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Some results on cohesive energies: approximation, lower semicontinuity, and quasistatic evolution

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Some results on cohesive energies: approximation, lower semicontinuity, and quasistatic evolution

Ph.D. Thesis

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Il presente lavoro costituisce la tesi presentata da Gianluca Orlando, sotto la direzione dei Proff. Gianni Dal Maso e Rodica Toader, al fine di ottenere l'attestato di ricerca post-universitaria Doctor Philosophiae presso la SISSA, Curriculum in Analisi Matematica, Modelli e Applicazioni, Area di Matematica. Ai sensi dell'art. 1, comma 4, dello Statuto della Sissa pubblicato sulla G.U. no. 36 del 13.02.2012, il predetto attestato è equipollente al titolo di Dottore di Ricerca in Matematica.

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Abstract

In this thesis, cohesive fracture is investigated under three different perspectives.

First we study the asymptotic behaviour of a variational model for damaged elasto-plastic materials in the case of antiplane shear. The energy functionals we consider depend on a small parameter ε, which forces damage concentration on regions of codimension one. We determine the Γ-limit as ε tends to zero and show that it contains an energy term involving the crack opening.

The second problem we consider is the lower semicontinuity of some free discontinuity functionals with linear growth defined on the space of functions with bounded deformation. The volume term is convex and depends only on the Euclidean norm of the symmetrised gradient. We introduce a suitable class of cohesive surface terms, which make the functional lower semicontinuous with respect to L^1 convergence.

Finally, we prove the existence of quasistatic evolutions for a cohesive fracture on a prescribed crack surface, in small-strain antiplane elasticity. The main feature of the model is that the density of the energy dissipated in the fracture process depends on the total variation of the amplitude of the jump. Thus, any change in the crack opening entails a loss of energy, until the crack is complete. In particular this implies a fatigue phenomenon, i.e., a complete fracture may be produced by oscillation of small jumps.

Hofstadter's Law: It always takes longer than you expect, even when you take into account Hofstadter's Law.

— Douglas R. Hofstadter

Con la stesura di questa tesi ho potuto riorganizzare il lavoro scientifico svolto durante il mio PhD. Ciò mi ha spontaneamente portato a ripensare – non lo nascondo, con un pizzico di nostalgia – alle molte esperienze vissute in questi tre anni. Chi mi è stato vicino sa che è raro sentirmi pronunciare parole di gratitudine, e pertanto approfitto di questo spazio per cercare di esprimere la mia riconoscenza per chi è stato essenziale per la realizzazione di questo traguardo.

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Contents

Introduction

This thesis is devoted to the variational modelling of cohesive fracture. Compared to models for brittle materials, cohesive zone models provide a more accurate description of the process of crack growth, as we explain below. However, the analysis of cohesive-type surface energies, which appear in the functionals involved in the modelling of fracture, entails many mathematical difficulties. The work presented here is devoted to some problems arising in the analysis of cohesive energies both in the static setting and in the evolutionary setting. Before exposing in detail the results contained in this thesis, we give a broad overview of the variational approach to Fracture Mechanics, underlining the main differences between the brittle case and the cohesive case.

Variational approach to fracture. At the basis of the mathematical formulation of quasistatic crack growth there is the idea, due to Griffith [53], that the propagation of a fracture is determined by the competition between the elastic energy released by the body when the crack grows and the energy dissipated to produce a new crack. Inspired by Griffith's principle, Francfort and Marigo proposed in [49] a variational approach to the quasistatic growth of fracture in brittle materials. Their model is based on a procedure of time discretisation, and is built around the construction of discrete-time evolutions obtained by solving time-incremental minimisation problems. These variational problems involve the total energy of the system, given by the sum of the mechanical energy and of the dissipated energy. In principle the continuous-time evolution, obtained by passing to the limit as the discretisation step goes to 0, satisfies two fundamental properties:

- *global stability:* at each time, the state of the system minimises the mechanical energy plus the energy dissipated to reach any other admissible state;
- *energy-dissipation balance:* the increment of the internal energy plus the dissipated energy equals the work of the external forces.

The two previous conditions characterise energetic solutions to rate-independent systems. (We refer to the book by Mielke and Roubíček [61] and the references therein for a general theory.)

The evolution of a brittle fracture also satisfies an *irreversibility* condition, i.e., the crack is nondecreasing with respect to the time variable and thus fracture is a completely unrecoverable process.

Brittle and cohesive fracture. The irreversibility condition is inherent in the brittle model itself. The main feature that characterises brittle materials is that the energy spent to produce a new crack in the body only depends on the geometry of the crack set. The expression of the energy dissipated has its simplest form in the situation of an homogeneous and isotropic material, in the setting of small-strain antiplane shear. In this case the reference configuration of the body is supposed to be an infinite cylinder $\Omega\times\mathbb{R}$, with $\Omega \subset \mathbb{R}^n$ ($n = 2$ being the physically relevant case), and the deformation $v\colon \Omega\times\mathbb{R} \to \Omega\times\mathbb{R}$ takes the form $v(x_1,\dots,x_{n+1}) = (x_1,\dots,x_n, x_{n+1}+u(x_1,\dots,x_n))$, where $u\colon \Omega \to \mathbb{R}$ is the vertical displacement. We sample the energy of the material by considering its intersection with two horizontal hyperplanes at unit distance. In this finite portion of the cylinder, the total energy, given by the sum of the elastic energy and the energy dissipated, reads

$$\frac{1}{2}\int_{\Omega\setminus K} |\nabla u|^2 \,\mathrm{d}x + \kappa\mathcal{H}^{n-1}(K)\,,$$

where $K \subset \Omega$ is the section of a crack $K\times\mathbb{R}$, $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure and $\kappa > 0$ is a constant representing the toughness of the material. Since the energy dissipated to create the fracture K just depends on the measure of the crack set, even for a small amplitude of the crack opening there is no interaction between the two sides of the fracture.

However, fracture should be regarded as a gradual process, where the material is considered completely cracked at a point only when the amplitude of the crack opening is sufficiently large. Indeed, experiments [45] show that even some materials commonly regarded as brittle (such as glass) go through the formation of microcracks, microvoids, or plastic strains in a zone nearby the crack tip. Cohesive zone models, introduced by BARENBLATT in [11], account for this behaviour by considering surface energies which also depend on the crack opening. To be precise, if $[u]$ denotes the difference between the traces of the displacement u on the two sides of a crack K, the prototypical cohesive surface energy is given by

$$\int_K g(|[u]|)\,\mathrm{d}\mathcal{H}^{n-1}.$$

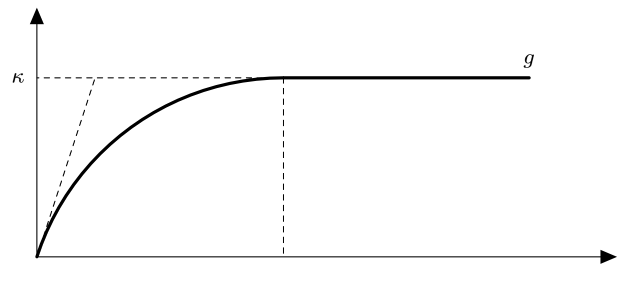

Figure 1: Typical profile of a cohesive energy density g.

Typically, the function $g\colon [0,+\infty) \to [0,+\infty)$ is continuous, it satisfies $g(0) = 0$, it is nondecreasing, and it becomes constant after a critical length. Thus, when the crack opening gradually increases, some energy is dissipated, until the opening overcomes a certain threshold. Moreover, $g'(|[u]|)$ gives the force per unit area acting between the lips of the crack: since this force decreases as the crack opening increases, the function g is assumed to be concave. Usually, the force between the two sides of the crack tends to a finite limit as the crack opening tends to zero, i.e., $\sigma_{\mathrm{Y}} := g'(0^+) \in (0,+\infty)$. The finite value σ_Y has a physical relevance: it represents the maximal stress that the material can withstand before rupture occurs. Unfortunately, this last assumption on g leads to the first mathematical difficulty in the study of cohesive models: if the crack is not constrained to lie on a prescribed path, in the minimisation of the total energy a relaxation process occurs, and microcracks with small openings can accumulate, causing a diffuse macroscopic effect (cf. e.g. [13] and the discussions in Chapter 3). Furthermore, an additional issue in the modelling of cohesive fracture is which irreversibility condition to prescribe: in the cohesive setting there is no obvious threshold that marks the unrecoverable advance of the crack.

In contrast, the relaxation process does not take place in the brittle setting, and in this case the irreversibility condition is unambiguous. In fact, the approach proposed by Francfort and Marigo turned out to be successful for the mathematical theory of quasistatic evolution in brittle materials, and it motivated the development of refined mathematical tools. The first rigorous proof of the existence of a continuous-time evolution was given by DAL MASO and TOADER [39], in the case of antiplane linearised elasticity in dimension two, and with a restriction on the number of the connected components of the crack set. FRANCFORT and LARSEN [48] removed the restrictions on the dimension and on the topology of the crack set, by setting the problem in the space SBV of special functions with bounded variation introduced by DE GIORGI and AMBROSIO [41]. Some results were also extended to the case of plane linearised elasticity in [21] and in the very recent paper [50]. The latter setting is more involved because of the many mathematical difficulties that arise from the use of functions of bounded deformation (cf. [67] or Subsection 1.3.4 for the definition and the general properties of the space BD). Moreover, some existence results were obtained in the case of nonlinear elasticity [32, 33] and also for some models of finite elasticity with noninterpenetration [35, 59].

At the present time, there are no analogue results for the cohesive case and this is one of the mathematical challenges of the theory of crack growth. The literature about quasistatic evolution of cohesive fracture is not so rich as the one for brittle fracture, and the results about the existence of evolutions have been all obtained under the assumption that the region where the crack occurs is a prescribed surface Γ. (See e.g. [40, 19, 20, 28].) Removing the restriction of a prescribed crack set seems to be by now out of reach.

Phase-field approximation of brittle fracture. All the mathematical results concerning the evolution of brittle fracture must rely on formulations based on spaces of functions which allow for discontinuities. As for the numerical simulations, instead, finite element methods do not cope well with discontinuities. The basic idea to circumvent this problem is to regularise the discontinuous displacement by introducing an auxiliary variable that concentrates around the discontinuities (cf. the book [15] and the references therein). This idea is an application of a renown mathematical result due to AMBROSIO and TORTORELLI [8], which we summarily describe below.

Let $u\colon \Omega \to \mathbb{R}$ be the vertical displacement of the body, in the antiplane shear setting described above, and let us consider an additional phase-field variable $\alpha\colon \Omega \to [0,1]$. The result obtained in [8] concerns the asymptotic behaviour of the functionals defined by

$$AT_\varepsilon(u,\alpha) := \frac{1}{2}\int_\Omega \alpha|\nabla u|^2\,\mathrm{d}x + \int_\Omega \Big[\frac{W(\alpha)}{\varepsilon} + \varepsilon|\nabla\alpha|^2\Big]\,\mathrm{d}x\,, \quad \text{for } u \in H^1(\Omega),\ \alpha \in H^1(\Omega)\,,$$

under the constraint $\alpha \geq \delta_\varepsilon > 0$, where $\delta_\varepsilon/\varepsilon \to 0$ as $\varepsilon \to 0$. The function $W\colon [0,1] \to \mathbb{R}$ is continuous, strictly decreasing, and it satisfies $W(1) = 0$. Functionals of this kind were initially introduced in order to solve the problem of finding effective algorithms for computing the minimisers of free-discontinuity energies, with applications to liquid crystals [9], phase-transition [62], and image processing [63].

However, the functional AT_ε is also interesting from the mechanical point of view, since it is meaningful for the variational approach to damage (cf. [64, 65]). Indeed, it is possible to give a physical meaning to the auxiliary variable α. First of all, the stored elastic energy decreases when the variable α decreases. Therefore, in the regions where $\alpha = 1$, the material is completely sound and exhibits an elastic behaviour; whereas, in the regions where $\alpha = \delta_\varepsilon$, the material has suffered the maximum possible damage, and thus it is allowed to deform massively without storing much elastic energy. On the other hand, the term $\int_\Omega \frac{W(\alpha)}{\varepsilon}\,\mathrm{d}x$ is a dissipative energy, and the gradient term $\varepsilon|\nabla\alpha|^2$ has a regularising effect. For all the previous reasons, α is interpreted as an internal variable of the system which indicates the damage in the material.

To illustrate the result obtained in [8], let us consider for every $\varepsilon > 0$ a minimiser $(u_\varepsilon, \alpha_\varepsilon)$ of the functional AT_ε (with suitable prescribed boundary conditions). The term $1/\varepsilon$ in the integral of $W(\alpha_\varepsilon)$ implies that $\alpha_\varepsilon \to 1$ a.e. in Ω as $\varepsilon \to 0$, so that no diffuse damage can be seen in the limit. Nevertheless, to make the elastic energy small, it might be convenient to force the damage variable α_ε to be close to 0 around some lower dimensional set K, which in the limit can be interpreted as a fracture set (cf. Figure 2). By means of the variational notion of Γ-convergence (cf. [29, 17]), Ambrosio and Tortorelli rigorously proved in [8] that the asymptotic behaviour of the functionals AT_ε as $\varepsilon \to 0$ is described by the Mumford-Shah functional defined by

$$MS(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \kappa_W \mathcal{H}^{n-1}(J_u)\,, \quad \text{for } u \in SBV(\Omega)\,, \tag{1}$$

where ∇u is the approximate gradient and J_u is the jump set of the displacement u. (We refer to the book [7] or to Subsection 1.3.1 for the fine properties of BV functions.) The constant κ_W appearing in (1) is given by

$$\kappa_W := 4\int_0^1 \sqrt{W(s)}\,\mathrm{d}s \tag{2}$$

and is due to the competition between the dissipative term and the regularising term of AT_ε. At the limit, the concentration of damage, which approaches the value 0 around the set J_u, results in a cost per unit surface given by κ_W.

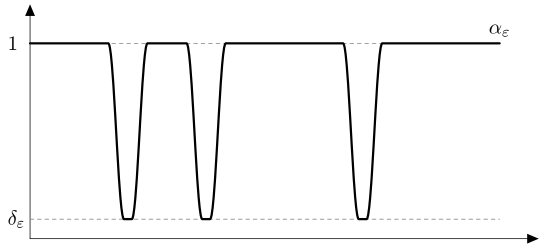

Figure 2: A sequence of damage variables α_ε approaching the value 0 on a lower dimensional set.

At least from a static point of view, the gradient damage model described by AT_ε converges to the total energy of a fractured brittle material with toughness κ_W. To complete the picture, GIACOMINI proved in [51] that the quasistatic evolution of the gradient damage model AT_ε actually converges to a quasistatic evolution of brittle fracture in the sense of [48].

The result by Ambrosio and Tortorelli [8] has been extended to the case of vector-valued functions [46, 47] and to the setting of linearized elasticity [22, 23, 55]. However, these analyses are only suited for the brittle setting. In [34, 56, 57], analogous results are obtained under different assumptions on the limit of $\delta_\varepsilon/\varepsilon$ as $\varepsilon \to 0$. In the regime $\delta_\varepsilon \sim \varepsilon$, the surface energy in the Γ-limit functional depends on the crack opening $|[u]|$ through a density $g(|[u]|)$. Still, an activation energy is present, i.e., $g(0) > 0$, and the force $g'(|[u]|)$ does not vanish for large values of the crack opening. Instead, in the recent paper [24], CONTI, FOCARDI, and IURLANO have studied the Γ-limit of a gradient damage model proposed in [64, 65], where the elastic energy depends in a nontrivial way on the damage variable: the limit functional they obtain is characterised by a surface energy with a density $g(|[u]|)$, with g satisfying the appropriate features of a cohesive model mentioned above.

Cohesive models as limit of gradient damage models coupled with plasticity. In the second chapter of this thesis, we explore the possibility to obtain cohesive energies starting from models in which damage and plasticity interact. The perfect plasticity model itself accounts for the formation of slips in the material; however, in the standard model, the maximal tensile stress on the slip surfaces is always constant. In cohesive fracture, instead, the maximal tension along a crack should decrease as the amplitude of the slip increases. Therefore, by coupling the plastic strain with a softening damage variable, in principle one should be able to catch the behaviour of a cohesive fracture as the damage variable is forced to concentrate on hypersurfaces. This idea is made rigorous by means of the result presented in Chapter 2, based on a work in collaboration with Gianni Dal Maso and Rodica Toader [37].

Our analysis has its basis on a model recently proposed by ALESSI, MARIGO, and VIDOLI [3, 4], and further analysed in [25, 27], which describes the evolution of an elasto-plastic material which undergoes a damage process. Here we illustrate the model in the setting of antiplane shear.

Following the approach of [31] to the modelling of elasto-plastic materials, we decompose the gradient of the displacement $u\colon \Omega \to \mathbb{R}$ as $\nabla u = e + p$, where e and p are vector functions, representing the elastic and the plastic part of the strain, respectively. The main feature of elasto-plasticity is that the stress σ is only determined by the elastic part of the strain e and is constrained to lie in a prescribed compact and convex set, whose boundary is referred to as the yield surface.

The scalar damage variable included in the model is denoted by α. We assume that the stress σ depends on the elastic part of the strain through the formula $\sigma = \alpha e$. The stored elastic energy is thus given by

$$\frac{1}{2}\int_\Omega \sigma \cdot e \,\mathrm{d}x = \frac{1}{2}\int_\Omega \alpha |e|^2 \,\mathrm{d}x \,.$$

Hence, for a prescribed elastic strain, the stored elastic energy decreases when the damage variable α decreases. In order to avoid complete damage, we assume that

$\alpha \geq \delta_\varepsilon > 0$, where $\delta_\varepsilon/\varepsilon \to 0$ as $\varepsilon \to 0$, as in the setting of the Ambrosio-Tortorelli result.

The coupling between damage and plasticity is expressed through a dependence of the yield surface on the damage variable. To be precise, the stress constraint is given by $|\sigma| \leq \sigma_{\mathrm{Y}}(\alpha)$, where $\sigma_{\mathrm{Y}}\colon [0,1] \to \mathbb{R}$ is a continuous nondecreasing function with $0 \leq \sigma_{\mathrm{Y}}(0) \leq \sigma_{\mathrm{Y}}(1) < +\infty$, and $\sigma_{\mathrm{Y}}(\beta) > 0$ for $\beta > 0$. The monotonicity of σ_{Y} with respect to the damage variable entails a softening behaviour. It follows that the plastic potential, which is related to the energy dissipated by the plastic strain, is given by

$$\int_\Omega \sigma_{\mathrm{Y}}(\alpha)|p|\,\mathrm{d}x\,.$$

We will focus our attention on the total energy of the system, given by

$$\mathcal{E}_\varepsilon(e,p,\alpha) := \frac{1}{2}\int_\Omega \alpha|e|^2\,\mathrm{d}x + \int_\Omega \sigma_{\mathrm{Y}}(\alpha)|p|\,\mathrm{d}x + \int_\Omega \Big[\frac{W(\alpha)}{\varepsilon} + \varepsilon|\nabla\alpha|^2\Big]\,\mathrm{d}x\,,$$

under the constraints $\nabla u = e + p$ and $\delta_\varepsilon \leq \alpha \leq 1$. In the last term of the total energy, the function $W\colon [0,1] \to \mathbb{R}$ is continuous, strictly decreasing, and it satisfies $W(1) = 0$. The functional $\mathcal{E}_\varepsilon$ formally corresponds to the Ambrosio-Tortorelli functional AT_ε in the case where $\sigma_{\mathrm{Y}}(\beta) = +\infty$ for every $0 \leq \beta \leq 1$, i.e., when the material is purely elastic. Since the functional $\mathcal{E}_\varepsilon$ has linear growth in p, it is convenient to extend it to the space of vector-valued bounded measures $\mathcal{M}_b(\Omega;\mathbb{R}^n)$, which has better compactness properties, by setting

$$\mathcal{E}_\varepsilon(e,p,\alpha) := \frac{1}{2}\int_\Omega \alpha|e|^2\,\mathrm{d}x + \int_\Omega \sigma_{\mathrm{Y}}(\widetilde{\alpha})\,\mathrm{d}|p| + \int_\Omega \Big[\frac{W(\alpha)}{\varepsilon} + \varepsilon|\nabla\alpha|^2\Big]\,\mathrm{d}x\,,$$

where $\widetilde{\alpha}$ denotes the quasicontinuous representative of $\alpha \in H^1(\Omega)$ (cf. Subsection 1.2.1) and $|p|$ is the total variation of the vector measure $p \in \mathcal{M}_b(\Omega;\mathbb{R}^n)$. This implies that the displacement u belongs to the space $BV(\Omega)$ of functions of bounded variation in Ω. The distributional gradient of u will be thus decomposed as $Du = e\mathcal{L}^n \llcorner \Omega + p$, with $e \in L^2(\Omega;\mathbb{R}^n)$ and $p \in \mathcal{M}_b(\Omega;\mathbb{R}^n)$. For the sake of simplicity, in the thesis we will use the shorthand notation $Du = e + p$.

To describe the asymptotic behaviour of $\mathcal{E}_\varepsilon$ as $\varepsilon \to 0$, it is convenient to introduce the functionals $\mathcal{F}_\varepsilon$, depending only on the displacement u and on the damage variable α, defined by

$$\mathcal{F}_\varepsilon(u,\alpha) := \min_{e,p}\{\mathcal{E}_\varepsilon(e,p,\alpha) : e \in L^2(\Omega;\mathbb{R}^n),\ p \in \mathcal{M}_b(\Omega;\mathbb{R}^n),\ Du = e + p\}$$

under the constraint $\delta_\varepsilon \leq \alpha \leq 1$. The functional $\mathcal{F}_\varepsilon$ represents the energy of the optimal additive decomposition of the displacement gradient Du.

The main result presented in Chapter 2 concerns the Γ-limit, as $\varepsilon \to 0$, of the functionals $\mathcal{F}_\varepsilon$ with respect to the $L^1(\Omega)\times L^1(\Omega)$ topology. We show that the resulting limit functional is the total energy of an elasto-plastic material with a cohesive crack. Specifically, the asymptotic behaviour is described in terms of a functional $\mathcal{F}$ defined on the space $GBV(\Omega)$ of generalized functions of bounded variation (cf. Subsection 1.3.2).

To keep the presentation simple, in this introduction we provide the expression of $\mathcal{F}$ only for $u \in BV(\Omega)$ (for the corresponding expression in $GBV(\Omega)$, see (2.8)):

$$\mathcal{F}(u) = \min_{e,p} \left\{ \frac{1}{2} \int_\Omega |e|^2 \,\mathrm{d}x + \sigma_{\mathrm{Y}}(1)|p|(\Omega \setminus J_u) + \int_{J_u} g(|[u]|) \,\mathrm{d}\mathcal{H}^{n-1} \right\}, \tag{3}$$

where the minimum is taken among all $e \in L^2(\Omega;\mathbb{R}^n)$ and $p \in \mathcal{M}_b(\Omega;\mathbb{R}^n)$ such that $Du = e + p$. Some comments about the function g appearing in (3) are in order.

As in the Ambrosio-Tortorelli result, the limit functional does not depend on the damage variable and hence the material described in the limit is linearly elastic/perfectly plastic outside the crack J_u. On the other hand, a concentration of damage occurs on J_u. Differently from the standard Ambrosio-Tortorelli approximation, in this case it is relevant to account also the regions where sequences of damage variables approach a value $\beta \in [0,1)$ (as in Figure 3). In the limit, this damage concentration results in a cost per unit surface area given by

$$\kappa_W(\beta) := 4 \int_\beta^1 \sqrt{W(s)} \,\mathrm{d}s\,.$$

Notice that the integral above is defined on the interval $[\beta, 1]$, differently from (2).

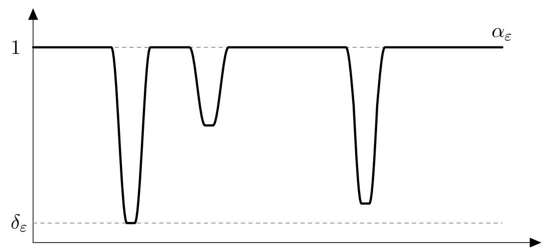

Figure 3: A sequence of damage variables α_ε approaching different values in the interval $[0,1)$.

A crack opening $|[u]|$ can be thus approximated through the plastic strain variable, by paying an energy which amounts to $\sigma_{\mathrm{Y}}(\beta)|[u]|$. With this in mind, it comes at no surprise that the function g admits an explicit expression in terms of the constitutive functions σ_{Y} and W. In fact, for a fixed crack opening $|[u]|$, we have

$$g(|[u]|) = \min \left\{ \min_{0\le\beta\le1} \left[\sigma_{\mathrm{Y}}(\beta)|[u]| + \kappa_W(\beta)\right], \kappa_W(0) \right\}. \tag{4}$$

The additional competition with the value $\kappa_W(0)$ in (4) is the result (in the case where $\sigma_{\mathrm{Y}}(0) > 0$) of the approximation of the crack opening $|[u]|$ by means of the elastic strain, in the regions where damage approaches the value 0. From the explicit formula (4) it turns out that g satisfies the appropriate features of a cohesive model: it is concave, $g(0) = 0$, $g'(0) = \sigma_{\mathrm{Y}}(1) \in (0, +\infty)$, and $g(t) = \kappa_W(0)$ for $\sigma_{\mathrm{Y}}(0)t \ge \kappa_W(0)$.

A Γ-convergence theorem would be idle without a joined compactness result for the minimisers of the approximating functionals. Therefore, we conclude Chapter 2 by presenting some results concerning the asymptotic behaviour of solutions to minimum problems associated to the functionals $\mathcal{F}_\varepsilon$. (See Theorems 2.1.2 and 2.5.3.)

The Γ-convergence result presented in Chapter 2 has been obtained in the setting of antiplane shear. In this case, the displacement variable is a scalar function in $BV(\Omega)$. However, in the general setting, the displacement is a vector-valued function, and therefore it belongs to the space $BD(\Omega)$ of functions of bounded deformation. A proof of the Γ-convergence result in the general case is still missing. It is even nontrivial to propose a candidate limit functional. The reason is essentially technical and it is related to the lack of appropriate lower semicontinuity and relaxation results in the space $BD(\Omega)$. In Chapter 3 we delve deeper into this kind of issues.

Lower semicontinuity of functionals defined on BD. Motivated by these difficulties, in Chapter 3 we study the lower semicontinuity of a class of free discontinuity functionals with linear growth defined on the space $BD(\Omega)$. The results presented in Chapter 3 are based on the work [38], in collaboration with Gianni Dal Maso and Rodica Toader.

To introduce the problem, we start with the following observation: for every $u \in BV(\Omega)$, the functional $\mathcal{F}$ obtained in (3) can be recast in the following integral form

$$\mathcal{F}(u) = \int_\Omega f(|\nabla u|)\,\mathrm{d}x + \sigma_{\mathrm{Y}}(1)|D^c u|(\Omega) + \int_{J_u} g(|[u]|)\,\mathrm{d}\mathcal{H}^{n-1}, \tag{5}$$

where

$$f(t) = \begin{cases} \frac{1}{2}t^2 & \text{if } 0 \le t \le \sigma_{\mathrm{Y}}(1)\,, \\ \sigma_{\mathrm{Y}}(1)t - \frac{\sigma_{\mathrm{Y}}(1)^2}{2} & \text{if } t \ge \sigma_{\mathrm{Y}}(1)\,, \end{cases}$$

(cf. Figure 4) and $D^c u$ is the Cantor part of the distributional gradient Du.

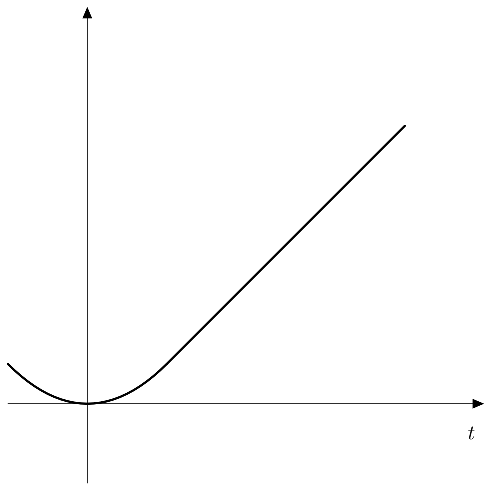

Figure 4: Profile of the function $f(t)$.

Therefore the functional $\mathcal{F}$ belongs to the general class of functionals of the form

$$\int_\Omega f(|\nabla u|)\,\mathrm{d}x + C|D^c u|(\Omega) + \int_{J_u} g([u])\,\mathrm{d}\mathcal{H}^{n-1}, \tag{6}$$

where the function $f\colon [0,+\infty) \to [0,+\infty)$ is convex and nondecreasing, the function

$g\colon \mathbb{R} \to [0,+\infty)$ is even and subadditive, and

$$\lim_{t\to+\infty} \frac{f(t)}{t} = \lim_{t\to 0^+} \frac{g(t)}{t} = C \in (0,+\infty)\,. \tag{7}$$

The main issue with functionals of the form (6) is that there is a strong interaction among the three terms of the functional. For instance, because of the linear growth assumption, a jump can be obtained as the limit of absolutely continuous functions; conversely, an absolutely continuous function can be approximated by means of pure-jump functions. Actually, this interplay occurs in all possible ways among the volume, the Cantor, and the cohesive surface term. (See Section 3.3 for discussions about related problems.) This explains why condition (7) is necessary for the lower semicontinuity of (6) with respect to the weak* convergence in BV. BOUCHITTÉ, BRAIDES, and BUTTAZZO proved in [13] that the assumptions on f, g, and C listed above are also sufficient for the lower semicontinuity of (6).

The purpose of Chapter 3 is to extend this lower semicontinuity result to functionals defined in the space $BD(\Omega)$ of functions of bounded deformation. *Prima facie*, it might seem natural to choose as an extension of (6) to $BD(\Omega)$ the functional

$$\int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} g(|[u]|)\,\mathrm{d}\mathcal{H}^{n-1}, \tag{8}$$

where $Eu = \frac{1}{2}(Du + Du^T)$ is the symmetric part of the distributional gradient of u, $\mathcal{E}u$ is the density of the absolutely continuous part of Eu, while $E^c u$ is the Cantor part of Eu (cf. [6] for the fine properties of functions of bounded deformation). Here and in the rest of the thesis, the space $\mathbb{M}^{n\times n}_{\mathrm{sym}}$ of $n\times n$ symmetric matrices is endowed with the Euclidean (or Frobenius) norm defined by

$$|A| = \Big(\sum_{i,j=1}^n A_{ij}^2\Big)^{\frac{1}{2}}$$

and the variation of the measure $|E^c u|$ is defined accordingly.

However, the functional (8) cannot be lower semicontinuous. The main reason is that the cohesive term in (8) does not take into account the orientation of the jump set J_u (cf. Proposition 3.3.1).

A possible way to overcome this drawback is to consider the restriction to J_u of the measure Eu which is given by

$$Eu \,\llcorner\, J_u = [u] \odot \nu_u \mathcal{H}^{n-1} \,\llcorner\, J_u\,,$$

where ν_u is the approximate unit normal to J_u and, for every pair of vectors $a, b \in \mathbb{R}^n$, $a \odot b$ is the matrix whose components are $\frac{1}{2}(a_i b_j + a_j b_i)$. The matrix $[u] \odot \nu_u$ encodes the behavior of the jump of u, taking into account also the orientation of the jump set. This suggests that a natural extension of (6) to $BD(\Omega)$ is

$$\int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} g(|[u]\odot\nu_u|)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{9}$$

In general, even the functional (9) is not lower semicontinuous. Indeed, we shall see in Proposition 3.3.2 that, if $g(t) = (C|t|)\wedge 1$, then the functional given by (9) is not lower semicontinuous because the 1-homogeneous extension of the function $\nu \mapsto g(|z \odot \nu|)$ is not convex on $\mathbb{R}^n$.

The functional we propose as extension of (6) to $BD(\Omega)$ has the form

$$\int_\Omega f(|\mathcal{E}u|)\, dx + C|E^c u|(\Omega) + \int_{J_u} G([u], \nu_u)\, d\mathcal{H}^{n-1}, \tag{10}$$

where the function $G(z, \nu)$ has a specific structure and, in general, does not depend only on $|z \odot \nu|$. To explain the hypothesis we will consider on G, it is convenient to anticipate the technique of the proof. This will be based on a slicing argument that relies on the following well-known formula for the Euclidean norm of a symmetric $n\times n$ matrix A:

$$|A|^2 = \sup_{(\xi^1,\dots,\xi^n)} \sum_{i=1}^n |A\xi^i \cdot \xi^i|^2, \tag{11}$$

where the supremum is taken over all orthonormal bases $(\xi^1, \dots, \xi^n)$ of $\mathbb{R}^n$. This method suggests that the lower semicontinuity of (10) can be proved when G satisfies the following condition: there exists a function $g\colon \mathbb{R} \mapsto [0, +\infty)$ such that

$$G(z, \nu) = \sup_{(\xi^1,\dots,\xi^n)} \Big(\sum_{i=1}^n g(z\cdot \xi^i)^2 |\nu \cdot \xi^i|^2 \Big)^{\frac{1}{2}}, \tag{12}$$

where the supremum is taken over all orthonormal bases $(\xi^1, \dots, \xi^n)$ of $\mathbb{R}^n$. Notice that, in the particular case $g(t) = C|t|$, the function in (12) takes the form $G(z, \nu) = C|z \odot \nu|$. Notice that in the case where the surface energy density coincides with the recession function of the integrand in the volume term, more general results have been obtained in [66, 43]. The difficulty in our setting is precisely the presence of a surface term different from the recession function.

The main result contained in Chapter 3 is that the functional (10) is L^1-lower semicontinuous on $BD(\Omega)$ under the following assumptions: f is convex and nondecreasing, G is given by (12), with g even and subadditive, and

$$\lim_{t\to+\infty} \frac{f(t)}{t} = \lim_{t\to 0^+} \frac{g(t)}{t} = C \in (0, +\infty), \quad \liminf_{t\to+\infty} g(t) > 0. \tag{13}$$

The last assumption on g in (13) is used only to prove the semicontinuity in L^1 rather than in the weak* topology of BD.

The lower semicontinuity result is then used to prove a relaxation theorem for functionals of the form

$$\int_\Omega f(|\mathcal{E}u|)\, dx + C|E^c u|(\Omega) + \int_{J_u} \psi([u], \nu_u)\, d\mathcal{H}^{n-1},$$

assuming that f is convex and nondecreasing and that there exists an even and subadditive function $g\colon \mathbb{R} \mapsto [0, +\infty)$ satisfying (13) such that

$$\psi(z, \nu) \geq \Big(\sum_{i=1}^n g(z\cdot \xi^i)^2 |\nu \cdot \xi^i|^2 \Big)^{\frac{1}{2}},$$

for every orthonormal basis $(\xi^1,\dots,\xi^n)$ of $\mathbb{R}^n$. In this case, the lower semicontinuous envelope takes the form

$$\int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} \overline{\psi}([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1},$$

for a suitable function $\overline{\psi}$ (cf. Theorem 3.5.2 for more details). Note that the function f and the constant C do not change in the relaxation process.

Chapter 3 concludes our considerations about cohesive energies in the static setting.

Quasistatic evolution for an irreversible cohesive model. In Chapter 4 we focus our attention on the evolutionary framework. As explained earlier in the introduction, the choice of an irreversibility condition is crucial in order to give a consistent notion of evolution. In contrast with the brittle case, irreversibility is not just a geometric issue in cohesive zone models; indeed, in the cohesive setting, the response of the system depends on the history of the deformations occurred during the whole evolution up to the present time. Thus, in order to describe the evolution, the definition of a suitable memory variable is needed.

A possible assumption may be that the energy dissipated depends on the maximal crack opening reached during the evolution. This choice, adopted in the mathematical literature so far, allows to describe a possible behaviour of the system when the amplitude of a crack is not monotone in time: in the model studied by DAL MASO and ZANINI [40], no energy is recovered when the crack opening decreases; instead, CAGNETTI and TOADER assume in [20] that the dissipated energy is partially recovered in the unloading phase.

However, the previous models do not take into account energy dissipation in the decreasing phases of the crack opening. Nonetheless, this kind of response is suited for mechanical systems in which the repeated relative surface motion induces deterioration in the material. For instance, this is a plausible behaviour for the cohesive energy (3) examined in Chapter 2. Indeed, the surface density $g(|[u]|)$ appearing in the limit model is the result of a plastic slip occurring in regions where partial damage is concentrated (cf. formula (4)). Thus all slips, whatever their signs, should entail a loss of energy, until the cracked region is completely damaged. Chapter 4, based on a work in collaboration with Vito Crismale and Giuliano Lazzaroni [28], is devoted to the analysis of a cohesive model characterised by an energy dissipation that depends on the cumulated opening of the crack.

We describe here the model in the setting of small-strain antiplane elasticity, under the assumption that the body may present cracks of the form $K\times\mathbb{R}$, where K is contained in a prescribed $(n-1)$-dimensional manifold $\Gamma\subset\mathbb{R}^n$. As mentioned above, the energy dissipated during the fracture process depends on the evolution of the amplitude of the jump $[u(t)]\colon\Gamma\to\mathbb{R}$, where $t\in[0,T]$ is the time variable. To describe the response of the system to loading, we start by considering the situation where $[u(0)]=0$ on Γ and $t\mapsto[u(t)]$ is increasing on Γ in a time interval $[0,t_1]$. In this case, the energy dissipated in $[0,t_1]$ is

$$\int_\Gamma g\big(\big|[u(t_1)]\big|\big)\,\mathrm{d}\mathcal{H}^{n-1},$$

where $g\colon [0,+\infty) \to [0,+\infty)$ is concave, $g(0)=0$, $g'(0)=\sigma_{\mathrm{Y}} \in (0,+\infty)$, and it reaches the constant κ after a critical length. (See Chapter 4 for more general assumptions on g.) If, afterwards, $t \to [u(t)]$ is decreasing in the interval $[t_1,t_2]$, the energy dissipated in $[0,t_2]$ amounts to (cf. also Figure 5)

$$\int_\Gamma g\big(\big|[u(t_1)]\big| + \big|[u(t_2)] - [u(t_1)]\big|\big)\, \mathrm{d}\mathcal{H}^{n-1}.$$

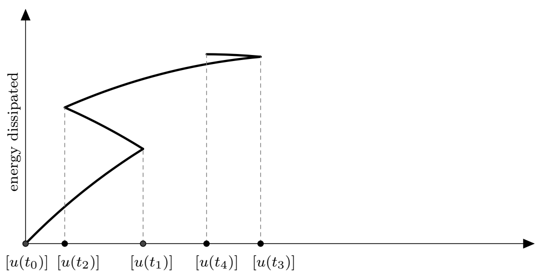

Figure 5: Energy dissipated by a jump $t \mapsto [u(t)]$ with a non-monotone history in a time interval $[t_0,t_4]$: $t \mapsto [u(t)]$ increases in $[t_0,t_1]$ and in $[t_2,t_3]$, whereas it decreases in $[t_1,t_2]$ and in $[t_3,t_4]$.

Therefore, in order to define the energy dissipated by a crack, we choose the cumulated opening of the crack as the relevant memory variable. If $t \mapsto u(t)$ is an absolutely continuous evolution, then the cumulated opening is the function $V_u(t)$ defined on Γ by

$$V_u(t) := \int_0^t \big|[\dot{u}(s)]\big|\, \mathrm{d}s\,.$$

The energy dissipated by the system is expressed in terms of the function $V_u(t)$ (cf. also Figure 6) and is given by

$$\int_\Gamma g\big(V_u(t)\big)\, \mathrm{d}\mathcal{H}^{n-1}.$$

Since $V_u(t)$ is nondecreasing, cohesive fracture is a unidirectional process in this model. More precisely, if in a subinterval $[t_1,t_2] \subset [0,T]$ the jump $t \mapsto [u(t)]$ is not constant in a part of Γ, then the evolution in $[t_1,t_2]$ is irreversible and the state of the system at t_1 is never recovered later on, even if $u(t_1)=u(t_2)$; indeed the maximal tensile stress has decreased. As a consequence, this leads to a fatigue phenomenon, i.e., a complete fracture (corresponding to $g=\kappa$) may occur not only after a large crack opening, but even after oscillations of small jumps (e.g. by a cyclic loading).

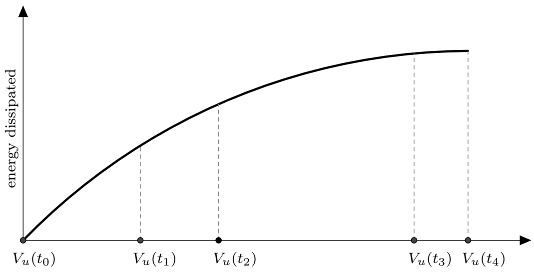

Figure 6: Energy dissipated as a function of the variation of the jumps $V_u(t)$ corresponding to a jump history as in Figure 5. Notice that the variation $V_u(t)$ is nondecreasing in time.

To prove the existence of a quasistatic evolution, we start from the following discrete-time problem, which is a generalisation of the incremental scheme proposed in [1]. Given an initial condition $u(0) = u_0$ and a time-dependent Dirichlet datum $w(t)$ on $\partial_D\Omega \subset \partial\Omega$, for every $k \in \mathbb{N}$ we fix a subdivision $0 = t_k^0 < t_k^1 < \cdots < t_k^{k-1} < t_k^k = T$ and we define recursively u_k^i and V_k^i by

$$u_k^i \in \operatorname*{argmin}_u \left\{ \frac{1}{2} \int\limits_{\Omega\setminus\Gamma} |\nabla u|^2 \,\mathrm{d}x + \int\limits_\Gamma g\Big(V_k^{i-1} + \big|[u] - [u_k^{i-1}]\big|\Big) \,\mathrm{d}\mathcal{H}^{n-1} \ : \ u = w(t_k^i) \text{ on } \partial_D\Omega \right\},$$
$$V_k^i := V_k^{i-1} + \big|[u_k^i] - [u_k^{i-1}]\big|, \tag{14}$$

where $u_k^0 := u_0$ and $V_k^0 = \big|[u_0]\big| = 0$. The function V_k^i describes the cumulated jump of the approximate evolutions at each point of Γ.

Following the general approach for proving the existence of energetic (or globally minimising) solutions to rate-independent systems (cf. [61]), we define $u_k(t)$ and $V_k(t)$ as the piecewise constant interpolations of u_k^i and V_k^i in time, respectively. We shall pass to the limit as $k \to \infty$ and prove that the resulting continuous-time evolution $u(t)$ satisfies the usual properties of quasistatic processes:

- *global stability*: for every $t \in [0, T]$

$$\frac{1}{2} \int\limits_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x + \int\limits_\Gamma g\big(V_u(t)\big) \,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2} \int\limits_{\Omega\setminus\Gamma} |\nabla \hat{u}|^2 \,\mathrm{d}x + \int\limits_\Gamma g\big(V_u(t) + \big|[\hat{u}] - [u(t)]\big|\big) \,\mathrm{d}\mathcal{H}^{n-1},$$

  for any admissible competitor $\hat{u}$;

- *energy-dissipation balance*: for every $t \in [0, T]$

$$\frac{1}{2} \int\limits_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x + \int\limits_\Gamma g\big(V_u(t)\big) \,\mathrm{d}\mathcal{H}^{n-1} = \frac{1}{2} \int\limits_{\Omega\setminus\Gamma} |\nabla u_0|^2 \,\mathrm{d}x + \int_0^t \int\limits_{\Omega\setminus\Gamma} \nabla u(s) \cdot \nabla \dot{w}(s) \,\mathrm{d}x \,\mathrm{d}s.$$

The main difficulty in the passage to limit as $k \to \infty$ is the lack of controls on $V_k(t)$. In fact, from (14) we can only infer that $\int_\Gamma g(V_k(t))\,\mathrm{d}\mathcal{H}^{n-1}$ is uniformly bounded, but this gives no information on the equi-integrability of $V_k(t)$, since g is bounded. In the first instance, in order to pass to the limit as $k \to \infty$, the only chance is to employ compactness properties of the wider class of Young measures (as already done in [20]). Indeed, because of the monotonicity of $V_k(t)$, a Helly-type selection principle guarantees that $V_k(t)$ generates a Young measure $\nu(t) = (\nu^x(t))_{x\in\Gamma}$ for every t, up to a subsequence independent of t.

As for the displacements, from the uniform a priori bounds we obtain that there is a subsequence $u_{k_j}(t)$ weakly converging to a function $u(t)$. Yet the subsequence $k_j = k_j(t)$ may depend on t. We explain below the reason why this is a technical inconvenience.

The irreversibility of the fracture process is encoded in the fact that the energy dissipation depends on the variation of the jumps $V_k(t)$. Let us notice that $V_k(t)$ satisfies the condition

$$V_k(t) \geq V_k(s) + \big|[u_k(t)] - [u_k(s)]\big| \quad \text{for any } s \leq t\,.$$

Unfortunately, it is not immediate to pass in the limit in the previous condition in order to infer an analogous property for the Young measure $\nu(t)$. Indeed, $u_k(t)$ and $u_k(s)$ *a priori* converge along different subsequences! This difficulty is solved by rewriting the previous inequality as a system of two inequalities

$$\begin{aligned} V_k(t) + [u_k(t)] &\geq V_k(s) + [u_k(s)] \quad \text{for any } s \leq t\,,\\ V_k(t) - [u_k(t)] &\geq V_k(s) - [u_k(s)] \quad \text{for any } s \leq t\,. \end{aligned}$$

In fact, we can pass to the limit in these relations by means of a Helly-type theorem, extracting a further subsequence (not relabelled) independent of t and exploiting the monotonicity of $V_k(t) \pm [u_k(t)]$. Moreover, thanks to this trick it turns out that we can identify the limit jump $[u(t)]$ without extracting further subsequences. This last property implies that also the displacement $u(t)$ is the limit of the whole sequence $u_k(t)$, since $u(t)$ is the solution of a minimum problem among functions with prescribed jump $[u(t)]$ (cf. Proposition 4.4.6).

At this point of the analysis, we can pass to the limit in the global stability and in the energy balance, obtaining that $(u(t), \nu(t))$ fulfils a weak notion of quasistatic evolution, where the variation of jumps $V_u(t)$ is replaced by the Young measure $\nu(t)$. (See Theorem 4.4.4.)

Finally, we conclude the chapter by improving the existence result (Theorem 4.2.9). Indeed, we show that $(u(t), V_u(t))$ satisfies the properties of global stability and energy balance: this proves the existence of a quasistatic evolution in a stronger formulation that does not employ Young measures. Furthermore, we prove that for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$, either the measure $\nu^x(t)$ is concentrated on $V_u(t;x)$, or it is supported where g is constant, i.e., where the energy is not dissipated any longer. Therefore also the limit of the discrete variations $V_k(t)$ is characterised.

Structure of the thesis. In Chapter 1 we fix the notation and we recall some preliminary results.

In Chapter 2 we carry out a static Ambrosio-Tortorelli-type analysis for a variational model for damaged elasto-plastic materials, in the case of antiplane shear. We find out that the limit functional exhibits a cohesive surface energy. The results contained in this chapter are based on the work [37], in collaboration with Dal Maso and Toader.

Chapter 3 is devoted to functionals with linear growth defined on the space of functions of bounded deformation. The main result presented here is based on a work in collaboration with Dal Maso and Toader [38]. It concerns the lower semicontinuity of a class of functionals defined on BD with nontrivial surface energies.

The evolution of cohesive fracture is discussed in Chapter 4, which contains a result obtained in collaboration with Crismale and Lazzaroni [28]. We prove the existence of a quasistatic evolution for a cohesive model in which the energy dissipated depends on the cumulated opening of the jump.

Finally, we conclude the thesis by discussing some possible future developments of the three problems treated in Chapters 2–4.

CHAPTER 1

Notation and preliminary results

In this chapter we fix the notation and we recall some preliminary results useful for the sequel. For the reader's convenience, Section 1.1 comprises a list of the symbols adopted throughout the thesis.

In Section 1.2 we recall the notions of convergence for measures used in this thesis and we recall the definition of quasicontinuous representatives of Sobolev functions. The latter notion is needed to define the plastic potential appearing in the energy studied in Chapter 2.

Section 1.3 is devoted to functions of bounded variation and to functions of bounded deformation: we list the main properties of the spaces $BV(\Omega)$ and $BD(\Omega)$ and we recall the main results about one-dimensional slicing. The slicing method shall be adopted in Chapter 2 in the setting of functions of bounded variation; in Chapter 3 we shall instead employ the slicing technique for functions of bounded deformation.

The definition of Γ-convergence of a sequence of functionals and of lower semicontinuity of a functional is given in Section 1.4. Moreover, we recall two integral representation results for local functionals: the integral representation result for functionals defined on $BV(\Omega)$ will be applied in Chapter 2 to prove the Γ-limsup inequality; the integral representation result for the surface term of functionals defined on $BD(\Omega)$ will be employed to prove a relaxation result in Chapter 3.

We conclude this chapter by fixing the notation for Young measures in Section 1.5. In particular, we recall a Helly-type selection principle for Young measures which will be applied for the proof of existence of evolutions in Chapter 4.

1.1 Notation

Basic notation:

$\alpha \wedge \beta$ / $\alpha \vee \beta$	minimum between α and β / maximum between α and β
$a \cdot b$	scalar product between $a, b \in \mathbb{R}^n$
$\mathrm{O}(n)$	group of $n{\times}n$ orthogonal matrices
$\mathbb{M}^{n\times n}_{\mathrm{sym}}$	space of $n{\times}n$ symmetric matrices
$a \odot b$	symmetrised tensor product between $a, b \in \mathbb{R}^n$
$\lvert\cdot\rvert$	modulus, Euclidean norm of vectors, Frobenius norm of matrices

$\mathbb{S}^{n-1}$	$(n-1)$-dimensional sphere in $\mathbb{R}^n$
$B_\rho(x)$	ball of centre x and radius ρ
Q_ρ^ν	cube of side ρ centred at 0, with two faces orthogonal to $\nu \in \mathbb{S}^{n-1}$
$A \Subset B$	$\overline{A} \subset B$ and $\overline{A}$ compact
$\mathscr{A}(\Omega)$	class of all open subsets contained in Ω

Functions spaces: Let Ξ be a metric space and let Ω be an open set in $\mathbb{R}^n$.

$\mathcal{C}_c(\Xi;\mathbb{R}^n)$	space of $\mathbb{R}^m$-valued continuous functions with compact support in Ξ
$\mathcal{C}_0(\Xi;\mathbb{R}^n)$	closure of $\mathcal{C}_c(\Xi;\mathbb{R}^n)$ with respect to the supremum norm
$\mathcal{C}_b(\Xi;\mathbb{R}^n)$	space of continuous and bounded functions
$L^p(\Xi;\mathbb{R}^n)$	space of functions $f\colon \Xi \to \mathbb{R}^n$ with $\|f\|_{L^p(\Xi;\mathbb{R}^n)} < +\infty$
$\langle\cdot,\cdot\rangle_{L^2(\Xi;\mathbb{R}^n)}$	scalar product in $L^2(\Xi;\mathbb{R}^n)$ (simply $\langle\cdot,\cdot\rangle_{L^2}$, if clear from context)
$H^1(\Omega)$	Sobolev space

In the spaces above, $\mathbb{R}^n$ is omitted when $n=1$.

Measure theory: Let Ξ be a metric space.

$\mathscr{B}(\Xi)$	class of Borel sets contained in Ξ
$\mathcal{L}^n$	Lebesgue measure in $\mathbb{R}^n$
$\lvert A\rvert$	Lebesgue measure of the set A
$\mathcal{H}^k$	k-dimensional Hausdorff measure
$\mathcal{M}_b(\Xi;\mathbb{R}^n)$	space of $\mathbb{R}^n$-valued finite Radon measures on Ξ
$\lvert\mu\rvert$	total variation of the measure μ
$\mu \llcorner A$	restriction of the measure μ to the set $A \subset \Xi$
$\varphi_\#\mu$	push-forward of μ through the function φ
$\mathcal{P}(\Xi)$	probability measures on Ξ
$\langle f,\mu\rangle$	duality between $f \in \mathcal{C}_b(\Xi)$ and $\mu \in \mathcal{P}(\Xi)$

BV functions and BD functions: Let Ω be a bounded open set in $\mathbb{R}^n$.

$BV(\Omega)$	space of scalar functions of bounded variation
Du	distributional gradient of u
D^au / D^su	absolutely continuous part of Du / singular part of Du
D^cu / D^ju	Cantor part of Du / jump part of Du
∇u	approximate gradient of u
J_u / ν_u	jump set of u / normal to J_u
u^+, u^- / $[u]$	traces of u on J_u / jump of u given by $u^+ - u^-$
$\widetilde{u}$	precise representative of u, defined $\mathcal{H}^{n-1}$-a.e.
$BD(\Omega)$	space of vector-valued functions of bounded deformation
Eu	symmetric part of Du for $u \in BD(\Omega)$
E^au / E^su	absolutely continuous part of Eu / singular part of Eu
E^cu / E^ju	Cantor part of Eu / jump part of Eu
$\mathcal{E}u$	symmetric part of the approximate gradient ∇u

Slicing: Let $E \subset \mathbb{R}^n$ and let Ω be a bounded open set in $\mathbb{R}^n$.

Π^ξ	hyperplane orthogonal to $\xi \in \mathbb{S}^{n-1}$
E^ξ_y	$\{t \in \mathbb{R} \ : \ y + t\xi \in E\}$, where $\xi \in \mathbb{S}^{n-1}$ and $y \in \Pi^\xi$
w^ξ_y	slice of $w\colon \Omega \to \mathbb{R}$, defined by $w^\xi_y(t) := w(y + t\xi)$
$\hat{v}^\xi_y$	slice of $v\colon \Omega \to \mathbb{R}^n$, defined by $\hat{v}^\xi_y(t) := v(y + t\xi) \cdot \xi$

1.2 Measures

Given a metric space Ξ, we denote by $\mathcal{M}_b(\Xi;\mathbb{R}^n)$ the space of bounded $\mathbb{R}^n$-valued Radon measures on Ξ. For every $\mu \in \mathcal{M}_b(\Xi;\mathbb{R}^n)$, $|\mu|$ is the *total variation* of μ. We denote by $\mathcal{M}_b(\Xi)$ the space of bounded scalar measures, by $\mathcal{M}_b^+(\Xi)$ the set of positive bounded measures, and by $\mathcal{P}(\Xi)$ the set of probability measures.

If Ξ is a separable metric space and $\mu \in \mathcal{M}_b(\Xi;\mathbb{R}^n)$, the support of μ is the smallest closed subset of Ξ where the measure μ is concentrated, i.e.,

$$\operatorname{supp}(\mu) := \bigcap \{C \ : \ C \text{ closed}, \ \mu(\Xi \setminus C) = 0\} .$$

Different notions of convergence can be considered in the space $\mathcal{M}_b(\Xi;\mathbb{R}^n)$. In this thesis we will mainly deal with the weak* convergence and with the narrow convergence. We say that a sequence $\mu_k \in \mathcal{M}_b(\Xi;\mathbb{R}^n)$ converges *weakly** to $\mu \in \mathcal{M}_b(\Xi;\mathbb{R}^n)$ if

$$\int_\Xi f \cdot \mathrm{d}\mu_k \to \int_\Xi f \cdot \mathrm{d}\mu , \quad \text{for every } f \in \mathcal{C}_0(\Xi;\mathbb{R}^n) .$$

We say that μ_k converges *narrowly* to μ if

$$\int_\Xi f \cdot \mathrm{d}\mu_k \to \int_\Xi f \cdot \mathrm{d}\mu , \quad \text{for every } f \in \mathcal{C}_b(\Xi;\mathbb{R}^n) .$$

Let Ξ_1 and Ξ_2 be two metric spaces, let $\varphi : \Xi_1 \to \Xi_2$ be a Borel map, and let $\mu \in \mathcal{M}_b(\Xi_1;\mathbb{R}^n)$. The *push-forward* of μ through the map φ is the measure $\varphi_\#\mu \in \mathcal{M}_b(\Xi_2)$ defined by $\varphi_\#\mu(A) := \mu(\varphi^{-1}(A))$ for every $A \in \mathscr{B}(\Xi_2)$.

In Chapter 4, we shall denote the duality between $\mathcal{P}(\Xi)$ and $\mathcal{C}_b(\Xi)$ by

$$\langle f, \mu \rangle := \int_\Xi f(\xi)\mu(\mathrm{d}\xi) = \int_\Xi f(\xi)\,\mathrm{d}\mu(\xi) , \tag{1.1}$$

for every $\mu \in \mathcal{P}(\Xi)$ and $f \in \mathcal{C}_b(\Xi)$.

1.2.1 Integrals of quasicontinuous representatives

Let Ω be a bounded open set in $\mathbb{R}^n$. In general, if $\sigma\colon \mathbb{R} \to \mathbb{R}$ is a continuous function, $\mu \in \mathcal{M}_b(\Omega)$, and $\alpha\colon \Omega \to \mathbb{R}$ is a function defined $\mathcal{L}^n$-a.e., then the integral

$$\int_\Omega \sigma(\alpha)\,\mathrm{d}\mu \tag{1.2}$$

is not well defined, since μ may charge sets of Lebesgue measure zero. However, if μ vanishes on $\mathcal{H}^{n-1}$-negligible sets and $\alpha \in H^1(\Omega)$, it is possible to give a meaning to (1.2) in the sense that we specify below.

We recall that the *capacity* of a set $E \subset \Omega$ (we refer, e.g., to [44, 54, 60, 69] for a general theory) is defined by

$$\mathrm{Cap}(E) := \inf\Big\{ \int_\Omega |\nabla w|^2 \, \mathrm{d}x \ : \ w \in H^1_0(\Omega)\,,\ w \geq 1 \text{ a.e. in a neighbourhood of } E \Big\}\,.$$

A property is said to hold Cap-*quasi everywhere* (abbreviated as Cap-q.e.) if it holds except for a subset of capacity zero. A function $\beta : \Omega \to \mathbb{R}$ is Cap-*quasicontinuous* if for every $\varepsilon > 0$ there exists a set E_ε with $\mathrm{Cap}(E_\varepsilon) < \varepsilon$ such that $\beta|_{\Omega \setminus E_\varepsilon}$ is continuous. For every function $\alpha \in H^1(\Omega)$ there exists a Cap-*quasicontinuous representative* $\widetilde{\alpha}$, i.e., a Cap-quasicontinuous function $\widetilde{\alpha}$ such that $\widetilde{\alpha} = \alpha$ $\mathcal{L}^n$-a.e. in Ω. The Cap-quasicontinuous representative is essentially unique, that is, if β is another Cap-quasicontinuous representative of α, then $\beta = \widetilde{\alpha}$ Cap-q.e. in Ω. Moreover it can be proved that (see [44, Theorem 4.8.1])

$$\lim_{\rho \to 0^+} \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |\alpha(y) - \widetilde{\alpha}(x)| \, \mathrm{d}y = 0 \qquad \text{for Cap-a.e. } x \in \Omega\,. \tag{1.3}$$

We recall that if $E \subset \mathbb{R}^n$ is such that $\mathrm{Cap}(E) = 0$, then its s-dimensional Hausdorff measure $\mathcal{H}^s(E)$ vanishes for every $s > n-2$. As a consequence, the Cap-quasicontinuous representative of a function $\alpha \in H^1(\Omega)$ is well defined $\mathcal{H}^{n-1}$-a.e. in Ω. Therefore for every $\alpha \in H^1(\Omega)$ the integral

$$\int_\Omega \sigma(\widetilde{\alpha}) \, \mathrm{d}\mu$$

makes sense for every measure $\mu \in \mathcal{M}_b(\Omega)$ which vanishes on $\mathcal{H}^{n-1}$-negligible sets.

1.3 BV functions and BD functions

For the general theory regarding the space of functions of bounded variation we refer to the books [7, 44], while we refer to the book [67] for the definition of functions of bounded deformation and to [6] for their fine properties. In this section we recall the definitions and the properties relevant for the results presented in this thesis. Throughout the chapter, Ω is a bounded open set in $\mathbb{R}^n$.

1.3.1 Functions of bounded variation

A function $u \in L^1(\Omega)$ belongs to the space $BV(\Omega)$ of *functions of bounded variation* if $Du \in \mathcal{M}_b(\Omega;\mathbb{R}^n)$, where Du is the distributional gradient of u. A function u belongs to $BV_{loc}(\Omega)$ if $u \in BV(U)$ for every open set U such that $U \Subset \Omega$.

A sequence $u_k \in BV(\Omega)$ *converges weakly** to a function $u \in BV(\Omega)$ (and we denote it by $u_k \xrightharpoonup{*} u$) if $u_k \to u$ in $L^1(\Omega)$ and $Du_k \xrightharpoonup{*} Du$ weakly* in $\mathcal{M}_b(\Omega;\mathbb{R}^n)$.

If $u \in BV(\Omega)$, then its distributional gradient Du can be decomposed as

$$Du = D^a u + D^s u$$

where $D^a u$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^n$ and $D^s u$ is singular with respect to $\mathcal{L}^n$. It is possible to prove that $D^a u = \nabla u \, \mathcal{L}^n \llcorner \Omega$, where ∇u is the *approximate gradient* of u. Moreover, the singular measure D^s can be further decomposed as

$$D^s u = D^c u + D^j u = D^c u + [u] \, \nu_u \, \mathcal{H}^{n-1} \llcorner J_u$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure, J_u is the *jump set* of u, ν_u is the normal to the $\mathcal{H}^{n-1}$-rectifiable set J_u, $[u] = u^+ - u^-$ is the *jump* of u, and $D^c u$ is the *Cantor part* of Du, which is a singular measure with respect to the Lebesgue measure and vanishes on all Borel sets $B \subset \mathbb{R}^n$ with $\mathcal{H}^{n-1}(B) < +\infty$.

Every function $u \in BV(\Omega)$ admits a *precise representative* $\widetilde{u}(x)$ of u, defined for $\mathcal{H}^{n-1}$-a.e. x in $\Omega \setminus J_u$. For $\mathcal{H}^{n-1}$-a.e. x in $\Omega \setminus J_u$ we have

$$\lim_{\rho \to 0^+} \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |u(y) - \widetilde{u}(x)| \, \mathrm{d}y = 0 \, .$$

Hence a function of bounded variation u is approximately continuous $\mathcal{H}^{n-1}$-a.e. in $\Omega \setminus J_u$. In particular, if $u, v \in BV(\Omega)$, then $\mathcal{H}^{n-1}(J_{u+v} \setminus (J_u \cup J_v)) = 0$.

1.3.2 Generalised functions of bounded variation

Generalised functions of bounded variation arise as limits of functions of bounded variation with lack of controls on the L^∞ norm. A function u is in the space $GBV(\Omega)$ of *generalised functions of bounded variation* if the truncated functions

$$u_M := ((-M) \vee u) \wedge M$$

belong to $BV_{loc}(\Omega)$ for every $M > 0$.

The structure of a generalised function of bounded variation is similar to that of a function of bounded variation (cf. [7, Theorem 4.34]). For the sequel, we recall the following fine properties.

1. The weak approximate gradient $\nabla u(x)$ exists for $\mathcal{L}^n$-a.e. $x \in \Omega$ and it satisfies

$$\nabla u(x) = \nabla(u_M)(x) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \{|u| \leq M\} \, .$$

2. The weak approximate limits u^+ and u^- satisfy

$$u^+(x) = \lim_{M \to +\infty} (u_M)^+(x) \, , \quad u^-(x) = \lim_{M \to +\infty} (u_M)^-(x) \, .$$

   Moreover, the set J_u of weak approximate jump points satisfies $J_u = \bigcup_{M>0} J_{u_M}$ and it is $\mathcal{H}^{n-1}$-rectifiable.

In general, the function ∇u is not locally $\mathcal{L}^n$-integrable and $[u] = u^+ - u^-$ is not locally $(\mathcal{H}^{n-1} \llcorner J_u)$-integrable.

The following proposition shows that we can define the Cantor part of the gradient of a GBV function under suitable assumptions. An alternative proof of this result can be found in [5].

Proposition 1.3.1. *Let $u \in GBV(\Omega)$ be such that*

$$\sup_{M>0} |D^c u_M|(\Omega) < +\infty, \tag{1.4}$$

where $u_M := ((-M) \vee u) \wedge M$. Then there exists a unique $\mathbb{R}^n$-valued Borel measure on $\Omega \setminus J_u$, denoted by $D^c u$, such that

$$(D^c u)(B) = (D^c u_M)(B) \tag{1.5}$$

for every $M > 0$ and for every Borel set $B \subset \{|\widetilde{u}| < M\} \setminus J_u$. Moreover, we have

$$|D^c u|(B) = \sup_{M>0} |D^c u_M|(B \cap \{|\widetilde{u}| < M\}) \tag{1.6}$$

for every Borel set $B \subset \Omega \setminus J_u$.

Proof. For every Borel set $B \subset \Omega$ we define

$$\mu(B) := \sup_{M>0} |D^c u_M|(B).$$

By (1.4), the set function μ is a bounded Borel measure on Ω.

We observe that for every $M > 0$ we have $\widetilde{u_M} = \widetilde{u}$ $\mathcal{H}^{n-1}$-a.e. on $\{|\widetilde{u}| < M\} \setminus J_u$. Applying [7, Proposition 3.92-(c)], for every $0 < M < M^*$, we have $(D^c u_M)(B) = (D^c u_{M^*})(B)$ for every Borel set $B \subset \{|\widetilde{u}| < M\} \setminus J_u$. For these sets we can define $(D^c u)(B) := (D^c u_M)(B)$ and the definition does not depend on M. Since μ is a bounded measure, we have $\mu(\{M \le |\widetilde{u}| < +\infty\} \setminus J_u) \to 0$ as $M \to +\infty$. By the Cauchy criterion, this implies that the limit

$$\lim_{M\to+\infty} (D^c u_M)(B \cap \{|\widetilde{u}| < M\}) \tag{1.7}$$

exists and is finite for every Borel set $B \subset \Omega \setminus J_u$. We now define $(D^c u)(B)$ by (1.7). Using again the upper bound $|D^c u_M|(B) \le \mu(B)$, we can prove that $D^c u$ is a bounded $\mathbb{R}^n$-valued Borel measure on $\Omega \setminus J_u$. Equality (1.6) follows easily from (1.7), while the uniqueness is a consequence of the fact that $\mu(\{x \in \Omega \setminus J_u \ : \ |\widetilde{u}(x)| = +\infty\}) = 0$. □

We conclude this subsection with a remark about the one-dimensional GBV functions. In dimension one, a control on the weak approximate gradient and on the Cantor part is enough to guarantee that a GBV function is actually a BV function, as explained in the following proposition. Even in dimension one we keep the notation ∇u for the approximate gradient of u.

Proposition 1.3.2. *Let $\Omega \subset \mathbb{R}$ be a bounded open set. For every $u \in GBV(\Omega)$ and for every open set $A \subset \Omega$ let*

$$\Psi(u; A) := \int_A |\nabla u| \,\mathrm{d}x + |D^c u|(A) + \sum_{x \in (J_u \setminus J_u^1) \cap A} |[u](x)| + \mathcal{H}^0(J_u^1 \cap A),$$

where $J_u^1 := \{x \in J_u : |[u](x)| \ge 1\}$. Let $u \in GBV(\Omega) \cap L^1(\Omega)$ be such that $\Psi(u; \Omega) < +\infty$. Then $u \in BV(\Omega)$.

Proof. Step 1: Let us assume that Ω is a bounded interval and that all the jumps of u are smaller than 1, i.e., $J_u^1 = \emptyset$. Then for every $M > 0$, the truncated functions u_M belong to $BV(\Omega)$ and $|Du_M|(\Omega) \leq \Psi(u;\Omega)$, which implies that $u \in BV(\Omega)$ and $|Du|(\Omega) \leq \Psi(u;\Omega)$.

Step 2: Let us assume that Ω is a bounded interval. Since $\Psi(u;\Omega) < +\infty$, the set J_u^1 is finite. Therefore $\Omega \setminus J_u^1$ is the union of a finite number of open intervals Ω_i. By *Step 1*, we have $u \in BV(\Omega_i)$ with $|Du|(\Omega_i) \leq \Psi(u;\Omega_i)$, because all jump points of u in Ω_i are smaller than 1. We conclude that $u \in BV(\Omega)$ and $|Du|(\Omega) \leq \Psi(u;\Omega) + \sum_{x\in J_u^1} |[u](x)|$.

Step 3: Let us assume that Ω is a bounded open set in $\mathbb{R}$. Then Ω is the union of a family of pairwise disjoint open intervals Ω_i. Since $\Psi(u;\Omega) < +\infty$, the set J_u^1 is finite, hence there exists a finite set of indices I such that $J_u^1 \subset \bigcup_{i\in I} \Omega_i$. Arguing as in *Step 1* for $i \notin I$ and as in *Step 2* for $i \in I$, we get that $u \in BV(\Omega_i)$ for every i and

$$|Du|(\Omega) \leq \sum_{i\notin I} \Psi(u;\Omega_i) + \sum_{i\in I} \Psi(u;\Omega_i) + \sum_{x\in J_u^1} |[u](x)| = \Psi(u;\Omega) + \sum_{x\in J_u^1} |[u](x)| < +\infty\,,$$

hence $u \in BV(\Omega)$. □

1.3.3 Slicing of functions of bounded variation

In Chapter 2, we shall use a slicing argument to reduce the problem from the n-dimensional to the one-dimensional case. For all the details about the slicing of BV functions, we refer to [7, Section 3.11]. Here we fix the notation and we recall the main properties.

For every $\xi \in \mathbb{S}^{n-1}$ (playing the role of the slicing direction) we consider the hyperplane orthogonal to ξ

$$\Pi^\xi := \{y \in \mathbb{R}^n : y \cdot \xi = 0\}$$

and for every set $B \subset \mathbb{R}^n$ we define the slice of the set B by

$$B_y^\xi := \{t \in \mathbb{R} : y + t\xi \in B\}\,,$$

for every $y \in \Pi^\xi$.

If $w\colon \Omega \to \mathbb{R}$ is a scalar function and $v\colon \Omega \to \mathbb{R}^n$ is a vector function, we define their slices $w_y^\xi\colon \Omega_y^\xi \to \mathbb{R}$ and $\hat{v}_y^\xi\colon \Omega_y^\xi \to \mathbb{R}$ by

$$w_y^\xi(t) := w(y + t\xi) \quad \text{and} \quad \hat{v}_y^\xi := (v \cdot \xi)_y^\xi\,,$$

respectively. If u_k is a sequence in $L^1(\Omega)$ such that $u_k \to u$ in $L^1(\Omega)$, using Fubini Theorem we can prove that for every $\xi \in \mathbb{S}^{n-1}$ there exists a subsequence u_{k_j} such that $(u_{k_j})_y^\xi \to u_y^\xi$ in $L^1(\Omega_y^\xi)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$.

We recall that a function $u \in L^1(\Omega)$ belongs to $BV(\Omega)$ if and only if, for every direction $\xi \in \mathbb{S}^{n-1}$, we have

$$u_y^\xi \in BV(\Omega_y^\xi) \ \text{ for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi \quad \text{and} \quad \int_{\Pi^\xi} |Du_y^\xi|(\Omega_y^\xi)\, \mathrm{d}\mathcal{H}^{n-1}(y) < +\infty. \tag{1.8}$$

Moreover, we have that $(\nabla u \cdot \xi)_y^\xi$ coincides $\mathcal{L}^1$-a.e. in Ω_y^ξ with the density ∇u_y^ξ of the absolutely continuous part of the distributional derivative of u_y^ξ; as for the Cantor part, we have

$$D^c u(B)\cdot\xi = \int_{\Pi^\xi} D^c u^\xi_y(B^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}$$

for every Borel set $B\subset\Omega$; finally, for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$ we have that $(J_u)^\xi_y = J_{u^\xi_y}$ and $[u](y+t\xi)=[u^\xi_y](t)$.

1.3.4 Functions of bounded deformation

A vector-valued function $u\in L^1(\Omega;\mathbb{R}^n)$ belongs to the space $BD(\Omega)$ of *functions of bounded deformation* if $Eu\in\mathcal{M}_b(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$, where $Eu := \frac{1}{2}(Du^T+Du)$ is the symmetric part of distributional gradient of u.

A sequence $u_k\in BD(\Omega)$ *converges weakly** to a function $u\in BD(\Omega)$ (and we denote it by $u_k \overset{*}{\rightharpoonup} u$) if $u_k\to u$ in $L^1(\Omega;\mathbb{R}^n)$ and $Eu_k\overset{*}{\rightharpoonup} Eu$ weakly* in $\mathcal{M}_b(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$.

If $u\in BD(\Omega)$, then the measure Eu can be decomposed as

$$Eu = E^a u + E^s u$$

where $E^a u$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^n$ and $E^s u$ is singular with respect to $\mathcal{L}^n$. It is possible to prove that $E^a u = \mathcal{E}u\,\mathcal{L}^n \llcorner \Omega$, where $\mathcal{E}u$ is the symmetric part of the approximate gradient of u. Moreover, the singular measure $E^s u$ can be further decomposed as

$$E^s u = E^c u + E^j u = E^c u + [u]\odot\nu_u\,\mathcal{H}^{n-1}\llcorner J_u$$

where J_u is the *jump set* of u, ν_u is the normal to J_u, $[u]=u^+-u^-$ is the *jump* of u, and $E^c u$ is the *Cantor part* of Eu, which is a singular measure with respect to the Lebesgue measure and vanishes on all Borel sets $B\subset\mathbb{R}^n$ with $\mathcal{H}^{n-1}(B)<+\infty$.

In contrast to the case of BV functions, it is still unknown whether a function $u\in BD(\Omega)$ is approximately continuous $\mathcal{H}^{n-1}$-a.e. in $\Omega\setminus J_u$. Nonetheless, it is possible to prove the following property for the jump set of the sum of BD functions.

Proposition 1.3.3. *For every $u,v\in BD(\Omega)$ we have that $\mathcal{H}^{n-1}(J_{u+v}\setminus(J_u\cup J_v))=0$.*

Proof. Let S_u be the set of points in which u is not approximately continuous. While it is easy to see that $J_u\subset S_u$, it is not known whether $\mathcal{H}^{n-1}(S_u\setminus J_u)=0$. However, it is proved in [6, Remark 6.3] that $S_u\setminus J_u$ is purely $(\mathcal{H}^{n-1},n-1)$-unrectifiable, i.e., $\mathcal{H}^{n-1}((S_u\setminus J_u)\cap M)=0$ for every $(\mathcal{H}^{n-1},n-1)$-rectifiable set M.

Since $J_{u+v}\subset S_{u+v}\subset S_u\cup S_v$, we have that $J_{u+v}\setminus(J_u\cup J_v)\subset(S_u\setminus J_u)\cup(S_v\setminus J_v)$. Since $J_{u+v}\setminus(J_u\cup J_v)$ is $(\mathcal{H}^{n-1},n-1)$-rectifiable, by the pure $(\mathcal{H}^{n-1},n-1)$-unrectifiability of $S_u\setminus J_u$ and $S_v\setminus J_v$ we conclude that $\mathcal{H}^{n-1}(J_{u+v}\setminus(J_u\cup J_v))=0$. □

As a consequence of the previous result, we obtain the following triangle inequality, which will be employed in Chapter 3.

Corollary 1.3.4. *Let $u,v\in BD(\Omega)$, let $\varphi\in\mathcal{C}^\infty(\Omega)$ with $0\le\varphi\le 1$, and let $w:=\varphi u+(1-\varphi)v$. Then*

$$\int_{J_w}|[w]|\wedge 1\,\mathrm{d}\mathcal{H}^{n-1} \le \int_{J_u}|[u]|\wedge 1\,\mathrm{d}\mathcal{H}^{n-1} + \int_{J_v}|[v]|\wedge 1\,\mathrm{d}\mathcal{H}^{n-1}.$$

Proof. Let us define $[u]^*\colon J_w \to \mathbb{R}^n$ by $[u]^* = [u]$ on $J_w \cap J_u$ and $[u]^* = 0$ on $J_w \setminus J_u$. Similarly we define $[v]^*\colon J_w \to \mathbb{R}^n$ by $[v]^* = [v]$ on $J_w \cap J_v$ and $[v]^* = 0$ on $J_w \setminus J_v$. Using the sets S_u and S_v considered in the proof of Proposition 1.3.3 and taking into account the pure $(\mathcal{H}^{n-1}, n{-}1)$-unrectifiability of $S_u \setminus J_u$ and $S_v \setminus J_v$, we obtain that $\mathcal{H}^{n-1}(J_w \cap (S_u \setminus J_u)) = \mathcal{H}^{n-1}(J_w \cap (S_v \setminus J_v)) = 0$. This implies that $[w] = \varphi[u]^* + (1-\varphi)[v]^*$ $\mathcal{H}^{n-1}$-a.e. on J_w. The conclusion follows easily. □

1.3.5 Slicing of functions of bounded deformation

For all the details about slicing of BD functions, we refer to [6]. Here we recall that a function $u \in L^1(\Omega;\mathbb{R}^n)$ belongs to $BD(\Omega)$ if and only if we have

$$\hat{u}^\xi_y \in BV(\Omega^\xi_y) \ \text{ for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi \quad \text{and} \quad \int_{\Pi^\xi} |D\hat{u}^\xi_y|(\Omega^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}(y) < +\infty\,,$$

for every direction $\xi \in \mathbb{S}^{n-1}$ (cf. Subsection 1.3.3 for the notation). Moreover, we have that $(\mathcal{E}u\xi\cdot\xi)^\xi_y$ coincides $\mathcal{L}^1$-a.e. in Ω^ξ_y with the density $\nabla\hat{u}^\xi_y$ of the absolutely continuous part of the distributional derivative of $\hat{u}^\xi_y$; as for the Cantor part, we have

$$E^c u(B)\xi\cdot\xi = \int_{\Pi^\xi} D^c\hat{u}^\xi_y(B^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}$$

for every Borel set $B \subset \Omega$; finally, for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have that $(J^\xi_u)^\xi_y = J_{\hat{u}^\xi_y}$ and $[u](y+t\xi)\cdot\xi = [\hat{u}^\xi_y](t)$, where $J^\xi_u = \{x \in J_u : [u](x)\cdot\xi \neq 0\}$.

1.4 Γ-convergence and lower semicontinuity

We recall now the definition of Γ-convergence and some basic results of the general theory. For more details about Γ-convergence we refer to the books [29, 17]. In the following, Ξ denotes a metric space.

1.4.1 Γ-convergence and convergence of minimisers

Given a sequence of functionals $F_k\colon \Xi \to [0,+\infty]$, for every $u \in \Xi$ we define

$$\Gamma\text{-}\liminf_{k\to+\infty} F_k(u) := \inf\Big\{\liminf_{k\to+\infty} F_k(u_k) \ :\ u_k \to u\Big\}\,,$$
$$\Gamma\text{-}\limsup_{k\to+\infty} F_k(u) := \inf\Big\{\limsup_{k\to+\infty} F_k(u_k) \ :\ u_k \to u\Big\}\,.$$

We say that F_k Γ*-converges* to $F\colon \Xi \to [0,+\infty]$ if for every $u \in \Xi$ we have

$$F(u) = \Gamma\text{-}\liminf_{k\to+\infty} F_k(u) = \Gamma\text{-}\limsup_{k\to+\infty} F_k(u)\,.$$

Equivalently, F_k Γ-converges to F if and only if for every $u \in \Xi$ the following two properties hold true:

- for every sequence u_k such that $u_k \to u$, we have $F(u) \le \liminf_{k\to+\infty} F_k(u_k)$;

- there exists a sequence u_k such that $u_k \to u$ and $\limsup\limits_{k\to+\infty} F_k(u_k) \le F(u)$.

The Γ-limit of a sequence of functionals is always lower semicontinuous. We recall that a functional $F : \Xi \to [0,+\infty]$ is *lower semicontinuous* if for every sequence $u_k \in \Xi$ such that $u_k \to u$ we have

$$F(u) \le \liminf_{k\to+\infty} F(u_k)\,.$$

As shown in the following theorem, the Γ-convergence of functionals guarantees the convergence of minima and minimisers of the functionals. For every $\eta > 0$, we say that $u \in \Xi$ is an η-minimiser of $F\colon \Xi \to [0,+\infty]$ if $F(u) < \inf\limits_{\Xi} F + \eta$.

Theorem 1.4.1. *Let $F_k\colon \Xi \to [0,+\infty]$ be a sequence of functionals and let $\eta_k \searrow 0$. For every $k \in \mathbb{N}$, let u_k be an η_k-minimiser of F_k. If u is a cluster point of the sequence u_k, then u is a minimiser of F and $F(u) = \limsup_k F_k(u_k)$. Moreover, if k_j is a subsequence such that $u_{k_j} \to u$, then $F(u) = \lim_j F_{k_j}(u_{k_j})$.*

1.4.2 $\overline{\Gamma}$-convergence and integral representation results

We recall here the notion of $\overline{\Gamma}$-convergence for sequences of increasing functionals.

Let us fix an open set $\Omega \subset \mathbb{R}^n$ and let us denote by $\mathscr{A}(\Omega)$ the class of all open subsets contained in Ω. We say that a functional $F\colon \Xi\times\mathscr{A}(\Omega) \to [0,+\infty]$ is *increasing* if $F(u;A) \le F(u;B)$ for every $u \in \Xi$ and for every $A, B \in \mathscr{A}(\Omega)$ such that $A \subset B$.

Let $F_k\colon \Xi\times\mathscr{A}(\Omega) \to [0,+\infty]$ be a sequence of increasing functionals and let us define

$$F'(\cdot;A) := \Gamma\text{-}\liminf F_k(\cdot;A)\,, \quad F''(\cdot;A) := \Gamma\text{-}\limsup F_k(\cdot;A)\,,$$

for every $A \in \mathscr{A}(\Omega)$.

We say that the sequence F_k $\overline{\Gamma}$*-converges* to a functional $F\colon \Xi\times\mathscr{A}(\Omega) \to [0,+\infty]$ if F coincides with the inner regular envelope of both functionals F' and F'', i.e.,

$$\begin{aligned} F(u;A) &= \sup\{F'(u;U) \ : \ U \in \mathscr{A}(\Omega)\,,\ U \Subset A\}\,,\\ &= \sup\{F''(u;U) \ : \ U \in \mathscr{A}(\Omega)\,,\ U \Subset A\}\,. \end{aligned}$$

The following compactness theorem holds.

Theorem 1.4.2. *Assume that Ξ is a separable metric space. Then every sequence $F_k\colon \Xi\times\mathscr{A}(\Omega) \to [0,+\infty]$ of increasing functionals has a $\overline{\Gamma}$-convergent subsequence.*

Remark 1.4.3. Let $F_k\colon \Xi\times\mathscr{A}(\Omega) \to [0,+\infty]$ be a sequence of increasing functionals that $\overline{\Gamma}$-converges to F. Then

- $F(u;\cdot)$ is inner regular for every $u \in \Xi$;
- $F(\cdot;A)$ is lower semicontinuous for every $u \in \Xi$;
- if $F_k(u;\cdot)$ is superadditive for every k and for every $u \in \Xi$, then $F(u;\cdot)$ is superadditive for every $u \in \Xi$ (cf. [29, Proposition 16.12]).

In general, $F(u;\cdot)$ is not subadditive, even if $F_k(u;\cdot)$ is subadditive for every k (cf. [29, Example 16.14]). This is one of the main difficulties when proving that $F(u;\cdot)$ is a measure. (See Lemma 2.4.3 and Lemma 3.5.3 for the cases studied in this thesis.)

In the case where $\Xi = L^1(\Omega;\mathbb{R}^m)$, we say that a functional $F\colon L^1(\Omega;\mathbb{R}^m)\times\mathscr{A}(\Omega)\to[0,+\infty]$ is *local* if for every $A\in\mathscr{A}(\Omega)$ we have $F(u;A)=F(v;A)$ for every $u,v\in L^1(\Omega;\mathbb{R}^m)$ such that $u=v$ a.e. in Ω. Given a sequence of increasing functionals $F_k\colon L^1(\Omega;\mathbb{R}^m)\times\mathscr{A}(\Omega)\to[0,+\infty]$ that $\overline{\Gamma}$-converges to F, we have that

- if F_k is local for every k, then F is local (cf. [29, Proposition 16.15]).

One of the main difficulties in the applications of $\overline{\Gamma}$-convergence is the explicit determination of the limit functional. We recall here an integral representation result proved by BOUCHITTÉ, FONSECA, and MASCARENHAS in [14] concerning functionals defined on the space $BV(\Omega)$. We shall apply this result in Section 2.4.

Theorem 1.4.4. *Assume that $\mathcal{G}\colon BV(\Omega)\times\mathscr{A}(\Omega)\to[0,+\infty)$ satisfies the following properties:*

(a) $\mathcal{G}$ is local;

(b) $\mathcal{G}(\cdot;A)$ is L^1-lower semicontinuous, for every $A\in\mathscr{A}(\Omega)$;

(c) there exists a constant $c>0$ such that $\frac{1}{c}|Du|(A)\le\mathcal{G}(u;A)\le c(|Du|(A)+\mathcal{L}^n(A))$, for every $u\in BV(\Omega)$ and for every $A\in\mathscr{A}(\Omega)$;

(d) for every $u\in BV(\Omega)$, $\mathcal{G}(u;\cdot)$ is the restriction to open sets of a Radon measure;

(e) $\mathcal{G}(u(\cdot-x_0)+b;x_0+A)=\mathcal{G}(u;A)$ for all $b\in\mathbb{R}$ and $x_0\in\mathbb{R}^n$ such that $x_0+A\subset\Omega$.

Then there exists three Borel functions $\overline{f}\colon\mathbb{R}^n\to[0,+\infty)$, $\overline{h}\colon\mathbb{R}^n\to[0,+\infty)$, and $\overline{g}\colon\mathbb{R}\times\mathbb{S}^{n-1}\to[0,+\infty)$ such that

$$\mathcal{G}(u;A)=\int_A\overline{f}(\nabla u)\,\mathrm{d}x+\int_A\overline{h}\Big(\frac{\mathrm{d}D^cu}{\mathrm{d}|D^cu|}\Big)\,\mathrm{d}|D^cu|+\int_{A\cap J_u}\overline{g}([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{1.9}$$

Remark 1.4.5. In [14], the authors provide an explicit formula for the functions $\overline{f},\overline{h}$, and $\overline{g}$ appearing in (1.9). In particular, we recall that the surface term can be characterized by means of minimum problems related to the pure jump functions $u_{\nu,a}$ defined by

$$u_{\nu,a}(x):=\begin{cases}a & \text{if } x\cdot\nu>0\,,\\ 0 & \text{if } x\cdot\nu<0\,,\end{cases} \tag{1.10}$$

for $a\in\mathbb{R}$ and $\nu\in\mathbb{S}^{n-1}$. More precisely, let Q_ρ^ν be a cube of side ρ centred at the origin and with a face orthogonal to ν. Then we have that

$$\overline{g}(a,\nu)=\limsup_{\rho\to0^+}\Big[\frac{1}{\rho^{n-1}}\inf\{\mathcal{G}(v;Q_\rho^\nu):v\in BV(Q_\rho^\nu)\,,\ v(x)=u_{\nu,a}(x)\text{ for }y\in\partial Q_\rho^\nu\}\Big]\,.$$

In Section 3.5 we shall use a similar result to characterise the surface term of functionals defined on the space $BD(\Omega)$. The result was originally stated by BARROSO, FONSECA, and TOADER in [12] for the relaxation of a functional defined on the Sobolev space $W^{1,1}(\Omega;\mathbb{R}^n)$. In the following statement, $u_{\nu,z}$ is the pure jump function defined as in (1.10) with the vector $z\in\mathbb{R}^n$ replacing the scalar $a\in\mathbb{R}$.

Theorem 1.4.6. *Assume that $\mathcal{G}\colon BD(\Omega)\times\mathscr{A}(\Omega)\to[0,+\infty)$ satisfies the following properties:*

(a) $\mathcal{G}$ *is local;*

(b) $\mathcal{G}(\cdot;A)$ *is* L^1*-lower semicontinuous, for every* $A \in \mathscr{A}(\Omega)$*;*

(c) there exists a constant $c > 0$ *such that* $\frac{1}{c}|Eu|(A) \leq \mathcal{G}(u;A) \leq c(|Eu|(A)+\mathcal{L}^n(A))$*, for every* $u \in BD(\Omega)$ *and for every* $A \in \mathscr{A}(\Omega)$*;*

(d) for every $u \in BD(\Omega)$*,* $\mathcal{G}(u;\cdot)$ *is the restriction to* $\mathscr{A}(\Omega)$ *of a Radon measure;*

(e) $\mathcal{G}(u(\cdot - x_0)+b; x_0+A) = \mathcal{G}(u;A)$ *for all* $b \in \mathbb{R}^n$ *and* $x_0 \in \mathbb{R}^n$ *such that* $x_0+A \subset \Omega$*.*

Then, for every $u \in BD(\Omega)$ *and for every* $A \in \mathscr{A}(\Omega)$*, we have that*

$$\mathcal{G}(u; J_u \cap A) = \int_{J_u \cap A} \overline{\psi}([u], \nu_u)\, \mathrm{d}\mathcal{H}^{n-1}\,, \tag{1.11}$$

where

$$\overline{\psi}(z,\nu) = \limsup_{\rho \to 0^+} \Big[\frac{1}{\rho^{n-1}} \inf\{\mathcal{G}(v;Q^\nu_\rho) : v \in BD(Q^\nu_\rho)\,,\ v(x) = u_{\nu,z}(x) \text{ on } \partial Q^\nu_\rho\}\Big]\,. \tag{1.12}$$

Proof. A careful inspection of the proof of [12, Proposition 5.1] shows that the integral representation result still holds for a functional satisfying properties (a)–(e). In particular [12, Lemma 3.10], only uses assumptions (a), (c), and (d). Under assumptions (a)–(d) it is easy to check that [12, Lemma 3.11] holds for $\mathcal{G}$. Using these results, the proof of the integral representation [12, Proposition 5.1] for $\mathcal{G}$ can be easily extended from the case $u \in SBD(\Omega)$ to the case $u \in BD(\Omega)$. As for (1.12), it is a consequence of the formula for the integrand given in [12, Proposition 5.1] and of the invariance properties due to (e). □

1.4.3 A localisation lemma

We conclude this section by recalling a useful tool of measure theory. We shall employ this result in Section 2.4 and in Section 3.4 to optimise locally the lower bounds of functionals. The proof of the lemma can be found, e.g., in [17, Lemma 15.2].

Lemma 1.4.7. *Let* Λ *be a function defined on the family of open subsets of* Ω*, which is superadditive on open sets with disjoint compact closure. Let* λ *be a positive measure on* Ω*, let* φ_j*,* $j \in \mathbb{N}$*, be nonnegative Borel functions such that*

$$\int_K \varphi_j \,\mathrm{d}\lambda \leq \Lambda(A)$$

for every open set $A \subset \Omega$*, for every compact set* $K \subset A$*, and for every* $j \in \mathbb{N}$*. Then*

$$\int_K \sup_j \varphi_j \,\mathrm{d}\lambda \leq \Lambda(A)$$

for every open set $A \subset \Omega$ *and for every compact set* $K \subset A$*. Moreover, if* $A \subset \Omega$ *is an open set with* $\Lambda(A) < +\infty$*, then*

$$\int_K \sup_j \varphi_j \,\mathrm{d}\lambda = \sup\Big\{\sum_{j=1}^r \int_{K^j} \varphi_j \,\mathrm{d}\lambda : (K^j)_{j=1}^r \text{ disjoint compact subsets of } K,\ r \in \mathbb{N}\Big\}$$

for every compact set $K \subset A$*.*

1.5 Young measures

For an introduction to the general theory of Young measures we refer, e.g., to [68]. Here we recall some basic notions and properties.

1.5.1 Definition and notion of convergence

Let us fix a σ-compact locally compact metric space Ξ and a compact metric space Γ endowed with a positive bounded measure μ. The reader may think of Ξ as $\mathbb{R}$ or $[-\infty, \infty]$. Moreover, in Chapter 4, μ will be the measure $\mathcal{H}^{n-1}$ and Γ will be the intersection of a $(n-1)$-dimensional manifold with $\overline{\Omega}$, where Ω is a bounded open set in $\mathbb{R}^n$.

Definition 1.5.1. The collection of *Young measures* on $\Gamma\times\Xi$ with respect to the measure μ is the set

$$\mathcal{Y}(\Gamma;\Xi) := \{\nu \in \mathcal{M}_b^+(\Gamma\times\Xi) : \pi^\Gamma_\# \nu = \mu\}\,,$$

where $\pi^\Gamma \colon \Gamma\times\Xi \to \Gamma$ is the projection on Γ.

Remark 1.5.2. We recall that a family $(\nu^x)_{x\in\Gamma}$ of probability measures $\nu^x \in \mathcal{P}(\Xi)$ parametrised on Γ is said to be *measurable* if the function $x \mapsto \nu^x(A)$ is μ-measurable for every $A \in \mathscr{B}(\Xi)$. By the Disintegration Theorem (see [68, Corollary A5] or [7, Theorem 2.28]), it is always possible to associate a measurable family of probability measures $(\nu^x)_{x\in\Gamma}$ with a Young measure $\nu \in \mathcal{Y}(\Gamma;\Xi)$ in such a way that

$$\int\limits_{\Gamma\times\Xi} f(x,\xi)\,\mathrm{d}\nu(x,\xi) = \int\limits_{\Gamma}\int\limits_{\Xi} f(x,\xi)\,\mathrm{d}\nu^x(\xi)\,\mathrm{d}\mu(x) \quad \text{for every } f \in L^1_\nu(\Gamma\times\Xi)\,. \tag{1.13}$$

Moreover, the family $(\nu^x)_{x\in\Gamma}$ is unique up to μ-negligible sets, i.e., if $(\hat{\nu}^x)_{x\in\Gamma}$ is any other measurable family of probability functions satisfying (1.13), then $\hat{\nu}^x = \nu^x$ for μ-a.e. $x \in \Gamma$.

If $\nu = (\nu^x)_{x\in\Gamma} \in \mathcal{Y}(\Gamma;\Xi)$, for every $f \in \mathcal{C}_b(\Gamma\times\Xi)$ the duality between ν and f reads

$$\int\limits_{\Gamma\times\Xi} f(x,\xi)\,\mathrm{d}\nu(x,\xi) = \int\limits_{\Gamma}\int\limits_{\Xi} f(x,\xi)\,\mathrm{d}\nu^x(\xi)\,\mathrm{d}\mu(x) = \int\limits_{\Gamma}\langle f(x,\cdot),\nu^x\rangle\,\mathrm{d}\mu(x)\,.$$

Example 1.5.3. The simplest example of a Young measure is obtained by fixing a measurable function $v \colon \Gamma \to \Xi$ and by considering the Young measure *concentrated* on the graph of the function v, identified by the measurable family of Dirac deltas $\delta_v := (\delta_{v(x)})_{x\in\Gamma}$.

We will consider the space $\mathcal{Y}(\Gamma;\Xi)$ endowed with the *narrow* topology.

Definition 1.5.4. We say that ν_j converges *narrowly* to ν (and denote $\nu_j \rightharpoonup \nu$) if and only if

$$\int\limits_{\Gamma}\langle f(x,\cdot),\nu_j^x\rangle\,\mathrm{d}\mu(x) \to \int\limits_{\Gamma}\langle f(x,\cdot),\nu^x\rangle\,\mathrm{d}\mu(x)\,, \tag{1.14}$$

for every $f \in \mathcal{C}_b(\Gamma\times\Xi)$.

If $v_j \colon \Gamma \to \Xi$ is a sequence of measurable functions such that $\delta_{v_j} \rightharpoonup \nu$, we also say that the sequence v_j *generates* the Young measure ν.

We say that $f\colon \Gamma\times\Xi \to \mathbb{R}$ is a *Carathéodory integrand* if f is a measurable function such that $f(x,\cdot) \in \mathcal{C}_b(\Xi)$ for μ-a.e. $x \in \Gamma$ and such that $x \mapsto \|f(x,\cdot)\|_\infty$ belongs to $L^1_\mu(\Gamma)$.

Remark 1.5.5. If Ξ is a compact metric space, by [68, Theorem 2] the convergence in (1.14) also holds for every Carathéodory integrand f.

The narrow convergence for concentrated Young measures is characterised in the following proposition. For the proof, we refer to [68, Proposition 6].

Proposition 1.5.6. *Assume that Ξ is a compact metric space. Let $v_j, v\colon \Gamma \to \Xi$ be measurable functions. Then $\delta_{v_j} \rightharpoonup \delta_v$ if and only if $v_j \to v$ in measure.*

The following compactness result holds (cf. [68, Theorem 2]).

Theorem 1.5.7. *Assume that Ξ is a compact metric space. Then $\mathcal{Y}(\Gamma;\Xi)$, endowed with the narrow topology, is sequentially compact.*

Remark 1.5.8. The assumption on the compactness of the space Ξ is crucial to guarantee the compactness of $\mathcal{Y}(\Gamma;\Xi)$ with respect to the narrow convergence. For instance, if $\Xi = \mathbb{R}$, it may happen that a sequence $\nu_j \in \mathcal{Y}(\Gamma;\mathbb{R})$ has some mass escaping to infinity.

1.5.2 Young measures on the extended real line

In Chapter 4 we shall work with Young measures that are generated by functions taking values in the extended real line $[-\infty,\infty]$. In this subsection we fix some notation for this setting.

The set $[-\infty,\infty]$ is endowed with the metric induced by an increasing homeomorphism

$$\phi\colon [-\infty,\infty] \to [-1,1]\,, \tag{1.15}$$

e.g. $\phi(\xi) := \frac{2}{\pi}\arctan(\xi)$. Probability measures in $\mathcal{P}([-\infty,\infty])$ are in duality with bounded continuous functions $f \in \mathcal{C}_b([-\infty,\infty])$, i.e., continuous functions with a finite limit at $\pm\infty$.

We also recall that for every probability measure $\lambda \in \mathcal{P}([-\infty,\infty])$ we can define the *cumulative distribution function* $F_\lambda\colon [-\infty,\infty] \to [0,1]$ by

$$F_\lambda(\xi) := \lambda([-\infty,\xi]) \quad \text{for every } \xi \in [-\infty,\infty]\,. \tag{1.16}$$

By the right continuity of F_λ, it is possible to define its *pseudo-inverse* $F_\lambda^{[-1]}\colon [0,1] \to [-\infty,\infty]$ by

$$F_\lambda^{[-1]}(m) := \min\{\xi \in \mathbb{R} : F_\lambda(\xi) \geq m\}\,. \tag{1.17}$$

To deal with Young measures in $\mathcal{Y}(\Gamma;[-\infty,\infty])$, it is convenient to introduce the map

$$\Phi\colon \Gamma\times[-\infty,\infty] \to \Gamma\times[-1,1]\,, \quad \Phi(x,\xi) := (x,\phi(x))\,, \tag{1.18}$$

where ϕ is the homeomorphism defined in (1.15). Thus, for every $\nu \in \mathcal{Y}(\Gamma;[-\infty,\infty])$ we have $\Phi_\#\nu \in \mathcal{Y}(\Gamma;[-1,1])$. The elements of $\mathcal{Y}(\Gamma;[-\infty,\infty])$ are in duality with functions $f \in \mathcal{C}_b(\Gamma\times[-\infty,\infty])$, i.e., such that $f\circ\Phi^{-1} \in \mathcal{C}_b(\Gamma\times[-1,1])$.

Translation. We now recall how to shift Young measures. For every measurable function $\gamma\colon \Gamma \to \mathbb{R}$ we define the translation map $\mathcal{S}^\gamma\colon \Gamma\times[-\infty,\infty] \to \Gamma\times[-\infty,\infty]$ by $\mathcal{S}^\gamma(x,\xi) := (x, \xi+\gamma(x))$, with the usual convention that $a\pm\infty = \pm\infty$ for every $a\in\mathbb{R}$. For every $\nu\in\mathcal{Y}(\Gamma;[-\infty,\infty])$ we set

$$\nu\oplus\gamma := \mathcal{S}^\gamma_\#\nu \in \mathcal{Y}(\Gamma;[-\infty,\infty])\,, \tag{1.19}$$

$$\nu\ominus\gamma := \mathcal{S}^{(-\gamma)}_\#\nu \in \mathcal{Y}(\Gamma;[-\infty,\infty])\,. \tag{1.20}$$

Remark 1.5.9. Let $\nu_j,\nu\in\mathcal{Y}(\Gamma;[-\infty,\infty])$ be such that $\nu_j \rightharpoonup \nu$ and let $\gamma\colon\Gamma\to\mathbb{R}$ be a measurable function. By Remark 1.5.5 we have $\nu_j\oplus\gamma \rightharpoonup \nu\oplus\gamma$.

Moreover, if $\gamma,\gamma_j\colon\Gamma\to\mathbb{R}$ are such that $\gamma_j\to\gamma$ in measure, then it is easy to see that $\nu_j\oplus\gamma_j \rightharpoonup \nu\oplus\gamma$.

Truncation. We now introduce the notion of truncation of Young measures. This will be employed in Section 4.5. Given a Young measure $\nu\in\mathcal{Y}(\Gamma;[-\infty,\infty])$ and a measurable function $\theta\colon\Gamma\to[-\infty,\infty]$, we consider the map $\mathcal{T}^\theta\colon \Gamma\times[-\infty,\infty]\to\Gamma\times[-\infty,\infty]$ given by

$$\mathcal{T}^\theta(x,\xi) := (x, \xi\wedge\theta(x)) \tag{1.21}$$

and we say that $\mathcal{T}^\theta_\#\nu$ is the *truncation* of ν by θ.

Remark 1.5.10. In this case, the cumulative distribution functions of the measures $(\mathcal{T}^\theta_\#\nu)^x$ are given by

$$F_{(\mathcal{T}^\theta_\#\nu)^x}(\xi) = \begin{cases} F_{\nu^x}(\xi) & \text{if } \xi<\theta(x)\,,\\ 1 & \text{if } \xi\ge\theta(x)\,,\end{cases}$$

for μ-a.e. $x\in\Gamma$. Moreover, if $\nu_j\rightharpoonup\nu$ in $\mathcal{Y}(\Gamma;[-\infty,\infty])$, then by Remark 1.5.5 we have $\mathcal{T}^\theta_\#\nu_j \rightharpoonup \mathcal{T}^\theta_\#\nu$ in $\mathcal{Y}(\Gamma;[-\infty,\infty])$.

Partial order. Following [20, Definition 3.10], we introduce a partial order in the space of Young measures on $\Gamma\times\mathbb{R}$. We recall here the definition of this order and its main properties.

Definition 1.5.11. Let $\nu_1 = (\nu_1^x)_{x\in\Gamma}$, $\nu_2 = (\nu_2^x)_{x\in\Gamma} \in \mathcal{Y}(\Gamma;\mathbb{R})$. We say that $\nu_1\preceq\nu_2$ if one of the following equivalent conditions is satisfied:

(i) for every Carathéodory integrand $f\colon\Gamma\times\mathbb{R}\to\mathbb{R}$ nondecreasing with respect to the second variable we have

$$\int_\Gamma \langle f(x,\cdot),\nu_1^x\rangle\,\mathrm{d}\mu(x) \le \int_\Gamma \langle f(x,\cdot),\nu_2^x\rangle\,\mathrm{d}\mu(x)\,;$$

(ii) $F_{\nu_1^x}(\xi)\ge F_{\nu_2^x}(\xi)$ for μ-a.e. $x\in\Gamma$ and for every $\xi\in\mathbb{R}$.

Remark 1.5.12. If ν_1 and ν_2 are concentrated on some measurable functions γ_1 and γ_2, respectively, then

$$\nu_1\preceq\nu_2 \qquad \text{if and only if} \qquad \gamma_1(x)\le\gamma_2(x) \text{ for } \mu\text{-a.e. } x\in\Gamma\,.$$

The partial order $\preceq$ is naturally extended to Young measures $\mathcal{Y}(\Gamma;[-\infty,\infty])$ by employing the homeomorphism $\Phi\colon \Gamma\times[-\infty,\infty]\to\Gamma\times[-1,1]$ defined in (1.18). Namely, for every ν_1, $\nu_2\in\mathcal{Y}(\Gamma;[-\infty,\infty])$ we have $\nu_1\preceq\nu_2$ if and only if $\Phi_\#\nu_1\preceq\Phi_\#\nu_2$.

In the following we recall the definition of supremum of a family of Young measures. (See [20, Proposition 3.16] for the existence of such a Young measure.)

Definition 1.5.13. Let $(\nu_i)_{i\in I}$ be a family of Young measures in $\mathcal{Y}(\Gamma;[-\infty,\infty])$. We say that $\overline{\nu}\in\mathcal{Y}(\Gamma;[-\infty,\infty])$ is the *supremum* over $i\in I$ of the family $(\nu_i)_{i\in I}$, and we write

$$\overline{\nu}=\sup_{i\in I}\nu_i\,,$$

if the following two conditions hold:

(i) $\overline{\nu}\succeq\nu_i$ for every $i\in I$;

(ii) if $\nu\in\mathcal{Y}(\Gamma;[-\infty,\infty])$ such that $\nu\succeq\nu_i$ for every $i\in I$, then $\nu\succeq\overline{\nu}$.

Remark 1.5.14. In the case where ν_i are concentrated on measurable functions $v_i\colon\Gamma\to[-\infty,\infty]$, $i\in I$, we have

$$\sup_{i\in I}\delta_{v_i}=\delta_{\overline{v}}\,,$$

where $\overline{v}=\operatorname*{ess\,sup}_{i\in I}v_i$ (cf. [20, Remark 3.17]).

Remark 1.5.15. If a map $t\mapsto\nu(t)$ from $[0,T]$ to $\mathcal{Y}(\Gamma;[-\infty,\infty])$ is nondecreasing with respect to $\preceq$, then there exists a countable set $E\subset[0,T]$ such that $t\mapsto\nu(t)$ is continuous in $[0,T]\setminus E$. The proof of this fact is an easy consequence of [20, Lemma 3.19].

We conclude this section by recalling the Helly Selection Principle for Young measures [20, Theorem 3.20], a key tool for the proof of our result. Notice that [20, Theorem 3.20] is stated for Young measures with values in $\mathbb{R}$ instead of $[-\infty,\infty]$.

Theorem 1.5.16. *Let $t\mapsto\nu_k(t)$, $k\in\mathbb{N}$, be functions defined on $[0,T]$ with values in $\mathcal{Y}(\Gamma;[-\infty,\infty])$ that are nondecreasing with respect to $\preceq$. Then there exists a subsequence ν_{k_j}, independent of t, and a nondecreasing map $t\mapsto\nu(t)$ from $[0,T]$ to $\mathcal{Y}(\Gamma;[-\infty,\infty])$ such that $\nu_{k_j}(t)\rightharpoonup\nu(t)$, as $j\to\infty$, for every $t\in[0,T]$.*

Proof. The result follows from a straightforward application of [20, Theorem 3.20] to the sequence of nondecreasing maps $\Phi_\#\nu_k(t)\in\mathcal{Y}(\Gamma;[-1,1])$, where Φ is the homeomorphism Φ defined in (1.18). □

CHAPTER 2

Γ-LIMIT OF GRADIENT DAMAGE MODELS COUPLED WITH PLASTICITY

2.1 Overview of the chapter

In this chapter we study the Γ-limit of gradient damage models coupled with plasticity, in the case of antiplane shear. The results presented in this chapter have been published in the work [37], in collaboration with Dal Maso and Toader.

We briefly recall the functional setting for the model proposed in [3, 4]. Given a bounded open set $\Omega \subset \mathbb{R}^n$, the total energy that describes the system is given by

$$\mathcal{E}_\varepsilon(e,p,\alpha) := \frac{1}{2}\int_\Omega \alpha|e|^2\,\mathrm{d}x + \int_\Omega \sigma_{\mathrm{Y}}(\widetilde{\alpha})\,\mathrm{d}|p| + \int_\Omega \Big[\frac{W(\alpha)}{\varepsilon} + \varepsilon|\nabla\alpha|^2\Big]\,\mathrm{d}x\,, \tag{2.1}$$

for $e \in L^2(\Omega;\mathbb{R}^n)$, $p \in \mathcal{M}_b(\Omega;\mathbb{R}^n)$, $\alpha \in H^1(\Omega)$. In (2.1), the function $\widetilde{\alpha}$ is the quasicontinuous representative of α (cf. Subsection 1.2.1) and $|p|$ is the total variation of the measure p. We make the following constitutive assumptions:

- $e + p = Du$ for $u \in BV(\Omega)$;
- $\delta_\varepsilon \le \alpha \le 1$ $\mathcal{L}^n$-a.e. in Ω, where $\delta_\varepsilon/\varepsilon \to 0$ as $\varepsilon \to 0$;
- $\sigma_{\mathrm{Y}}\colon [0,1] \to \mathbb{R}$ is a continuous nondecreasing function with $0 \le \sigma_{\mathrm{Y}}(0) \le \sigma_{\mathrm{Y}}(1) < +\infty$, and $\sigma_{\mathrm{Y}}(\beta) > 0$ for $\beta > 0$;
- the function $W\colon [0,1] \to \mathbb{R}$ is continuous, strictly decreasing, and it satisfies $W(1) = 0$.

It is convenient to write the energy as

$$\mathcal{E}_\varepsilon(e,p,\alpha) := \mathcal{Q}(e,\alpha) + \mathcal{H}(p,\alpha) + \mathcal{W}_\varepsilon(\alpha)\,,$$

where

$$\mathcal{Q}(e,\alpha) := \frac{1}{2}\int_\Omega \alpha|e|^2\,\mathrm{d}x$$

is the elastic energy,

$$\mathcal{H}(p,\alpha) := \int_\Omega \sigma_{\mathrm{Y}}(\widetilde{\alpha})\,\mathrm{d}|p|$$

is the plastic potential, and

$$\mathcal{W}_\varepsilon(\alpha) := \int_\Omega \Big[\frac{W(\alpha)}{\varepsilon} + \varepsilon|\nabla\alpha|^2\Big]\,\mathrm{d}x \tag{2.2}$$

is the energy dissipated by the damage variable.

To describe the asymptotic behaviour of $\mathcal{E}_\varepsilon$ as $\varepsilon \to 0$, we define the functionals $\mathcal{F}_\varepsilon \colon BV(\Omega)\times H^1(\Omega) \to [0,+\infty]$ by

$$\mathcal{F}_\varepsilon(u,\alpha) := \min_{e,p}\{\mathcal{E}_\varepsilon(e,p,\alpha) : e \in L^2(\Omega;\mathbb{R}^n),\ p \in \mathcal{M}_b(\Omega;\mathbb{R}^n),\ Du = e+p\} \tag{2.3}$$

if $\delta_\varepsilon \le \alpha \le 1$ $\mathcal{L}^n$-a.e. in Ω, and $\mathcal{F}_\varepsilon(u,\alpha) = +\infty$ otherwise. Note (see Proposition 2.2.1) that the minimum in (2.3) is achieved at a unique pair (e,p), and that $\mathcal{F}_\varepsilon$ can be written explicitly in an integral form as

$$\mathcal{F}_\varepsilon(u,\alpha) = \int_\Omega f_\varepsilon(\alpha,|\nabla u|)\,\mathrm{d}x + \int_\Omega \sigma_{\mathrm{Y}}(\widetilde{\alpha})\,\mathrm{d}|D^s u| + \mathcal{W}_\varepsilon(\alpha)\,. \tag{2.4}$$

In order to define the integrand f_ε which appears in (2.4), we first introduce the function f (cf. Figure 2.1) defined for every $\beta \in (0,1]$ and $t \ge 0$ by

$$f(\beta,t) := \min_{0\le s\le t}\Big\{\frac{1}{2}\beta s^2 + \sigma_{\mathrm{Y}}(\beta)(t-s)\Big\} = \begin{cases} \dfrac{1}{2}\beta t^2 & \text{if } t \le \dfrac{\sigma_{\mathrm{Y}}(\beta)}{\beta}\,, \\[2ex] \sigma_{\mathrm{Y}}(\beta)t - \dfrac{\sigma_{\mathrm{Y}}(\beta)^2}{2\beta} & \text{if } t \ge \dfrac{\sigma_{\mathrm{Y}}(\beta)}{\beta}\,, \end{cases} \tag{2.5}$$

and then we set $f_\varepsilon(\beta,t) := f(\beta,t)$ if $\delta_\varepsilon \le \beta \le 1$ and $f_\varepsilon(\beta,t) := +\infty$ otherwise.

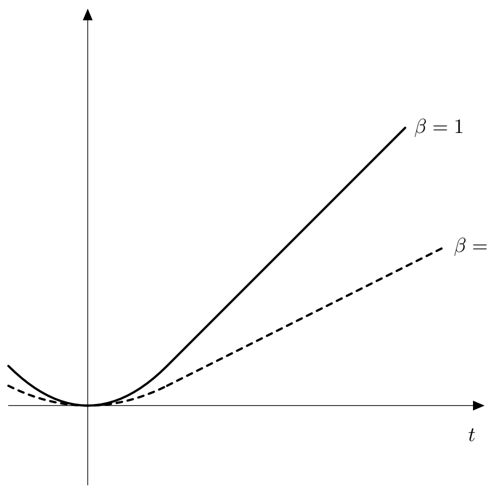

Figure 2.1: Profile of the functions $f(1,t)$ and $f(\frac{1}{2},t)$ for $\sigma_{\mathrm{Y}}(\beta) = \beta$.

The asymptotic behaviour of the functionals $\mathcal{F}_\varepsilon$ is obtained by studying their Γ-limit in the space $L^1(\Omega)\times L^1(\Omega)$ as $\varepsilon \to 0$. The choice of the topology is suggested by the compactness properties of sequences $(u_\varepsilon,\alpha_\varepsilon)$ with equibounded energies $\mathcal{F}_\varepsilon(u_\varepsilon,\alpha_\varepsilon)$ (see Theorem 2.5.2). Therefore the functionals $\mathcal{F}_\varepsilon$ defined in (2.3) are extended to $L^1(\Omega)\times L^1(\Omega)$ by setting $\mathcal{F}_\varepsilon(u,\alpha) := +\infty$ if $u \notin BV(\Omega)$ or $\alpha \notin H^1(\Omega)$.

In order to define the Γ-limit of the functionals $\mathcal{F}_\varepsilon$, we introduce the functional $\mathcal{F}$ defined by

$$\mathcal{F}(u) := \int_\Omega f(1, |\nabla u|)\,\mathrm{d}x + \sigma_{\mathrm{Y}}(1)|D^c u|(\Omega) + \int_{J_u} g(|[u]|)\,\mathrm{d}\mathcal{H}^{n-1}, \tag{2.6}$$

for $u \in GBV(\Omega)$, where for every $t \geq 0$

$$g(t) := \min\Big\{ \min_{0\leq\beta\leq 1} \big[\sigma_{\mathrm{Y}}(\beta)t + \kappa_W(\beta)\big], \kappa_W(0)\Big\}, \quad \text{with} \tag{2.7}$$

$$\kappa_W(\beta) := 4\int_\beta^1 \sqrt{W(s)}\,\mathrm{d}s, \quad \beta \in [0,1]\,.$$

Notice that for every $u \in GBV(\Omega)$, the functional $\mathcal{F}$ can be written as

$$\mathcal{F}(u) = \min_{e,p}\Big\{ \frac{1}{2}\int_\Omega |e|^2\,\mathrm{d}x + \sigma_{\mathrm{Y}}(1)|p|(\Omega\setminus J_u) + \int_{J_u} g(|[u]|)\,\mathrm{d}\mathcal{H}^{n-1}\Big\}, \tag{2.8}$$

where the minimum is taken among all $e \in L^2(\Omega;\mathbb{R}^n)$, $p \in \mathcal{M}_b(\Omega;\mathbb{R}^n)$ such that $\nabla u\mathcal{L}^n + D^c u = e + p$ as measures on $\Omega \setminus J_u$ (see Proposition 2.2.2).

The function g in (2.7) satisfies the following properties (see Figure 2.2):

- g is concave, nondecreasing, and $g(t) > 0$ for $t > 0$;
- $g(1)\min\{t,1\} \leq g(t) \leq \min\{\sigma_{\mathrm{Y}}(1)t, \kappa_W(0)\}$;
- $g'(0) = \sigma_{\mathrm{Y}}(1)$;
- $g(t) = \kappa_W(0)$ if $\sigma_{\mathrm{Y}}(0)t \geq \kappa_W(0)$.

Since the force between the crack lips is given by the derivative of g, the above properties show that this force is always present when the crack opening is small and vanishes when the crack opening is large enough, provided $\sigma_{\mathrm{Y}}(0) > 0$. We refer to Subsection 2.2.3 for a detailed description of the behaviour of g when $\sigma_{\mathrm{Y}}(0) = 0$.

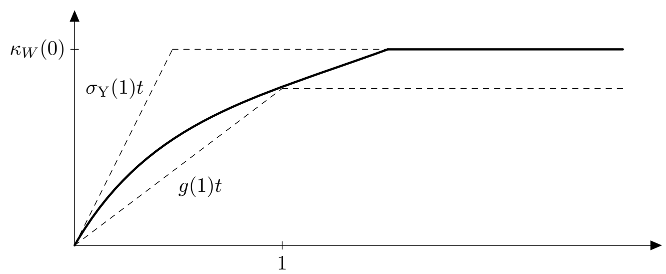

Figure 2.2: Graph of the crack energy density $g(t)$.

To state the main result of this chapter, we define $\mathcal{F}_0\colon L^1(\Omega)\times L^1(\Omega) \to [0,+\infty]$ by

$$\mathcal{F}_0(u,\alpha) = \begin{cases} \mathcal{F}(u) & \text{if } u \in GBV(\Omega) \text{ and } \alpha = 1\ \mathcal{L}^n\text{-a.e. in } \Omega\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

Theorem 2.1.1. *The functionals $\mathcal{F}_\varepsilon$ Γ-converge to $\mathcal{F}_0$ in $L^1(\Omega)\times L^1(\Omega)$ as $\varepsilon \to 0$.*

The proof of Theorem 2.1.1 is presented in Section 2.3 in the case $n = 1$, where we can give a more precise description of the behaviour of the sequence of functions α_ε in a neighbourhood of each point of the domain Ω. The extension to the antiplane case with $n > 1$ is obtained in Section 2.4 by a slicing argument. Unfortunately this approach is not enough to deal with the full three-dimensional model introduced in [4], because in that case $\mathcal{H}(p, \alpha)$ is $+\infty$ whenever the matrix-valued plastic strain p is not trace-free.

In Section 2.5 we apply 2.1.1 to study minimisers of the functionals $\mathcal{F}_\varepsilon$. To impose Dirichlet boundary conditions, we assume in addition that Ω has a Lipschitz boundary and we fix a relatively open subset $\partial_D\Omega$ of $\partial\Omega$, where we prescribe the displacement.

We would like to analyse the asymptotic behaviour of solutions to the minimum problems

$$\min\left\{\mathcal{F}_\varepsilon(u,\alpha) : u \in BV(\Omega),\ \alpha \in H^1(\Omega),\ u = w,\ \alpha = 1\ \ \mathcal{H}^{n-1}\text{-a.e. on } \partial_D\Omega\right\}, \quad (2.9)$$

where $w \in L^\infty(\partial_D\Omega)$. Unfortunately these problems, in general, have no solutions. As for many other variational problems with linear growth in Du, the difficulty is given by the attainment of the boundary condition $u = w$ $\mathcal{H}^{n-1}$-a.e. on $\partial_D\Omega$.

However, for every $\eta > 0$, it is always possible to consider an η-minimiser of (2.9), defined as a pair $(u_\varepsilon, \alpha_\varepsilon) \in BV(\Omega)\times H^1(\Omega)$, with $u_\varepsilon = w$ and $\alpha_\varepsilon = 1$ $\mathcal{H}^{n-1}$-a.e. on $\partial_D\Omega$, such that

$$\mathcal{F}_\varepsilon(u_\varepsilon, \alpha_\varepsilon) < \mathcal{I}_\varepsilon + \eta\,,$$

where $\mathcal{I}_\varepsilon$ is the infimum in (2.9).

Since the functional $\mathcal{F}_\varepsilon(\cdot, \alpha)$ decreases by truncation, for every $w \in L^\infty(\partial_D\Omega)$ and for every $\eta > 0$ the minimum problem (2.9) always has an η-minimiser $(u_\varepsilon, \alpha_\varepsilon)$ satisfying

$$||u_\varepsilon||_{L^\infty(\Omega)} \le ||w||_{L^\infty(\partial_D\Omega)}\,. \quad (2.10)$$

In Section 2.5, we obtain the following result.

Theorem 2.1.2. *Let $w \in L^\infty(\partial_D\Omega)$ and let $\eta_\varepsilon \searrow 0$. For every $\varepsilon > 0$, let $(u_\varepsilon, \alpha_\varepsilon) \in BV(\Omega)\times H^1(\Omega)$ be a η_ε-minimiser of problem (2.9) satisfying (2.10). Then $\alpha_\varepsilon \to 1$ in $L^1(\Omega)$ and a subsequence of u_ε converges in $L^1(\Omega)$ to a minimiser $u \in BV(\Omega)$ of the problem*

$$\min\left\{\mathcal{F}(u) + \int_{\partial_D\Omega} g(|u-w|)\, d\mathcal{H}^{n-1} : u \in BV(\Omega)\right\}. \quad (2.11)$$

Note that in the limit problem the boundary condition $u = w$ $\mathcal{H}^{n-1}$-a.e. on $\partial_D\Omega$ is relaxed. Indeed, it is replaced by the term $\int_{\partial_D\Omega} g(|u-w|)\, d\mathcal{H}^{n-1}$, which penalises the non attainment of the prescribed boundary value. This is a typical feature of functionals with linear growth in the gradient. The set $\{x \in \partial_D\Omega : u(x) \neq w(x)\}$ can be interpreted as a crack on the Dirichlet part of the boundary of Ω and the integral on $\partial_D\Omega$ in (2.11) is the corresponding dissipated energy.

The chapter is organized as follows. In Section 2.2 we list some useful properties of the function f and we describe in detail the density g of the crack energy of the limit problem. Section 2.3 is devoted to the proof of the main theorem in the one-dimensional case. The general case is studied in Section 2.4, where the Γ-liminf inequality is proved by a slicing argument, whereas the Γ-limsup inequality is obtained by using an integral representation result. Finally, in Section 2.5 we establish the convergence of minimisers of some model problems.

Since in Theorems 2.1.1, 2.1.2 it is enough to prove the result along every sequence $\varepsilon_k \to 0$, we fix once and for all a sequence $\varepsilon_k \to 0$ and we use the shorthand notation $\delta_k := \delta_{\varepsilon_k}$, $f_k := f_{\varepsilon_k}$, $\mathcal{F}_k := \mathcal{F}_{\varepsilon_k}$, $\mathcal{W}_k := \mathcal{W}_{\varepsilon_k}$, and $\mathcal{E}_k := \mathcal{E}_{\varepsilon_k}$.

2.2 Properties of the energies

2.2.1 The energy of the optimal decomposition

We provide here an explicit expression for the minimum value in (2.3).

Proposition 2.2.1. *Let $\mathcal{F}_k$ be the functional defined in* (2.3)*. Then for every $u \in BV(\Omega)$ and for every $\alpha \in H^1(\Omega)$, with $\delta_k \le \alpha \le 1$, there exists a unique pair (e,p) with $e \in L^2(\Omega;\mathbb{R}^n)$ and $p \in \mathcal{M}_b(\Omega;\mathbb{R}^n)$ such that $Du = e + p$ and*

$$\mathcal{F}_k(u,\alpha) = \mathcal{E}_k(e,p,\alpha)\,.$$

Moreover

$$\mathcal{F}_k(u,\alpha) = \int_\Omega f(\alpha,|\nabla u|)\,\mathrm{d}x + \int_\Omega \sigma_{\mathrm{Y}}(\widetilde{\alpha})\,\mathrm{d}|D^s u| + \mathcal{W}_k(\alpha)\,,$$

where f is the function defined in (2.5).

Proof. The proof of the existence of a minimizing pair (e,p) is straightforward, and the uniqueness follows from the strict convexity of the L^2 norm.

Let us prove the integral formula for $\mathcal{F}_k$. The inequality

$$\mathcal{F}_k(u,\alpha) \ge \int_\Omega f(\alpha,|\nabla u|)\,\mathrm{d}x + \int_\Omega \sigma_{\mathrm{Y}}(\widetilde{\alpha})\,\mathrm{d}|D^s u| + \mathcal{W}_k(\alpha)$$

is trivial. To prove the opposite inequality, we fix $u \in BV(\Omega)$, $\alpha \in H^1(\Omega)$ with $\delta_k \le \alpha \le 1$, and we define

$$e(x) := \begin{cases} \nabla u(x) & \text{if } |\nabla u(x)| \le \dfrac{\sigma_{\mathrm{Y}}(\alpha(x))}{\alpha(x)}\,, \\ \dfrac{\sigma_{\mathrm{Y}}(\alpha(x))}{\alpha(x)} \dfrac{\nabla u(x)}{|\nabla u(x)|} & \text{if } |\nabla u(x)| \ge \dfrac{\sigma_{\mathrm{Y}}(\alpha(x))}{\alpha(x)}\,, \end{cases}$$

so that $e \in L^2(\Omega;\mathbb{R}^n)$ and

$$\frac{1}{2}\alpha(x)|e(x)|^2 + \sigma_{\mathrm{Y}}(\alpha(x))|\nabla u(x) - e(x)| = f(\alpha(x),|\nabla u(x)|)$$

for a.e. $x \in \Omega$. Let $p := Du - e \in \mathcal{M}_b(\Omega;\mathbb{R}^n)$, whose Lebesgue decomposition is

$$p = (\nabla u - e)\mathcal{L}^n + D^s u\,.$$

We have

$$\begin{aligned}\mathcal{F}_k(u,\alpha) &\leq \frac{1}{2}\int_\Omega \alpha|e|^2\,\mathrm{d}x + \int_\Omega \sigma_{\mathrm{Y}}(\widetilde{\alpha})\,\mathrm{d}|p| + \mathcal{W}_k(\alpha)\\ &= \frac{1}{2}\int_\Omega \alpha|e|^2 + \sigma_{\mathrm{Y}}(\alpha)|\nabla u - e|\,\mathrm{d}x + \int_\Omega \sigma_{\mathrm{Y}}(\widetilde{\alpha})\,\mathrm{d}|D^s u| + \mathcal{W}_k(\alpha)\\ &= \int_\Omega f(\alpha,|\nabla u|)\,\mathrm{d}x + \int_\Omega \sigma_{\mathrm{Y}}(\widetilde{\alpha})\,\mathrm{d}|D^s u| + \mathcal{W}_k(\alpha)\,.\end{aligned}$$

This concludes the proof. □

The same argument can be used to prove the following characterization of the functional $\mathcal{F}$.

Proposition 2.2.2. *Let $\mathcal{F}$ be the functional defined in* (2.6). *Then for every $u \in GBV(\Omega)$ with $\mathcal{F}(u) < +\infty$ we have*

$$\mathcal{F}(u) = \min_{e,p}\left\{\frac{1}{2}\int_\Omega |e|^2\,\mathrm{d}x + \sigma_{\mathrm{Y}}(1)|p|(\Omega\setminus J_u) + \int_{J_u} g(|[u]|)\,\mathrm{d}\mathcal{H}^{n-1}\right\}, \tag{2.12}$$

where the minimum in (2.12) *is taken among all $e \in L^2(\Omega;\mathbb{R}^n)$, $p \in \mathcal{M}_b(\Omega;\mathbb{R}^n)$ such that $\nabla u\mathcal{L}^n + D^c u = e + p$ as measures on $\Omega\setminus J_u$. Moreover, the minimum is attained at a unique pair (e,p).*

We conclude with some remarks on the function f used in Proposition 2.2.1. From the very definition of f (see (2.5)) it follows that $f(\beta,t)$ is increasing with respect to β and convex with respect to t. Moreover, from the explicit formula it is immediate to deduce that there exists a constant $C > 0$ such that

$$\frac{1}{C}t - C \leq f(1,t) \leq Ct \tag{2.13}$$

for all $t \geq 0$. Finally, we notice that

$$f(\beta,\lambda t) \leq \lambda^2 f(\beta,t) \tag{2.14}$$

for every $\lambda \geq 1$, $\beta \in (0,1]$, and $t \geq 0$.

2.2.2 Semicontinuity of the energies

In the next result, for every k we discuss the semicontinuity properties of the functional $\mathcal{F}_k$ introduced in (2.3).

Proposition 2.2.3. *Let $u_j, u \in BV(\Omega)$ and $\alpha_j, \alpha \in H^1(\Omega)$, $\delta_k \leq \alpha_j \leq 1$ be such that*

$$\begin{aligned} u_j &\to u \quad \text{strongly in } L^1(\Omega)\,,\\ \alpha_j &\rightharpoonup \alpha \quad \text{weakly in } H^1(\Omega)\,,\end{aligned}$$

as $j \to +\infty$. Then

$$\mathcal{F}_k(u,\alpha) \leq \liminf_{j\to+\infty} \mathcal{F}_k(u_j,\alpha_j)\,. \tag{2.15}$$

Proof. In a first instance, let us prove the theorem in the case $||u_j||_{L^\infty(\Omega)} \le M$. Moreover, let us assume that σ_{Y} is a Lipschitz function.

We may assume that the liminf in (2.15) is finite and, up to extracting a subsequence, that $\mathcal{F}_k(u_j, \alpha_j)$ is equibounded with respect to j. Let us fix $e_j \in L^2(\Omega;\mathbb{R}^n)$ and $p_j \in \mathcal{M}_b(\Omega;\mathbb{R}^n)$ such that $Du_j = e_j + p_j$ and $\mathcal{F}_k(u_j,\alpha_j) = \mathcal{E}_k(e_j,p_j,\alpha_j)$ (see Proposition 2.2.1). Since e_j is bounded in $L^2(\Omega;\mathbb{R}^n)$ and p_j is bounded in $\mathcal{M}_b(\Omega;\mathbb{R}^n)$, we have that

$$\begin{aligned} e_j &\rightharpoonup e \quad \text{weakly in } L^2(\Omega;\mathbb{R}^n)\,, \\ p_j &\overset{*}{\rightharpoonup} p \quad \text{weakly* in } \mathcal{M}_b(\Omega;\mathbb{R}^n)\,, \end{aligned}$$

up to a subsequence. This implies that $Du = e + p$. It is not restrictive to assume that $\alpha_j \to \alpha$ $\mathcal{L}^n$-a.e. in Ω. Then, since the sequence α_j is uniformly bounded in $L^\infty(\Omega)$, we have that

$$\sqrt{\alpha_j} e_j \rightharpoonup \sqrt{\alpha} e \quad \text{weakly in } L^2(\Omega;\mathbb{R}^n)$$

as $j \to +\infty$. This implies

$$\int_\Omega \alpha |e|^2 \,\mathrm{d}x \le \liminf_{j\to+\infty} \int_\Omega \alpha_j |e_j|^2 \,\mathrm{d}x\,.$$

Thus, to conclude the proof of (2.15), it suffices to show that

$$\int_\Omega \sigma_{\mathrm{Y}}(\widetilde{\alpha}) \,\mathrm{d}|p| \le \liminf_{j\to+\infty} \int_\Omega \sigma_{\mathrm{Y}}(\widetilde{\alpha}_j) \,\mathrm{d}|p_j|\,,$$

since the other terms of the functional can be treated in a simple way. In order to prove this inequality, we just need to show that

$$\sigma_{\mathrm{Y}}(\widetilde{\alpha}_j) p_j \overset{*}{\rightharpoonup} \sigma_{\mathrm{Y}}(\alpha) p \quad \text{weakly* in } \mathcal{M}_b(\Omega;\mathbb{R}^n)\,. \tag{2.16}$$

Let us start by noticing that $\sigma_{\mathrm{Y}}(\alpha_j)u_j, \sigma_{\mathrm{Y}}(\alpha)u \in BV(\Omega)$ and

$$\begin{aligned} D(\sigma_{\mathrm{Y}}(\alpha_j)u_j) &= \nabla(\sigma_{\mathrm{Y}}(\alpha_j))u_j + \sigma_{\mathrm{Y}}(\widetilde{\alpha}_j) Du_j\,, \\ D(\sigma_{\mathrm{Y}}(\alpha)u) &= \nabla(\sigma_{\mathrm{Y}}(\alpha))u + \sigma_{\mathrm{Y}}(\widetilde{\alpha}) Du\,. \end{aligned}$$

Indeed, since u_j is bounded in L^∞ and σ_{Y} is a Lipschitz function, the formulas above are true if α_j and α are $\mathcal{C}^1$ functions. Then they can be extended to the case $\alpha_j, \alpha \in H^1(\Omega)$ by an approximation argument, based on the fact that strong convergence in $H^1(\Omega)$ implies Cap-q.e. pointwise convergence (for a subsequence) of the quasicontinuous representatives, which implies Du_j-a.e. and Du-a.e. convergence, respectively.

The measures $\sigma_{\mathrm{Y}}(\widetilde{\alpha}_j) Du_j$ are uniformly bounded in $\mathcal{M}_b(\Omega;\mathbb{R}^n)$ and $\nabla(\sigma_{\mathrm{Y}}(\alpha_j)) \rightharpoonup \nabla(\sigma_{\mathrm{Y}}(\alpha))$ weakly in $L^2(\Omega;\mathbb{R}^n)$, which implies that $\nabla(\sigma_{\mathrm{Y}}(\alpha_j))u_j \sigma_{\mathrm{Y}} \nabla(\sigma_{\mathrm{Y}}(\alpha))u$, weakly in $L^2(\Omega;\mathbb{R}^n)$. Hence the measures $D(\sigma_{\mathrm{Y}}(\alpha_j)u_j)$ are uniformly bounded in $\mathcal{M}_b(\Omega;\mathbb{R}^n)$.

Since $\sigma_{\mathrm{Y}}(\alpha_j)u_j \to \sigma_{\mathrm{Y}}(\alpha)u$ in $L^1(\Omega)$, we have that $D(\sigma_{\mathrm{Y}}(\alpha_j)u_j) \overset{*}{\rightharpoonup} D(\sigma_{\mathrm{Y}}(\alpha)u)$ weakly* in $\mathcal{M}_b(\Omega;\mathbb{R}^n)$, and therefore, by difference, $\sigma_{\mathrm{Y}}(\widetilde{\alpha}_j)Du_j \overset{*}{\rightharpoonup} \sigma_{\mathrm{Y}}(\widetilde{\alpha})Du$ weakly* in $\mathcal{M}_b(\Omega;\mathbb{R}^n)$. We conclude that (2.16) holds, taking into account that $\sigma_{\mathrm{Y}}(\alpha_j)e_j \rightharpoonup \sigma_{\mathrm{Y}}(\alpha)e$ weakly in $L^2(\Omega;\mathbb{R}^n)$.

To remove the assumption that $\sigma_{\rm Y}$ is Lipschitz, we approximate $\sigma_{\rm Y}$ from below with Lipschitz functions $\sigma_{\rm Y}^h \nearrow \sigma_{\rm Y}$. By applying the previous step, we deduce that

$$\int_\Omega \sigma_{\rm Y}^h(\widetilde{\alpha})\,{\rm d}|p| \le \liminf_{j\to+\infty} \int_\Omega \sigma_{\rm Y}^h(\widetilde{\alpha}_j)\,{\rm d}|p_j| \le \liminf_{j\to+\infty} \int_\Omega \sigma_{\rm Y}(\widetilde{\alpha}_j)\,{\rm d}|p_j|$$

and then we pass to the limit in h. This concludes the proof of (2.15) when u_j is bounded in $L^\infty(\Omega)$.

The extension to the unbounded case is obtained by a truncation argument. □

2.2.3 The density of the crack energy

In this subsection we study the main qualitative properties of the function g defined in (2.7). It is convenient to introduce the function $\gamma\colon [0,+\infty) \to \mathbb{R}$ defined by

$$\gamma(t) := \min_{0\le\beta\le1} \left[\sigma_{\rm Y}(\beta)t + \kappa_W(\beta)\right], \tag{2.17}$$

so that

$$g(t) = \min\{\gamma(t), \kappa_W(0)\}\,. \tag{2.18}$$

Since $\sigma_{\rm Y}(\beta) > 0$ for $\beta > 0$ and $\kappa_W(0) > 0$, we have $\gamma(t) \ge g(t) > 0$ for every $t > 0$. Since γ and g are obtained as minimum of nondecreasing affine functions, they are concave and nondecreasing. Therefore the inequality $g(t) > 0$ implies that

$$g(1)\min\{t,1\} \le g(t)\,. \tag{2.19}$$

For every $\beta \in [0,1]$ we have $\sigma_{\rm Y}(0)t \le \sigma_{\rm Y}(\beta)t + \kappa_W(\beta)$, hence $\sigma_{\rm Y}(0)t \le \gamma(t)$. Moreover, the equality $\kappa_W(1) = 0$ implies that $\gamma(t) \le \sigma_{\rm Y}(1)t$. Therefore we have

$$\sigma_{\rm Y}(0)t \le \gamma(t) \le \sigma_{\rm Y}(1)t \quad \text{for every } t \ge 0\,, \tag{2.20}$$

which gives

$$\min\{\sigma_{\rm Y}(0)t, \kappa_W(0)\} \le g(t) \le \min\{\sigma_{\rm Y}(1)t, \kappa_W(0)\} \quad \text{for every } t \ge 0\,. \tag{2.21}$$

In particular, if $\sigma_{\rm Y}(0) > 0$, then

$$g(t) = \kappa_W(0) \quad \text{for} \quad t \ge \frac{\kappa_W(0)}{\sigma_{\rm Y}(0)}\,. \tag{2.22}$$

When $\sigma_{\rm Y}(0) = 0$, we always have

$$\gamma(t) \le \sigma_{\rm Y}(0)t + \kappa_W(0) = \kappa_W(0)\,,$$

so that, in this case,

$$g(t) = \gamma(t) \quad \text{for every } t \ge 0\,. \tag{2.23}$$

In the following proposition we show that, in any case, the function g approaches the value $\kappa_W(0)$ at infinity.

Proposition 2.2.4. *We have that*

$$\lim_{t\to+\infty} g(t) = \kappa_W(0)\,. \tag{2.24}$$

Proof. Since g is nondecreasing, it suffices to prove the proposition when $g(t) < \kappa_W(0)$ for every $t \geq 0$. In this case

$$g(t) = \gamma(t) = \sigma_{\mathrm{Y}}(\beta_t)t + \kappa_W(\beta_t)$$

for some $\beta_t \in (0,1]$. Let us prove that $\beta_t \to 0$ as $t \to +\infty$. If $\limsup_t \beta_t =: \ell > 0$, then there would exist a sequence $t_j \to +\infty$ such that $\beta_{t_j} \geq \ell/2$, in turn implying that

$$\sigma_{\mathrm{Y}}(\ell/2)t_j \leq \sigma_{\mathrm{Y}}(\beta_{t_j})t_j + \kappa_W(\beta_{t_j}) = g(t_j) < \kappa_W(0)\,.$$

This would lead to a contradiction as $j \to +\infty$, and therefore $\beta_t \to 0$ as $t \to +\infty$. Since

$$\kappa_W(\beta_t) \leq \sigma_{\mathrm{Y}}(\beta_t)t + \kappa_W(\beta_t) \leq \sup_{s\geq 0} g(s) \leq \lim_{s\to+\infty} g(s)\,,$$

by letting $t \to +\infty$ we obtain $\kappa_W(0) \leq \lim_{s\to+\infty} g(s)$. □

In general, when $\sigma_{\mathrm{Y}}(0) = 0$, it may happen that $g(t) < \kappa_W(0)$ for every $t \geq 0$, as the following example shows.

Example 2.2.5. Let us consider the functions

$$\sigma_{\mathrm{Y}}(\beta) := \beta^2 \quad \text{and} \quad W(\beta) := \frac{(1-\beta)^2}{4}\,.$$

In this way $\kappa_W(\beta) = (1-\beta)^2$. Then it is immediate to see that

$$g(t) = \min_{0\leq\beta\leq 1} \big[(1+t)\beta^2 - 2\beta + 1\big] = \frac{t}{1+t} < 1 = \kappa_W(0)\,.$$

Nevertheless, if $\sigma_{\mathrm{Y}}(\beta)$ tends to zero slowly enough as $\beta \to 0$, we still have $g(t) = \kappa_W(0)$ for some $t > 0$, as shown in the following proposition.

Proposition 2.2.6. *Assume that*

$$\liminf_{\beta\to 0^+} \frac{\sigma_{\mathrm{Y}}(\beta)}{\beta} > 0\,. \tag{2.25}$$

Then there exists t_0 such that $g(t) = \kappa_W(0)$ for $t \geq t_0$.

Proof. Suppose, by contradiction, that for every $j \in \mathbb{N}$ there exists $\beta_j \in (0,1]$ such that

$$\sigma_{\mathrm{Y}}(\beta_j)j + \kappa_W(\beta_j) < \kappa_W(0)\,. \tag{2.26}$$

Arguing as in the proof of Proposition 2.2.4, we get that $\beta_j \to 0$ as $j \to +\infty$.

From (2.26) it follows that

$$j \leq \frac{\kappa_W(0) - \kappa_W(\beta_j)}{\sigma_{\mathrm{Y}}(\beta_j)}\,,$$

which implies, by (2.25), that

$$\begin{aligned}
+\infty &= \limsup_{j\to+\infty} \frac{\kappa_W(0) - \kappa_W(\beta_j)}{\sigma_{\mathrm{Y}}(\beta_j)} \leq \limsup_{\beta\to 0^+} \frac{\kappa_W(0) - \kappa_W(\beta)}{\beta}\,\frac{\beta}{\sigma_{\mathrm{Y}}(\beta)} \\
&= \kappa_W'(0) \limsup_{\beta\to 0^+} \frac{\beta}{\sigma_{\mathrm{Y}}(\beta)} = 4\sqrt{W(0)}\,\limsup_{\beta\to 0^+} \frac{\beta}{\sigma_{\mathrm{Y}}(\beta)} < +\infty\,.
\end{aligned}$$

This contradiction concludes the proof of the proposition. □

We now investigate the regularity properties of γ and g. Since these functions are concave they admit left and right derivatives at every every point. The following proposition provides the connection between σ_{Y} and the derivatives of γ. For every function $\psi(t)$, the left and right derivatives are denoted by $\psi'_-(t)$ and $\psi'_+(t)$, respectively.

Proposition 2.2.7. *Let $t \in [0, +\infty)$ and let $\beta_t^{\min}, \beta_t^{\max} \in [0,1]$ be the smallest and the greatest solution of the minimum problem* (2.17) *which defines $\gamma(t)$. Then*

$$\gamma'_+(t) = \sigma_{\mathrm{Y}}(\beta_t^{\min}) \quad \text{for } t \geq 0 \quad \text{and} \quad \gamma'_-(t) = \sigma_{\mathrm{Y}}(\beta_t^{\max}) \quad \text{for } t > 0\,. \tag{2.27}$$

If $\beta_t^{\min} = \beta_t^{\max}$, then γ is differentiable at t and $\gamma'(t) = \sigma_{\mathrm{Y}}(\beta_t^{\min})$. If $\beta_t^{\min} < \beta_t^{\max}$, then γ is not differentiable at t.

Proof. Let us fix $t > 0$ and let β_t be such that $\gamma(t) = \sigma_{\mathrm{Y}}(\beta_t)t + \kappa_W(\beta_t)$. First of all we prove that

$$\gamma'_+(t) \leq \sigma_{\mathrm{Y}}(\beta_t) \leq \gamma'_-(t)\,. \tag{2.28}$$

Indeed, by the definition of γ, for every $s \geq 0$ we have $\gamma(s) \leq \sigma_{\mathrm{Y}}(\beta_t)s + \kappa_W(\beta_t)$. By the choice of β_t, this implies $\gamma(s) \leq \sigma_{\mathrm{Y}}(\beta_t)(s-t) + \gamma(t)$, which leads immediately to (2.28).

To prove the first equality in (2.27), let us now fix $t \geq 0$. Since γ is concave, there exists a decreasing sequence $t_j \to t$ such that γ is differentiable at every t_j. Let $\beta_j \in [0,1]$ be such that $\gamma(t_j) = \sigma_{\mathrm{Y}}(\beta_j)t_j + \kappa_W(\beta_j)$. A subsequence of β_j converges to some β^*. Passing to the limit in the previous equality, by the continuity of σ_{Y}, γ, and κ_W we get $\gamma(t) = \sigma_{\mathrm{Y}}(\beta^*)t + \kappa_W(\beta^*)$, which implies $\beta_t^{\min} \leq \beta^*$, and hence $\sigma_{\mathrm{Y}}(\beta_t^{\min}) \leq \sigma_{\mathrm{Y}}(\beta^*)$. As $\gamma'_-(t_j) = \gamma'_+(t_j)$, by (2.28) we have that $\gamma'(t_j) = \sigma_{\mathrm{Y}}(\beta_j) \to \sigma_{\mathrm{Y}}(\beta^*)$. Using the monotonicity of the difference quotients of γ, it is easy to prove that $\gamma'(t_j) \to \gamma'_+(t)$ as $j \to +\infty$. This implies that $\gamma'_+(t) = \sigma_{\mathrm{Y}}(\beta^*)$. Therefore, the inequality $\sigma_{\mathrm{Y}}(\beta_t^{\min}) \leq \sigma_{\mathrm{Y}}(\beta^*)$ together with (2.28) gives $\gamma'_+(t) = \sigma_{\mathrm{Y}}(\beta_t^{\min})$, which concludes the proof of the first part of (2.27). The proof of the second part is analogous.

The statement about the differentiability of γ is an obvious consequence of (2.27). As for the last statement, if $\beta_t^{\min} < \beta_t^{\max}$ we have $\sigma_{\mathrm{Y}}(\beta_t^{\min})t + \kappa_W(\beta_t^{\min}) = \sigma_{\mathrm{Y}}(\beta_t^{\max})t + \kappa_W(\beta_t^{\max})$. Since κ_W is injective, we have also $\kappa_W(\beta_t^{\min}) \neq \kappa_W(\beta_t^{\max})$, which excludes the case $t = 0$ and implies $\sigma_{\mathrm{Y}}(\beta_t^{\min}) \neq \sigma_{\mathrm{Y}}(\beta_t^{\max})$. Then (2.27) gives $\gamma'_+(t) < \gamma'_-(t)$, hence γ is not differentiable at t. □

Remark 2.2.8. If $\sigma_{\mathrm{Y}}(0) > 0$, by (2.22) there exists $t_0 > 0$ such that $g(t) = \gamma(t) < \kappa_W(0)$ for $0 \leq t < t_0$ and $\gamma(t) = \kappa_W(0)$ for $t \geq t_0$. It is clear that $g'_-(t_0) = \gamma'_-(t_0)$ and $g'_+(t_0) = 0$. By Proposition 2.2.7 we have that $\gamma'_-(t_0) = \sigma_{\mathrm{Y}}(\beta_{t_0}^{\max}) > 0$, so that $g'_-(t_0) > g'_+(t_0)$.

We also provide an example in which $\sigma_{\mathrm{Y}}(0) = 0$ and g, equal to γ by (2.23), is not everywhere differentiable.

Example 2.2.9. Let us define

$$\sigma_{\mathrm{Y}}(\beta) := \sqrt{\beta} \quad \text{and} \quad W(\beta) := \frac{(1-\beta)^2}{4},$$

so that $\kappa_W(\beta) = (1-\beta)^2$. For $0 < t < \frac{8}{3\sqrt{3}}$, the function $\Theta_t(\beta) := \sigma_{\mathrm{Y}}(\beta)t + \kappa_W(\beta)$ has exactly two local minimum points in $[0,1]$: the first one is 0, whereas the second one is a point $\alpha_t \in (0,1)$. For $0 < t < \frac{4\sqrt{2}}{3\sqrt{3}}$, the global minimum of Θ_t is attained only at α_t; for $t > \frac{4\sqrt{2}}{3\sqrt{3}}$, the global minimum of Θ_t is attained only at 0 (see Figure 2.3). For $t_0 = \frac{4\sqrt{2}}{3\sqrt{3}}$, there are two different global minimum points: 0 and α_{t_0}. By the last statement of Proposition 2.2.7, γ is not differentiable at t_0. The previous analysis shows that $\gamma(t) = \Theta_t(\alpha_t) < \Theta_t(0) = \kappa_W(0)$ for $t < t_0$, while $\gamma(t) = \Theta_t(0) = \kappa_W(0)$ for $t \geq t_0$. So, in this example, the function γ is not differentiable at the first point where it attains the constant value $\kappa_W(0)$.

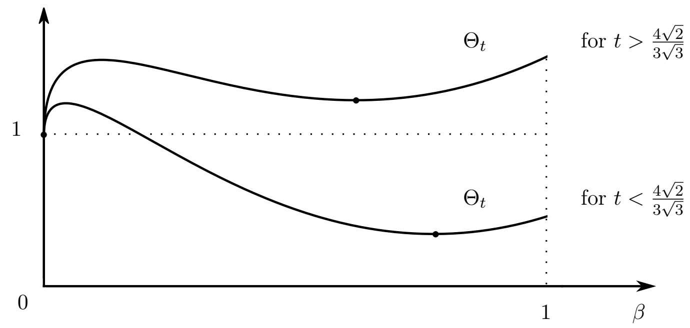

Figure 2.3: Graph of $\Theta_t(\beta)$ for different values of t.

Remark 2.2.10. If for every $t \geq 0$ the minimum problem (2.17) in the definition of γ has a unique solution, then γ is differentiable everywhere. Since it is concave, we conclude that it is of class $\mathcal{C}^1([0,+\infty))$. The uniqueness of the solution of (2.17) is always satisfied if σ_Y is convex. Indeed κ_W is strictly convex, because its derivative $-4\sqrt{W}$ is increasing. If σ_Y is convex, $\sigma_Y(0) = 0$, and $\sigma'_Y(0) > 0$, then $g = \gamma$ by (2.23), g is differentiable by the previous analysis, and by Proposition 2.2.6 there exists $t_0 > 0$ such that $g(t) < \kappa_W(0)$ for $t < t_0$ and $g(t) = \kappa_W(0)$ for $t \geq t_0$. Note that in this case g is differentiable at the first point in which it attains the constant value $\kappa_W(0)$.

2.3 Proof of the Γ-convergence result in dimension one

In this section we prove Theorem 2.1.1 when $n = 1$. We recall that in dimension one all Sobolev functions have a continuous representative. Without specifying it further again, we will always identify a function $\alpha \in H^1(\Omega)$ with its continuous representative.

Γ-liminf inequality: dimension one. We start with the proof of the Γ-liminf inequality. Let us fix a sequence (u_k, α_k) in $L^1(\Omega)\times L^1(\Omega)$ and $u \in L^1(\Omega)$ such that

$$(u_k, \alpha_k) \to (u, 1) \quad \text{in} \quad L^1(\Omega)\times L^1(\Omega)\,. \tag{2.29}$$

We want to prove that

$$\mathcal{F}_0(u, 1) \leq \liminf_{k\to+\infty} \mathcal{F}_k(u_k, \alpha_k)\,. \tag{2.30}$$

It is not restrictive to assume that the liminf in (2.30) is finite, hence

$$u_k \in BV(\Omega), \quad \alpha_k \in H^1(\Omega), \quad \text{and } \delta_k \leq \alpha_k \leq 1\,, \tag{2.31}$$

where $\delta_k = \delta_{\varepsilon_k} > 0$ is the sequence fixed in the introduction such that $\delta_k/\varepsilon_k \to 0$.

To obtain an estimate from below of the liminf, we will carry out a careful analysis of the regions on which the damage is concentrating as $\varepsilon_k \to 0$. To do this, we will study the Γ-convergence of the sequence of functions α_k defined on the space Ω endowed with the topology induced by $\mathbb{R}$. This notion will be denoted by $\Gamma(\mathbb{R})$-convergence.

It is enough to prove (2.30) when Ω is an interval, since the liminf is superadditive. Let $e_k \in L^2(\Omega)$ and $p_k \in \mathcal{M}_b(\Omega)$ be two sequences such that

$$Du_k = e_k + p_k \quad \text{in } \Omega\,. \tag{2.32}$$

We will prove that $u \in BV(\Omega)$ and

$$\mathcal{F}(u) \le \liminf_{k\to+\infty} \mathcal{E}_k(e_k, p_k, \alpha_k)\,. \tag{2.33}$$

We may assume that the liminf in (2.33) is finite and, up to extracting a subsequence, that it is actually a limit, so that

$$\mathcal{E}_k(e_k, p_k, \alpha_k) \le c \quad \text{for every } k\,, \tag{2.34}$$

for some $c \in \mathbb{R}$. We now extract a subsequence of α_k, not relabeled, such that

$$\alpha_k\ \Gamma(\mathbb{R})\text{-converges to some function } \alpha\colon \Omega \to [0,1]\,. \tag{2.35}$$

Remark 2.3.1. For every $\lambda \in [0,1)$, the set $\{\alpha \le \lambda\}$ is finite. Indeed, let $E = \{x_1, \dots, x_r\}$ be any finite subset of $\{\alpha \le \lambda\}$ and let $\varsigma > 0$ be such that the intervals $[x_i - \varsigma, x_i + \varsigma]$, $i = 1, \dots, r$, are pairwise disjoint and contained in Ω. Since α_k $\Gamma(\mathbb{R})$-converges to α, for every i there exists a recovery sequence $x_k^i \in (x_i - \varsigma/2, x_i + \varsigma/2)$ converging to x_i and such that $\alpha_k(x_k^i) \to \alpha(x_i)$ as $k \to +\infty$. Moreover, since $\alpha_k(x) \to 1$ for a.e. $x \in \Omega$, it is possible to find $x_i - \varsigma < y_1^i < x_k^i < y_2^i < x_i + \varsigma$ such that $\alpha_k(y_1^i) \to 1$, $\alpha_k(y_2^i) \to 1$. Using Young's inequality, from (2.34) we deduce that

$$\begin{aligned}
c &\ge \sum_{i=1}^r \int_{x_i-\varsigma}^{x_i+\varsigma} \Big[\frac{W(\alpha_k)}{\varepsilon_k} + \varepsilon_k |\nabla \alpha_k|^2\Big]\,\mathrm{d}x \\
&\ge 2\sum_{i=1}^r \Big[\int_{y_1^i}^{x_k^i} \sqrt{W(\alpha_k)}|\nabla\alpha_k|\,\mathrm{d}x + \int_{x_k^i}^{y_2^i} \sqrt{W(\alpha_k)}|\nabla\alpha_k|\,\mathrm{d}x\Big] \\
&= 2\sum_{i=1}^r \Big[\int_{\alpha_k(x_k^i)}^{\alpha_k(y_1^i)} \sqrt{W(s)}\,\mathrm{d}s + \int_{\alpha_k(x_k^i)}^{\alpha_k(y_2^i)} \sqrt{W(s)}\,\mathrm{d}s\Big]
\end{aligned}$$

and letting $k \to +\infty$

$$c \ge 4\sum_{i=1}^r \int_{\alpha(x_i)}^1 \sqrt{W(s)}\,\mathrm{d}s = \sum_{i=1}^r \kappa_W(\alpha(x_i)) \ge \mathcal{H}^0(E)\kappa_W(\lambda)\,,$$

since $\mathcal{H}^0$ is the counting measure. It follows that

$$\mathcal{H}^0(E) \le \frac{c}{\kappa_W(\lambda)}\,. \tag{2.36}$$

Since E was an arbitrary finite subset of $\{\alpha \le \lambda\}$ and the right hand side of the estimate (2.36) does not depend on E, we conclude that $\{\alpha \le \lambda\}$ is finite.

Remark 2.3.2. Let $\lambda \in [0,1)$ and let K be a compact set such that $K \subset \{\alpha > \lambda\}$. Since α is lower semicontinuous, we have that $\lambda < \min_K \alpha \leq \liminf_k \min_K \alpha_k$, where the last inequality follows from the lower semicontinuity with respect to the $\Gamma(\mathbb{R})$-convergence of the minimum on compact sets. It follows that $\alpha_k > \lambda$ on K for k large enough.

Lemma 2.3.3. *Assume* (2.29), (2.31), (2.32), (2.34), *and* (2.35). *The function u belongs to $BV(\Omega)$ and there exist a subsequence of (e_k, p_k) (not relabeled), a function $e \in L^2(\Omega)$, and a measure $p \in \mathcal{M}_b(\Omega)$ such that*

$$Du = e + p \quad \textit{in } \Omega\,, \tag{2.37}$$

$$e_k \rightharpoonup e \quad \textit{weakly in } L^2(A)\,, \tag{2.38}$$

$$p_k \overset{*}{\rightharpoonup} p \quad \textit{weakly* in } \mathcal{M}_b(A)\,, \tag{2.39}$$

for every open set $A \Subset \{\alpha > 0\}$.

Proof. Since $\{\alpha \leq \frac{1}{2}\}$ is finite by Remark 2.3.1, we can find $\lambda \in (0, \frac{1}{2})$ such that

$$\lambda < \min\{\alpha(x) : \alpha(x) \leq \tfrac{1}{2},\ \alpha(x) > 0\}\,. \tag{2.40}$$

From (2.40) it follows that $\{\alpha \leq \lambda\} = \{\alpha = 0\}$. Let us consider a sequence of open sets A_j such that

$$A_j \Subset A_{j+1}\,, \quad \bigcup_{j=1}^{+\infty} A_j = \{\alpha > \lambda\}\,. \tag{2.41}$$

Fix $j \geq 1$. By Remark 2.3.2, $\alpha_k > \lambda$ on A_j for k large enough. From (2.34) we deduce

$$c \geq \int_{A_j} \alpha_k |e_k|^2 \,\mathrm{d}x \geq \lambda \int_{A_j} |e_k|^2 \,\mathrm{d}x$$

and hence $||e_k||^2_{L^2(A_j)} \leq c/\lambda$. Therefore there exists a subsequence, which we do not relabel, such that

$$e_k \rightharpoonup e^j \quad \text{weakly in } L^2(A_j)\,,$$

and by a diagonal argument it is possible to extract a subsequence, not depending on j, such that

$$e_k \rightharpoonup e^j \quad \text{weakly in } L^2(A_j) \quad \text{for every } j \geq 1\,.$$

By the lower semicontinuity of the norm, we have $||e^j||^2_{L^2(A_j)} \leq c/\lambda$. Therefore there exists a function $e \in L^2(\Omega)$ such that $e = e^j$ on A_j, for every j. It follows that

$$e_k \rightharpoonup e \quad \text{weakly in } L^2(A_j)\,. \tag{2.42}$$

On the other hand, since σ_{Y} is nondecreasing and by (2.34),

$$c \geq \int_{A_j} \sigma_{\mathrm{Y}}(\alpha_k) \,\mathrm{d}|p_k| \geq \sigma_{\mathrm{Y}}(\lambda)|p_k|(A_j)\,,$$

from which it follows that p_k is bounded in $\mathcal{M}_b(A_j)$. Thus there exists a subsequence (which we do not relabel) and a measure $p^j \in \mathcal{M}_b(A_j)$ such that

$$p_k \overset{*}{\rightharpoonup} p^j \quad \text{weakly* in } \mathcal{M}_b(A_j)\,.$$

By a diagonal argument, there exists a subsequence, not depending on j, such that

$$p_k \overset{*}{\rightharpoonup} p^j \quad \text{weakly* in } \mathcal{M}_b(A_j) \quad \text{for every } j \geq 1\,.$$

By the lower semicontinuity of the total variation, it follows that $|p^j|(A_j) \leq c/\sigma_Y(\lambda)$, and hence there exists a measure $p \in \mathcal{M}_b(\{\alpha > 0\})$ such that $p \llcorner A_j = p^j$, for every j. This yields

$$p_k \overset{*}{\rightharpoonup} p \quad \text{weakly* in } \mathcal{M}_b(A_j)\,. \tag{2.43}$$

From (2.42) and (2.43), it follows that $u \in BV(A_j)$, $Du = e + p$ in A_j, and $Du_k \overset{*}{\rightharpoonup} Du$ in $\mathcal{M}_b(A_j)$, for every $j \geq 1$. Since

$$||e||^2_{L^2(A_j)} \leq \frac{c}{\lambda} \quad \text{and} \quad |p|(A_j) \leq \frac{c}{\sigma_Y(\lambda)} \quad \text{for every } j \geq 1\,,$$

we deduce that $u \in BV(\{\alpha > 0\})$, with $Du = e + p$ in the open set $\{\alpha > 0\}$. Since the set $\{\alpha = 0\}$ is finite and the right and left limits u^+ and u^- are well defined and finite on each point of $\{\alpha = 0\}$, we conclude that $u \in BV(\Omega)$. The measure $p \in \mathcal{M}_b(\Omega)$ extended to Ω by

$$p := p \llcorner \{\alpha > 0\} + (u^+ - u^-)\mathcal{H}^0 \llcorner \{\alpha = 0\}$$

satisfies (2.39) and (2.37). □

Remark 2.3.4. If $\{\alpha = 0\} \neq \emptyset$, the assumptions of Lemma 2.3.3 do not imply that the sequence e_k is bounded in $L^2(\Omega)$, as the following example shows. Let Ω be the interval $(-1,1)$, $\varepsilon_k = \frac{1}{k}$,

$$u_k := \begin{cases} 0 & \text{in } (-1, -\frac{1}{2k})\,, \\ kx + \frac{1}{2} & \text{in } [-\frac{1}{2k}, \frac{1}{2k}]\,, \\ 1 & \text{in } (\frac{1}{2k}, 1)\,, \end{cases}$$

$$\alpha_k := \begin{cases} 1 & \text{in } (-1, -\frac{1}{k}) \cup (\frac{1}{k}, 1)\,, \\ \delta_k & \text{in } (-\frac{1}{2k}, \frac{1}{2k})\,, \\ -2k(1-\delta_k)(x + \frac{1}{2k}) + \delta_k & \text{in } [-\frac{1}{k}, -\frac{1}{2k}]\,, \\ 2k(1-\delta_k)(x - \frac{1}{2k}) + \delta_k & \text{in } [\frac{1}{2k}, \frac{1}{k}]\,. \end{cases}$$

$e_k := Du_k$ in $(-1,1)$, and $p_k := 0$ in $(-1,1)$. Then

$$u = \begin{cases} 0 & \text{in } (-1,0)\,, \\ 1 & \text{in } (0,1)\,, \end{cases} \qquad \alpha = \begin{cases} 1 & \text{in } (-1,0) \cup (0,1)\,, \\ 0 & \text{in } \{0\}\,, \end{cases}$$

and it is easy to see that the assumptions of Lemma 2.3.3 are satisfied, while $e_k = \frac{1}{2}k$ on $(-\frac{1}{k}, \frac{1}{k})$, hence it is unbounded in $L^2(\Omega)$.

Remark 2.3.5. Assume $\sigma_Y(0) > 0$. By (2.34) and (2.1), we obtain that $|p_k|(\Omega)$ is bounded uniformly with respect to k. This implies that there exists a subsequence (not relabeled) and $q \in \mathcal{M}_b(\Omega)$ such that p_k converges to q weakly* in $\mathcal{M}_b(\Omega)$. It is easy to see that $q \llcorner \{\alpha > 0\} = p \llcorner \{\alpha > 0\}$, but, in general, $q \llcorner \{\alpha = 0\} \neq p \llcorner \{\alpha = 0\} = (u^+ - u^-)\mathcal{H}^0 \llcorner \{\alpha = 0\}$. Indeed, in the example of the previous remark, Lemma 2.3.3 gives $e = 0$ in $(-1,1)$ and $p = \mathcal{H}^0 \llcorner \{0\}$ in $(-1,1)$. On the other hand, the weak* limit q of p_k is identically zero, which is obviously different from p on $\{\alpha = 0\} = \{0\}$.

We are now able to prove (2.33).

Proposition 2.3.6. *Let* $e \in L^2(\Omega)$ *and* $p \in \mathcal{M}_b(\Omega)$ *be given by Lemma 2.3.3, in such a way that* (2.37), (2.38), *and* (2.39) *hold. Then*

$$\frac{1}{2}\int_{\Omega} |e|^2 \,\mathrm{d}x \le \liminf_{k\to+\infty} \frac{1}{2}\int_{\Omega} \alpha_k |e_k|^2 \,\mathrm{d}x\,, \tag{2.44}$$

$$\sigma_{\mathrm{Y}}(1)|p|(\Omega\setminus J_u) + \sum_{x\in J_u} g(|[u](x)|) \le \liminf_{k\to+\infty} [\mathcal{H}(p_k,\alpha_k) + \mathcal{W}_k(\alpha_k)]\,. \tag{2.45}$$

Moreover, (2.33) *holds.*

Proof. Let us fix $\eta \in (0,1]$. By Remark 2.3.1, the set $\{\alpha \le 1-\eta\}$ is finite, hence we can write $\{\alpha \le 1-\eta\} = \{x_1,\dots,x_r\}$ with $x_1 < \dots < x_r$. Moreover, let $\partial\Omega = \{x_0, x_{r+1}\}$. Finally, let $\varsigma_0 > 0$ be such that the intervals $[x_i - \varsigma_0, x_i + \varsigma_0]$, $i = 0,\dots,r+1$, are pairwise disjoint. For $\varsigma \in (0,\varsigma_0)$, let

$$A_\varsigma := \Omega \setminus \Big(\bigcup_{i=0}^{r+1} [x_i - \varsigma, x_i + \varsigma]\Big)\,.$$

Since $A_\varsigma \Subset \{\alpha > 1-\eta\}$, we have $\alpha_k > 1-\eta$ for k large enough, by Remark 2.3.2. Moreover (2.38) and (2.39) hold with $A = A_\varsigma$. By the lower semicontinuity of the norm in $L^2(A_\varsigma)$ and in $\mathcal{M}_b(A_\varsigma)$, it follows that

$$\frac{1-\eta}{2}\int_{A_\varsigma} |e|^2 \,\mathrm{d}x \le \liminf_{k\to+\infty} \frac{1-\eta}{2}\int_{A_\varsigma} |e_k|^2 \,\mathrm{d}x \le \liminf_{k\to+\infty} \frac{1}{2}\int_{\Omega} \alpha_k |e_k|^2 \,\mathrm{d}x\,, \tag{2.46}$$

$$\sigma_{\mathrm{Y}}(1-\eta)|p|(A_\varsigma) \le \liminf_{k\to+\infty} \sigma_{\mathrm{Y}}(1-\eta)|p_k|(A_\varsigma) \le \liminf_{k\to+\infty} \int_{A_\varsigma} \sigma_{\mathrm{Y}}(\alpha_k)\,\mathrm{d}|p_k|\,. \tag{2.47}$$

Let $i = 1,\dots,r$. Arguing as in Remark 2.3.1, we can find a sequence $x_k^i \to x_i$, with $\alpha_k(x_k^i) \to \alpha(x_i)$, and $x_i - \varsigma < y_1^i < x_k^i < y_2^i < x_i + \varsigma$ such that $\alpha_k(y_1^i) \to 1$, $\alpha_k(y_2^i) \to 1$, yielding

$$\begin{aligned}
&\liminf_{k\to+\infty} \int_{x_i-\varsigma}^{x_i+\varsigma} \Big[\frac{W(\alpha_k)}{\varepsilon_k} + \varepsilon_k |\nabla\alpha_k|^2\Big]\,\mathrm{d}x \\
&\ge \liminf_{k\to+\infty} 2\Big[\int_{y_1^i}^{x_k^i} \sqrt{W(\alpha_k)}|\nabla\alpha_k|\,\mathrm{d}x + \int_{x_k^i}^{y_2^i} \sqrt{W(\alpha_k)}|\nabla\alpha_k|\,\mathrm{d}x\Big] \ge \kappa_W(\alpha(x_i))\,.
\end{aligned} \tag{2.48}$$

If $\alpha(x_i) = 0$, the only estimate from below we can obtain is

$$\liminf_{k\to+\infty} \int_{[x_i-\varsigma,x_i+\varsigma]} \sigma_{\mathrm{Y}}(\alpha_k)\,\mathrm{d}|p_k| \ge 0\,. \tag{2.49}$$

Indeed, the example in Remark 2.3.5 shows that we cannot get a better estimate, even if $\sigma_{\mathrm{Y}}(0) > 0$.

If, instead, $\alpha(x_i) > 0$, we can fix $\omega > 0$ such that $\alpha(x_i) - \omega > 0$. Since α_k $\Gamma(\mathbb{R})$-converges to α, then

$$\alpha(x_i) = \sup_{\rho>0} \liminf_{k\to+\infty} \inf_{|x-x_i|<\rho} \alpha_k(x)\,,$$

and therefore there exists $\rho_i > 0$ such that

$$\alpha(x_i) - \omega < \liminf_{k\to+\infty} \inf_{|x-x_i|<\rho_i} \alpha_k(x)\,,$$

from which it follows that for k large enough

$$\alpha(x_i) - \omega < \inf_{|x-x_i|<\rho_i} \alpha_k(x)\,.$$

Hence, if $\varsigma_0 = \varsigma_0(\omega) > 0$ is small enough, by (2.39) we obtain

$$\begin{aligned} &\sigma_{\mathrm{Y}}(\alpha(x_i)-\omega)|p(\{x_i\})| \le \sigma_{\mathrm{Y}}(\alpha(x_i)-\omega)|p|((x_i-\varsigma, x_i+\varsigma)) \\ &\quad \le \liminf_{k\to+\infty} \sigma_{\mathrm{Y}}(\alpha(x_i)-\omega)|p_k|((x_i-\varsigma, x_i+\varsigma)) \le \liminf_{k\to+\infty} \int_{[x_i-\varsigma, x_i+\varsigma]} \sigma_{\mathrm{Y}}(\alpha_k)\,\mathrm{d}|p_k|\,, \end{aligned} \tag{2.50}$$

for every $\varsigma \in (0, \varsigma_0)$. Summing (2.47)–(2.50), by the superadditivity of the liminf we deduce that

$$\begin{aligned} \sigma_{\mathrm{Y}}(1-\eta)|p|(A_\varsigma) + &\sum_{x\in\{0<\alpha\le 1-\eta\}} \sigma_{\mathrm{Y}}(\alpha(x)-\omega)|p(\{x\})| + \sum_{x\in\{\alpha\le 1-\eta\}} \kappa_W(\alpha(x)) \\ &\le \liminf_{k\to+\infty} [\mathcal{H}(p_k, \alpha_k) + \mathcal{W}_k(\alpha_k)]\,. \end{aligned} \tag{2.51}$$

Letting $\varsigma \to 0^+$, $\omega \to 0^+$, and then $\eta \to 0^+$ in (2.46) and (2.51), we obtain (2.44) and

$$\begin{aligned} &\sigma_{\mathrm{Y}}(1)|p|(\{\alpha=1\}) + \sum_{x\in\{0<\alpha<1\}} \sigma_{\mathrm{Y}}(\alpha(x))|p(\{x\})| + \sum_{x\in\{\alpha<1\}} \kappa_W(\alpha(x)) \\ &\quad \le \liminf_{k\to+\infty} [\mathcal{H}(p_k, \alpha_k) + \mathcal{W}_k(\alpha_k)]\,. \end{aligned} \tag{2.52}$$

By (2.37) and by the general properties of the Cantor part of Du, we have $p(B \setminus J_u) = 0$ for every countable set B. Since $\{\alpha < 1\}$ is countable, using the definition of g (see (2.7)) and the inequality $g(z) \le \sigma_{\mathrm{Y}}(1)|z|$, we get

$$\begin{aligned} &\sigma_{\mathrm{Y}}(1)|p|(\Omega\setminus J_u) + \sum_{x\in J_u} g(|[u](x)|) \\ &= \sigma_{\mathrm{Y}}(1)|p|(\{\alpha=1\}) + \sum_{x\in\{\alpha<1\}} g(|p(\{x\})|) - \sigma_{\mathrm{Y}}(1)|p|(J_u\cap\{\alpha=1\}) + \sum_{x\in J_u\cap\{\alpha=1\}} g(|[u](x)|) \\ &\le \sigma_{\mathrm{Y}}(1)|p|(\{\alpha=1\}) + \sum_{x\in\{\alpha<1\}} g(|p(\{x\})|) \\ &\le \sigma_{\mathrm{Y}}(1)|p|(\{\alpha=1\}) + \sum_{x\in\{0<\alpha<1\}} \sigma_{\mathrm{Y}}(\alpha(x))|p(\{x\})| + \sum_{x\in\{\alpha<1\}} \kappa_W(\alpha(x))\,, \end{aligned}$$

which, together with (2.52), gives (2.45).

By (2.12) we have

$$\mathcal{F}(u) \leq \frac{1}{2}\int_\Omega |e|^2\,\mathrm{d}x + \sigma_{\mathrm{Y}}(1)|p|(\Omega\setminus J_u) + \sum_{x\in J_u} g(|[u](x)|)\,,$$

so that (2.44) and (2.45) yield (2.33). This concludes the proof. □

Remark 2.3.7. With respect to (2.45), inequality (2.52) proved in Proposition 2.3.6 gives a more precise estimate from below, which takes into account the asymptotic values of the damage variable on sets of Lebesgue measure zero. Unfortunately, it is not clear how to extend this result to dimension $n > 1$.

Inequality (2.30) now simply follows from (2.33) by choosing $e_k \in L^2(\Omega)$ and $p_k \in \mathcal{M}_b(\Omega)$ such that $Du_k = e_k + p_k$ in Ω and

$$\mathcal{F}_k(u_k,\alpha_k) = \mathcal{E}_k(e_k,p_k,\alpha_k)\,.$$

Γ-limsup inequality: dimension one. We now prove the Γ-limsup inequality. We start with the following preliminary result concerning the domain of the limit functional in the one-dimensional setting.

Proposition 2.3.8. *Let $\Omega\subset\mathbb{R}$ be a bounded open set. Let $u\in GBV(\Omega)\cap L^1(\Omega)$ be such that $\mathcal{F}(u) < +\infty$. Then $u\in BV(\Omega)$.*

Proof. For every open set $A\subset\Omega$ we define

$$\Psi(u;A) := \int_A |\nabla u|\,\mathrm{d}x + |D^c u|(A) + \sum_{x\in(J_u\setminus J_u^1)\cap A} |[u](x)| + \mathcal{H}^0(J_u^1\cap A)\,,$$

where $J_u^1 := \{x\in J_u : |[u](x)| \geq 1\}$. By (2.13) and (2.19), there exists a constant $c > 0$ such that $\Psi(u;\Omega) \leq c(\mathcal{F}(u)+1) < +\infty$. By Proposition 1.3.2 we conclude the proof. □

We now construct a recovery sequence. More precisely, we prove the following result.

Proposition 2.3.9. *For every $u\in BV(\Omega)$ with $\mathcal{F}(u) < +\infty$, there exist $u_k\in BV(\Omega)$, $e_k\in L^2(\Omega)$, $p_k\in\mathcal{M}_b(\Omega)$, and $\alpha_k\in H^1(\Omega)$ such that*

$$\begin{gathered}(u_k,\alpha_k)\to(u,1) \quad \text{in } L^1(\Omega)\times L^1(\Omega)\,,\\ Du_k = e_k + p_k \quad \text{in } \Omega\,,\\ \limsup_{k\to+\infty}\mathcal{E}_k(e_k,p_k,\alpha_k) \leq \mathcal{F}(u)\,.\end{gathered} \tag{2.53}$$

Proof. Let us fix $u\in BV(\Omega)$ with $\mathcal{F}(u) < +\infty$. By Proposition 2.2.2 there exist $e\in L^2(\Omega)$ and $p\in\mathcal{M}_b(\Omega)$ such that $Du = e + p$ in $\Omega\setminus J_u$ and

$$\mathcal{F}(u) = \frac{1}{2}\int_\Omega |e|^2\,\mathrm{d}x + \sigma_{\mathrm{Y}}(1)|p|(\Omega\setminus J_u) + \sum_{x\in J_u} g(|[u](x)|)\,. \tag{2.54}$$

For every $\lambda > 0$, the set

$$J_u^\lambda := \{x\in J_u : |[u](x)| \geq \lambda\}\,,$$

is finite. Let $\eta > 0$ and let us choose $\lambda > 0$ such that

$$\sigma_{\mathrm{Y}}(1)|p|(\Omega \setminus J_u^\lambda) \leq \sigma_{\mathrm{Y}}(1)|p|(\Omega \setminus J_u) + \eta\,. \tag{2.55}$$

For simplicity, let us assume for the moment that $J_u^\lambda = \{x_0\}$. From the definition of g in (2.7), we have that there exists a value $\alpha_0 \in [0,1]$ such that

$$g(|[u](x_0)|) = \begin{cases} \sigma_{\mathrm{Y}}(\alpha_0)|[u](x_0)| + \kappa_W(\alpha_0) & \text{if } \alpha_0 > 0\,, \\ \kappa_W(0) & \text{if } \alpha_0 = 0\,. \end{cases}$$

If $\alpha_0 = 1$, then we have trivially

$$\limsup_{k \to +\infty} \mathcal{E}_k(e, p, 1) \leq \mathcal{F}(u)\,,$$

since $g(|[u](x_0)|) = \sigma_{\mathrm{Y}}(1)|[u](x_0)|$.

Let us discuss the case $\alpha_0 < 1$. We define now a suitable infinitesimal sequence τ_k, as in the proof of [56, Theorem 3.3]. Let $h_1(\tau) := W(1-\tau)$, $h_2(\tau) := (\int_{\alpha_0}^{1-\tau} W(s)^{-\frac{1}{2}}\,\mathrm{d}s)^{-1}$. The function $(h_1 h_2)^{\frac{1}{2}}$ is strictly increasing and infinitesimal in 0, and h_1/h_2 is infinitesimal in 0. Indeed, since W is decreasing,

$$\frac{h_1(\tau)}{h_2(\tau)} = W(1-\tau) \int_{\alpha_0}^{1-\tau} W(s)^{-\frac{1}{2}}\,\mathrm{d}s \leq (1-\tau) W(1-\tau)^{\frac{1}{2}} \to 0 \qquad \text{as } \tau \to 0\,.$$

Let τ_k be such that $(h_1(\tau_k)h_2(\tau_k))^{\frac{1}{2}} = \varepsilon_k$. In this way

$$\frac{W(1-\tau_k)}{\varepsilon_k} = \frac{h_1(\tau_k)}{\varepsilon_k} \to 0 \quad \text{and} \quad \zeta_k := \varepsilon_k \int_{\alpha_0+\delta_k}^{1-\tau_k} W(s)^{-\frac{1}{2}}\,\mathrm{d}s = \frac{\varepsilon_k}{h_2(\tau_k)} + o(\varepsilon_k) \to 0 \tag{2.56}$$

as $k \to +\infty$.

Let us consider the solution ψ_k of the differential equation

$$\begin{cases} \psi_k' = \dfrac{1}{\varepsilon_k}\sqrt{W(\psi_k)}\,, \\ \psi_k(0) = \alpha_0 + \delta_k\,. \end{cases} \tag{2.57}$$

The solution of (2.57) is given by the inverse of the function

$$z \in [\alpha_0 + \delta_k, 1 - \tau_k] \mapsto \varepsilon_k \int_{\alpha_0+\delta_k}^{z} W(s)^{-\frac{1}{2}}\,\mathrm{d}s \in [0, \zeta_k]\,.$$

Moreover, let ς_k be an infinitesimal sequence such that

$$\frac{\varsigma_k}{\varepsilon_k} \to 0 \quad \text{and} \quad \frac{\delta_k}{\varsigma_k} \to 0\,. \tag{2.58}$$

Let $A_k := [x_0 - \varsigma_k, x_0 + \varsigma_k]$ and $B_k := [x_0 - \varsigma_k - \zeta_k, x_0 - \varsigma_k] \cup [x_0 + \varsigma_k, x_0 + \varsigma_k + \zeta_k]$. It is not restrictive to assume that $\partial A_k \cap J_u = \emptyset$ for every k, so that the precise values $\widetilde{u}(x_0 - \varsigma_k)$

and $\widetilde{u}(x_0+\varsigma_k)$ are well defined. Let $u_k \in BV(\Omega)$ be the affine interpolation between $\widetilde{u}(x_0-\varsigma_k)$ and $\widetilde{u}(x_0+\varsigma_k)$ on A_k, while $u_k := u$ out of A_k. Finally, let $\alpha_k \in H^1(\Omega)$ be defined as

$$\alpha_k(x) := \begin{cases} 1-\tau_k & \text{if } x \in \Omega \setminus (A_k \cup B_k)\,, \\ \alpha_0+\delta_k & \text{if } x \in A_k\,, \\ \psi_k(|x-x_0|-\varsigma_k) & \text{if } x \in B_k\,. \end{cases}$$

Let us notice that $\delta_k \leq \alpha_k \leq 1$.

Let us discuss the case $\alpha_0 > 0$ first. In this case, let $e_k \in L^2(\Omega)$, $p_k \in \mathcal{M}_b(\Omega)$ be defined by

$$e_k := \begin{cases} e & \text{in } \Omega \setminus A_k\,, \\ 0 & \text{in } A_k\,, \end{cases} \qquad p_k := \begin{cases} p & \text{in } \Omega \setminus A_k\,, \\ \nabla u_k \mathcal{L}^1 & \text{in } A_k\,. \end{cases}$$

Let us estimate $\mathcal{F}_k(u_k,\alpha_k)$:

$$\int_\Omega \alpha_k |e_k|^2 \,\mathrm{d}x = \int_{\Omega\setminus A_k} \alpha_k |e|^2 \,\mathrm{d}x \leq \int_{\Omega\setminus A_k} |e|^2 \,\mathrm{d}x\,. \tag{2.59}$$

$$\int_{\Omega\setminus A_k} \sigma_{\mathrm{Y}}(\alpha_k)\,\mathrm{d}|p_k| \leq \sigma_{\mathrm{Y}}(1)|p|(\Omega\setminus A_k)\,. \tag{2.60}$$

Since u_k is linear in A_k, we have

$$\begin{aligned} \int_{A_k} \sigma_{\mathrm{Y}}(\alpha_k)\,\mathrm{d}|p_k| &= \sigma_{\mathrm{Y}}(\alpha_0+\delta_k) \int_{A_k} |\nabla u_k(x)|\,\mathrm{d}x \\ &= \sigma_{\mathrm{Y}}(\alpha_0+\delta_k) \int_{A_k} \Big|\frac{\widetilde{u}(x_0+\varsigma_k)-\widetilde{u}(x_0-\varsigma_k)}{2\varsigma_k}\Big|\,\mathrm{d}x \\ &= \sigma_{\mathrm{Y}}(\alpha_0+\delta_k)|\widetilde{u}(x_0+\varsigma_k)-\widetilde{u}(x_0-\varsigma_k)|\,. \end{aligned} \tag{2.61}$$

Moreover

$$\int_{\Omega\setminus(A_k\cup B_k)} \Big[\frac{W(\alpha_k)}{\varepsilon_k} + \varepsilon_k|\nabla\alpha_k|^2\Big]\,\mathrm{d}x = \frac{W(1-\tau_k)}{\varepsilon_k}\mathcal{L}^1(\Omega)\,, \tag{2.62}$$

is infinitesimal as $k \to +\infty$ by (2.56), and

$$\int_{A_k} \Big[\frac{W(\alpha_k)}{\varepsilon_k} + \varepsilon_k|\nabla\alpha_k|^2\Big]\,\mathrm{d}x = \int_{A_k} \Big[\frac{W(\alpha_0+\delta_k)}{\varepsilon_k}\Big]\,\mathrm{d}x = W(\alpha_0+\delta_k)\frac{2\varsigma_k}{\varepsilon_k} \tag{2.63}$$

goes to 0 as $k \to +\infty$ by (2.58).

Finally, from the definition of α_k in B_k it turns out that the equality in Young's inequality holds, and hence

$$\int_{B_k} \Big[\frac{W(\alpha_k)}{\varepsilon_k} + \varepsilon_k|\nabla\alpha_k|^2\Big]\,\mathrm{d}x = 2\int_{B_k} \sqrt{W(\alpha_k)}|\nabla\alpha_k|\,\mathrm{d}x = 4\int_{\alpha_0+\delta_k}^{1-\tau_k} \sqrt{W(s)}\,\mathrm{d}s\,. \tag{2.64}$$

By (2.58), summing (2.59)–(2.64), and passing to the limsup we obtain

$$\begin{aligned}\limsup_{k\to+\infty}\mathcal{E}_k(e_k,p_k,\alpha_k)&\le\frac12\int_\Omega|e|^2\,\mathrm{d}x+\sigma_{\mathrm{Y}}(1)|p|(\Omega\setminus\{x_0\})+\sigma_{\mathrm{Y}}(\alpha_0)|[u](x_0)|+\kappa_W(\alpha_0)\\&=\frac12\int_\Omega|e|^2\,\mathrm{d}x+\sigma_{\mathrm{Y}}(1)|p|(\Omega\setminus\{x_0\})+g(|[u](x_0)|)\,.\end{aligned}$$

Let us discuss the case $\alpha_0=0$. Let us define this time

$$e_k(x):=\begin{cases}e(x)&\text{if }x\in\Omega\setminus A_k\,,\\\nabla u_k(x)&\text{if }x\in A_k\,,\end{cases}\qquad p_k:=\begin{cases}p&\text{in }\Omega\setminus A_k\,,\\0&\text{in }A_k\,.\end{cases}$$

The term

$$\int_{\Omega\setminus A_k}\alpha_k|e_k|^2\,\mathrm{d}x\le\int_{\Omega\setminus A_k}|e|^2\,\mathrm{d}x \tag{2.65}$$

can be treated as in (2.59). Moreover

$$\begin{aligned}\int_{A_k}\alpha_k|e_k|^2\,\mathrm{d}x=\int_{A_k}\alpha_k|\nabla u_k|^2\,\mathrm{d}x&\le\int_{A_k}\delta_k\Big|\frac{\widetilde u(x_0+\varsigma_k)-\widetilde u(x_0-\varsigma_k)}{2\varsigma_k}\Big|^2\,\mathrm{d}x\\&\le\frac{\delta_k}{2\varsigma_k}|\widetilde u(x_0+\varsigma_k)-\widetilde u(x_0-\varsigma_k)|^2\,.\end{aligned}\tag{2.66}$$

By (2.58), by summing (2.60), (2.62)–(2.66), and passing to the limsup, we obtain

$$\begin{aligned}\limsup_{k\to+\infty}\mathcal{E}_k(e_k,p_k,\alpha_k)&\le\frac12\int_\Omega|e|^2\,\mathrm{d}x+\sigma_{\mathrm{Y}}(1)|p|(\Omega\setminus\{x_0\})+\kappa_W(0)\\&=\frac12\int_\Omega|e|^2\,\mathrm{d}x+\sigma_{\mathrm{Y}}(1)|p|(\Omega\setminus\{x_0\})+g(|[u](x_0)|)\,.\end{aligned}$$

Arguing in this way for all the elements of J_u^λ, by the choice of λ made in (2.55), and by (2.54) we get

$$\begin{aligned}\limsup_{k\to+\infty}\mathcal{E}_k(e_k,p_k,\alpha_k)&\le\frac12\int_\Omega|e|^2\,\mathrm{d}x+\sigma_{\mathrm{Y}}(1)|p|(\Omega\setminus J_u^\lambda)+\sum_{x\in J_u^\lambda}g(|[u](x)|)\\&\le\mathcal{F}(u)+\eta\,,\end{aligned}$$

which yields (2.53) by letting $\eta\to0$. □

2.4 Proof of the Γ-convergence result in the general case

To study the n-dimensional case, we shall use the localized version of the functionals introduced in (2.2) and (2.4): they are defined for every open set $A\subset\Omega$, for every $u\in BV(A)$, and for every $\alpha\in H^1(A)$ by

$$\mathcal{W}_k(\alpha;A):=\int_A\Big[\frac{W(\alpha)}{\varepsilon_k}+\varepsilon_k|\nabla\alpha|^2\Big]\,\mathrm{d}x\,,$$

$$\mathcal{F}_k(u,\alpha;A):=\int_A f_k(\alpha,|\nabla u|)\,\mathrm{d}x+\int_A\sigma_{\mathrm{Y}}(\widetilde\alpha)\,\mathrm{d}|D^su|+\mathcal{W}_k(\alpha;A)\,,$$

and extended to $+\infty$ otherwise in $L^1(\Omega)$ and $L^1(\Omega)\times L^1(\Omega)$ respectively. For the localized version of the Γ-limits, we adopt the notation

$$\mathcal{F}'(\cdot,\cdot;A) := \Gamma\text{-}\liminf_{k\to+\infty} \mathcal{F}_k(\cdot,\cdot;A) \quad \text{and} \quad \mathcal{F}''(\cdot,\cdot;A) := \Gamma\text{-}\limsup_{k\to+\infty} \mathcal{F}_k(\cdot,\cdot;A)\,.$$

We omit the indication of the set when $A=\Omega$.

Γ-liminf inequality: the general case. We start by proving the Γ-liminf inequality by employing a slicing argument.

Proposition 2.4.1. *For every $u \in L^1(\Omega)$ we have $\mathcal{F}_0(u,1) \le \mathcal{F}'(u,1)$.*

Proof. We first prove the proposition under the additional assumption that $||u||_{L^\infty(\Omega)} \le M$ for some constant $M>0$. Let us consider a sequence $(u_k,\alpha_k) \in L^1(\Omega)\times L^1(\Omega)$ and $u \in L^1(\Omega)$ such that $(u_k,\alpha_k) \to (u,1)$ in $L^1(\Omega)\times L^1(\Omega)$ and

$$\mathcal{F}'(u,1) = \liminf_{k\to+\infty} \mathcal{F}_k(u_k,\alpha_k)\,. \tag{2.67}$$

We can always assume that the liminf in (2.67) is a limit and that $\mathcal{F}_k(u_k,\alpha_k)$ is bounded, and hence $u_k \in BV(\Omega)$, $\alpha_k \in H^1(\Omega)$, and $\delta_k \le \alpha_k \le 1$. Let $e_k \in L^2(\Omega;\mathbb{R}^n)$ and $p_k \in \mathcal{M}_b(\Omega;\mathbb{R}^n)$ be such that $Du_k = e_k + p_k$ and

$$\mathcal{E}_k(e_k,p_k,\alpha_k) = \mathcal{F}_k(u_k,\alpha_k) \le c\,. \tag{2.68}$$

Let us fix $\xi \in \mathbb{S}^{n-1}$. Then there exists a subsequence (not relabeled), possibly depending on ξ, such that

$$((u_k)^\xi_y,(\alpha_k)^\xi_y) \to (u^\xi_y,1) \quad \text{in } L^1(\Omega^\xi_y)\times L^1(\Omega^\xi_y) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi.$$

Since $u_k \in BV(\Omega)$, we know that $(u_k)^\xi_y \in BV(\Omega^\xi_y)$ for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$ and that the measures $Du_k\cdot\xi$ and $|Du_k\cdot\xi|$ are decomposed as

$$Du_k\cdot\xi(B) = \int_{\Pi^\xi} (D(u_k)^\xi_y)(B^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}(y),$$
$$|Du_k\cdot\xi|(B) = \int_{\Pi^\xi} |D(u_k)^\xi_y|(B^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\,,$$

for every Borel set $B\subset\Omega$. Since $Du_k = e_k + p_k$, it is immediate to deduce that

$$p_k\cdot\xi(B) = \int_{\Pi^\xi} (\hat{p}_k)^\xi_y(B^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}(y),$$
$$|p_k\cdot\xi|(B) = \int_{\Pi^\xi} |(\hat{p}_k)^\xi_y|(B^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\,,$$

where the measures $(\hat{p}_k)^\xi_y \in \mathcal{M}_b(\Omega^\xi_y)$ are defined by $(\hat{p}_k)^\xi_y := D(u_k)^\xi_y - (\hat{e}_k)^\xi_y$.

To apply the results of the one dimensional case, we first have to check that $(\widetilde{\alpha}_k)^\xi_y$ coincides with the continuous representative of $(\alpha_k)^\xi_y \in H^1(\Omega^\xi_y)$ for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$.

Indeed, $\widetilde{\alpha}_k$ is the precise representative of α_k, in the sense of (1.3) and this implies, by [7, Theorem 3.108], that for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ the function $(\widetilde{\alpha}_k)^\xi_y$ is a good representative of $(\alpha_k)^\xi_y$, meaning that its pointwise total variation coincides with the total variation of $(\alpha_k)^\xi_y$. This implies that $(\widetilde{\alpha}_k)^\xi_y$ must be the continuous representative of $(\alpha_k)^\xi_y$.

From the Fubini Theorem it follows that

$$\frac{1}{2}\int_\Omega \alpha_k|e_k|^2\,\mathrm{d}x \geq \frac{1}{2}\int_\Omega \alpha_k|e_k\cdot\xi|^2\,\mathrm{d}x = \frac{1}{2}\int_{\Pi^\xi}\int_{\Omega^\xi_y}(\alpha_k)^\xi_y|(\hat{e}_k)^\xi_y|^2\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^{n-1}(y)\,,$$

and

$$\begin{aligned}&\int_\Omega \sigma_{\mathrm{Y}}(\widetilde{\alpha}_k)\,\mathrm{d}|p_k| + \int_\Omega\Big[\frac{W(\alpha_k)}{\varepsilon_k}+\varepsilon_k|\nabla\alpha_k|^2\Big]\,\mathrm{d}x\\ &\geq \int_\Omega \sigma_{\mathrm{Y}}(\widetilde{\alpha}_k)\,\mathrm{d}|p_k\cdot\xi| + \int_\Omega\Big[\frac{W(\alpha_k)}{\varepsilon_k}+\varepsilon_k|\nabla\alpha_k\cdot\xi|^2\Big]\,\mathrm{d}x\\ &= \int_{\Pi^\xi}\Big\{\int_{\Omega^\xi_y}\sigma_{\mathrm{Y}}((\widetilde{\alpha}_k)^\xi_y))\,\mathrm{d}|(\hat{p}_k)^\xi_y| + \int_{\Omega^\xi_y}\Big[\frac{W((\alpha_k)^\xi_y)}{\varepsilon_k}+\varepsilon_k|\nabla(\alpha_k)^\xi_y|^2\Big]\,\mathrm{d}t\Big\}\,\mathrm{d}\mathcal{H}^{n-1}(y)\,.\end{aligned}$$

Summing the previous inequalities and using (2.68) we obtain that

$$\int_{\Pi^\xi}\mathcal{E}^{\xi,y}_k((\hat{e}_k)^\xi_y,(\hat{p}_k)^\xi_y,(\alpha_k)^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}(y) \leq \mathcal{F}_k(u_k,\alpha_k)\leq c\,, \tag{2.69}$$

where $\mathcal{E}^{\xi,y}_k$ is defined by

$$\mathcal{E}^{\xi,y}_k(e,p,\alpha) := \frac{1}{2}\int_{\Omega^\xi_y}\alpha|e|^2\,\mathrm{d}t + \int_{\Omega^\xi_y}\sigma_{\mathrm{Y}}(\alpha)\,\mathrm{d}|p| + \int_{\Omega^\xi_y}\Big[\frac{W(\alpha)}{\varepsilon_k}+\varepsilon_k|\nabla\alpha|^2\Big]\,\mathrm{d}t$$

for every $e\in L^2(\Omega^\xi_y)$, $p\in\mathcal{M}_b(\Omega^\xi_y)$, and $\alpha\in H^1(\Omega^\xi_y)$ with $\delta_k\leq\alpha\leq 1$. By the Fatou Lemma we have that

$$\liminf_{k\to+\infty}\mathcal{E}^{\xi,y}_k((\hat{e}_k)^\xi_y,(\hat{p}_k)^\xi_y,(\alpha_k)^\xi_y) < +\infty \tag{2.70}$$

for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$.

Let us fix $y\in\Pi^\xi$ such that (2.70) holds. Up to a subsequence, possibly depending on y, we can suppose that the liminf in (2.70) is actually a limit. By Lemma 2.3.3 and Proposition 2.3.6, we have that $u^\xi_y\in BV(\Omega^\xi_y)$ and there exist $e_{\xi,y}\in L^2(\Omega^\xi_y)$, $p_{\xi,y}\in\mathcal{M}_b(\Omega^\xi_y)$ such that $Du^\xi_y = e_{\xi,y}+p_{\xi,y}$,

$$\frac{1}{2}\int_{\Omega^\xi_y}|e_{\xi,y}(t)|^2\,\mathrm{d}t \leq \liminf_{k\to+\infty}\frac{1}{2}\int_{\Omega^\xi_y}(\alpha_k)^\xi_y(t)|(\hat{e}_k)^\xi_y(t)|^2\,\mathrm{d}t\,, \tag{2.71}$$

and

$$\begin{aligned}&\sigma_{\mathrm{Y}}(1)|p_{\xi,y}|(\Omega^\xi_y\setminus J_{u^\xi_y}) + \sum_{t\in J_{u^\xi_y}} g(|[u^\xi_y](t)|)\\ &\leq \liminf_{k\to+\infty}\Big\{\int_{\Omega^\xi_y}\sigma_{\mathrm{Y}}((\alpha_k)^\xi_y))\,\mathrm{d}|(\hat{p}_k)^\xi_y| + \int_{\Omega^\xi_y}\Big[\frac{W((\alpha_k)^\xi_y)}{\varepsilon_k}+\varepsilon_k|\nabla(\alpha_k)^\xi_y|^2\Big]\,\mathrm{d}t\Big\}\,.\end{aligned} \tag{2.72}$$

We now prove that $u \in BV(\Omega)$ by showing that (1.8) holds. From the additive decomposition of Du_y^ξ we get

$$
\begin{aligned}
|Du_y^\xi|(\Omega_y^\xi) &\le \int_{\Omega_y^\xi} |e_{\xi,y}|\,\mathrm{d}t + |p_{\xi,y}|(\Omega_y^\xi) \\
&\le \frac{1}{2}\mathcal{L}^1(\Omega_y^\xi) + \frac{1}{2}\int_{\Omega_y^\xi} |e_{\xi,y}|^2\,\mathrm{d}t + |p_{\xi,y}|(\Omega_y^\xi \setminus J_{u_y^\xi}) + |p_{\xi,y}|(J_{u_y^\xi})\,.
\end{aligned}
\tag{2.73}
$$

Let us estimate the last term in the sum. By (2.19), using the a priori bound $||u||_{L^\infty(\Omega)} \le M$, we obtain

$$
\begin{aligned}
|p_{\xi,y}|(J_{u_y^\xi}) &= \sum_{t\in\{|[u_y^\xi]|<1\}} |[u_y^\xi](t)| + \sum_{t\in\{|[u_y^\xi]|\ge 1\}} |[u_y^\xi](t)| \\
&\le \sum_{t\in\{|[u_y^\xi]|<1\}} \frac{1}{g(1)} g(|[u_y^\xi](t)|) + \sum_{t\in\{|[u_y^\xi]|\ge 1\}} \frac{2M}{g(1)} g(|[u_y^\xi](t)|) \le c \sum_{t\in J_{u_y^\xi}} g(|[u_y^\xi](t)|)\,.
\end{aligned}
\tag{2.74}
$$

By (2.71)–(2.74), by Fatou Lemma, and by (2.69) it follows that

$$
\begin{aligned}
\int_{\Pi^\xi} |Du_y^\xi|(\Omega_y^\xi)\,\mathrm{d}\mathcal{H}^{n-1}(y) &\le C\Big[1 + \int_{\Pi^\xi} \liminf_{k\to+\infty} \mathcal{E}_k^{\xi,y}((\hat{e}_k)_y^\xi, (\hat{p}_k)_y^\xi, (\alpha_k)_y^\xi)\,\mathrm{d}\mathcal{H}^{n-1}(y)\Big] \\
&\le C\Big[1 + \liminf_{k\to+\infty} \int_{\Pi^\xi} \mathcal{E}_k^{\xi,y}((\hat{e}_k)_y^\xi, (\hat{p}_k)_y^\xi, (\alpha_k)_y^\xi)\,\mathrm{d}\mathcal{H}^{n-1}(y)\Big] \\
&\le C\Big[1 + \liminf_{k\to+\infty} \mathcal{F}_k(u_k, \alpha_k)\Big] < +\infty\,.
\end{aligned}
$$

This proves that $u \in BV(\Omega)$.

We can now go back to the proof of the estimate from below $\mathcal{F}(u) \le \mathcal{F}'(u, 1)$. Summing (2.71) and (2.72), we obtain that

$$
\begin{aligned}
&\int_{\Omega_y^\xi} f(1, |\nabla u_y^\xi|)\,\mathrm{d}t + \sigma_{\mathrm{Y}}(1)|D^c u_y^\xi|(\Omega_y^\xi) + \sum_{t\in J_{u_y^\xi}} g(|[u_y^\xi](t)|) \\
&\qquad \le \liminf_{k\to+\infty} \mathcal{E}_k^{\xi,y}((\hat{e}_k)_y^\xi, (\hat{p}_k)_y^\xi, (\alpha_k)_y^\xi)\,.
\end{aligned}
$$

Integrating the inequality above with respect to $y \in \Pi^\xi$ and using the Fatou Lemma, from (2.69) and (2.67) we obtain

$$
\int_\Omega f(1, |\nabla u \cdot \xi|)\,\mathrm{d}t + \sigma_{\mathrm{Y}}(1)|D^c u \cdot \xi|(\Omega) + \int_{J_u} g(|[u]|)|\nu_u \cdot \xi|\,\mathrm{d}\mathcal{H}^{n-1} \le \mathcal{F}'(u, 1)\,. \tag{2.75}
$$

To get rid of ξ, we use a localization argument. Let $(\xi_i)_i$ be a dense sequence in $\mathbb{S}^{n-1}$ and let

$$
\mu := \mathcal{L}^n + |D^c u| + \mathcal{H}^{n-1} \llcorner J_u\,.
$$

Let Σ be a Borel set containing J_u such that $\mathcal{L}^n(\Sigma) = 0$ and $|D^s u|(\Omega \setminus \Sigma) = 0$. For every ξ, we define the function

$$
\varphi_\xi := f(1, |\nabla u \cdot \xi|)1_{\Omega\setminus\Sigma} + \sigma_{\mathrm{Y}}(1)|\chi_u \cdot \xi|1_{\Sigma\setminus J_u} + g(|[u]|)|\nu_u \cdot \xi|1_{J_u},
$$

where $\chi_u = \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}$ is the density of the measure $D^c u$ with respect to its total variation. It is immediate to obtain estimate (2.75) on every open set contained in Ω. This implies that

$$\int_{A_i} \varphi_{\xi_i}\, \mathrm{d}\mu \leq \mathcal{F}'(u,1;A_i)$$

for every i and for every open set $A_i \subset \Omega$. Since $\mathcal{F}'(u,1;\cdot)$ is superadditive, we obtain

$$\sum_i \int_{A_i} \varphi_{\xi_i}\, \mathrm{d}\mu \leq \sum_i \mathcal{F}'(u,1;A_i) \leq \mathcal{F}'(u,1)$$

for every sequence A_i of pairwise disjoint open sets contained in Ω. By Lemma 1.4.7, the supremum of the left hand side is given by

$$\int_\Omega \sup_i \varphi_{\xi_i}\, \mathrm{d}\mu\,.$$

Since

$$\sup_i f(1,|\nabla u\cdot \xi_i|) = f(1,|\nabla u|)\,, \quad \sup_i |\chi_u \cdot \xi_i| = 1\,, \quad \sup_i |\nu_u \cdot \xi_i| = 1\,,$$

this concludes the proof in the case $||u||_{L^\infty(\Omega)} \leq M$.

The general case is treated with a truncation argument. Let $M>0$ be any positive constant. Let us consider the functions

$$u_{k,M} := (-M \vee u_k)\wedge M \quad \text{and} \quad u_M := (-M\vee u)\wedge M\,.$$

Notice that $u_{k,M} \to u_M$ in $L^1(\Omega)$, $\alpha_k \to 1$ in $L^1(\Omega)$, and

$$\mathcal{F}_k(u_{k,M},\alpha_k) \leq \mathcal{F}_k(u_k,\alpha_k) \leq c\,,$$

since the functionals are decreasing by truncation. From the bounded case, it follows that $u_M \in BV(\Omega)$ and

$$\mathcal{F}(u_M) \leq \liminf_{k\to+\infty} \mathcal{F}_k(u_{k,M},\alpha_k) \leq \liminf_{k\to+\infty} \mathcal{F}_k(u_k,\alpha_k) = \mathcal{F}'(u,1)\,.$$

By letting $M\to+\infty$ we conclude that $u\in GBV(\Omega)$ and $\mathcal{F}(u)\leq \mathcal{F}'(u,1)$. □

Γ-limsup inequality: the general case. To prove the Γ-limsup inequality, we shall apply an integral representation result to the limit functional. In order to do this, we use the notion of $\overline{\Gamma}$-convergence, for which we refer to Subsection 1.4.2.

We start with a rough estimate of the Γ-limsup.

Proposition 2.4.2. *There exists a constant $C>0$ such that for all $u\in BV(\Omega)$ and for every open set A we have*

$$\mathcal{F}''(u,1;A) \leq C|Du|(A)\,.$$

Proof. Let us choose $u_k = u$ and $\alpha_k = 1$ for every k. In this way

$$\begin{aligned}\mathcal{F}''(u,1;A) &\leq \limsup_{k\to+\infty} \mathcal{F}_k(u,1;A) \\ &\leq \int_A f(1,|\nabla u|)\,\mathrm{d}x + \sigma_{\mathrm{Y}}(1)|D^s u|(A) \leq C|Du|(A)\,,\end{aligned} \tag{2.76}$$

where we used (2.13) in the last inequality. □

We now use a slicing and averaging argument due to De Giorgi in order to prove the weak subadditivity of the Γ-limsup.

Lemma 2.4.3. *Let $u \in L^1(\Omega)$, let A', A, B be open subset of Ω with $A' \Subset A$. Then*

$$\mathcal{F}''(u,1;A'\cup B) \leq \mathcal{F}''(u,1;A) + \mathcal{F}''(u,1;B)\,.$$

Proof. Let $(u_k^A,\alpha_k^A),(u_k^B,\alpha_k^B) \in L^1(\Omega)\times L^1(\Omega)$ be such that

$$(u_k^A,\alpha_k^A),(u_k^B,\alpha_k^B) \to (u,1) \quad \text{in } L^1(\Omega)\times L^1(\Omega)$$

and

$$\limsup_{k\to+\infty} \mathcal{F}_k(u_k^A,\alpha_k^A;A) = \mathcal{F}''(u,1;A), \quad \limsup_{k\to+\infty} \mathcal{F}_k(u_k^B,\alpha_k^B;B) = \mathcal{F}''(u,1;B)\,.$$

We can assume that both $\mathcal{F}''(u,1;A)$ and $\mathcal{F}''(u,1;B)$ are finite, otherwise the statement is trivial. In particular $u_k^A \in BV(A), u_k^B \in BV(B)$, $\alpha_k^A \in H^1(A), \alpha_k^B \in H^1(B)$, and $\delta_k \leq \alpha_k^A, \alpha_k^B \leq 1$. Let $d := \mathrm{dist}(A',\partial A) > 0$ and let $h \in \mathbb{N}$. Let $A_0 := A'$ and $A_{h+1} := A$. We consider a chain of open sets $A_1,\dots,A_h$ such that $A_i \Subset A_{i+1}$ and $\mathrm{dist}(A_i,\partial A_{i+1}) \geq d/(h+1)$ for every $0 \leq i \leq h-1$. Let $\varphi_i \in \mathcal{C}_c^1(\Omega)$ be a cut-off function between A_i and A_{i+1}, i.e., $0 \leq \varphi_i \leq 1$, $\mathrm{supp}(\varphi_i) \subset A_{i+1}$, and $\varphi_i = 1$ in a neighborhood of $\overline{A}_i$. We assume in addition that $||\nabla\varphi_i||_{L^\infty(\Omega)} \leq 2(h+1)/d$. We set

$$u_k^i := \varphi_i u_k^A + (1-\varphi_i)u_k^B \in BV(A'\cup B)\,,$$

and we define the functions $\alpha_k^i \in H^1(A'\cup B)$ as in [24, Lemma 6.2]:

$$\alpha_k^i := \begin{cases} \varphi_{i-1}\alpha_k^A + (1-\varphi_{i-1})(\alpha_k^A \wedge \alpha_k^B) & \text{in } A_i\,, \\ \alpha_k^A \wedge \alpha_k^B & \text{in } A_{i+1}\setminus A_i\,, \\ \varphi_{i+1}(\alpha_k^A \wedge \alpha_k^B) + (1-\varphi_{i+1})\alpha_k^B & \text{in } \Omega\setminus A_{i+1}\,. \end{cases}$$

Let us notice that $\delta_k \leq \alpha_k^i \leq 1$. Let $1 \leq i \leq h-1$. We estimate $\mathcal{F}_k$ on $A'\cup B$ in the following way

$$\begin{aligned}\mathcal{F}_k(u_k^i,\alpha_k^i;A'\cup B) &\leq \mathcal{F}_k(u_k^i,\alpha_k^i;(A'\cup B)\cap A_{i-1}) + \mathcal{F}_k(u_k^i,\alpha_k^i;B\setminus A_{i+2}) \\ &\quad + \mathcal{F}_k(u_k^i,\alpha_k^i;B\cap(A_{i+2}\setminus A_{i-1})) \\ &\leq \mathcal{F}_k(u_k^A,\alpha_k^A;A_{i-1}) + \mathcal{F}_k(u_k^B,\alpha_k^B;B\setminus A_{i+2}) \\ &\quad + \mathcal{F}_k(u_k^i,\alpha_k^i;B\cap(A_{i+2}\setminus A_{i-1}))\,.\end{aligned} \tag{2.77}$$

We need only to bound the last term:

$$\mathcal{F}_k(u_k^i,\alpha_k^i;B\cap(A_{i+2}\setminus A_{i-1})) \leq \mathcal{F}_k(u_k^i,\alpha_k^i;S_{i+1}) + \mathcal{F}_k(u_k^i,\alpha_k^i;S_i) + \mathcal{F}_k(u_k^i,\alpha_k^i;S_{i-1})\,,$$

where $S_i = B \cap (A_{i+1} \setminus A_i)$ for $0 \le i \le h-1$. Since $\alpha_k^i \ge \alpha_k^A \wedge \alpha_k^B$, we have

$$\int_{S_{i+1}} \frac{W(\alpha_k^i)}{\varepsilon_k}\,\mathrm{d}x \le \int_{S_{i+1}} \frac{W(\alpha_k^A \wedge \alpha_k^B)}{\varepsilon_k}\,\mathrm{d}x \le \mathcal{F}_k(u_k^A, \alpha_k^A; S_{i+1}) + \mathcal{F}_k(u_k^B, \alpha_k^B; S_{i+1}).$$

Moreover

$$\begin{aligned}
&\int_{S_{i+1}} |\nabla \alpha_k^i|^2\,\mathrm{d}x \\
&\quad = \int_{S_{i+1}} |\nabla \varphi_{i+1}((\alpha_k^A \wedge \alpha_k^B) - \alpha_k^B) + \varphi_{i+1}\nabla(\alpha_k^A \wedge \alpha_k^B) + (1-\varphi_{i+1})\nabla \alpha_k^B|^2\,\mathrm{d}x \\
&\quad \le \int_{S_{i+1}} 2\|\nabla \varphi_{i+1}\|^2_{L^\infty(\Omega)}|(\alpha_k^A \wedge \alpha_k^B) - \alpha_k^B)|^2 + 2|\nabla(\alpha_k^A \wedge \alpha_k^B)|^2 + 2|\nabla \alpha_k^B|^2\,\mathrm{d}x \\
&\quad \le \frac{c(h+1)^2}{d^2} \int_{S_{i+1}} |\alpha_k^A - \alpha_k^B|^2\,\mathrm{d}x + c\int_{S_{i+1}} |\nabla \alpha_k^A|^2\,\mathrm{d}x + c\int_{S_{i+1}} |\nabla \alpha_k^B|^2\,\mathrm{d}x
\end{aligned}$$

and hence, using the fact that $\alpha_k^i \le \alpha_k^B$ (and $\widetilde{\alpha}_k^i \le \widetilde{\alpha}_k^B$) in S_{i+1} and the monotonicity of σ_{Y} and of f with respect to the first variable, we get

$$\begin{aligned}
\mathcal{F}_k(u_k^i, \alpha_k^i; S_{i+1}) &= \int_{S_{i+1}} f(\alpha_k^i, |\nabla u_k^i|)\,\mathrm{d}x + \int_{S_{i+1}} \sigma_{\mathrm{Y}}(\widetilde{\alpha}_k^i)\,\mathrm{d}|D^s u_k^i| + \mathcal{W}(\alpha_k^i; S_{i+1}) \\
&\le \int_{S_{i+1}} f(\alpha_k^B, |\nabla u_k^B|)\,\mathrm{d}x + \int_{S_{i+1}} \sigma_{\mathrm{Y}}(\widetilde{\alpha}_k^B)\,\mathrm{d}|D^s u_k^B| + \mathcal{W}(\alpha_k^i; S_{i+1}) \\
&\le c\big[\mathcal{F}_k(u_k^A, \alpha_k^A; S_{i+1}) + \mathcal{F}_k(u_k^B, \alpha_k^B; S_{i+1})\big] \\
&\qquad + \frac{c(h+1)^2}{d^2}\varepsilon_k \int_{S_{i+1}} |\alpha_k^A - \alpha_k^B|^2\,\mathrm{d}x\,.
\end{aligned} \tag{2.78}$$

In the same way, we estimate

$$\begin{aligned}
\mathcal{F}_k(u_k^i, \alpha_k^i; S_{i-1}) \le c\big[&\mathcal{F}_k(u_k^A, \alpha_k^A; S_{i-1}) + \mathcal{F}_k(u_k^B, \alpha_k^B; S_{i-1})\big] \\
&+ \frac{c(h+1)^2}{d^2}\varepsilon_k \int_{S_{i-1}} |\alpha_k^A - \alpha_k^B|^2\,\mathrm{d}x.
\end{aligned} \tag{2.79}$$

It remains to bound $\mathcal{F}_k(u_k^i, \alpha_k^i; S_i)$. This time we use the fact that in S_i we have

$$Du_k^i = \nabla \varphi_i (u_k^A - u_k^B) + \varphi_i D u_k^A + (1-\varphi_i) D u_k^B,$$

from which it follows that

$$\begin{aligned}
\nabla u_k^i &= \nabla \varphi_i (u_k^A - u_k^B) + \varphi_i \nabla u_k^A + (1-\varphi_i)\nabla u_k^B, \\
D^s u_k^i &= \varphi_i D^s u_k^A + (1-\varphi_i) D^s u_k^B.
\end{aligned}$$

Using the convexity of f with respect to the second variable and (2.14), this implies that

$$\begin{aligned}\mathcal{F}_k(u_k^i,\alpha_k^i;S_i) &= \int_{S_i} f(\alpha_k^i,|\nabla u_k^i|)\,\mathrm{d}x + \int_{S_i}\sigma_{\mathrm{Y}}(\widetilde{\alpha}_k^i)\,\mathrm{d}|D^s u_k^i| + \mathcal{W}(\alpha_k^i;S_i)\\ &\le \int_{S_i} 2f(\alpha_k^A\wedge\alpha_k^B,|\nabla\varphi_i(u_k^A-u_k^B)|)\,\mathrm{d}x + \int_{S_i}2f(\alpha_k^A,|\nabla u_k^A|)\,\mathrm{d}x\\ &\quad + \int_{S_i}2f(\alpha_k^B,|\nabla u_k^B|)\,\mathrm{d}x + \int_{S_i}\sigma_{\mathrm{Y}}(\widetilde{\alpha}_k^A)\,\mathrm{d}|D^s u_k^A| + \int_{S_i}\sigma_{\mathrm{Y}}(\widetilde{\alpha}_k^B)\,\mathrm{d}|D^s u_k^B|\\ &\quad + \mathcal{W}(\alpha_k^A;S_i)+\mathcal{W}(\alpha_k^B;S_i)\\ &\le c\big[F_k(u_k^A,\alpha_k^A;S_i)+F_k(u_k^B,\alpha_k^B;S_i)\big] + \frac{c(h+1)}{d}\int_{S_i}|u_k^A-u_k^B|\,\mathrm{d}x\,,\end{aligned} \tag{2.80}$$

where we used (2.13). Summing (2.77)–(2.80), we obtain

$$\begin{aligned}\mathcal{F}_k(u_k^i,\alpha_k^i;A'\cup B) &\le \mathcal{F}_k(u_k^A,\alpha_k^A;A)+\mathcal{F}_k(u_k^B,\alpha_k^B;B)\\ &+ c\big[\mathcal{F}_k(u_k^A,\alpha_k^A;B\cap(A_{i+2}\setminus A_{i-1}))+\mathcal{F}_k(u_k^B,\alpha_k^B;B\cap(A_{i+2}\setminus A_{i-1}))\big]\\ &+ \frac{c(h+1)^2}{d^2}\varepsilon_k\int_{B\cap(A_{i+2}\setminus A_{i-1})}|\alpha_k^A-\alpha_k^B|^2\,\mathrm{d}x + \frac{c(h+1)}{d}\int_{B\cap(A_{i+2}\setminus A_{i-1})}|u_k^A-u_k^B|\,\mathrm{d}x.\end{aligned}$$

Now, summing on i between 1 and $h-1$ and taking the average, we obtain that for every k there exists an index i_k such that

$$\begin{aligned}\mathcal{F}_k(u_k^{i_k},\alpha_k^{i_k};A'\cup B) &\le \mathcal{F}_k(u_k^A,\alpha_k^A;A)+\mathcal{F}_k(u_k^B,\alpha_k^B;B)\\ &+ \frac{c}{h-1}\big[\mathcal{F}_k(u_k^A,\alpha_k^A;B\cap(A\setminus A'))+\mathcal{F}_k(u_k^B,\alpha_k^B;B\cap(A\setminus A'))\big]\\ &+ \frac{c(h+1)^2}{d^2(h-1)}\varepsilon_k\int_{B\cap(A\setminus A')}|\alpha_k^A-\alpha_k^B|^2\,\mathrm{d}x + \frac{c(h+1)}{d(h-1)}\int_{B\cap(A\setminus A')}|u_k^A-u_k^B|\,\mathrm{d}x.\end{aligned}$$

We conclude by letting $k\to+\infty$ and then $h\to+\infty$. □

Proposition 2.4.4. *Let $\mathcal{F}_{k_j}$ be a subsequence of $\mathcal{F}_k$ $\overline{\Gamma}$-converging to some functional $\hat{\mathcal{F}} : L^1(\Omega)\times L^1(\Omega)\times\mathscr{A}(\Omega)\to[0,+\infty]$. Then for every $u\in BV(\Omega)$ the set function $\hat{\mathcal{F}}(u,1;\cdot)$ is the restriction to open sets of a Radon measure on Ω. Moreover, $\hat{\mathcal{F}}$ is local, i.e., for every open set $A\subset\Omega$ we have $\hat{\mathcal{F}}(u,1;A)=\hat{\mathcal{F}}(v,1;A)$ if $u=v$ a.e. in A.*

Proof. We have already observed that $\hat{\mathcal{F}}(u,1;\cdot)$ is increasing, inner regular and superadditive. Subadditivity follows from Lemma 2.4.3, taking inner regularity into account. We can now apply an extension theorem (see [42] and [29, Theorem 14.23]) to construct a Borel measure which coincides with $\hat{\mathcal{F}}(u,1;\cdot)$ on all open sets. This measure is bounded thanks to Proposition 2.4.2. The locality of $\hat{\mathcal{F}}$ is trivial. □

Proposition 2.4.5. *For every $u\in L^1(\Omega)$ we have $\mathcal{F}''(u,1)\le\mathcal{F}_0(u,1)$.*

Proof. Let us fix a subsequence of $\mathcal{F}_k$, which we do not relabel, $\overline{\Gamma}$-converging to some functional $\hat{\mathcal{F}} : L^1(\Omega)\times L^1(\Omega)\times\mathscr{A}(\Omega) \to [0,+\infty]$. By Proposition 2.4.4, for every $u \in BV(\Omega)$ the set function $\hat{\mathcal{F}}(u,1;\cdot)$ is the restriction to open sets of a Radon measure on Ω. We notice that $\hat{\mathcal{F}}$ coincides with the Γ-limit of the sequence $\mathcal{F}_k$ on the space $BV(\Omega)$. Indeed, let $u \in BV(\Omega)$ and $A \in \mathscr{A}(\Omega)$. Given $\varepsilon > 0$, we can find a compact set $K \subset A$ such that $C|Du|(A \setminus K) < \varepsilon$. Let us choose $A', A'' \in \mathscr{A}(\Omega)$ such that $K \subset A' \Subset A'' \Subset A$. Then, by Lemma 2.4.3 and Proposition 2.4.2, we have

$$\begin{aligned}\mathcal{F}''(u,1;A) = \mathcal{F}''(u,1;A' \cup (A\setminus K)) &\le \mathcal{F}''(u,1;A'') + \mathcal{F}''(u,1;A\setminus K)\\ &\le \hat{\mathcal{F}}(u,1;A) + C|Du|(A\setminus K) \le \hat{\mathcal{F}}(u,1;A) + \varepsilon\,.\end{aligned}$$

Letting $\varepsilon \to 0$, we conclude that $\mathcal{F}'(u,1;A) = \mathcal{F}''(u,1;A) = \hat{\mathcal{F}}(u,1;A)$.

Let us now prove that for every $u \in BV(\Omega)$ we have $\hat{\mathcal{F}}(u,1) \le \mathcal{F}(u)$. Let us define the functional $\mathcal{G}\colon BV(\Omega)\times\mathscr{A}(\Omega) \to [0,+\infty)$ by

$$\mathcal{G}(u;A) := \hat{\mathcal{F}}(u,1;A)\,.$$

The functional $\mathcal{G}$ satisfies the following properties for every $u \in BV(\Omega)$ and for every $A \in \mathscr{A}(\Omega)$:

(a) $\mathcal{G}(\cdot;A)$ is L^1-lower semicontinuous on $BV(\Omega)$;

(b) $\mathcal{G}$ is local;

(c) $\mathcal{G}(u;A) = \hat{\mathcal{F}}(u,1;A) \le C|Du|(A)$;

(d) $\mathcal{G}(u;\cdot)$ is the restriction to open sets of a Radon measure;

(e) $\mathcal{G}(u(\cdot - z) + b; z + A) = \mathcal{G}(u;A)$ for all $b \in \mathbb{R}$ and $z \in \mathbb{R}^n$ such that $z + A \subset \Omega$.

We now want to apply Theorem 1.4.4, which requires also an estimate from below. To this aim, for every $\lambda > 0$ we consider the functional

$$\mathcal{G}_\lambda(u;A) := \mathcal{G}(u;A) + \lambda|Du|(A)\,.$$

By Theorem 1.4.4, there exist three Borel functions $\overline{f}_\lambda\colon \mathbb{R}^n \to [0,+\infty)$, $\overline{h}_\lambda\colon \mathbb{R}^n \to [0,+\infty)$, and $\overline{g}_\lambda\colon \mathbb{R}\times\mathbb{S}^{n-1} \to [0,+\infty)$ such that

$$\mathcal{G}_\lambda(u;A) = \int_A \overline{f}_\lambda(\nabla u)\,\mathrm{d}x + \int_A \overline{h}_\lambda\Big(\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\,\mathrm{d}|D^c u| + \int_{J_u\cap A} \overline{g}_\lambda([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}$$

for every $u \in BV(\Omega)$ and for every $A \in \mathscr{A}(\Omega)$.

By (2.76), we have that

$$\mathcal{G}_\lambda(u;A) \le \int_A (f(1,|\nabla u|)+\lambda|\nabla u|)\,\mathrm{d}x + (\sigma_{\mathrm{Y}}(1)+\lambda)|D^c u|(A) + \int_{A\cap J_u} (\sigma_{\mathrm{Y}}(1)+\lambda)|[u]|\,\mathrm{d}\mathcal{H}^{n-1},$$

from which it follows in particular that

$$\overline{f}_\lambda(\xi) \le f(1,|\xi|) + \lambda|\xi|, \quad \overline{h}_\lambda(\xi) \le \sigma_{\mathrm{Y}}(1)|\xi| + \lambda|\xi|\,, \tag{2.81}$$

for every $\xi \in \mathbb{R}^n$.

As for the surface term, by Remark 1.4.5 we have that

$$\begin{aligned}\overline{g}_\lambda(a,\nu) &= \lim_{\rho\to 0^+} \frac{\inf\{\mathcal{G}_\lambda(v;Q^\nu_\rho) : v\in BV(Q^\nu_\rho),\ v=u_{\nu,a} \text{ on } \partial Q^\nu_\rho\}}{\rho^{n-1}} \\ &\le \limsup_{\rho\to 0^+} \frac{\mathcal{G}_\lambda(u_{\nu,a};Q^\nu_\rho)}{\rho^{n-1}} \le \limsup_{\rho\to 0^+} \frac{\mathcal{G}(u_{\nu,a};Q^\nu_\rho)}{\rho^{n-1}} + \lambda|a|\,.\end{aligned}$$

We claim that

$$\limsup_{\rho\to 0^+} \frac{\mathcal{G}(u_{\nu,a};Q^\nu_\rho)}{\rho^{n-1}} \le g(|a|)\,. \tag{2.82}$$

This will conclude the proof of the Γ-limsup inequality when $u\in BV(\Omega)$. Indeed, combining (2.81) and (2.82), we obtain that

$$\mathcal{G}(u;\Omega) \le \mathcal{G}_\lambda(u;\Omega) \le \mathcal{F}(u) + \lambda|Du|(\Omega)$$

and the result follows by letting $\lambda\to 0^+$.

To prove (2.82), we construct a suitable approximating sequence. Without loss of generality, let us assume that $\nu = e_n$, so that Q^ν_ρ is the cube Q_ρ of side ρ centred at the origin with faces orthogonal to the axes. The corresponding function $u_{\nu,a}$ will be denoted simply by u_a. Let τ_k, ζ_k, ς_k, and ψ_k be as in the construction in the one-dimensional case, i.e., as in (2.56)–(2.58). Let

$$\begin{aligned}A_k &:= \{x_n = 0\}\times(-\varsigma_k,\varsigma_k)\,,\\ B_k &:= \{x_n = 0\}\times((-\varsigma_k-\zeta_k,-\varsigma_k)\cup(\varsigma_k,\varsigma_k+\zeta_k))\,.\end{aligned}$$

We define u_k as u_a outside A_k, and by linking linearly the values 0 and a inside A_k. Let $\alpha_0\in[0,1]$ be such that

$$g(|a|) = \begin{cases} \sigma_{\mathrm{Y}}(\alpha_0)|a| + \kappa_W(\alpha_0) & \text{if } \alpha_0 > 0\,,\\ \kappa_W(0) & \text{if } \alpha_0 = 0\,.\end{cases}$$

If $\alpha_0 = 1$, we simply put $\alpha_k = 1$. Otherwise, let

$$\alpha_k(x',x_n) := \begin{cases} 1-\tau_k & \text{if } |x_n| \ge \zeta_k+\varsigma_k\,,\\ \psi_k(|x_n|-\varsigma_k) & \text{if } t\in(-\varsigma_k-\zeta_k,-\varsigma_k)\cup(\varsigma_k,\varsigma_k+\zeta_k)\,,\\ \alpha_0+\delta_k & \text{if } |x_n|\le\varsigma_k\,,\end{cases}$$

where $x' = (x_1,\dots,x_{n-1})$.

In the case $0<\alpha_0<1$, we define $e_k = 0$ and

$$p_k := \begin{cases} 0 & \text{in } Q_\rho\setminus A_k\,,\\ \nabla u_k\mathcal{L}^n & \text{in } Q_\rho\cap A_k\,.\end{cases}$$

Let us estimate all the terms in $\mathcal{F}_k(u_k,\alpha_k;Q_\rho)$:

$$\int_{Q_\rho\cap A_k} \sigma_{\mathrm{Y}}(\alpha_k)\,\mathrm{d}|p_k| = \sigma_{\mathrm{Y}}(\alpha_0+\delta_k)\int_{Q_\rho\cap A_k} |\nabla u_k|\,\mathrm{d}x = \sigma_{\mathrm{Y}}(\alpha_0+\delta_k)|a|\rho^{n-1},$$

$$\int_{Q_\rho\setminus(A_k\cup B_k)} \Big[\frac{W(\alpha_k)}{\varepsilon_k}+\varepsilon_k|\nabla\alpha_k|^2\Big]\,\mathrm{d}x \le \frac{W(1-\tau_k)}{\varepsilon_k}\rho^n,$$

$$\int_{Q_\rho\cap A_k} \Big[\frac{W(\alpha_k)}{\varepsilon_k}+\varepsilon_k|\nabla\alpha_k|^2\Big]\,\mathrm{d}x = \int_{Q_\rho\cap A_k} \frac{W(\alpha_0+\delta_k)}{\varepsilon_k}\,\mathrm{d}x = \frac{2\varsigma_k}{\varepsilon_k}W(\alpha_0+\delta_k)\rho^{n-1},$$

$$\int_{Q_\rho\cap B_k} \Big[\frac{W(\alpha_k)}{\varepsilon_k}+\varepsilon_k|\nabla\alpha_k|^2\Big]\,\mathrm{d}x = \int_{Q'_\rho}\int_0^{\zeta_k} 4\sqrt{W(\psi_k(t))}|\psi'_k(t)|\,\mathrm{d}t\,\mathrm{d}x' = 4\rho^{n-1}\int_{\alpha_0+\delta_k}^{1-\tau_k} \sqrt{W(s)}\,\mathrm{d}s\,,$$

where Q'_ρ is the corresponding cube in $\mathbb{R}^{n-1}$. Summing all the above inequalities and letting $k\to+\infty$, we obtain

$$\begin{aligned}\mathcal{G}(u_a;Q_\rho) = \hat{\mathcal{F}}(u_a,1;Q_\rho) &\le \limsup_{k\to+\infty}\mathcal{F}_k(u_k,\alpha_k;Q_\rho)\\ &\le (\sigma_{\mathrm{Y}}(\alpha_0)|a|+\kappa_W(\alpha_0))\rho^{n-1} = g(|a|)\rho^{n-1},\end{aligned}$$

from which (2.82) follows.

In the case $\alpha_0=0$, let $p_k=0$ and

$$e_k(x) := \begin{cases} 0 & \text{if } x\in Q_\rho\setminus A_k\,,\\ \nabla u_k(x) & \text{if } x\in Q_\rho\cap A_k\,,\end{cases}$$

With this choice,

$$\int_{Q_\rho\cap A_k} \alpha_k|e_k|^2\,\mathrm{d}x \le \int_{Q_\rho\cap A_k} \delta_k|\nabla u_k|^2\,\mathrm{d}x = \frac{\delta_k}{2\varsigma_k}|a|^2\rho^{n-1}$$

and therefore

$$\mathcal{G}(u_a;Q_\rho) = \hat{\mathcal{F}}(u_a,1;Q_\rho) \le \limsup_{k\to+\infty}\mathcal{F}_k(u_k,1;Q_\rho) \le \kappa_W(0)\rho^{n-1} = g(|a|)\rho^{n-1},$$

from which (2.82) follows.

We have proved that $\mathcal{F}''(u,1) = \hat{\mathcal{F}}(u,1) \le \mathcal{F}(u)$ for all $u\in BV(\Omega)$. Assume now that $u\in GBV(\Omega)$. For every M we consider the truncated functions $u_M := (-M\vee u)\wedge M\in BV_{loc}(\Omega)$. We want to prove that

$$\mathcal{F}''(u_M,1)\le\mathcal{F}(u_M)\,. \tag{2.83}$$

It is not restrictive to assume that $\mathcal{F}(u_M)<+\infty$. From (2.13) and (2.19), we obtain

$$\int_\Omega|\nabla u_M|\,\mathrm{d}x + |D^c u_M|(\Omega) + \int_{J_{u_M}\setminus J^1_{u_\lambda}} |[u_M]|\,\mathrm{d}\mathcal{H}^{n-1} + \mathcal{H}^{n-1}(J^1_{u_M}) < +\infty\,, \tag{2.84}$$

where $J^1_{u_M} := \{|[u_M]| \geq 1\}$. Since $||u_M||_{L^\infty(\Omega)} \leq M$, we have

$$|Du_M|(J_{u_M}) \leq \int_{J_{u_M}\setminus J^1_{u_M}} |[u_M]|\, \mathrm{d}\mathcal{H}^{n-1} + 2M\mathcal{H}^{n-1}(J^1_{u_M}),$$

so that (2.84) implies $|Du_M|(\Omega) < +\infty$. Therefore $u_M \in BV(\Omega)$ and (2.83) follows from the previous step of the proof. Letting $M \to +\infty$, we obtain $\mathcal{F}''(u,1) \leq \mathcal{F}(u)$, thanks to the lower semicontinuity of $\mathcal{F}''(\cdot,1)$. □

2.5 Asymptotic behaviour of minimisers

In this section we study the convergence of η_ε-minimisers of problem (2.9) with Dirichlet boundary conditions. To this aim, for every $w \in L^\infty(\partial_D\Omega)$ we introduce the functionals $\mathcal{F}^w_k$, $\mathcal{F}^w_0$ defined on the space $L^1(\Omega)\times L^1(\Omega)$ by

$$\mathcal{F}^w_k(u,\alpha) := \begin{cases} \mathcal{F}_k(u,\alpha) & \text{if } u = w \text{ and } \alpha = 1 \ \ \mathcal{H}^{n-1}\text{-a.e. on } \partial_D\Omega\,, \\ +\infty & \text{otherwise}\,, \end{cases} \tag{2.85}$$

$$\mathcal{F}^w_0(u,\alpha) := \begin{cases} \mathcal{F}(u) + \displaystyle\int_{\partial_D\Omega} g(|u-w|)\, \mathrm{d}\mathcal{H}^{n-1} & \text{if } u \in GBV(\Omega) \text{ and} \\ & \alpha = 1 \ \mathcal{L}^n\text{-a.e. in } \Omega\,, \\ +\infty & \text{otherwise}\,. \end{cases} \tag{2.86}$$

We begin by proving the following result.

Theorem 2.5.1. *Let $w \in L^\infty(\partial_D\Omega)$. Then the functionals $\mathcal{F}^w_k$ Γ-converge to $\mathcal{F}^w_0$, as $k \to +\infty$ in $L^1(\Omega)\times L^1(\Omega)$.*

Proof. Let us prove the Γ-liminf inequality. Given a sequence (u_k,α_k) converging to $(u,1)$ in $L^1(\Omega)\times L^1(\Omega)$, we want to show that

$$\mathcal{F}^w_0(u,1) \leq \liminf_{k\to+\infty} \mathcal{F}^w_k(u_k,\alpha_k)\,, \tag{2.87}$$

where $\mathcal{F}^w_0$ is defined by (2.86). By Gagliardo's Theorem (see [52, Theorem 2.16]), there exists a function $v \in W^{1,1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ whose trace on $\partial_D\Omega$ coincides with w. We can assume that the liminf is finite and it is actually a limit, hence $u_k \in BV(\Omega)$ with $u_k = w$ $\mathcal{H}^{n-1}$-a.e. on $\partial_D\Omega$, and $\alpha_k \in H^1(\Omega)$ with $\delta_k \leq \alpha_k \leq 1$ and $\alpha_k = 1$ $\mathcal{H}^{n-1}$-a.e. on $\partial_D\Omega$. Since $\partial_D\Omega$ is relatively open in $\partial\Omega$, there exists a bounded open set $U \subset \mathbb{R}^n$ such that $\partial_D\Omega = U \cap \partial\Omega$. Let $\widetilde{\Omega} := \Omega \cup U$. We can extend the functions u_k and α_k to $\widetilde{\Omega}$ by putting $u_k := v$ and $\alpha_k := 1$ in $U \setminus \Omega$, respectively. Moreover, we extend u to $\widetilde{\Omega}$ by defining $u := v$ in $U \setminus \Omega$. Since $(u_k,\alpha_k) \to (u,1)$ in $L^1(\widetilde{\Omega})\times L^1(\widetilde{\Omega})$ and the functionals $\mathcal{F}_k(\cdot,\cdot;\widetilde{\Omega})$ Γ-converge to $\mathcal{F}_0(\cdot,\cdot;\widetilde{\Omega})$ by Theorem 2.1.1 (applied to $\widetilde{\Omega}$), we have that $u \in GBV(\widetilde{\Omega})$ and

$$\mathcal{F}(u;\widetilde{\Omega}) \leq \liminf_{k\to+\infty} \mathcal{F}_k(u_k,\alpha_k;\widetilde{\Omega})\,.$$

On the other hand

$$\mathcal{F}(u;\widetilde{\Omega}) = \mathcal{F}(u;\Omega) + \int_{\partial_D\Omega} g(|u-w|)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{U\setminus\Omega} f(1,|\nabla v|)\,\mathrm{d}x\,,$$

$$\mathcal{F}_k(u_k,\alpha_k;\widetilde{\Omega}) = \mathcal{F}_k(u_k,\alpha_k;\Omega) + \int_{U\setminus\Omega} f(1,|\nabla v|)\,\mathrm{d}x\,,$$

and therefore

$$\mathcal{F}(u;\Omega) + \int_{\partial_D\Omega} g(|u-w|)\,\mathrm{d}\mathcal{H}^{n-1} \le \liminf_{k\to+\infty} \mathcal{F}_k(u_k,\alpha_k;\Omega)\,.$$

This concludes the proof of (2.87).

To prove the Γ-limsup inequality, it is enough to consider the case $u \in BV(\Omega)$. Indeed, if $u \in GBV(\Omega)$, we can argue by approximation as in the proof of Proposition 2.4.5. We now construct a sequence (u_k,α_k) converging to $(u,1)$ in $L^1(\Omega)\times L^1(\Omega)$ and satisfying the boundary conditions $u_k = w$, $\alpha_k = 1$ $\mathcal{H}^{n-1}$-a.e on $\partial_D\Omega$. We extend the function w to the whole boundary $\partial\Omega$ by putting w equal to the trace of u on $\partial\Omega\setminus\partial_D\Omega$. By [52, Theorem 2.16], there exists a function $v\in W^{1,1}(\mathbb{R}^n)$ whose trace on $\partial\Omega$ is w. By [36, Proposition 1.2], for every $\eta>0$ it is possible to find a $\mathcal{C}^\infty$ function $r_\eta:\mathbb{R}^n\to\mathbb{R}^n$ such that $r_\eta(\overline{\Omega})\subset\Omega$, $r_\eta - Id$ has compact support, and $r_\eta - Id\to 0$ in $\mathcal{C}^\infty_c(\mathbb{R}^n;\mathbb{R}^n)$ as $\eta\to 0$, where Id is the identity map. Let us fix $\eta>0$ and let us consider the function u_η defined by

$$u_\eta(x) := \begin{cases} u(x) & \text{if } x\in\Omega_\eta := r_\eta(\Omega)\,,\\ v(x) & \text{if } x\in\Omega\setminus\Omega_\eta\,.\end{cases}$$

Let us fix $\hat{\Omega}$ such that $\Omega_\eta \Subset \hat{\Omega} \Subset \Omega$. By Proposition 2.4.5, there exists a recovery sequence $(\hat{u}_k,\hat{\alpha}_k)\to(u_\eta,1)$ in $L^1(\hat{\Omega})\times L^1(\hat{\Omega})$ such that

$$\mathcal{F}(u_\eta;\hat{\Omega}) = \limsup_{k\to+\infty}\mathcal{F}_k(\hat{u}_k,\hat{\alpha}_k;\hat{\Omega})\,.$$

We now modify the sequence $(\hat{u}_k,\hat{\alpha}_k)$ using the De Giorgi slicing and averaging argument in such a way that the boundary conditions are satisfied. Let $d := \mathrm{dist}(\Omega_\eta,\partial\hat{\Omega})$. As in the proof of Lemma 2.4.3, we consider a finite chain of open sets $\Omega_\eta = A_0 \Subset A_1 \Subset \ldots \Subset A_h \Subset A_{h+1} = \hat{\Omega}$ such that $\mathrm{dist}(A_i,\partial A_{i+1}) \ge d/(h+1)$. Then we consider $\varphi_i\in\mathcal{C}^1_c(\mathbb{R}^n)$ such that $0\le\varphi_i\le 1$, $\mathrm{supp}(\varphi_i)\subset A_{i+1}$, $\varphi_i = 1$ on an open neighborhood of $\overline{A}_i$ and $||\nabla\varphi_i||_{L^\infty(\Omega)} \le 2(h+1)/d$ and we define

$$u_k^i := \varphi_i\hat{u}_k + (1-\varphi_i)v,\quad \alpha_k^i := \varphi_{i+1}\hat{\alpha}_k + (1-\varphi_{i+1})\,.$$

We have that $u_k^i = w$ and $\alpha_k^i = 1$ $\mathcal{H}^{n-1}$-a.e. on $\partial_D\Omega$. With computations similar to those made in the proof of Lemma 2.4.3, it is possible to deduce the following estimate

$$\begin{aligned}\mathcal{F}_k(u_k^i,\alpha_k^i;\Omega) \le{}& \mathcal{F}_k(\hat{u}_k,\hat{\alpha}_k;\hat{\Omega}) + \mathcal{F}_k(v,1;\Omega\setminus\Omega_\eta)\\ &+ c[\mathcal{F}_k(\hat{u}_k,\hat{\alpha}_k;A_{i+2}\setminus A_i) + \mathcal{F}_k(v,1;A_{i+2}\setminus A_i)]\\ &+ \frac{c(h+1)}{d}\int_{A_{i+2}\setminus A_i}|\hat{u}_k - v|\,\mathrm{d}x + \frac{c(h+1)^2}{d^2}\varepsilon_k\int_{A_{i+2}\setminus A_i}|\hat{\alpha}_k - 1|^2\,\mathrm{d}x\end{aligned}$$

for every $i \in \{0, \dots, h-1\}$, and therefore, by taking averages, there exists $i_k \in \{0, \dots, h-1\}$ such that

$$\begin{aligned}\mathcal{F}_k(u_k^{i_k}, \alpha_k^{i_k}; \Omega) \leq \mathcal{F}_k(\hat{u}_k, \hat{\alpha}_k; \hat{\Omega}) + \mathcal{F}_k(v, 1; \Omega \setminus \Omega_\eta) \\ + \frac{c}{h}[\mathcal{F}_k(\hat{u}_k, \hat{\alpha}_k; \hat{\Omega} \setminus \Omega_\eta) + \mathcal{F}_k(v, 1; \hat{\Omega} \setminus \Omega_\eta)] \\ + \frac{c(h+1)}{dh} \int_{\hat{\Omega}\setminus\Omega_\eta} |\hat{u}_k - v| \, \mathrm{d}x + \frac{c(h+1)^2}{d^2 h} \varepsilon_k \int_{\hat{\Omega}\setminus\Omega_\eta} |\hat{\alpha}_k - 1|^2 \, \mathrm{d}x \,.\end{aligned}$$

Letting $k \to +\infty$ and then $h \to +\infty$, we obtain

$$\limsup_{k\to+\infty} \mathcal{F}_k(u_k^{i_k}, \alpha_k^{i_k}; \Omega) \leq \mathcal{F}(u_\eta; \hat{\Omega}) + \mathcal{F}(v; \Omega \setminus \Omega_\eta) \,.$$

By the arbitrariness of $\hat{\Omega}$, we have

$$(\Gamma\text{-}\limsup_{k\to+\infty} \mathcal{F}_k^w)(u_\eta, 1) \leq \limsup_{k\to+\infty} \mathcal{F}_k(u_k^{i_k}, \alpha_k^{i_k}; \Omega) \leq \mathcal{F}(u_\eta; \overline{\Omega}_\eta) + \mathcal{F}(v; \Omega \setminus \Omega_\eta) = \mathcal{F}(u_\eta; \Omega) \,.$$

By the lower semicontinuity of the Γ-limsup, to conclude the proof it is enough to show that

$$\mathcal{F}(u_\eta; \Omega) \to \mathcal{F}(u; \Omega) + \int_{\partial_D\Omega} g(|u - w|) \, \mathrm{d}\mathcal{H}^{n-1} \quad \text{as } \eta \to 0 \,. \tag{2.88}$$

We observe that

$$\mathcal{F}(u_\eta; \Omega) = \mathcal{F}(u; \Omega_\eta) + \int_{\partial\Omega_\eta} g(|u_{\Omega_\eta} - v|) \, \mathrm{d}\mathcal{H}^{n-1} + \mathcal{F}(v; \Omega \setminus \Omega_\eta) \,,$$

where u_{Ω_η} is the trace on $\partial\Omega_\eta$ of $u|_{\Omega_\eta}$. Since $\mathcal{F}(v; \Omega\setminus\Omega_\eta) \to 0$ and $\mathcal{F}(u; \Omega_\eta) \to \mathcal{F}(u; \Omega)$, to prove (2.88) we only need to show that

$$\int_{\partial\Omega_\eta} g(|u_{\Omega_\eta} - v|) \, \mathrm{d}\mathcal{H}^{n-1} \to \int_{\partial\Omega} g(|u - w|) \, \mathrm{d}\mathcal{H}^{n-1} = \int_{\partial_D\Omega} g(|u - w|) \, \mathrm{d}\mathcal{H}^{n-1}. \tag{2.89}$$

By making the change of variables $z = r_\eta(x)$, we obtain

$$\int_{r_\eta(\partial\Omega)} g(|u_{\Omega_\eta}(z) - v(z)|) \, \mathrm{d}\mathcal{H}^{n-1}(z) = \int_{\partial\Omega} g(|(u_\eta^*)_\Omega(x) - v_\eta^*(x)|)(1 + \omega_\eta(x)) \, \mathrm{d}\mathcal{H}^{n-1}(x) \tag{2.90}$$

where $u_\eta^* := u \circ r_\eta$ and $v_\eta^* := v \circ r_\eta$. The term $(1 + \omega_\eta(x))$ is due to the Generalised Area Formula (see [7, Theorem 2.91]) and $\omega_\eta \to 0$ uniformly since r_η is converging to the identity map in $\mathcal{C}^\infty$. Since $v \in W^{1,1}(\mathbb{R}^n)$, it is easy to see that $v_\eta^* \to v$ in $L^1(\partial\Omega)$. To prove the same result for u_η^* we start by computing its total variation. If u is $\mathcal{C}^1$, we have

$$\begin{aligned}|Du_\eta^*|(\Omega) &= \int_\Omega |\nabla u_\eta^*(x)| \, \mathrm{d}x = \int_\Omega |\nabla u(r_\eta(x)) \nabla r_\eta(x)| \, \mathrm{d}x \\ &= \int_{r_\eta(\Omega)} |\nabla u(z) \nabla r_\eta(r_\eta^{-1}(z))| \frac{1}{|\det(\nabla r_\eta(r_\eta^{-1}(z)))|} \, \mathrm{d}z \\ &\leq (1 + \omega'_\eta) \int_{\Omega_\eta} |\nabla u| \, \mathrm{d}z \leq (1 + \omega'_\eta) |Du|(\Omega_\eta) \,,\end{aligned} \tag{2.91}$$

with $\omega'_\eta \to 0$. By approximation we obtain that (2.91) holds for an arbitrary $u \in BV(\Omega)$. Formula (2.91) in particular implies that

$$\limsup_{\eta\to 0} |Du^*_\eta|(\Omega) \le |Du|(\Omega)\,.$$

From the convergence $u^*_\eta \to u$ in $L^1(\Omega)$, we conclude that $|Du^*_\eta|(\Omega) \to |Du|(\Omega)$. Since the trace is continuous with respect to this kind of convergence, we deduce that $(u^*_\eta)_\Omega \to u_\Omega$ in $L^1(\partial\Omega)$. Therefore we can pass to the limit in (2.90) and eventually obtain (2.89). This concludes the proof. □

Another ingredient in the proof of the convergence of η_ε-minimisers with Dirichlet boundary conditions is the following compactness result.

Theorem 2.5.2. *Let $M, c > 0$ and let $(u_k, \alpha_k) \in BV(\Omega)\times H^1(\Omega)$. Assume that $||u_k||_{L^\infty(\Omega)} \le M$ and*

$$\mathcal{F}_k(u_k, \alpha_k) \le c\,.$$

Then $\alpha_k \to 1$ in $L^1(\Omega)$ and there exists a subsequence of u_k and a function $u \in BV(\Omega)$ such that $u_k \to u$ in $L^1(\Omega)$.

Proof. Let us start with the proof of the theorem in the case $n = 1$. As in the proof of Lemma 2.3.3, we extract a subsequence from α_k such that α_k $\Gamma(\mathbb{R})$-converges to some function α, and we consider the set $\{\alpha = 0\}$, which is finite by Remark 2.3.1. Let A_j, $j \ge 1$, be open sets as in (2.41). By repeating the proof of Lemma 2.3.3, we obtain that the sequence u_k is bounded in $BV(A_j)$, uniformly with respect to k and j. Therefore, by a diagonal argument, it is possible to extract a subsequence from u_k converging to some $u \in L^1(\Omega)$ strongly in $L^1(\Omega)$. Moreover $u \in BV(\Omega)$.

To prove the theorem in the case $n > 1$, we make use of [2, Theorem 6.6] to reduce the problem to the one dimensional case. In order to apply that result, we consider the family $\mathcal{U} = (u_k)$, which is by hypotheses equibounded in $L^\infty(\Omega)$. To prove that $\mathcal{U}$ is relatively compact in $L^1(\Omega)$, it suffices to prove that there exist n linearly independent vectors ξ satisfying the following property: for every $\eta > 0$, there exists an equibounded subset $\mathcal{U}_\eta$ of $L^\infty(\Omega)$ lying in a η-neighborhood of $\mathcal{U}$ with respect to the $L^1(\Omega)$ topology, and such that $(\mathcal{U}_\eta)^\xi_y := \{w^\xi_y : w \in \mathcal{U}_\eta\}$ is relatively compact in $L^1(\Omega^\xi_y)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$. To prove this, we fix $\xi \in \mathbb{R}^n$ and we consider the set

$$A_k := \{y \in \Pi^\xi : \mathcal{F}^{\xi,y}_k((u_k)^\xi_y, (\alpha_k)^\xi_y) \le L\}\,,$$

where $\mathcal{F}^{\xi,y}_k \colon BV(\Omega^\xi_y)\times H^1(\Omega^\xi_y) \to [0, +\infty]$ is the one-dimensional functional defined by

$$\mathcal{F}^{\xi,y}_k(u,\alpha) := \int_{\Omega^\xi_y} f_k(\alpha, |\nabla u|)\,\mathrm{d}t + \sigma_{\mathrm{Y}}(1)|D^s u|(\Omega^\xi_y) + \int_{\Omega^\xi_y} \Big[\frac{W(\alpha)}{\varepsilon_k} + \varepsilon_k|\nabla\alpha|^2\Big]\,\mathrm{d}t\,,$$

and L is a suitable constant that we will choose later. By the Chebyshev Inequality, we have

$$L\mathcal{H}^{n-1}(\Omega^\xi \setminus A_k) \le \int_{\Omega^\xi\setminus A_k} \mathcal{F}^{\xi,y}_k((u_k)^\xi_y, (\alpha_k)^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}(y) \le \mathcal{F}_k(u_k,\alpha_k) \le c\,,$$

where Ω^ξ is the projection of Ω on Π^ξ. Let us define the function w_k in such a way that

$$(w_k)^\xi_y := \begin{cases} (u_k)^\xi_y & \text{if } y \in A_k\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Letting $\mathcal{U}_\eta := (w_k)$, we have that $\mathcal{U}_\eta$ lies in a η-neighborhood of $\mathcal{U}$ for a suitable choice of L, since

$$||w_k - u_k||_{L^1(\Omega)} = \int\limits_{\Omega^\xi \setminus A_k} \int\limits_{\Omega^\xi_y} |(u_k)^\xi_y| \,\mathrm{d}t \,\mathrm{d}\mathcal{H}^{n-1}(y) \le \frac{c}{L} \mathrm{diam}(\Omega) M \le \eta\,,$$

if $L \ge \eta^{-1} c\, \mathrm{diam}(\Omega) M$. Moreover $(\mathcal{U}_\eta)^\xi_y$ is relatively compact in $L^1(\Omega^\xi_y)$ by the previous step. This proves that $\mathcal{U}$ is relatively compact and therefore there exists a subsequence of u_k converging to some $u \in L^1(\Omega)$. Following the proof of Proposition 2.4.1, we deduce that $u \in BV(\Omega)$. □

Proof of Theorem 2.1.2. The result is an immediate consequence of Theorem 2.5.1, Theorem 2.5.2, and of the Theorem 1.4.1. □

We conclude this section with an application in which the limit problem is actually defined on the space $GBV(\Omega)$ and not just on $BV(\Omega)$. We omit the proof, since it follows the arguments in [34] with obvious modifications.

Theorem 2.5.3. *Let $q > 1$ and let $\psi \in L^q(\Omega)$. For every $k \in \mathbb{N}$, let $(u_k, \alpha_k) \in BV(\Omega) \times H^1(\Omega)$ be a minimiser of the problem*

$$\min \Big\{ \mathcal{F}_k(u, \alpha) + \int\limits_\Omega |u - \psi|^q \,\mathrm{d}x : u \in BV(\Omega),\ \alpha \in H^1(\Omega),\ \delta_k \le \alpha \le 1 \Big\}\,.$$

Then $\alpha_k \to 1$ in $L^1(\Omega)$ and a subsequence of u_k converges in $L^q(\Omega)$ to a minimiser $u \in GBV(\Omega)$ of the problem

$$\min \Big\{ \mathcal{F}(u) + \int\limits_\Omega |u - \psi|^q \,\mathrm{d}x : u \in GBV(\Omega) \Big\}\,.$$

CHAPTER 3

SEMICONTINUITY OF A CLASS OF FUNCTIONALS DEFINED ON BD

3.1 Overview of the chapter

In this chapter we deal with the lower semicontinuity of functionals with linear growth defined on the space of functions of bounded deformation. The results presented here have been published in the work [38], in collaboration with Dal Maso and Toader.

After recalling some technical preliminaries useful for the sequel (Section 3.2), we delve into the analysis of functionals of the form

$$\int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} G([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}, \quad u \in BD(\Omega)\,, \tag{3.1}$$

where Ω is a bounded open set in $\mathbb{R}^n$ and $|\cdot|$ denotes the Euclidean (or Frobenius) norm defined by

$$|A| = \Big(\sum_{i,j=1}^n A_{ij}^2\Big)^{\frac{1}{2}},$$

for every $A = (A_{ij}) \in \mathbb{M}^{n\times n}_{\mathrm{sym}}$. In order to give an insight into the issues implied by the presence of the cohesive surface energy G, in Section 3.3 we present two examples of functionals defined on BD that are not lower semicontinuous. Specifically, we shall study the functionals $\mathcal{G}_1, \mathcal{G}_2 \colon BD(\Omega) \to [0,+\infty)$ defined by

$$\mathcal{G}_1(u) := \int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} g(|[u]|)\,\mathrm{d}\mathcal{H}^{n-1} \tag{3.2}$$

and

$$\mathcal{G}_2(u) := \int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} (C|[u]\odot\nu_u|)\wedge 1\,\mathrm{d}\mathcal{H}^{n-1}, \tag{3.3}$$

respectively. We shall see in Proposition 3.3.1 that the functional $\mathcal{G}_1$ cannot be lower semicontinuous, since the surface term in (3.2) does not take into account the orientation of the jump set J_u. On the other hand, the surface term in the functional given by (3.3) depends on the normal ν_u, but $\mathcal{G}_2$ fails to be lower semicontinuous because the 1-homogeneous extension of the function $\nu \mapsto (C|z\odot\nu|)\wedge 1$ is not convex on $\mathbb{R}^n$ (Proposition 3.3.2).

Section 3.4 contains the main result of the chapter. We prove the lower semicontinuity of the functional $\mathcal{F}\colon BD(\Omega)\to[0,+\infty)$ defined by

$$\mathcal{F}(u) := \int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} G([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}, \tag{3.4}$$

under the following assumptions:

(H1) $f\colon[0,+\infty)\to[0,+\infty)$ is a convex nondecreasing function;

(H2) there exists an even subadditive function $g\colon\mathbb{R}\to[0,+\infty)$ such that the function $G\colon\mathbb{R}^n\times\mathbb{S}^{n-1}\to[0,+\infty)$ can be written as

$$G(z,\nu) = \sup_{(\xi^1,\dots,\xi^n)} \Big(\sum_{i=1}^n g(z\cdot\xi^i)^2|\nu\cdot\xi^i|^2\Big)^{\frac{1}{2}} \quad \text{for every } z\in\mathbb{R}^n, \nu\in\mathbb{S}^{n-1}, \tag{3.5}$$

where the supremum is taken over all orthonormal bases $(\xi^1,\dots,\xi^n)$ of $\mathbb{R}^n$;

(H3) $0<C<+\infty$ and

$$\lim_{t\to+\infty}\frac{f(t)}{t} = \lim_{t\to0^+}\frac{g(t)}{t} = C\,, \quad \liminf_{t\to+\infty} g(t) > 0\,.$$

The lower semicontinuity result is applied in Section 3.5 to prove a relaxation theorem for functionals of the form

$$\mathcal{F}(u) = \int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} \psi([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}.$$

Under suitable assumptions on f, C, and ψ, the lower semicontinuous envelope $\mathrm{sc}^-\mathcal{F}$ takes the form

$$\mathrm{sc}^-\mathcal{F}(u) = \int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} \overline{\psi}([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1},$$

for a suitable function $\overline{\psi}$. In particular, the function f and the constant C do not change in the relaxation process.

Finally, in Section 3.6 we compute explicitly the function $G(z,\nu)$ given by (3.5) when $g(t)=\min\{|t|,1\}$ and $n=2$. In particular, we find that in this case $G(z,\nu)=|z\odot\nu|$ if $|z|\le1$, $G(z,\nu)=1$ if $|z|\ge\sqrt{2}$, while there is a region in the annulus $1<|z|<\sqrt{2}$ where $G(z,\nu)<\min\{|z\odot\nu|,1\}$.

3.2 Preliminary results

3.2.1 Characterisation of the Euclidean norm on matrices

We start by proving a formula for the Euclidean norm on matrices which will be useful for the slicing argument used in Section 3.4.

Proposition 3.2.1. *For every $n \times n$ symmetric matrix A we have*

$$|A| = \sup_{(\xi^1,\ldots,\xi^n)} \Big(\sum_{i=1}^n |A\xi^i \cdot \xi^i|^2 \Big)^{\frac{1}{2}}, \tag{3.6}$$

where the supremum is taken over all orthonormal bases $(\xi^1, \ldots, \xi^n)$ of $\mathbb{R}^n$.

Proof. Let A be a symmetric matrix, let $(\xi^1, \ldots, \xi^n)$ be an orthonormal basis and let $R \in \mathrm{O}(n)$ be a rotation such that $\xi^i = Re_i$, where $(e_1, \ldots, e_n)$ is the standard basis in $\mathbb{R}^n$. Then

$$\sum_{i=1}^n |A\xi^i \cdot \xi^i|^2 = \sum_{i=1}^n |ARe_i \cdot Re_i|^2 = \sum_{i=1}^n |R^T ARe_i \cdot e_i|^2 \leq |R^T AR|^2 = |A|^2.$$

To show that the supremum in (3.6) is attained, let $S \in \mathrm{O}(n)$ be a rotation such that $S^T AS$ is a diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$, and let $\zeta^i := Se_i$ for $i = 1, \ldots, n$. Then $(\zeta^1, \ldots, \zeta^n)$ is an orthonormal basis of $\mathbb{R}^n$ and we have that

$$\sum_{i=1}^n |A\zeta^i \cdot \zeta^i|^2 = \sum_{i=1}^n |ASe_i \cdot Se_i|^2 = \sum_{i=1}^n |S^T ASe_i \cdot e_i|^2 = \sum_{i=1}^n \lambda_i^2 = |S^T AS|^2 = |A|^2.$$

This concludes the proof. □

Remark 3.2.2. We note that taking the supremum over all orthonormal bases $(\xi^1, \ldots, \xi^n)$ of $\mathbb{R}^n$ is equivalent to taking the supremum over the columns of all rotations $R \in \mathrm{O}(n)$. Therefore the supremum in (3.6) does not change if we consider only a countable dense family in $\mathrm{O}(n)$.

3.2.2 Properties of even subadditive functions

We recall that a function $g \colon \mathbb{R} \to [0, +\infty)$ is subadditive if

$$g(s + t) \leq g(s) + g(t) \quad \text{for every } s, t \in \mathbb{R}.$$

Remark 3.2.3. It is known (see, for instance, [58, Theorem 16.3.3]) that for a subadditive function we have

$$\lim_{t \to 0^+} \frac{g(t)}{t} = \sup_{t>0} \frac{g(t)}{t}$$

provided the right-hand side is finite. Moreover, if g is even, subadditive, and the right-hand side of the previous formula is finite, then $g(0) = 0$ and g is continuous at 0, hence at every point of $\mathbb{R}$ (see, for instance, [58, Theorem 16.2.1]).

Remark 3.2.4. If g is a subadditive function satisfying

$$\lim_{t \to 0^+} \frac{g(t)}{t} = C \in (0, +\infty), \quad \liminf_{t \to +\infty} g(t) > 0, \tag{3.7}$$

and $0 < c < C$, then there exists a constant $b > 0$ such that

$$g(t) \geq \min\{ct, b\} \quad \text{for every } t \geq 0. \tag{3.8}$$

Indeed, the first assumption in (3.7) implies that there exists $\delta > 0$ such that $g(t) \geq ct$, for every $t \in [0, \delta]$. The second assumption in (3.7) implies that there exist $\eta > 0$ and $M > \delta$ such that $g(t) \geq \eta$ for every $t \geq M$. We claim that

$$\inf_{t \in [\delta, M]} g(t) > 0 . \tag{3.9}$$

To prove the claim, we fix an integer $n \geq \frac{M}{\delta}$ and a constant $\varepsilon > 0$ such that $n\varepsilon < \eta$. If (3.9) does not hold, then there exists $t \in [\delta, M]$ such that $g(t) < \varepsilon$. By subadditivity we have $g(nt) \leq n\varepsilon < \eta$. On the other hand $nt \geq n\delta \geq M$, hence $g(nt) \geq \eta$. This contradiction proves (3.9). To obtain (3.8) it is enough to take a constant b less than the infimum in (3.9) and with $0 < b < \min\{c\delta, \eta\}$.

Remark 3.2.5. Let g be an even subadditive function satisfying (3.7) and let $a \in [0, C)$. Let us define the function

$$g^a(t) := \inf_{s \in \mathbb{R}} [\, g(s) + a|t - s|\,] .$$

It is easy to see that the function g^a is even, subadditive, and that $g^a \nearrow g$ as $a \nearrow C$. Moreover, using Remark 3.2.4, we can prove that there exists $\delta_a > 0$ such that $g^a(t) = at$ for every $t \in [0, \delta_a]$.

3.2.3 Lower semicontinuity of functionals in dimension one

We now recall some lower semicontinuity results about one dimensional functionals defined on the space $BV(U)$, where U is a bounded open subset of $\mathbb{R}$. Let us consider the functional $\Psi : BV(U) \times \mathscr{A}(U) \to [0, +\infty)$ defined by

$$\Psi(u; A) := \int_A f(|\nabla u|)\, \mathrm{d}t + C|D^c u|(A) + \sum_{t \in J_u \cap A} g([u](t)) , \tag{3.10}$$

for every $u \in BV(U)$ and for every open set A contained in U. It is well known that the functional Ψ is lower semicontinuous with respect to the weak* topology in $BV(U)$ under the following assumptions: f is convex and nondecreasing, g is even and subadditive, and

$$\lim_{t \to +\infty} \frac{f(t)}{t} = \lim_{t \to 0^+} \frac{g(t)}{t} = C . \tag{3.11}$$

For a proof of this result we refer to [16] and [7, Theorem 5.2]. If, in addition, g satisfies also

$$\liminf_{t \to +\infty} g(t) > 0 , \tag{3.12}$$

then the functional Ψ is also lower semicontinuous with respect to the L^1 topology, which is weaker than the weak* topology in $BV(U)$. When g is nondecreasing on $[0, +\infty)$, this result can be obtained easily by a truncation argument. For the reader's convenience we give here a complete proof in the general case.

Proposition 3.2.6. *Let $f \colon [0, +\infty) \to [0, +\infty)$ be a convex nondecreasing function, let $g \colon \mathbb{R} \to [0, +\infty)$ be an even subadditive function, and assume that* (3.11) *and* (3.12) *hold. Then the functional $\Psi(\cdot; U)$ defined in* (3.10) *is L^1-lower semicontinuous on $BV(U)$, i.e.,*

$$\Psi(u; U) \leq \liminf_{k \to +\infty} \Psi(u_k; U) , \tag{3.13}$$

for every $u_k, u \in BV(U)$ *such that* $u_k \to u$ *in* $L^1(U)$.

Proof. It is enough to prove the result when U is a bounded open interval, denoted by I. Let us fix $u_k, u \in BV(I)$ such that $u_k \to u$ in $L^1(I)$. Up to extracting a subsequence, we can assume that $u_k \to u$ a.e. in Ω, that the liminf in (3.13) is finite, and that it is actually a limit. Therefore

$$\Psi(u_k; I) \leq M \tag{3.14}$$

for some positive constant M.

We start by proving that the number of large jumps of the functions u_k is equi-bounded. By Remark 3.2.4, there exist a constant $c > 0$ such that $g(t) \geq c \min\{t, 1\}$. By (3.14), this implies that there exists a constant $M' > 0$ such that

$$|Du_k|(I \setminus J^1_{u_k}) + \mathcal{H}^0(J^1_{u_k}) \leq M', \tag{3.15}$$

where $J^1_{u_k} := \{t \in J_{u_k} : |[u_k](t)| \geq 1\}$ and $\mathcal{H}^0$ is the counting measure. Hence, up to a subsequence, we can assume that there exists an integer $m \geq 1$ such that $J^1_{u_k} = \{t^k_1, \dots, t^k_m\}$, with $t^k_1 < \dots < t^k_m$. We can also assume that $t^k_i \to t_i$ as $k \to +\infty$, for $i = 1, \dots, m$, where $t_1 \leq \dots \leq t_m$. Let us consider $s_1 < \dots < s_\ell$ such that $\{s_1, \dots, s_\ell\} = \{t_1, \dots, t_m\}$. Let us fix $\delta > 0$ such that the following properties are satisfied:

- the intervals $[s_i - \delta, s_i + \delta]$, $i = 1, \dots, \ell$, are pairwise disjoint;
- $s_i - \delta$ and $s_i + \delta$ do not belong to $\bigcup_k J_{u_k} \cup J_u$;
- $u_k(s_i - \delta) \to u(s_i - \delta)$ and $u_k(s_i + \delta) \to u(s_i + \delta)$ as $k \to +\infty$, for $i = 1, \dots, \ell$.

Let us consider the open set $A_\delta := I \setminus \bigcup_{i=1}^{\ell} [s_i - \delta, s_i + \delta]$. First of all, we notice that, for k large enough, we have that $|[u_k](t)| < 1$ for all $t \in J_{u_k} \cap A_\delta$, i.e., $J^1_{u_k} \cap A_\delta = \emptyset$. Hence, by (3.15), we have that $|Du_k|(A_\delta) \leq M'$ for all k. This implies that $u_k \overset{*}{\rightharpoonup} u$ in $BV(A_\delta)$, and by the lower semicontinuity of $\Psi(\cdot; A_\delta)$ with respect to the weak* convergence of $BV(A_\delta)$, we deduce that

$$\Psi(u; A_\delta) \leq \liminf_{k \to +\infty} \Psi(u_k; A_\delta). \tag{3.16}$$

Let us now fix $i \in \{1, \dots, \ell\}$ and let $I^i_\delta := (s_i - \delta, s_i + \delta)$. By (3.11), given $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for every $t \geq t_\varepsilon$ we have that

$$f(t) \geq (C - \varepsilon)t. \tag{3.17}$$

We claim that

$$g(u_k(s_i + \delta) - u_k(s_i - \delta)) \leq \frac{C}{C - \varepsilon} \Psi(u_k; I^i_\delta) + 2C\delta t_\varepsilon. \tag{3.18}$$

First of all we observe that

$$u_k(s_i + \delta) - u_k(s_i - \delta) = Du_k(I^i_\delta) = \int_{I^i_\delta} \nabla u_k \, \mathrm{d}t + D^c u_k(I^i_\delta) + \sum_{t \in J_{u_k} \cap I^i_\delta} [u_k](t).$$

By the subadditivity and the continuity of g and by the inequality $g(t) \leq C|t|$, we have that

$$\begin{aligned} g(u_k(s_i+\delta)-u_k(s_i-\delta)) &\leq g\Big(\int_{I^i_\delta} \nabla u_k \,\mathrm{d}t\Big) + g(D^c u_k(I^i_\delta)) + \sum_{t\in J_{u_k}\cap I^i_\delta} g([u_k](t)) \\ &\leq C\int_{I^i_\delta} |\nabla u_k|\,\mathrm{d}t + C|D^c u_k|(I^i_\delta) + \sum_{t\in J_{u_k}\cap I^i_\delta} g([u_k](t))\,. \end{aligned} \tag{3.19}$$

By (3.19) and (3.17) we get

$$\begin{aligned} \frac{C}{C-\varepsilon}\Psi(u_k; I^i_\delta) &\geq \frac{C}{C-\varepsilon}\int_{I^i_\delta} f(|\nabla u_k|)\,\mathrm{d}t + C|D^c u_k|(I^i_\delta) + \sum_{t\in J_{u_k}\cap I^i_\delta} g([u_k](t)) \\ &\geq \frac{C}{C-\varepsilon}\int_{I^i_\delta} f(|\nabla u_k|)\,\mathrm{d}t + g(u_k(s_i+\delta)-u_k(s_i-\delta)) - C\int_{I^i_\delta}|\nabla u_k|\,\mathrm{d}t \\ &\geq \frac{C}{C-\varepsilon}\int_{\{|\nabla u_k|\geq t_\varepsilon\}\cap I^i_\delta} f(|\nabla u_k|)\,\mathrm{d}t + g(u_k(s_i+\delta)-u_k(s_i-\delta)) \\ &\quad - C\int_{\{|\nabla u_k|\geq t_\varepsilon\}\cap I^i_\delta} |\nabla u_k|\,\mathrm{d}t - C\int_{\{|\nabla u_k|< t_\varepsilon\}\cap I^i_\delta} |\nabla u_k|\,\mathrm{d}t \\ &\geq g(u_k(s_i+\delta)-u_k(s_i-\delta)) - C\mathcal{L}^1(I^i_\delta)t_\varepsilon\,. \end{aligned}$$

Since $\mathcal{L}^1(I^i_\delta) = 2\delta$, this proves (3.18).

Letting $k \to +\infty$ in (3.18) we obtain

$$g(u(s_i+\delta)-u(s_i-\delta)) \leq \liminf_{k\to+\infty} \frac{C}{C-\varepsilon}\Psi(u_k; I^i_\delta) + 2C\delta t_\varepsilon\,. \tag{3.20}$$

Summing (3.16) and (3.20) for $i = 1, \dots, \ell$, it follows that

$$\Psi(u; A_\delta) + \sum_{i=1}^{\ell} g(u(s_i+\delta)-u(s_i-\delta)) \leq \frac{C}{C-\varepsilon}\liminf_{k\to+\infty}\Psi(u_k; I) + 2\ell C\delta t_\varepsilon\,.$$

Letting $\delta \to 0$ and then $\varepsilon \to 0$, we conclude the proof of (3.13). □

3.3 Examples of non lower semicontinuous functionals

In this section we show that, in general, the functionals defined in (3.2) and in (3.3) are not L^1-lower semicontinuous on $BD(\Omega)$.

We start by studying the functional $\mathcal{G}_1\colon BD(\Omega) \to [0, +\infty)$ defined by

$$\mathcal{G}_1(u) := \int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} g(|[u]|)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{3.21}$$

where $f\colon [0, +\infty) \to [0, +\infty)$ is a convex nondecreasing function such that

$$0 < \lim_{t\to+\infty} \frac{f(t)}{t} < +\infty\,, \tag{3.22}$$

$C \in (0,+\infty)$, and $g\colon \mathbb{R} \to [0,+\infty)$ is a Borel function. As we shall see in the following proposition, the reason why the functional $\mathcal{G}_1$ fails to be lower semicontinuous is the fact that the surface density only depends on $|[u]|$.

Proposition 3.3.1. *The functional* $\mathcal{G}_1$ *defined in* (3.21) *is not* L^1*-lower semicontinuous on* $BD(\Omega)$.

Proof. For the sake of simplicity, we give the proof only when Ω is the unit cube in $\mathbb{R}^n$ centred at the origin, i.e., $\Omega = (-\frac{1}{2}, \frac{1}{2})^n$. Let Q' be the unit cube in $\mathbb{R}^{n-1}$, i.e., $Q' = (-\frac{1}{2}, \frac{1}{2})^{n-1}$. For every $x \in \Omega$, let x' be the vector in Q' with components $(x_1, \dots, x_{n-1})$. Let us assume, by contradiction, that the functional $\mathcal{G}_1$ is L^1-lower semicontinuous on $BD(\Omega)$.

Let us start by proving that

$$\liminf_{s\to 0^+} \frac{g(s)}{s} \geq \lim_{t\to+\infty} \frac{f(t)}{t}\,. \tag{3.23}$$

Let us fix $t > 2$, $z \in \mathbb{R}^n$, $z \neq 0$, and let us define the function u which connects linearly the vector 0 and the vector z in the rectangle $Q' \times [0, \frac{1}{t}]$:

$$u(x', x_n) := \begin{cases} z & \text{if } \frac{1}{t} \leq x_n < \frac{1}{2}\,, \\ tzx_n & \text{if } 0 < x_n < \frac{1}{t}\,, \\ 0 & \text{if } -\frac{1}{2} < x_n \leq 0\,. \end{cases}$$

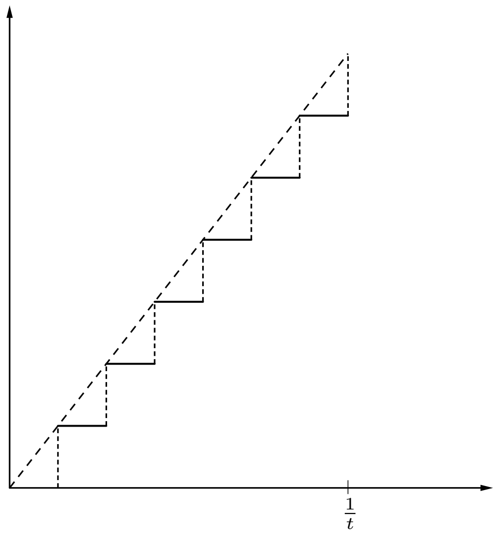

Figure 3.1: Graph of the function α_k.

We now define a sequence of pure jump functions u_k which approximate in L^1 the function u. Let $t_k \to 0^+$ be a sequence such that the liminf in (3.23) is equal to $\lim_k \frac{g(t_k|z|)}{t_k|z|}$. For every $k \in \mathbb{N}$ let $h_k \in \mathbb{N}$ be such that $h_k \leq \frac{1}{t_k} < h_k + 1$. We define the function

$$\alpha_k(s) := \sum_{j=1}^{h_k} (j-1) t_k \mathbb{1}_{\left(\frac{j-1}{h_k t}, \frac{j}{h_k t}\right)}(s)\,, \quad s \in (0, \tfrac{1}{t})\,,$$

where $\mathbb{1}_I$ is the indicator function of the interval I. Let

$$u_k(x', x_n) := \begin{cases} z & \text{if } \frac{1}{t} \le x_n < \frac{1}{2}\,, \\ z\alpha_k(x_n) & \text{if } 0 < x_n < \frac{1}{t}\,, \\ 0 & \text{if } -\frac{1}{2} < x_n \le 0\,. \end{cases}$$

It is easy to see that $u_k \to u$ in $L^1(\Omega; \mathbb{R}^n)$. Therefore, by the lower semicontinuity of $\mathcal{G}_1$ we get

$$\begin{aligned} \frac{f(t|z \odot e_n|)}{t} &= \int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x = \mathcal{G}_1(u) \le \liminf_{k\to+\infty} \mathcal{G}_1(u_k) = \liminf_{k\to+\infty} \int_{J_{u_k}} g(|[u_k]|)\,\mathrm{d}\mathcal{H}^{n-1} \\ &= \liminf_{k\to+\infty} h_k g(t_k|z|) = \lim_{k\to+\infty} \frac{g(t_k|z|)}{t_k} = |z| \liminf_{s\to0^+} \frac{g(s)}{s}\,. \end{aligned}$$

Letting $t \to +\infty$ in the inequality above, by (3.22) we get

$$|z \odot e_n| \lim_{t\to+\infty} \frac{f(t)}{t} \le |z| \liminf_{s\to0^+} \frac{g(s)}{s}\,.$$

If we choose $z = e_n$, this proves (3.23).

Let us now prove that

$$\limsup_{s\to0^+} \frac{g(s)}{s} \le \frac{1}{\sqrt{2}} \lim_{t\to+\infty} \frac{f(t)}{t}\,. \tag{3.24}$$

Taking (3.22) into account, this contradicts (3.23). To prove (3.24) we fix $z \in \mathbb{R}^n$, with $z \neq 0$, and we consider the pure jump function v defined by

$$v(x', x_n) = \begin{cases} z & \text{if } 0 < x_n < \frac{1}{2}\,, \\ 0 & \text{if } -\frac{1}{2} < x_n \le 0\,. \end{cases}$$

We now construct a sequence of piecewise affine functions v_k which approximate v in L^1. For every $k \in \mathbb{N}$, $k \ge 2$, let

$$v_k(x', x_n) := \begin{cases} z & \text{if } \frac{1}{k} \le x_n < \frac{1}{2}\,, \\ kzx_n & \text{if } 0 < x_n < \frac{1}{k}\,, \\ 0 & \text{if } -\frac{1}{2} < x_n \le 0\,. \end{cases}$$

By the lower semicontinuity of $\mathcal{G}_1$ and by (3.22), we have that

$$\begin{aligned} g(|z|) &= \int_{J_v} g(|[v]|)\,\mathrm{d}\mathcal{H}^{n-1} = \mathcal{G}_1(v) \le \liminf_{k\to+\infty} \mathcal{G}_1(v_k) = \liminf_{k\to+\infty} \int_\Omega f(|\mathcal{E}v_k|)\,\mathrm{d}x \\ &= \lim_{k\to+\infty} \frac{f(k|z \odot e_n|)}{k} = |z \odot e_n| \lim_{t\to+\infty} \frac{f(t)}{t}\,. \end{aligned}$$

By choosing z of the form $z = \delta e_1 = (\delta, 0, \dots, 0)$ we get

$$\frac{g(|\delta e_1|)}{\delta} \le |e_1 \odot e_n| \lim_{t\to+\infty} \frac{f(t)}{t} = \frac{1}{\sqrt{2}} \lim_{t\to+\infty} \frac{f(t)}{t}\,,$$

and therefore, by letting $\delta \to 0^+$, we obtain (3.24). This concludes the proof. $\square$

Let us now consider the functional $\mathcal{G}_2\colon BD(\Omega) \to [0,+\infty)$ defined by

$$\mathcal{G}_2(u) := \int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} g(|[u] \odot \nu_u|)\,\mathrm{d}\mathcal{H}^{n-1}, \tag{3.25}$$

where $f\colon [0,+\infty) \to [0,+\infty)$ is a convex nondecreasing function such that

$$\lim_{t\to+\infty} \frac{f(t)}{t} = C\,, \tag{3.26}$$

with $0 < C < +\infty$, and $g(t) = (C|t|) \wedge 1$. In the next proposition, we prove that $\mathcal{G}_2$ is not lower semicontinuous. In this case, the main issue is the fact that the surface density does not satisfy a necessary condition for the lower semicontinuity of the functional. Indeed, the function

$$\psi(z,\nu) := g(|z \odot \tfrac{\nu}{|\nu|}|)|\nu|$$

is not convex in the variable ν.

Proposition 3.3.2. *The functional $\mathcal{G}_2$ defined in* (3.25) *is not L^1-lower semicontinuous on $BD(\Omega)$.*

Proof. For simplicity, we assume $C = 1$. Let us show that ψ is not convex with respect to the variable ν, i.e., there exist $z, \nu_0, \nu_1, \nu_2 \in \mathbb{R}^n$ such that $\nu_0 = \lambda\nu_1 + (1-\lambda)\nu_2$ for some $0 < \lambda < 1$ and

$$\psi(z,\nu_0) > \lambda\psi(z,\nu_1) + (1-\lambda)\psi(z,\nu_2)\,. \tag{3.27}$$

Indeed, let $z = \rho e_1 = (\rho, 0, \ldots, 0)$, with $\rho > 0$. Then, if $\nu = (a_1, a_2, 0, \ldots, 0)$, we have that

$$\psi(z,\nu)^2 = \min\{\rho^2(a_1^2 + \tfrac{1}{2}a_2^2), a_1^2 + a_2^2\}\,.$$

For $1 < \rho < \sqrt{2}$, the set $\psi(z,\nu) \le 1$ is not convex, and it is possible to find $\nu_0, \nu_1, \nu_2 \in \mathbb{R}^n$ such that $\nu_0 = \lambda\nu_1 + (1-\lambda)\nu_2$ for some $0 < \lambda < 1$, $\psi(z,\nu_1) = \psi(z,\nu_2) = 1$ and $\psi(z,\nu_0) > 1$. (See Figure 3.2.) This concludes the proof of (3.27).

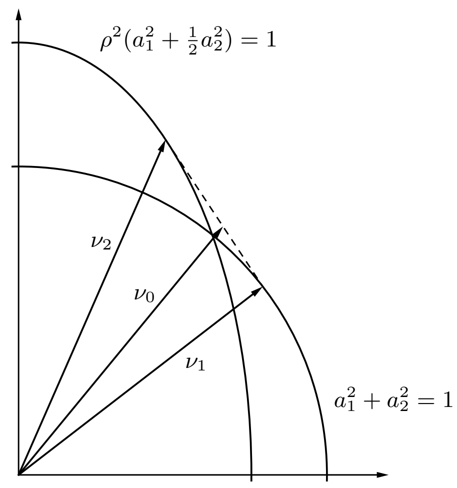

Figure 3.2: Construction of the vectors ν_0, ν_1, ν_2.

Let $E_0 := \{(x_1, x_2) \in \Omega : x_2 \geq 0\}$ and let us define the pure jump function

$$u(x) := \begin{cases} z & \text{if } x \in E_0\,, \\ 0 & \text{if } x \in \Omega \setminus E_0\,. \end{cases}$$

Note that the jump set of u is orthogonal to $\frac{\nu_0}{|\nu_0|}$. We now define a sequence of pure jump functions u_k such that their jump set is oriented with normals $\frac{\nu_1}{|\nu_1|}$ and $\frac{\nu_2}{|\nu_2|}$ in the following way. Let T_1 be the triangle with one side given by $(-\frac{1}{2}, \frac{1}{2}) \times \{0\}$ and the other two sides orthogonal to $\frac{\nu_1}{|\nu_1|}$, $\frac{\nu_2}{|\nu_2|}$ having length $\lambda\frac{|\nu_1|}{|\nu_0|}$, $(1-\lambda)\frac{|\nu_2|}{|\nu_0|}$, respectively. For every $k \in \mathbb{N}$, $k \geq 2$, let T_k be the set contained in Ω formed by k copies of T_1 scaled by a factor $\frac{1}{k}$. Finally, let $E_k := E_0 \setminus T_k$ (see Figure 3.3) and let

$$u_k(x) := \begin{cases} z & \text{if } x \in E_k \\ 0 & \text{if } x \in \Omega \setminus E_k\,. \end{cases}$$

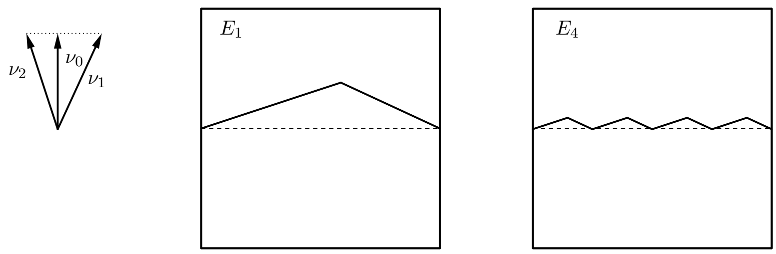

Figure 3.3: Construction of the approximating sequence u_k.

Since $u_k \to u$ in $L^1(\Omega; \mathbb{R}^2)$, by the lower semicontinuity of $\mathcal{G}_2$ we obtain

$$\begin{aligned} \tfrac{1}{|\nu_0|}\psi(z, \nu_0) = g(|z \odot \tfrac{\nu_0}{|\nu_0|}|) &= \int_{J_u} g(|[u] \odot \nu_u|)\, \mathrm{d}\mathcal{H}^{n-1} \\ &= \mathcal{G}_2(u) \leq \liminf_{k\to+\infty} \mathcal{G}_2(u_k) = \liminf_{k\to+\infty} \int_{J_{u_k}} g(|[u_k] \odot \nu_{u_k}|)\, \mathrm{d}\mathcal{H}^{n-1} \\ &= \liminf_{k\to+\infty} \sum_{j=1}^{k} \tfrac{1}{k}\big[\lambda\tfrac{|\nu_1|}{|\nu_0|} g(|z \odot \tfrac{\nu_1}{|\nu_1|}|) + (1-\lambda)\tfrac{|\nu_2|}{|\nu_0|} g(|z \odot \tfrac{\nu_2}{|\nu_2|}|)\big] \\ &= \tfrac{1}{|\nu_0|}[\lambda\psi(z, \nu_1) + (1-\lambda)\psi(z, \nu_2)]\,. \end{aligned}$$

This contradicts (3.27), and therefore $\mathcal{G}_2$ cannot be lower semicontinuous. □

3.4 Semicontinuity by slicing

In this section we prove the following theorem.

Theorem 3.4.1. *Under assumptions* (H1)–(H3), *the functional* $\mathcal{F}$ *defined in* (3.4) *is* L^1*-lower semicontinuous on* $BD(\Omega)$, *i.e.,*

$$\mathcal{F}(u) \leq \liminf_{k\to+\infty} \mathcal{F}(u_k)$$

for every sequence $u_k \in BD(\Omega)$ *and* $u \in BD(\Omega)$ *such that* $u_k \to u$ *in* $L^1(\Omega; \mathbb{R}^n)$.

In the proof of Theorem 3.4.1 it is convenient to consider the functional

$$\mathcal{F}\colon BD(\Omega)\times\mathscr{A}(\Omega)\to[0,+\infty)$$

defined by

$$\mathcal{F}(u;A):=\int_A f(|\mathcal{E}u|)\,\mathrm{d}x+C|E^cu|(A)+\int_{J_u\cap A}G([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} \tag{3.28}$$

for every $u\in BD(\Omega)$ and $A\in\mathscr{A}(\Omega)$. Clearly we have $\mathcal{F}(u)=\mathcal{F}(u;\Omega)$.

The proof is based on a slicing argument, which allows us to reduce the problem to the one-dimensional setting.

The first step of the proof of Theorem 3.4.1 is a result about the lower semicontinuity of the functional $\mathcal{F}_\xi\colon BD(\Omega)\times\mathscr{A}(\Omega)\to[0,+\infty)$ defined for every $\xi\in\mathbb{S}^{n-1}$ by

$$\mathcal{F}_\xi(u;A):=\int_A f(|\mathcal{E}u\xi\cdot\xi|)\,\mathrm{d}x+C|E^cu\xi\cdot\xi|(A)+\int_{J_u\cap A}g([u]\cdot\xi)|\nu_u\cdot\xi|\,\mathrm{d}\mathcal{H}^{n-1}$$

for every $u\in BD(\Omega)$ and for every open set $A\subset\Omega$. In the previous formula, $|E^cu\xi\cdot\xi|$ is the total variation of the scalar measure $E^cu\xi\cdot\xi$ defined by $(E^cu\xi\cdot\xi)(B):=E^cu(B)\xi\cdot\xi$ for every Borel set $B\subset\Omega$.

Remark 3.4.2. Let us consider, for every $\xi\in\mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$, the one dimensional functional $\Psi_{\xi,y}\colon BV(\Omega^\xi_y)\times\mathscr{A}(\Omega^\xi_y)\to[0,+\infty)$ defined by

$$\Psi_{\xi,y}(v;U):=\int_U f(|\nabla v|)\,\mathrm{d}t+C|D^cv|(U)+\sum_{t\in J_v\cap U}g([v](t)) \tag{3.29}$$

for every $v\in BV(\Omega^\xi_y)$ and for every open set $U\subset\Omega^\xi_y$. Using the Coarea Formula and the slicing properties mentioned in Subsection 1.3.5, it is easy to see that

$$\mathcal{F}_\xi(u;A)=\int_{\Pi^\xi}\Psi_{\xi,y}(\hat{u}^\xi_y;A^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}(y) \tag{3.30}$$

for every $u\in BD(\Omega)$ and for every open set $A\subset\Omega$.

Lemma 3.4.3. *Let $\xi\in\mathbb{S}^{n-1}$ and let $u_k,u\in BD(\Omega)$ be such that $u_k\to u$ in $L^1(\Omega;\mathbb{R}^n)$. Assume that* (H1)–(H3) *hold. Then*

$$\mathcal{F}_\xi(u;A)\le\liminf_{k\to+\infty}\mathcal{F}_\xi(u_k;A) \tag{3.31}$$

for every open set $A\subset\Omega$.

Proof. Let A be an open set contained in Ω. Up to a subsequence, we can assume that the liminf in (3.31) is actually a limit and that $(\hat{u}_k)^\xi_y\to\hat{u}^\xi_y$ in $L^1(\Omega^\xi_y)$ for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$. Since by Proposition 3.2.6 the one dimensional functional $\Psi_{\xi,y}$ defined in (3.29) is L^1-lower semicontinuous, we obtain that

$$\Psi_{\xi,y}(\hat{u}^\xi_y;A^\xi_y)\le\liminf_{k\to+\infty}\Psi_{\xi,y}((\hat{u}_k)^\xi_y;A^\xi_y) \tag{3.32}$$

for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$. Integrating (3.32) with respect to y, by the Fatou Lemma we deduce that

$$\int_{\Pi^\xi} \Psi_{\xi,y}(\hat{u}^\xi_y; A^\xi_y)\, \mathrm{d}\mathcal{H}^{n-1}(y) \le \liminf_{k\to+\infty} \int_{\Pi^\xi} \Psi_{\xi,y}((\hat{u}_k)^\xi_y; A^\xi_y)\, \mathrm{d}\mathcal{H}^{n-1}(y)\,. \tag{3.33}$$

Inequality (3.31) simply follows from the inequality above and from (3.30). □

We prove now a lower semicontinuity result for functionals which are less than or equal to the original functional $\mathcal{F}$, but which have a much simpler structure. For every $a \in [0, C]$, we consider the function

$$g^a(t) := \inf_{s\in\mathbb{R}} [\, g(s) + a|t - s|\,]$$

and we define the function $G^a\colon \mathbb{R}^n\times\mathbb{S}^{n-1} \to [0, +\infty)$ by

$$G^a(y, \nu) := \sup_{(\xi^1,\dots,\xi^n)} \Big(\sum_{i=1}^n g^a(y\cdot\xi^i)^2 |\nu\cdot\xi^i|^2\Big)^{\frac{1}{2}}, \tag{3.34}$$

where the supremum is taken over all orthonormal bases $(\xi^1, \dots, \xi^n)$ of $\mathbb{R}^n$. In the following two lemmas, we shall study the functional

$$a\int_A |\mathcal{E}u|\,\mathrm{d}x + a|E^c u|(A) + \int_{J_u\cap A} G^a([u], \nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\,. \tag{3.35}$$

Note that, if $b \ge 0$ is such that $at - b \le f(t)$ for every $t \in [0, +\infty)$, then

$$\int_A (a|\mathcal{E}u| - b)\,\mathrm{d}x + a|E^c u|(A) + \int_{J_u\cap A} G^a([u], \nu_u)\,\mathrm{d}\mathcal{H}^{n-1} \le \mathcal{F}(u; A)$$

for every open set A contained in Ω. We will deduce the lower semicontinuity of $\mathcal{F}$ from the lower semicontinuity of the functional defined in (3.35) by passing to the supremum among all possible $a, b \ge 0$ such that $at - b \le f(t)$.

We start with a technical lemma.

Lemma 3.4.4. *Let $a \in [0, C]$, let $(\xi^1, \dots, \xi^n)$ be an orthonormal basis of $\mathbb{R}^n$, and let $u \in BD(\Omega)$. Then, for every open set $A \subset \Omega$,*

$$a\Big(\sum_{i=1}^n \Big(\int_A |\mathcal{E}u\xi^i\cdot\xi^i|\,\mathrm{d}x\Big)^2\Big)^{\frac{1}{2}} \le a\int_A |\mathcal{E}u|\,\mathrm{d}x\,, \tag{3.36}$$

$$a\Big(\sum_{i=1}^n |E^c u\xi^i\cdot\xi^i|(A)^2\Big)^{\frac{1}{2}} \le a|E^c u|(A)\,, \tag{3.37}$$

$$\Big(\sum_{i=1}^n \Big(\int_{J_u\cap A} g^a([u]\cdot\xi^i)|\nu_u\cdot\xi^i|\,\mathrm{d}\mathcal{H}^{n-1}\Big)^2\Big)^{\frac{1}{2}} \le \int_{J_u\cap A} G^a([u], \nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\,. \tag{3.38}$$

Proof. Let us prove (3.36). By the Hölder inequality with respect to the measure $|\mathcal{E}u|\mathcal{L}^n$, we get

$$\Big(\int_A |\mathcal{E}u\xi^i\cdot\xi^i|\,\mathrm{d}x\Big)^2 = \Big(\int_A \Big|\frac{\mathcal{E}u}{|\mathcal{E}u|}\xi^i\cdot\xi^i\Big||\mathcal{E}u|\,\mathrm{d}x\Big)^2 \le \int_A \Big|\frac{\mathcal{E}u}{|\mathcal{E}u|}\xi^i\cdot\xi^i\Big|^2|\mathcal{E}u|\,\mathrm{d}x \int_A |\mathcal{E}u|\,\mathrm{d}x\,.$$

Summing with respect to i, from (3.6) it follows that

$$\Big(\sum_{i=1}^{n}\Big(\int_A |\mathcal{E}u\xi^i\cdot\xi^i|\,\mathrm{d}x\Big)^2\Big)^{\frac12} \le \Big(\int_A\sum_{i=1}^{n}\Big|\frac{\mathcal{E}u}{|\mathcal{E}u|}\xi^i\cdot\xi^i\Big|^2|\mathcal{E}u|\,\mathrm{d}x\Big)^{\frac12}\Big(\int_A|\mathcal{E}u|\,\mathrm{d}x\Big)^{\frac12} \le \int_A|\mathcal{E}u|\,\mathrm{d}x\,,$$

which proves (3.36).

To prove (3.37), we use the Hölder inequality with respect to the measure $|E^cu|$ and we obtain

$$|E^cu\xi^i\cdot\xi^i|(A)^2 = \Big(\int_A\Big|\frac{\mathrm{d}E^cu}{\mathrm{d}|E^cu|}\xi^i\cdot\xi^i\Big|\,\mathrm{d}|E^cu|\Big)^2 \le \Big(\int_A\Big|\frac{\mathrm{d}E^cu}{\mathrm{d}|E^cu|}\xi^i\cdot\xi^i\Big|^2\,\mathrm{d}|E^cu|\Big)|E^cu|(A)\,.$$

Therefore, by (3.6),

$$\Big(\sum_{i=1}^{n}|E^cu\xi^i\cdot\xi^i|(A)^2\Big)^{\frac12} \le \Big(\int_A\sum_{i=1}^{n}\Big|\frac{\mathrm{d}E^cu}{\mathrm{d}|E^cu|}\xi^i\cdot\xi^i\Big|^2\,\mathrm{d}|E^cu|\Big)^{\frac12}|E^cu|(A)^{\frac12} \le |E^cu|(A)\,,$$

since $\Big|\frac{\mathrm{d}E^cu}{\mathrm{d}|E^cu|}\Big| = 1$ $|E^cu|$-a.e. in Ω.

The strategy to prove (3.38) is the same. By the Hölder inequality with respect to the measure $G^a([u],\nu_u)\mathcal{H}^{n-1}$ we have

$$\begin{aligned}\Big(\int_{J_u\cap A} g^a([u]\cdot\xi^i)|\nu_u\cdot\xi^i|\,\mathrm{d}\mathcal{H}^{n-1}\Big)^2 &= \Big(\int_{J_u\cap A}\frac{g^a([u]\cdot\xi^i)|\nu_u\cdot\xi^i|}{G^a([u],\nu_u)}G^a([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\Big)^2\\ &\le \int_{J_u\cap A}\frac{g^a([u]\cdot\xi^i)^2|\nu_u\cdot\xi^i|^2}{G^a([u],\nu_u)^2}G^a([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\int_{J_u\cap A}G^a([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\end{aligned}$$

and hence, by (3.34) we obtain

$$\begin{aligned}&\Big(\sum_{i=1}^{n}\Big(\int_{J_u\cap A} g^a([u]\cdot\xi^i)|\nu_u\cdot\xi^i|\,\mathrm{d}\mathcal{H}^{n-1}\Big)^2\Big)^{\frac12}\\ &\le \Big(\int_{J_u\cap A}\sum_{i=1}^{n}\frac{g^a([u]\cdot\xi^i)^2|\nu_u\cdot\xi^i|^2}{G^a([u],\nu_u)^2}G^a([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\Big)^{\frac12}\Big(\int_{J_u\cap A}G^a([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\Big)^{\frac12}\\ &\le \int_{J_u\cap A}G^a([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned}$$

This concludes the proof. □

In the following lemma, we prove a preliminary result which is strongly connected to the lower semicontinuity of the functional defined in (3.35). The main idea of the proof is based on the following remark: by Proposition 3.2.1 we have that

$$|\mathcal{E}u| = \sup_j\Big(\sum_{i=1}^{n}|\mathcal{E}u\xi_j^i\cdot\xi_j^i|^2\Big)^{\frac12},$$

where $\{(\xi_j^1,\dots,\xi_j^n) : j \in \mathbb{N}\}$ is a suitable countable collection of orthonormal bases of $\mathbb{R}^n$. Therefore we can apply a localization argument based on Lemma 1.4.7 which leads to

$$\int_K |\mathcal{E}u|\,\mathrm{d}x = \sup\Big\{\sum_{j=1}^r \int_{K^j} \Big(\sum_{i=1}^n |\mathcal{E}u\xi_j^i \cdot \xi_j^i|^2\Big)^{\frac{1}{2}}\,\mathrm{d}x\Big\},$$

where the supremum is taken among all families $(K^j)_{j=1}^r$ of disjoint compact subsets of K and $r \in \mathbb{N}$. This will allow us to use the semicontinuity result already proved for $\mathcal{F}^a_{\xi_j^i}$ defined in (3.43) below.

Lemma 3.4.5. *Let $a \in [0,C]$, let $u_k, u \in BD(\Omega)$ be such that $u_k \to u$ in $L^1(\Omega;\mathbb{R}^n)$, and let*

$$\Lambda(A) := \liminf_{k\to+\infty}\Big[a\int_A |\mathcal{E}u_k|\,\mathrm{d}x + a|E^c u_k|(A) + \int_{J_{u_k}\cap A} G^a([u_k],\nu_{u_k})\,\mathrm{d}\mathcal{H}^{n-1}\Big] \tag{3.39}$$

for every open set $A \subset \Omega$. Then

$$a\int_K |\mathcal{E}u|\,\mathrm{d}x \le \Lambda(A), \tag{3.40}$$

$$a|E^c u|(K) \le \Lambda(A), \tag{3.41}$$

$$\int_{J_u\cap K} G^a([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} \le \Lambda(A) \tag{3.42}$$

for every compact set K and for every open set A such that $K \subset A \subset \Omega$.

Proof. Let us fix an orthonormal basis $(\xi^1,\dots,\xi^n)$ of $\mathbb{R}^n$. For every $i = 1,\dots,n$, let us consider the functional $\mathcal{F}^a_{\xi^i}\colon BD(\Omega)\times\mathscr{A}(\Omega) \to [0,+\infty)$ defined by

$$\mathcal{F}^a_{\xi^i}(u;A) := a\int_A |\mathcal{E}u\xi^i\cdot\xi^i|\,\mathrm{d}x + a|E^c u\xi^i\cdot\xi^i|(A) + \int_{J_u\cap A} g^a([u]\cdot\xi^i)|\nu_u\cdot\xi^i|\,\mathrm{d}\mathcal{H}^{n-1}, \tag{3.43}$$

for every $u \in BD(\Omega)$ and for every open set $A \subset \Omega$. Since $\mathcal{F}^a_{\xi^i}$ satisfies the hypotheses of Lemma 3.4.3 (see Remark 3.2.5), we have that

$$\mathcal{F}^a_{\xi^i}(u;A) \le \liminf_{k\to+\infty}\mathcal{F}^a_{\xi^i}(u_k;A), \tag{3.44}$$

for every $i = 1,\dots,n$ and for every open set $A \subset \Omega$.

In order to prove (3.40), we observe that by (3.44)

$$a\int_K |\mathcal{E}u\xi^i\cdot\xi^i|\,\mathrm{d}x \le \mathcal{F}^a_{\xi^i}(u;A) \le \liminf_{k\to+\infty}\mathcal{F}^a_{\xi^i}(u_k;A)$$

for every compact set $K \subset A$. From this inequality and from the superadditivity of the liminf, it follows that

$$a\Big(\sum_{i=1}^n\Big(\int_K |\mathcal{E}u\xi^i\cdot\xi^i|\,\mathrm{d}x\Big)^2\Big)^{\frac{1}{2}} \le \liminf_{k\to+\infty}\Big(\sum_{i=1}^n \mathcal{F}^a_{\xi^i}(u_k;A)^2\Big)^{\frac{1}{2}}. \tag{3.45}$$

By the triangle inequality of the Euclidean norm in $\mathbb{R}^n$ and by Lemma 3.4.4, we obtain

$$\begin{aligned}
&\Big(\sum_{i=1}^{n} \mathcal{F}^a_{\xi^i}(u_k;A)^2\Big)^{\frac{1}{2}} \\
&\quad\le a\Big(\sum_{i=1}^{n}\Big(\int_A |\mathcal{E}u_k\xi^i\cdot\xi^i|\,\mathrm{d}x\Big)^2\Big)^{\frac{1}{2}} + a\Big(\sum_{i=1}^{n}|E^c u_k\xi^i\cdot\xi^i|(A)^2\Big)^{\frac{1}{2}} \\
&\qquad+\Big(\sum_{i=1}^{n}\Big(\int_{J_{u_k}\cap A} g^a([u_k]\cdot\xi^i)|\nu_{u_k}\cdot\xi^i|\,\mathrm{d}\mathcal{H}^{n-1}\Big)^2\Big)^{\frac{1}{2}} \\
&\quad\le a\int_A|\mathcal{E}u_k|\,\mathrm{d}x + a|E^c u_k|(A) + \int_{J_{u_k}\cap A} G^a([u_k],\nu_{u_k})\,\mathrm{d}\mathcal{H}^{n-1}\,.
\end{aligned} \tag{3.46}$$

Hence, by (3.45) and (3.46) it follows that

$$a\Big(\sum_{i=1}^{n}\Big(\int_K|\mathcal{E}u\xi^i\cdot\xi^i|\,\mathrm{d}x\Big)^2\Big)^{\frac{1}{2}} \le \Lambda(A)$$

for every compact set K, for every open set A such that $K\subset A\subset\Omega$, and for every orthonormal basis $(\xi^1,\dots,\xi^n)$.

Let us fix a sequence R_j dense in $\mathrm{O}(n)$ and let $\xi^1_j,\dots,\xi^n_j$ be the column vectors of R_j. Let us define the vector functions $\varphi^j=(\varphi^j_1,\dots,\varphi^j_n)$ with components given by $\varphi^j_i=|\mathcal{E}u\xi^i_j\cdot\xi^i_j|$, $i=1,\dots,n$. By the previous inequality, under the same assumptions on K and A, we have

$$a\Big|\int_K\varphi^j\,\mathrm{d}x\Big|\le\Lambda(A)$$

for every j. Since Λ is superadditive, we obtain

$$\begin{aligned}
&a\int_K|\varphi^j|\,\mathrm{d}x \\
&\quad=\sup\Big\{\sum_{h=1}^{r} a\Big|\int_{K^h}\varphi^j\,\mathrm{d}x\Big| : (K^h)_{h=1}^r \text{ disjoint compact subsets of } K,\ r\in\mathbb{N}\Big\} \\
&\quad\le\sup\Big\{\sum_{h=1}^{r}\Lambda(A^h) : (A^h)_{h=1}^r \text{ disjoint open subsets of } A,\ r\in\mathbb{N}\Big\}\le\Lambda(A)
\end{aligned} \tag{3.47}$$

for every compact set K and for every open set A such that $K\subset A\subset\Omega$. By Lemma 1.4.7 we deduce that

$$a\int_K\sup_j|\varphi^j|\,\mathrm{d}x\le\Lambda(A)\,.$$

On the other hand, by Proposition 3.2.1 and by Remark 3.2.2, we have that $\sup_j|\varphi^j| = |\mathcal{E}u|$. Together with the previous inequality, this concludes the proof of (3.40).

Let us now prove (3.41). Arguing as in the first part of the proof, we obtain that

$$a\Big(\sum_{i=1}^{n}|E^c u\xi^i\cdot\xi^i|(K)^2\Big)^{\frac{1}{2}}\le\Lambda(A)$$

for every compact set K, for every open set A such that $K \subset A \subset \Omega$, and for every orthonormal basis $(\xi^1, \dots, \xi^n)$. Let $(\xi^1_j, \dots, \xi^n_j)$ be the sequence of orthonormal bases introduced above. We now define a sequence of vector functions $\varphi^j = (\varphi^j_1, \dots, \varphi^j_n)$ with components given by

$$\varphi^j_i = \Big| \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|} \xi^i_j \cdot \xi^i_j \Big| .$$

The inequality above gives, under the same assumptions on K and A,

$$a \Big| \int_K \varphi^j \, \mathrm{d}|E^c u| \Big| \leq \Lambda(A)$$

for every j. As in (3.47), we obtain that

$$a \int_K |\varphi^j| \, \mathrm{d}|E^c u| \leq \Lambda(A) ,$$

hence, by Lemma 1.4.7, we deduce that

$$a \int_K \sup_j |\varphi^j| \, \mathrm{d}|E^c u| \leq \Lambda(A)$$

for every compact set K and for every open set A such that $K \subset A \subset \Omega$. On the other hand, since $\big| \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|} \big| = 1$ $|E^c u|$-a.e. in Ω, we have $\sup_j |\varphi^j| = 1$ $|E^c u|$-a.e. in Ω, by Proposition 3.2.1 and by Remark 3.2.2. Together with the previous inequality, this concludes the proof of (3.41).

The proof of (3.42) follows the same steps. Arguing as in the first part of the proof we obtain that

$$\Big(\sum_{i=1}^n \Big(\int_{J_u \cap K} g^a([u] \cdot \xi^i) |\nu_u \cdot \xi^i| \, \mathrm{d}\mathcal{H}^{n-1} \Big)^2 \Big)^{\frac{1}{2}} \leq \Lambda(A)$$

for every compact set K, for every open set A such that $K \subset A \subset \Omega$, and for every orthonormal basis $(\xi^1, \dots, \xi^n)$. We now continue as in the previous step, replacing the measure $|E^c u|$ by $\mathcal{H}^{n-1} \llcorner J_u$ and defining $\varphi^j_i := g^a([u] \cdot \xi^i_j) |\nu_u \cdot \xi^i_j|$. Since now $\sup_j |\varphi^j| = G^a([u], \nu_u)$, we finally obtain (3.42). □

Remark 3.4.6. In order to treat separately the three terms of the functional $\mathcal{F}$, given $u \in BD(\Omega)$, it is useful to consider a partition of Ω into three Borel sets B_1, B_2, B_3 such that

$$\mathcal{L}^n(\Omega \setminus B_1) = 0 , \tag{3.48}$$

$$|E^c u|(\Omega \setminus B_2) = 0 , \tag{3.49}$$

$$\mathcal{H}^{n-1}((J_u \setminus B_3) \cup (B_3 \setminus J_u)) = 0 . \tag{3.50}$$

Let $\varepsilon > 0$ and let K_1, K_2, K_3 be three pairwise disjoint compact sets such that $K_h \subset B_h$

for $h = 1, 2, 3$ and

$$\int_{B_1 \setminus K_1} f(|\mathcal{E}u|)\,\mathrm{d}x < \varepsilon\,, \tag{3.51}$$

$$C|E^c u|(B_2 \setminus K_2) < \varepsilon\,, \tag{3.52}$$

$$\int_{B_3 \setminus K_3} G([u], \nu_u)\,\mathrm{d}\mathcal{H}^{n-1} < \varepsilon\,. \tag{3.53}$$

Finally, let A_1, A_2, A_3 be three pairwise disjoint open subsets of Ω such that $K_h \subset A_h$ for $h = 1, 2, 3$.

We now exploit the property that a convex function can be written as the supremum of affine functions, which combined with Lemma 3.4.5 gives the following result.

Lemma 3.4.7. *Let $u_k, u \in BD(\Omega)$ be such that $u_k \to u$ in $L^1(\Omega; \mathbb{R}^n)$ and let K_1, K_2, K_3 and A_1, A_2, A_3 be as in Remark 3.4.6. Then*

$$\int_{K_1} f(|\mathcal{E}u|)\,\mathrm{d}x \leq \liminf_{k \to +\infty} \mathcal{F}(u_k; A_1)\,, \tag{3.54}$$

$$C|E^c u|(K_2) \leq \liminf_{k \to +\infty} \mathcal{F}(u_k; A_2)\,, \tag{3.55}$$

$$\int_{K_3} G([u], \nu_u)\,\mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{k \to +\infty} \mathcal{F}(u_k; A_3)\,. \tag{3.56}$$

Proof. Without loss of generality, we can assume $f(0) = 0$. Since f is convex and nonnegative, there exists a sequence of functions $f_j(t) = (a_j t - b_j)^+ = \max\{a_j t - b_j, 0\}$, with $a_j, b_j \geq 0$, such that $f_j \leq f$ and $f(t) = \sup_j f_j(t)$, for every $t \in [0, +\infty)$. Hence, by Lemma 1.4.7, given $\delta > 0$ there exists a finite family of disjoint compact sets $K_1^1, \ldots, K_1^r$ contained in K_1 such that

$$\int_{K_1} f(|\mathcal{E}u|)\,\mathrm{d}x \leq \sum_{j=1}^{r} \int_{K_1^j} (a_j |\mathcal{E}u| - b_j)^+\,\mathrm{d}x + \delta\,. \tag{3.57}$$

For every $j = 1, \ldots, r$, let us fix a compact set $\widetilde{K}_1^j \subset K_1^j \cap \{a_j |\mathcal{E}u| - b_j \geq 0\}$ such that

$$\int_{K_1^j} (a_j |\mathcal{E}u| - b_j)^+\,\mathrm{d}x \leq \int_{\widetilde{K}_1^j} (a_j |\mathcal{E}u| - b_j)\,\mathrm{d}x + \frac{\delta}{r}\,.$$

Let us consider a family of pairwise disjoint open sets $A_1^1, \ldots, A_1^r$ such that $\widetilde{K}_1^j \subset A_1^j \subset A_1$ and $b_j \mathcal{L}^n(A_1^j \setminus \widetilde{K}_1^j) \leq \delta / r$.

Note that by (H3) we have that $a_j \leq C$, since $a_j t - b_j \leq f(t)$. Therefore, we can apply (3.40) to $\widetilde{K}_1^j$ and A_1^j to obtain

$$a_j \int_{\widetilde{K}_1^j} |\mathcal{E}u|\,\mathrm{d}x \leq \liminf_{k \to +\infty} \Big[a_j \int_{A_1^j} |\mathcal{E}u_k|\,\mathrm{d}x + a_j |E^c u_k|(A_1^j) + \int_{J_{u_k} \cap A_1^j} G^{a_j}([u_k], \nu_{u_k})\,\mathrm{d}\mathcal{H}^{n-1} \Big]$$

for every $j=1,\dots,r$, and therefore

$$\int_{K_1^j}(a_j|\mathcal{E}u|-b_j)^+\,\mathrm{d}x \le \int_{\widetilde{K}_1^j}(a_j|\mathcal{E}u|-b_j)\,\mathrm{d}x+\frac{\delta}{r}\le \liminf_{k\to+\infty}\mathcal{F}(u_k;A_1^j)+\frac{2\delta}{r}\,,$$

where we have used the inequality $b_j\mathcal{L}^n(A_1^j\setminus\widetilde{K}_1^j)\le\delta/r$. This implies, by (3.57) and by the superadditivity of the liminf, that

$$\int_{K_1} f(|\mathcal{E}u|)\,\mathrm{d}x\le\liminf_{k\to+\infty}\mathcal{F}(u_k;A_1)+3\delta\,.$$

Letting $\delta\to 0$, we conclude the proof of (3.54).

To prove (3.55) and (3.56), first of all we note that $\sup_j a_j=C$. Indeed, given $\delta>0$, by (H3) there exists $T>0$ such that $C<f(T)/T+\delta$. Since $f(T)=\sup_j f_j(T)$, there exists a j such that $f(T)<f_j(T)+\delta T\le a_jT+\delta T$. Therefore $C<a_j+2\delta$. Letting $\delta\to 0$, we conclude that $\sup_j a_j=C$.

Since $\mathcal{L}^n(K_2)=0$, for every $\delta>0$ and for every j there exists an open set A_2^j such that $K_2\subset A_2^j\subset A_2$ and $b_j\mathcal{L}^n(A_2^j)\le\delta$. Applying (3.41) we have that

$$\begin{aligned} a_j|E^cu|(K_2) &\le \liminf_{k\to+\infty}\Big[a_j\int_{A_2^j}|\mathcal{E}u_k|\,\mathrm{d}x+a_j|E^cu_k|(A_2^j)+\int_{J_{u_k}\cap A_2^j}G^{a_j}([u_k],\nu_{u_k})\,\mathrm{d}\mathcal{H}^{n-1}\Big]\\ &\le \liminf_{k\to+\infty}\mathcal{F}(u_k;A_2^j)+b_j\mathcal{L}^n(A_2^j)\le\liminf_{k\to+\infty}\mathcal{F}(u_k;A_2)+\delta\,.\end{aligned}$$

Recalling that $\sup_j a_j=C$, we obtain

$$C|E^cu|(K_2)\le\liminf_{k\to+\infty}\mathcal{F}(u_k;A_2)+\delta$$

and by the arbitrariness of δ, we deduce (3.55).

Finally, let us prove (3.56). Arguing as in the previous step, for every $\delta>0$ and for every j we have that

$$\int_{K_3}G^{a_j}([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\le\liminf_{k\to+\infty}\mathcal{F}(u_k;A_3)+\delta\,.$$

Letting $\delta\to 0$, we obtain

$$\int_{K_3}G^{a_j}([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\le\liminf_{k\to+\infty}\mathcal{F}(u_k;A_3)\,. \tag{3.58}$$

Since $\sup_j a_j=C$, either there exists j_0 such that $a_{j_0}=C$ or there exists a strictly increasing subsequence a_{j_h} converging to C. In the first case, we have $g^{a_{j_0}}=g$, hence $G^{a_{j_0}}=G$, and (3.58) with j_0 coincides with (3.56). In the other case, $g^{a_{j_h}}$ is an increasing sequence and converges to g (see Remark 3.2.5). Consequently, $G^{a_{j_h}}$ is an increasing sequence and converges to G. Therefore, we can pass to the limit in (3.58) along the sequence j_h using the monotone convergence theorem, and we obtain (3.56). □

Theorem 3.4.1 is now a simple consequence of Lemma 3.4.7, thanks to the choice of K_1, K_2, K_3 made in Remark 3.4.6.

Proof of Theorem 3.4.1. Let us fix $u_k, u \in BD(\Omega)$ such that $u_k \to u$ in $L^1(\Omega;\mathbb{R}^n)$. Let us consider the three disjoint Borel sets B_1, B_2, B_3, the three disjoint compact sets K_1, K_2, K_3, and the three disjoint open sets A_1, A_2, A_3 as in Remark 3.4.6. By Lemma 3.4.7 and by the superadditivity of the liminf, we have

$$\int_{K_1} f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(K_2) + \int_{K_3} G([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} \le \liminf_{k\to+\infty} \mathcal{F}(u_k;\Omega)\,.$$

From this inequality and from (3.48)–(3.53) we obtain

$$\begin{aligned}\mathcal{F}(u;\Omega) &= \int_{B_1} f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(B_2) + \int_{B_3} G([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\le \int_{K_1} f(|\mathcal{E}u|)\,\mathrm{d}x + \varepsilon + C|E^c u|(K_2) + \varepsilon + \int_{K_3} G([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \varepsilon\\ &\le \liminf_{k\to+\infty} \mathcal{F}(u_k;\Omega) \, + 3\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$, we conclude the proof of Theorem 3.4.1. □

3.5 A relaxation result for functionals defined on BD

The aim of this section is to obtain an integral representation for the relaxation of the functional $\mathcal{F}\colon BD(\Omega) \to [0,+\infty)$ defined by

$$\mathcal{F}(u) := \int_{\Omega} f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} \psi([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}, \tag{3.59}$$

for every $u \in BD(\Omega)$. We assume that:

(H1′) $f\colon [0,+\infty) \to [0,+\infty)$ is a convex nondecreasing function;

(H2′) $\psi\colon \mathbb{R}^n \times \mathbb{S}^{n-1} \to [0,+\infty)$ is a Borel function and there exist a constant $c_1 > 0$ and an even subadditive function $g\colon \mathbb{R} \to [0,+\infty)$ such that for every orthonormal basis $(\xi^1,\dots,\xi^n)$

$$\Big(\sum_{i=1}^n g(z\cdot\xi^i)^2|\nu\cdot\xi^i|^2\Big)^{\frac{1}{2}} \le \psi(z,\nu) \le c_1(|z|\wedge 1)\,,$$

for every $z \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$;

(H3′) $0 < C < +\infty$ and

$$\lim_{t\to+\infty} \frac{f(t)}{t} = \lim_{t\to 0^+} \frac{g(t)}{t} = C\,, \quad \liminf_{t\to+\infty} g(t) > 0\,.$$

Remark 3.5.1. Note that, by assumption (H3′), there exist two constants $\alpha, \beta > 0$ such that

$$\alpha t - \beta \leq f(t) \leq \beta(t+1) \tag{3.60}$$

for every $t \geq 0$. Moreover, we claim that there exists a constant $c_2 > 0$ such that

$$c_2(|z| \wedge 1) \leq \psi(z, \nu) .$$

Indeed, let $c > 0$ be such that $g(t) \geq c(|t| \wedge 1)$ (see Remark 3.2.4). Let ξ^1 be the unit vector lying on the plane spanned by z and ν with the direction of the bisector of the angle in $[0, \frac{\pi}{2}]$ between the directions $\pm\nu$ and $\frac{z}{|z|}$. Note that $|z \cdot \xi^1| \geq \frac{\sqrt{2}}{2}|z|$ and $|\nu \cdot \xi^1| \geq \frac{\sqrt{2}}{2}$. Let $\xi^2, \dots, \xi^n \in \mathbb{S}^{n-1}$ be such that $(\xi^1, \dots, \xi^n)$ is an orthonormal basis of $\mathbb{R}^n$. Then, by (H2′), we have

$$\begin{aligned} \psi(z,\nu) &\geq \Big(\sum_{i=1}^{n} g(z \cdot \xi^i)^2 |\nu \cdot \xi^i|^2 \Big)^{\frac{1}{2}} \geq c(|z \cdot \xi^1| \wedge 1)|\nu \cdot \xi^1| \\ &\geq c(\tfrac{\sqrt{2}}{2}|z| \wedge 1)\tfrac{\sqrt{2}}{2} \geq c_2(|z| \wedge 1) , \end{aligned}$$

for a suitable constant $c_2 > 0$.

The L^1-lower semicontinuous envelope of $\mathcal{F}$ is the functional $\mathrm{sc}^-\mathcal{F} \colon BD(\Omega) \to [0, +\infty)$ defined by

$$\mathrm{sc}^-\mathcal{F}(u) := \inf\{\liminf_{k\to+\infty} \mathcal{F}(u_k) : \ u_k \in BD(\Omega), \ u_k \to u \text{ in } L^1(\Omega;\mathbb{R}^n)\} .$$

We are now in a position to state our relaxation result.

Theorem 3.5.2. *Assume that* (H1′)–(H3′) *hold. Then there exists a Borel function* $\overline{\psi} \colon \mathbb{R}^n \times \mathbb{S}^{n-1} \to [0, +\infty)$ *such that*

$$\mathrm{sc}^-\mathcal{F}(u) = \int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} \overline{\psi}([u], \nu_u)\,\mathrm{d}\mathcal{H}^{n-1}, \tag{3.61}$$

for every $u \in BD(\Omega)$.

To prove the theorem, we consider the localised functional $\mathcal{F} \colon L^1(\Omega;\mathbb{R}^n) \times \mathscr{A}(\Omega) \to [0, +\infty]$ defined by

$$\mathcal{F}(u; A) := \int_A f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(A) + \int_{J_u \cap A} \psi([u], \nu_u)\,\mathrm{d}\mathcal{H}^{n-1},$$

if $u|_A \in BD(A)$, and $\mathcal{F}(u; A) := +\infty$ otherwise in $L^1(\Omega;\mathbb{R}^n)$. Its L^1-lower semicontinuous envelope is the functional $\mathrm{sc}^-\mathcal{F} \colon L^1(\Omega;\mathbb{R}^n) \times \mathscr{A}(\Omega) \to [0, +\infty]$ defined by

$$\mathrm{sc}^-\mathcal{F}(u; A) := \inf\{\liminf_{k\to+\infty} \mathcal{F}(u_k; A) : \ u_k \to u \text{ in } L^1(\Omega;\mathbb{R}^n)\} ,$$

for every $u \in L^1(\Omega;\mathbb{R}^n)$ and $A \in \mathscr{A}(\Omega)$. Let us notice that $\mathrm{sc}^-\mathcal{F}(u;\Omega) = \mathrm{sc}^-\mathcal{F}(u)$ for every $u \in BD(\Omega)$.

One of the main tools used to prove Theorem 3.5.2 is Theorem 1.4.6. We will show that the lower semicontinuous envelope of $\mathcal{F}$ satisfies the assumptions of Theorem 1.4.6. We start by proving a weak form of the subadditivity condition for $\mathrm{sc}^-\mathcal{F}$.

Lemma 3.5.3. *Let $A, B \in \mathscr{A}(\Omega)$ and $A' \in \mathscr{A}(\Omega)$ such that $A' \Subset A$. Then*

$$\mathrm{sc}^- \mathcal{F}(u; A' \cup B) \le \mathrm{sc}^- \mathcal{F}(u; A) + \mathrm{sc}^- \mathcal{F}(u; B)\,, \tag{3.62}$$

for every $u \in L^1(\Omega; \mathbb{R}^n)$.

Proof. Let us fix $u \in L^1(\Omega; \mathbb{R}^n)$ and let us consider two sequences u_k^A and u_k^B in $L^1(\Omega; \mathbb{R}^n)$ converging in $L^1(\Omega; \mathbb{R}^n)$ to the function u such that

$$\mathrm{sc}^- \mathcal{F}(u; A) = \lim_{k\to+\infty} \mathcal{F}(u_k^A; A) \quad \text{and} \quad \mathrm{sc}^- \mathcal{F}(u; B) = \lim_{k\to+\infty} \mathcal{F}(u_k^B; B)\,. \tag{3.63}$$

It suffices to prove (3.62) when the right hand side is finite. Hence we can assume that $\mathcal{F}(u_k^A; A)$ and $\mathcal{F}(u_k^B; B)$ are equibounded sequences. In particular, we have that $u_k^A \in BD(A)$ and $u_k^B \in BD(B)$.

We now use the De Giorgi slicing and averaging argument to construct a suitable sequence $u_k \in BD(A\cup B)$ converging to u in $L^1(\Omega; \mathbb{R}^n)$. Let $d := \mathrm{dist}(A', \partial A) > 0$ and let $h \in \mathbb{N}$. Let $A_0 := A'$ and $A_{h+1} := A$. We consider a chain of open sets $A_1, \dots, A_h$ such that $A_i \Subset A_{i+1}$ and $\mathrm{dist}(A_i, \partial A_{i+1}) \ge d/(h+1)$ for every $0 \le i \le h$. Let $\varphi_i \in \mathcal{C}_c^1(\Omega)$ be such that $0 \le \varphi_i \le 1$, $\mathrm{supp}(\varphi_i) \subset A_{i+1}$, and $\varphi_i = 1$ in a neighborhood of $\overline{A}_i$. We assume in addition that $||\nabla\varphi_i||_{L^\infty(\Omega)} \le 2(h+1)/d$. We set

$$u_k^i := \varphi_i u_k^A + (1-\varphi_i) u_k^B \in BD(A \cup B)\,, \tag{3.64}$$

for $i = 0, \dots, h$. By the locality of $\mathcal{F}$, we obtain

$$\begin{aligned} \mathcal{F}(u_k^i; A' \cup B) &\le \mathcal{F}(u_k^i; A_i) + \mathcal{F}(u_k^i; B \cap (A_{i+1} \setminus A_i)) + \mathcal{F}(u_k^i; B \setminus A_{i+1}) \\ &= \mathcal{F}(u_k^A; A_i) + \mathcal{F}(u_k^i; B \cap (A_{i+1} \setminus A_i)) + \mathcal{F}(u_k^B; B \setminus A_{i+1}) \\ &\le \mathcal{F}(u_k^A; A) + \mathcal{F}(u_k^i; B \cap (A_{i+1} \setminus A_i)) + \mathcal{F}(u_k^B; B)\,, \end{aligned} \tag{3.65}$$

since $u_k^i = u_k^A$ on A_i and $u_k^i = u_k^B$ in a neighborhood of $\mathbb{R}^n \setminus A_{i+1}$. Let $S_i := B \cap (A_{i+1} \setminus A_i)$ and let us estimate $\mathcal{F}(u_k^i; S_i)$. From (3.64) we deduce that

$$Eu_k^i = \varphi_i Eu_k^A + (1-\varphi_i) Eu_k^B + \nabla\varphi_i \odot (u_k^A - u_k^B)$$

and therefore

$$\begin{aligned} \mathcal{F}(u_k^i; S_i) = &\int_{S_i} f(|\varphi_i \mathcal{E}u_k^A + (1-\varphi_i)\mathcal{E}u_k^B + \nabla\varphi_i \odot (u_k^A - u_k^B)|)\,\mathrm{d}x + \\ &+ C|\varphi_i E^c u_k^A + (1-\varphi_i) E^c u_k^B|(S_i) + \int_{J_{u_k^i} \cap S_i} \psi([u_k^i], \nu_{u_k^i})\,\mathrm{d}\mathcal{H}^{n-1}. \end{aligned} \tag{3.66}$$

By (3.60) we have that

$$\begin{aligned} &\int_{S_i} f(|\varphi_i \mathcal{E}u_k^A + (1-\varphi_i)\mathcal{E}u_k^B + \nabla\varphi_i \odot (u_k^A - u_k^B)|)\,\mathrm{d}x \\ &\le \beta \int_{S_i} |\varphi_i \mathcal{E}u_k^A + (1-\varphi_i)\mathcal{E}u_k^B + \nabla\varphi_i \odot (u_k^A - u_k^B)|\,\mathrm{d}x + \beta\mathcal{L}^n(S_i) \\ &\le \beta\Big[\int_{S_i} |\mathcal{E}u_k^A|\,\mathrm{d}x + \int_{S_i} |\mathcal{E}u_k^B|\,\mathrm{d}x + \int_{S_i} |\nabla\varphi_i \odot (u_k^A - u_k^B)|\,\mathrm{d}x + \mathcal{L}^n(S_i)\Big] \\ &\le c\Big[\int_{S_i} f(|\mathcal{E}u_k^A|)\,\mathrm{d}x + \int_{S_i} f(|\mathcal{E}u_k^B|)\,\mathrm{d}x + (h+1)\int_{S_i} |u_k^A - u_k^B|\,\mathrm{d}x + \mathcal{L}^n(S_i)\Big]\,, \end{aligned} \tag{3.67}$$

where $c > 0$ is a suitable constant. Moreover

$$C|\varphi_i E^c u_k^A + (1-\varphi_i)E^c u_k^B|(S_i) \le C|E^c u_k^A|(S_i) + C|E^c u_k^B|(S_i)\,. \tag{3.68}$$

Finally, using the bounds $c_2(|z| \wedge 1) \le \psi(z,\nu) \le c_1(|z| \wedge 1)$, from (3.64) and Corollary 1.3.4, we deduce that

$$\begin{aligned}\int\limits_{J_{u_k^i}\cap S_i} \psi([u_k^i],\nu_{u_k^i})\,\mathrm{d}\mathcal{H}^{n-1} &\le c_1 \int\limits_{J_{u_k^i}\cap S_i} |[u_k^i]| \wedge 1\,\mathrm{d}\mathcal{H}^{n-1}\\ &\le c_1\Big[\int\limits_{J_{u_k^A}\cap S_i} |[u_k^A]| \wedge 1\,\mathrm{d}\mathcal{H}^{n-1} + \int\limits_{J_{u_k^B}\cap S_i} |[u_k^B]| \wedge 1\,\mathrm{d}\mathcal{H}^{n-1}\Big]\\ &\le c\Big[\int\limits_{J_{u_k^A}\cap S_i} \psi([u_k^A],\nu_{u_k^A})\,\mathrm{d}\mathcal{H}^{n-1} + \int\limits_{J_{u_k^B}\cap S_i} \psi([u_k^B],\nu_{u_k^B})\,\mathrm{d}\mathcal{H}^{n-1}\Big]\,,\end{aligned} \tag{3.69}$$

where $c > 0$ is a suitable constant. Summing (3.67)–(3.69), by (3.65) and (3.66) we get

$$\begin{aligned}\mathcal{F}(u_k^i; A' \cup B) \le \mathcal{F}(u_k^A; A) + \mathcal{F}(u_k^B; B) + c[\mathcal{F}(u_k^A; S_i) + \mathcal{F}(u_k^B; S_i) + \mathcal{L}^n(S_i)]\\ + c(h+1)\int\limits_{S_i} |u_k^A - u_k^B|\,\mathrm{d}x\,.\end{aligned}$$

Summing the inequality above for $i = 0, \dots, h$ and taking the average with respect to i, we obtain that there exists an index i_k such that

$$\begin{aligned}\mathcal{F}(u_k^{i_k}; A' \cup B) &\le \mathcal{F}(u_k^A; A) + \mathcal{F}(u_k^B; B)\\ &\quad + \frac{c}{h+1}[\mathcal{F}(u_k^A; B \cap (A \setminus A')) + \mathcal{F}(u_k^B; B \cap (A \setminus A')) + \mathcal{L}^n(B \cap (A \setminus A'))]\\ &\quad + c \int\limits_{B\cap(A\setminus A')} |u_k^A - u_k^B|\,\mathrm{d}x\\ &\le \mathcal{F}(u_k^A; A) + \mathcal{F}(u_k^B; B) + \frac{c}{h+1}[\mathcal{F}(u_k^A; A) + \mathcal{F}(u_k^B; B) + \mathcal{L}^n(\Omega)]\\ &\quad + c \int\limits_{B\cap(A\setminus A')} |u_k^A - u_k^B|\,\mathrm{d}x\,.\end{aligned}$$

Let us define $u_k := u_k^{i_k}$. Letting $k \to +\infty$ and $h \to +\infty$ in the inequality above, we conclude that

$$\mathrm{sc}^-\mathcal{F}(u; A' \cup B) \le \liminf_{k\to+\infty} \mathcal{F}(u_k; A' \cup B) \le \lim_{k\to+\infty} \mathcal{F}(u_k^A; A) + \lim_{k\to+\infty} \mathcal{F}(u_k^B; B)\,.$$

By (3.63), this gives (3.62). □

We are now able to prove that the functional $\mathrm{sc}^-\mathcal{F}$ satisfies all the hypotheses of Theorem 1.4.6.

Lemma 3.5.4. *The functional* $\mathrm{sc}^-\mathcal{F}$ *satisfies the following properties:*

(a) $\mathrm{sc}^-\mathcal{F}$ *is local;*

(b) $\mathrm{sc}^-\mathcal{F}(\cdot;A)$ *is* L^1*-lower semicontinuous, for every* $A \in \mathscr{A}(\Omega)$*;*

(c) $\mathrm{sc}^-\mathcal{F}(u;A) \leq C|Eu|(A) + f(0)\mathcal{L}^n(A)$*, for every* $u \in BD(\Omega)$ *and for every* $A \in \mathscr{A}(\Omega)$*;*

(d) for every $u \in BD(\Omega)$*,* $\mathrm{sc}^-\mathcal{F}(u;\cdot)$ *is the restriction to* $\mathscr{A}(\Omega)$ *of a Radon measure;*

(e) $\mathrm{sc}^-\mathcal{F}(u(\cdot - x_0) + b; x_0 + A) = \mathrm{sc}^-\mathcal{F}(u;A)$ *for all* $b \in \mathbb{R}^n$ *and* $x_0 \in \mathbb{R}^n$ *such that* $x_0 + A \subset \Omega$*.*

Proof. The proofs of the lower semicontinuity, of the upper bound, and of the translation invariance are immediate. The functional $\mathrm{sc}^-\mathcal{F}$ is local by [29, Proposition 16.15].

In order to prove that $\mathrm{sc}^-\mathcal{F}(u;\cdot)$ is a measure, it is convenient to introduce the inner regular envelope of $\mathrm{sc}^-\mathcal{F}$, i.e., the functional $\overline{\mathcal{F}}\colon L^1(\Omega;\mathbb{R}^n)\times\mathscr{A}(\Omega) \to [0,+\infty)$ defined by

$$\overline{\mathcal{F}}(u;A) := \sup\{(\mathrm{sc}^-\mathcal{F})(u;U) : U \in \mathscr{A}(\Omega),\ U \Subset A\}\,,$$

for every $u \in L^1(\Omega;\mathbb{R}^n)$ and $A \in \mathscr{A}(\Omega)$. Note that $\overline{\mathcal{F}}(u;A) \leq \mathrm{sc}^-\mathcal{F}(u;A)$. Let us fix $u \in BD(\Omega)$. We claim that

$$\overline{\mathcal{F}}(u;A) = \mathrm{sc}^-\mathcal{F}(u;A)\,, \tag{3.70}$$

for every $A \in \mathscr{A}(\Omega)$. Indeed, let $\varepsilon > 0$. There exists a compact set $K \subset A$ such that $\mathcal{F}(u, A \setminus K) < \varepsilon$. Let us fix $A', A'' \in \mathscr{A}(\Omega)$ such that $K \subset A' \Subset A'' \Subset A$. Then, by Lemma 3.5.3, we have that

$$\begin{aligned}\mathrm{sc}^-\mathcal{F}(u;A) &= \mathrm{sc}^-\mathcal{F}(u;A' \cup (A\setminus K)) \leq \mathrm{sc}^-\mathcal{F}(u;A'') + \mathrm{sc}^-\mathcal{F}(u;A\setminus K)\\ &\leq \overline{\mathcal{F}}(u;A) + \mathcal{F}(u;A\setminus K) \leq \overline{\mathcal{F}}(u;A) + \varepsilon\,.\end{aligned}$$

Letting $\varepsilon \to 0$, we conclude the proof of (3.70). This implies that $\mathrm{sc}^-\mathcal{F}(u;\cdot)$ is inner regular on $\mathscr{A}(\Omega)$, for every $u \in BD(\Omega)$. By [29, Proposition 16.12], $\mathrm{sc}^-\mathcal{F}(u;\cdot)$ is superadditive. The subadditivity of $\mathrm{sc}^-\mathcal{F}(u;\cdot)$ follows from Lemma 3.5.3 and from the inner regularity. Hence, we can apply [29, Theorem 14.23] to extend $\mathrm{sc}^-\mathcal{F}(u;\cdot)$ to a Borel measure. Actually, it is a bounded measure thanks to the upper bound $\mathrm{sc}^-\mathcal{F}(u;A) \leq C|Eu|(A) + f(0)\mathcal{L}^n(A)$. ☐

In order to prove Theorem 3.5.2, it is useful to bound from below the functional $\mathcal{F}$ with the functional $\mathcal{F}_G\colon L^1(\Omega;\mathbb{R}^n)\times\mathscr{A}(\Omega) \to [0,+\infty]$ defined by

$$\mathcal{F}_G(u;A) := \int_A f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(A) + \int_{J_u\cap A} G([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} \tag{3.71}$$

if $u \in BD(A)$, and $\mathcal{F}_G(u;A) := +\infty$ otherwise, where

$$G(z,\nu) = \sup_{(\xi^1,\dots,\xi^n)} \Big(\sum_{i=1}^n g(z\cdot\xi^i)^2|\nu\cdot\xi^i|^2\Big)^{\frac{1}{2}}.$$

Note that, by Theorem 3.4.1, for every $A \in \mathscr{A}(\Omega)$ the functional $\mathcal{F}_G(\cdot;A)$ is L^1-lower semicontinuous in $BD(A)$.

Proof of Theorem 3.5.2. Let us fix $u \in BD(\Omega)$. By $(\mathrm{H2}')$, we have that $G \leq \psi$ and hence

$$\mathcal{F}_G(\cdot; A) \leq \mathcal{F}(\cdot; A)\,,$$

for every open set $A \subset \Omega$. Therefore, by the lower semicontinuity of $\mathcal{F}_G(\cdot; A)$,

$$\mathcal{F}_G(u; A) \leq \mathrm{sc}^-\mathcal{F}(u; A) \leq \mathcal{F}(u; A)\,,$$

for every open set $A \subset \Omega$. By Lemma 3.5.4, $\mathrm{sc}^-\mathcal{F}(u;\cdot)$ is the restriction to $\mathscr{A}(\Omega)$ of a Radon measure defined on $\mathscr{B}(\Omega)$, still denoted by $\mathrm{sc}^-\mathcal{F}(u;\cdot)$. Hence

$$\mathcal{F}_G(u; B) \leq \mathrm{sc}^-\mathcal{F}(u; B) \leq \mathcal{F}(u; B)\,,$$

for every Borel set $B \subset \Omega$. Let us consider the sets $B_1, B_2, B_3 \in \mathscr{B}(\Omega)$ as in Remark 3.4.6. We recall that B_1, B_2, B_3 are pairwise disjoint and $B_1 \cup B_2 \cup B_3 = \Omega$. Moreover, $\mathcal{L}^n$ is concentrated on B_1, $E^c u$ is concentrated on B_2, and $\mathcal{H}^{n-1}((J_u \setminus B_3) \cup (B_3 \setminus J_u)) = 0$. Then

$$\int_{B_1} f(|\mathcal{E}u|)\,\mathrm{d}x = \mathcal{F}_G(u; B_1) \leq \mathrm{sc}^-\mathcal{F}(u; B_1) \leq \mathcal{F}(u; B_1) = \int_{B_1} f(|\mathcal{E}u|)\,\mathrm{d}x\,,$$
$$C|E^c u|(B_2) = \mathcal{F}_G(u; B_2) \leq \mathrm{sc}^-\mathcal{F}(u; B_2) \leq \mathcal{F}(u; B_2) = C|E^c u|(B_2)$$

and therefore

$$\mathrm{sc}^-\mathcal{F}(u; B_1) = \int_{\Omega} f(|\mathcal{E}u|)\,\mathrm{d}x\,, \tag{3.72}$$

$$\mathrm{sc}^-\mathcal{F}(u; B_2) = C|E^c u|(\Omega)\,. \tag{3.73}$$

On the other hand, for every $\lambda > 0$ we can apply Theorem 1.4.6 to the functional $\mathcal{G}_\lambda(u; A) := \mathrm{sc}^-\mathcal{F}(u; A) + \lambda|Eu|(A)$. Thus we have

$$\mathrm{sc}^-\mathcal{F}(u; B_3) + \lambda|Eu|(B_3) = \mathrm{sc}^-\mathcal{F}(u; J_u) + \lambda|Eu|(J_u) = \int_{J_u} \overline{\psi}_\lambda([u], \nu_u)\,\mathrm{d}\mathcal{H}^{n-1},$$

where $\overline{\psi}_\lambda$ is given by (1.12), i.e.,

$$\overline{\psi}_\lambda(z,\nu) = \limsup_{\rho\to 0^+} \left[\frac{1}{\rho^{n-1}} \inf\{\mathcal{G}_\lambda(v; Q^\nu_\rho) : v \in BD(Q^\nu_\rho),\ v(x) = u_{\nu,z}(x) \text{ on } \partial Q^\nu_\rho\}\right]. \tag{3.74}$$

Defining $\overline{\psi}(z,\nu) := \lim_{\lambda\searrow 0} \overline{\psi}_\lambda(z,\nu)$, we infer

$$\mathrm{sc}^-\mathcal{F}(u; B_3) = \int_{J_u} \overline{\psi}([u], \nu_u)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{3.75}$$

Summing (3.72), (3.73), and (3.75) we obtain (3.61). □

3.6 An example of surface density

In this section we provide the explicit expression of the surface density $G(z,\nu)$ defined in (3.5) when $g(t)=|t|\wedge 1$, i.e.,

$$G(z,\nu) := \sup_{(\xi^1,\dots,\xi^n)} \Big(\sum_{i=1}^n (|z\cdot\xi^i|\wedge 1)^2 |\nu\cdot\xi^i|^2\Big)^{\frac12}, \tag{3.76}$$

where the supremum is taken over all orthonormal bases $(\xi^1,\dots,\xi^n)$ of $\mathbb{R}^n$.

First of all we prove that the function G in (3.76) is invariant under rotations.

Lemma 3.6.1. *Let G be the function defined in (3.76). Then for every $z\in\mathbb{R}^n$, $\nu\in\mathbb{S}^{n-1}$, and $R\in \mathrm{O}(n)$ we have that $G(z,\nu)=G(Rz,R\nu)$. Moreover $G(z,\nu)=G(-z,\nu)=G(z,-\nu)$.*

Proof. Let us fix $z\in\mathbb{R}^n$, $\nu\in\mathbb{S}^{n-1}$, and $R\in\mathrm{O}(n)$. For every orthonormal basis $(\xi^1,\dots,\xi^n)$ we have that $(R^T\xi^1,\dots,R^T\xi^n)$ is an orthonormal basis. Moreover, for every orthonormal basis $(\zeta^1,\dots,\zeta^n)$ there exists an orthonormal basis $(\xi^1,\dots,\xi^n)$ such that $\zeta^i=R^T\xi^i$, $i=1,\dots,n$. Therefore

$$\begin{aligned} G(Rz,R\nu) &= \sup_{(\xi^1,\dots,\xi^n)} \Big(\sum_{i=1}^n g(Rz\cdot\xi^i)^2|R\nu\cdot\xi^i|^2\Big)^{\frac12} \\ &= \sup_{(\xi^1,\dots,\xi^n)} \Big(\sum_{i=1}^n g(z\cdot R^T\xi^i)^2|\nu\cdot R^T\xi^i|^2\Big)^{\frac12} \\ &= \sup_{(\zeta^1,\dots,\zeta^n)} \Big(\sum_{i=1}^n g(z\cdot\zeta^i)^2|\nu\cdot\zeta^i|^2\Big)^{\frac12} \\ &= G(z,\nu)\,. \end{aligned}$$

The symmetry of the function G stated in the lemma is a straightforward consequence of (3.76). ☐

We study the function G in the two dimensional case, i.e., when $n=2$. Let us fix $\nu\in\mathbb{S}^1$. Thanks to Lemma 3.6.1, we can reduce to the case $\nu=(1,0)$ by applying a suitable rotation. To study the function $z\mapsto G(z,\nu)$, it is convenient to express the vector $z\in\mathbb{R}^2$ in polar coordinates. Let ρ be the norm of z and let φ be the angle between ν and z. By Lemma 3.6.1, it is enough to study the case $\varphi\in[0,\frac{\pi}{2}]$. In this way $z=(\rho\cos\varphi,\rho\sin\varphi)$.

Proposition 3.6.2. *Let $\nu=(1,0)$. Let $\rho>0$, $\varphi\in[0,\frac{\pi}{2}]$, and $z=(\rho\cos\varphi,\rho\sin\varphi)$. Then*

$$G(z,\nu)=\begin{cases} \rho\Big(\cos^4(\frac{\varphi}{2})+\sin^4(\frac{\varphi}{2})\Big)^{\frac12} & \textit{if } \rho\cos(\frac{\varphi}{2})\le 1\,,\\ 1 & \textit{if } \rho\cos\varphi\ge 1 \textit{ or } \rho\ge\sqrt{2}\,,\\ \Big(\cos^2(\overline{\theta})+(\rho^2-1)\sin^2(\overline{\theta})\Big)^{\frac12} & \textit{otherwise}\,, \end{cases} \tag{3.77}$$

where $\overline{\theta}=\varphi-\arccos(\frac{1}{\rho})\in[0,\frac{\varphi}{2}]$.

Proof. By (3.76) we have that

$$G(z,\nu)^2 = \sup_{(\xi^1,\xi^2)} \left(g(z\cdot\xi^1)^2|\nu\cdot\xi^1|^2 + g(z\cdot\xi^2)^2|\nu\cdot\xi^2|^2\right), \tag{3.78}$$

where the supremum is taken over all orthonormal bases of $\mathbb{R}^2$. If we write ξ^1 in polar coordinates, it is easy to see that (3.78) is equivalent to

$$G(z,\nu)^2 = \sup_{\varphi-\frac{\pi}{2}<\theta\le\varphi} \gamma(\theta). \tag{3.79}$$

where

$$\gamma(\theta) := \min\{\rho|\cos(\varphi-\theta)|,1\}^2\cos^2(\theta) + \min\{\rho|\sin(\varphi-\theta)|,1\}^2\sin^2(\theta).$$

Indeed, it is sufficient to take the supremum in (3.78) for θ ranging in an interval of length $\frac{\pi}{2}$. Note that $G(z,\nu)\le 1$, because $\gamma(\theta)\le 1$ for $\varphi-\frac{\pi}{2}<\theta\le\varphi$.

Let us assume that $\rho\cos(\frac{\varphi}{2})\le 1$. Then, for $\varphi-\frac{\pi}{2}<\theta\le\varphi$, we have that

$$\begin{aligned}\gamma(\theta) &\le \rho^2\cos^2(\varphi-\theta)\cos^2(\theta) + \rho^2\sin^2(\varphi-\theta)^2\sin^2(\theta)\\ &= \tfrac14\rho^2\cos(2\varphi-4\theta) + \rho^2\cos^4(\tfrac{\varphi}{2}) + \rho^2\sin^4(\tfrac{\varphi}{2}) - \tfrac14\rho^2.\end{aligned} \tag{3.80}$$

Since $-\pi\le -2\varphi\le 2\varphi-4\theta<2\pi-2\varphi\le 2\pi$, the function $\theta\mapsto\cos(2\varphi-4\theta)$ attains its maximum at $\theta=\frac{\varphi}{2}$. Since $\rho\cos(\frac{\varphi}{2})\le 1$ and $0\le\frac{\varphi}{2}\le\frac{\pi}{4}$, we also have that $\rho\sin(\frac{\varphi}{2})\le 1$. This implies that $\gamma(\theta)$ attains its maximum at $\theta=\frac{\varphi}{2}$ and therefore $G(z,\nu)^2=\rho^2\cos^4(\frac{\varphi}{2})+\rho^2\sin^4(\frac{\varphi}{2})$. This concludes the study of the case $\rho\cos(\frac{\varphi}{2})\le 1$.

We therefore suppose that $\rho\cos(\frac{\varphi}{2})>1$ in what follows.

Let us assume that $\rho\cos\varphi\ge 1$ first. We simply note that in this case the maximum of $\gamma(\theta)$ is attained at $\theta=0$ and $\gamma(0)=1$.

Hence, let $\rho\cos\varphi<1$ and $\rho\le\sqrt{2}$. We claim that the maximum in (3.79) is attained at $\theta=\overline{\theta}=\varphi-\arccos(\frac{1}{\rho})$. Note that $0<\overline{\theta}<\frac{\varphi}{2}$, since $\rho\cos\varphi<1$ and $\rho\cos(\frac{\varphi}{2})>1$. For $\overline{\theta}\le\theta\le\varphi$, we have that $\rho\cos(\varphi-\theta)\ge 1$, and therefore $\rho^2\sin^2(\varphi-\theta)\le\rho^2-1\le 1$. This implies that

$$\begin{aligned}\gamma(\theta) &\le \cos^2(\theta)+\rho^2\sin^2(\varphi-\theta)^2\sin^2(\theta)\le\cos^2(\theta)+(\rho^2-1)\sin^2(\theta)\\ &= (\rho^2-1)+(2-\rho^2)\cos^2\theta.\end{aligned} \tag{3.81}$$

The function $\theta\mapsto(\rho^2-1)+(2-\rho^2)\cos^2\theta$ is nonincreasing for $\theta\in[\overline{\theta},\varphi]$, and therefore its maximum in the interval $[\overline{\theta},\varphi]$ is attained at $\theta=\overline{\theta}$. Since $\rho\cos(\varphi-\overline{\theta})=1$ and $\rho\sin(\varphi-\theta)\le 1$, we get that the maximum of the function $\gamma(\theta)$ in the interval $[\overline{\theta},\varphi]$ is attained at $\theta=\overline{\theta}$. For $\varphi-\frac{\pi}{2}<\theta\le\overline{\theta}$, the maximum of the function $\theta\mapsto\cos(2\varphi-4\theta)$ is attained at $\theta=\overline{\theta}$, since $0\le 2\varphi-4\overline{\theta}\le 2\varphi-4\theta<2\pi-2\varphi\le 2\pi$ and $\overline{\theta}>0$. Hence, by inequality (3.80), we have that the maximum of the function $\gamma(\theta)$ in the interval $(\varphi-\frac{\pi}{2},\overline{\theta}]$ is attained at $\theta=\overline{\theta}$. This concludes the study of the case $\rho\cos\varphi<1$ and $\rho\le\sqrt{2}$.

We conclude the proof by observing that if $\rho\cos\varphi<1$ and $\rho>\sqrt{2}$, then $\gamma(\overline{\theta})=1$. Indeed, $\rho\cos(\varphi-\overline{\theta})=1$ and $\rho^2\sin^2(\varphi-\overline{\theta})=\rho^2-\rho^2\cos^2(\varphi-\overline{\theta})=\rho^2-1\ge 1$. □

Remark 3.6.3. Let us fix $\nu\in\mathbb{S}^{n-1}$. For every $z\in\mathbb{R}^n$ we have that

$$|z\odot\nu| = |z|\Big(\cos^4(\tfrac{\varphi}{2})+\sin^4(\tfrac{\varphi}{2})\Big)^{\frac12},$$

where $\varphi \in [0, \frac{\pi}{2}]$ is the angle formed by the directions $\pm\nu$ and $\frac{z}{|z|}$. Hence, by formula (3.77), for $|z|\cos(\frac{\varphi}{2}) \leq 1$, i.e., in the region colored in light gray in Figure 3.4, we have that $G(z,\nu) = |z \odot \nu|$. For $|z| \geq \sqrt{2}$ or $|z|\cos(\varphi) \geq 1$, i.e., outside the colored regions in Figure 3.4, by (3.77) we obtain $G(z,\nu) = 1$. In the remaining part of $\mathbb{R}^2$, i.e., in the region colored in dark gray in Figure 3.4, the function $z \mapsto G(z,\nu)$ makes a transition between the function $z \mapsto |z \odot \nu|$ and the function with constant value 1.

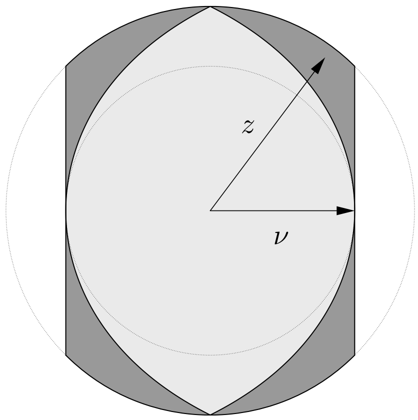

Figure 3.4: Behaviour of the function $z \mapsto G(z,\nu)$.

CHAPTER 4

Quasistatic evolution for irreversible cohesive fracture

4.1 Overview of the chapter

In this chapter we prove the existence of quasistatic evolutions for a cohesive zone model, whose main feature is that the density of the energy dissipated in the fracture process depends on the total variation of the amplitude of the jump. The results presented here are based on the work [28], in collaboration with Crismale and Lazzaroni.

The model is described in detail in Section 4.2, where we specify all the assumptions on the initial data, on the time-dependent boundary conditions that drive the evolution, and on the surface energy density g. Moreover, we present the notion of quasistatic evolution (Definition 4.2.6) and some results on the strong formulation that is satisfied by the energetic solutions under suitable regularity assumptions (Propositions 4.2.7 and 4.2.8). Theorem 4.2.9 is the main result of the chapter: it concerns the existence of quasistatic evolutions and their approximation by means of the discrete-time evolutions. For the reader's convenience, the final part of Section 4.2 contains a short outline of the existence proof, which is presented in more detail in the remaining part of the chapter.

In Section 4.3 we study the discrete-time evolutions obtained by solving the incremental minimum problems. In particular, in Proposition 4.3.1 we show that the approximate evolutions satisfy a discrete global stability and a discrete energy-dissipation inequality.

We pass to the continuous-time limit in Section 4.4, obtaining a weak notion of quasistatic evolution based on Young measures. Specifically, in the dissipated energy which appears in the global stability and in the energy-dissipation balance, the variation of the jumps is replaced by a Young measure.

Finally, in Section 4.5 we employ the results obtained in Section 4.4 to prove the existence of quasistatic evolutions according to the notion based on functions. Moreover, we mention some possible generalisations of the model that account for asymmetric responses to loading and unloading.

4.2 Assumptions on the model and statement of the main result

Reference configuration and boundary conditions. Throughout the chapter, Ω is a bounded, Lipschitz, open set in $\mathbb{R}^n$ representing the cross-section of a cylindrical body in the reference configuration (in the setting of antiplane shear). The cracks of the body will be contained in a prescribed crack surface Γ, where Γ is a $(n-1)$-dimensional Lipschitz manifold in $\mathbb{R}^n$ with $0 < \mathcal{H}^{n-1}(\Gamma \cap \overline{\Omega}) < \infty$. Moreover, we assume that $\Omega \setminus \Gamma = \Omega^+ \cup \Omega^-$, where Ω^+ and Ω^- are disjoint open connected sets with Lipschitz boundary. The normal $\nu(x) = \nu_\Gamma(x)$ to the surface Γ is chosen in such a way that it coincides with the outer normal to $\partial\Omega^-$.

We consider evolutions driven by a time-dependent boundary condition assigned on the Dirichlet part of the boundary $\partial_D\Omega$. We assume that $\partial_D\Omega$ is a relatively open set of $\partial\Omega$ and that $\mathcal{H}^{n-1}(\partial_D\Omega \cap \partial\Omega^\pm) > 0$, in order to apply the Poincaré Inequality separately in Ω^+ and Ω^-. We denote by $\partial_N\Omega$ the remaining part of the boundary, i.e., $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$.

For every $w \in H^1(\Omega)$, we define the set of *admissible displacements* corresponding to w by

$$\mathcal{A}dm(w) := \{u \in H^1(\Omega \setminus \Gamma) \ : \ u = w \text{ on } \partial_D\Omega\}. \tag{4.1}$$

We assign a function $t \mapsto w(t)$ defined on $[0,T]$ with values in $H^1(\Omega)$ and we assume that

$$t \mapsto w(t) \quad \text{belongs to} \quad AC([0,T]; H^1(\Omega)). \tag{4.2}$$

For simplicity in this thesis we do not consider volume or boundary forces, which may be included in the model with minor modifications.

Variation of jumps and initial data. In order to present the notion of quasistatic evolution, we introduce a function $V_u(t)$ describing the variation of the jumps on Γ of an evolution $s \mapsto u(s)$ in a time interval $[0,t]$. To define $V_u(t)$ without regularity assumptions on $s \mapsto u(s)$, we employ the notion of essential variation.

First we recall the definition of the essential supremum of a family of measurable functions, that is the least upper bound in the sense of a.e. inequality. We give this definition in the case of functions defined on the measure space $(\Gamma; \mathcal{H}^{n-1})$. Indeed, this will be the relevant setting for our model.

Definition 4.2.1. Let $(v_i)_{i\in I}$ be a family of measurable functions from Γ to $[-\infty, \infty]$. Let $\overline{v}\colon \Gamma \to [-\infty, \infty]$ be a measurable function such that

(i) for every $i \in I$ we have $\overline{v} \geq v_i$ $\mathcal{H}^{n-1}$-a.e. on Γ;

(ii) if $v\colon \Gamma \to [-\infty, \infty]$ is a measurable function such that for every $i \in I$ we have $v \geq v_i$ $\mathcal{H}^{n-1}$-a.e. on Γ, then $v \geq \overline{v}$ $\mathcal{H}^{n-1}$-a.e. on Γ.

We say that $\overline{v}$ an *essential supremum* of the family $(v_i)_{i\in I}$.

Remark 4.2.2. Given a family of measurable functions $(v_i)_{i\in I}$, there exists a unique (up to $\mathcal{H}^{n-1}$-a.e. equivalence) essential supremum $\overline{v}$ of the family $(v_i)_{i\in I}$. We denote it by $\operatorname*{ess\,sup}_{i\in I} v_i := \overline{v}$.

We now define the essential variation, namely the variation for a time-dependent family of measurable functions, in the sense of a.e. inequality. As done for the essential supremum, we give this definition in the case of functions defined on the measure space $(\Gamma; \mathcal{H}^{n-1})$.

Definition 4.2.3. Let us consider a function $t \mapsto \gamma(t)$, with $\gamma(t)\colon \Gamma \to \mathbb{R}$ measurable for every $t \in [0,T]$. For every $0 \le t_1 \le t_2 \le T$, the *essential variation* of γ in $[t_1, t_2]$ is the function $\operatorname{ess\,Var}(\gamma; t_1, t_2)\colon \Gamma \to [0,\infty]$ defined by

$$\operatorname{ess\,Var}(\gamma; t_1, t_2) := \operatorname{ess\,sup} \Big\{ \sum_{i=1}^{j} |\gamma(s_i) - \gamma(s_{i-1})| \Big\} ,$$

where the essential supremum is taken among all $j \in \mathbb{N}$ and all partitions $t_1 = s_0 < s_1 < \cdots < s_{j-1} < s_j = t_2$ of the interval $[t_1, t_2]$.

Remark 4.2.4. The essential variation satisfies the usual property that

$$\operatorname{ess\,Var}(\gamma; t_1, t_3) = \operatorname{ess\,Var}(\gamma; t_1, t_2) + \operatorname{ess\,Var}(\gamma; t_2, t_3) \quad \mathcal{H}^{n-1}\text{-a.e. on } \Gamma ,$$

for any $0 \le t_1 < t_2 < t_3 \le t$.

Given a function $t \mapsto u(t)$ defined on $[0,T]$ with values in $H^1(\Omega \setminus \Gamma)$, we define the variation $V_u(t)\colon \Gamma \to [0,\infty]$ of its jumps on Γ with initial condition V_0 by

$$V_u(t) := \operatorname{ess\,Var}([u]; 0, t) + V_0 , \tag{4.3}$$

for every $t \in [0,T]$, where $V_0 \colon \Gamma \to [0,\infty]$ is an assigned measurable function.

Initial data. We fix an initial displacement

$$u_0 \in \mathcal{A}dm(w(0)) \tag{4.4}$$

and a function $V_0 \colon \Gamma \to [0,\infty]$ accounting for the variation of previous jumps until the initial time $t = 0$. Indeed we assume that

$$V_0(x) \ge \big|[u_0(x)]\big| \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma . \tag{4.5}$$

If $V_0 = \big|[u_0]\big|$, a monotone crack opening has occurred before the initial time $t = 0$. In general, the crack opening may have oscillated before the initial time in such a way that its variation in time equals V_0.

The surface energy density. We assume that the surface energy density g depends on the point on Γ and on the history of the jump. More precisely, $g\colon \Gamma \times [0,\infty) \to [0,\infty)$ satisfies the following assumptions:

($g1$) g is a Carathéodory integrand, i.e., $g(x,\cdot)$ is continuous for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$ and $g(\cdot,\xi)$ is $\mathcal{H}^{n-1}$-measurable for every $\xi \in [0,\infty)$;

($g2$) $g(x,0) = 0$ and $g(x,\cdot)$ is concave for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$;

($g3$) $\lim_{\xi \to \infty} g(x,\xi) = \kappa(x) \in [\kappa_1, \kappa_2]$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$, where $\kappa_1, \kappa_2 \in (0,\infty)$;

($g4$) the limit

$$\lim_{\xi \to 0^+} \frac{g(x,\xi)}{\xi} =: g'(x,0)$$

exists for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$ and $g'(\cdot,0) \in L^\infty(\Gamma)$.

In particular, for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$ it turns out that $g(x,\cdot)$ is nondecreasing and can be extended to a function in $\mathcal{C}_b([0,\infty])$ by setting $g(x,\infty) := \kappa(x)$.

It will be convenient to introduce a measurable function $\theta\colon \Gamma \to [0,\infty]$ that represents the threshold after which the function $g(x,\cdot)$ becomes constant, i.e.,

$$\theta(x) := \inf\{\xi > 0 \ : \ g(x,\xi) = \kappa(x)\} \in (0,\infty]\,. \tag{4.6}$$

The function $g(x,\cdot)$ is strictly increasing if and only if $\theta(x) = \infty$.

Notice that the set $\Gamma_N(0) := \{V_0 \geq \theta(x)\}$ represents the part of Γ which is already completely broken at the beginning of the process.

As already discussed in the Introduction, the energy dissipated by the crack opening is a function of the variation of the jump $V_u(t)$ defined in (4.3):

$$\int_\Gamma g\big(x, V_u(t;x)\big)\, \mathrm{d}\mathcal{H}^{n-1}(x)\,.$$

Remark 4.2.5. In the cohesive models studied in [40] and [20], the dissipated energy depends the supremum of the jumps reached during the evolution. There it is assumed that, when the crack opening decreases, no energy is dissipated or some dissipated energy is recovered. This behaviour complies with models where the cohesive phenomenon is due to an interplay between elasticity and damage [24, 10]. In contrast, the behaviour described here is motivated by the limit obtained in Chapter 2, where the cohesive energy is due to the interplay between plasticity and damage. For this reason we expect a dissipation of energy even when the crack decreases, which entails the irreversibility of evolutions.

Definition of quasistatic evolution and strong formulation. We are now in a position to give the definition of quasistatic evolution.

Definition 4.2.6. Let w, u_0, and V_0 be as in (4.2)–(4.5). Let $t \mapsto u(t)$ be a function defined on $[0,T]$ with values in $H^1(\Omega\setminus\Gamma)$ and let $V_u(t)$ be the variation of its jumps on Γ, defined in (4.3). We say that $t \mapsto u(t)$ is a *quasistatic evolution* with initial conditions (u_0, V_0) and boundary datum w if u satisfies $u(0) = u_0$ and the following conditions:

(GS) *Global stability*: For every $t \in [0,T]$ we have $u(t) \in \mathcal{A}dm(w(t))$ and

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x + \int_\Gamma g\big(x, V_u(t)\big)\,\mathrm{d}\mathcal{H}^{n-1} \leq \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \hat u|^2\,\mathrm{d}x + \int_\Gamma g\big(x,\, V_u(t) + \big|[\hat u] - [u(t)]\big|\big)\,\mathrm{d}\mathcal{H}^{n-1}\,,$$

for every $\hat u \in \mathcal{A}dm(w(t))$.

(EB) *Energy-dissipation balance*: For every $t \in [0,T]$

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2\,\mathrm{d}x + \int_\Gamma g\big(x, V_u(t)\big)\,\mathrm{d}\mathcal{H}^{n-1} = \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_0|^2\,\mathrm{d}x + \int_\Gamma g\big(x, V_0\big)\,\mathrm{d}\mathcal{H}^{n-1} + \int_0^t \langle \nabla u(s), \nabla \dot w(s)\rangle_{L^2}\,\mathrm{d}s\,.$$

In order to give an insight into the strong formulation of the model studied here, we state two results regarding necessary conditions satisfied by a quasistatic evolution. For simplicity, we derive these differential conditions under the assumption that $g(x,\cdot)$ is of class $\mathcal{C}^1$. We denote by $g'(x,\xi)$ the derivative of $g(x,\xi)$ with respect to ξ.

Proposition 4.2.7. *Assume that $g(x,\cdot)$ is of class $\mathcal{C}^1$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$. Let $t \mapsto u(t)$ be a function defined on $[0,T]$ with values in $H^1(\Omega \setminus \Gamma)$ and satisfying* (GS). *Then for every $t \in [0,T]$ the following hold:*

(i) *The function $u(t)$ is a weak solution to the problem*

$$\begin{cases} \Delta u(t) = 0 & \text{in } \Omega \setminus \Gamma\,, \\ u(t) = w(t) & \text{on } \partial_D \Omega\,, \\ \partial_\nu u(t) = 0 & \text{in } H^{-\frac{1}{2}}(\partial_N \Omega)\,. \end{cases}$$

(ii) *Let $u(t)^+ := u(t)|_{\Omega^+}$ and $u(t)^- := u(t)|_{\Omega^-}$. Then $\partial_\nu u(t)^+ = \partial_\nu u(t)^-$ in $H^{-\frac{1}{2}}(\Gamma)$.*

(iii) *Let $\partial_\nu u(t) := \partial_\nu u(t)^+ = \partial_\nu u(t)^-$. Then $\partial_\nu u(t) \in L^\infty(\Gamma)$ and*

$$|\partial_\nu u(t;x)| \leq g'(x, V_u(t;x)) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma\,. \tag{4.7}$$

To keep the presentation clear, the proof of Proposition 4.2.7 is given in Section 4.5.

Condition (iii) in Proposition 4.2.7 expresses the fact that the surface tension on Γ due to the displacement is constrained to stay below a suitable threshold. The material exhibits an irreversible softening behaviour on Γ, since this threshold decreases in time. Indeed $g'(x,\cdot)$ is nonincreasing and $V_u(\cdot\,;x)$ is nondecreasing in time. However, this condition is static and is not enough to characterise an evolution.

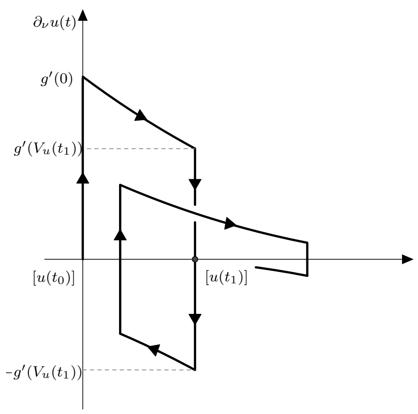

Figure 4.1: Crack opening versus surface tension.

Nonetheless, in the following proposition we employ the energy-dissipation balance to show that the evolution satisfies a flow rule: at the points where a crack opening grows, the surface tension actually must reach the maximal threshold. (See Figure 4.1 for a possible evolution of the surface tension.) The result is proved under regularity assumptions on the evolution $t \mapsto u(t)$. To make the statement concise, we denote by Sign the multifunction given by

$$\mathrm{Sign}(\xi) := \begin{cases} 1 & \text{if } \xi > 0\,, \\ [-1,1] & \text{if } \xi = 0\,, \\ -1 & \text{if } \xi < 0\,. \end{cases}$$

Proposition 4.2.8. *Assume that $g(x,\cdot)$ is of class $\mathcal{C}^1$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$. Let $t \mapsto u(t)$ be a quasistatic evolution in the sense of Definition 4.2.6 and assume that $u \in AC([0,T]; H^1(\Omega \setminus \Gamma))$. Then*

$$\partial_\nu u(t;x) \in g'(x, V_u(t;x))\,\mathrm{Sign}\big([\dot u(t;x)]\big) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma \text{ and a.e. } t \in [0,T]\,,$$

where $[\dot u(t)]$ is the derivative in time of $[u(t)]$ with respect to the strong topology in $L^2(\Gamma)$.

Proposition 4.2.8 is proved in Section 4.5.

Statement of the main result. We now introduce the tools needed to state our main result, which concerns the existence of a quasistatic evolution and the approximation by means of discrete-time evolutions.

As usual in the proof of existence of quasistatic evolutions for rate-independent systems, we construct discrete-time evolutions by solving incremental minimum problems. For every $k \in \mathbb{N}$, let us consider a subdivision of the time interval $[0,T]$ given by $k+1$ nodes

$$0 = t_k^0 < t_k^1 < \cdots < t_k^{k-1} < t_k^k = T, \qquad \lim_{k\to\infty} \max_{1\le i\le k} |t_k^i - t_k^{i-1}| = 0\,,$$

and let us define $w_k^i := w(t_k^i)$.

We assume that the initial condition (u_0, V_0) is globally stable, namely

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_0|^2\,\mathrm{d}x + \int_\Gamma g\big(x, V_0\big)\,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \hat u|^2\,\mathrm{d}x + \int_\Gamma g\big(x,\, V_0 + \big|[\hat u]-[u_0]\big|\big)\,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{4.8}$$

for every $\hat u \in \mathcal{Adm}(w(0))$.

As the first step of the incremental process, we set $u_k^0 := u_0$ and $V_k^0 := V_0$. Let $i \in \{1,\dots,k\}$ and assume that we know u_k^h and V_k^h for $h = 0,\dots,i-1$. Then we define u_k^i as a solution to the problem

$$\min_u \left\{ \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u|^2\,\mathrm{d}x + \int_\Gamma g\big(x,\, V_k^{i-1} + \big|[u]-[u_k^{i-1}]\big|\big)\,\mathrm{d}\mathcal{H}^{n-1} \;:\; u \in \mathcal{Adm}(w_k^i) \right\}, \tag{4.9}$$

and we set

$$V_k^i := V_k^{i-1} + \big|[u_k^i]-[u_k^{i-1}]\big| = V_0 + \sum_{j=1}^{i} \big|[u_k^j]-[u_k^{j-1}]\big|\,. \tag{4.10}$$

The existence of a solution to (4.9) is obtained by employing the direct method of the Calculus of Variations.

The discrete-time evolutions are then defined as piecewise constant interpolations of the solutions to the incremental problems. Namely, we set

$$u_k(t) := u_k^i\,, \quad V_k(t) := V_k^i\,, \quad w_k(t) := w_k^i \quad \text{for } t_k^i \le t < t_k^{i+1} \tag{4.11}$$

and $u_k(T) := u_k^k$, $V_k(T) := V_k^k$, $w_k(T) := w(T)$.

Passing to the limit as $k \to \infty$, we prove that u_k converges to a quasistatic evolution u. A major point of our result is that the convergence holds for a subsequence independent of t. We also provide a convergence result for the variations of the jumps. Specifically, the truncated functions $V_k(t) \wedge \theta$ converge to $V_u(t) \wedge \theta$, where θ is as in (4.6). We remark that when $V_u(t;x)$ overcomes the threshold $\theta(x)$, we have no control on $V_u(t;x)$, which may increase without further dissipation of energy. Moreover, we obtain that $t \mapsto u(t)$ and $t \mapsto V_u(t)$ are continuous (in a suitable sense), except for countably many times.

These results are stated in the following theorem, whose proof is given in Section 4.5.

Theorem 4.2.9 (Existence and approximation of quasistatic evolutions)**.** *Assume that g satisfies $(g1)$–$(g4)$. Let w, u_0, and V_0 be as in* (4.2)–(4.5) *and assume that (u_0, V_0) is globally stable in the sense of* (4.8). *Consider the piecewise constant evolutions $t \mapsto u_k(t)$ and the piecewise constant variations $t \mapsto V_k(t)$ defined in* (4.11). *Then there exist a subsequence (independent of t and not relabelled) and a quasistatic evolution $t \mapsto u(t)$ with initial conditions (u_0, V_0) and boundary datum w such that, for every $t \in [0,T]$,*

$$u_k(t) \to u(t) \qquad \textit{strongly in } H^1(\Omega \setminus \Gamma)\,, \tag{4.12}$$

$$V_k(t) \wedge \theta \to V_u(t) \wedge \theta \quad \textit{in measure}\,, \tag{4.13}$$

where $V_u(t)$ is the function defined in (4.3) *and θ is given in* (4.6).

Moreover, there exists a set $E \subset [0,T]$, at most countable, such that, for every $t \in [0,T] \setminus E$ and every $s \to t$,

$$u(s) \to u(t) \qquad \textit{strongly in } H^1(\Omega \setminus \Gamma)\,. \tag{4.14}$$

$$V_u(s) \wedge \theta \to V_u(t) \wedge \theta \quad \textit{in measure}\,. \tag{4.15}$$

We underline that, if $\theta(x)$ is finite and $V_u(t;x) \ge \theta(x)$, the material is completely broken at x. Therefore $V_u(t) \wedge \theta$, appearing in the theorem above, is the relevant state variable for the system.

Remark 4.2.10. If $\theta \in L^\infty(\Gamma)$, then the convergence in (4.13) and (4.15) is also strong in $L^p(\Gamma)$ for every $p \in [1, \infty)$. In contrast, if $\theta \equiv \infty$ (that is $g(x, \cdot)$ is strictly increasing for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$), then $V_k(t) \to V_u(t)$ in measure as $k \to \infty$ and $V_u(s) \to V_u(t)$ in measure as $s \to t$.

Guidelines for the proof of the main result. The main difficulty in the passage to the continuous-time limit as $k \to \infty$ is that we lack of controls on $V_k(t)$. In fact, by (4.9), we can only infer that $\int_\Gamma g(x, V_k(t))\, \mathrm{d}\mathcal{H}^{n-1}$ is uniformly bounded, but this gives no information on $V_k(t)$, since g is bounded. For this reason we resort to a weaker notion of quasistatic evolution, where the variation of jumps on Γ is replaced by a Young measure. Notwithstanding, after establishing the properties of such an evolution, we are able to show that the Young measure found in the limit is concentrated on a function. Eventually, we obtain a quasistatic evolution in the sense of Definition 4.2.6. We describe here the strategy followed to prove Theorem 4.2.9.

Following the scheme of the proof of existence of solutions to rate-independent systems [61], the starting point of our analysis is to obtain a global stability and an energy-dissipation inequality for the discrete-time evolutions $t \mapsto u_k(t)$ (Proposition 4.3.1). As usual, the energy-dissipation inequality provides a priori bounds in $H^1(\Omega \setminus \Gamma)$ for the functions $u_k(t)$, independently of k and t. In order to study the limit of the functions $V_k(t)$, it is convenient to introduce the Young measures concentrated on the graph of $V_k(t)$, namely

$$\nu_k(t) := \delta_{V_k(t)} \in \mathcal{Y}(\Gamma; [0, \infty]) \quad \text{for every } t \in [0, T]\,. \tag{4.16}$$

We refer to Section 1.5 for the notation and the basic properties of Young measures. Since the functions $V_k(t)$ are nondecreasing with respect to t, we can apply a Helly-type selection principle (proved in [20]) to infer that the Young measures $\nu_k(t)$ converge narrowly to a Young measure $\nu(t) \in \mathcal{Y}(\Gamma; [0, \infty])$ on a subsequence independent of t. Thanks to the a priori bounds on $u_k(t)$, it is possible to extract a subsequence $k_j(t)$ (depending on t) such that $u_{k_j(t)}(t)$ converges to $u(t)$ weakly in $H^1(\Omega \setminus \Gamma)$. These convergences allow us to pass to the limit in the global stability of the discrete-time evolutions (Proposition 4.3.4), and thus to deduce that $t \mapsto (u(t), \nu(t))$ satisfies a suitable notion of global stability (condition (GSY) in Definition 4.4.1).

Afterwards, we show that $t \mapsto (u(t), \nu(t))$ satisfies an energy-dissipation balance (condition (EBY) in Definition 4.4.1). One inequality in this balance is a consequence of the energy-dissipation inequality of the discrete-time evolutions $t \mapsto u_k(t)$. On the contrary, the proof of the opposite inequality requires a thorough analysis. The main reason is that the Helly Selection Principle adopted before does not give any information about the relation between the Young measure $\nu(t)$ and $V_u(t)$. This relation is though encoded in a property satisfied by $t \mapsto \nu(t)$ (the *irreversibility* condition (IRY) in Definition 4.4.1), that we derive from the analogous condition $(\text{IRY})_k$ for the approximating Young measures $t \mapsto \nu_k(t)$. This property relates $\nu(t)$ to $[u(t)]$ and allows us to conclude the proof of the other inequality in the energy-dissipation balance by employing the global stability.

In addition, we prove that $u_k(t)$ actually converges to $u(t)$ strongly in $H^1(\Omega \setminus \Gamma)$ on a subsequence independent of t. This convergence result is proved in Section 4.4 by showing that the jump $\gamma(t) := [u(t)]$ is determined *de facto* independently of t (cf. equation (4.39)). Indeed this implies that the function $u(t)$ is the unique solution of a minimum problem among functions with a prescribed jump $\gamma(t)$ (Proposition 4.4.6). With similar arguments, we prove that $t \mapsto u(t)$ is continuous in t except for a countable set $E \subset [0, T]$.

Finally, in Section 4.5 we prove that u is actually a quasistatic evolution in the sense of Definition 4.2.6. Notice that for this step we need the assumption on the concavity of $g(x, \cdot)$. Moreover, this allows us to prove that the Young measure $\nu(t)$ (suitably truncated with θ) is concentrated on the function $V_u(t)$. As a consequence of this fact, we are able to deduce also the convergences in (4.13) and (4.15) in Theorem 4.2.9.

4.3 Discrete-time evolutions

We study here the discrete-time evolutions already introduced in Section 4.2.

Let $u_k(t)$, $V_k(t)$, and $w_k(t)$ be the piecewise constant interpolations given in (4.11). Let $\nu_k(t) \in \mathcal{Y}(\Gamma;[0,\infty])$ be the Young measures concentrated on $V_k(t)$ defined in (4.16). In the following proposition we state the main properties satisfied by such approximate evolutions and we provide a priori bounds for $u_k(t)$.

Proposition 4.3.1. *The discrete evolutions $t \mapsto u_k(t)$ defined in* (4.11) *satisfy the following conditions:*

$(\mathrm{GS})_k$ *Global stability: For every $t \in [0,T]$ we have $u_k(t) \in \mathcal{A}dm(w_k(t))$ and*

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_k(t)|^2 \,\mathrm{d}x + \int_\Gamma g\big(x, V_k(t)\big)\,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \hat u|^2 \,\mathrm{d}x + \int_\Gamma g\big(x,\, V_k(t) + \big|[\hat u]-[u_k(t)]\big|\big)\,\mathrm{d}\mathcal{H}^{n-1},$$

for every $\hat u \in \mathcal{A}dm(w_k(t))$.

$(\mathrm{EI})_k$ *Energy-dissipation inequality: There exists a sequence η_k with $\eta_k \to 0$ as $k \to \infty$ such that for every $t \in [0,T]$ we have*

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_k(t)|^2 \,\mathrm{d}x + \int_\Gamma g\big(x, V_k(t)\big)\,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_0|^2 \,\mathrm{d}x + \int_\Gamma g\big(x, V_0\big)\,\mathrm{d}\mathcal{H}^{n-1} + \int_0^{t_k^i} \langle \nabla u_k(s), \nabla \dot w(s)\rangle_{L^2}\,\mathrm{d}s + \eta_k\,,$$

where $i \in \{0,\dots,k\}$ is the largest integer such that $t_k^i \le t$.

Moreover, there exists a constant $C>0$ independent of k and t such that

$$\|u_k(t)\|_{H^1(\Omega\setminus\Gamma)} \le C \quad \text{for every } k \in \mathbb{N} \text{ and } t \in [0,T]\,. \tag{4.17}$$

Proof. In order to prove the global stability $(\mathrm{GS})_k$, we notice that if i is the largest integer such that $t_k^i \le t$, then by (4.10) we get that

$$V_k(t) + \big|[\hat u] - [u_k(t)]\big| = V_k^i + \big|[\hat u] - [u_k^i]\big| = V_k^{i-1} + \big|[u_k^i] - [u_k^{i-1}]\big| + \big|[\hat u] - [u_k^i]\big| \ge V_k^{i-1} + \big|[\hat u] - [u_k^{i-1}]\big|\,.$$

Then we infer $(\mathrm{GS})_k$ by the fact that $u_k(t) = u_k^i$ is a solution to (4.9) and by the monotonicity of $g(x,\cdot)$.

Let us prove the energy-dissipation inequality $(\mathrm{EI})_k$. Let us fix $t \in [0,T]$, $k \in \mathbb{N}$, and $i \in \{1,\dots,k\}$ as in the statement (the case $i=0$ being trivial). For $1 \le h \le i$, the function $u_k^{h-1} - w_k^{h-1} + w_k^h$ is an admissible competitor for the minimum problem (4.9) solved by u_k^h. Hence

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_k^h|^2\,\mathrm{d}x + \int_{\Gamma} g\big(x, V_k^h\big)\,\mathrm{d}\mathcal{H}^{n-1}\\
&\quad\le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_k^{h-1}|^2\,\mathrm{d}x + \int_{\Gamma} g\big(x, V_k^{h-1}\big)\,\mathrm{d}\mathcal{H}^{n-1}\\
&\qquad + \int_{\Omega\setminus\Gamma} \nabla u_k^{h-1}\cdot(\nabla w_k^h - \nabla w_k^{h-1})\,\mathrm{d}x + \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla w_k^h - \nabla w_k^{h-1}|^2\,\mathrm{d}x\\
&\le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_k^{h-1}|^2\,\mathrm{d}x + \int_{\Gamma} g\big(x, V_k^{h-1}\big)\,\mathrm{d}\mathcal{H}^{n-1}\\
&\qquad + \int_{t_k^{h-1}}^{t_k^h} \langle \nabla u_k(s), \nabla \dot w(s)\rangle_{L^2}\,\mathrm{d}s + \frac{1}{2}\Big(\int_{t_k^{h-1}}^{t_k^h} \|\nabla \dot w(s)\|_{L^2}\,\mathrm{d}s\Big)^2,
\end{aligned}
\tag{4.18}
$$

where we used our assumption (4.2) on w to deduce that

$$
\nabla w_k^h - \nabla w_k^{h-1} = \int_{t_k^{h-1}}^{t_k^h} \nabla \dot w(s)\,\mathrm{d}s\,,
$$

as a Bochner integral in L^2. Summing up the inequalities given by (4.18) for $h = 1, \dots, i$, we get $(\mathrm{EI})_k$ with

$$
\eta_k := \frac{1}{2}\Big(\max_{1\le h\le k} \int_{t_k^{h-1}}^{t_k^h} \|\nabla \dot w(s)\|_{L^2}\,\mathrm{d}s\Big)\Big(\int_0^T \|\nabla \dot w(s)\|_{L^2}\,\mathrm{d}s\Big).
$$

In particular, from $(\mathrm{EI})_k$ we readily deduce that there exists a constant $C > 0$ independent of k and t such that $\|\nabla u_k(t)\|_{L^2} \le C$. Then, by the Poincaré inequality, we get (4.17) (up to changing the name of the constant). □

Remark 4.3.2. It is convenient to express the properties satisfied by $u_k(t)$ also in terms of the Young measures $\nu_k(t) \in \mathcal{Y}(\Gamma;[0,\infty])$ defined in (4.16). In Section 4.4, we will pass to the limit in these conditions.

$(\mathrm{IRY})_k$ *Irreversibility*: $\nu_k(t) \succeq \nu_k(s) \oplus \big|[u_k(t)] - [u_k(s)]\big|$ for every $s, t \in [0,T]$ with $s \le t$.

$(\mathrm{GSY})_k$ *Global stability*: For every $t \in [0,T]$ we have $u_k(t) \in \mathcal{A}dm(w_k(t))$ and

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_k(t)|^2\,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \nu_k^x(t)\rangle\,\mathrm{d}\mathcal{H}^{n-1}\\
&\qquad \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \hat u|^2\,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \hat\nu_k^x\rangle\,\mathrm{d}\mathcal{H}^{n-1},
\end{aligned}
$$

for every $\hat u \in \mathcal{A}dm(w_k(t))$, where $\hat\nu_k := \nu_k(t) \oplus \big|[\hat u] - [u_k(t)]\big| \in \mathcal{Y}(\Gamma;[0,\infty])$.

$(\mathrm{EIY})_k$ *Energy-dissipation inequality*: For every $t \in [0,T]$

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_k(t)|^2\,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu_k^x(t)\rangle\,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_0|^2\,\mathrm{d}x + \int_\Gamma g\big(x,V_0\big)\,\mathrm{d}\mathcal{H}^{n-1} + \int_0^{t_k^i} \langle \nabla u_k(s), \nabla \dot{w}(s)\rangle_{L^2}\,\mathrm{d}s + \eta_k\,,$$

where $i \in \{0,\dots,k\}$ is the largest integer such that $t_k^i \le t$.

Notice that $(\mathrm{GSY})_k$ trivially implies that

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_k(t)|^2\,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu_k^x(t)\rangle\,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \hat{u}|^2\,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \hat{\nu}^x\rangle\,\mathrm{d}\mathcal{H}^{n-1},$$

for every $\hat{u} \in \mathcal{A}dm(w_k(t))$ and for every $\hat{\nu} \in \mathcal{Y}(\Gamma;[0,\infty])$ with $\hat{\nu} \succeq \nu_k(t) \oplus \big|[\hat{u}] - [u_k(t)]\big|$.

Remark 4.3.3. By passing to the limit as $k \to \infty$ in $(\mathrm{IRY})_k$, we may formally obtain the irreversibility condition for the continuous-time quasistatic evolution. (See Definition 4.4.1 in Section 4.4 below.) Unfortunately, it is not immediate to rigorously pass to the limit in $(\mathrm{IRY})_k$: as we shall see below, in the construction of the continuous-time evolution the jumps $[u_k(t)]$ converge to $[u(t)]$ on subsequences possibly depending on t, thus precluding the possibility to have convergence on the same subsequence for both $[u_k(t)]$ and $[u_k(s)]$ in $(\mathrm{IRY})_k$. For this reason, we reformulate $(\mathrm{IRY})_k$ in a more convenient way. We start by noticing that the condition

$$V_k(t) \ge V_k(s) + \big|[u_k(t)] - [u_k(s)]\big| \quad \text{for every } s,t \in [0,T] \text{ with } s \le t\,,$$

is equivalent to the system of inequalities

$$V_k(t) + [u_k(t)] \ge V_k(s) + [u_k(s)] \quad \text{for every } s,t \in [0,T] \text{ with } s \le t\,, \tag{4.19}$$

$$V_k(t) - [u_k(t)] \ge V_k(s) - [u_k(s)] \quad \text{for every } s,t \in [0,T] \text{ with } s \le t\,. \tag{4.20}$$

Let us notice that since $V_0 \ge \big|[u_0]\big|$ by (4.5), we have $V_k(t) + [u_k(t)] \ge 0$ and $V_k(t) - [u_k(t)] \ge 0$ for every $t \in [0,T]$. In terms of the Young measures ν_k, the inequalities (4.19) and (4.20) are equivalent to stating that the functions

$$t \mapsto \nu_k(t) \oplus [u_k(t)] =: \lambda_k^{\oplus}(t) \in \mathcal{Y}(\Gamma;[0,\infty])\,, \tag{4.21}$$

$$t \mapsto \nu_k(t) \ominus [u_k(t)] =: \lambda_k^{\ominus}(t) \in \mathcal{Y}(\Gamma;[0,\infty]) \tag{4.22}$$

are nondecreasing with respect to t. Thanks to the Helly Selection Principle for Young measures (Theorem 1.5.16), (4.21) and (4.22) are easier to handle than $(\mathrm{IRY})_k$, as we shall see later in Section 4.4.

We conclude this section with the following proposition, which shall be used to pass to the limit in $(\mathrm{GSY})_k$ as $k \to \infty$.

Proposition 4.3.4. *Let $w_k \rightharpoonup w$ weakly in $H^1(\Omega)$. Let $v_k \in \mathcal{A}dm(w_k)$ and $v \in H^1(\Omega\setminus\Gamma)$ be such that $v_k \rightharpoonup v$ weakly in $H^1(\Omega\setminus\Gamma)$ and let $\mu_k, \mu \in \mathcal{Y}(\Gamma;[0,\infty])$ be such that $\mu_k \rightharpoonup \mu$. Let us assume that for every $k \in \mathbb{N}$*

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla v_k|^2\,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \mu_k^x\rangle\,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \hat{v}|^2\,\mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \hat{\mu}_k^x\rangle\,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{4.23}$$

for every $\hat{v} \in \mathcal{A}dm(w_k)$, *where* $\hat{\mu}_k := \mu_k \oplus \big|[\hat{v}] - [v_k]\big| \in \mathcal{Y}(\Gamma; [0,\infty])$. *Then* $v \in \mathcal{A}dm(w)$ *and*

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla v|^2 \,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \mu^x\rangle \,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \hat{v}|^2 \,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \hat{\mu}^x\rangle \,\mathrm{d}\mathcal{H}^{n-1}, \tag{4.24}$$

for every $\hat{v} \in \mathcal{A}dm(w)$, *where* $\hat{\mu} := \mu \oplus \big|[\hat{v}] - [v]\big| \in \mathcal{Y}(\Gamma; [0,\infty])$.

Proof. By the continuity of the trace operator on $\partial_D\Omega$ with respect to the weak convergence in $H^1(\Omega\setminus\Gamma)$ we have $v \in \mathcal{A}dm(w)$. To prove (4.24), fix $\hat{v} \in \mathcal{A}dm(w)$. Define $\hat{\mu} := \mu \oplus \big|[\hat{v}] - [v]\big| \in \mathcal{Y}(\Gamma; [0,\infty])$ and

$$\begin{aligned} &\hat{v}_k := v_k + \hat{v} - v \in \mathcal{A}dm(w_k)\,, \\ &\hat{\mu}_k := \mu_k \oplus \big|[\hat{v}] - [v]\big| = \mu_k \oplus \big|[\hat{v}_k] - [v_k]\big|\,. \end{aligned} \tag{4.25}$$

Since $v_k \rightharpoonup v$ and $\mu_k \rightharpoonup \mu$, by Remark 1.5.9 we have

$$\hat{v}_k \rightharpoonup \hat{v} \quad \text{weakly in } H^1(\Omega\setminus\Gamma)\,, \tag{4.26}$$

$$\hat{\mu}_k \rightharpoonup \hat{\mu} \quad \text{narrowly.} \tag{4.27}$$

From (4.23) we get that

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla v_k|^2 \,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \mu_k^x\rangle \,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \hat{v}_k|^2 \,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \hat{\mu}_k^x\rangle \,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.28}$$

We now use a classical quadratic trick. By (4.25), we infer that

$$\begin{aligned} \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla v_k|^2 \,\mathrm{d}x - \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \hat{v}_k|^2 \,\mathrm{d}x &= \frac{1}{2}\int_{\Omega\setminus\Gamma} (\nabla v_k - \nabla \hat{v}_k)\cdot(\nabla v_k + \nabla \hat{v}_k)\,\mathrm{d}x \\ &= \frac{1}{2}\int_{\Omega\setminus\Gamma} (\nabla v - \nabla \hat{v})\cdot(2\nabla v_k + \nabla \hat{v} - \nabla v)\,\mathrm{d}x\,. \end{aligned} \tag{4.29}$$

Thanks to (4.27) we deduce that

$$\int_{\Gamma} \langle g(x,\cdot), \hat{\mu}_k^x\rangle \,\mathrm{d}\mathcal{H}^{n-1} \to \int_{\Gamma} \langle g(x,\cdot), \hat{\mu}^x\rangle \,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.30}$$

Since $v_k \rightharpoonup v$ and $\mu_k \rightharpoonup \mu$, by (4.28)–(4.30) we have

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} (\nabla v - \nabla \hat{v})\cdot(\nabla v + \nabla \hat{v})\,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \mu^x\rangle \,\mathrm{d}\mathcal{H}^{n-1} \le \int_{\Gamma} \langle g(x,\cdot), \hat{\mu}^x\rangle \,\mathrm{d}\mathcal{H}^{n-1},$$

from which we easily conclude that (4.24) holds. □

4.4 Quasistatic evolution in the setting of Young measures

In this section we study the continuous-time limit of the discrete evolutions $u_k(t)$ constructed in Section 4.3. The limit of the sequence of (Young measures concentrated on) functions $\nu_k(t)$ defined in (4.16) can only be found in the space of Young measures $\mathcal{Y}(\Gamma;[0,\infty])$. For this reason we require a definition of quasistatic evolution in a generalised sense.

Definition 4.4.1. Let w, u_0, and V_0 be as in (4.2)–(4.5). A *quasistatic evolution in the sense of Young measures* with initial conditions (u_0, V_0) and boundary datum w is a function $t \mapsto (u(t), \nu(t))$ defined in $[0,T]$ with values in $H^1(\Omega\setminus\Gamma)\times\mathcal{Y}(\Gamma;[0,\infty])$ that satisfies $u(0)=u_0$, $\nu(0)=\delta_{V_0}$, and the following conditions:

(IRY) *Irreversibility*: $\nu(t) \succeq \nu(s) \oplus \big|[u(t)]-[u(s)]\big|$ for every $s,t\in[0,T]$ with $s\le t$.

(GSY) *Global stability*: For every $t\in[0,T]$, $u(t)\in \mathcal{A}dm(w(t))$ and

$$\frac{1}{2}\int\limits_{\Omega\setminus\Gamma} |\nabla u(t)|^2\,\mathrm{d}x + \int\limits_{\Gamma}\langle g(x,\cdot),\nu^x(t)\rangle\,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2}\int\limits_{\Omega\setminus\Gamma} |\nabla \hat{u}|^2\,\mathrm{d}x + \int\limits_{\Gamma}\langle g(x,\cdot),\hat{\nu}^x\rangle\,\mathrm{d}\mathcal{H}^{n-1},$$

for every $\hat{u}\in\mathcal{A}dm(w(t))$, where $\hat{\nu} := \nu(t)\oplus\big|[\hat{u}]-[u(t)]\big| \in \mathcal{Y}(\Gamma;[0,\infty])$.

(EBY) *Energy-dissipation balance*: For every $t\in[0,T]$

$$\frac{1}{2}\int\limits_{\Omega\setminus\Gamma} |\nabla u(t)|^2\,\mathrm{d}x + \int\limits_{\Gamma}\langle g(x,\cdot),\nu^x(t)\rangle\,\mathrm{d}\mathcal{H}^{n-1} = \frac{1}{2}\int\limits_{\Omega\setminus\Gamma} |\nabla u_0|^2\,\mathrm{d}x + \int\limits_{\Gamma} g(x,V_0)\,\mathrm{d}\mathcal{H}^{n-1} + \int_0^t \langle \nabla u(s), \nabla\dot{w}(s)\rangle_{L^2}\,\mathrm{d}s\,.$$

Remark 4.4.2. In order to recognise the connection with the classical notion of quasistatic evolution, we notice that $t\mapsto u(t)$ is a quasistatic evolution (Definition 4.2.6) if and only if $t\mapsto (u(t),\delta_{V_u(t)})$ is a quasistatic evolution in the sense of Young measures (Definition 4.4.1), where $V_u(t)$ is the function defined in (4.3). Indeed, the irreversibility condition (IRY) of Definition 4.4.1 automatically holds for $t\mapsto\delta_{V_u(t)}$ by definition of essential variation. Moreover, (GS) and (EB) correspond to (GSY) and (EBY), since the Young measure considered in this case is concentrated on $V_u(t)$.

Remark 4.4.3. Notice that (GSY) trivially implies that

$$\frac{1}{2}\int\limits_{\Omega\setminus\Gamma} |\nabla u(t)|^2\,\mathrm{d}x + \int\limits_{\Gamma}\langle g(x,\cdot),\nu^x(t)\rangle\,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2}\int\limits_{\Omega\setminus\Gamma} |\nabla \hat{u}|^2\,\mathrm{d}x + \int\limits_{\Gamma}\langle g(x,\cdot),\hat{\nu}^x\rangle\,\mathrm{d}\mathcal{H}^{n-1},$$

for every $\hat{u}\in\mathcal{A}dm(w(t))$ and for every $\hat{\nu}\in\mathcal{Y}(\Gamma;[0,\infty])$ with $\hat{\nu}\succeq\nu(t)\oplus\big|[\hat{u}]-[u(t)]\big|$.

Moreover we underline that (IRY) is a stronger condition than the monotonicity of $t\mapsto\nu(t)$ and dictates a relationship between ν and $[u]$.

In the following theorem we prove the existence of a quasistatic evolution in the sense of Young measures. As explained in Section 4.2, this result will be then improved in Section 4.5 by showing that the truncated Young measures $\mathcal{T}^{\theta}_{\#}\nu(t)$ are concentrated on the function $V_u(t) \wedge \theta$ which represents the cumulation of the jumps on Γ.

Theorem 4.4.4 (Existence of quasistatic evolutions in the sense of Young measures)**.** *Assume that g satisfies $(g1)$–$(g4)$ and let w, u_0, and V_0 be as in* (4.2), (4.4), *and* (4.5). *Assume that the pair (u_0, δ_{V_0}) is globally stable, i.e.,* (4.8) *holds. Then there exists a quasistatic evolution in the sense of Young measures $t \mapsto (u(t), \nu(t))$ with initial conditions (u_0, V_0) and boundary datum w.*

In the rest of this section, we give a proof of Theorem 4.4.4.

Construction of the evolution. Let us consider the Young measures $\nu_k(t)$ defined in (4.16). The starting point of the proof is the construction of a limit of $\nu_k(t)$ as $k \to \infty$. Since the functions $t \mapsto \nu_k(t) \in \mathcal{Y}(\Gamma; [0, \infty])$ are increasing with respect to the order $\preceq$, we can apply Theorem 1.5.16 to deduce that there exists a subsequence (independent of t and still denoted by ν_k) and an increasing function $t \mapsto \nu(t)$ from $[0, T]$ to $\mathcal{Y}(\Gamma; [0, \infty])$ such that

$$\nu_k(t) \rightharpoonup \nu(t) \quad \text{narrowly for every } t \in [0, T]\,. \tag{4.31}$$

Unfortunately, the convergence in (4.31) is not enough to guarantee that the irreversibility condition (IRY) holds for $\nu(t)$. In other words, it is nontrivial to pass to the limit in the discrete version of the irreversibility condition $(\text{IRY})_k$. Nonetheless, by Remark 4.3.3, we know that the functions $t \mapsto \lambda_k^{\oplus}(t)$ and $t \mapsto \lambda_k^{\ominus}(t)$ defined in (4.21) and (4.22) are increasing. Hence we can apply again Theorem 1.5.16 and deduce that there exists a subsequence independent of t (not relabelled) and two increasing functions $t \mapsto \lambda_{\oplus}(t) \in \mathcal{Y}(\Gamma; [0, \infty])$ and $t \mapsto \lambda_{\ominus}(t) \in \mathcal{Y}(\Gamma; [0, \infty])$ such that

$$\lambda_k^{\oplus}(t) \rightharpoonup \lambda_{\oplus}(t) \quad \text{narrowly for every } t \in [0, T]\,, \tag{4.32}$$

$$\lambda_k^{\ominus}(t) \rightharpoonup \lambda_{\ominus}(t) \quad \text{narrowly for every } t \in [0, T]\,. \tag{4.33}$$

The monotonicity of both the functions $\lambda_{\oplus}$ and $\lambda_{\ominus}$ encodes the irreversibility of the process in the continuous-time evolution.

We are now in a position to construct a limit of the sequence $u_k(t)$. Thanks to (4.17), we have $\|u_k(t)\|_{H^1(\Omega\setminus\Gamma)} \leq C$, where the constant C is independent of k and t. Let $t \in [0, T]$ and let $k_j(t)$ be a subsequence of k such that

$$u_{k_j(t)}(t) \rightharpoonup u(t) \quad \text{weakly in } H^1(\Omega \setminus \Gamma)\,, \tag{4.34}$$

for some function $u(t) \in H^1(\Omega \setminus \Gamma)$.

Remark 4.4.5. A priori, the function $u(t)$ depends on the subsequence $k_j(t)$ such that (4.34) holds. Nevertheless, we shall prove that

$$u_k(t) \to u(t) \quad \text{strongly in } H^1(\Omega \setminus \Gamma) \tag{4.35}$$

on the whole sequence (independent of t) found by the Helly Selection Principle (cf. the convergences in (4.31)–(4.33)).

We remark that also the topology of the convergence is improved. The convergence in (4.35) will be proved later in this section by showing that the function $u(t)$ is characterised as the unique solution to a minimum problem (Proposition 4.4.6). The convergence with respect to the strong topology of $H^1(\Omega\setminus\Gamma)$ will be a consequence of the energy-dissipation balance (EBY).

Proof of irreversibility. We can now infer (IRY) from the monotonicity of the functions $\lambda_\oplus$ and $\lambda_\ominus$ obtained in (4.32) and (4.33). Indeed, from (4.34) we deduce that $[u_{k_j(t)}] \to [u(t)]$ strongly in $L^2(\Gamma)$. By (4.31) and by Remark 1.5.9 this implies that $\lambda^\oplus_{k_j(t)}(t) = \nu_{k_j(t)}(t) \oplus [u_{k_j(t)}(t)] \rightharpoonup \nu(t) \oplus [u(t)]$. Thus, from (4.32) we deduce that

$$\lambda_\oplus(t) = \nu(t) \oplus [u(t)]\,, \tag{4.36}$$

and therefore that the function $t \mapsto \nu(t) \oplus [u(t)]$ is increasing. Similarly one can prove that $\lambda_\ominus(t) = \nu(t) \ominus [u(t)]$ and that $t \mapsto \nu(t) \ominus [u(t)]$ is increasing. Therefore, for every $s,t \in [0,T]$ with $s \leq t$ we have

$$\begin{aligned}\nu(t) \oplus [u(t)] &\succeq \nu(s) \oplus [u(s)]\,,\\ \nu(t) \ominus [u(t)] &\succeq \nu(s) \ominus [u(s)]\,.\end{aligned}$$

It is immediate to see that the previous inequalities imply (IRY).

In order to prove (4.35), it is convenient to make the following key observations:

- the Young measures $\lambda_\oplus(t)$ and $\nu(t)$ are obtained as limits of a sequence independent of t;
- the jump $[u(t)]$ can be recovered just from $\lambda_\oplus(t)$ and $\nu(t)$ thanks to (4.36).

We now make precise the previous statements. We start by observing that if $x \in \Gamma$ is such that $\lambda^x_\oplus(t) = \nu^x(t) = \delta_\infty$, then $[u(t;x)]$ is not uniquely determined by (4.36). For this reason we introduce the set

$$\Gamma_N(t) := \{x \in \Gamma \;:\; \nu^x(t) \succeq \delta_{\theta(x)}\}\,, \tag{4.37}$$

which corresponds to the subset of Γ where the material is completely fractured. For $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma\setminus\Gamma_N(t)$ there exists a mass $m_x \in (0,1]$ such that $F^{[-1]}_{\nu^x(t)}(m_x) \in [0,\theta(x))$, where $F^{[-1]}_{\nu^x(t)}$ is the pseudo-inverse of the cumulative distribution function $F_{\nu^x(t)}$ of $\nu^x(t)$ (cf. (1.16) and (1.17)). In particular, we have that $F^{[-1]}_{\nu^x(t)}(m_x)$ is finite. By (4.36) and by the definition of pseudo-inverse, it is easy to see that

$$F^{[-1]}_{\lambda^x_\oplus(t)}(m_x) - F^{[-1]}_{\nu^x(t)}(m_x) = [u(t;x)] \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma\setminus\Gamma_N(t)\,. \tag{4.38}$$

(We remark that, if instead $x \in \Gamma_N(t)$, it may happen that $\nu^x(t) = \delta_\infty$, and thus $F^{[-1]}_{\nu^x(t)}(m) = \infty$ for every $m \in (0,1]$. This does not allow us to infer (4.38).) Therefore, we can define a measurable function $\gamma(t)\colon \Gamma\setminus\Gamma_N(t) \to \mathbb{R}$ by

$$\gamma(t;x) := F^{[-1]}_{\lambda^x_\oplus(t)}(m_x) - F^{[-1]}_{\nu^x(t)}(m_x)\,, \tag{4.39}$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma\setminus\Gamma_N(t)$. We stress that the function $\gamma(t)$ is obtained independently of the subsequence $k_j(t)$. The proof of (4.35) will be continued after the proof of (GSY) and (EBY).

Proof of global stability. The global stability (GSY) directly follows from Proposition 4.3.4, since $u_{k_j(t)}(t)$ and $\nu_{k_j(t)}(t)$ satisfy condition $(\mathrm{GSY})_k$ and by (4.34) and (4.31).

In general, the function $u(t)$ is not uniquely determined by (GSY), because $u(t)$ appears both in the left-hand side and in the right-hand side of (GSY); specifically, $\hat{\nu}$ depends on $u(t)$. However, we have shown that the jump of $u(t)$ is given by the function $\gamma(t)$ defined in (4.39) independently of the subsequence $k_j(t)$. This allows us to prove the following result.

Proposition 4.4.6. *The function $u(t)$ obtained in (4.34) is the unique solution to the minimum problem*

$$\min_{\hat{u}}\Big\{\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla\hat{u}|^2\,\mathrm{d}x\ :\ \hat{u}\in\mathcal{A}dm(w(t)),\ [\hat{u}]=\gamma(t)\ \ \mathcal{H}^{n-1}\text{-a.e. on }\Gamma\setminus\Gamma_N(t)\Big\}\,,\tag{4.40}$$

where $\Gamma_N(t)$ is the set defined in (4.37) and $\gamma(t)$ is the function defined in (4.39).

Remark 4.4.7. Notice that Proposition 4.4.6 holds true also when $\mathcal{H}^{n-1}(\Gamma\setminus\Gamma_N(t))=0$, i.e., when the material is completely fractured on the whole surface Γ. In this case, the competitors in (4.40) are all functions $\hat{u}\in\mathcal{A}dm(w(t))$ (without any constraint on the jump).

Proof of Proposition 4.4.6. We have already observed (see (4.38)) that $[u(t)]=\gamma(t)$ $\mathcal{H}^{n-1}$-a.e. on $\Gamma\setminus\Gamma_N(t)$. Let us fix $\hat{u}\in\mathcal{A}dm(w(t))$ such that $[\hat{u}]=\gamma(t)=[u(t)]$ $\mathcal{H}^{n-1}$-a.e. on $\Gamma\setminus\Gamma_N(t)$. Setting $\hat{\nu}:=\nu(t)\oplus\big|[\hat{u}]-[u(t)]\big|$, by (GSY) we have

$$\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla u(t)|^2\,\mathrm{d}x+\int_{\Gamma}\langle g(x,\cdot),\nu^x(t)\rangle\,\mathrm{d}\mathcal{H}^{n-1}\le\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla\hat{u}|^2\,\mathrm{d}x+\int_{\Gamma}\langle g(x,\cdot),\hat{\nu}^x\rangle\,\mathrm{d}\mathcal{H}^{n-1}.\tag{4.41}$$

Since $\hat{\nu}^x\succeq\nu^x(t)\succeq\delta_{\theta(x)}$ for $\mathcal{H}^{n-1}$-a.e. $x\in\Gamma_N(t)$ and since $g(x,\xi)=\kappa(x)$ for every $\xi\in[\theta(x),\infty]$, we deduce that

$$\int_{\Gamma_N(t)}\langle g(x,\cdot),\nu^x(t)\rangle\,\mathrm{d}\mathcal{H}^{n-1}=\int_{\Gamma_N(t)}\langle g(x,\cdot),\hat{\nu}^x\rangle\,\mathrm{d}\mathcal{H}^{n-1}=\int_{\Gamma_N(t)}\kappa(x)\,\mathrm{d}\mathcal{H}^{n-1}(x)\,.$$

Therefore (4.41) is equivalent to

$$\begin{aligned}&\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla u(t)|^2\,\mathrm{d}x+\int_{\Gamma\setminus\Gamma_N(t)}\langle g(x,\cdot),\nu^x(t)\rangle\,\mathrm{d}\mathcal{H}^{n-1}\\&\qquad\le\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla\hat{u}|^2\,\mathrm{d}x+\int_{\Gamma\setminus\Gamma_N(t)}\langle g(x,\cdot),\hat{\nu}^x\rangle\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned}$$

Since $[\hat{u}]=[u(t)]$ $\mathcal{H}^{n-1}$-a.e. on $\Gamma\setminus\Gamma_N(t)$, we have $\hat{\nu}^x=\nu^x(t)$ for $\mathcal{H}^{n-1}$-a.e. $x\in\Gamma\setminus\Gamma_N(t)$, hence the previous inequality reads

$$\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla u(t)|^2\,\mathrm{d}x\le\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla\hat{u}|^2\,\mathrm{d}x\,.$$

This proves that $u(t)$ is a solution to (4.40).

The argument to prove uniqueness is standard: if u_1 and u_2 were two different solutions to (4.40), then $\hat{u} := \frac{1}{2}(u_1 + u_2)$ would be an admissible competitor; by strict convexity,

$$\begin{aligned}\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \hat{u}|^2 \,\mathrm{d}x &= \frac{1}{2}\int_{\Omega\setminus\Gamma} \Big|\frac{\nabla u_1 + \nabla u_2}{2}\Big|^2 \,\mathrm{d}x \\ &< \frac{1}{4}\int_{\Omega\setminus\Gamma} |\nabla u_1|^2 \,\mathrm{d}x + \frac{1}{4}\int_{\Omega\setminus\Gamma} |\nabla u_2|^2 \,\mathrm{d}x = \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_1|^2 \,\mathrm{d}x\,.\end{aligned}$$

This contradicts the minimality. □

Remark 4.4.8. The minimum problem (4.40) is independent of the subsequence $k_j(t)$. As a consequence, we have shown that if $k_j(t)$ is such that $u_{k_j(t)} \rightharpoonup u(t)$, then $u(t)$ is the unique solution to (4.40). Thus $u(t)$ does not depend on $k_j(t)$, and this implies that

$$u_k(t) \rightharpoonup u(t) \quad \text{weakly in } H^1(\Omega \setminus \Gamma) \quad \text{for every } t \in [0,T] \tag{4.42}$$

on the whole sequence (independent of t) found by the Helly Selection Principle (cf. the convergences in (4.31)–(4.33)). In particular, by (4.17) we have

$$\|u(t)\|_{H^1(\Omega\setminus\Gamma)} \leq C\,. \tag{4.43}$$

After proving the energy-dissipation balance, it will turn out that the convergence is strong in $H^1(\Omega \setminus \Gamma)$.

Proof of energy-dissipation balance. Before proving (EBY), we show that the function $t \mapsto u(t)$ is continuous with respect to the weak topology for almost every time. This result allows for a simple proof of the energy-dissipation balance.

Lemma 4.4.9. *There exists a countable set $E \subset [0,T]$ such that for every $t \in [0,T] \setminus E$*

$$u(s) \rightharpoonup u(t) \quad \textit{weakly in } H^1(\Omega \setminus \Gamma)\,, \tag{4.44}$$

$$\nu(s) \rightharpoonup \nu(t) \quad \textit{narrowly in } \mathcal{Y}(\Gamma;[0,\infty])\,. \tag{4.45}$$

as $s \to t$.

Proof. Since the functions $t \mapsto \lambda_\oplus(t)$ and $t \mapsto \nu(t)$ are nondecreasing, we can find a countable set $E \subset [0,T]$ such that both $\lambda_\oplus$ and ν are continuous (with respect to the narrow topology) in t for every $t \in [0,T] \setminus E$. (See Remark 1.5.15.) Thus, given $t \in [0,T] \setminus E$ and a sequence $s_k \to t$, we have

$$\lambda_\oplus(s_k) \rightharpoonup \lambda_\oplus(t)\,, \quad \nu(s_k) \rightharpoonup \nu(t)\,. \tag{4.46}$$

Thanks to (4.43), we can extract a subsequence (not relabelled) such that

$$u(s_k) \rightharpoonup u^* \quad \text{weakly in } H^1(\Omega \setminus \Gamma) \tag{4.47}$$

for some $u^* \in H^1(\Omega \setminus \Gamma)$. By Proposition 4.3.4, we infer that $u^* \in \mathcal{A}dm(w(t))$ and

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u^*|^2 \,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \nu^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1} \leq \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \hat{u}|^2 \,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \hat{\nu}^x\rangle \,\mathrm{d}\mathcal{H}^{n-1},$$

for every $\hat{u} \in \mathcal{A}dm(w(t))$, where $\hat{\nu} = \nu(t) \oplus \big|[\hat{u}] - [u^*]\big|$.

On the other hand, by (4.36), we have $\lambda_\oplus(s_k) = \nu(s_k) \oplus [u(s_k)]$. By (4.46), (4.47), and Remark 1.5.9 we deduce that $\lambda_\oplus(t) = \nu(t) \oplus [u^*]$. Hence, by (4.39), we obtain that $[u^*(x)] = \gamma(t;x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma \setminus \Gamma_N(t)$. Therefore, arguing as in the proof of Proposition 4.4.6, we infer that u^* is a solution to the minimum problem (4.40). By uniqueness of the solution we get $u^* = u(t)$, which concludes the proof. □

Remark 4.4.10. Lemma 4.4.9 will be improved in Proposition 4.4.12 below by showing that the continuity actually holds with respect to the strong topology.

Let us now prove (EBY). We start by proving the inequality

$$\begin{aligned} &\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \nu^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1} \\ &\quad\leq \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_0|^2 \,\mathrm{d}x + \int_{\Gamma} g(x, V_0) \,\mathrm{d}\mathcal{H}^{n-1} + \int_0^t \langle \nabla u(s), \nabla \dot{w}(s)\rangle_{L^2} \,\mathrm{d}s\,. \end{aligned} \tag{4.48}$$

By (4.31), (4.42), and by $(\mathrm{EIY})_k$, for every $t \in [0,T]$ we have

$$\begin{aligned} &\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \nu^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1} \\ &\quad\leq \liminf_{k\to\infty}\left[\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_k(t)|^2 \,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \nu_k^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1}\right] \\ &\quad\leq \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_0|^2 \,\mathrm{d}x + \int_{\Gamma} g(x, V_0) \,\mathrm{d}\mathcal{H}^{n-1} + \limsup_{k\to\infty}\int_0^{t_k^i} \langle \nabla u_k(s), \nabla \dot{w}(s)\rangle_{L^2} \,\mathrm{d}s\,, \end{aligned} \tag{4.49}$$

where $i \in \{0,\dots,k\}$ is the largest integer such that $t_k^i \leq t$. Thanks to (4.42) we know that

$$\langle \nabla u_k(s), \nabla \dot{w}(s)\rangle_{L^2} \to \langle \nabla u(s), \nabla \dot{w}(s)\rangle_{L^2} \quad \text{for every } s \in [0,t]\,.$$

Moreover, from (4.17) we deduce that

$$\langle \nabla u_k(s), \nabla \dot{w}(s)\rangle_{L^2} \leq \|\nabla u_k(s)\|_{L^2}\|\nabla \dot{w}(s)\|_{L^2} \leq C\|\nabla \dot{w}(s)\|_{L^2}\,,$$

for every $s \in [0,T]$. By our assumption (4.2) on w, the function $t \mapsto \nabla \dot{w}(t)$ is $L^1([0,T];L^2(\Omega\setminus\Gamma))$, so we can apply the Dominated Convergence Theorem to infer that the function $s \mapsto \langle \nabla u(s), \nabla \dot{w}(s)\rangle_{L^2}$ belongs to $L^1([0,t])$ and

$$\begin{aligned} \limsup_{k\to\infty}\int_0^{t_k^i} \langle \nabla u_k(s), \nabla \dot{w}(s)\rangle_{L^2} \,\mathrm{d}s &= \lim_{k\to\infty}\int_0^{t} \langle \nabla u_k(s), \nabla \dot{w}(s)\rangle_{L^2} \,\mathrm{d}s \\ &= \int_0^{t} \langle \nabla u(s), \nabla \dot{w}(s)\rangle_{L^2} \,\mathrm{d}s\,. \end{aligned} \tag{4.50}$$

Together with (4.49), the previous inequality yields (4.48).

We now exploit the global stability to prove, for a fixed $t \in [0,T]$, the opposite inequality

$$\begin{aligned} \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \nu^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1} \\ \geq \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_0|^2 \,\mathrm{d}x + \int_{\Gamma} g(x,V_0)\,\mathrm{d}\mathcal{H}^{n-1} + \int_0^t \langle \nabla u(s), \nabla \dot w(s)\rangle_{L^2}\,\mathrm{d}s\,. \end{aligned} \tag{4.51}$$

For every $k \in \mathbb{N}$, let us consider the subdivision of the time interval $[0,t]$ given by the $k{+}1$ equispaced nodes

$$s_k^h := \tfrac{h}{k} t \quad \text{for } h = 0,\dots,k\,.$$

Let $h \in \{1,\dots,k\}$. By the irreversibility condition (IRY), we have $\nu(s_k^h) \succeq \nu(s_k^{h-1}) \oplus \big|[u(s_k^h)] - [u(s_k^{h-1})]\big| =: \hat\nu_h$. Since $u(s_k^h) - w(s_k^h) + w(s_k^{h-1}) \in \mathcal{Adm}(w(s_k^{h-1}))$, by (GSY) we obtain

$$\begin{aligned} &\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(s_k^{h-1})|^2 \,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \nu^x(s_k^{h-1})\rangle \,\mathrm{d}\mathcal{H}^{n-1} \\ &\quad\leq \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(s_k^h)|^2 \,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \hat\nu_h^x\rangle \,\mathrm{d}\mathcal{H}^{n-1} \\ &\quad\quad - \int_{\Omega\setminus\Gamma} \nabla u(s_k^h)\cdot(\nabla w(s_k^h) - \nabla w(s_k^{h-1}))\,\mathrm{d}x + \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla w(s_k^h) - \nabla w(s_k^{h-1})|^2\,\mathrm{d}x \\ &\quad\leq \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(s_k^h)|^2 \,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \nu^x(s_k^h)\rangle \,\mathrm{d}\mathcal{H}^{n-1} \\ &\quad\quad - \int_{s_k^{h-1}}^{s_k^h} \langle \nabla \overline{u}^k(s), \nabla \dot w(s)\rangle_{L^2}\,\mathrm{d}s + \frac{1}{2}\bigg(\int_{s_k^{h-1}}^{s_k^h} \|\nabla \dot w(s)\|_{L^2}\,\mathrm{d}s\bigg)^2, \end{aligned} \tag{4.52}$$

where

$$\overline{u}^k(s) := u(s_k^h) \quad \text{for every } s \in (s_k^{h-1}, s_k^h]\,.$$

Summing up the inequalities given by (4.52) for $h = 1,\dots,k$, we get

$$\begin{aligned} \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \nu^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1} \\ \geq \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_0|^2 \,\mathrm{d}x + \int_{\Gamma} g(x,V_0)\,\mathrm{d}\mathcal{H}^{n-1} + \int_0^t \langle \nabla \overline{u}_k(s), \nabla \dot w(s)\rangle_{L^2}\,\mathrm{d}s - \overline{\eta}_k\,, \end{aligned}$$

where

$$\overline{\eta}_k := \frac{1}{2}\bigg(\max_{1\leq h\leq k} \int_{s_k^{h-1}}^{s_k^h} \|\nabla \dot w(s)\|_{L^2}\,\mathrm{d}s\bigg)\bigg(\int_0^T \|\nabla \dot w(s)\|_{L^2}\,\mathrm{d}s\bigg)\,.$$

In order to infer (4.51), we notice that by Lemma 4.4.9 we have $\overline{u}^k(s) \rightharpoonup u(s)$ for almost every $s \in [0,t]$, and therefore

$$\lim_{k\to\infty} \int_0^t \langle \nabla \overline{u}^k(s), \nabla \dot{w}(s)\rangle_{L^2} \, \mathrm{d}s = \int_0^t \langle \nabla u(s), \nabla \dot{w}(s)\rangle_{L^2} \, \mathrm{d}s \,,$$

by the Dominated Convergence Theorem. This concludes the proof of (EBY) and of Theorem 4.4.4.

Approximation of the evolution and continuity for almost every time. Thanks to (EBY), we prove the convergence of the approximating evolutions (4.35) and we improve Lemma 4.4.9.

Proposition 4.4.11. *We have*

$$u_k(t) \to u(t) \quad \textit{strongly in } H^1(\Omega \setminus \Gamma)$$

on the whole sequence (independent of t) such that (4.31)–(4.33) *hold.*

Proof. By (4.31) and (4.42), for every $t \in [0,T]$ we have

$$\begin{aligned} &\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \, \mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu^x(t)\rangle \, \mathrm{d}\mathcal{H}^{n-1} \\ &\qquad \le \liminf_{k\to\infty} \left[\frac{1}{2}\int_\Omega |\nabla u_k(t)|^2 \, \mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu^x_k(t)\rangle \, \mathrm{d}\mathcal{H}^{n-1}\right]. \end{aligned} \tag{4.53}$$

On the other hand, by (4.50), (EBY), and $(\mathrm{EIY})_k$ we get

$$\begin{aligned} &\limsup_{k\to\infty} \left[\frac{1}{2}\int_\Omega |\nabla u_k(t)|^2 \, \mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu^x_k(t)\rangle \, \mathrm{d}\mathcal{H}^{n-1}\right] \\ &\le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_0|^2 \, \mathrm{d}x + \int_\Gamma g(x, V_0) \, \mathrm{d}\mathcal{H}^{n-1} + \int_0^t \langle \nabla u(s), \nabla \dot{w}(s)\rangle_{L^2} \, \mathrm{d}s \\ &= \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \, \mathrm{d}x + \int_\Gamma \langle g(x,\cdot), \nu^x(t)\rangle \, \mathrm{d}\mathcal{H}^{n-1}. \end{aligned} \tag{4.54}$$

Thus all inequalities in (4.53) and (4.54) are equalities. Since

$$\int_\Gamma \langle g(x,\cdot), \nu^x_k(t)\rangle \, \mathrm{d}\mathcal{H}^{n-1} \to \int_\Gamma \langle g(x,\cdot), \nu^x(t)\rangle \, \mathrm{d}\mathcal{H}^{n-1},$$

we have $\|\nabla u_k(t)\|_{L^2} \to \|\nabla u(t)\|_{L^2}$. Thanks to (4.42), this concludes the proof. □

Proposition 4.4.12. *There exists a countable set $E \subset [0,T]$ such that for every $t \in [0,T]\setminus E$*

$$u(s) \to u(t) \quad \textit{strongly in } H^1(\Omega \setminus \Gamma)\,, \tag{4.55}$$

$$\nu(s) \rightharpoonup \nu(t) \quad \textit{narrowly in } \mathcal{Y}(\Gamma; [0,\infty])\,. \tag{4.56}$$

as $s \to t$.

Proof. By (EBY) we have for every $s,t \in [0,T]$

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \nu^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1} = \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(s)|^2 \,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \nu^x(s)\rangle \,\mathrm{d}\mathcal{H}^{n-1} + \int_s^t \langle \nabla u(r), \nabla \dot{w}(r)\rangle_{L^2} \,\mathrm{d}r\,.$$

Thus, if t is a continuity point for the nondecreasing function $s \mapsto \nu(s)$, we have $\|\nabla u(s)\|_{L^2} \to \|\nabla u(t)\|_{L^2}$ as $s \to t$, since $r \mapsto \langle \nabla u(r), \nabla w(r)\rangle_{L^2}$ is in $L^1([0,T])$ by (4.2) and (4.43). By Lemma 4.4.9, this gives the desired convergence. □

4.5 Quasistatic evolution in the setting of functions

This section is devoted to the proof of Theorem 4.2.9. Besides, we also give a proof of Proposition 4.2.7 and of Proposition 4.2.8 regarding the strong formulation of the quasistatic evolution. Finally, we mention some possible extensions of the model studied in this chapter.

In Section 4.4 we have shown the existence of a quasistatic evolution $(u(t), \nu(t))$ in the sense of Young measures. We will now exploit the concavity of $g(x,\cdot)$ to prove that the very same displacement $t \mapsto u(t)$ found in Section 4.4 is also a quasistatic evolution in the sense of Definition 4.2.6. We recall that $g(x,\cdot)$ is strictly increasing in the interval $[0,\theta(x)]$, where $\theta(x)$ is the threshold defined in (4.6). This allows us to prove that the Young measure $\nu(t)$ truncated by θ (see (1.21) for the definition) is actually concentrated on $V_u(t) \wedge \theta$, i.e., $V_u(t) \wedge \theta$ is the limit of $V_k(t) \wedge \theta$.

Proof of Theorem 4.2.9. By Theorem 4.4.4 and Proposition 4.4.11, we know that there exists a quasistatic evolution in the sense of Young measures $t \mapsto (u(t), \nu(t))$ such that, for every $t \in [0,T]$, we have (4.12) and

$$\delta_{V_k(t)} = \nu_k(t) \rightharpoonup \nu(t) \quad \text{in } \mathcal{Y}(\Gamma; [0,\infty])\,, \tag{4.57}$$

up to a subsequence independent of t (not relabelled).

In order to prove (GS), we first prove that

$$\nu(t) \succeq \delta_{V_u(t)} \quad \text{for every } t \in [0,T]\,. \tag{4.58}$$

By definition of $V_u(t)$ and Remark 1.5.14, it is enough to show that for any partition P of $[0,t]$, $P = \{0 = s_0 < s_1 < \dots < s_{j-1} < s_j = t\}$, we have

$$\nu(t) \succeq \delta_{V^P(t)}\,, \tag{4.59}$$

where

$$V^P(t) := V_0 + \sum_{i=1}^{j} \big|[u(s_i)] - [u(s_{i-1})]\big|\,.$$

The irreversibility condition (IRY) satisfied by $s \mapsto \nu(s)$ yields

$$\nu(s_i) \succeq \nu(s_{i-1}) \oplus \big|[u(s_i)] - [u(s_{i-1})]\big| \quad \text{for } i = 1, \dots, j\,. \tag{4.60}$$

Employing (4.60) inductively, we obtain the chain of inequalities

$$
\begin{aligned}
\nu(t) = \nu(s_j) &\succeq \nu(s_{j-1}) \oplus \big|[u(s_j)] - [u(s_{j-1})]\big| \\
&\succeq \nu(s_{j-2}) \oplus \big(\big|[u(s_{j-1})] - [u(s_{j-2})]\big| + \big|[u(s_j)] - [u(s_{j-1})]\big|\big) \succeq \dots \\
&\succeq \nu(s_1) \oplus \sum_{i=2}^{j} \big|[u(s_i)] - [u(s_{i-1})]\big| \\
&\succeq \nu(0) \oplus \sum_{i=1}^{j} \big|[u(s_i)] - [u(s_{i-1})]\big| = \delta_{V^P(t)} ,
\end{aligned}
$$

and thus (4.58) holds true.

Recalling the definition of cumulative distribution function (1.16), for every $\xi < V_u(t;x)$ we have $F_{\delta_{V_u(t;x)}}(\xi) = 0$. Thus, by (ii) in Definition 1.5.11, we deduce that

$$
\operatorname{supp} \nu^x(t) \subset [V_u(t;x), \infty] \tag{4.61}
$$

for every $t \in [0,T]$ and for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$.

We are now in a position to prove that $t \mapsto u(t)$ satisfies the global stability condition (GS). We start by fixing $t \in [0,T]$ and $\hat{u} \in \mathcal{Adm}(w(t))$, and by setting

$$
\hat{\nu} := \nu(t) \oplus \big|[\hat{u}] - [u(t)]\big| . \tag{4.62}
$$

Condition (GSY) for $t \mapsto (u(t), \nu(t))$ gives

$$
\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \nu^x(t)\rangle \,\mathrm{d}\mathcal{H}^{n-1} \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \hat{u}|^2 \,\mathrm{d}x + \int_{\Gamma} \langle g(x,\cdot), \hat{\nu}^x\rangle \,\mathrm{d}\mathcal{H}^{n-1},
$$

and thus (GS) follows if we show that

$$
\begin{aligned}
&\int_{\Gamma} \Big(\langle g(x,\cdot), \hat{\nu}^x\rangle - \langle g(x,\cdot), \nu^x(t)\rangle\Big) \,\mathrm{d}\mathcal{H}^{n-1} \\
&\qquad \le \int_{\Gamma} \Big(g\big(x,\, V_u(t) + \big|[\hat{u}]-[u(t)]\big|\big) - g\big(x, V_u(t)\big)\Big) \,\mathrm{d}\mathcal{H}^{n-1}.
\end{aligned} \tag{4.63}
$$

In order to prove (4.63), notice that by (4.61) and (4.62) we have

$$
\begin{aligned}
\langle g(x,\cdot), \hat{\nu}^x\rangle - \langle g(x,\cdot), \nu^x(t)\rangle &= \int_{[0,\infty]} \Big(g\big(x,\, \xi + \big|[\hat{u}(x)]-[u(t;x)]\big|\big) - g(x,\xi)\Big)\nu^x(t)(\mathrm{d}\xi) \\
&= \int_{[V_u(t;x),\infty]} \Big(g\big(x,\, \xi + \big|[\hat{u}(x)]-[u(t;x)]\big|\big) - g(x,\xi)\Big)\nu^x(t)(\mathrm{d}\xi) ,
\end{aligned} \tag{4.64}
$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$. Since $g(x,\cdot)$ is a concave function, for every $\xi \ge V_u(t;x)$ it holds

$$
g\big(x,\, \xi + \big|[\hat{u}(x)]-[u(t;x)]\big|\big) - g(x,\xi) \le g\big(x,\, V_u(t;x) + \big|[\hat{u}(x)]-[u(t;x)]\big|\big) - g\big(x, V_u(t;x)\big) . \tag{4.65}
$$

Let us observe that the right hand side in the inequality above does not depend on ξ. Therefore, by (4.64), (4.65), and recalling that $\nu^x(t)$ is a probability measure for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$, we deduce (4.63). This completes the proof of (GS).

Let us now prove that $t \mapsto u(t)$ satisfies (EB). Arguing as in the proof of (4.51), using (GS) it is possible to see that

$$\begin{aligned}\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 \,\mathrm{d}x + \int_\Gamma g\big(x, V_u(t)\big)\,\mathrm{d}\mathcal{H}^{n-1} \\ \geq \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_0|^2 \,\mathrm{d}x + \int_\Gamma g\big(x, V_0\big)\,\mathrm{d}\mathcal{H}^{n-1} + \int_0^t \langle \nabla u(s), \nabla \dot w(s)\rangle_{L^2}\,\mathrm{d}s\,.\end{aligned}$$

On the other hand, the opposite inequality follows immediately from (EBY) since by (4.58) we have

$$\int_\Gamma g\big(x, V_u(t)\big)\,\mathrm{d}\mathcal{H}^{n-1} \leq \int_\Gamma \langle g(x,\cdot), \nu^x(t)\rangle\,\mathrm{d}\mathcal{H}^{n-1}.$$

Therefore, $t \mapsto u(t)$ is a quasistatic evolution in the sense of Definition 4.2.6.

We now claim that the truncation $\mathcal{T}^\theta_\#\nu(t)$ (see (1.21) for the definition) is concentrated on $V_u(t)\wedge\theta$. To this end, we compare (EB) and (EBY), and deduce that for every $t \in [0,T]$

$$\int_\Gamma g\big(x, V_u(t)\big)\,\mathrm{d}\mathcal{H}^{n-1} = \int_\Gamma \langle g(x,\cdot), \nu^x(t)\rangle\,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.66}$$

Since by (4.58) and Definition 1.5.11 we have $g\big(x, V_u(t;x)\big) \leq \langle g(x,\cdot), \nu^x(t)\rangle$, equality (4.66) implies that

$$g\big(x, V_u(t;x)\big) = \langle g(x,\cdot), \nu^x(t)\rangle \tag{4.67}$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$. Let us now fix t and let x be such that (4.61) holds. To prove the claim, we need to show that if $V_u(t;x) < \theta(x)$, then $\nu^x(t)\big((V_u(t;x),\infty]\big) = 0$. Let us assume, on the contrary, that $\nu^x(t)\big((V_u(t;x),\infty]\big) = c \in (0,1]$. By (4.61) we know that

$$\langle g(x,\cdot), \nu^x(t)\rangle = g(x, V_u(t;x))(1-c) + \int_{(V_u(t;x),\infty]} g(x,\xi)\,\nu^x(t)(\mathrm{d}\xi)\,,$$

and thus

$$\langle g(x,\cdot), \nu^x(t)\rangle - g\big(x, V_u(t;x)\big) = \int_{(V_u(t;x),\infty]} \Big(g(x,\xi) - g\big(x, V_u(t;x)\big)\Big)\,\nu^x(t)(\mathrm{d}\xi)\,. \tag{4.68}$$

Since $g(x,\cdot)$ is strictly increasing in $[0,\theta(x)]$ and $\nu^x(t)\,((V_u(t;x),\infty]) > 0$, we get that the right-hand side in (4.68) is strictly positive. This contradicts (4.67), and therefore we have proved that $\mathcal{T}^\theta_\#\nu(t)$ is concentrated on $V_u(t)\wedge\theta$.

Eventually, using also (4.57) and Remark 1.5.10, we deduce that

$$\delta_{V_k(t)\wedge\theta} = \mathcal{T}^\theta_\#\nu_k(t) \rightharpoonup \mathcal{T}^\theta_\#\nu(t) = \delta_{V_u(t)\wedge\theta} \quad \text{in } \mathcal{Y}(\Gamma;[0,\infty])\,. \tag{4.69}$$

By Proposition 1.5.6, (4.69) is equivalent to (4.13).

As for the proof of (4.14) and (4.15), we notice that by Proposition 4.4.12 there exists a set E, at most countable, such that we have (4.14) and $\nu(s) \rightharpoonup \nu(t)$ in $\mathcal{Y}(\Gamma;[0,\infty])$, for $t \in [0,T] \setminus E$ and $s \to t$. The convergence in (4.15) then follows with an argument analogous to the one used to show (4.13).

This concludes the proof. $\square$

Remark 4.5.1. In the proof of Theorem 4.2.9, we have shown that $\mathcal{T}^{\theta}_{\#}\nu(t) = \delta_{V_u(t)\wedge\theta}$. In particular, this allows us to rewrite the set $\Gamma_N(t)$ introduced in (4.37) (corresponding to the part of Γ where the material is completely fractured) in terms of the variation of the jumps $V_u(t)$ and the threshold θ. Namely, we have

$$\Gamma_N(t) = \{x \in \Gamma \ : \ V_u(t;x) \geq \theta(x)\} .$$

We now give the proof of the results concerning the strong formulation of the quasistatic evolution discussed in Section 4.2. The derivation of the Euler-Lagrange conditions follows by standard arguments illustrated below.

Proof of Proposition 4.2.7. Let consider the set $\Gamma_N(t) = \{x \in \Gamma \ : \ V_u(t;x) \geq \theta(x)\}$. Let $\psi \in H^1(\Omega \setminus \Gamma)$ with $\psi = 0$ on $\partial_D\Omega$ and let $\varepsilon \in \mathbb{R}$. Since

$$\int_{\Gamma_N(t)} g\big(x, V_u(t)\big)\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\Gamma_N(t)} \kappa(x)\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\Gamma_N(t)} g\big(x,\, V_u(t) + \big|\varepsilon[\psi]\big|\big)\,\mathrm{d}\mathcal{H}^{n-1}$$

and $u(t) + \varepsilon\psi \in \mathcal{A}dm(w(t))$, by (GS) we have

$$\begin{aligned}\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2\,\mathrm{d}x + &\int_{\Gamma\setminus\Gamma_N(t)} g\big(x, V_u(t)\big)\,\mathrm{d}\mathcal{H}^{n-1} \\ &\leq \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t) + \varepsilon\nabla\psi|^2\,\mathrm{d}x + \int_{\Gamma\setminus\Gamma_N(t)} g\big(x,\, V_u(t) + \big|\varepsilon[\psi]\big|\big)\,\mathrm{d}\mathcal{H}^{n-1} .\end{aligned}$$

Since g is of class $\mathcal{C}^1$, differentiating the previous inequality with respect to ε for $\varepsilon > 0$ and $\varepsilon < 0$, we get

$$-\int_{\Gamma\setminus\Gamma_N(t)} g'\big(x, V_u(t)\big)\big|[\psi]\big|\,\mathrm{d}\mathcal{H}^{n-1} \leq \int_{\Omega\setminus\Gamma} \nabla u(t)\cdot\nabla\psi\,\mathrm{d}x \leq \int_{\Gamma\setminus\Gamma_N(t)} g'\big(x, V_u(t)\big)\big|[\psi]\big|\,\mathrm{d}\mathcal{H}^{n-1}.$$

Using the fact that $g'(x,\xi) = 0$ for $\xi \geq \theta(x)$, we also get

$$-\int_{\Gamma} g'\big(x, V_u(t)\big)\big|[\psi]\big|\,\mathrm{d}\mathcal{H}^{n-1} \leq \int_{\Omega\setminus\Gamma} \nabla u(t)\cdot\nabla\psi\,\mathrm{d}x \leq \int_{\Gamma} g'\big(x, V_u(t)\big)\big|[\psi]\big|\,\mathrm{d}\mathcal{H}^{n-1}. \quad (4.70)$$

By (4.70) for arbitrary $\psi \in H^1(\Omega)$ with $\psi = 0$ on $\partial_D\Omega$ and $\psi = 0$ in Ω^-, we infer that $\Delta u(t) = 0$ in Ω^+ and $\partial_\nu u(t) = 0$ in $H^{-\frac{1}{2}}(\partial_N\Omega \cap \partial\Omega^+)$. With similar arguments, we obtain analogous properties in Ω^- and we eventually deduce (i).

Let us prove (ii). Since ν_Γ is chosen in such a way that it coincides with the outer normal to $\partial\Omega^-$, by definition of normal derivative of the function $u(t)^+ = u(t)|_{\Omega^+}$ on Γ we have that $\partial_\nu u(t)^+ \in H^{-\frac{1}{2}}(\Gamma)$ is given by

$$\langle \partial_\nu u(t)^+, \psi^+ \rangle = -\int_{\Omega^+} \nabla u(t)\cdot\nabla\psi^+\,\mathrm{d}x ,$$

for every $\psi^+ \in H^1(\Omega^+)$ with $\psi^+ = 0$ on $\partial_D\Omega \cap \partial\Omega^+$. Similarly, the normal derivative $\partial_\nu u(t)^- \in H^{-\frac{1}{2}}(\Gamma)$ is given by

$$\langle \partial_\nu u(t)^-, \psi^- \rangle = \int_{\Omega^-} \nabla u(t) \cdot \nabla \psi^- \,\mathrm{d}x\,,$$

for every $\psi^- \in H^1(\Omega^-)$ with $\psi^- = 0$ on $\partial_D\Omega \cap \partial\Omega^-$. Hence, by testing (4.70) with functions $\psi \in H^1(\Omega \setminus \Gamma)$ with $\psi = 0$ on $\partial_D\Omega$ and $[\psi] = 0$ on Γ, we infer

$$-\langle \partial_\nu u(t)^+, \psi \rangle + \langle \partial_\nu u(t)^-, \psi \rangle = 0\,,$$

which implies (ii) by the arbitrariness of ψ.

In order to prove (iii), we note that since $g'(x,\xi) \le g'(x,0)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$ and for every $\xi \in [0,\infty]$, by inequality (4.70) we get

$$\big|\langle \partial_\nu u(t), [\psi] \rangle\big| \le \|g'(\cdot,0)\|_{L^\infty} \|[\psi]\|_{L^1}\,,$$

for every $\psi \in H^1(\Omega \setminus \Gamma)$ with $\psi = 0$ on $\partial_D\Omega$. Thus $\partial_\nu u(t)$ is a linear and continuous operator on the space $X := \{[\psi] \,:\, \psi \in H^1(\Omega\setminus\Gamma) \text{ such that } \psi = 0 \text{ on } \partial_D\Omega\}$. By density of X in $L^1(\Gamma)$, this implies that $\partial_\nu u(t)$ can be extended to a linear and continuous operator on $L^1(\Gamma)$, and hence $\partial_\nu u(t) \in L^\infty(\Gamma)$. From (4.70) we deduce that

$$-\int_\Gamma g'\big(x, V_u(t)\big)|z| \,\mathrm{d}\mathcal{H}^{n-1} \le -\int_\Gamma \partial_\nu u(t) z \,\mathrm{d}\mathcal{H}^{n-1} \le \int_\Gamma g'\big(x, V_u(t)\big)|z| \,\mathrm{d}\mathcal{H}^{n-1},$$

for every $z \in L^1(\Gamma)$. This concludes the proof of (iii). □

In order to give a proof of Proposition 4.2.8, we need to prove the following lemma regarding the differentiability in time of the essential variation of a function that is absolutely continuous in time with values in $L^2(\Gamma)$.

Lemma 4.5.2. *Let* $\gamma \in AC([0,T]; L^2(\Gamma))$. *Then* $\operatorname{ess\,Var}(\gamma;0,\cdot) \in AC([0,T]; L^2(\Gamma))$ *and*

$$\lim_{s\to t} \frac{\operatorname{ess\,Var}(\gamma;s,t)}{t-s}(x) = |\dot\gamma(t;x)| \quad \textit{for } \mathcal{H}^{n-1}\textit{-a.e. } x \in \Gamma \textit{ and for a.e. } t \in [0,T]\,, \tag{4.71}$$

where the limit and the derivative $\dot\gamma$ *are defined with respect to the strong topology in* $L^2(\Gamma)$.

Proof. We fix $s,t \in [0,T]$ with $s < t$ and we consider a partition of the interval $[s,t]$, namely $s = s_0 < \dots < s_j = t$. By the absolute continuity of γ, for every $i = 1,\dots,j$ we have

$$|\gamma(s_i;x) - \gamma(s_{i-1};x)| = \left| \int_{s_{i-1}}^{s_i} \dot\gamma(\tau;x)\,\mathrm{d}\tau \right| \le \int_{s_{i-1}}^{s_i} \big|\dot\gamma(\tau;x)\big|\,\mathrm{d}\tau \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma\,,$$

where the integrals are Bochner integrals and $\dot\gamma(\tau)$ is the derivative in $L^2(\Gamma)$ of $\gamma(\tau)$. Summing up the previous inequalities for $i = 1,\dots,j$, we obtain

$$\sum_{i=1}^{j} |\gamma(s_i;x) - \gamma(s_{i-1};x)| \le \int_s^t |\dot\gamma(\tau;x)|\,\mathrm{d}\tau \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma\,. \tag{4.72}$$

By Definition 4.2.3, inequality (4.72) implies that

$$\operatorname{ess\,Var}(\gamma; s, t)(x) \leq \int_s^t |\dot{\gamma}(\tau; x)| \, \mathrm{d}\tau \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma . \tag{4.73}$$

In particular, choosing $s = 0$ in (4.73) we deduce that $\operatorname{ess\,Var}(\gamma; 0, t)$ belongs to $L^2(\Gamma)$, for every $t \in [0, T]$. By taking the L^2 norm in (4.73) we infer

$$\| \operatorname{ess\,Var}(\gamma; s, t) \|_{L^2} \leq \int_s^t \| \dot{\gamma}(\tau) \|_{L^2} \, \mathrm{d}\tau .$$

Since the function $\tau \mapsto \|\dot{\gamma}(\tau)\|_{L^2}$ belongs to $L^1([0, T]; \mathbb{R})$, we conclude that the function $\operatorname{ess\,Var}(\gamma; 0, \cdot)$ belongs to $AC([0, T]; L^2(\Gamma))$.

We now compute the derivative of $\operatorname{ess\,Var}(\gamma; 0, \cdot)$. Since $\frac{1}{t-s} \int_s^t |\dot{\gamma}(\tau)| \, \mathrm{d}\tau \to |\dot{\gamma}(t)|$ strongly in $L^2(\Gamma)$ as $s \to t$, dividing all terms in (4.73) by $t - s$ and letting $s \to t$ we deduce that

$$\lim_{s \to t} \frac{\operatorname{ess\,Var}(\gamma; s, t)}{t - s}(x) \leq |\dot{\gamma}(t; x)|$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$. On the other hand, since $\{s, t\}$ is a particular partition of the interval $[s, t]$, by definition of essential variation we have

$$|\gamma(t; x) - \gamma(s; x)| \leq \operatorname{ess\,Var}(\gamma; s, t)(x) ,$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$. Dividing by $t - s$ and letting $s \to t$ in the inequality above, we obtain (4.71). □

We are now in a position to prove Proposition 4.2.8.

Proof of Proposition 4.2.8. Since by assumption $u \in AC([0, T]; H^1(\Omega \setminus \Gamma))$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega \setminus \Gamma} |\nabla u(t)|^2 \, \mathrm{d}x = \int_{\Omega \setminus \Gamma} \nabla u(t) \cdot \nabla \dot{u}(t) \, \mathrm{d}x . \tag{4.74}$$

Moreover we claim that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma} g(x, V_u(t)) \, \mathrm{d}\mathcal{H}^{n-1} = \int_{\Gamma} g'\big(x, V_u(t)\big) \big|[\dot{u}(t)]\big| \, \mathrm{d}\mathcal{H}^{n-1}. \tag{4.75}$$

Let us prove (4.75). The absolute continuity of u implies that $[u] \in AC([0, T]; L^2(\Gamma))$. Let us consider the set $\Gamma_N(0) = \{x \in \Gamma \ : \ V_0(x) \geq \theta(x)\}$. Thanks to Lemma 4.5.2 and by the definition (4.3) of $V_u(t)$, for every $t \in [0, T]$ we have $V_u(t; x) < \infty$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma \setminus \Gamma_N(0)$. Then, since $g(x, \xi) = \kappa(x)$ for $\xi \in [\theta(x), \infty]$, since $g(x, \cdot)$ is monotone, and since $V_u(t)$ is monotone in t,

$$\int_{\Gamma} \frac{g\big(x, V_u(t + h)\big) - g\big(x, V_u(t)\big)}{h} \, \mathrm{d}\mathcal{H}^{n-1} = \int_{\Gamma \setminus \Gamma_N(0)} \frac{g\big(x, V_u(t + h)\big) - g\big(x, V_u(t)\big)}{h} \, \mathrm{d}\mathcal{H}^{n-1}.$$

Since $V_u(t + h; x) - V_u(t; x) = \operatorname{ess\,Var}([u]; t, t + h)(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma \setminus \Gamma_N(0)$ and $g'(x, V_u(t; x)) = 0$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma_N(0)$, by taking the limit as $h \to 0^+$

in the previous equality, by Lemma 4.5.2, and since g is of class $\mathcal{C}^1$, we eventually deduce (4.75).

The equalities (4.74) and (4.75) combined with (EB) imply that

$$\int_{\Omega\setminus\Gamma} \nabla u(t)\cdot\nabla(\dot u(t)-\dot w(t))\,\mathrm{d}x + \int_\Gamma g'(x,V_u(t))\big|[\dot u(t)]\big|\,\mathrm{d}\mathcal{H}^{n-1} = 0\,.$$

Since $\dot u(t)-\dot w(t)=0$ on $\partial_D\Omega$, by definition of $\partial_\nu u(t)$ we obtain

$$\int_\Gamma \partial_\nu u(t)[\dot u(t)]\,\mathrm{d}\mathcal{H}^{n-1} = \int_\Gamma g'(x,V_u(t))\big|[\dot u(t)]\big|\,\mathrm{d}\mathcal{H}^{n-1},$$

and thus

$$\int_{\{[\dot u(t)]\neq 0\}} \big(g'(x,V_u(t))\,\mathrm{Sign}([\dot u(t)]) - \partial_\nu u(t)\big)[\dot u(t)]\,\mathrm{d}\mathcal{H}^{n-1} = 0\,.$$

By (iii) in Proposition 4.2.7, this proves the claim. □

Remark 4.5.3. The method presented in this chapter can be used to treat other models where the system exhibits asymmetric responses to loading and unloading. For instance, we can study a model where some energy is dissipated when the jump $[u(t)]$ increases in time, while no energy is dissipated when $[u(t)]$ decreases. Notice that here we are considering $[u(t)]$ and not $|[u(t)]|$. Accordingly, Definition 4.2.6 is modified by replacing the total variation $V_u(t)$ of $t\mapsto[u(t)]$ by its positive variation given by

$$P_u(t) = \operatorname{ess\,sup}\Big\{\sum_{i=1}^{j}\big[\big([u(s_i)]-[u(s_{i-1})]\big)\vee 0\big]\Big\}\,.$$

where the essential supremum is taken among all $j\in\mathbb{N}$ and all partitions $t_1=s_0<s_1<\cdots<s_{j-1}<s_j=t_2$ of the interval $[t_1,t_2]$. Specifically, the dissipation is given by

$$\int_\Gamma g\big(x,P_u(t)\big)\,\mathrm{d}\mathcal{H}^{n-1}.$$

For a related model, see [15, Section 5.2].

In order to prove the existence of a quasistatic evolution in this case, we pass to the limit in the approximate evolutions obtained by solving the incremental minimisation problems

$$u_k^i \in \underset{u\in\mathcal{A}dm(w(t_k^i))}{\operatorname{argmin}}\Big\{\frac12\int_\Omega|\nabla u|^2\,\mathrm{d}x + \int_\Gamma g\Big(x,\ P_k^{i-1}+\big([u]-[u_k^{i-1}]\big)\vee 0\Big)\,\mathrm{d}\mathcal{H}^{n-1}\Big\},$$
$$P_k^i := P_k^{i-1}+\big([u_k^i]-[u_k^{i-1}]\big)\vee 0\,.$$

As above, we denote by $u_k(t)$ and $P_k(t)$ the piecewise constant interpolations of u_k^i and P_k^i, respectively.

Here we do not provide the details of the proof, which follows the lines of the proof of Theorem 4.2.9. Let us only mention that, in order to pass to the limit in the irreversibility relation

$$P_k(t)\geq P_k(s)+\big([u_k(t)]-[u_k(s)]\big)\vee 0 \quad \text{for any } s\leq t\,,$$

we rewrite it as a system of two inequalities

$$P_k(t) - [u_k(t)] \geq P_k(s) - [u_k(s)]\,,$$
$$P_k(t) \geq P_k(s)\,.$$

By Helly's Selection Principle, in the limit as $k \to +\infty$, $P_k(t)$ generates a Young measure $\pi(t)$ and $P_k(t) - [u_k(t)]$ generates a Young measure $\lambda_\ominus(t)$, both nondecreasing in time. As in Section 4.4, we can pass to the limit in the two irreversibility relations and we characterise $[u(t)]$ as the limit of a subsequence of $[u_k(t)]$ extracted independently of t. At this stage of the analysis, the proof is concluded as in Section 4.5.

Other responses can be studied: for instance, one can assume that the dissipation is given by

$$\int_\Gamma g\big(x,\, \alpha\, P_u(t) + \beta\, N_u(t)\big)\, \mathrm{d}\mathcal{H}^{n-1},$$

where $\alpha, \beta > 0$ and $N_u(t)$ is the negative variation of $t \mapsto [u(t)]$.

Concluding remarks

The results presented in this thesis suggest how cohesive surface energies entail many mathematical difficulties in different situations. In this last chapter, we conclude by mentioning some possible open problems connected to the results discussed above.

In Chapter 2 we have presented a Γ-convergence result for gradient damage models coupled with plasticity. The analysis has been carried out in the simplified case of antiplane shear. We describe here the model in the case of linearised elasticity to explain what are the main difficulties arising in this general setting.

Let Ω be an open bounded set in $\mathbb{R}^n$ ($n = 3$ being the physically relevant case), representing the reference configuration of the body. The displacement of the body is a vector-valued function $u\colon \Omega \to \mathbb{R}^n$ and its symmetrised gradient Eu is is decomposed as $Eu = e + p$, where $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$ and $p \in \mathcal{M}_b(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$ are the elastic and plastic strain, respectively. The matrix p is assumed to be deviatoric (i.e., trace-free), as usual for materials which are insensitive to pressure. The stress σ is determined by the relation $\sigma = \mathbb{C}(\alpha)\, e$, where $\mathbb{C}(\alpha)$ is the elasticity tensor. Denoting by $A : B$ the scalar product between two $n\times n$ matrices A and B, we assume that $\mathbb{C}(\alpha)\, e : e$ is nondecreasing in α for every $e \in \mathbb{M}^{n\times n}_{\mathrm{sym}}$ and coercive in e for every $\alpha \in (0, 1]$. The deviatoric part of the stress $\sigma_D := \sigma - \frac{\mathrm{tr}(\sigma)}{n} I$ is constrained to lie in a convex and compact set $K(\alpha)$ of the space of deviatoric matrices $\mathbb{M}^{n\times n}_D$. The dependence on α of the constraint sets is assumed to be such that $K(\alpha_1) \subset K(\alpha_2)$, for every $\alpha_1 \leq \alpha_2$. In order to define the plastic potential, we consider the support function of $K(\alpha)$ defined by

$$H(\alpha, \xi) := \sup_{\zeta \in K(\alpha)} \xi : \zeta \,.$$

The total energy relative to a displacement $u \in BD(\Omega)$ is given by

$$\frac{1}{2} \int_\Omega \mathbb{C}(\alpha)\, e : e \,\mathrm{d}x + \int_\Omega H\Big(\widetilde{\alpha}, \frac{\mathrm{d}p}{\mathrm{d}|p|}\Big)\, \mathrm{d}|p| + \int_\Omega \Big[\frac{W(\alpha)}{\varepsilon} + \varepsilon |\nabla \alpha|^2\Big] \,\mathrm{d}x \,, \qquad \text{(C.1)}$$

for $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$ and $p \in \mathcal{M}_b(\Omega; \mathbb{M}^{n\times n}_D)$ such that $Eu = e + p$. This is a generalisation of the three-dimensional model proposed in [3, 4].

The quasistatic evolution model studied in [25, 27] is based on the minimisation of the energy defined in (C.1) (for ε fixed), with a slight different expression for the damage regularising term. Specifically, in [25, 27] the authors require a higher Sobolev regularity, which in turn implies the continuity of the damage variable. In particular, the stronger regularising term allows to prove the lower semicontinuity of the total energy. In contrast, a proof of the lower semicontinuity of (C.1) is still not available. The main reason is that the arguments of the proof of Proposition 2.2.3 cannot be repeated when

$u \in BD(\Omega)$. A first result in this direction was obtained in [26, Theorem 3.1], where damage is coupled with a strain gradient plasticity model, and thus p is not just a measure, but a matrix-valued BV function.

Some comments about the Γ-convergence analysis of (C.1) are now in order. The slicing method adopted in Chapter 2 cannot be applied in this case, since it is not possible to recover the integrands $\mathbb{C}(\alpha)\, e : e$ and $H(\alpha, p)$ from the one-dimensional slices. Instead, a blow-up technique would be more suited in this setting. Yet the latter method demands lower semicontinuity results for more general classes of free discontinuity functionals defined on $BD(\Omega)$, e.g. for those where the volume term depends on the whole matrix $\mathcal{E}u$ and not just on its norm $|\mathcal{E}u|$ (as in the discussion of Chapter 3).

An additional technical issue concerns the domain of the Γ-limit functional. In the limit process, the concentration of the damage variable causes a loss of control on jumps with large amplitude. In the antiplane setting, we proved that for this reason the vertical displacements related to the limit model belong to the set $GBV(\Omega)$ of generalised functions of bounded variation. This suggests that the limit of the functionals (C.1) should be finite for displacements belonging to the space $GBD(\Omega)$ of generalised functions of bounded deformation, introduced by DAL MASO in [30]. On account of this, we are motivated to further investigate into the structure of functions in this space.

The result obtained in Chapter 4 about the existence of a quasistatic evolution for a cohesive fracture model is quite satisfactory: indeed, despite the use of Young measures, the evolution found in Theorem 4.2.9 actually satisfies a formulation that does not require such an abstract tool. However, the technique adopted in Chapter 4 only works in the setting of a prescribed crack surface. The extension of the result to the case of free discontinuities is still an open problem; yet it is necessary in order to understand whether the Γ-convergence result obtained in Chapter 2 also holds in the evolutionary setting.

Besides, in Theorem 4.2.9 the evolution satisfies a global stability condition at every time. However, when nonconvex terms are present in the energy of the system, the global stability condition is questionable. Indeed, evolutions should rather satisfy only a stability condition, without imposing global minimality. For this reason, it would be interesting to look for other equilibria of the system, e.g. by following a vanishing viscosity approach.

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