle$ is abelian then from Lemma 2.3.1 we clearly have $F_\Gamma(n) \leq n^{|X|}$.

From Theorem 2.4.1 we can choose the optimal Følner sets and consider the cardinality:

Corollary 3.1.1. *Let $\Gamma = \langle L_1 \cup L_2 \rangle$ be a finitely generated group, L_1 and L_2 two finite disjoint subsets and respectively H_1 and H_2 the subgroups that they generate. Suppose that H_2 is amenable, $H_1^{H_2}$ is abelian and $\Gamma = H_1^{H_2} \rtimes H_2$, then:*

$$F_{\Gamma, L_1 \cup L_2}(n) \leq F_{H_2, L_2}(n) n^{|L_1| F_{H_2, L_2}(n)}.$$

For Kharlampovich groups $G(M)$ (see Section 1.7 and Section 2.4) we have:

$$F_{G(M)}(n) \leq |C_n(L_2) C_p(L_1^{C_n(L_2)}) C_p(L_0^{C_n(L_2) C_p(L_1^{C_n(L_2)})})| \leq$$

$$\leq n^{|L_2|} p^{|L_1| n^{|L_2|}} p^{|L_0| n^{|L_2|} p^{|L_1 n^{|L_2|}}}.$$

From Proposition 2.5.1 we have:

Corollary 3.1.2. *If Γ is finitely generated by X and $N \triangleleft \Gamma$ is an abelian normal subgroup, denoting with $\rho : \Gamma \to \Gamma/N$ the projection:*

$$F_\Gamma(n) \leq F_{\Gamma/N}(2n)(2n|X|F_{\Gamma/N}(2n)^2)^{|X|F_{\Gamma/N}(2n)^2}.$$

Proof. We consider $\rho(A) \in \mathfrak{F}\text{ø}l_{\Gamma/N}(2n)$ such that $|\rho(A)| = |A| = F_{\Gamma/N}(2n)$, recall that $S = A^{-1}XA$ and then $|S| \leq |A|^2|X|$. Finally observe that

$$AC_{2n|S|}(S) \subset AC_{2n|A|^2|X|}(A^{-1}XA).$$

$\square$

In particular, fixing two numbers $k, l \in \mathbb{N}$, there exists a recursive function $f : \mathbb{N} \to \mathbb{N}$ (asymptotically a k-iterated $n^{n^{\cdots^{n}}}$) such that for every group G solvable of step less than k and with less than l generators we have

$$F_G(n) \leq f(n).$$

We can say this, without any information on function f other than recursivity, directly by the solvability of word problem for free solvable groups (see Remark 2.5.2).

From Theorem 2.5.1 we obtain a bound. First we recall that for a group H generated by a finite set Y, if we are not interested in computability we can choose an optimal n-Følner set $A \in \mathfrak{F}\text{ø}l'_{H,Y}(n)$ that is right-connected (see Lemma 2.2.1) containing the identity 1 and then $|A|_Y \leq |A| \leq F_H(|Y \cup Y^{-1}|n)$.

Corollary 3.1.3. *Let* $N = \langle X|R_1\rangle$ *and* $H = \langle Y|R_2\rangle$ *be finitely generated groups and let* $\phi : H \to Aut(N)$ *be a homomorphism. and consider* $N \rtimes H = \langle X, Y|\ R_1,\ R_2,\ x^y = \phi_y(x),\ \forall x \in X, \forall y \in Y\rangle$*, then:*

$$F_{N\rtimes H}(n) \leq F_H(kn)F_N(nc^{F_H(kn)}),$$

where $c := \max\{|\phi_y(x)|_X :\ x \in X,\ y \in Y\}$ *and* $k := |Y \cup Y^{-1}|$*.*

Finally, from Theorem 2.5.2 and again using connectivity argument:

Corollary 3.1.4. *Let* N, Γ, K *finitely generated groups such that:*

$$1 \to N \to \Gamma \to K \to 1.$$

Then

$$F_\Gamma(n) \leq F_K(kn)F_N(knF_K(kn)^2\Delta_N^\Gamma(2F_K(kn)+1)),$$

where $k := 2|X \cup X^{-1}|$ *the finite generating set of* Γ*.*

This implies that if N and K have subrecursive Følner function and if Δ_N^Γ is subrecursive then Γ has subrecursive Følner function.

Coming back to our Question 1, it seems that apparently the property of subrecursivity of Følner function is more stable than computability of Følner sets. Or at least these stability properties are easier to prove.

3.2 Depth and LEF functions

We want to consider some quantifications of some other approximation properties that imply the soficity. First of all, the residual finiteness:

Definition 3.2.1. *Let* Γ *be a finitely generated residually finite group,*

$$R_\Gamma(n) := \max\{\min\{[\Gamma : N] :\ N \lhd \Gamma,\ g \notin N\}, |g| \leq n\} \quad n \in \mathbb{N}.$$

Bou-Rabee defines the function R_Γ in [5], we call it the *depth function* of Γ. He proves the independence on the generators (up to asymptotic equivalence) and computes the function for some groups. There are many works on the depth function, especially for the free groups [7, 27].

In [6] it is remarked that for finitely presented residually finite groups the depth function is computable and finitely generated residually finite groups with arbitrary large depth function were constructed. In [26] starting with particular Minsky machines with a computable halting problem and with the same construction of the groups with unsolvable word problem, they obtain finitely presented residually finite groups with arbitrary subrecursive large depth function.

A very closed property is the embeddability into finite groups (LEF), we first need an equivalent version, by Proposition 1.4.1:

Proposition 3.2.1. *Γ is LEF if and only if for every $n \in \mathbb{N}$ there exist a finite group F and $(f_1, \dots f_d) \in F^d$ such that :*

$$\omega(f_1, \dots, f_d) \begin{cases} = 1_F & \text{if } \omega \in B_n \cap \ker \pi \\ \neq 1_F, & \text{if } \omega \in B_n \setminus \ker \pi. \end{cases} \tag{3.1}$$

So we can consider the growth of the cardinality of these finite groups.

Definition 3.2.2. *For a LEF group Γ we define*

$$LEF_{\Gamma,X}(n) := \min\{|F| : \exists\, (f_1, \dots, f_d) \in F^d \text{ for which } (3.1) \text{ holds } \}, \quad n \in \mathbb{N}.$$

It is simple to see that the asymptotic behaviour does not depend on the presentation. We will see it in the next section for the function quantifying soficity, the proof is the same.

We know that residual finiteness implies LEF, we give a quantitative version of this fact:

Proposition 3.2.2.

$$LEF_\Gamma(n) \leq R_\Gamma(n)^{\xi(n)},$$

where $\xi(n)$ is the number of conjugacy classes of Γ intersecting $B_n(\Gamma)$.

Proof. Let $g_1, g_2, \ldots, g_{\xi(n)}$ be the representative elements of the non trivial conjugacy classes intersecting the ball of radius n, we denote with Q_g the smallest finite group such that there exists a surjective homomorphism ρ_g:

$$\rho_g : \Gamma \to Q_g$$

with $\rho_g(g) \neq 1$. We observe that also $\rho_g(hgh^{-1}) \neq 1$ for all $h \in \Gamma$. We define

$$\rho : \Gamma \longrightarrow \prod_{i=1}^{\xi(2n)} Q_{g_i} \quad \rho := \prod_{i=1}^{\xi(n)} \rho_{g_i}.$$

We want to prove that for $(\rho(x_1), \ldots, \rho(x_d))$ the relations (3.1) of Proposition 3.2.1 hold.

For $\omega \in B_n \cap \ker \pi$ we have $\omega(\rho(x_1), \ldots, \rho(x_d)) = \rho(\pi(\omega)) = 1$ since ρ is a homomorphism. If $\omega \in B_n \setminus \ker \pi$ then $\pi(\omega)$ is different from 1 so there exists i such that g_i is conjugate with $\pi(\omega)$ and so $\rho_{g_i}(\pi(\omega)) \neq 1$ that implies $\rho(\pi(\omega)) \neq 1$, but again $\omega(\rho(x_1), \ldots, \rho(x_d)) = \rho(\pi(\omega))$. We know that $|\prod_{i=1}^{\xi(n)} Q_{g_i}| \leq R_\Gamma(n)^{\xi(n)}$ so we have the thesis. $\square$

But we also know, in the case of finitely presented groups, that the local embeddability into finite groups implies the residual finiteness. This is the simple comparison of the associated growth functions:

Proposition 3.2.3. *Suppose that R, the set of relations in the presentation of Γ, is finite, then:*

$$R_\Gamma(n) \leq LEF_\Gamma(n), \quad \textit{for all } n \geq r,$$

where $r := \max\{|\omega| : \omega \in R\}$.

Proof. For $n \geq r$ the approximation $x_i \mapsto f_i$ gives a homomorphism from Γ to F, with $|F| = LEF_\Gamma(n)$ and for $g \in B_n(\Gamma)$, $g \neq 1_\Gamma$ we have a word $\omega_g \in B_n \setminus \ker \pi$ such that $\pi(\omega_g) = g$. $\square$

3.3 Sofic dimension growth

The aim of this last section is to describe the growth of the dimension of sofic approximations and the relation with algorithmic problems. As we see in Section 1.5 for soficity we need of (K, ϵ)-approximations, almost homomorphisms into a finite symmetric group with a sort of uniform property of injectivity. We might consider the minimum rank of the symmetric groups containing approximations, choosing the diagonal subsequence:

$$K'_\Gamma(n) = \min\{k \in \mathbb{N} : \exists\, \phi : \Gamma \to Sym(k), (B_n(\Gamma), \frac{1}{n}) - \text{approximation}\}.$$

It is very clear that a finitely generated group Γ is sofic if and only if the function K'_Γ assumes finite values.

But actually we want something closer to the word problem and more easily worked with computational tools.

The following notion of *sofic dimension growth* is due to G. Arzhantseva and P. A. Cherix in an unpublished work [2]:

Definition 3.3.1. *Suppose that Γ is sofic. For $n \in \mathbb{N}$ let $K_{\Gamma,\pi}(n)$ be the minimum integer k such that there exists $(\sigma_1, \sigma_2, \ldots \sigma_d) \in Sym(k)^d$ with the following property:*

$$\ell_H(\omega(\sigma_1, \ldots, \sigma_d)) \begin{cases} \leq \frac{1}{n}, & \textit{if } \omega \in B_n \cap \ker \pi \\ \geq 1 - \frac{1}{n}, & \textit{if } \omega \in B_n \setminus \ker \pi \end{cases} \tag{3.2}$$

In the language of the work of Arzhantseva and Paunescu [4], $K_{\Gamma,\pi}(n)$ is the minimum rank among $\frac{1}{n}-$strong solutions of R, the set of relations of Γ. We denote as *n-approximations* the permutations $\sigma_1, \sigma_2, \dots \sigma_d$.

For the sake of completeness we give the easy proofs of some propositions that will appear in [2] in the more general setting of metric approximations. We prove them independently but we know by private communications that they define the sofic dimension growth and many other growth functions and describe the comparisons between them (including what in this text are Propositions 3.3.1; 3.3.3; 3.3.4) and some quantifications of stability properties of soficity (including what in this text is Proposition 3.3.6).

The first thing to see is that Γ is sofic if and only if the function K_Γ assumes finite values. Again this is a particular case of Proposition 1.4.1. But we give directly a comparison with the function K'_Γ.

Proposition 3.3.1.

$$K'_\Gamma(n) \leq K_\Gamma(3n), \qquad K_\Gamma(n) \leq K'_\Gamma(n(n+2)), \qquad \forall n \in \mathbb{N}.$$

Proof. First inequality:
For every $n \in \mathbb{N}$ consider the $3n$-approximations $\sigma_1, \dots, \sigma_d \in S_{K(4n)}$, so the (3.2) of Definition 3.3.1 holds for $4n$. For all $g \in \Gamma$ we choose $w_g \in \mathbb{F}_d$ such that $g = \pi(w_g)$ and $|w_g| = |g|$. This defines a map:

$$\begin{aligned} \phi\ : \Gamma \longrightarrow & S_{K(n)} \\ g \longmapsto & w_g(\sigma_1, \dots, \sigma_d). \end{aligned}$$

For $g, h \in B_n(\Gamma)$,

$$\begin{aligned} d_H(\phi(gh), \phi(g)\phi(h)) &= \ell_H(w_{gh}(\sigma_1, \ldots, \sigma_d)^{-1} w_g(\sigma_1, \ldots) w_h(\sigma_1, \ldots)) \\ &= \ell_H(w(\sigma_1, \ldots, \sigma_d)) \end{aligned}$$

where $w := w_{gh}^{-1} w_g w_h$.

But $|w| \leq |w_{gh}| + |w_g| + |w_h| = 2|g| + 2|h| \leq 4n$, moreover $\pi(w) = e$ so $w \in B_{4n}(\Gamma) \cap \ker \pi$ and then $d_H(\phi(gh), \phi(g)\phi(h)) \leq \frac{1}{4n}$.

For $g, h \in B_n(\Gamma)$, $g \neq h$,

$$d_H(\phi(g), \phi(h)) = \ell_H(w_g^{-1} w_h(\sigma_1, \ldots, \sigma_d)) \geq 1 - \frac{1}{4n}$$

beacuse $w_g^{-1} w_h \in B_{4n} \backslash \ker \pi$. And so the map ϕ is a $(B_n(\Gamma), \frac{1}{n})$-approximation.

Second inequality:

For each $N \in \mathbb{N}$ consider $\phi : \Gamma \to Sym(K'_\Gamma(N))$, a $(B_N(\Gamma), \frac{1}{N})$-approximation. We want to find N such that the permutations $\sigma_1 := \phi(x_1)$, $\sigma_2 := \phi(x_2), \ldots,$ $\sigma_d := \phi(x_d)$ are n-approximations.

We have:

$$\frac{1}{N} \geq d_H(\phi(1_\Gamma), \phi(1_\Gamma)\phi(1_\Gamma)) = \ell_H(\phi(1_\Gamma)),$$

For all $w \in B_n$, $w(x_1, x_2, \ldots, x_d) \in B_N(\Gamma)$ if $N \geq n$, then by the triangle inequality:

$$d_H(w(\phi(x_1), \phi(x_2), \ldots, \phi(x_d)), \phi(w(x_1, x_2, \ldots, x_d))) \leq n\frac{1}{N}.$$

If $w \in \ker \pi$ then $w(x_1, x_2, \ldots, x_d) = \pi(w) = 1_\Gamma$ then

$$\ell_H(w(\phi(x_1), \phi(x_2), \ldots, \phi(x_d))) \leq (n+1)\frac{1}{N}.$$

If $w \notin \ker \pi$:

$$\ell_H\big(w(\phi(x_1), \ldots, \phi(x_d))\big) \geq d_H\big(w(\phi(x_1), \ldots, \phi(x_d)), \phi(1_\Gamma)\big) - \frac{1}{N} \geq$$

$$d_H\big(\phi(w(x_1, \ldots, x_d)), \phi(1_\Gamma)\big) - d_H\big(w(\phi(x_1), \ldots, \phi(x_d)), \phi(w(x_1, \ldots, x_d))\big) - \frac{1}{N}$$

$$\geq 1 - (n+2)\frac{1}{N}.$$

So if we choose $N = n(n+2)$ we have that $\sigma_1, \sigma_2, \ldots, \sigma_d$ are n-approximations.

$\square$

So in particular K_Γ is finite valued if and only if K'_Γ is finite valued and then if and only if Γ is sofic.

3.3.1 Asymptotic independence on presentation

We recall the four types of Tietze transformations on the presentations $\langle X|R\rangle$ of the group Γ.

R^+ Add a relator:

Replace $\Gamma = \langle X|R\rangle$ by $\Gamma = \langle X|R \cup w\rangle$, for some $w \in \ker \pi$.

R^- Remove a relator:

Replace $\Gamma = \langle X|R\rangle$ by $\Gamma = \langle X|R \setminus \{w\}\rangle$, for some $w \in R$ such that $R^\Gamma\rangle = (R \setminus \{w\})^\Gamma$.

X^+ Add a new generator:

Replace $\Gamma = \langle X|R\rangle$ by $\Gamma = \langle X \cup \{y\}|R \cup \{yw^{-1}\}\rangle$, for any $w \in \mathbb{F}_X$.

X^- Remove a generator:

Replace $\Gamma = \langle X|R\rangle$ by $\Gamma = \langle X \setminus \{y\}|R \setminus \{w\}\rangle$, if $yw^{-1} \in \ker \pi$ and if all $s \in R \setminus \{w\}$ do not contain y or y^{-1}.

Now we compute the variation of the function K after these transformations.

$$R^+, R^- :$$

The kernel $\ker\pi$ of the canonical homomorphism is the same in the two presentations. The word length does not depend by the relations in the presentations, $B_n \cap \ker\pi$ is the same after these transformations so, by definition, the function $K(n)$ is the same.

$$X^- :$$

For simplicity suppose that we remove the last generator.

Before: $\langle x_1, x_2, \ldots, x_k \mid R\rangle$, $\ker\pi_1 \triangleleft \mathbb{F}_k$ the kernel of the canonical homomorphism;

after: $\langle x_1, x_2, \ldots, x_{k-1} \mid R \setminus \{w_0\}\rangle$ $\ker\pi_2 \triangleleft \mathbb{F}_{k-1}$ the kernel of the canonical homomorphism.

We can consider $\mathbb{F}_{k-1}$ as subgroup of $\mathbb{F}_k$ and then: $\ker\pi_2 = \{w \in \ker\pi_1 : x_k \text{ and } x_k^{-1} \text{ are not subword of } w\} = \ker\pi_1 \cap \mathbb{F}_{k-1}$

If $\sigma_1, \sigma_2, \ldots, \sigma_k \in S_{K_1(n)}$ are n-approximations for the first presentation, so $\sigma_1, \sigma_2, \ldots, \sigma_{k-1}$ are n-approximations for the second presentation.

The inclusion of $\mathbb{F}_{k-1}$ in $\mathbb{F}_k$ is isometric then $\ker\pi_2 \cap B_n(\mathbb{F}_{k-1}) = \mathbb{F}_{k-1} \cap \ker\pi_1 \cap B_n(\mathbb{F}_k)$ but clearly for $w \in \mathbb{F}_{k-1}$ simply $w(\sigma_1, \sigma_2, \ldots, \sigma_k) = w(\sigma_1, \sigma_2, \ldots, \sigma_{k-1})$. Then $K_2(n) \leq K_1(n)$.

$$X^+ :$$

Before: $\langle x_1, x_2, \ldots, x_k | R\rangle$, $\ker\pi_1 \triangleleft \mathbb{F}_k$ the kernel of the canonical homomorphism;

after: $\langle x_1, x_2, \ldots, x_{k+1} | R \cup \{x_{k+1} w_0^{-1}\}\rangle$ $\ker\pi_2 \triangleleft \mathbb{F}_{k+1}$ the kernel of the canonical homomorphism.

Let m be the length of w_0, consider $\sigma_1, \sigma_2, \dots, \sigma_k \in S_{K_1(nm)}$ the nm-approximations for the first presentation.

Set $\sigma_{k+1} := w_0(\sigma_1, \sigma_2, \dots, \sigma_k)$, now we show that $\sigma_1, \sigma_2, \dots, \sigma_k, \sigma_{k+1}$ are n-approximations for the second presentation.

Let $w \in B_n(\mathbb{F}_{k+1})$, we can replace every x_{k+1} and x_{k+1}^{-1} with respectively w_0 and w_0^{-1} and obtain a new word $w' \in \mathbb{F}_k \cap B_{nm}$. It's clear that w is in $\ker \pi_2$ if and only if w' is in $\ker \pi_1$. But $w(\sigma_1, \sigma_2, \dots, \sigma_k, \sigma_{k+1}) = w'(\sigma_1, \sigma_2, \dots, \sigma_k)$ so we have:

$$\ell_H(w(\sigma_1, \dots, \sigma_k, \sigma_{k+1})) \begin{cases} \leq \frac{1}{nm} \leq \frac{1}{n}, w \in \ker \pi_2 \\ \geq 1 - \frac{1}{nm} \geq 1 - \frac{1}{n}, w \notin \ker \pi_2. \end{cases}$$

and so $K_2(n) \leq K_1(nm)$.

And so finally, by Theorem 1.1.1:

Proposition 3.3.2. *Let $\pi : \mathbb{F}_d \twoheadrightarrow \Gamma$ and $\pi' : \mathbb{F}_{d'} \twoheadrightarrow \Gamma$ be two presentations of the same group Γ, then there exists $C > 0$ such that:*

$$K_{\Gamma,\pi}(C^{-1}n) \leq K_{\Gamma,\pi'}(n) \leq K_{\Gamma,\pi}(Cn) \quad \forall n \in \mathbb{N}$$

3.3.2 Comparisons with Følner function and with LEF function

Amenable groups and residually finite groups are sofic. Actually the soficity was born as the class of groups that contains both. So it is interesting and very simple to quantify these inclusions.

Proposition 3.3.3.

$$K_\Gamma(n) \leq F_\Gamma(n^2).$$

Proof. For every $N \in \mathbb{N}$ we consider N-Følner set Y such that $F_\Gamma(N) = |Y|$. Now consider $\sigma_1, \sigma_2, \ldots, \sigma_d \in Sym(Y)$ bijections of Y such that:

$$\sigma_i(y) = x_i y$$

for all $y \in Y \cap x_i^{-1}Y$, for $i = 1, 2, \ldots, d$. Let $w = z_1 z_2 \ldots z_n$ be a word such that $|w| \leq n$, with $z_i \in \{1, x_1, \ldots, x_n, x_1^{-1} \ldots, x_n^{-1}\}$.
If $y \in Y$ is such that

$$w(\sigma_1, \sigma_2, \ldots, \sigma_d)y \neq w(x_1, x_2 \ldots, x_d)y$$

then

$$y \notin Y \cap z_n^{-1}Y \text{ or } z_n y \notin Y \cap z_{n-1}^{-1}Y \text{ or ... or } z_2 z_3 \ldots z_n y \notin Y \cap z_1^{-1}Y;$$

that is:

$$y \in (Y \backslash z_n^{-1}Y) \cup (z_n^{-1}Y \backslash z_n^{-1}z_{n-1}^{-1}Y) \cup \ldots \cup (z_n^{-1}z_{n-1}^{-1} \ldots z_2^{-1}Y \backslash z_n^{-1}z_{n-1}^{-1} \ldots z_2^{-1}z_1^{-1}Y).$$

But

$$\frac{1}{|Y|}|(Y \backslash z_n^{-1}Y) \cup (z_n^{-1}Y \backslash z_n^{-1}z_{n-1}^{-1}Y) \cup \ldots \cup (z_n^{-1}z_{n-1}^{-1} \ldots z_2^{-1}Y \backslash z_n^{-1}z_{n-1}^{-1} \ldots z_2^{-1}z_1^{-1}Y)| \leq$$

$$\sum_{j=1}^{n} \frac{|Y \setminus z_j^{-1}Y|}{|Y|} \leq \frac{n}{N}.$$

We choose $N \geq n^2$: for $\omega \in \ker \pi$,

$$\ell_H(\omega(\sigma_1, \sigma_2, \ldots, \sigma_d)) = \frac{|\{y \in Y : \omega(\sigma_1, \sigma_2, \ldots, \sigma_d)y \neq y\}|}{|Y|} \leq$$

$$\frac{|\{y \in Y : \omega(\sigma_1, \sigma_2, \ldots, \sigma_d)y \neq \omega(x_1, x_2 \ldots, x_d)y\}|}{|Y|} \leq \frac{1}{n}$$

and if $\omega \notin \ker \pi$ with the analogous inequality we have:

$$\ell_H(\omega(\sigma_1, \sigma_2, \ldots, \sigma_d)) \geq \frac{|\{y \in Y : \omega(\sigma_1, \sigma_2, \ldots, \sigma_d)y = \omega(x_1, x_2 \ldots, x_d)y\}|}{|Y|} \geq 1 - \frac{1}{n}.$$

So if $N = n^2$, we have the n-approximations. □

It is very natural and sharp the n^2 inside the Følner function. Actually, remembering Lemma 2.2.2, we know that $F_\Gamma(n^2)$ is the cardinality of optimal Følner sets $\frac{1}{n}$-invariant for all the words of length less or equal to n.
This is one of the differences between amenability and soficity, also in finitely presented groups: suppose we have for all ϵ permutations $\sigma_1, \dots \sigma_d$ such that

$$\ell_H(w(\sigma_1, \dots, \sigma_k)) \begin{cases} \leq \epsilon, \ w \in \ker \pi \\ \geq 1 - \epsilon, \ w \notin \ker \pi \end{cases}$$

for all w in $\mathbb{F}_d$ with $|w| < n$. Then, if $\epsilon \leq n^{-1}$ we have n-approximations, but for smaller ϵ, a priori, we don't have approximation for longer words. In fact if $n \geq |r|, \forall r \in R$ we can say that for $w \in \ker \pi$ we have $\ell_H(w(\sigma_1, \dots, \sigma_k)) \leq \epsilon Dehn(|w|)$ so we have a (in general non computable) control, but for the second condition we have no chance because we haven't a lower bound for Hamming length of the product of permutations.

Question 2. *Under the hypothesis of amenability, is it possible to give a reverse inequality between K_Γ and F_Γ? Namely, for any amenable group Γ is there a recursive function $P : \mathbb{N} \longrightarrow \mathbb{N}$ such that*

$$F_X(n) \leq K_X(P(n))?$$

The idea comes from [17, Proposition 2.8]: we know that sofic approximations for amenable groups are in some sense conjugate with the ones given by Følner sets. By private communication we know that in [2] there is a bound for subexponential growth groups, it could be interesting to analyze the recursivity of that bound.

For LEF groups the comparison is direct and linear:

Proposition 3.3.4.

$$K_\Gamma(n) \leq LEF_\Gamma(n)$$

Proof. For each $n \in \mathbb{N}$ we consider the finite group F, with $LEF_\Gamma(n) = |F|$ and $f_1, \dots f_d \in F$ such that the (3.1) of Proposition 3.2.1 holds. We know that f_i describes a permutation σ_i on F by left multiplication.

Clearly $\omega(\sigma_1, \dots, \sigma_d)$ is the permutation associated to the left multiplication of $\omega(f_1, \dots, f_d)$ on F, so the normalized Hamming length is 0 if $\omega \in \ker \pi$ (because it implies $\omega(f_1, \dots, f_d) = 1_F$), and is 1 if $\omega \notin \ker \pi$ (because it implies $\omega(f_1, \dots, f_d) \neq 1_F$).

□

A simple remark is that for free groups $\mathbb{F}_d$ we have also a sort of reverse inequality: if $\sigma_1, \dots \sigma_d \in Sym(k)$ are n-approximations for $\mathbb{F}_d$ in particular they are elements of the finite group $Sym(k)$ and respect the relation (3.1) of Proposition 3.2.1 because if the kernel $\ker \pi$ is empty, the relations (3.2) of Definition 3.3.1 imply the relation (3.1) of Proposition 3.2.1 , and then

$$LEF_{\mathbb{F}_d}(n) \leq K_{\mathbb{F}_d}(n)!\,.$$

We want to generalize this in the case that R is not empty but it is weakly stable: more precisely we want to quantify part of Theorem 7.2 in the work [4], that is a generalization of the one in [20]. To do that we first try to quantify the weak stability of a set of relations.

Definition 3.3.2. *Let $\Gamma = \langle X, R \rangle$, for every $n \in \mathbb{N}$ we define $Stb_R(n)$ as, if it exists, the smallest $m \in \mathbb{N}$ such that for every $\sigma_1, \dots \sigma_d$ which are m-approximations for Γ, there exist $\sigma'_1, \dots \sigma'_d$ which are solutions of R and $d_H(\sigma_i, \sigma'_i) \leq \frac{1}{n}$ for $i = 1, \dots d$. In particular if $Stb_R(n)$ is defined, R is* weakly stable.

And then:

Proposition 3.3.5. *Let $\Gamma = \langle x_1, \dots x_d | R \rangle$ be a finitely presented group, $r = \max\{|\omega|, \omega \in R\}$.*

$$LEF_\Gamma(n) \le K_\Gamma(\max\{Stb_R(2n), n\})! \quad \textit{for } n \ge r.$$

Proof. Consider $m = \max\{Stb_R(2n), n\}$, the soficity gives us $\sigma_1, \dots \sigma_d$ which are m-approximation of R in $Sym(K(m))$. So there exist $\sigma'_1, \dots, \sigma'_d$ solution of R such that $d_H(\sigma_i, \sigma'_i) \le \frac{1}{2n}$, $i = 1, \dots d$. So $x_i \mapsto \sigma'_i$ is the right approximation for $LEF_\Gamma(n)$ because for $\omega \in B_n \cap \ker \pi$, $\omega(\sigma_1, \dots, \sigma_d) = 1$ (if $m > r$) and for $\omega \in B_n \setminus \ker \pi$ and $m > n$:

$$\ell_H(\omega(\sigma'_1, \dots \sigma'_d)) \ge \ell_H(\omega(\sigma_1, \dots \sigma_d)) - \frac{n}{2n} \ge \frac{1}{2} - \frac{1}{m} > 0.$$

□

Following Theorem 1 in [20] we can see that for any $k \in \mathbb{N}$ the relation $R = \{x^k\}$ is stable and then weakly stable and then Stb_R is finite valued. If we look at the proof of that theorem we can see that actually Stb_R is subrecursive: we have an effective bound. The result [4, Theorem 7.2] implies that the relations of an amenable residually finite group are weakly stable. In particular for each recursive function f the groups presented in [26] that are finitely presented, solvable of step 3, residually finite with depth function greater than f have weakly stable relations. Remembering Proposition 3.2.3 we know that LEF function is (eventually) greater than f, but we also have a uniform bound for the Følner function of this kind of groups (see Corollary 3.1.1) and then by Proposition 3.3.3 we have a bound for the function K, and then:

Corollary 3.3.1. *For any recursive function $f : \mathbb{N} \to \mathbb{N}$ there exists a set of weakly stable relations R such that $Stb_R(n) > f(n)$.*

In particular the problem of computing the solutions of R close to the n-approximations, even if solvable, can be arbitrarily complex.

However it is interesting that we can have arbitrary fast (but recursive) growth for LEF function with a fixed upper bound for the sofic dimension growth.

3.3.3 Computability of sofic approximations

Following what we have done in Section 2.1 with amenability we want do the same with soficity: we want an effective definition of soficity.

Definition 3.3.3. *A finitely generated group $\Gamma = \langle x_1, \ldots, x_d | R \rangle$ has* computable sofic approximations *if there exists an algorithm with:*
INPUT: $n \in \mathbb{N}$;
OUTPUT: $k \in \mathbb{N}$, $\sigma_1, \ldots, \sigma_d \in Sym(k)$ that are n-approximations.

But the situation in this case of soficity is very clear:

Theorem 3.3.1. *A finitely generated group Γ has computable sofic approximations if and only if Γ is sofic and has a solvable word problem.*

Proof. (enough for word problem)
For given $w \in \mathbb{F}_d$, consider $n = \max\{3, |w|\}$ as input, we have $k \in \mathbb{N}$, $\sigma_1, \ldots, \sigma_d \in Sym(k)$ that are n-approximations.

The number $\frac{\#fix\ w(\sigma_1,\ldots,\sigma_d)}{k}$ is computable and therefore looking at (3.2) of Definition 3.3.1 we can say if w is in $\ker \pi$ or not.

(necessary for word problem)
First we can construct an algorithm with:

INPUT: $k \in \mathbb{N}$, $\sigma_1, \ldots, \sigma_d \in Sym(k)$;

OUTPUT: answer to "are $\sigma_1, \ldots, \sigma_d \in Sym(k)$ an n-approximation for Γ?"

It is possible because the number $\frac{\#fix\ w(\sigma_1,\ldots,\sigma_d)}{k}$ is computable, the set B_n is finite and the word problem gives us an algorithm to say if $\omega \in B_n$ is in $\ker \pi$ or not, so with every input the algorithm will stop in a finite time.

The set $\{(k, \sigma) : \ k \in \mathbb{N},\ \sigma \in Sym(k)^d\}$ of possible INPUT is countable so we list the set, for each element we start the algorithm and stop when we obtain a positive answer. The group is sofic so the algorithm will stop. $\square$

A finitely generated group Γ with computable sofic approximations has subrecursive sofic dimension growth. It is natural ask whether the converse is true: does a recursive upper bound for K_Γ imply the word problem?

The answer is given again by the Kharlampovich group with unsolvable word problem and subrecursive, by Proposition 3.3.3, sofic dimension growth.

Corollary 3.3.2. *There exist groups with subrecursive sofic dimension growth that haven't computable sofic approximations.*

But we don't know what happens if we have exactly $K_{\Gamma,X}$ computable:

Question 3. *Does the computability of the function $K_{\Gamma,X}(n)$ imply the computability of the sofic approximation and then the word problem?*

Consider first an analogy: let Γ be finitely presented, $\gamma_{\Gamma,X}(n) := |B_n(\Gamma)|$ the growth function for Γ. We know that $\gamma_{\Gamma,X}$ is always subrecursive, because we have an exponential bound given by the cardinality of the ball in free groups. But if we have $\gamma_{\Gamma,X}$ recursive we obtain the solvability of the word problem, in the following way.

Given a word $\omega \in \mathbb{F}_d$, compute $\gamma_{\Gamma,X}(|\omega|)$ and by subtraction $z = |B_{|\omega|} \cap$

$\ker\pi|$. Then we enumerate $\ker\pi$ and stop or if ω appears or if after that z different words in $B_{|\omega|}\cap\ker\pi$ appear.

We follow the same idea. Suppose that for n we can compute $k = K_{\Gamma,X}(n)$. We know that in $Sym(k)^d$ there are n-approximations. We can consider just the set $Sym(k)^d_{|n} := \{\Sigma \in Sym(k)^d : \ell_H(\omega(\Sigma)) \in [0,\frac{1}{n}] \cup [\frac{n-1}{n},1], \forall\omega \in B_n\}$, in fact we know that we can found n-approximation in $Sym(k)^d_{|n}$. For $\Sigma \in Sym(k)^d_{|n}$ we define $N_\Sigma := \{\omega \in B_n : \ell_H(\omega(\Sigma)) \leq \frac{1}{n}\}$. Then Σ is n-approximation if and only if $N_\Sigma = \ker\pi \cap B_n$. Now we know that for $k' < k$ in $Sym(k')^d_{|n}$ there are no n-approximations, so we can compute the following set:

$$N := \{N_\Sigma,\ \Sigma \in Sym(k)^d_{|n}\} \setminus \bigcup_{i=1}^{k-1}\{N_\Sigma,\ \Sigma \in Sym(i)^d_{|n}\}$$

and we know that $\ker\pi \cap B_n \in N$. We have an order relation on N by the inclusion, it is easy to see that:

$$\ker\pi \cap B_n \text{ maximal in } N \ \Rightarrow \text{computable } n\text{-approximations}$$

This happens because we can recursively enumerate the words $\ker\pi$, consider only the one with length less than or equal to n and delete all elements in N which do not contain these words. In case of maximality of $\ker\pi \cap B_n$ at one point it remains only this set.
But this maximality is a very strong condition and we don't know if it holds in some class of sofic groups (other than $\mathbb{Z}$).

Question 4. *Is it possible for finitely presented or just for finitely generated groups to have sofic dimension growth faster than any recursive function?*

This is an (apparently) stronger version of the same question of Gromov about Følner function in [23, p.578]. In fact for Proposition 3.3.3 the Følner funtion is bounded from below by sofic dimension growth. A positive answer to Question 2 would establish the equivalence between subrecursivity of F_Γ and K_Γ. For finitely generated not finitely presented it is known that the Følner function can be non subrecursive [18], and the groups considered are with intermediate growth. A positive answer to 2 just for groups with subexponential growth (as announced for [2], if the bound is recursive) implies the existence of finitely generated groups with non subrecursive sofic dimension growth.

3.3.4 Stability properties

In this section we quantify some stability property of sofic groups, more or less is a sort of translation of the results in [16].

The direct product of sofic groups is sofic, simply we notice that in a grid the permutations of the rows commute with the permutations of the columns. The quantification version has exactly the same proof in [16] or in [8], the difference of settings is just illusory.

Proposition 3.3.6. *Let $G = \langle x_1, x_2, \ldots, x_l | R_G\rangle$ and $H = \langle y_1, y_2, \ldots, y_k | R_H\rangle$, be finitely generated groups. Consider the direct product with canonical presentation*

$$G\times H = \langle x_1, x_2, \ldots, x_l, y_1, y_2, \ldots, y_k | R_G, R_H, [x_i, y_j]\ i = 1, \ldots, l,\ j = 1, \ldots, k\rangle,$$

then:

$$K_{\Gamma_1\times\Gamma_2}(n) \leq K_{\Gamma_1}(2n)K_{\Gamma_2}(2n).$$

Proof. Let $\sigma_1, \sigma_2 \ldots, \sigma_l$ be m-approximations for G acting on the finite set A of cardinality $K_G(m)$ and and $\tau_1, \tau_2, \ldots, \tau_k$ be m-approximations for H acting on the finite set B of cardinality $K_H(m)$.

Denoting $C = A \times B$ we can consider the two following (Hamming)-isometric homomorphisms

$$i_\alpha : Sym(A) \hookrightarrow Sym(C),$$

$$i_\beta : Sym(B) \hookrightarrow Sym(C)$$

$$i_\alpha(\sigma)(a,b) := (\sigma a, b), \;\; \sigma \in Sym(A); \quad i_\beta(\tau)(a,b) = (a, \tau b) \;\; \tau \in Sym(B).$$

It is clear that for every $\sigma \in Sym(A)$ and $\tau \in Sym(B)$

$$i_\alpha(\sigma) i_\beta(\tau) = i_\beta(\tau) i_\alpha(\sigma).$$

Denoting with ℓ_C the normalized Hamming length on $Sym(C)$ and the same with A and B, we have:

$$\begin{aligned}\ell_C(i_\alpha(\sigma) i_\beta(\tau)) = 1 - \frac{|\{(a,b) \in C : (\sigma a, \tau b) = (a,b)\}|}{|C|} = \\ = 1 - \frac{|\{a \in A : \sigma a = a\}|}{|A|} \frac{|\{b \in B : \tau b = b\}|}{|B|} \\ = 1 - (1 - \ell_A(\sigma))(1 - \ell_B(\tau)) = \ell_A(\sigma) + \ell_B(\tau) - \ell_A(\sigma)\ell_B(\tau)\end{aligned}$$

then:

$$\ell_C(i_\alpha(\sigma) i_\beta(\tau)) \begin{cases} \leq \ell_A(\sigma) + \ell_B(\tau) \\ = \ell_A(\sigma) + \ell_B(\tau)(1 - \ell_A(\sigma)) \geq \ell_A(\sigma) \\ = \ell_B(\sigma) + \ell_A(\tau)(1 - \ell_B(\tau)) \geq \ell_B(\tau). \end{cases} \tag{3.3}$$

We define the $l + k$ permutations in $Sym(C)$ for the sofic approximation of $G \times H$:

$$\Sigma_i = i_\alpha(\sigma_i), \;\; i = 1, 2, \ldots, l; \qquad \mathcal{T}_j = i_\beta(\tau_j), \;\; j = 1, 2, \ldots, k.$$

Given a word $\omega \in \mathbb{F}_{(l+k)} = \mathbb{F}_l * \mathbb{F}_k$ we can associate a word $\varpi = \varpi_l \varpi_k$ with $\varpi_l \in \mathbb{F}_l$ and $\varpi_k \in \mathbb{F}_k$ obtained switching the generators of $\mathbb{F}_l$ with the generators of $\mathbb{F}_k$.

In particular $\omega(\Sigma_1,\dots,\Sigma_l,\mathcal{T}_1,\dots,\mathcal{T}_k)=\varpi(\Sigma_1,\dots,\Sigma_l,\mathcal{T}_1,\dots,\mathcal{T}_k)$. It's clear that

$$\omega\in\ker\pi_{G\times H}\Longleftrightarrow\varpi_l\in\ker\pi_G \text{ and } \varpi_k\in\ker\pi_H$$

Suppose $\omega\in\ker\pi_{G\times H}$, $|\omega|\le n$, using (3.3)

$$\ell_C(\omega(\Sigma_1,\dots,\Sigma_l,\mathcal{T}_1,\dots,\mathcal{T}_k))=\ell_C(\varpi(\Sigma_1,\dots,\Sigma_l,\mathcal{T}_1,\dots,\mathcal{T}_k))$$
$$\le\ell_A(\varpi_l(\sigma_1,\sigma_2\dots,\sigma_l))+\ell_B(\varpi_k(\tau_1,\tau_2,\dots,\tau_k))\le\frac{2}{m}.$$

If $\omega\notin\ker\pi_{G\times H}$ then $\varpi_l\notin\ker\pi_G$ or $\varpi_k\notin\ker\pi_H$. Suppose $\varpi_l\notin\ker\pi_G$, using (3.3):

$$\ell_C(\omega(\Sigma_1,\dots,\Sigma_l,\mathcal{T}_1,\dots,\mathcal{T}_k))=\ell_C(\varpi(\Sigma_1,\dots,\Sigma_l,\mathcal{T}_1,\dots,\mathcal{T}_k))$$
$$\ge\ell_A(\varpi_l(\sigma_1,\sigma_2\dots,\sigma_l))\ge1-\frac{1}{m}$$

then if $m=2n$ the permutations $\Sigma_1,\dots,\Sigma_l,\mathcal{T}_1,\dots,\mathcal{T}_k$ are n-approximations for $G\times H$ $\square$

We know that amenable extensions of sofic groups are sofic. For now we have the relative bound just when the extension splits:

Theorem 3.3.2. *Let $N=\langle X|R_1\rangle$ and $H=\langle Y|R_2\rangle$ be finitely generated groups and let $\phi:H\to Aut(N)$ be homomorphism. Let $c:=\max\{|\phi_y(x)|_X : x\in X,\ y\in Y\}$ and consider $N\rtimes H=\langle X,Y|R_1,\,R_2,\,x^y=\phi_y(x)\ \forall x\in X,\,\forall y\in Y\rangle$ then:*

$$K_{N\rtimes H}(n)\le F_H(kn^3)K_N(2c^{F_H(kn^3)+n}),$$

where $k:=|Y\cup Y^{-1}|$.

Proof. We denote $G = N \rtimes H$, $G = \langle X \cup Y \rangle$, $X = x_1, x_2 \dots x_d$, $Y = y_1, y_2 \dots y_k$ so we have:

$$\begin{aligned}
&\pi_N : \mathbb{F}_X \to N, && \pi_H : \mathbb{F}_Y \to H, \\
&\pi_G : \mathbb{F}_{X \cup Y} \to G, && \\
&\mathbb{F}_{X \cup Y} = \mathbb{F}_X * \mathbb{F}_Y, && \pi_G = \pi_N * \pi_H.
\end{aligned}$$

Moreover we need of a function $[*] : H \to \mathbb{F}_Y$ to associate to an element $h \in H$ one of the shortest words in the free group, that is:
$\pi([h]) = h$ and $|[h]| = |h|$.
For $x \in X$ and $y \in Y$ we fix $\phi_y(x) \in \mathbb{F}_X$ one of the shortest word in $\mathbb{F}_X$ such that $\pi_N(\phi_y(x)) = \pi_G(x^y)$.

We extend the definition of ϕ recursively: $\phi_y(x_1x_2) = \phi_y(x_1)\phi_y(x_2)$ and $\phi_{y_1y_2}(x) = \phi_{y_2}(\phi_{y_1}(x))$. In this way for $\mathcal{X} \in \mathbb{F}_X$ and $\mathcal{Y} \in \mathbb{F}_Y$ we have $|\phi_{\mathcal{Y}}(\mathcal{X})| \leq |\mathcal{X}c^{|\mathcal{Y}}$ and such that $\pi_N(\phi_{\mathcal{Y}}(\mathcal{X}^)) = \pi_G(\mathcal{X}^{\mathcal{Y}})$.

For every $M_N \in \mathbb{N}$ there exist M_N-approximations $\sigma_1, \sigma_2, \dots, \sigma_d \in Sym(B)$, $|B| = K_N(M_N)$.
For every $M_H \in \mathbb{N}$ there exists a right-connected M_H-Følner $A \subset H$ with $e \in A$ and $|A| \leq F_H(|Y \cup Y^{-1}|M_H)$. Let

$$\tau_1, \tau_2, \dots \tau_t \quad \in Sym(A),$$

be the permutations as in proof of Proposition 3.2.2 such that for $a \in A \cap y_i^{-1}A$ we have $\tau_i a = y_i a$.

We want to define

$$\Sigma_1, \Sigma_2, \dots \Sigma_d, T_1, T_2, \dots T_k \in Sym(A \times B)$$

such that they are n- approximations for G.

So for $i = 1, \dots d$ and $j = 1, \dots k$ we set:

$$\Sigma_i(a,b) = (a, \phi_{[a]}(x_i)(\sigma_1, \sigma_2, \dots \sigma_d)b)$$

$$T_j(a,b) = (\tau_j a, b).$$

We can write a general word $\omega \in \mathbb{F}_{X \cup Y}$, with $|\omega| \leq n$, in this form: $\omega = \mathcal{X}_1\mathcal{Y}_1\mathcal{X}_2\mathcal{Y}_2 \dots \mathcal{X}_p\mathcal{Y}_p$ with $\mathcal{X}_i \in \mathbb{F}_X \setminus \{e\}$, $\mathcal{Y}_j \in \mathbb{F}_Y \setminus \{e\}$ for $i = 2, \dots p$; $j = 1, \dots p-1$ and $\mathcal{X}_1 \in \mathbb{F}_X$ $\mathcal{Y}_p \in \mathbb{F}_Y$ and $p \leq \frac{n}{2}$. But also:

$$\omega = \prod_{i=1}^{p} \mathcal{X}_i^{(\mathcal{Y}_1\mathcal{Y}_2\dots\mathcal{Y}_{i-1})^{-1}} \mathcal{Y}_1\mathcal{Y}_2\dots\mathcal{Y}_p$$

So we can define:

$$\overline{\omega} := \omega_X\omega_Y = \prod_{i=1}^{p} \phi_{(\mathcal{Y}_1\mathcal{Y}_2\dots\mathcal{Y}_{i-1})^{-1}}(\mathcal{X}_i)\mathcal{Y}_1\mathcal{Y}_2\dots\mathcal{Y}_p$$

with $\omega_X \in \mathbb{F}_X$, $\omega_Y \in \mathbb{F}_Y$ and ω and $\overline{\omega}$ represent the same element in group G.
$|\omega_Y| \leq n$,
$|\omega_X| \leq \sum_{i=1}^{p} |\phi_{(\mathcal{Y}_1\mathcal{Y}_2\dots\mathcal{Y}_{i-1})}(\mathcal{X}_i)^{-1})| \leq nc^n$.

We want to prove that $\omega(\Sigma, \mathcal{T})$ is close to $\overline{\omega}(\Sigma, \mathcal{T})$. We start evaluating normalized Hamming distance between $\mathcal{X}^{\mathcal{Y}}(\Sigma, \mathcal{T})$ and $\phi_{\mathcal{Y}}(\mathcal{X})(\Sigma, \mathcal{T})$.

$$d_H(\mathcal{X}^{\mathcal{Y}}(\Sigma, \mathcal{T}), \phi(\mathcal{X}^{\mathcal{Y}})(\Sigma, \mathcal{T})) = \frac{|\{(a,b) \in A \times B : \mathcal{X}^{\mathcal{Y}}(\Sigma, \mathcal{T})(a,b) \neq \phi_{\mathcal{Y}}(\mathcal{X})(\Sigma, \mathcal{T})(a,b)\}|}{|A||B|},$$

$$\mathcal{X}^{\mathcal{Y}}(\Sigma, \mathcal{T})(a,b) = \mathcal{Y}^{-1}(\mathcal{T}_1, \dots \mathcal{T}_k)\mathcal{X}(\Sigma_1, \dots \Sigma_d)\mathcal{Y}^{-1}(\mathcal{T}_1, \dots \mathcal{T}_k)(a,b) =$$

$$= (a, \phi_{[\mathcal{Y}(\tau_1,\dots\tau_k)a]}X(\sigma_1,\dots\sigma_k)b).$$

From the other side:

$$\phi_{\mathcal{Y}}(\mathcal{X})(\Sigma,\mathcal{T})(a,b) = (a, \phi_{\mathcal{Y}[a]}(\mathcal{X})(\sigma_1,\dots\sigma_k)b).$$

So if $a \in A$ is such that $\mathcal{Y}(\tau_1,\dots\tau_k)a = \pi_H(\mathcal{Y})a$ we have that $\phi_{\mathcal{Y}[a]}(\mathcal{X})$ and $\phi_{[\mathcal{Y}(\tau_1,\dots\tau_k)a]}(\mathcal{X})$ project in N in the same element, then:

$$\omega_0 := \phi_{\mathcal{Y}[a]}(\mathcal{X})^{-1}\phi_{[\mathcal{Y}(\tau_1,\dots\tau_k)a]}(\mathcal{X}) \in \ker_{\pi_N}.$$

Then:

$$\{(a,b) \in A \times B : \mathcal{X}^{\mathcal{Y}}(\Sigma,\mathcal{T})(a,b) \neq \phi_{\mathcal{Y}}(\mathcal{X})(\Sigma,\mathcal{T})(a,b)\} \subset$$

$$\{a \in A : \mathcal{Y}(\tau_1,\dots\tau_k)a \neq \pi_H(\mathcal{Y})a\} \times B \cup$$

$$\cup\, A \times \{b \in B : \omega_0(\sigma_1,\dots\sigma_k)b \neq b\},$$

if $M_N \geq |\omega_0|$ we have:

$$d_H(\mathcal{X}^{\mathcal{Y}}(\Sigma,\mathcal{T}), \phi_{\mathcal{Y}}(\mathcal{X})(\Sigma,\mathcal{T})) \leq \frac{|\mathcal{Y}|}{M_H} + \frac{1}{M_N}.$$

If we consider $M_N \geq 2d^{|A|+n}$ we have (for n big enough):

$$M_N \geq |\phi_{(\mathcal{Y}_1\mathcal{Y}_2\dots\mathcal{Y}_{i-1})^{-1}[a]}(\mathcal{X})^{-1}\phi_{[(\mathcal{Y}_1\mathcal{Y}_2\dots\mathcal{Y}_{i-1})^{-1}(\tau_1,\dots\tau_k)a]}(\mathcal{X})|, \quad i = 2,\dots p,$$

and then by invariance of the Hamming distance we have:

$$d_H(\omega(\Sigma,\mathcal{T}), \overline{\omega}(\Sigma,\mathcal{T})) \leq \sum_{i=1}^{p}\left(\frac{|\mathcal{Y}_1\mathcal{Y}_2\dots\mathcal{Y}_{i-1}|}{M_H} + \frac{1}{M_N}\right) \leq \frac{n^2}{2M_H} + \frac{n}{2M_N}.$$

As in the proof of Proposition 3.3.6, for word $\omega_Y \in \mathbb{F}_Y$ we have:

$$\ell_{A\times B}(\omega_Y(\mathcal{T}_1,\dots\mathcal{T}_k))=\ell_A(\omega_y(\tau_1,\dots,\tau_k)),$$

but for $\omega_X\in\mathbb{F}_X$ we have:

$$\ell_{A\times B}(\omega_X(\Sigma_1,\dots\Sigma_d))=$$

$$\frac{|\{(a,b)\in A\times B:(a,\phi_{[a]}(\omega_X)(\sigma_1,\dots,\sigma_k)b)\neq(a,b)\}|}{|A||B|}=\sum_{a\in A}\ell_B(\phi_{[a]}(\omega_X)(\sigma_1,\dots\sigma_d)).$$

But we also have an analogue of inequality (3.3) of the proof of Proposition 3.3.6:

$$\ell_{A\times B}(\omega_X(\Sigma)\omega_Y(\mathcal{T}))\begin{cases}\leq\ell_B(\omega_X(\Sigma))+\ell_A(\omega_Y(\mathcal{T}))\\=\ell_B(\omega_X(\Sigma))+\ell_A(\omega_Y(\mathcal{T}))(1-\ell_B(\omega_X(\Sigma))\geq\ell_B(\omega_X(\Sigma))\\=\ell_A(\omega_Y(\mathcal{T}))+\ell_B(\tau)(1-\ell_A(\omega_Y(\mathcal{T})))\geq\ell_A(\omega_Y(\mathcal{T})).\end{cases}$$

Clearly if $\omega\in\ker\pi_G$ we have: $\overline{\omega}\in\ker\pi_G$, $\omega_X\in\ker\pi_N$, $\omega_Y\in\ker\pi_H$, $\phi_{[a]}(\omega_X)\in\ker\pi_N$ for all $a\in A$. Moreover $|\phi_{[a]}(\omega_X)|\leq M_N$ then:

$$\ell(\omega(\Sigma,\mathcal{T}))\leq\ell(\overline{\omega}(\Sigma,\mathcal{T})+\frac{n^2}{2M_H}+\frac{n}{2M_N}\leq$$
$$\leq\ell(\omega_X(\Sigma_1,\dots,\Sigma_k))+\ell(\omega_Y(\mathcal{T}_1,\dots,\mathcal{T}_t)+\frac{n^2}{2M_H}+\frac{n}{2M_N}$$
$$\leq(n+n/2)\frac{1}{M_N}+(n+n^2/2)\frac{1}{M_H}.$$

If $\omega\notin\ker\pi_G$ clearly $\overline{\omega}\notin\ker\pi_G$ and then $\omega_X\notin\ker\pi_N$ or $\omega_Y\notin\ker\pi_H$, that implies $\phi_{[a]}(\omega_X)\notin\ker\pi_N$ for all $a\in A$, again $|\phi_{[a]}(\omega_X)|<M_N$ then:

$$\ell(\omega(\Sigma,\mathcal{T}))\geq\ell(\overline{\omega}(\Sigma,\mathcal{T}))-\frac{n^2}{2M_H}-\frac{n}{2M_N}\geq\ell(\omega_Y(\tau))-\frac{n^2}{2M_H}-\frac{n}{2M_N}$$

$$\geq 1 - \frac{n}{2M_N} - (n + n^2/2)\frac{1}{M_H}$$

or

$$\ell(\omega(\Sigma, \mathcal{T})) \geq \ell(\overline{\omega}(\Sigma, \mathcal{T})) - \frac{n^2}{2M_H} - \frac{n}{2M_N} \geq \ell(\omega_X(\Sigma) - \frac{n^2}{2M_H} - \frac{n}{2M_N}$$

$$\geq 1 - (n + n/2)\frac{1}{M_N} - \frac{n^2}{2M_H}.$$

So finally setting $M_H = n^3$ and $M_N = c^{|A|+n}$ we have that $\Sigma_1 \dots \Sigma_d, \mathcal{T}_1 \dots \mathcal{T}_k$ are n-approximation for G.

□

For the free product we want again a sort of independence on the permutations associated to the two starting groups, but in this case we don't want the commutativity. We need a different setting, in fact if we go to the hypothesis of permutations without fixed points as in [17] the soficity holds but we lost information about the rank of permutations.

Theorem 3.3.3. *Let $G = \langle x_1, x_2, \dots, x_l | R_G\rangle$ and $H = \langle y_1, y_2, \dots, y_k | R_H\rangle$, finitely generated group. Consider the free product with canonical presentation*

$$G * H = \langle x_1, x_2, \dots, x_l, y_1, y_2, \dots, y_k | R_G, R_H\rangle,$$

then:

$$K_{G*H}(n) \leq K_G(2n^2)K_H(2n^2)LEF_{\mathbb{F}_{K_G(2n^2)K_H(2n^2)}}(n).$$

Proof. Let $\sigma_1, \sigma_2 \dots, \sigma_l$ be m-approximations for G acting on the finite set A of cardinality $K_G(m)$ and and $\tau_1, \tau_2, \dots, \tau_k$ be m-approximations for H acting on the finite set B of cardinality $K_H(m)$.

Now consider the elements of $A \times B$ as generators of the free group $\mathbb{F}_{A\times B}$ and for $r \in \mathbb{N}$ (to fix later) denote with V one of the smallest finite groups such that there exists $\phi((a,b)) \in V$, $\forall (a,b) \in A \times B$ for which the (3.1) of Proposition 3.2.1 holds with $2r$ and then $|V| = LEF_{\mathbb{F}_{A\times B}}(2r)$.

In $C := A \times B \times V$ we can build two partitions, α and β.

$$\alpha = \{A[b,v],\ b \in B, v \in V\},\ \text{where}\ A[b,v] = \{(a,b,v),\ a \in A\};$$

$$\beta = \{B[a,w],\ a \in A, w \in V\},\ \text{where}\ B[a,w] = \{(a,b,w\phi((a,b))),\ b \in B\}.$$

Remember that the incidence graph between two partitions α and β is the bipartite, non-oriented graph with vertex $\alpha \cup \beta$ and with an edge between $a \in \alpha$ and $b \in \beta$ if and only if $a \cap b \neq \emptyset$. We observe that:

Lemma 3.3.1.

a: $|A[b,v]| = |A| \ \ \forall b \in B\ \forall v \in V$

b: $|B[a,w]| = |B| \ \ \forall a \in A\ \forall w \in V$

c: $A[b,v] \cap B[a,w] = \begin{cases} \{(a,b,v)\}, & \text{if } v = w\phi((a,b)) \\ \emptyset, & \text{otherwise.} \end{cases}$

d: The incidence graph of α and β has no cycles of length less than or equal to $2r$.

Proof. The claims a, b, c are clear. For d, by contradiction, suppose we have a cycle of length less than or equal to $2r$. Without loss of generality we can consider that all elements in the cycle are distinct (if they aren't, we can consider a subcycle):

$$A[b_1,v_1], B[a_2,w_2], A[b_3,v_3], B[a_4,w_4], \ldots, B[a_{2r},w_{2r}]$$

and again $A[b_1,v_1]$, then we would have the following equations:

$$v_1 = w_2\phi((a_2, b_1))$$

$$v_3 = w_2\phi((a_2, b_3))$$

$$v_{2i-1} = w_{2i}\phi((a_{2i}, b_{2i-1}))$$

$$v_{2i+1} = w_{2i}\phi((a_{2i}, b_{2i+1}))$$

$i = 1, 2, \dots, r$ (indices mod $2r$). So we have:

$$1 = \phi((a_{2r}, b_1))^{-1}\phi((a_{2r}, b_{2r-1}))\dots\phi((a_2, b_3))^{-1}\phi((a_2, b_1)).$$

The word $((a_{2r}, b_1))^{-1}((a_{2r}, b_{2r-1}))\dots((a_2, b_3))^{-1}((a_2, b_1))$ must be equal to the identity in $\mathbb{F}_{A\times B}$ because its length is less than or equal to $2r$ and then the (3.1) of Proposition 3.2.1 holds.
So there exists $i \in \{1, 2, \dots, r\}$ such that

$$(a_{2i}, b_{2i-1}) = (a_{2i}, b_{2i+1}) \quad \text{or} \quad (a_{2i}, b_{2i+1}) = (a_{2(i+1)}, b_{2i+1}).$$

This implies:

$$b_{2i-1} = b_{2i+1}, \;\; v_{2i-1} = v_{2i+1} \quad \text{or} \quad a_{2i} = a_{2(i+1)} \;\; w_{2i} = w_{2(i+1)}$$

that is $A[b_{2i-1}, v_{2i-1}] = A[b_{2i+1}, v_{2i+1}]$ or $B[a_{2i}, w_{2i}] = B[a_{2i+2}, w_{2i+2}]$, contradicting the hypothesis. □

We consider the two following (Hamming)-isometric homomorphisms:

$$i_\alpha : Sym(A) \hookrightarrow Sym(C),$$

$$i_\beta : Sym(B) \hookrightarrow Sym(C)$$

$$i_\alpha(\sigma)(a, b, v) := (\sigma a, b, v), \;\; \sigma \in Sym(A);$$

$$i_\beta(\tau)(a,b,w\phi((a,b))) = (a,\tau b, w\phi((a,\tau b)) \;\; \tau \in Sym(B).$$

The action of $Sym(A)$ and $Sym(B)$ on C preserves respectively the partition α and the partition β.

We define the $l+k$ permutations in $Sym(C)$ for the sofic approximation of $G * H$:

$$\Sigma_i = i_\alpha(\sigma_i), \;\; i = 1,2,\ldots,l; \qquad \mathcal{T}_j = i_\beta(\tau_j), \;\; j = 1,2,\ldots,k.$$

It is possible to write every non trivial word $\omega \in \mathbb{F}_l * \mathbb{F}_k = \mathbb{F}_{l+k}$ as:

$$\omega = \omega_s \ldots \omega_1,$$

$\omega_i \in \mathbb{F}_l \setminus \{e\}$ if i is odd, $\omega_i \in \mathbb{F}_k \setminus \{e\}$ if i is even, or viceversa.

We can associate to ω a new word ϖ deleting the subwords ω_i belonging to $\ker \pi_G$ or to $\ker \pi_H$ and rename the subwords at every step. In less than s steps we obtain either the identity or a word $\varpi = \varpi_s \ldots \varpi_1$ such that ϖ_i neither is in $\ker \pi_G$ nor in $\ker \pi_H$.

But we can estimate the distance between ω and ϖ applied to our permutations. We write simply $\omega(\Sigma, \mathcal{T})$ to indicate $\omega(\Sigma_1, \ldots, \Sigma_l, \mathcal{T}_1, \ldots, \mathcal{T}_k)$. At every step, if $\omega_i \in \ker \pi_G$, by left and right invariance of the Hamming distance:

$$d_C(\omega_s \ldots \omega_{i+1}\omega_i\omega_{i-1}\omega_1(\Sigma,\mathcal{T}), \omega_s \ldots \omega_{i+1}\omega_{i-1}\omega_1(\Sigma,\mathcal{T})) =$$
$$= d_C(\omega_i(\Sigma,\mathcal{T}),1) = l_C(\omega_i(\Sigma_1,\ldots,\Sigma_l)) = l_A(\omega_i(\sigma_1,\ldots,\sigma_l)) \leq \frac{1}{m}$$

and similarly if $\omega_i \in \ker \pi_H$, so if $|\omega| \leq n$ iterating we obtain

$$d_C(\omega(\Sigma,\mathcal{T}), \varpi(\Sigma,\mathcal{T})) \leq \frac{s}{m} \leq \frac{n}{m}.$$

It is clear that $\omega \in \ker \pi_{G*H}$ if and only if $\varpi = e$, so if $\omega \in \ker \pi_{G*H} \cap B_n$ then

$$l_C(\omega(\Sigma,\mathcal{T})) \leq \frac{n}{m}.$$

In the other case, $\omega \notin \ker \pi_{G*H}$ we have $\varpi = \varpi_s \dots \varpi_1$ such that ϖ_i is out of $\ker \pi_G$ and out of $\ker \pi_H$. We want to estimate the number of fixed point of $\varpi(\Sigma, \mathcal{T})$ in terms of the number of fixed point of $\varpi_i(\Sigma, \mathcal{T})$. If $c \in C$ is a fixed point of $\varpi(\Sigma, \mathcal{T})$ we denote:

$$c_0 := c,\ c_1 = \varpi_1(\Sigma, \mathcal{T})c_0,\ \dots,\ c = c_s := \varpi_s(\Sigma, \mathcal{T})c_{s-1}$$

The action of the subwords ϖ_i preserves the partitions, so c_i and c_{i+1} belong to the same set in α or in β, for example if $\varpi_1 \in \mathbb{F}_l$:

$$c_0, c_1 \in A_1;\ \ c_1, c_2 \in B_2 \dots;\quad A_i \in \alpha, B_j \in \beta.$$

So there is a cycle $A_1, B_2, \dots A_1$ shorter than or equal to n in the incidence graph, for the property d in Lemma 3.3.1 if $n \leq 2r$ there is at least one return point, that is there exists i such that either $A_i = A_{i+2}$ or $B_i = B_{i+2}$. But this means that c_i and c_{i+1} belong to the intersection of the same two parts, for example in the first case:

$$c_i \in A_i \cap B_{i+1},\ \ c_{i+1} \in A_{i+2} \cap B_{i+1} = A_i \cap B_{i+1} :$$

for the property c in Lemma 3.3.1 we have $c_i = c_{i+1}$. So for every fixed point of $\varpi(\Sigma, \mathcal{T})$ we have at least an i and a fixed point of $\varpi_i(\Sigma, \mathcal{T})$ in C. If x, y are two fixed points of $\varpi(\Sigma, \mathcal{T})$, we have i and j such that $\varpi_i(\Sigma, \mathcal{T})x_i = x_i$ and $\varpi_j(\Sigma, \mathcal{T})y_j = y_j$. But if $i = j$ and $x_i = y_j$ then $x = y$, so there is an injection from the fixed points of $\varpi(\Sigma, \mathcal{T})$ and the union of the fixed points of the $\varpi_i(\Sigma, \mathcal{T})$'s. Finally we have:

$$|\{c \in C : \varpi(\Sigma, \mathcal{T})c = c\}| \leq \sum_{i=1}^{s} |\{c \in C : \varpi_i(\Sigma, \mathcal{T})c = c\}|.$$

$$1 - l_C(\varpi(\Sigma, \mathcal{T}) \leq \sum_{i=1}^{s}(1 - l_C(\varpi_i(\Sigma, \mathcal{T}))$$

$$l_C(\varpi(\Sigma, \mathcal{T}) \geq 1 - s + \sum_{i=1}^{s}(l_C(\varpi_i(\Sigma, \mathcal{T}))$$

but if $\varpi_i \in \mathbb{F}_l$:

$$l_C(\varpi_i(\Sigma, \mathcal{T}) = l_A(\varpi_i(\sigma_1, \ldots, \sigma_l) \geq 1 - \frac{1}{m}$$

if $\varpi_i \in \mathbb{F}_k$:

$$l_C(\varpi_i(\Sigma, \mathcal{T}) = l_B(\varpi_i(\tau_1, \ldots, \tau_k) \geq 1 - \frac{1}{m}.$$

Since $s \leq n$, $l_C(\varpi(\Sigma, \mathcal{T}) \geq 1 - s + s(1 - \frac{1}{m}) \geq 1 - \frac{n}{m}$

Finally, for every $\omega \notin \ker \pi_{G*H}$ (ϖ the associated reduced word):

$$l_C(\omega(\Sigma, \mathcal{T})) \geq l_C(\varpi(\Sigma, \mathcal{T}) - \frac{n}{m} \geq 1 - \frac{2n}{m}.$$

So if $m = 2n^2$ and $r = n$ we have our approximation. Finally

$$|C| = K_G(2n^2)K_H(2n^2)LEF_{\mathbb{F}_{K_G(2n^2)K_H(2n^2)}}(n).$$

$\square$

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